APPROXIMATE RESOLUTION OF TRANSCENDENT EQUATIONS - PART 3

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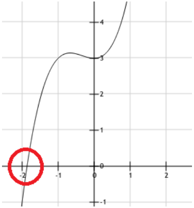
About the previous classes ...



- Introduction to solving transcendental equations $F(x) = 0$:
  1. **Dimensioning**; .
  2. **Separation**;
  3. **Approximate resolution**.
- Bisection Method
- Regula Falsi Method
- Secant Method
- The Newton-Raphson method
  1. What is the principle of each method?
  2. Which of the methods is always convergent?
  3. Which of the methods is faster?
  4. In which cases can there be no convergence?

Fixed point iteration

Definition

Let g be any function. A fixed point of g is any point p such that $g(p) = p$.

Consider a function f:

- Finding a raíz of $f(x)$ is equivalent to finding a fixed point of $g(x) = f(x) + x$;
  Suppose that the fixed point of $g(x) = f(x) + x$ is p:

  $$\Rightarrow g(p) = p \Rightarrow g(p) = f(p) + p = p \Rightarrow f(p) = 0 \Rightarrow$$

  p is the root of f.
- Finding a fixed point of $f(x)$ is equivalent to finding a ra ' i z of $g(x) = f(x) - x$.
  Suppose that the root of $g(x) = f(x) - x$ is p:

  $$\Rightarrow g(p) = 0 \Rightarrow g(p) = f(p) - p = 0 \Rightarrow f(p) = p \Rightarrow$$

  p is a fixed point of f.

Existence and uniqueness of fixed points

Theorem

Let $g : [a, b] \to [a, b]$ *be a continuous function. So there is at least one fixed point of* g *at* $[a, b]$.

Theorem

Let $g : [a, b] \to [a, b]$ *be a continuous function. Suppose there exists* $0 < k < 1$ *such that* $|g'(x)| \leq k$ *for all* $x \in (a, b)$. *So there exists a single fixed point of* g *at* $[a, b]$.

Functional iteration. Convergence.

Theorem

Let $g : [a, b] \to \mathbb{R}$ *be a continuous function. Let* $p_0 \in [a, b]$ *and consider the sequence*

$$p_n = g(p_{n-1})$$

Suppose there exists $0 < k < 1$ *such that* $|g(x) - g(y)| \leq k(|x - y|)$ *for any* $x, y \in [a, b]$*. Then the sequence* p_n *converges to a fixed point* p*.*

Functional iteration. Convergence.

Corollary

Let $g : [a, b] \to \mathbb{R}$ *be a continuous function differentiable on* (a, b)*, and let* $p_0 \in [a, b]$*. Suppose there exists* $0 < k < 1$ *such that for every* $x \in [a, b]$*,* $|g'(x)| \leq k \; 0 < k < 1$*. Then the iterated sequence defined by* $p_{n+1} = g(p_n)$ *converges to a fixed point.*

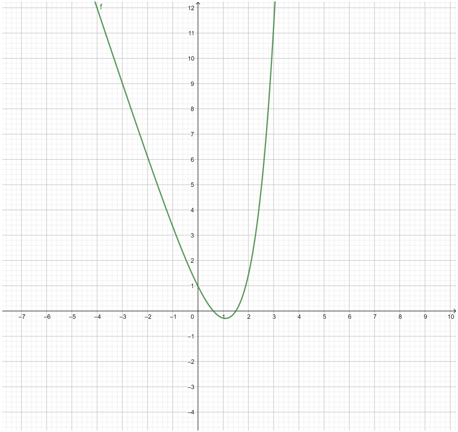
Example

- Finding a root of a function f is equivalent to finding a fixed point of $g(x) = f(x) + x$.
- The iterated sequence defined by $p_{n+1} = g(p_n)$ converges to a fixed point, ie:

  $$x_{n+1} = g(x_n) = f(x_n) + x_n$$

  converges to a fixed point of g which is a root of f.
- Find a root of $f(x) = e^x - 3x$ with a maximum tolerance of 0.001.
  Solution: Find the fixed point
  of$g(x) = f(x) + x = e^x - 2x \Rightarrow x_{n+1} = e^{x_n} - 2x_n$

Example



A root of $f(x) = e^x - 3x$ is s fix point of $g(x) = f(x) + x = e^x - 2x$

$$x_{n+1} = e^{x_n} - 2x_n$$

Example

$$x_{n+1} = e^{x_n} - 2x_n$$

n	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	0	1	1
1	1	...	
...	...	...	...

