Solutions Manual for

How to Read and Do Proofs

An Introduction to Mathematical Thought Processes

Sixth Edition

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Solutions to Exercises

1.1 (a), (c), and (e) are statements.

1.2 (a), (c), and (d) are statements.

1.3 a. Hypothesis: The right triangle XYZ with sides of lengths x and y and hypotenuse of length z has an area of $z^2/4$.
Conclusion: The triangle XYZ is isosceles.
b. Hypothesis: n is an even integer.
Conclusion: n^2 is an even integer.
c. Hypothesis: a, b, c, d, e, and f are real numbers for which $ad - bc \neq 0$.
Conclusion: The two linear equations $ax + by = e$ and $cx + dy = f$ can be solved for x and y.

1.4 a. Hypothesis: r is a real number that satisfies $r^2 = 2$.
Conclusion: r is irrational.
b. Hypothesis: p and q are positive real numbers such that $\sqrt{pq} \neq (p+q)/2$.
Conclusion: $p \neq q$.
c. Hypothesis: $f(x) = 2^{-x}$ for all real numbers x.
Conclusion: There exists a real number x such that $0 \leq x \leq 1$ and $f(x) = x$.

1.5 a. Hypothesis: A, B and C are sets of real numbers with $A \subseteq B$.
Conclusion: $A \cap C \subseteq B \cap C$.

b. Hypothesis: For a positive integer n, the function f defined by:

$$f(n) = \begin{cases} n/2, & \text{if } n \text{ is even} \\ 3n+1, & \text{if } n \text{ is odd} \end{cases}$$

For an integer $k \geq 1$, $f^k(n) = f^{k-1}(f(n))$, and $f^1(n) = f(n)$.
Conclusion: For any positive integer n, there is an integer $k > 0$ such that $f^k(n) = 1$.

c. Hypothesis: x is a real number.
Conclusion: The minimum value of $x(x-1) \geq -1/4$.

1.6 Jack's statement is true. This is because the hypothesis that Jack did not get his car fixed is false. Therefore, according to rows 3 and 4 of Table 1.1, the if/then statement is true, regardless of the truth of the conclusion.

1.7 Jack's statement is false. This is because the hypothesis, getting his car fixed, is true while the conclusion, not missing the interview, is false. Therefore, according to row 2 of the Table 1.1, the if/then statement is false.

1.8 Jack won the contest. This is because the hypothesis that Jack is younger than his father is true, and, because the if/then statement is true, row 1 of Table 1.1 is applicable. Therefore, the conclusion that Jack will not lose the contest is also true.

1.9 a. True because $A : 2 > 7$ is false (see rows 3 and 4 of Table 1.1).
b. True because $B : 1 < 2$ is true (see rows 1 and 3 of Table 1.1).

1.10 a. True because $1 < 3$ is true (see rows 1 and 3 of Table 1.1).
b. True if $x \neq 3$ (see rows 3 and 4 of Table 1.1).
False when $x = 3$ because then the hypothesis is true and the conclusion $1 > 2$ is false (see row 2 of Table 1.1).

1.11 If you want to prove that "A implies B" is true and you know that B is false, then A should also be false. The reason is that, if A is false, then it does not matter whether B is true or false because Table 1.1 ensures that "A implies B" is true. On the other hand, if A is true and B is false, then "A implies B" would be false.

1.12 When B is true, rows 1 and 3 of Table 1.1 indicate that the statement "A implies B" is true. You therefore need only consider the case when B is false. In this case, for "A implies B" to be true, it had better be that A is false so that row 4 of Table 1.1 is applicable. In other words, you can assume B is false; your job is to show that A is false.

1.13 (T = true, F = false)

A	B	C	$B \Rightarrow C$	$A \Rightarrow (B \Rightarrow C)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

1.14 (T = true, F = false)

A	B	C	$A \Rightarrow B$	$(A \Rightarrow B) \Rightarrow C$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	F

1.15 (T = true, F = false)

A	B	$B \Rightarrow A$	$A \Rightarrow B$
T	T	T	T
T	F	T	F
F	T	F	T
F	F	T	T

From this table, $B \Rightarrow A$ is not always true at the same time $A \Rightarrow B$ is true.

1.16 From row 2 of Table 1.1, you must show that A is true and B is false.

1.17 a. For A to be true and B to be false, it is necessary to find a real number $x > 0$ such that $\log_{10}(x) \leq 0$. For example, $x = 0.1 > 0$, while $\log_{10}(0.1) = -1 \leq 0$. Thus, $x = 0.1$ is a desired counterexample. (Any value of x such that $0 < x \leq 1$ would provide a counterexample.)

b. For A to be true and B to be false, it is necessary to find an integer $n > 0$ such that $n^3 < n!$. For example, $n = 6 > 0$, while $6^3 = 216 < 720 = 6!$. Thus, $n = 6$ is a desired counterexample. (Any integer $n \geq 6$ would provide a counterexample.)

1.18 a. For A to be true and B to be false, it is necessary to find a positive integer n such that $3^n < n!$. For example, $n = 6 > 0$, while $3^6 = 729 < 720 = 6!$. Thus, $n = 6$ is a desired counterexample. (Any integer $n \geq 6$ would produce a valid counterexample.)

b. For A to be true and B to be false, it is necessary to find a real number x between 0 and 1 such that the first three decimal digits of x are the same as the first three decimal digits of 2^{-x}. Using trial-and-error, the first three decimal digits of $x = 0.641$ are the same as the first three decimal digits of $2^{-0.641} \approx 0.641268$. Thus, $x = 0.641$ is a desired counterexample.

2

Solutions to Exercises

2.1 The forward process makes use of the information contained in the hypothesis A. The backward process tries to find a chain of statements leading to the fact that the conclusion B is true.

With the backward process, you start with the statement B that you are trying to conclude is true. By asking and answering key questions, you derive a sequence of new statements with the property that, if the sequence of new statements is true, then B is true. The backward process continues until you obtain the statement A or until you can no longer ask and/or answer the key question.

With the forward process, you begin with the statement A that you assume is true. You then derive from A a sequence of new statements that are true as a result of A being true. Every new statement derived from A is directed toward linking up with the last statement obtained in the backward process. The last statement of the backward process acts as the guiding light in the forward process, just as the last statement in the forward process helps you choose the right key question and answer.

2.2 When formulating the key question you should look at the last statement in the backward process. When answering the key question you should be guided by the last statement in the forward process.

2.3 a. This question is incorrect because it asks how to prove the hypothesis, not the conclusion. This key question is also incorrect because it uses specific notation from the problem.
b. This question is incorrect because it asks how to prove the hypothesis, not the conclusion.
c. This question is incorrect because it uses the specific notation given in the problem.
d. This question is correct.

2.4 (c) is incorrect because it uses the specific notation given in the problem.

2.5 a. This question is incorrect because it asks how to prove the hypothesis, not the conclusion. This key question is also incorrect because it uses specific notation from the problem.
b. This question is incorrect because it uses the specific notation given in the problem.
c. This question is correct.

2.6 a. This question is incorrect because it uses the specific notation given in the problem.
b. This question is incorrect because it asks how to prove the hypothesis, not the conclusion.
c. This question is correct.

