# APPROXIMATE RESOLUTION OF TRANSCENDENT EQUATIONS

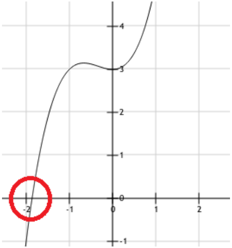

Francisco Javier Fernández

Universidad Pública de Navarra

Depto. Estadística, Informática y Matemáticas
Pamplona

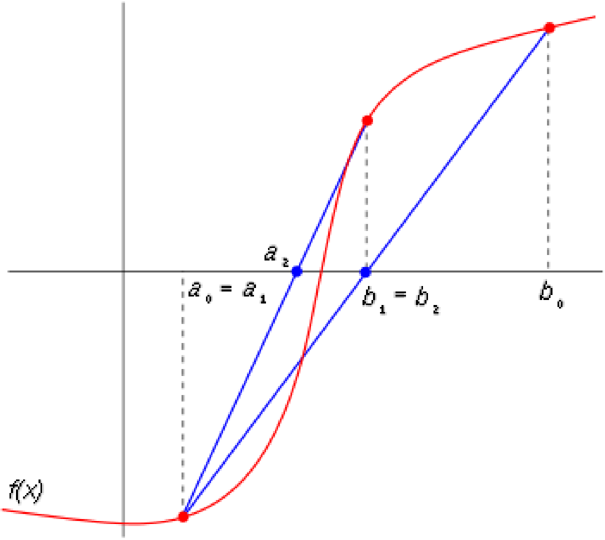

Computing 2024

# About the previous class ...

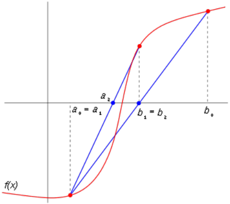



- Introduction to solving transcendent equations $F(x) = 0$:
  1. **Dimensioning**; .
  2. **Separation**;
  3. **Approximate resolution**.
- Bisection Method
  1. Approximate the root of the equation by the intermediate points $p$ of the interval $[m, M]$ that contains the root, narrowing the interval until the error between two successive approximations $a$ and $b$ is $\frac{b-a}{2} < \epsilon$ or $|f(p)| < \epsilon$.
  2. How do you narrow the interval at each iteration?

# Method of Regula falsi or false position

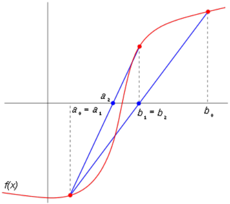


# Method of Regula falsi or false position




- As in the bisection method, we start from an initial interval $[a_0, b_0]$ with $f(a_0)$ and $f(b_0)$ of opposite signs, which guarantees that there are at least a root (Bolzano's theorem) inside of this interval.
- The algorithm successively obtains in each step a smaller interval $[a_{n-1}, b_{n-1}]$ that continues to include a root of the function $f$ .
- To do this, we draw a secant line defined by the points of the function at the endpoints of the interval $[a_0, b_0]$ and take the cross point with the X axis as the root approximation.

# Method of Regula falsi or false position

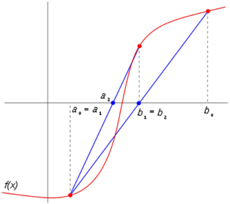



- From an interval $[a_{n-1}, b_{n-1}]$ calculate an interior point $p$ by drawing a secant line between these two points, that is, obtaining the equation of the line that goes through the two points:

$$p = a_{n-1} - \frac{f(a_{n-1})(b_{n-1} - a_{n-1})}{f(b_{n-1}) - f(a_{n-1})}$$

- Repeat the method with a new smaller interval containing the root: if $f(p)$ and $f(b_{n-1})$ have opposite signs, then we made $a_n = p$ and $b_n = b_{n-1}$. In the opposite case, $a_n = a_{n-1}$ and $b_n = p$.
- This process is carried out successively until the tangent line of the given function is reached; therefore, the point of tangency $p$ is the root or an approximation of the root when $| f(p) | <$ tolerance.