Example

$$x_{n+1} = e^{x_n} - 2x_n$$

n	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	0	1	1
1	1	0.718	0.282
2	0.718	...	
...	...	...	

Example

$$x_{n+1} = e^{x_n} - 2x_n$$

n	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	0	1	1
1	1	0.718	0.282
2	0.718	0.614	0.104
3	0.614	...	
...	...	...	

Example

A root of $f(x) = e^x - 3x$ is a fix point of
$g(x) = f(x) + x = e^x - 2x$

$$x_{n+1} = e^{x_n} - 2x_n$$

n	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	0	1	1
1	1	0.718	0.282
2	0.718	0.614	0.104
3	0.614	0.620	0.006
4	0.620	0.619	0.001
5	0.619	0.619	Fix point of g = root of f

Functional iteration. Convergence.

Corollary

Let $g : [a, b] \to \mathbb{R}$ *be a continuous function differentiable on* (a, b)*, and let* $p_0 \in [a, b]$*. Suppose there exists* $0 < k < 1$ *such that for every* $x \in [a, b]$*,* $|g'(x)| \leq k \; 0 < k < 1$*. Then, an error bound for the approximation by means of the sequence* $\{p_n\}$ *is given by*

$$|p - p_n| \leq \frac{k^n}{1 - k} |p - p_0|$$

Functional iteration. Convergence.

Proof. We have that

$$|p - p_n| = |g(p) - g(p_{n-1})| \leq k|p - p_{n-1}| \leq \ldots k^n|p - p_0|$$

Pero

$$|p - p_0| \leq |p - p_1| + |p_1 - p_0| \leq k|p - p_0| + |p_1 - p_0|$$

de donde

$$|p - p_0| \leq \frac{1}{1-k}|p_1 - p_0|$$

$$|p - p_n| = \frac{k^n}{1-k}|p_1 - p_0| \square$$

Functional iteration. Convergence.

Corollary

Under the same conditions as the previous result, if $k \leq \frac{1}{2}$, we have that

$$|p_n - p| \leq |p_n - p_{n-1}|$$

Proof. We have that

$$|p - p_n| \leq k|p - p_{n-1}|$$

and also

$$|p - p_{n-1}| \leq |p - p_n| + |p_n - p_{n-1}| \leq k|p - p_{n-1}| + +|p_n - p_{n-1}|$$

then

$$|p - p_n| \leq \frac{k}{1-k}|p_n - p_{n-1}| \leq |p_n - p_{n-1}| \square$$

Order of convergence.

Definition

The order of convergence of a convergent iterative method $(p_n \to p)$ *is the largest real number q such that*

$$\lim \frac{|p_{n+1} - p|}{|p_n - p|^q} = C \neq 0$$

Order of convergence.

Theorem

Let g be a function of class C^k on the interval $(p-\delta, p+\delta)$. Suppose that p_n converges to the fixed point p. Let q be a real number such that

$$g^{i)}(p) = 0 \text{ para todo } i = 1, 2, \ldots, q-1$$

y

$$g^{q)}(p) \neq 0$$

Then the order of convergence of the iterative method is at least q.

Polynomial equations

A polynomial of degree n is an expression of the form:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

with $a_0, \ldots, a_n \in \mathbb{R}^n$ y $a_n \neq 0$.

Theorem

Every polynomial of degree ≥ 1 has at least one root (real or complex)

Polynomial equations

Corollary

Let $P(x)$ be a polynomial of degree $n \geq 1$. Then there are k different numbers (real or complex) $s_1, \ldots, s_k$ and k integers $m_1, \ldots, m_k$ such that $\sum_{j=1}^{n} m_k = n$ y is verified

$$P(x) = a_n (x - s_1)^{m_1} \ldots (x - s_k)^{m_k}$$

Corollary

Let $P(x)$ and $Q(x)$ be two polynomials of degree at the most n. Suppose that k numbers $x_1, \ldots, x_k$ (with $k > n$) exist so that $P(x_j) = Q(x_j)$ for all $j = 1, \ldots, k$. Then $P(x) = Q(x)$.

Horner's Method: Synthetic Division

- For N-R, we need to evaluate the polynomial $P(x)$ and its derivative $P'(x)$.
- The most efficient way to do this is through nesting.
- Horner's method uses nesting to evaluate a polynomial at a point.
- To do this, it requires n multiplications and n sums for a polynomial of degree n.

