Given two vectors

$$\bar{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$$
$$\bar{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$$

we can make

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \circ (b_1 \; b_2 \; b_3) = \begin{pmatrix} \min(a_1,b_1) & \min(a_1,b_2) & \min(a_1,b_3) \\ \min(a_2,b_1) & \min(a_2,b_2) & \min(a_2,b_3) \\ \min(a_3,b_1) & \min(a_3,b_2) & \min(a_3,b_3) \end{pmatrix}$$

so we can represent a 3x3 matrix with just 2 vectors of dimension 3.

PROBLEM

Given

$$A = \begin{pmatrix} 0.9 & 0.7 & 0.2 \\ 0 & 0.1 & 0.5 \\ 0.4 & 0.6 & 0.3 \end{pmatrix}$$

find two vectors $\bar{a} = (a_1, a_2, a_3)$ and $\bar{b} = (b_1, b_2, b_3)$ such that

$$\bar{a} \circ \bar{b} = A$$

We are going to use the GSA to solve this problem.
First we must decide how we build the particles
We need to determine $\underline{\underline{6}}$ number, 3 for $\vec{a}$ and 3 for $\bar{b}$, so we can take

$$\bar{X} = (x_1, x_2, x_3, x_4, x_5, x_6)$$

From $\bar{X}$ we can build a matrix $\tilde{A}_X$ as follows

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \circ (x_4 \; x_5 \; x_6) = \begin{pmatrix} \min(x_1,x_4) & \min(x_1,x_5) & \min(x_1,x_6) \\ \min(x_2,x_4) & \min(x_2,x_5) & \min(x_2,x_6) \\ \min(x_3,x_4) & \min(x_3,x_5) & \min(x_3,x_6) \end{pmatrix}$$

Given a particle $\bar{X}$, we should define its fitness. This fitness function must measure how good or bath a given vector $\bar{X}=(x_1, \dots, x_6)$ is as a solution. As the matrix defined by $(x_1, \dots, x_6)$ is $\tilde{A}_X$, we can take as fitness the quadratic difference between $\tilde{A}_X$ and A. That is:

$$fit(\bar{X}) = \Big((\min(x_1,x_4) - 0.9)^2 + (\min(x_1,x_5) - 0.7)^2 + (\min(x_1,x_6) - 0.2)^2$$
$$+ (\min(x_2,x_4) - 0)^2 + (\min(x_2,x_5) - 0.1)^2 + (\min(x_2,x_6) - 0.5)^2$$
$$+ (\min(x_3,x_4) - 0.4)^2 + (\min(x_3,x_5) - 0.6)^2 + (\min(x_3,x_6) - 0.3)^2 \Big)$$

Note that

* $fit(\bar{X}) \geq 0 \quad \forall \bar{X}$
* $fit(\bar{X}) = 0$ if and only if $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \circ (x_4 \; x_5 \; x_6) = A$

We start with the algorithm.

1) We are going to initialize (randomly) 4 particles and their velocities.

$\bar{X}_1 = (0.2, 0.3, 0.4, 0.1, 0, 0.6)$ $\quad\quad$ $\bar{V}_1 = (0.1, 0.2, 0.2, 0, 0, 0.3)$

$\bar{X}_2 = (0, 0.9, 0.1, 0.4, 0.7, 0.2)$ $\quad\quad$ $\bar{V}_2 = (0, 0.1, 0.1, 0.3, 0.2, 0.1)$

$\bar{X}_3 = (0.2, 0.4, 0.3, 0.8, 0.1, 0.6)$ $\quad\quad$ $\bar{V}_3 = (0.1, 0.2, 0.3, 0.1, 0, 0.4)$

$\bar{X}_4 = (0.2, 0.2, 0.5, 0.6, 0.1, 0.3)$ $\quad\quad$ $\bar{V}_4 = (0.1, 0.1, 0.4, 0.5, 0, 0.2)$

2.- We calculate the fitness of each particle.

$$fit(\bar{X}_1) = (0.1-0.9)^2 + (0-0.7)^2 + (0.2-0.2)^2 + (0.1-0)^2 + (0-0.1)^2 + (0.3-0.5)^2$$
$$+ (0.1-0.4)^2 + (0-0.6)^2 + (0.4-0.3)^2 = 1.65$$

$$fit(\bar{X}_2) = \ldots = 2.33$$

$$fit(\bar{X}_3) = \ldots = 1.56$$

$$fit(\bar{X}_4) = \ldots = 1.24$$

$\hookrightarrow$ Since we are minimizing, the particle with the best fitness is $\bar{X}_4$, and the particle with the worst fitness is $\bar{X}_2$. So, since we are in iteration 0, we see that

$$best(0) = 1.24$$
$$worst(0) = 2.33$$

3.- We calculate now the mass of each particle, which is given by

$$m_i(0) = \frac{fit(\bar{X}_i) - worst(0)}{best(0) - worst(0)}$$

So

$$m_1(0) = \frac{fit(\bar{X}_1) - worst(0)}{best(0) - worst(0)} = \frac{1.65 - 2.33}{1.24 - 2.33} = \frac{-0.68}{-1.09} = 0.62$$

$$m_2(0) = \quad \ldots \quad = \frac{2.33 - 2.33}{1.24 - 2.33} = 0$$

$$m_3(0) = \quad \ldots \quad = \frac{1.56 - 2.33}{1.24 - 2.33} = 0.71$$

$$m_4(0) = \quad \ldots \quad = \frac{1.24 - 2.33}{1.24 - 2.33} = 1$$

Note that the particle with the best fitness has mass 1, and the particle with worst fitness has mass 0.