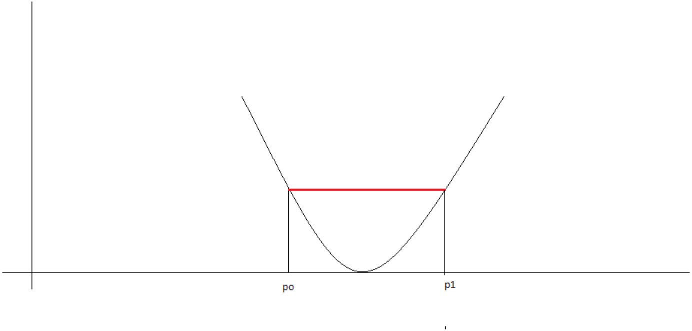
# Convergence of the Method of Regula Falsi




**Theorem**

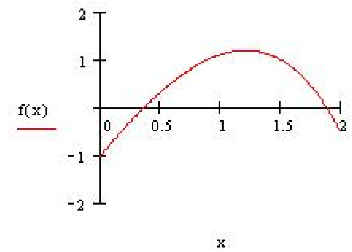
*Let $f : [a, b] \to \mathbb{R}$ with $f(p) = 0$, $f'(p) \neq 0$ and $f''$ continuous on $(p - \delta, p + \delta)$. If $f(x_n) \neq 0$ for all $n \geq 1$, the sequence obtained by the Regula Falsi method converges.*

# Example of when the Regula Falsi method does not converge



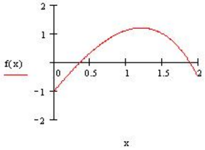

- If the root $p$ is of even multiplicity, we have that $f'(p) = 0$ and we can arrive at an interval $(p - \delta, p + \delta)$ that contains $p$.

# Example: Regula Falsi Method




- Find a root of $3x + \sin(x) - \exp(x) = 0$.
- Studying the function and its derivatives, it is clear that there is a root between 0 and 0.5 and also another root between 1.5 and 2.0.
- Now let us consider the function $f(x)$ in the interval $[0, 0.5]$ where $f(0) \cdot f(0.5) < 0$ and use the method of regula falsi to obtain the zero of $f(x) = 0$, with tolerance $\epsilon = 0.001$

## Example: regula falsi method

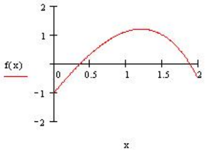



- Find a root of $3x + \sin(x) - \exp(x) = 0$, interval $[0, 0.5]$, tolerance $\epsilon = 0.001$

$$p = a_{n-1} - \frac{f(a_{n-1})(b_{n-1} - a_{n-1})}{f(b_{n-1}) - f(a_{n-1})}$$

| a | b | p | f (a) | f (b) | f (p) |
|---|---|---|---|---|---|
| 0 | 0.5 | | -1 | 0.33 | |
| ... | ... | ... | ... | ... | ... |

# Example: regula falsi method

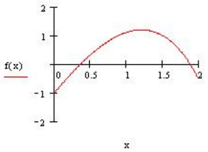



- Find a root of $3x + \sin(x) - \exp(x) = 0$, interval $[0, 0.5]$, tolerance $\epsilon = 0.001$

$$p = a_{n-1} - \frac{f(a_{n-1})(b_{n-1} - a_{n-1})}{f(b_{n-1}) - f(a_{n-1})}$$

| a | b | p | f (a) | f (b) | f (p) |
|---|---|---|---|---|---|
| 0 | 0.5 | 0.376 | -1 | 0.33 | 0.04 |
| ... | ... | ... | ... | ... | ... |
| ... | ... | ... | ... | ... | ... |

# Example: regula falsi method

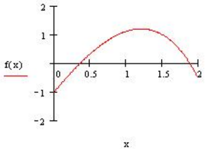



- Find a root of $3x + \sin(x) - \exp(x) = 0$, interval $[0, 0.5]$, tolerance $\epsilon = 0.001$

$$p = a_{n-1} - \frac{f(a_{n-1})(b_{n-1} - a_{n-1})}{f(b_{n-1}) - f(a_{n-1})}$$

| a | b | p | f (a) | f (b) | f (p) |
|---|---|---|---|---|---|
| 0 | 0.5 | 0.376 | -1 | 0.33 | 0.04 |
| 0 | 0.376 | | -1 | 0.04 | ... |
| ... | ... | ... | ... | ... | ... |

# Example: regula falsi method

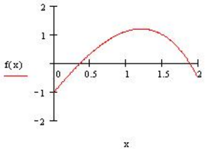



- Find a root of $3x + \sin(x) - \exp(x) = 0$, interval $[0, 0.5]$, tolerance $\epsilon = 0.001$

$$p = a_{n-1} - \frac{f(a_{n-1})(b_{n-1} - a_{n-1})}{f(b_{n-1}) - f(a_{n-1})}$$

| a | b | p | f (a) | f (b) | f (p) |
|---|---|---|---|---|---|
| 0 | 0.5 | 0.376 | -1 | 0.33 | 0.04 |
| 0 | 0.376 | 0.361 | -1 | 0.04 | 0.0014 |
| ... | ... | ... | ... | ... | ... |