2.7 All of these questions are valid.

2.8 (a) is correct because it asks, without symbols, how to prove that the statement B is true. Questions (b) and (c) are not valid because they use the specific notation in the problem. Question (d) is an incorrect question for this conclusion, which involves the concept of a subset.

2.9 Any answer to a key question for a statement B that results in a new statement $B1$ must have the property that, if $B1$ is true, then B is true. In this case, the answer that "$B1$: the integer is odd" does not mean that "B : the integer is prime." For example, the odd integer $9 = 3(3)$ is not prime.

2.10 (d) is incorrect because the fact that the two lines lie on opposite sides of a quadrilateral does not ensure that the two lines are parallel.

2.11 (c) is incorrect because the fact that the squares of two integers are equal does not guarantee that the integers are equal. For example, the square of the integers -5 and 5 are both 25, yet the two integers are not equal.

2.12 In addition to (c) being incorrect, the answer in part (d) would also be incorrect. For example, the absolute value of the difference of the real numbers 1.1 and 1.3 is $0.2 < 1$, yet these two real numbers are not equal.

2.13 a. How can I show that two lines are parallel?
How can I show that two lines do not intersect?
How can I show that two lines tangent to a circle are parallel?
How can I show that two tangent lines passing through the endpoints of the diameter of a circle are parallel?
b. How can I show that a function is a polynomial?
How can I show that the sum of two functions is a polynomial?
How can I show that the sum of two polynomials is a polynomial?

2.14 a. How can I show that an integer (namely, n^2) is even?
How can I show that the square of an integer (namely, n) is even?
b. How can I show that two equations have an integer solution?
How can I show that an integer (namely, n) satisfies a given equation (namely, $2n^2 - 3n = -2$)?
How can I show that an integer (namely, n) is the root of a quadratic equation (namely, $2n^2 - 3n + 2 = 0$)?

2.15 a. How can I show that a number (namely, $a^2 + b^2$) is less than or equal to another number (namely, $(a + b)^2$)?
How can I show that the sum of the squares of two numbers (namely, a and b) is less than or equal to the square of the sum of the two numbers?
b. How can I show that two lines are perpendicular?
How can I show that two lines with slopes whose product is -1 are perpendicular?
How can I show that two lines intersect at an angle of 90^0?

2.16 a. How can I show that two numbers (namely, $\overline{RS}$ and $\overline{ST}$) are equal?
How can I show that two sides of a triangle (namely, RS and ST) have equal length?
b. How can I show that a set (namely, R intersect T) is not empty?
How can I show that the intersection of two sets (namely, R and T) contains at least one element?

2.17 a. Show that one number is $\leq$ the other number and vice versa.
Show that the ratio of the two (nonzero) numbers is 1.
Show that the two numbers are both equal to a third number.
b. Show that the elements of the two sets are identical.
Show that each set is a subset of the other.
Show that both sets are equal to a third set.

2.18 a. Show that the two lines do not intersect.
Show that the two lines are both perpendicular to a third line.
Show that the two lines are both vertical or have equal slopes.
Show that the two lines are each parallel to a third line.
Show that the equations of the two lines are identical or have no common solution.
b. Show that their corresponding side-angle-sides are equal.
Show that their corresponding angle-side-angles are equal.
Show that their corresponding side-side-sides are equal.
Show that they are both congruent to a third triangle.

2.19 a. Show that the two lines intersect with an angle of 90 degrees.
Show that the two lines have slopes whose product is -1.
Show that the one line is perpendicular (parallel) to a third line that is parallel (perpendicular) to the other line.
b. Show that all the sides are equal.
Show that all angles are equal.
Show that two of the angles are 60 degrees.
Show that two of the exterior angles are 120 degrees.
Show that it is an isosceles triangle and has an angle of 60 degrees.
Show that it is similar to another equilateral triangle.

2.20 a. Show that the integer, n, is the product of two other integers strictly between 1 and n.
Show that the integer can be divided evenly by some integer m with $1 < m < n$.
b. (Suppose you are trying to show that $S \subseteq T$.)
Show that every element of S is also in T.
Show that $S \cap T = T$.
Show that S is a subset of a set U and $U \subseteq T$.

2.21 a. Show that the first integer minus the second integer is less than 0.
Show that the ratio of the first integer to the second (nonzero) integer is less than 1.
Show that there is a real number larger than the first integer and smaller than the second integer.
b. Show that the integer can be divided evenly by 2.
Show that the remainder on dividing the integer by 2 is 0.
Show that the integer is not odd.
Show that the integer is equal to another integer that is even.

2.22 (1) How can I show that the solution to a quadratic equation is positive?
(2) Show that the quadratic formula gives a positive solution.
(3) Show that the solution $-b/2a$ is positive.

2.23 (1) How can I show that a triangle is equilateral?
(2) Show that the three sides have equal length (or show that the three angles are equal).
(3) Show that $\overline{RT} = \overline{ST} = \overline{SR}$ (or show that $\angle R = \angle S = \angle T$).

2.24 (1) How can I show that two lines are parallel?
(2) Show that the slopes of the lines are equal.
(3) Show that $-a/b = -c/d$.

2.25 (1) How can I show that a quadratic equation has no real solution?
(2) Show that when the quadratic formula is applied, the discriminant (that is, the expression under the square root) is negative.
(3) Show that the expression $3^2 - 4d$ is negative.

2.26 The given statement is obtained by working forward from the conclusion rather than the hypothesis.

2.27 The definition of subset is not applied correctly. Specifically, if R is a subset of S, it is not true that every element of S is also an element of R but rather, every element of R is also an element of S.

2.28 The incorrect statement is $2n + 1 < 2^n$. To see why, consider the counterexample of $n = 1$. In this case, $2(1) + 1 \not< 2^1$. For this proof to be valid, it is necessary to include in the hypothesis that $n > 2$, for then $2n + 1 < 2^n$.

2.29 a. $(x - 2)(x - 1) < 0$.
$x(x - 3) < -2$.
$-x^2 + 3x - 2 > 0$.
b. $x/z = 1/\sqrt{2}$.
Angle X is a 45-degree angle.
$\cos(X) = 1/\sqrt{2}$.
c. The circle has its center at (3,2).
The circle has a radius of 5.
The circle crosses the y-axis at (0,6) and $(0, -2)$.
$x^2 - 6x + 9 + y^2 - 4y + 4 = 25$.

2.30 a. Each side of the rectangle has the same length.
The length of a diagonal is $\sqrt{2}$ times the length of one side.
The rectangle's area is the square of the length of one of its sides.
b. n^2 is even.
n is even.
$(n + 1)(n - 1)$ is odd.
$n + 1$ and $n - 1$ are odd.

c. There is a value of x that satisfies $3x - 1 = x^2 + x$.
The line intersects the function at the point $x = 1$ and $y = 2$.
The derivative of x^2+x at the point where the tangent line touches the graph is 3.

2.31 (d) is not valid because "$x \neq 5$" is not stated in the hypothesis and so, if $x = 5$, it will not be possible to divide by $x - 5$.

