Programming Search

How can we control the search in a finite domain programming solver

Overview

- Finite Domain Search
- Variable Selection
- Value Selection
- Splitting
- Complex Search Strategies
- Autonomous Search

Search with finite domain prop.

- search(F_0, F_n, D)

  D := isolv(F_0, F_n, D)

  if (D is a false domain) **return** *false*

  if (D is not a valuation domain)

  choose $\{c_1,..,c_m\}$ where $C \wedge D$

  implies $c_1 \vee \ldots \vee c_m$

  for (i in $1..m$)

  if (search(F_0 union F_n, prop(c_i), D))

  return *true*

  return *false*

  return *true*

Choice

- choose $\{c_1,..,c_m\}$ where
  - $C \wedge D$ implies $c_1 \vee \ldots \vee c_m$
- Usually (Labelling):
  - select a variable v
  - select a value d
  - $c_1 \approx v = d, c_2 \approx v \neq d$
- Although sometimes (Splitting):
  - $c_1 \approx v \leq d, c_2 \approx v > d$
- Rarely, something more complex
  - value set: $c_1 \approx v_1 = d, c_2 \approx v_2 = d, \ldots, c_n \approx v_n = d$
  - constraint split: $c_1 \approx v_1 = v_2, c_2 \approx v_1 \neq v_2$

Labelling in MiniZinc

- We can add solver specific information to MiniZinc models using annotations

```
include "all_different.mzn";
var 1..9: S;
var 0..9: E;
var 0..9: N;
var 0..9: D;
var 1..9: M;
var 0..9: O;
var 0..9: R;
var 0..9: Y;

constraint         1000 * S + 100 * E + 10 * N + D
                 + 1000 * M + 100 * O + 10 * R + E
               = 10000 * M + 1000 * O + 100 * N + 10 * E + Y;

constraint all_different([S,E,N,D,M,O,R,Y]);

solve satisfy;
```

- `solve :: int_search([S,E,N,D,M,O,R,Y], input_order, indomain_min, complete)`
  `satisfy;`
- Label the variables [S,E,N,D,M,O,R,Y] in order (input_order) trying the lowest value first (indomain_min), ignoring fixed variables

Labelling example

after initial propagation

S = 9, E in 4..7, N in 5..8, D in 2..8,
M = 1, O = 0, R in 2..8, Y in 2..8

E = 4

False domain

E ≠ 4

S = 9, E in 5..7, N in 6..8, D in 2..8,
M = 1, O = 0, R in 2..8, Y in 2..8

E = 5

S = 9, E = 5, N = 6, D = 7,
M = 1, O = 0, R = 8, Y = 2

E ≠ 5

S = 9, E in 6..7, N in 7..8, D in 2..8,
M = 1, O = 0, R in 2..8, Y in 2..8

E = 6

False domain

E ≠ 6

False domain

Variable selection

- int_search($vars$, var_select, $choice$, $explore$)
- Variable selection strategies
  - input_order: in the given order
  - first_fail: choose the variable v with smallest domain
  - smallest: choose the variable v with smallest value in domain
  - largest: choose the variable v with largest value in domain
  - max_regret: choose the variable v with largest difference between the two smallest values in its domain

First Fail Labelling

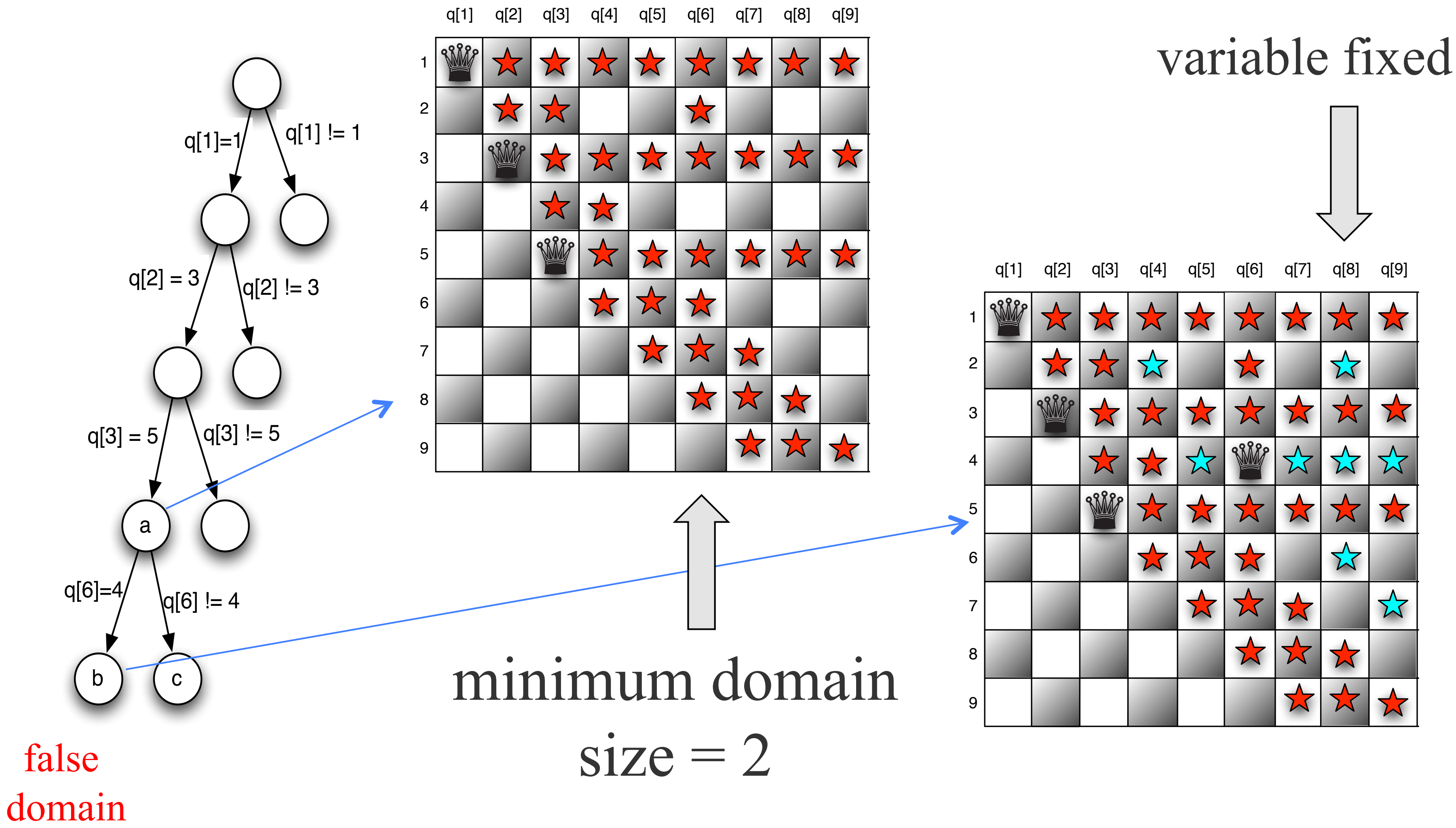
- One useful heuristic is the **first-fail principle**

  "To succeed, try first where you are most likely to fail"

- At each step choose the variable with the smallest domain.
- Do this dynamically based on the domain size after propagation.