# Example: regula falsi method

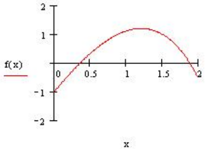



- Find a root of $3x + \sin(x) - \exp(x) = 0$, interval $[0, 0.5]$, tolerance $\epsilon = 0.001$

$$p = a_{n-1} - \frac{f(a_{n-1})(b_{n-1} - a_{n-1})}{f(b_{n-1}) - f(a_{n-1})}$$

| a | b | p | f (a) | f (b) | f (p) |
|---|---|---|---|---|---|
| 0 | 0.5 | 0.376 | -1 | 0.33 | 0.04 |
| 0 | 0.376 | 0.361 | -1 | 0.04 | 0.0014 |
| 0 | 0.361 | ... | -1 | 0.0014 | ... |

# Example: regula falsi method



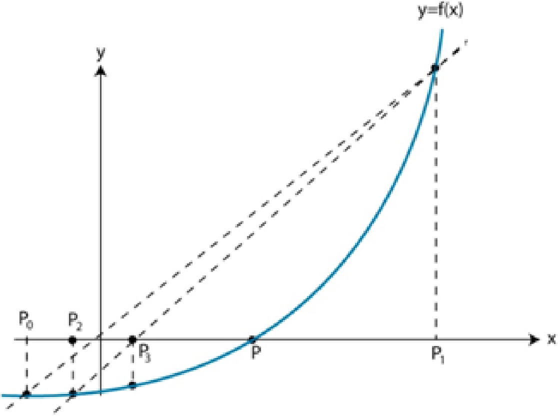

- Find a root of $3x + \sin(x) - \exp(x) = 0$, interval $[0, 0.5]$, tolerance $\epsilon = 0.001$

$$p = a_{n-1} - \frac{f(a_{n-1})(b_{n-1} - a_{n-1})}{f(b_{n-1}) - f(a_{n-1})}$$

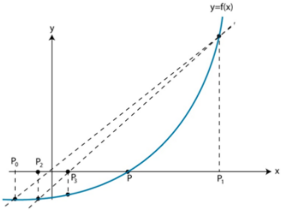
| a | b | p | f (a) | f (b) | f (p) |
|---|---|---|---|---|---|
| 0 | 0.5 | 0.376 | -1 | 0.33 | 0.04 |
| 0 | 0.376 | 0.361 | -1 | 0.04 | 0.0014 |
| 0 | 0.361 | 0.36 | -1 | 0.0014 | 0.001 |

The root is 0.36!

# Secant method

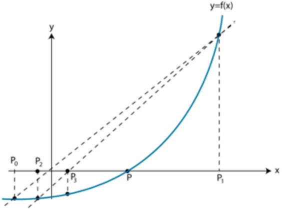


# Secant method




- This method follows the same principle as the Falsi Rule method, but differs the latter since the initial interval $[p_0, p_1]$ does not need to contain a root.
- The algorithm successively obtains in each step, given an interval $[p_{n-1}, p_n]$, a new approximation $p_{n+1}$ increasingly close to a root of the function $f$.
- To do this, we draw a secant line defined by the points of the function at the endpoints of the interval $[p_0, p_1]$ and take the intersection point with the X axis as an approximation of the root.

# Secant method




- From an interval $[p_{n-1}, p_n]$ calculate a point $p_{n+1}$ using the equation of the line that passes through these two points:

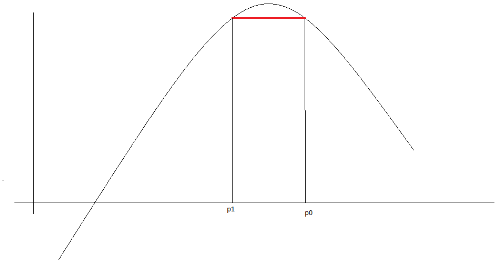
$$p_{n+1} = p_n - \frac{f(p_n)(p_{n-1} - p_n)}{f(p_{n-1}) - f(p_n)}$$

- Repeat the method with a new interval making $p_{n-1} \leftarrow p_n$ and $p_n \leftarrow p_{n+1}$.
- This process is carried out successively until the tangent line of the given function is reached; therefore, the point of tangency $p$ is the root or an approximation of the root when $\mid f(p) \mid < \text{tolerance}$.