2.32 (c) is not valid because n can be less than or equal to 1. For example, when $n = 1$, c is greater than b because c is the hypotenuse, therefore $c > b$ and $c^{1-2} < b^{1-2}$.

2.33 The third sentence, in which it says that $\sqrt{b^2 - 4ac} = b-2a$, is incorrect. This is because taking the positive square root of $b^2 - 4ac = (b - 2a)^2$ fails to take into account the second possible solution, namely $2a - b$; that is, $\sqrt{b^2 - 4ac} = \pm(b - 2a)$.

2.34 For sentence 1: The fact that $c^n = c^2c^{n-2}$ follows by algebra. The author then substitutes $c^2 = a^2 + b^2$, which is true from the Pythagorean theorem applied to the right triangle.
For sentence 2: For a right triangle, the hypotenuse c is longer than either of the two legs a and b so, $c > a$, $c > b$. Because $n > 2$, $c^{n-2} > a^{n-2}$ and $c^{n-2} > b^{n-2}$ and so, from sentence 1, $c^n = a^2c^{n-2} + b^2c^{n-2} > a^2(a^{n-2}) + b^2(b^{n-2})$.
For sentence 3: Algebra from sentence 2.

2.35 Key Question: How can I show that a real number is 0?
Key Answer: Show that the number is less than or equal to 0 and that the number is greater than or equal to 0.

2.36 a. **Analysis of Proof.** A key question associated with the conclusion is, "How can I show that a real number (namely, x) is 0?" To show that $x = 0$, it will be established that

B1: $x \leq 0$ and $x \geq 0$.

Working forward from the hypothesis immediately establishes that

A1: $x \geq 0$.

To see that $x \leq 0$, it will be shown that

B2: $x = -y$ and $-y \leq 0$.

Both of these statements follow by working forward from the hypotheses that

A2: $x + y = 0$ (so $x = -y$) and
A3: $y \geq 0$ (so $-y \leq 0$).

It remains only to show that

B3: $y = 0$,

which follows by working forward from the fact that

A4: $x = 0$

and the hypothesis that

A5: $x + y = 0$,

so,

A6: $0 = x + y = 0 + y = y$.

b. **Proof.** To see that both $x = 0$ and $y = 0$, it will first be shown that $x \geq 0$ (which is given in the hypothesis) and $x \leq 0$. The latter is accomplished by showing that $x = -y$ and that $-y \leq 0$. To see that $x = -y$, observe that the hypothesis states that $x + y = 0$. Similarly, $-y \leq 0$ because the hypothesis states that $y \geq 0$. Thus, $x = 0$. To see that $y = 0$, one can substitute $x = 0$ in the hypothesis $x + y = 0$ to reach the desired conclusion. $\square$

2.37 **Analysis of Proof.** A key question associated with the conclusion B is, "How can I show that a triangle (namely, SUR) is congruent to another triangle (namely, SUT)?" One answer is to use the side-angle-side theorem to show that

B1: $\overline{RU} = \overline{UT}$, $\angle RUS = \angle TUS$, and $\overline{SU} = \overline{SU}$.

The first of these follows from the hypothesis that SU is a perpendicular bisector of RT, so

A1: $\overline{RU} = \overline{UT}$.

The second part of $B1$ follows from SU being a perpendicular bisector of RT, so

A2: $\angle RUS = \angle TUS = 90^o$.

Finally, it is obvious that $\overline{SU} = \overline{SU}$, which, combined with $A1$ and $A2$, is $B1$, thus completing the proof.

2.38 **Analysis of Proof.** A key question associated with the conclusion is, "How can I show that an implication (namely, "A implies C") is true?" According to Table 1.1, one answer is to assume that

A1: A is true,

for which you must show that

B1: C is true.

Working forward from the hypothesis that "A implies B" is true, because A is true (see $A1$), it follows from Table 1.1 that

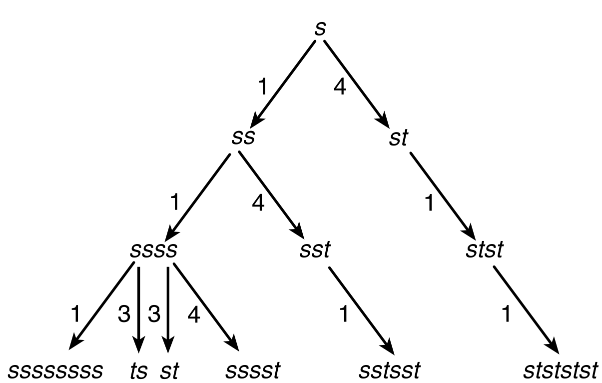
A2: B is true.

Similarly, working forward from the hypothesis that "B implies C" is true, because B is true (see $A2$), it follows from Table 1.1 that

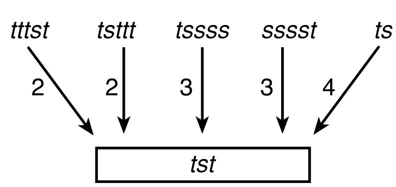
A3: C is true.

The proof is now complete because the last statement in the forward process ($A3$) is the same as the last statement in the backward process ($B1$).

2.39 a. The number to the left of each line in the following figure indicates which rule is used.



b. The number to the left of each line in the following figure indicates which rule is used.



c.

A:	s	given
A1:	ss	rule 1
A2:	$ssss$	rule 1
B1:	$sssst$	rule 4
B:	tst	rule 3

2.40 **Analysis of Proof.** In this problem one has:

A: The right triangle XYZ is isosceles.
B: The area of triangle XYZ is $z^2/4$.

A key question for B is, "How can I show that the area of a triangle is equal to a particular value?" One answer is to use the formula for computing the area of a triangle to show that

B1: $z^2/4 = xy/2$.

Working forward from the hypothesis that triangle XYZ is isosceles,

A1: $x = y$, so
A2: $x - y = 0$.

Because XYZ is a right triangle, from the Pythagorean theorem,

A3: $z^2 = x^2 + y^2$.

Squaring both sides of the equality in $A2$ and performing algebraic manipulations yields

A4: $(x - y)^2 = 0$.
A5: $x^2 - 2xy + y^2 = 0$.
A6: $x^2 + y^2 = 2xy$.

Substituting $A3$ in $A6$ yields

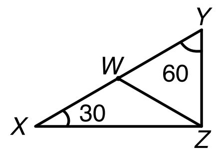
A7: $z^2 = 2xy$.

Dividing both sides by 4 finally yields the desired result:

A8: $z^2/4 = xy/2$.

Proof. From the hypothesis, $x = y$, or equivalently, $x - y = 0$. Performing algebraic manipulations yields $x^2 + y^2 = 2xy$. By the Pythagorean theorem, $z^2 = x^2 + y^2$ and on substituting z^2 for $x^2 + y^2$, one obtains $z^2 = 2xy$, or, $z^2/4 = xy/2$. From the formula for the area of a right triangle, the area of $XYZ = xy/2$. Hence $z^2/4$ is the area of the triangle. □

2.41 The analysis and the proof refer to the following figure:



Analysis of Proof. The conclusion is reached by showing that

B1: Triangle XWZ is isosceles and triangle WYZ is equilateral.