First fail labelling: Ex. N queens

solve :: int_search(q, first_fail, indomain_min, complete) satisfy;



Regret Based Search

- max_regret: choose the variable v with largest difference between the two smallest values in its domain
- Usually tied with indomain_min
- Used when selecting to minimize costs
- pw[i] = profit from worker *I*
- max regret search
  - pw1 (regret 3)
  - pw2 (regret 4)
  - pw3 (regret 3)
- Total cost = 10

	$p1$	$p2$	$p3$	$p4$
$w1$	7	2	5	
$w2$	8		5	1
$w3$	4		7	
$w4$	3		3	

Smallest Search

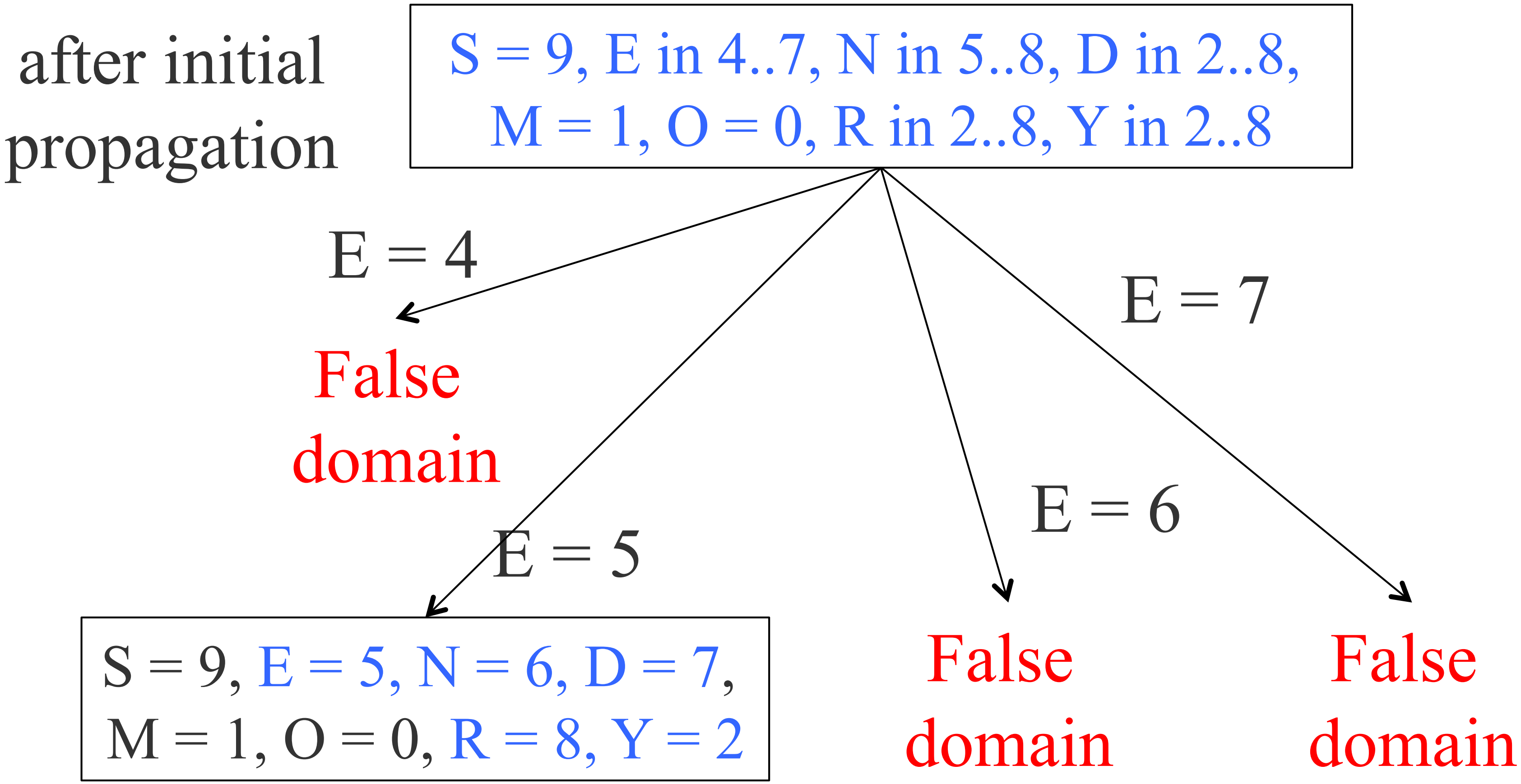
- smallest: choose the variable *v* with smallest value in its domain
- Again usually tied with indomain_min
- Used when selecting to minimize costs
- pw[i] = profit from worker *I*
- smallest search
  - pw2 (smallest 1)
  - pw4 (smallest 1)
  - pw3 (smallest 4)
- Total cost = 11

	$p1$	$p2$	$p3$	$p4$
$w1$	7		5	
$w2$	8		5	1
$w3$	4		7	
$w4$	3	1	3	

Value selection

- int_search(*vars*, *var_select*, *choice*, *explore*)
- Value selection strategies:
  - indomain_min: *d* = smallest value in domain
  - indomain_man: *d* = largest value in domain
  - indomain_median: d = median domain value
  - indomain_random: d is a random value from the domain
  - indomain: try all values in order lowest to highest
    - value set search, not a labelling search

indomain labelling example




```
solve :: int_search([S,E,N,D,M,O,R,Y], input_order, indomain, complete)
        satisfy;
```