Horner's Method: Synthetic Division

Theorem

Consider $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ *y* $x_0 \in \mathbb{R}$. *Define* $b_n = a_n$ *and* $b_k = a_k + b_{k+1}x_0$ *for* $k = n-1, \ldots, 1$. *Then*

1. $b_0 = P(x_0)$;
2. $P(x) = (x - x_0)Q(x) + b_0$, *where* $Q(x) = b_n x^{n-1} + \cdots + b_2 x + b_1$.

Horner's Method: example

$b_n = a_n$, $b_k = a_k + b_{k+1}x_0$, $b_0 = P(x_0)$,
$Q(x) = b_n x^{n-1} + \cdots + b_2 x + b_1$, $P(x) = (x - x_0)Q(x) + b_0$

$$P(x) = 5x^3 + 7x^2 + 2x + 3$$

Suppose $x_0 = 0$:
$b_0 = P(x_0) = P(0) = 3$, $b_3 = a_3 = 5$, $b_2 = a_2 + b_3 x_0 = 7 + 0 = 7$
, $b_1 = a_1 + b_2 x_0 = 2 + 0 = 2$
$Q(x) = 5x^2 + 7x + 2 \Rightarrow P(x) = (5x^2 + 7x + 2)x + 3$

$$P_1(x) = 5x^2 + 7x + 2$$

$b_0 = P_1(x_0) = P_1(0) = 2$, $b_2 = a_2 = 5$, $b_1 = a_1 + b_2 x_0 = 7 + 0 = 7$
$Q_1(x) = 5x + 7 \Rightarrow P(x) = ((5x + 7)x + 2)x + 3$

Derivative evaluation

As $P(x) = (x - x_0)Q(x) + b_0$, we have that

$$P'(x) = Q(x) + (x - x_0)Q'(x) \rightarrow P'(x_0) = Q(x_0)$$

then it is enough to apply Horner twice.

Deflation

Let x_1 be an approximation to a zero of $P(x)$. Then

$$P(x) = (x - x_1)Q_1(x) + P(x_1) \simeq (x - x_1)Q_1(x)$$

By Newton, we can calculate another zero for Q. In this way, we can find all the zeros of P.
Problem When approaching a zero of Q_k, we generally do not approach a zero of P. The inaccuracy grows as k increases.
Possible solution Take the zero of Q_k as the starting value to apply N-R to P directly.

Bounding of roots of polynomials

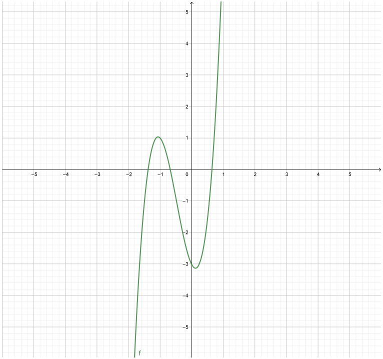
Theorem

If all the roots of the equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ *are real, then they are all contained in the interval* $[-M, M]$*, where*

$$M = \sqrt{(\frac{a_{n-1}}{a_n})^2 - 2\frac{a_{n-2}}{a_n}}$$

M is called the Cardano-Vieta bound.

Bounding of roots of polynomials: Example



$$P(x) = 5x^3 + 7x^2 - 2x - 3$$

$$M = \sqrt{(\frac{7}{5})^2 - 2\frac{-2}{5}} = 1.66 \Rightarrow [-1.66, 1.66]$$

Laguerre-Thibault bound: Upper bound of positive roots

Proposition

Let $M > 0$ and let $P(x)$ be a polynomial of degree $n \geq 1$. If by dividing $P(x)$ by $x - M$ we obtain that:

1. *the rest is positive.*
2. *all coefficients of the quotient polynomial are positive or null.*

then M is an upper bound of the positive roots of the polynomial $P(x)$.

Laguerre-Thibault bound: Lower bound of positive roots

Let $P(x)$ be a polynomial. We define

$$Q(y) = P(\frac{1}{x})$$

If from the previous proposition we obtain as an upper bound of the numerator M', then $m = \frac{1}{M}$ is a lower bound of the positive roots of $P(x)$.

Laguerre-Thibault dimension: bounds of negative roots

To calculate the lower bound, just make the change $Q(y) = P(-x)$. If M is the bound obtained for the roots of Q, then $-M$ is the lower bound sought.
To calculate the upper bound, just set $Q(y) = P(-\frac{1}{x})$. If M is the upper bound obtained for the numerator of Q, then $m = -\frac{1}{M}$ is the upper bound of the negative roots we are looking for.