The key questions associated with $B1$ are, "How can I show that a triangle (namely, XWZ) is isosceles and a triangle (namely, WYZ) is equilateral?" One answer is to show that

B2: $\angle ZXW = \angle XZW$ and $\angle WYZ = \angle YZW = \angle ZWY$.

To that end, working forward from the hypothesis that W is the midpoint of the hypotenuse XY means that

A1: $\overline{XW} = \overline{WY}$.

Also, the fact that XYZ is a 30-60-90 degree triangle means that

A2: $\sin(\angle X) = \overline{YZ}/\overline{XY} = 1/2$, that is, $\overline{YZ} = \overline{XY}/2$.

Because $\overline{XY} = \overline{XW} + \overline{WY}$, you have from $A1$ that

A3: $\overline{XY} = 2\overline{XW}$.

Substituting $A3$ in $A2$ and using $A1$ yields

A4: $\overline{YZ} = \overline{XW} = \overline{WY}$.

The fact that $\overline{YZ} = \overline{WY}$ means that the angles opposite these sides in triangle WYZ are equal, that is,

A5: $\angle WZY = \angle ZWY$.

Furthermore, the sum of the angles of triangle WYZ is 180^o, so, using the hypothesis that $\angle ZYW = 60^o$, you have that

A6: $\angle WZY + \angle ZWY + 60^0 = 180^o$.

Combining $A5$ and $A6$ leads to the second part of $B2$, namely,

A7: $\angle WZY = \angle ZWY = \angle ZYW = 60^o$.

Finally, because $\angle XZY = 90^o$ and from $A7$, $\angle WZY = 60^o$, it follows that

A8: $\angle XZW = \angle XZY - \angle WZY = 90^o - 60^o = 30^0$.

Combining $A8$ with the hypothesis that $\angle ZXW = 30^0$ means that the first part of $B2$ is true, namely,

A9: $\angle ZXW = \angle XZW$.

The proof is now complete.

Proof. To see that triangle XWZ is isosceles and triangle WYZ is equilateral, it will be shown that $\angle ZXW = \angle XZW = 30^0$ and $\angle WYZ = \angle YZW = \angle ZWY = 60^0$. To that end, the hypothesis that W is the midpoint of XY means that $\overline{XW} = \overline{WY}$ and because triangle XYZ is a 30-60-90 degree triangle, $\overline{YZ} = \overline{XY}/2 = \overline{WY}$. But then, because $\overline{WY} = \overline{YZ}$, the angles opposite these sides in triangle WYZ must be equal. However, because $\angle WYZ = 60^o$ by hypothesis and the sum of the angles of triangle WYZ is 180^o, it follows that $\angle WYZ = \angle YZW = \angle ZWY = 60^0$, as desired.

It remains to show that $\angle XZW = 30^0$. However, this follows because $\angle XZY = 90^o$ by hypothesis and it has just been shown that $\angle YZW = 60^o$, so $\angle XZW = 30^0$, as desired, thus completing the proof. □

2.42 **Analysis of Proof.** A key question associated with the conclusion is, "How can I show that a triangle is equilateral?" One answer is to show that all three sides have equal length, specifically,

B1: $\overline{RS} = \overline{ST} = \overline{RT}$.

To see that $\overline{RS} = \overline{ST}$, work forward from the hypothesis to establish that

B2: Triangle RSU is congruent to triangle SUT.

Specifically, from the hypothesis, SU is a perpendicular bisector of RT, so

A1: $\overline{RU} = \overline{UT}$.

In addition,

A2: $\angle RUS = \angle SUT = 90^o$.
A3: $\overline{SU} = \overline{SU}$.

Thus the side-angle-side theorem states that the two triangles are congruent and so $B2$ has been established.

It remains (from $B1$) to show that

B3: $\overline{RS} = \overline{RT}$.

Working forward from the hypothesis you can obtain this because

A4: $\overline{RS} = 2\overline{RU} = \overline{RU} + \overline{UT} = \overline{RT}$.

Proof. To see that triangle RST is equilateral, it will be shown that $\overline{RS} = \overline{ST} = \overline{RT}$. To that end, the hypothesis that SU is a perpendicular bisector of RT ensures (by the side-angle-side theorem) that triangle RSU is congruent to triangle SUT. Hence, $\overline{RS} = \overline{ST}$. To see that $\overline{RS} = \overline{RT}$, by the hypothesis, one can conclude that $\overline{RS} = 2\overline{RU} = \overline{RU} + \overline{UT} = \overline{RT}$. □

2.43 **Analysis of Proof.** A key question associated with the conclusion is, "How can I show that the area of a triangle is equal to a particular number?" One answer is to use the formula of one-half the base times the height for the area. Doing so means that, in this case, you must show that

B1: $(\overline{RT})(\overline{SU})/2 = \sqrt{3}\left(\overline{RS}\right)^2/4.$

The idea now is to relate $\overline{RT}$ and $\overline{SU}$ to $\overline{RS}$. Specifically, because the hypothesis states that triangle RST is equilateral, it follows that

A1: $\overline{RT} = \overline{RS}.$

Also, from the hypothesis that SU is a perpendicular bisector of RT,

A2: Triangle RSU is a 30-60-90 degree triangle.

But then,

A3: $\overline{SU} = \sqrt{3}(\overline{RS})/2.$

Substituting $A1$ and $A3$ in the left side of $B1$, leads to the desired conclusion that

A4: $(\overline{RT})(\overline{SU})/2 = \overline{RS}(\sqrt{3}(\overline{RS})/2)/2 = \sqrt{3}\left(\overline{RS}\right)^2/4.$

Proof. The hypothesis that triangle RST is equilateral means that $\overline{RT} = \overline{RS}$. Furthermore, because SU is a perpendicular bisector of RT, RSU is a 30-60-90 degree triangle. Hence, $\overline{SU} = \sqrt{3}(\overline{RS})/2$. But then,

$$(\overline{RT})(\overline{SU})/2 = \overline{RS}(\sqrt{3}(\overline{RS})/2)/2 = \sqrt{3}\left(\overline{RS}\right)^2/4.$$

In other words, the area of triangle RST is $\sqrt{3}\left(\overline{RS}\right)^2/4$, thus completing the proof. □

3

Solutions to Exercises

3.1 a. Key Question: How can I show that an integer (namely, n^2) is odd?
Abstract Answer: Show that the integer equals two times some integer plus one.
Specific Answer: Show that $n^2 = 2k + 1$ for some integer k.

b. Key Question: How can I show that a real number (namely, s/t) is rational?
Abstract Answer: Show that the real number is equal to the ratio of two integers in which the denominator is not zero.
Specific Answer: Show that $s/t = p/q$, where p and q are integers and $q \neq 0$.

c. Key Question: How can I show that two pairs of real numbers (namely, (x_1, y_1) and (x_2, y_2)) are equal?
Abstract Answer: Show that the first and second elements of one pair of real numbers are equal to the corresponding elements of the other pair.
Specific Answer: Show that $x_1 = x_2$ and $y_1 = y_2$.