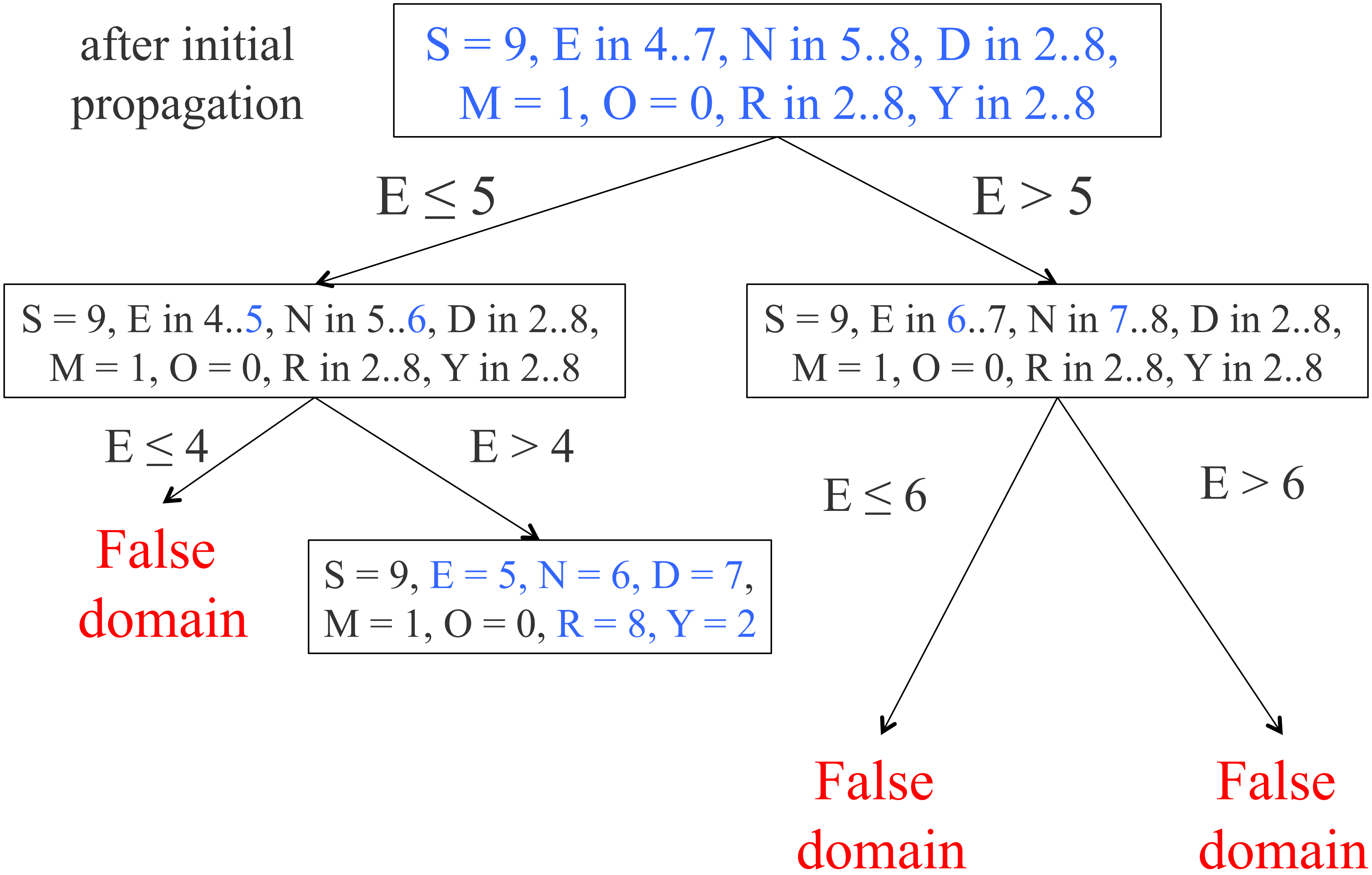
Value selection question

- What is the difference between
  - indomain, and
  - indomain_min ?

Splitting

- Particularly with strongly arithmetic variables it can be better to split the domain
- Splitting choice strategies:
  - indomain_split: $v \leq d \lor v > d$
    - where $d = (\min(D,v) + \max(D,v)) \text{ div } 2$
  - indomain_reverse_split: $v > d \lor v \leq d$
- Splitting doesn't make sense unless there are constraints that can propagate bounds

Splitting example



Search variables

- int_search(*vars*, *var_select*, *choice*, *explore*)
- The variables to be searched on are an important part of any search strategy
  - usually enough so that fixing them fixes all variables

```
include "all_different.mzn";
var 1..9: S;
var 0..9: E;
var 0..9: N;
var 0..9: D;
var 1..9: M;
var 0..9: O;
var 0..9: R;
var 0..9: Y;

var 0..1: C1;
var 0..1: C2;
var 0..1: C3;

constraint D + E = 10*C1 + Y;
constraint N + R = 10*C2 + E;
constraint E + O = 10*C3 + N;
constraint S + M = 10*M + O;

constraint all_different
      ([S,E,N,D,M,O,R,Y]);

solve :: int_search(
            [S,E,N,D,M,O,R,Y],
            input_order,
            indomain_min,
            complete)
      satisfy;
```

- The search does not need to fix the C1,C2,C3 vars
  - they are fixed when [S,E,N,D,M,O,R,Y] are fixed

Search Variables Example

allinterval problem: Find a sequence of numbers 1..n such that all the differences between adjacent numbers are also different

```
include "all_different.mzn";
int: n;
array[1..n] of var 1..n: x;       % sequence of numbers
array[1..n-1] of var 1..n-1: u;  % sequence of differences

constraint all_different(x);
constraint all_different(u)
constraint forall(i in 1..n-1)(u[i] = abs(x[i+1] - x[i])));

solve :: int_search(x, first_fail, indomain_min, complete)
     satisfy;
output ["x = ",show(x),"\n"];
```

Search on *x* variables is enough to fix *u* variables

Search Variables Example

A better search: search on which position each number is in

But how? Dual model with channeling!

```
include "inverse.mzn";
int: n;
array[1..n] of var 1..n: x;  % sequence of numbers
array[1..n-1] of var 1..n-1: u;  % sequence of differences
constraint forall(i in 1..n-1)(u[i] = abs(x[i+1] - x[i]));
array[1..n] of var 1..n: y;  % position of each number
array[1..n-1] of var 1..n-1: v; % position of difference I
constraint inverse(x,y);
constraint inverse(u,v);
constraint abs(y[1] - y[n]) = 1 /\ v[n-1] = min(y[1], y[n]); % redundant

solve :: int_search(y, first_fail, indomain_min, complete)  satisfy;

output ["x = ",show(x),"\n"];
```

For n = 10 this model requires 1714 choices for all sols vs 84598

Programming Search

- Variable selection can make a big difference
  - in size of search tree
  - The right variable order is thus very important
- Value selection just "reorders" the tree
  - moves solutions more to the left
  - "irrelevant" if finding all solutions
  - not irrelevant for optimization
    - finding good solutions early reduces search!

Comparing Searches: N Queens

- int_search(q, input_order, indomain_min, complete);
- int_search(q, input_order, indomain_median, complete);
- int_search(q, first_fail, indomain_min, complete);
- int_search(q, input_order, indomain_median, complete);

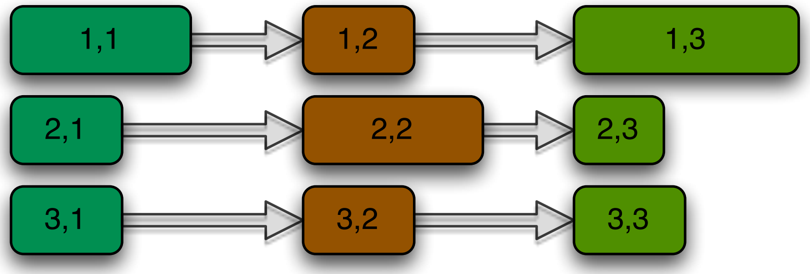
Number of choices to find first solution