# Convergence of the Secant Method

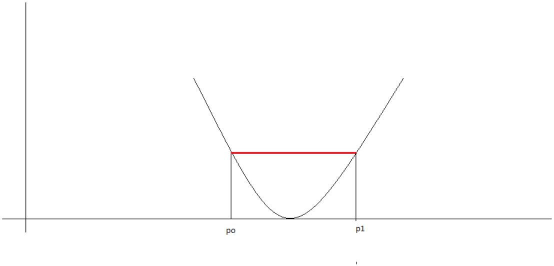
- Remember that with the secant method we do not have root bounding intervals.
- In case the initial approximations are too far or the root is not simple, the secant method may not converge.

# Convergence of the Secant Method




- P.ex., if the initial approximations are too far from the root, we can arrive at an interval $[p_{n-1}, p_n]$ that contains a point $p$ such that $f(p) \neq 0$, that is $p$ is not a root, but $f'(p) = 0$.

# Convergence of the Secant Method




- Also if the root $p$ is of even multiplicity, we have that $f'(p) = 0$ and we can arrive at an interval $[p_{n-1}, p_n]$ that contains $p$.

# Convergence of the secant method versus the Regula Falsi method

- Although the Secant method is divergent in many cases, when it converges it does faster than the Regula Falsi method.
- The inferiority of the Regula Falsi method is due to the fact that many times one of the ends of the interval is kept fixed, to guarantee that the interval contains the root.
- This property, although it guarantees the convergence of the method whenever $f'(p) \neq 0$, has a disadvantage in relation to the speed of convergence.

## Convergence of the Secant Method

- How to guarantee that the method converges?
- How to choose the initial values?

# Convergence of the Secant Method

**Theorem**

*Theorem (Local convergence of the secant method) If $f''$ is continuous in an environment of $p$ where $f(p) = 0$ and $f'(p) \neq 0$ then there is an interval $(p - \delta, p + \delta)$ for which the secant method converges for any initial values $x_0, x_1 \in (p - \delta, p + \delta)$.*

So we must know the graph of $f$ to determine this interval.

## Stop criteria

- As the method can not converge, we must analyze if $\mid p_{n+1} - p_n \mid <$ tolerance, and set the maximum number of iterations.

# Example of the secant method

The secant method applied to the polynomial $f(x) = x^7 - 3x^6 - 12x^5 + 12x^4 - 27x^3 + 34x^2 - 14x + 18$ with the initial interval $(-4, -3)$ previously determined and with $\varepsilon = 10^{-9}$, setting the maximum number of iterations at 10.

| n | p n-1 | p n | f(p n-1) | f(p n) | p n+1 | f(p n+1) | \| p n+1 - p n \| |
|---|---|---|---|---|---|---|---|
| 1 | -4.000000 | -3.000000 | -10966 | 609 | $\boxed{-3.052613}$ | 470.9336 | $\boxed{0.05261}$ |
| | | | | | | | |

$$p_{n+1} = p_n - \frac{f(p_n)(p_{n-1} - p_n)}{f(p_{n-1}) - f(p_n)}$$

# Example of the secant method

The secant method applied to the polynomial $f(x) = x^7 - 3x^6 - 12x^5 + 12x^4 - 27x^3 + 34x^2 - 14x + 18$ with the initial interval $(-4, -3)$ previously determined and with $\varepsilon = 10^{-9}$, setting the maximum number of iterations at 10.

| n | p n-1 | p n | f(p n-1) | f(p n) | p n+1 | f(p n+1) | \| p n+1 - p n \| |
|---|---|---|---|---|---|---|---|
| 1 | -4.000000 | -3.000000 | -10966 | 609 | -3.052613 | 470.9336 | 0.05261 |
| 2 | -3.000000 | -3.052613 | 609.00 | 470.933 | -3.232074 | -232.340 | 0.1794 |

$$p_{n+1} = p_n - \frac{f(p_n)(p_{n-1} - p_n)}{f(p_{n-1}) - f(p_n)}$$

# Example of the secant method

The secant method applied to the polynomial $f(x) = x^7 - 3x^6 - 12x^5 + 12x^4 - 27x^3 + 34x2 - 14x + 18$ with the initial interval $(-4, -3)$ previously determined and with $\varepsilon = 10^{-9}$, setting the maximum number of iterations at 10.