3.2 a. Key Question: How can I show that an integer (namely, n) is even?
Abstract Answer: Show that the integer equals two times some integer.
Specific Answer: Show that $n = 2k$ for some integer k.

b. Key Question: How can I show that an integer (namely, n) is prime?
Abstract Answer: Show that the integer is greater than 1 and can be divided only by itself and 1.
Specific Answer: Show that $n > 1$ and, if k is an integer that divides n, then $k = 1$ or $k = n$.

c. Key Question: How can I show that an integer (namely, 9) divides another integer (namely, $(n-1)^3 + n^3 + (n+1)^3$)?
Abstract Answer: Show that the second integer equals the product of the first integer with another integer.
Specific Answer: Show that the following expression is true for some integer $k : (n-1)^3 + n^3 + (n+1)^3 = 9k$.

3.3 a. **B1:** The only positive integers that divide m are 1 and m.
b. **B1:** $p^2 = 2k$, for some integer k.
c. **B1:** $\overline{AB} = \overline{BC} = \overline{CA}$ (or $\angle A = \angle B = \angle C$).
d. **B1:** There are integers p and q with $q \neq 0$ such that $\sqrt{n} = p/q$.

3.4 (A is the hypothesis and $A1$ is obtained by working forward one step.)
a. **A:** n is an odd integer.
A1: $n = 2k + 1$, where k is an integer.
b. **A:** s and t are rational numbers with $t \neq 0$.
A1: $s = p/q$, where p and q are integers with $q \neq 0$. Also, $t = a/b$, where $a \neq 0$ and $b \neq 0$ are integers.
c. **A:** $\sin(X) = \cos(X)$.
A1: $x/z = y/z$ (or $x = y$).
d. **A:** a, b, c are integers for which $a|b$ and $b|c$.
A1: $b = pa$ and $c = qb$, where p and q are both integers.

3.5 a. For every integer, k with $1 < k < 2^n - 1$, $\frac{2^n-1}{k}$ is not an integer (or k does not divide $2^n - 1$).
b. $R \subseteq S \cup T$ and $S \cup T \subseteq R$.
c. For all real numbers x and y and for every real number t with $0 \leq t \leq 1$, $(f+g)(tx+(1-t)y) \leq t(f+g)(x)+(1-t)(f+g)(y)$, that is, $f(tx+(1-t)y)+g(tx+(1-t)y) \leq t[f(x)+g(x)]+(1-t)[f(y)+g(y)]$.
d. For every element $x \in S \cap T$, $f(x) \geq g(x)$.

3.6 (T = true, F = false)
a. Truth Table for the Converse of "A Implies B."

A	B	"A Implies B"	"B Implies A"
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

b. Truth Table for the Inverse of "A Implies B."

A	B	$NOT\ A$	$NOT\ B$	"$NOT\ A$ Implies $NOT\ B$"
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

The converse and inverse of "A implies B" are equivalent. Both are true except when A is false and B is true.

3.7 (T = true, F = false)

a. Truth Table for "A AND B."

A	B	"$A\ AND\ B$"
T	T	T
T	F	F
F	T	F
F	F	F

b. Truth Table for "A AND NOT B"

A	B	$NOT\ B$	"$A\ AND\ NOT\ B$"
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

3.8 (T = true, F = false)

a. Truth Table for "A OR B."

A	B	"$A\ OR\ B$"
T	T	T
T	F	T
F	T	T
F	F	F

b. Truth Table for "($NOT\ A$) $OR\ B$."

A	$NOT\ A$	B	"($NOT\ A$) $OR\ B$"	"A Implies B"
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T

"A implies B" and "($NOT\ A$) $OR\ B$" are equivalent. Both are false when A is true and B is false and true otherwise.

3.9 a. If n is an odd integer, then n^2 is odd.
b. If r is a real number such that $r^2 \neq 2$, then r is rational.
c. If the quadrilateral $ABCD$ is a rectangle, then $ABCD$ is a parallelogram with one right angle.

3.10 a. If a, b, and c are integers for which $a|c$, then $a|b$ and $b|c$.
b. If $n > 1$ is an integer that is not prime, then $2^n - 1$ is not prime.
c. If r is not a rational number with $r \neq 0$, that is, if r is an irrational number or $r = 0$, then $1/r$ is irrational.

3.11 **Analysis of Proof.** Using the forward-backward method one is led to the key question, "How can I show that one statement (namely, A) implies another statement (namely, C)?" According to Table 1, the answer is to assume that the statement to the left of the word "implies" is true, and then reach the conclusion that the statement to the right of the word "implies" is true. In this case, you assume

A1: A is true,

and try to reach the conclusion that

B1: C is true.

Working forward from the information given in the hypothesis, because "A implies B" is true and A is true, by row 1 in Table 1, it must be that

A1: B is true.

Because B is true, and "B implies C" is true, it must also be that

A2: C is true.

Hence the proof is complete.

Proof. To conclude that "A implies C" is true, assume that A is true. By the hypothesis, "A implies B" is true, so B must be true. Finally, because "B implies C" is true, C is true, thus completing the proof. □

3.12 a. "D implies E".
b. "E implies C".

3.13 To show that A is equivalent to B, one key question is, "How can I show that two statements are equivalent?" By definition, you must show that

B1: "A implies B" and "B implies A".

The hypothesis states that "A implies B" so it remains to show that

B2: "B implies A".

This, however, follows from the hypothesis that "B implies C" and "C implies A" so, by the result in Exercise 3.12, "B implies A" and hence $B2$ is true.

Likewise, to show that A is equivalent to C, one is led to the key question, "How can I show that two statements are equivalent?" By definition, it must be shown that

B1: "A implies C" and "C implies A".

The hypothesis states that "C implies A" so it remains to show that

B2: "A implies C".

This, however, follows from the hypothesis that "A implies B" and "B implies C" so, by the result in Exercise 3.12, "A implies C" and hence $B2$ is true. The proof is now complete.

3.14 a. If the four statements in part (a) are true, then you can show that A is equivalent to any of the alternatives by using Exercise 3.13. For instance, to show that A is equivalent to D, you already know that "D implies A." By Exercise 3.13, because "A implies B," "B implies C," and "C implies D," you have that "A implies D."
b. The advantage of the approach in part (a) is that only four proofs are required ($A \Rightarrow B$, $B \Rightarrow C$, $C \Rightarrow D$, and $D \Rightarrow A$) as opposed to the six proofs ($A \Rightarrow B$, $B \Rightarrow A$, $A \Rightarrow C$, $C \Rightarrow A$, $A \Rightarrow D$, and $D \Rightarrow A$) required to show that A is equivalent to each of the three alternatives.

3.15 In this case, to use Proposition 3, it must be shown that the hypothesis of the question, the right triangle ABC with sides of lengths a and b and hypotenuse of length c has an area of $c^2/4$, implies the hypothesis of Proposition 3. Matching up the notation with $r = a$, $s = b$, and $t = c$, you must show that $c = \sqrt{2ab}$.

3.16 Because the hypotheses of Proposition 3 are true for the current triangle, it follows that the conclusion of Proposition 3 is true for the current triangle, that is, triangle ABC is isosceles.

3.17 a.

A:	s	given
A1:	tst	from 2.39 part (c)
A2:	$tsttst$	rule 1
A3:	$tsst$	rule 2
A4:	$tssttsst$	rule 1
B1:	$tssst$	rule 2
B:	$ttst$	rule 3

b.