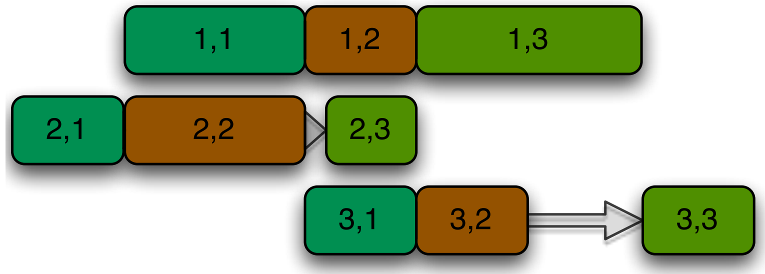
n	input-min	input-median	ff-min	ff-median
10	28	15	16	20
15	248	34	23	15
20	37330	97	114	43
25	7271	846	2637	80
30	—	385	1095	639
35	—	4831	—	240
40	—	—	—	236

Complex Searches

- Actually very many different complex search strategies have been used/defined for FD solvers
- MiniZinc only supports one complex search constructor: sequential search
  - seq_search([*search_ann*, …, *search_ann*])
- Complete the first search before starting the next one.

Jobshop scheduling






```
include "disjunctive.mzn";
int: jobs;                                                % no of jobs
int: tasks;                                               % no of tasks per job
array [1..jobs,1..tasks] of int: d;                       % task durations
int: total = sum(i in 1..jobs, j in 1..tasks) (d[i,j]);   % total duration
array [1..jobs,1..tasks] of var 0..total: s;              % start times
var 0..total: end;                                        % total end time
constraint %% ensure the tasks occur in sequence
   forall(i in 1..jobs) (  forall(j in 1..tasks-1)
                              (s[i,j] + d[i,j] <= s[i,j+1]) /\
                          s[i,tasks] + d[i,tasks] <= end      );
constraint %% ensure no overlap of tasks
   forall(j in 1..tasks) ( disjunctive([s[i,j] | i in 1..jobs], [d[i,j] | i in 1..jobs]) );
solve minimize end;
```

Jobshop search strategies

- seq_search([
    int_search([s[i.j]| i in 1..jobs, j in 1..tasks],
                    smallest, indomain_min, complete),
    int_search([end], input_order, indomain_min, complete)
  ])

  Place earliest tasks first, when finished set end to minimum time!

- seq_search([
    int_search([end], input_order, indomain_min, complete),
    int_search([s[i.j]| i in 1..jobs, j in 1..tasks],
                    smallest, indomain_min, complete)
  ])

  Optimistic search: Search for a solution with least end time, if that fails search for one higher. Search for solutions using earliest start time.

Annotations

- Annotations are how to communicate information to the solver from a MiniZinc model
  - first class object: type ann, annotation variables
  - can be defined in data files
  - you can create your own new annotations
    - annotation <ann-name> (<arg-def> .. <arg-def>)

```
ann: search;
ann: subsearch = int_search([s[i.j]| i in 1..jobs, j in 1..tasks],
                            smallest, indomain_min, complete);
solve :: search minimize end;
```

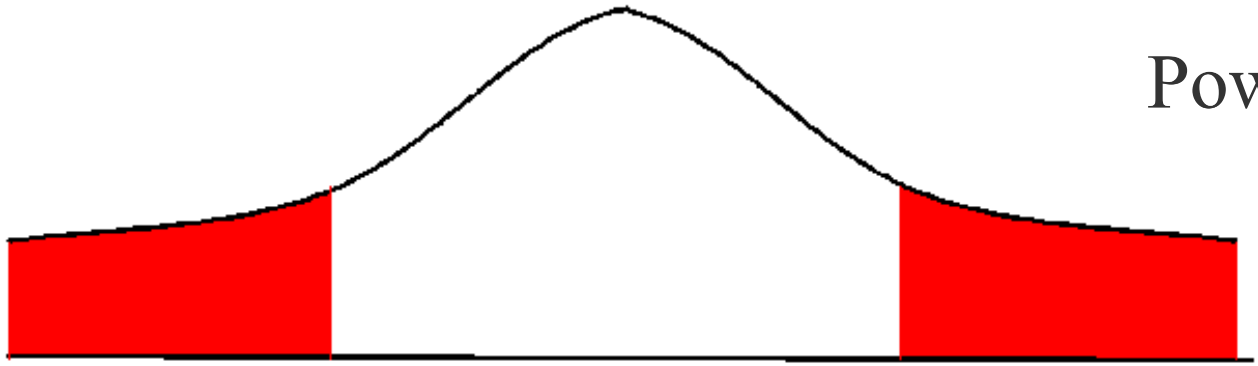
(data file 1) `search = subsearch;`

(data file 2) `search = seq_search([subsearch, int_search ([end], input_order, indomain_min, complete)]);`

Annotations apart from search

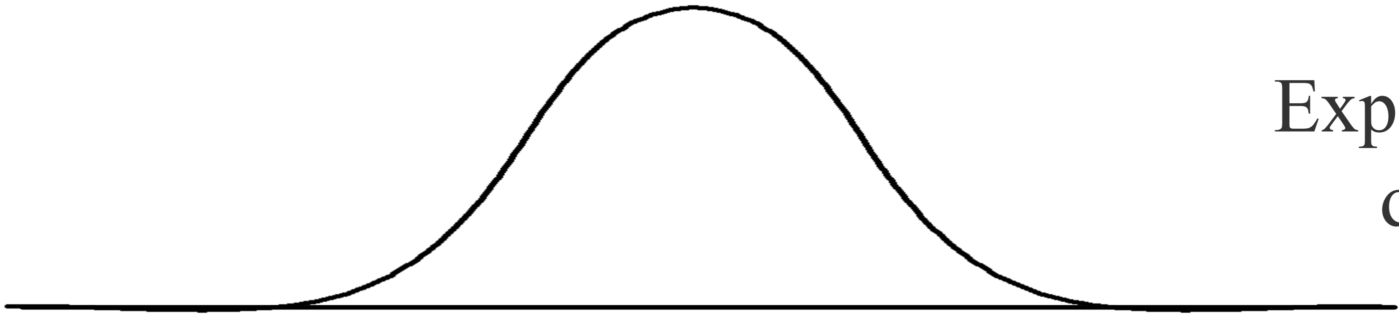
- Annotations can be used to transmit information to the solver by annotating variables and constraints
  - mzn2fzn adds annotations
    - :: is_defined_var variable is and introduced variable with defn
    - :: defines_var(x) this constraint defined variable
  - Possible variable annotations
    - :: bounds_only only store bounds for variable
    - :: bitdomain(32) store domain as bit string
  - Possible constraint annotations
    - :: bounds use bounds propagation
    - :: domain use domain propagation
- Dependent on solver, allowed to be ignored!