| n | p n-1 | p n | f(p n-1) | f(p n) | p n+1 | f(p n+1) | \| p n+1 - p n \| |
|---|---|---|---|---|---|---|---|
| 1 | -4.000000 | -3.000000 | -10966 | 609 | -3.052613 | 470.9336 | 0.05261 |
| 2 | -3.000000 | -3.052613 | 609.00 | 470.933 | -3.232074 | -232.340 | 0.1794 |
| 3 | -3.052613 | -3.232074 | 470.933 | -232.340 | -3.172785 | 44.3865 | 0.05928 |

$$p_{n+1} = p_n - \frac{f(p_n)(p_{n-1} - p_n)}{f(p_{n-1}) - f(p_n)}$$

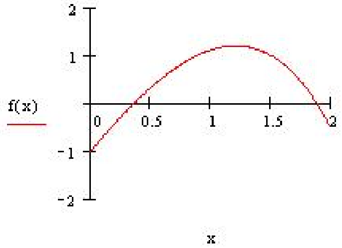
# Example of the secant method

The secant method applied to the polynomial $f(x) = x^7 - 3x^6 - 12x^5 + 12x^4 - 27x^3 + 34x2 - 14x + 18$ with the initial interval $(-4, -3)$ previously determined and with $\varepsilon = 10^{-9}$, setting the maximum number of iterations to 10.

| n | p n-1 | p n | f(p n-1) | f(p n) | p n+1 | f(p n+1) | \| p n+1 - p n \| |
|---|---|---|---|---|---|---|---|
| 1 | -4.000000 | -3.000000 | -10966 | 609 | -3.052613 | 470.9336 | 0.05261 |
| 2 | -3.000000 | -3.052613 | 609.00 | 470.933 | -3.232074 | -232.340 | 0.1794 |
| 3 | -3.052613 | -3.232074 | 470.933 | -232.340 | -3.172785 | 44.3865 | 0.05928 |
| 4 | -3.232074 | -3.172785 | -232.340 | 44.3865 | -3.182295 | 3.19515 | 0.0095 |
| 5 | -3.172785 | -3.182295 | 44.3865 | 3.19515 | -3.183033 | -0.04947 | 0.0007 |
| 6 | -3.182295 | -3.183033 | 3.1951 | -0.04946 | -3.183022 | 5.3810E-005 | 1.12468E-005 |
| 7 | -3.183033 | -3.183022 | -0.0494 | 5.38105E-005 | -3.183022 | 9.02993E-010 | 1.22203E-008 |

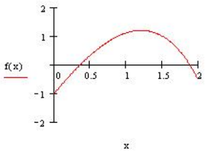
The root is $p = -3.18302162$.

# Example 2 of the secant method




- Find a root of $3x + \sin(x) - \exp(x) = 0$ (the same equation studied with the Falsi Rule Method).
- Let us consider the same interval $[0, 0.5]$ and we will use the Secant method to obtain the zero of $f(x) = 0$, with tolerance $\epsilon = 0.001$.

# Example 2 of the secant method

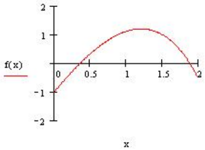



- Find a root of $3x + \sin(x) - \exp(x) = 0$, interval $[0, 0.5]$, tolerance $\epsilon = 0.001$

$$p_{n+1} = p_n - \frac{f(p_n)(p_{n-1} - p_n)}{f(p_{n-1}) - f(p_n)}$$

| $p_{n-1}$ | $p_n$ | $f(p_{n-1})$ | $f(p_n)$ | $p_{n+1}$ | $f(p_{n+1})$ | $\mid p_{n+1} - p_n \mid$ |
|---|---|---|---|---|---|---|
| 0 | 0.5 | -1 | 0.33 | 0.375 | 0.0362 | 0.1240 |
| ... | ... | ... | ... | ... | ... | |

# Example 2 of the secant method

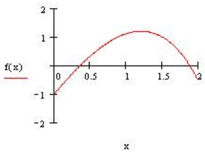



- Find a root of $3x + \sin(x) - \exp(x) = 0$, interval $[0, 0.5]$, tolerance $\epsilon = 0.001$

$$p_{n+1} = p_n - \frac{f(p_n)(p_{n-1} - p_n)}{f(p_{n-1}) - f(p_n)}$$

| $p_{n-1}$ | $p_n$ | $f(p_{n-1})$ | $f(p_n)$ | $p_{n+1}$ | $f(p_{n+1})$ |
|---|---|---|---|---|---|
| 0 | 0.5 | -1 | 0.33 | 0.375 | 0.0362 |
| textcolor[rgb]0.00,0.07,1.00 0.5 | 0.375 | 0.33 | 0.0362 | ... | ... |
| ... | ... | ... | ... | ... | ... |