A:	$stsss$	given
A1:	stt	rule 3
B1:	s	rule 2
B:	tst	from 2.39 part (c)

c. In part (a), a previous proposition is used in the forward process, and, because the hypothesis of that proposition is true for the current problem, the conclusion of the previous proposition becomes a new statement $A1$ in the forward process of the current problem. In part (b), a previous proposition is used in the backward process, and so it must be shown that the hypotheses of that proposition are true for the current problem (see $B1$).

3.18 Proposition 3 is used in the backward process to answer the key question, "How can I show that a triangle is isosceles?" Accordingly, it must be shown that the hypothesis of Proposition 3 is true for the current problem—that is, that $w = \sqrt{2uv}$. This is precisely what the author does by working forward from the hypothesis of the current proposition.

3.19 First, previous knowledge that a line tangent to the endpoint of a diameter of a circle is perpendicular to the diameter is used to conclude that L_1 and L_2 are both perpendicular to the diameter D of the circle. Then, previous knowledge—that two lines in a plane that are both perpendicular to a third line are parallel—is used to conclude that L_1 and L_2 are parallel.

3.20 **Analysis of Proof.** The key question for this problem is, "How can I show that a triangle is isosceles?" This proof answers this question by recognizing that the conclusion of Proposition 3 is the same as the conclusion you are trying to reach. So, if the current hypothesis implies that the hypothesis of Proposition 3 is true, then the triangle is isosceles. Because triangle UVW is a right triangle, on matching up the notation, all that remains to be shown is that $\sin(U) = \sqrt{u/2v}$ implies $w = \sqrt{2uv}$. To that end,

A: $\sin(U) = \sqrt{\frac{u}{2v}}$ (by hypothesis)

A1: $\sin(U) = \frac{u}{w}$ (by definition of sine)

A2: $\frac{u}{w} = \sqrt{\frac{u}{2v}}$ (from A and $A1$)

A3: $\frac{u}{2v} = \frac{u^2}{w^2}$ (square both sides of $A2$)

A4: $uw^2 = 2vu^2$ (cross-multiply $A3$)

A5: $w^2 = 2vu$ (divided $A4$ by u)

B: $w = \sqrt{2uv}$ (take the square root of both sides of $A5$)

It has been shown that the hypothesis of Proposition 3 is true, so the conclusion of Proposition 3 is also true. Hence triangle UVW is isosceles.

3.21 a. Proposition 2 is used in the last sentence of the proof. Specifically, a key question for the conclusion of the current proposition is, "How can I show that the square of an integer (namely, $m + n$) is even?" This question is answered using Proposition 2 because the conclusion of Proposition 2 is the same as the conclusion of this proof. Thus, all that needs to be shown is that the hypothesis of Proposition 2 is true—namely, that $m + n$ is even—which is what the author shows.

b. Proposition 2 is used in the first sentence of this proof. Specifically, the author works forward from the hypothesis of the current proposition that m and n are even integers to claim, by Proposition 2, that m^2 and n^2 are even integers. In other words, because m and n satisfy the hypothesis of Proposition 2 (namely, being even), m and n must also satisfy the conclusion of Proposition 2, that is, m^2 and n^2 are even.

3.22 **Analysis of Proof.** A key question for this problem is, "How can I show that an integer (namely, n^2) is odd?" By definition, you must show that

B1: There is an integer k such that $n^2 = 2k + 1$.

Turning to the forward process to determine the desired value of k in $B1$, from the hypothesis that n is odd, by definition,

A1: There is an integer p such that $n = 2p + 1$.

Squaring both sides of the equality in $A1$ and applying algebra results in

A2: $n^2 = (2p + 1)^2 = 4p^2 + 2p + 1 = 2(2p^2 + p) + 1$.

The proof is completed on noting from $A2$ that $k = 2p^2 + p$ satisfies $B1$.

Proof. Because n is odd, by definition, there is an integer p such that $n = 2p+1$. Squaring both sides of this equality and applying algebra, it follows that $n^2 = (2p+1)^2 = 4p^2+2p+1 = 2(2p^2+p)+1$. But this means that, for $k = 2p^2+p$, $n^2 = 2k+1$ and so n^2 is odd. □

3.23 **Analysis of Proof.** A key question for this problem is, "How can I show that the square of an integer (namely, $a+b$), is odd?" One answer is provided by the proposition in Exercise 3.22, whose conclusion is the same as the one in this problem. Thus, it must be shown that the hypothesis of that proposition is true. Matching up notation, this means it must be shown that

B1: $a+b$ is odd.

A key question associated with $B1$ is, "How can I show that an integer (namely, $a+b$) is odd?" By definition, it must be shown that

B2: There is an integer k such that $a+b = 2k+1$.

Turning to the forward process to determine the desired value of k in $B2$, from the hypothesis that a and b are consecutive integers, it follows that

A1: $b = a+1$, and so

A2: $a+b = a+a+1 = 2a+1$.

The proof is now complete because $B2$ is true.

Proof. Because a and b are consecutive integers, $b = a+1$. Thus, $a+b = 2a+1$ is odd. By the prop. in Exer. 3.22, $(a+b)^2$ is odd and the proof is done. □

3.24 **Analysis of Proof.** A key question for this problem is, "How can I show that the square of an integer (namely, $a+b$) is even?" One answer is provided by Proposition 2, whose conclusion is the same as the one in this problem. Thus, it must be shown that the hypothesis of that proposition is true. Matching up notation, this means it must be shown that

B1: $a+b$ is even.

A key question associated with $B1$ is, "How can I show that an integer (namely, $a+b$) is even?" By definition, it must be shown that

B2: There is an integer k such that $a+b = 2k$.

Turning to the forward process to determine the value of k in $B2$, from the hypothesis that a and b are odd, it follows that

A1: There are integers p and q such that $a = 2p+1$ and $b = 2q+1$ and so

A2: $a+b = 2p+1+2q+1 = 2(p+q+1)$.

The proof is now complete because $B2$ is true.

Proof. Because a and b are odd integers, there are integers p and q such that $a = 2p+1$ and $b = 2q+1$. Thus, $a+b = 2(p+q+1)$ is even. By Proposition 2, it follows that $(a+b)^2$ is even and so the proof is complete. □

3.25 a. To avoid overlapping notation, rewrite the already-proved proposition as follows: "If c and d are nonnegative real numbers, then $(c+d)/2 \geq \sqrt{cd}$." Now, to use this proposition to prove that, if a and b are real numbers satisfying the property that $b \geq 2|a|$, then $b \geq \sqrt{b^2 - 4a^2}$, match c to $b - 2a$ and d to $b + 2a$. Then for these values of c and d, the hypothesis of the already proved proposition holds because both $b - 2a \geq 0$ and $b + 2a \geq 0$ (as $b \geq 2|a|$ from the hypothesis of the current proposition). As such, the conclusion of the already-proved proposition, namely that $(c+d)/2 \geq \sqrt{cd}$, must hold for $c = b - 2a$ and $d = b + 2a$. That is,

$$\begin{array}{rcl} [(b-2a)+(b+2a)]/2 & \geq & \sqrt{(b-2a)(b+2a)} \\ b & \geq & \sqrt{b^2-4a^2}. \end{array}$$

b. **Analysis of Proof.** Working backward using the quadratic formula, it must be shown that

B1: Either $\frac{-b+\sqrt{b^2-4a^2}}{2a} \leq \frac{-b}{a}$ or $\frac{-b-\sqrt{b^2-4a^2}}{2a} \leq \frac{-b}{a}$.