Newton's dimension

Theorem

Let $P(x)$ be a polynomial of degree n. M is an upper bound of the roots of P if $P(M), P'(M), \ldots, P^{(n)}(M)$ are all positive values or null.

Sturm method

- The most effective method to isolate zeros is the Sturm's theorem
- The first thing to see is the Sturm polynomial system, we need it to make evaluations in it since the number of sign changes between the different evaluations will give us the number of zeros between the values

Sturm method

Let $P(x)$ be a polynomial. We want to know how many roots are in (a, b). We know that

1. If $P(a)P(b) < 0$ there is a number odd of roots (counting multiplicities).
2. If $P(a)P(b) > 0$ there is a number pair of roots (counting multiplicities).

Sturm method

Definition

Given a finite sequence of n real numbers $c_1, \ldots, c_n$*, it is said that there is a sign change for a pair of consecutive elements* c_k *and* c_{k+1} *if* $c_k \cdot c_{k*1} < 0$*.*

The number of sign changes in the sequence is called the total number of sign changes that occur in it.

Sturm's method: Sturm's sequence

- Let P be a polynomial of degree n, with real coefficients and with all real and simple roots. We define:
  $P_0(x) = P(x)$; $P_1(x) = P'(x)$; $P_2(x) = -\lambda_2 Remainder(\frac{P_0}{P_1})$;
  $P_3(x) = -\lambda_3 Remainder(\frac{P_1}{P_2})$...
  $P_m(x) = -\lambda_m Remainder(\frac{P_{m-2}}{P_{m-1}})$,
  where we continue until $P_m \in \mathbb{R}$ and $\lambda_j, j = 1..m$, are constants that serve to avoid fractions.
- If we take $x = c \in \mathbb{R}$, we have a sequence of real numbers: $P_0(c), P_1(c), ..., P_m(c)$ item N_c the number of sign changes in this sequence.

EXAMPLE 1 (Sturm polynomial system)

Calculate the Sturm polynomial system of
$P(x) = x^3 - 9x^2 + 24x - 36$.
Solution:

$$\begin{aligned} P_0(x) &= x^3 - 9x^2 + 24x - 36 \\ P_1(x) &= P'(x) = 3x^2 - 18^x + 24 \\ P_2(x) &= -\lambda_2 Remainder(\frac{P_0}{P_1}) = 18x + 108(\lambda_2 = 3^2) \end{aligned}$$

$$\begin{array}{r|l} 9x^3 - 81x^2 + 216x - 324 & \underline{3x^2 - 18x + 24} \\ \underline{-9x^3 + 54x^2 - 72x \qquad\quad} & 3x - 9 \\ -27x^2 + 144x - 324 & \\ \underline{27x^2 - 162x + 216} & \\ -18x - 108 & \end{array}$$

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P_2(x) &= -\lambda_2 Remainder(\frac{P_0}{P_1}) = 18x + 108 \\
P_3(x) &= -\lambda_3 Remainder(\frac{P_1}{P_2}) = -77760(\lambda_3 = 18^2)
\end{aligned}
$$

$$
\begin{array}{r|l}
972x^2 - 5832x + 7776 & 18x + 108 \\ \cline{2-2}
\underline{-972x^2 - 5832x } & 54x - 648 \\
-11664x + 7776 & \\
\underline{11664x + 69984} & \\
77760 &
\end{array}
$$

EXAMPLE 2 (Sturm polynomial system)

Calculate the Sturm polynomial system of $P(x) = x^3 + 4x^2 - 7$.
Solution:

$$\begin{aligned} P_0(x) &= x^3 + 4x^2 - 7 \\ P_1(x) &= P'(x) = 3x^2 + 8x \\ P_2(x) &= -\lambda_2 Remainder(\frac{P_0}{P_1}) = \end{aligned}$$

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$$\begin{array}{l|l} 9x^3 + 36x^2 + 0x - 63 & 3x^2 + 8x \\ \hline -9x^3 - 24x^2 & 3x + 4 \\ \hline \quad 12x^2 + 0x - 63 & \\ \quad -12x^2 - 32x & \\ \hline \qquad -32x - 63 & \end{array}$$

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P_3(x) &= -\lambda_3 Remainder(\frac{P_1}{P_2}) = ...(\lambda_3 = ...)
\end{aligned}$$

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P_3(x) &= -\lambda_3 Remainder(\frac{P_1}{P_2}) = 4221(\lambda_3 = 32^2)
\end{aligned}
$$

$$
\begin{array}{r|l}
3072x^2 + 8192x + 0 & \underline{32x + 63} \\
\underline{-3072x^2 - 6048x \quad} & 96x + 67 \\
2144x + 0 & \\
\underline{-2144x - 4221} & \\
-4221 &
\end{array}
$$

Sturm Theorem

Theorem

Let $P(x)$ be a polynomial without multiple roots and such that $P(a) \neq 0$ and $P(b) \neq 0$. Then, the number of real roots of $P(x)$ in (a, b) is equal to $|N_c(a) - N_c(b)|$.