# Example 2 of the secant method

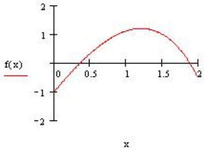



- Find a root of $3x + \sin(x) - \exp(x) = 0$, interval $[0, 0.5]$, tolerance $\epsilon = 0.001$

$$p_{n+1} = p_n - \frac{f(p_n)(p_{n-1} - p_n)}{f(p_{n-1}) - f(p_n)}$$

| $p_{n-1}$ | $p_n$ | $f(p_{n-1})$ | $f(p_n)$ | $p_{n+1}$ | $f(p_{n+1})$ | $\|$ |
|---|---|---|---|---|---|---|
| 0 | 0.5 | -1 | 0.33 | 0.375 | 0.0362 | |
| textcolor[rgb]0.00,0.07,1.00 0.5 | 0.375 | 0.33 | 0.0362 | 0.36 | -0.0011 | |
| ... | ... | ... | ... | ... | ... | |

# Example 2 of the secant method



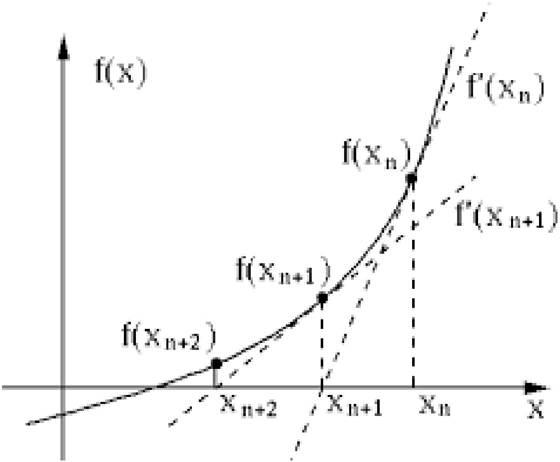

- Find a root of $3x + \sin(x) - \exp(x) = 0$, interval $[0, 0.5]$, tolerance $\epsilon = 0.001$

$$p_{n+1} = p_n - \frac{f(p_n)(p_{n-1} - p_n)}{f(p_{n-1}) - f(p_n)}$$

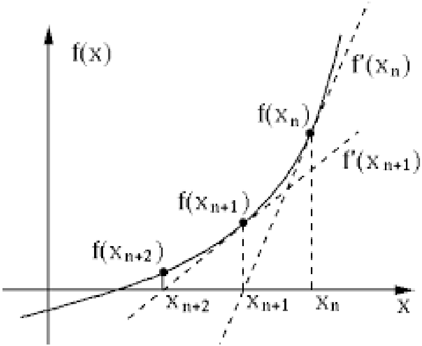
| $p_{n-1}$ | $p_n$ | $f(p_{n-1})$ | $f(p_n)$ | $p_{n+1}$ | $f(p_{n+1})$ |
|---|---|---|---|---|---|
| 0 | 0.5 | -1 | 0.33 | 0.375 | 0.0362 |
| textcolor[rgb]0.00,0.07,1.00 0.5 | 0.375 | 0.33 | 0.0362 | 0.36 | -0.0011 |
| 0.375 | 0.36 | 0.0362 | -0.0011 | 0.36 | -0.0011 |

The root is 0.36.

# Newton-Raphson (NR) method

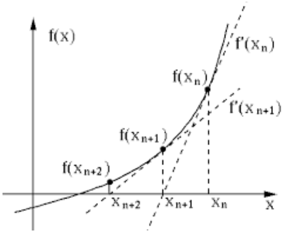


# Newton-Raphson (NR) method




- This method differs from the previous methods once it only needs an initial approximation $x_n$ and not an interval.
- It is about bringing the secant method to the limit and, therefore, in each iteration $n$, consider the tangent line to $f(x)$ at $(x_n, f(x_n))$ and take as the following approximation $x_{n+1}$ the intersection of the tangent with the X axis.
- The algorithm successively obtains in each step, given an approximation $x_n$ a new approximation $x_{n+1}$ increasingly closer to a root of the function $f$.

# NR method




- For this, draw a line tangent to the curve at the point $x_n$, taking into account that the equation of the tangent line to the graph of $f(x)$ at the point $(x_n, f(x_n))$ is

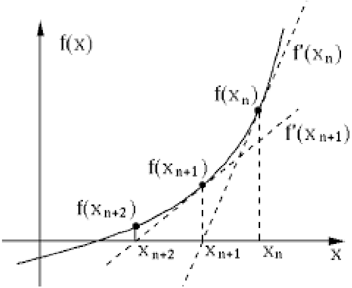
$$y - f(x_n) = f'(x_n)(x - x_n).$$

- It results is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

provided that $f'(x_n) \neq 0$.

# NR method



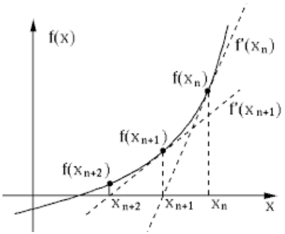

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

- Repeat the method with the new approximation $x_{n+1}$ and so on, until the point of tangency of the function is the root $p$ or an approximation of the root that respects the given tolerance.