In particular, it will be shown that

B2: $\frac{-b-\sqrt{b^2-4a^2}}{2a} \leq \frac{-b}{a}$.

To that end, from the hypothesis that $b \geq 2|a|$, it follows from the result in part (a) that

A1: $b \geq \sqrt{b^2-4a^2}$, that is, $b - \sqrt{b^2-4a^2} \geq 0$.

Subtracting $2b$ from both sides of $A1$ and then dividing by $2a < 0$ (from the hypothesis), results in the desired conclusion in $B2$ that

A2: $\frac{-b-\sqrt{b^2-4a^2}}{2a} \leq \frac{-b}{a}$.

Proof. It will be shown that

$$\frac{-b-\sqrt{b^2-4a^2}}{2a} \leq \frac{-b}{a}.$$

To that end, from the hypothesis that $b \geq 2|a|$, it follows from the result in part (a) that

$$b \geq \sqrt{b^2-4a^2}, \text{ that is, } b - \sqrt{b^2-4a^2} \geq 0.$$

The desired conclusion follows on subtracting $2b$ from both sides and then dividing by $2a < 0$, thus completing the proof. □

3.26 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show that a triangle is isosceles?" Using the definition of an isosceles triangle, you must show that two of its sides are equal, which, in this case, means you must show that

B1: $u = v$.

Working forward from the hypothesis, you have the following statements and reasons

Statement	**Reason**
A1: $\sin(U) = \sqrt{u/2v}$.	Hypothesis.
A2: $\sqrt{u/2v} = u/w$.	Definition of sine.
A3: $w^2 = 2uv$.	From $A2$ by algebra.
A4: $u^2 + v^2 = w^2$.	Pythagorean theorem.
A5: $u^2 + v^2 = 2uv$	Substituting w^2 from $A3$ in $A4$.
A6: $u^2 - 2uv + v^2 = 0$.	From $A5$ by algebra.
A7: $u - v = 0$.	Factoring $A6$ and taking square root.

Thus, $u = v$, completing the proof.

Proof. Because $\sin(U) = \sqrt{u/2v}$ and also $\sin(U) = u/w$, $\sqrt{u/2v} = u/w$, or, $w^2 = 2uv$. Now, from the Pythagorean theorem, $w^2 = u^2 + v^2$. On substituting $2uv$ for w^2 and then performing algebraic manipulations, one has $u = v$. □

3.27 **Analysis of Proof.** To verify the hypothesis of Proposition 1 for the current triangle UVW, it is necessary to match up the notation. Specifically, $x = u$, $y = v$, and $z = w$. Then it must be shown that

B1: $uv/2 = w^2/4$.

Working forward from the current hypothesis that $\sin(U) = \sqrt{u/2v}$, and because $\sin(U) = u/w$, one has

A1: $\sqrt{u/2v} = u/w$, or,
A2: $u/2v = u^2/w^2$, or,
A3: $w^2 = 2uv$.

On dividing both sides of the equality in $A3$ by 4 yields precisely $B1$, thus completing the proof. (Observe also that triangle UVW is a right triangle.)

Proof. By the hypothesis, $\sin(U) = \sqrt{u/2v}$ and from the definition of sine, $\sin(U) = u/w$, thus $\sqrt{u/2v} = u/w$. By applying algebraic manipulations, one obtains $uv/2 = w^2/4$. Hence the hypothesis of Proposition 1 holds for the current right triangle UVW and consequently the triangle is isosceles. □

4

Solutions to Exercises

4.1

	Object	Certain Property	Something Happens
(a)	two people	none	they have the same number of friends
(b)	integer x	none	$f(x) = 0$
(c)	a point (x, y)	$x \geq 0$ and $y \geq 0$	$y = m_1 x + b_1$ and $y = m_2 x + b_2$
(d)	angle t'	$0 \leq t' \leq \pi$	$\tan(t') > \tan(t)$
(e)	integers m and n	none	$am + bn = c$

4.2 a. There is an element s in the set S such that $s \geq 0$.

Object: element s.
Certain property: $s \geq 0$.
Something happens: s in the set S.

b. There is an element t in the set T.
Object: element t.
Certain property: none.
Something happens: t in the set T.

c. There is an integer $k > 0$ such that $x^2 - kx + 2 = 0$.
Object: integer k.
Certain property: k positive.
Something happens: $x^2 - kx + 2 = 0$.

4.3 a. Check that n is a positive integer and that $n! > 3^n$.
b. Check that p is an integer with $1 < p < n$ and that n divided by p is an integer.
c. Check that the roots $r_1, \cdots, r_n$ are complex numbers (note that every real number is in the set of complex numbers) and that $a_0 + a_1 r_i^1 + \cdots + a_n r_i^n = 0$ for each $i = 1, \ldots, n$.

4.4 b. Construct an integer x and show that $f(x) = 0$.
c. Construct a point (x, y) in the plane with $x \geq 0$ and $y \geq 0$ and show that $y = m_1 x + b_1$ and $y = m_2 x + b_2$
d. Construct an angle t' with $0 \leq t' \leq \pi$ and show that $\tan(t') > \tan(t)$.
e. Construct integers m and n and show that $am + bn = c$.

4.5 a. Yes, because the statement can be rewritten to contain the quantifier "there is" explicitly, as follows: There is a real number x such that $x^{71} - 4x^{44} + 11x - 3 = 0$.
b. No, because the conclusion requires you to show that there does *not* exist a positive integer x such that $ax^2 + bx + b - a = 0$, rather than there does exist something.
c. Yes, because the statement can be rewritten so that the conclusion contains the quantifier "there is" explicitly, as follows: If $ABCD$ is a square whose sides have length s, then there is a circle inscribed in $ABCD$ whose area is at least $3s^2/4$.

4.6 The construction method arises in the proof of Proposition 2 because the statement,

B1: n^2 can be expressed as two times some other integer

can be rewritten as follows to contain the quantifier "there is":

B2: There is an integer p such that $n^2 = 2p$.

The construction method is then used to produce the integer p for which $n^2 = 2p$. Specifically, using the fact that n is even, and hence that there is an integer k such that $n = 2k$, the desired value for p is constructed as $p = 2k^2$. This value for p is correct because $n^2 = (2k)^2 = 4k^2 = 2(2k^2) = 2p$.

4.7 **Analysis of Proof.** The appearance of the keywords "there is" in the conclusion suggests using the construction method to find an integer x for which $x^2 - 5x/2 + 3/2 = 0$. Factoring means that you want an integer x such that $(x-1)(x-3/2) = 0$. Thus, the desired value is $x = 1$. On substituting this value of x in $x^2 - 5x/2 + 3/2$ yields 0, so the quadratic equation is satisfied. This integer solution is unique because the only other solution is $x = 3/2$, which is not an integer.