4.- We normalize the masses using the expression

$$M_i(0) = \frac{m_i(0)}{\sum_{j=1}^{4} m_j(0)}$$

so

$$M_1(0) = \frac{0.62}{0.62+0.71+1} = 0.27$$

$$M_2(0) = 0$$

$$M_3(0) = 0.30$$

$$M_4(0) = 0.43$$

Now, we have that

$$M_1(0) + M_2(0) + M_3(0) + M_4(0) = 1.$$

5.) We calculate the gravitational force acting over each particle.

The force that particle j makes over particle i is given by:

$$\overline{F_{ij}} = G \cdot \frac{M_i \cdot M_j}{R_{ij} + \epsilon} (\overline{X_j} - \overline{X_i})$$

where:

* G is a constant (which may change from one iteration to the next one. We take $G \equiv 1$
* R_{ij} is the Euclidean distance between $\overline{X_i}$ and $\overline{X_j}$. That is, if

$$\overline{X_i} = (X_i^1, X_i^2, \dots, X_i^6) \quad \text{and} \quad \overline{X_j} = (X_j^1, X_j^2, \dots, X_j^6)$$

then:

$$R_{ij} = \sqrt{(X_i^1 - X_j^1)^2 + (X_i^2 - X_j^2)^2 + \dots + (X_i^6 - X_j^6)^2}.$$

*ϵ is a very small constant to avoid division by zero. We take $\epsilon = 0.01$.

The total force acting over particle i is:

$$\vec{F_i} = \sum_{j \neq i} \omega_j \vec{F_{ij}}$$

where $(\omega_1, \omega_2, \omega_3, \omega_4) \in [0,1]^4$ is a vector of random numbers uniformely distributed on $[0,1]$. For this example, we take

$$\omega_1 = \omega_2 = \omega_3 = \omega_4 = 1.$$

Note that

$$\vec{F_{ij}} = -\vec{F_{ji}}.$$

Let's calculate the forces acting over particle 1.

$$\vec{F_{12}} = \frac{M_1 \cdot M_2}{R_{12} + 0.01}(\bar{X_2} - \bar{X_1}) = \frac{0.27 \cdot 0}{1.12}(\bar{X_2} - \bar{X_1}) = \bar{0}$$

$$R_{12} = \sqrt{(0-0.2)^2 + (0.9-0.3)^2 + (0.1-0.4)^2 + (0.4-0.1)^2 + (0.7-0)^2 + (0.2-0.6)^2} = 1.11$$

$$\vec{F_{13}} = \frac{M_1 \cdot M_3}{R_{13} + 0.01}(\bar{X_3} - \bar{X_1}) = \frac{0.27 \cdot 0.30}{0.73}(0, 0.1, -0.1, 0.7, 0.1, 0) =$$

$R_{13} = 0.72$

$$= (0, 0.01, -0.01, 0.07, 0.01, 0)$$

$$\bar{F_{14}} = \frac{M_1 \cdot M_4}{R_{14} + 0.01}(\bar{X_4} - \bar{X_1}) = \frac{0.27 \cdot 0.43}{0.62}(0, -0.1, 0.1, 0.5, 0.1, -0.3) =$$

$$= (0, -0.02, 0.02, 0.1, 0.02, -0.06)$$

$R_{14} = 0.61$

So the total force acting over particle 1 is

$$\vec{F_1} = \bar{F_{12}} + \bar{F_{13}} + \bar{F_{14}} = (0, -0.01, 0.01, 0.17, 0.03, -0.06)$$

We make the same calculation for particle 2:

$$\overline{F_{21}} = -\overline{F_{12}} = \overline{0}$$

$$\overline{F_{23}} = \frac{M_2 \cdot M_3}{r_{23}+\epsilon}(\overline{X_3}-\overline{X_2}) = \overline{0}$$

$$\overline{F_{24}} = \overline{0}$$

In this case, as $M_2 = 0$, all the forces are $\overline{0}$, so, $\overline{F_2} = \overline{0}$.