Restarts + Heavy tails



Power series decay

HEAVY TAILED DISTRIBUTION
(infinite mean & variance)

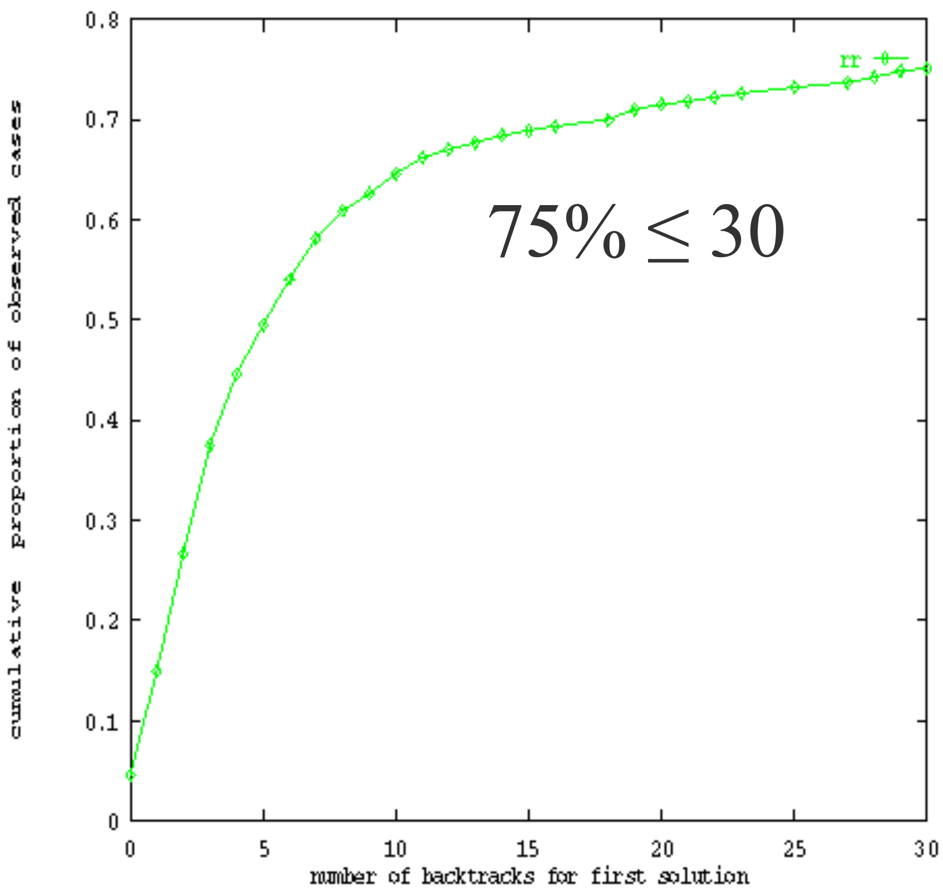


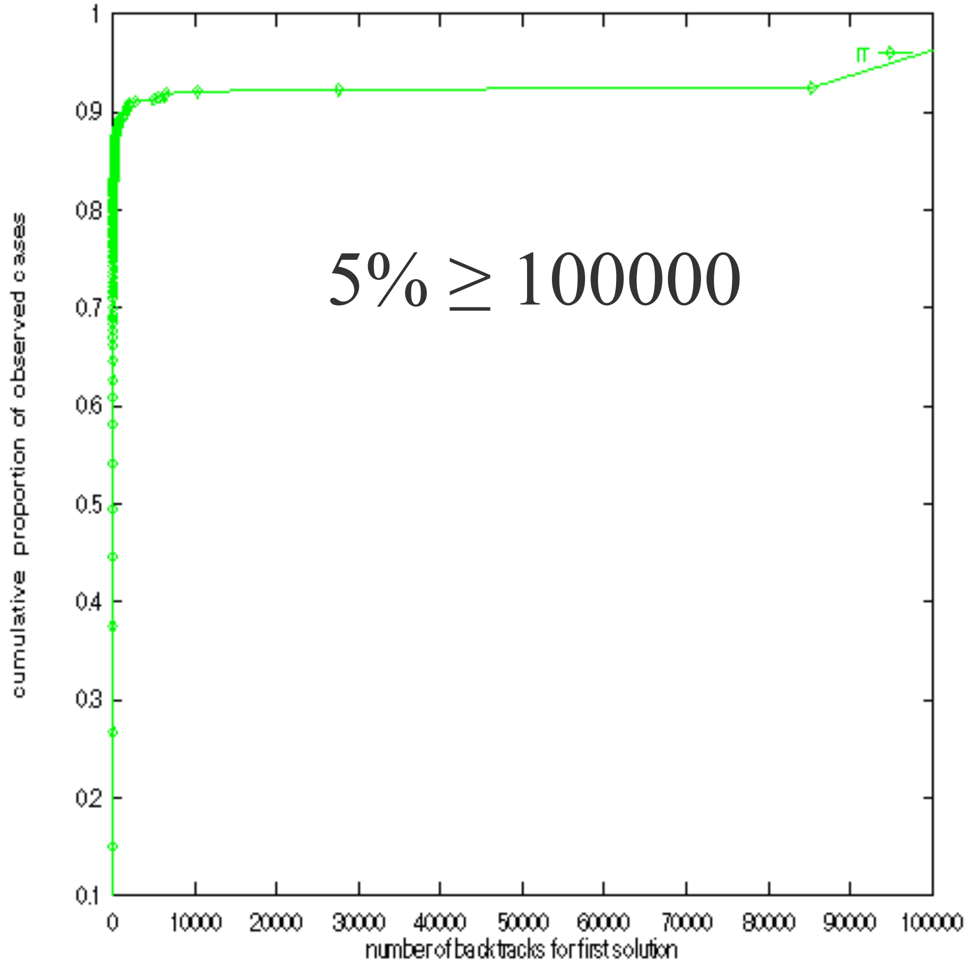
Exponential decay

Standard Distribution
(finite mean &
variance)

Heavy Tailed Behaviour

Searching for solutions to Quasigroup completion problems



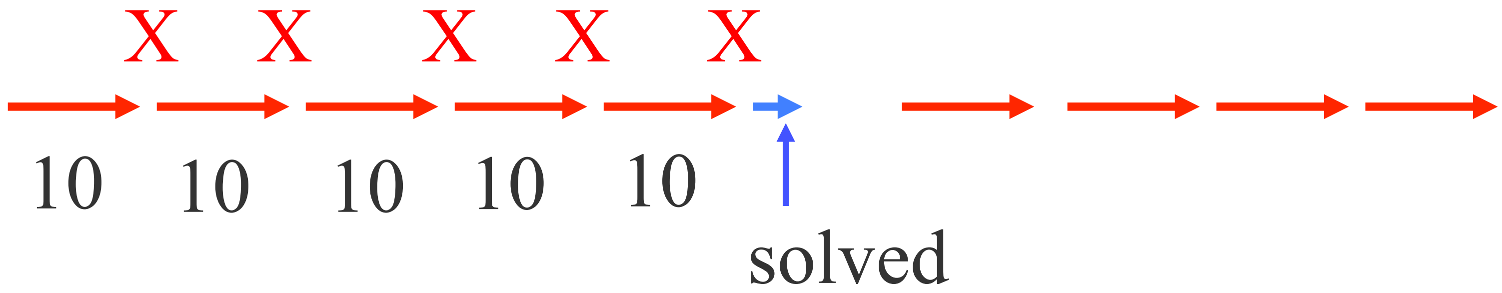


Heavy-Tailed Behavior

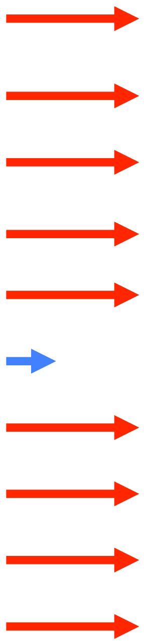
Restarts

- If 75% finish in 30 backtracks
  - after 50 backtracks why not start again
  - you might be in one of the 5% that require > 100,000
- Restarting conquers heavy tailed behaviour