# Convergence of the NR method

- How to guarantee that the method converges?
- How to choose the initial value?

# Convergence of Newton Raphson's Method



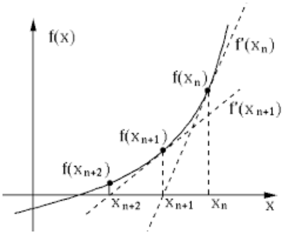

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

**Theorem**

*Let $f : [a, b] \to \mathbb{R}$, continuous and of class $C^2$ (it can be derived $n$ times and its $n$-th derivative is continuous). Let $p \in [a, b]$ such that $f(p) = 0$ and $f'(p) \neq 0$. Then, there exists $\delta > 0$ such that the sequence generated by Newton-Raphson with initial value $p_0 \in (p - \delta, p + \delta)$ converges.*

Then we must know the graph of $f$ to determine this interval and choose the initial value within this interval.

# Convergence of the Newton-Raphson method



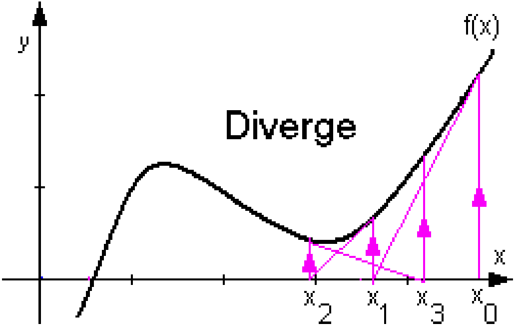

**Theorem**

*Let* $f \in C^2([a,b])$ *such that*

- $f(a)f(b) < 0$;
- $f'(x) \neq 0$ *for all* $x \in [a,b]$;
- $f''(x)f''(y) \geq 0$ *for all* $x, y \in [a,b]$.
- $\max\left(\frac{|f(a)|}{|f'(a)|}, \frac{|f(b)|}{|f'(b)|}\right) \leq b - a$

*Then, the Newton-Raphson method converges to the only zero that exists in the interval* $[a,b]$.

# When the Método NR does not converge




- P.ex., if the initial approximations are too far from the root, we can arrive at an environment that contains a point $p$ such that $f(p) \neq 0$, that is, $p$ is not a root, but $f'(p) = 0$.

# Annotation of the error for NR

**Theorem**

*Let* $f \in C^2([a,b])$ *such that*

- $f(a)f(b) < 0$;
- $f'(x) \neq 0$ *for all* $x \in [a,b]$;
- $f''(x) \neq 0$ *for all* $x \in [a,b]$.

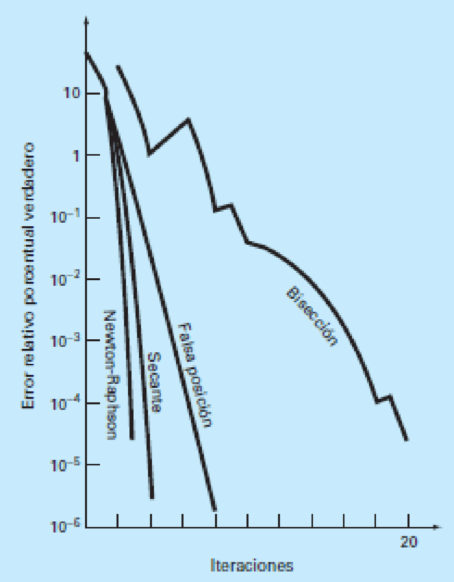
*So if there exist* $m, M \in (0,\infty)$ *such that* $|f'(x)| > m$ *and* $|f''(x)| < M$ *for all* $x \in [a,b]$, *we have to*

- $|p - p_n| \leq \frac{M}{2m}|p - p_{n-1}|^2$;
- $|p - p_n| \leq \frac{M}{2m}|p_n - p_{n-1}|^2$.

# Observations on the NR Method

- It is the fastest;
- You need to determine the derivative (a formula for each case), in practical applications, this can be a problem;
- Requires that $f'(n) \neq 0$ throughout the process;
- Choosing the initial approximation can be tricky (analyze the graph, derivatives, etc.), because if the initial value is too far from the root, the method can be divergent: it will not find the root, in fact, it will move away from it;
- The delimitation of the error is complicated;
- The most common stopping criterion is the repetition of approximations or $\mid x_{n+1} - x_n \mid <$ tolerance and we can set the maximum number of iterations for the case where it does not agree.