Now, particle 3:

$$\overline{F_{31}} = -\overline{F_{13}} = (0, -0.01, 0.01, -0.07, -0.01, 0)$$

$$\overline{F_{32}} = \overline{F_{23}} = \overline{0}$$

$$\overline{F_{34}} = \frac{M_3 \cdot M_4}{r_{34}+\epsilon}(\overline{X_4}-\overline{X_3}) = (0, -0.05, 0.05, -0.05, 0, -0.08)$$

So we have

$$\overline{F_3} = \overline{F_{31}} + \overline{F_{32}} + \overline{F_{34}} = (0, -0.06, 0.06, -0.12, -0.01, -0.08)$$

Finally, for particle 4:

$$\overline{F_4} = \overline{F_{41}} + \overline{F_{42}} + \overline{F_{43}} = (0, 0.07, -0.07, -0.05, -0.02, 0.14)$$

6) We now calculate the corresponding accelerations:

$$\overline{a_i} = \frac{\overline{F_i}}{M_i} \text{ if } M_i \neq 0$$

so

$$\overline{a_1} = \frac{\overline{F_1}}{M_1} = \frac{1}{0.27}\overline{F_1} = (0, -0.04, 0.04, 0.63, 0.11, -0.22)$$

$$\overline{a_3} = \frac{\overline{F_3}}{M_3} = \frac{1}{0.30}\overline{F_3} = (0, -0.17, 0.17, -0.17, 0, -0.27)$$

$$\overline{a_4} = \frac{\overline{F_4}}{M_4} = \frac{1}{0.43}\overline{F_4} = (0, 0.16, -0.16, -0.12, -0.05, -0.33)$$

What about $\bar{a}_2$? In this case, as $M_2 = 0$, we make

$$\bar{a}_2 = \sum_{j \neq 2} \frac{M_j}{R_{2j} + \epsilon} \left(\bar{X}_j - \bar{X}_2\right)$$

so we arrive at

$$\bar{a}_2 = \frac{0.27}{1.12}(0.2, -0.6, 0.3, -0.3, -0.7, 0.4) + \frac{0.30}{1.01}(0.2, -0.5, 0.2, 0.4, -0.6, 0.4)$$
$$+ \frac{0.43}{1.06}(0.2, -0.7, 0.4, 0.2, -0.6, 0.1)$$

$$= (0.19, -0.57, 0.29, 0.13, -0.59, 0.26)$$

7) Calculation of the new velocities

We use the expression:

$$\vec{V}_i(1) = p_i \bar{V}_i(0) + \bar{a}_i(0)$$

where the (1) refers to the fact that we move to iteration 1.

(p_1, p_2, p_3, p_4) is a vector of random numbers in $[0, 1]$ uniformely distributed

We take for this example

$$p_1 = \frac{1}{2} \quad p_2 = \frac{1}{3} \quad p_3 = p_4 = 1$$

Then

$$\bar{V}_1(1) = \frac{1}{2}\bar{V}_1(0) + \bar{a}_1(0) = \frac{1}{2}(0.1, 0.2, 0.2, 0, 0, 0.3) + (0, -0.04, 0.04, 0.63, 0.11, -0.22) =$$
$$= (0.05, 0.06, 0.14, 0.63, 0.11, -0.07)$$

$$\bar{V}_2(1) = \frac{1}{3}\bar{V}_2(0) + \bar{a}_2(0) = (0.19, -0.54, 0.32, 0.23, -0.52, 0.29)$$

$$\bar{V}_3(1) = \bar{V}_3(0) + \bar{a}_3(0) = (0.1, 0.03, 0.47, -0.07, 0, 0.13)$$

$$\bar{V}_4(1) = \bar{V}_4(0) + \bar{a}_4(0) = (0.1, 0.1, 0.4, 0.5, 0, 0.2) + (0, +0.16, -0.16, -0.12, -0.05, -0.33)$$
$$= (0.1, \cancel{0.03}\, 0.26, 0.24, 0.38, -0.05, -0.13)$$

8) Finally, we calculate the new positions of the particles:

$$\bar{X}_1(1) = \bar{X}_1(0) + \bar{V}_1(1) = (0.15, 0.36, 0.54, 0.73, 0.11, 0.53)$$

$$\bar{X}_2(1) = \bar{X}_2(0) + \bar{V}_2(1) = (0.19, 0.36, 0.42, 0.63, 0.18, 0.49)$$

$$\bar{X}_3(1) = \bar{X}_3(0) + \bar{V}_3(0) = (0.2, 0.4, 0.3, 0.8, 0.1, 0.6) + (0.1, 0.03, 0.47, -0.07, 0, 0.13)$$

$$= (0.3, 0.43, 0.77, 0.73, 0.1, 0.73)$$

$$\bar{X}_4(1) = \bar{X}_4(0) + \bar{V}_4(0) = (0.3, 0.46, 0.74, 0.98, 0.05, 0.17)$$

We now repeat steps 2) - 8) until:

- either we make more than T iterations (where T pre-fixed), or
- For some iteration t it holds that, $\forall i = 1, 2, 3, 4$

$$\sqrt{(V_i^1(t))^2 + (V_i^2(t))^2 + (V_i^3(t))^2 + (V_i^4(t))^2} < tol$$

where tol is a small positive number fixed beforehand.

Observe that:

$$fit(\bar{X}_1(1)) = 1.23$$
$$fit(\bar{X}_2(1)) = 0.78$$
$$fit(\bar{X}_3(1)) = 1.65$$
$$fit(\bar{X}_4(1)) = 0.94$$

so fitness in the first iteration have already improved ~~na~~ notably.