Super linear speedups



Sequential: 50 +1 = 51 seconds



Parallel: 10 machines --- 1 second
51 x speedup

Interleaved (1 machine): 10 x 1 = 10 seconds
5 x speedup

Restart Strategies

Policy for when to restart

- Constant restart – after using L resources
- Geometric restart
  - restart after using L resources, with new limit αL
- Luby restart
  - 1,1,2,1,1,2,4,1,1,2,1,1,2,4,8, …
  - "universally optimal" for randomized algorithms:
    - no worse than a log factor slower than optimal policy
    - not bettered by more than a constant factor by other universal policies

Limits + Restart in MiniZinc

- Not in MiniZinc 1.1.5 (but is on slippers2 ..)
- limit(<Measure>, <Limit>, <Search>)
  - <Measure> is one of fails, solutions, nodes, time
  - <Limit> is the limit where we fail
  - <Search> is the search we limit
- Examples

```
limit(time, 10,
      int_search(x, smallest, indomain, complete)

limit(time, 600,
        seq_search([
          int_search(x,input_order,indomain_random,complete),
           int_search(y, smallest, indomain_min, complete)
        ])
      )
```

Restarts in MiniZinc

- Geometric Restart only on fails
- restart_geometric(<IncrementF>, <LimitF>, <Search>)
  - <IncrementF> is float we multiply fail limit by
  - <LimitF> is initial (float) fail limit
  - <Search> is the search strategy
- Example (for n-queens)

```
restart_geometric(1.2, int2float(2 * n),
        int_search(q, first_fail, indomain_random, complete))
```

- Note restart makes no sense if nothing changes

Autonomous Search

- A highly active research area in constraint programming (all rely on restarting)
- Automatic search strategies examples
  - dom_w_deg: choose a variable with minimum
    - domain size / sum of failures caused by constraints it is in
  - impact: record for each $v = d$ constraint
    - the average change in product of domain sizes when this choice is made = impact of decision
    - choose the variable v with maximum impact
    - choose the value d for v with minimum impact
  - activity: record for $v = d$, $v \leq d$, $v \geq d$, $v \neq d$
    - when it is involved in a failure (requires tracking implications)
    - decay activities, to focus on more recent failures
    - choose the constraint with highest activity

Dom_w_deg

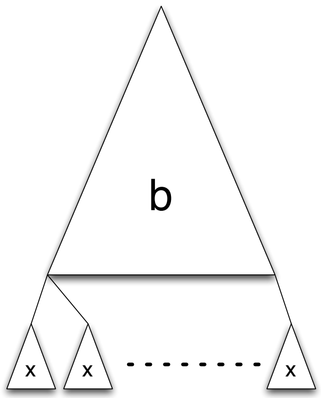
- Domain / weighted degree
  - degree in the number of constraints the var is in
- dom_w_deg: choose a variable with minimum
  - domain size / sum of failures by constraints it is in
- Each variable gets a fail count (= number of constraints initially)
- Each time a constraint detects failure
  - increment fail count for all variables involved
- Choose the variable with minimum
  - domain size / failcount

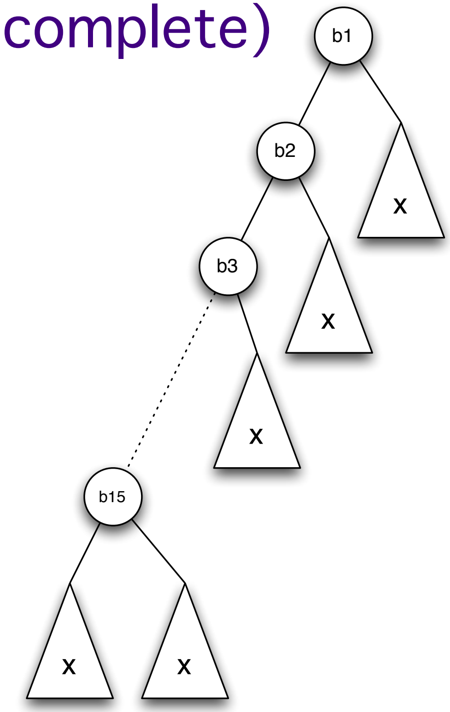
Dom_w_deg

- Why does it work

```
include "all_different.mzn";
array[1..15] of var 0..1: b;
array[1..4] of var 1..10: x;
constraint sum(b) >= 1 /\ exists([b[i] == 1 | i in 1..15]);
constraint all_different(x) /\ sum(i in 1..4)(x[i]) = 9;
solve :: int_search(b++x, first_fail, indomain_min, complete)
    satisfy;
```

  - 491504 choices to fail
- Change to dom_w_deg
  - 182 choices to fail
    - first branch choose *b*s then *x*s
    - since all failure is on *x*s we never rechoose a *b* on backtracking





Impact

- Measure the impact on total domain size of each decision
  - make decisions on variables with high impact
    - small search tree
  - take values with low impact
    - solutions more likely
- Raw search space $size(D) = \prod_{v \in \mathrm{var}(D)} | D(v) |$
- Impact(v=d) = $size(D)$ / $size(D')$ where D' is domain after propagation

Impact

- For each $v = d$
  - keep track of (log of) total impact
  - total number of times selected as choice
  - can determine average impact
- Impact of v
  - average impact of ($v = d$) for d in $D_{init}(v)$
- Simpler implementation
  - keep track of average impact
  - avimpact' = (avimpact + impact)/2