If $P(x)$ has multiple roots, we calculate the Sturm sequence of $P_0(x) = \frac{P(x)}{P_{mcd}(x)}$, where

$$P_{mcd}(x) = mcd(P(x), P'(x))$$

EXAMPLE 3 (Sturm's Theorem)

Isolate the zeros of $P(x) = x^3 - 9x^2 + 24x - 36$.
Solution:
We already know the Sturm system for this polynomial

$$\begin{aligned} P_0(x) &= x^3 - 9x^2 + 24x - 36 \\ P_1(x) &= P'(x) = 3x^2 - 18^x + 24 \\ P_2(x) &= 18x + 108 \\ P_3(x) &= -77760 \end{aligned}$$

The first thing to isolate the zeros is to check how many there are in total, so the system is evaluated in - ∞ y + ∞, we also evaluate in $x = 0$ to verify how many positive and how many negative zeros there are, let us see:

Signs	- ∞	0	+ ∞
$P_0(x)$	-	-	+
$P_1(x)$	+	+	+
$P_2(x)$	-	+	+
$P_3(x)$	-	-	-
Changes	2	2	1

EXAMPLE 3 (Sturm's Theorem)

Isolate the zeros of $P(x) = x^3 - 9x^2 + 24x - 36$.

Signs	- ∞	0	+ ∞
$P_0(x)$	-	-	+
$P_1(x)$	+	+	+
$P_2(x)$	-	+	+
$P_3(x)$	-	-	-
Changes	2	2	1

As in - ∞ there are 2 sign changes and in + ∞ there is only one so from - ∞ to + ∞ there are 2 - 1 = 1 zeros, that is, the polynomial only has a zero in R.
Now, from - ∞ to 0 there are no zeros because both have the same number of sign changes, so zero is found from 0 to + ∞ because when subtracting sign changes we get 2 - 1 = 1.
Now we can evaluate one by one starting from 0 until we find in which interval it is.

EXAMPLE 3 (Sturm's Theorem)

Isolate the zeros $P(x) = x^3 - 9x^2 + 24x - 36$.
We already know the Sturm system for this polynomial:

$$\begin{aligned} P_0(x) &= x^3 - 9x^2 + 24x - 36 \\ P_1(x) &= P'(x) = 3x^2 - 18^x + 24 \\ P_2(x) &= 18x + 108 \\ P_3(x) &= -77760 \end{aligned}$$

Signs	1	2	3	4	5	6
$P_0(x)$	-	-	-	-	-	0
$P_1(x)$	+	0	-	0	+	+
$P_2(x)$	+	+	+	+	+	+
$P_3(x)$	-	-	-	-	-	-
Changes	2	2	2	2	2	1

From here it is obtained that zero is between 5 and 6.
In fact, in this case it was directly found that $P(6) = 0$, that is, the only zero that the polynomial has is $x = 6$.
If zero had not been found directly, some approximation method would have had to be used on the interval]5, 6[.

EXAMPLE 3 (Sturm's Theorem)

Isolate the zeros of $P(x) = x^3 + 4x^2 - 7$.
We already know the Sturm system for this polynomial:

$$\begin{aligned} P_0(x) &= x^3 + 4x^2 - 7 \\ P_1(x) &= P'(x) = 3x^2 + 8x \\ P_2(x) &= 32x + 63 \\ P_3(x) &= 4221 \end{aligned}$$

Signs	$-\infty$	0	$+\infty$
$P_0(x)$			
$P_1(x)$			
$P_2(x)$			
$P_3(x)$			
Changes			

EXAMPLE 3 (Sturm's Theorem)

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Signs	- ∞	0	$+\infty$
$P_0(x)$	-	-	$+$
$P_1(x)$	$+$	0	$+$
$P_2(x)$	-	$+$	$+$
$P_3(x)$	$+$	$+$	$+$
Changes	3	1	0

Number of roots between- ∞ y 0: 3-1=2
Number of roots between 0 y $+\infty$: 1-0=1
Now we can go evaluating one by one starting from 0 until we find in which intervals the roots are found.