# Comparison between methods



# Multiple zeros

**Definition**

*A solution* $p$ *of the equation* $f(x) = 0$ *is a zero of multiplicity* $m$ *if there exists a function* $g(x)$ *such that* $\lim_{x \to p} g(x) \neq 0$ *so*

$$f(x) = (x - p)^m g(x)$$

**Theorem**

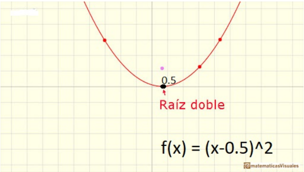
$f : [a, b] \to \mathbb{R}$ *has a single zero in* $p \in (a, b)$ *if and only if* $f(p) = 0$ *and* $f'(p) \neq 0$.

# Multiple zeros

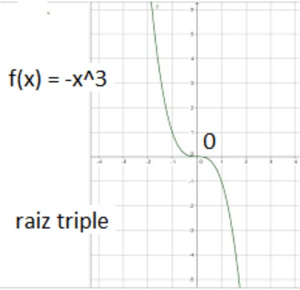
**Theorem**

*$f:[a,b]\to\mathbb{R}$ has a zero of multiplicity $m$ in $p\in(a,b)$ if and only if $0=f(p)=f'(p)=\cdots=f^{(m-1)}(p)$ and $f^{(m)}(p)\neq 0$.*

# Examples




$$f(0.5) = 0,\ f'(0.5) = 0,\ f''(0.5) = 2 \neq 0$$




$$f(0) = 0,\ f'(0) = 0,\ f''(0) = 0,\ f'''(0) = 6 \neq 0$$

# Finding multiple zeros with N-R

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

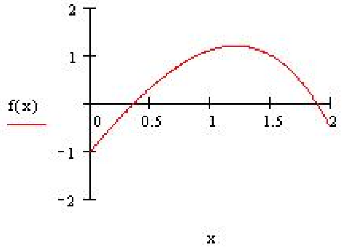
- When we arrive at a multiple root, we have that, for a given $n$, $f'(x_n) = 0$. And we cannot go on, because in this case $x_n$ is the root of the derivative as well.
- So we take $\mu(x_n) = \frac{f(x_n)}{f'(x_n)}$, and, by NR:

$$x_{n+1} = x_n - \frac{\mu(x_n)}{\mu'(x_n)},$$

resulting:

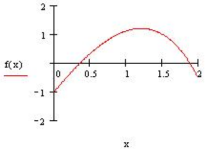
$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{(f'(x_n))^2 - f(x_n)f''(x_n)}$$

# Example of the NR method




- Find a root of $3x + \sin(x) - \exp(x) = 0$ (the same equation studied with the other methods).
- Let us consider the initial approximation $x_0 = 0$ and we will use the NR method to obtain the zero of $f(x) = 0$, with tolerance $\epsilon = 0.001$.

# Example of the NR method

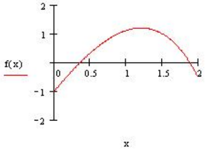



- Find a root of $f(x) = 3x + \sin(x) - \exp(x)$, interval $[0, 0.5]$, tolerance $\epsilon = 0.001$
- $f'(x) = 3 + \cos(x) - \exp(x)$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

| $x_n$ | $f(x_n)$ | $f'(x_n)$ | $x_{n+1}$ | $\mid x_{n+1} - x_n \mid$ |
|---|---|---|---|---|
| 0 | -1 | 3 | 0.333 | 0.333 |
| ... | ... | ... | ... | ... |

# Example of the NR method

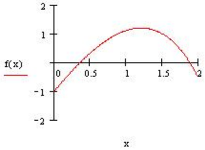



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x) = 3x + \sin(x) - \exp(x), \; f'(x) = 3 + \cos(x) - \exp(x)$$

| $x_n$ | $f(x_n)$ | $f'(x_n)$ | $x_{n+1}$ | $\mid x_{n+1} - x_n \mid$ |
|---|---|---|---|---|
| 0 | -1 | 3 | 0.333 | 0.333 |
| 0.333 | -0.0011 | 2.55 | 0.36 | 0.027 |
| ... | ... | ... | ... | ... |

# Example of the NR method




$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$f(x) = 3x + \sin(x) - \exp(x),\ f'(x) = 3 + \cos(x) - \exp(x)$

| $x_n$ | $f(x_n)$ | $f'(x_n)$ | $x_{n+1}$ | $\mid x_{n+1} - x_n \mid$ |
|---|---|---|---|---|
| 0 | -1 | 3 | 0.333 | 0.333 |
| 0.333 | -0.069 | 2.55 | 0.36 | 0.027 |
| 0.36 | -0.0011 | 2.503 | 0.36 | 0 |