Impact in MiniZinc

- Can use impact currently only with indomain_split
- Jobshop scheduling: schedule start times s[i,j]
- ```
  solve :: int_search([s[i,j] | i in 1..jobs, j in 1..tasks], impact,
                                  indomain_split, complete)
        minimize end;
  ```
- Will concentrate on tasks that cause the most change in domains
  - those which precede many tasks (since we set there start time)
  ```

Activity-based Search

- We will examine after we have studied
  - Boolean Satisfiability Search

  where it was devised.

Comparing Search Strategies

- Simple jobshop scheduling problem 5x5
  1. first_fail + indomain_min
  2. smallest + indomain_min
  3. dom_w_deg + indomain_min
  4. impact + indomain_split
  5. default (first_fail on all variables + indomain_min)

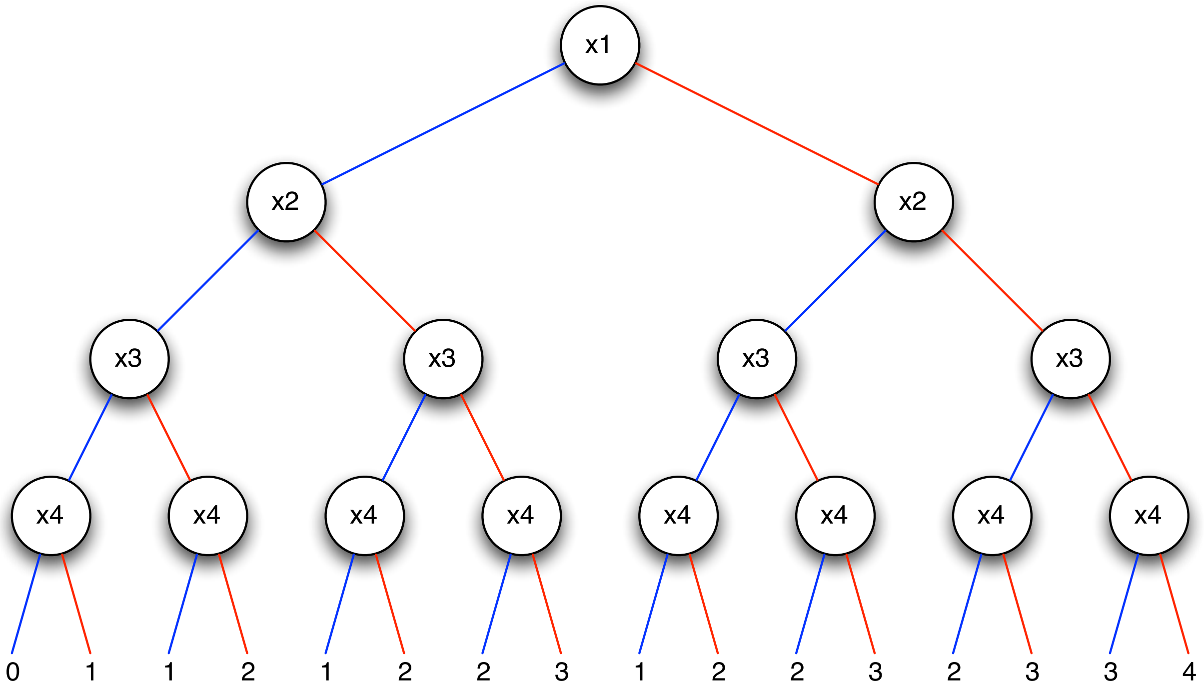
Search	Choices	Time (s)	Solns to Opt.
1	1116263	1m30	9
2	6493819	5m7	7
3	191	0.10	6
4	425	0.14	8
5	306	0.11	6

Limited Discrepancy Search

- Programmed search difficulties
  - most important decisions at top of tree
  - where least information is available
- Restarting fixes this to some degree
  - restart with better information
- Restarting usually changes the order of variables selected
- What about changing the order of values selected?

Limited Discrepancy Search

- Assume binary choice
  - assume left choice is good, right is discrepancy
- Search first
  - no discrepancies, 1 discrepancy, 2 discrepancy, …



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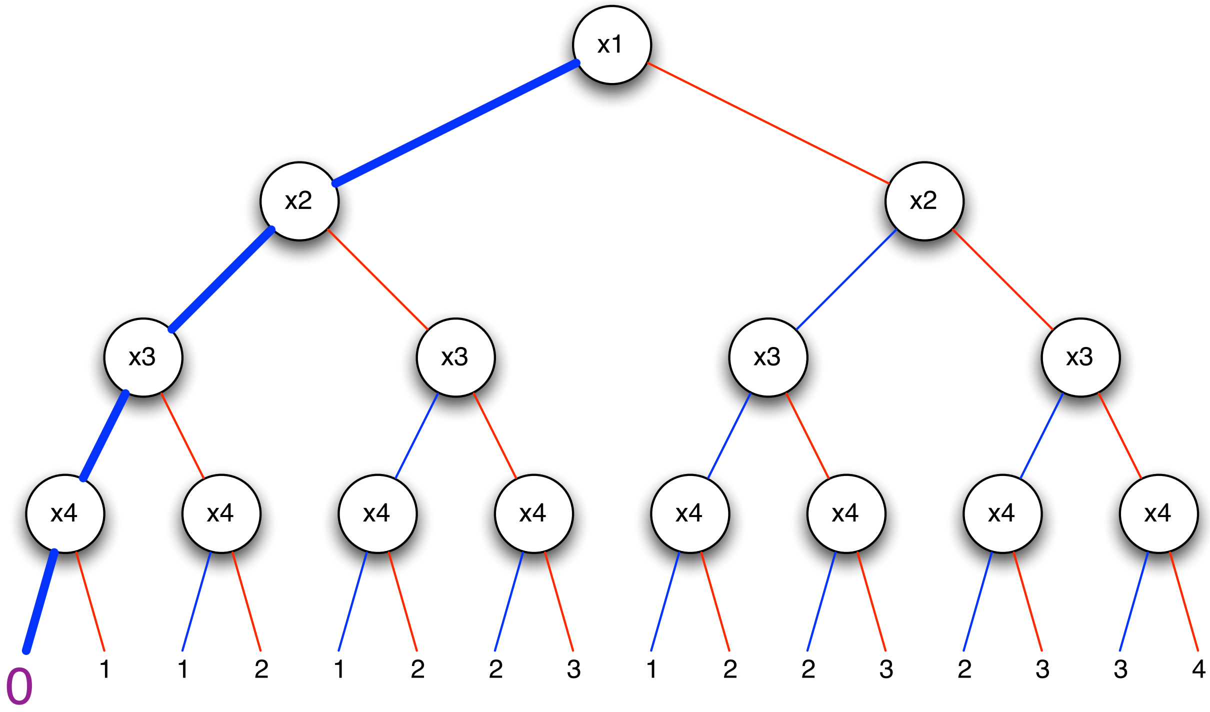

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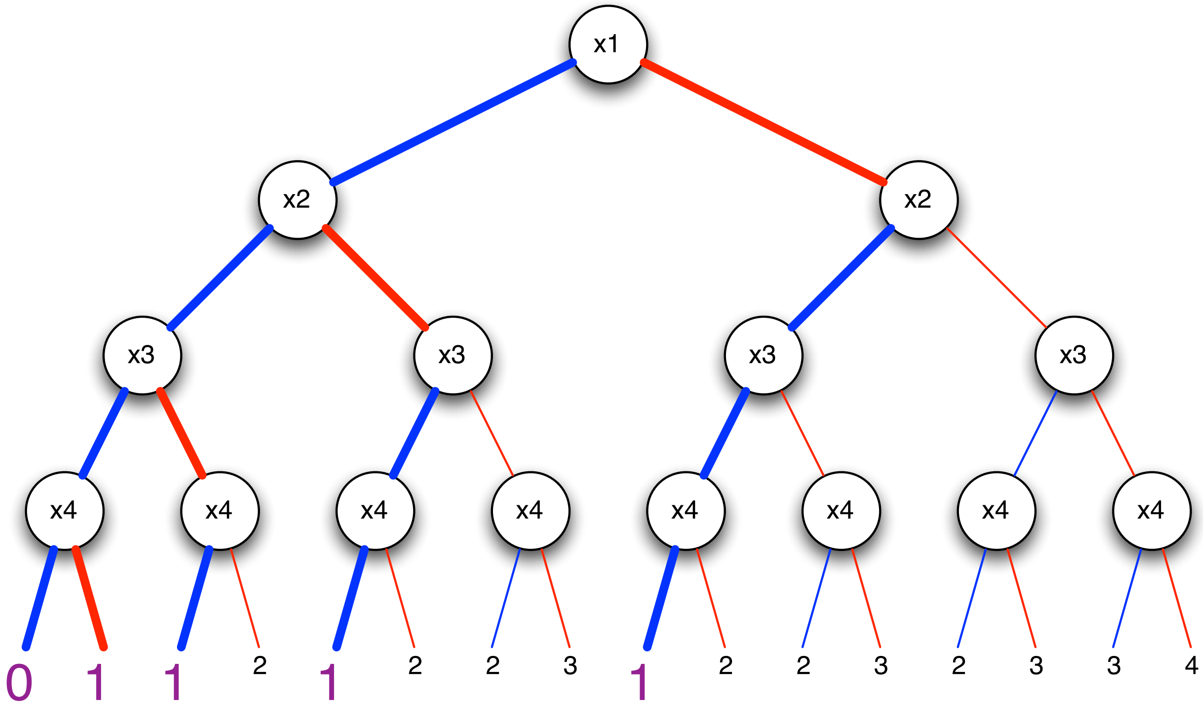

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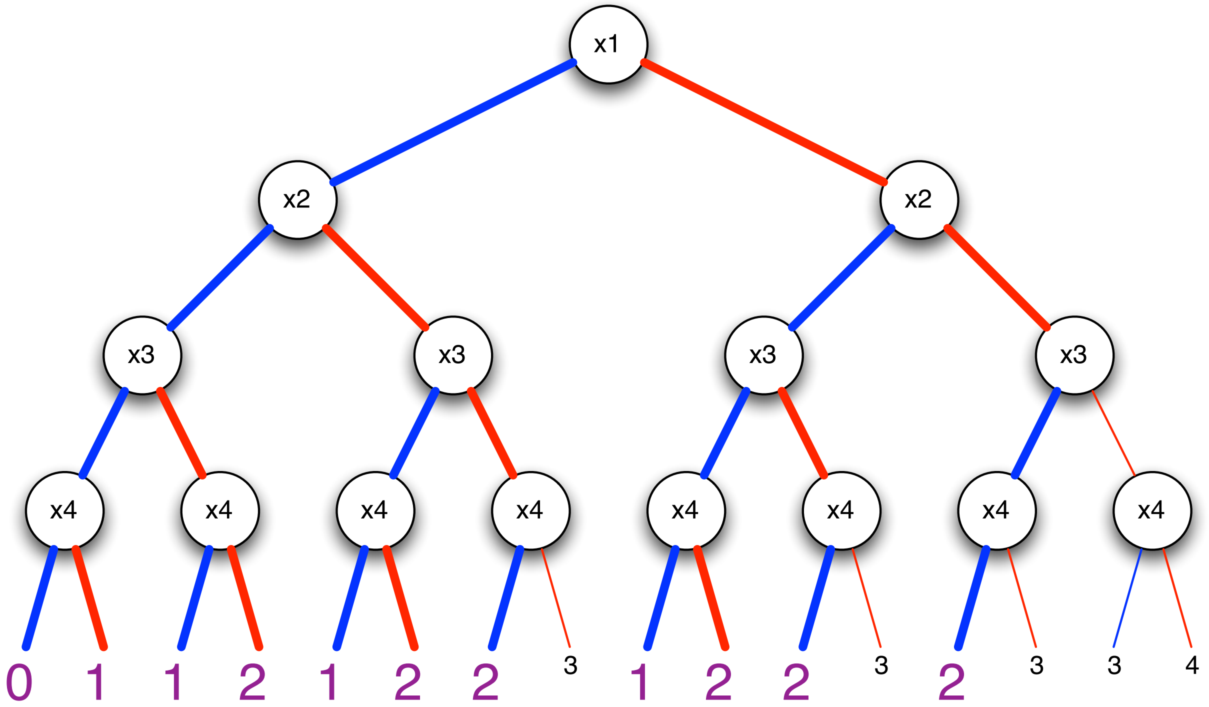

Limited Discrepancy Search

- Effectively reorders the way we visit leaves
- Implemented by restarting
- Note unless we know the depth of the tree
  - we have to visit all < *k* discrepancies to find all *k* discrepancies
- Simple jobshop scheduling 5x5:

smallest + indomain_min

LDS Limit	Best sol	Time (s)	Solns to Best
not lds	30	5m7	7
1	31	0.06	4
2	30	0.08	5
4	30	0.29	5
8	30	5.1	5

first_fail + indomain_min

LDS Limit	Best sol	Time (s)	Solns to Best
not lds	30	1m30	9
1	41	0.06	1
2	33	0.22	5
4	30	0.36	6
8	30	1.7	6

Summary

- Constraint programming techniques are based on backtracking search
- Reduce the search using consistency methods
  - incomplete but faster
  - node, arc, bound, generalized
- Optimization can be based on a branch & bound with a backtracking search
- Very general approach, not restricted to linear constraints.
- Programmer can add new global constraints and program their propagation behaviour.

Exercise 1: Send-most-money

- The send-most-money problem is to find different digits that make the cryptarithmetic problem:

  SEND + MOST = MONEY

  hold while maximizing MONEY (ie. 10000*M+ 1000*O+100*N_10*E+Y)
- Write a MiniZinc model and try out different search strategies to solve it. Which requires the least choices?

Comparison between CP and MIP

- What are the similarities?
- What are the strengths of MIP?
- What are the strengths of CP?
- Does it make sense to combine them?

Homework

- Read Chapter 3 of Marriott&Stuckey, 1998
- Solve the Australian Map Colouring problem by hand using simple backtracking, then with arc consistency and backtracking.
- Give propagation rules for constraints of form

$$a_1 X_1 + \ldots + a_n X_n \leq b_1 Y_1 + \ldots + b_m Y_m + c$$

where each a_i, $b_i > 0$.

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- MiniZinc provides decision variables which are sets of integer and normal set operations including cardinality. How would you
  - Represent sets?
  - Program these constraints using propagation rules?