Proof. Factoring $x^2 - 5x/2 + 3/2$ means that the only roots are $x = 1$ and $x = 3/2$. Thus, there exists an integer (namely, $x = 1$) such that $x^2 - 5x/2 + 3/2 = 0$. The integer is unique. □

4.8 **Analysis of Proof.** The appearance of the keywords "there is" in the conclusion suggests using the construction method to find a real number x such that $x^2 - 5x/2 + 3/2 = 0$. Factoring this equation means you want to find a real number x such that $(x-3/2)(x-1) = 0$. So the desired real number is either $x = 1$ or $x = 3/2$ which, when substituted in $x^2 - 5x/2 + 3/2$, yields 0. The real number is not unique as either $x = 1$ or $x = 3/2$ works.

Proof. Factoring $x^2 - 5x/2 + 3/2 = 0$ yields $(x - 3/2)(x - 1) = 0$, so $x = 1$ or $x = 3/2$. Thus, there exists a real number, namely, $x = 1$ or $x = 3/2$, such that $x^2 - 5x/2 + 3/2 = 0$. The real number is not unique. □

4.9 a. 7 is the smallest positive integer such that $n! > 3^n$. You can see that $7! = 5040 > 2187 = 3^7$. Furthermore, it is easy to check that for $n = 1, \ldots, 6$, $n! \leq 3^n$.

b. After 15 years at an interest rate of 5%, an investment will have more than doubled because $\left(1 + \frac{r}{100}\right)^n = (1 + 0.05)^{15} \approx 2.079 > 2$. Furthermore, after 14 years, $\left(1 + \frac{r}{100}\right)^n = (1 + 0.05)^{14} \approx 1.98 < 2$, so the money will not have doubled. Although one can check the value of the investment at each year $1, \ldots, 14$, that is not necessary. An investment with a positive rate of return can only increase in value. So, if the money has not more than doubled in year 14, it has not doubled before year 14 either. So 15 is smallest positive integer number of years such that an investment with 5% interest will at least double. The value of 15 can be obtained without trial-and-error by solving the equation $(1 + 0.05)^x = 2$ for x to obtain $x = \ln(2)/\ln(1.05) \approx 14.2$. So 15 is the smallest integer larger than 14.2.

c. The angle $x = 0.739$ satisfies $\cos(x) = x$ to three decimal digits. This value is obtained by trial-and-error.

4.10 **Analysis of Proof.** One answer to the key question, "How can I show that a number (namely, $s + t$) is rational?" is to use the definition and show that

B1: There are integers p and q with $q \neq 0$ such that $s + t = \frac{p}{q}$.

Because of the keywords "there are" in the backward statement $B1$, the author recognizes the need for the construction method. Specifically, the author works forward from the hypothesis, by definition of a rational number, to claim that

A1: There are integers a, b, c, d with $b, d \neq 0$ such that $s = \frac{a}{b}$ and $t = \frac{c}{d}$.

Adding the two fractions in $A1$ together results in

A2: $s + t = \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

From $A2$, the author constructs the values for the integers p and q in $B1$, as indicated when the author says, "Now set $p = ad + bc$ and $q = bd$." The remainder of this proof involves showing that these values of p and q satisfy the desired properties in $B1$. Specifically, $q = bd \neq 0$ because $b, d \neq 0$ (see $A1$) and $s + t = \frac{p}{q}$ from $A2$. The proof is now complete. □

4.11 **Analysis of Proof.** The conclusion of the proposition can be reworded as follows:

B: There is a rational root for the equation $ax^2 + bx + c = 0$.

In this form, the keywords "there is" suggests using the construction method to produce this rational root. Because this equation is quadratic with $a \neq 0$, the author works forward using the quadratic formula, so the roots of the equation are

A1: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

In the last sentence of the condensed proof, the author constructs the desired root as $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$. Therefore, to complete the construction method, it remains to be shown that

B1: $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ is rational.

A key question associated with $B1$ is, "How can I show that number (namely, x) is rational?" Using the definition to answer this question means it is necessary to show that

B2: There are integers p and q with $q \neq 0$ such that $x = \frac{p}{q}$.

Recognizing the keywords "there are" in the backward statement $B1$, the author uses the construction method to produce the desired integers p and q. Although the author never explicitly says so, looking at $B1$, the author constructs $p = -b + \sqrt{b^2 - 4ac}$ and $q = 2a$. It is clear that $q = 2a \neq 0$ because $a \neq 0$ from the hypothesis. However, it is necessary to be sure that the value of $p = -b + \sqrt{b^2 - 4ac}$ is an integer, which will be true if you can show that

B3: $\sqrt{b^2 - 4ac}$ is an integer.

Working forward from the hypothesis that $b^2 - 4ac$ is a square, by definition, this means that

> **A2:** There is an integer k such that $b^2 - 4ac = k^2$.

Taking the square root of both sides of the equality in $A2$ means that

> **A3:** $\sqrt{b^2 - 4ac} = \sqrt{k^2} = |k|$ is an integer.

The proof is now complete because $A3$ is the same as $B3$.

4.12 The error is in the statement that, if $n > 1/\log_2(r)$, then $\frac{1}{n} < \log_2(r)$. It is not necessarily true that, if a and b are non-zero numbers with $a > b$, then $1/a < 1/b$. For example, if $r = 1/2$, then $\log_2(r) = -1$. Then for $n = 2$, $n = 2 > -1 = 1/\log_2(r)$ but $\frac{1}{n} = \frac{1}{2} \not< -1 = \log_2(r)$.

4.13 a. p and q are defined in the first two sentences because the author is constructing a rational number $r = \frac{p}{q}$ that satisfies the conclusion of the proposition.

b. Yes because $0 < m < n$ and so $\frac{1}{n} < \frac{1}{m}$, that is, $\frac{1}{m} - \frac{1}{n} = \frac{n-m}{mn} > 0$. Now $n - m \geq 1$ because $n > m$ and n and m are integers. Then indeed for any integer $q > 2mn > 0$,

$$q\left(\frac{1}{m} - \frac{1}{n}\right) = q\left(\frac{n-m}{mn}\right) > 2(n-m) \geq 2 > 1.$$

c. Yes, because the author has just shown that $\frac{q}{m} > \frac{q}{n} + 1$. Therefore, there is at least one integer p between $\frac{q}{m}$ and $\frac{q}{n}$.

d. Yes, because the author has already shown that for the integer p,

$$\frac{q}{n} < p < \frac{q}{m}.$$

Dividing the foregoing inequalities through by $q > 0$ yields

$$\frac{1}{n} < \frac{p}{q} < \frac{1}{m}.$$

e. The author correctly states in the first sentence that q is a positive integer [see part (b)] and, as such, $q \neq 0$.

4.14 The error in the proof occurs when the author says to divide by p^2 because it will not be possible to do so if $p = 0$, which can happen.

4.15 The proof is not correct. The mistake occurs because the author uses the same symbol x for the element that is in $R \cap S$ and in $S \cap T$, where, in fact, the element in $R \cap S$ need not be the same as the element in $S \cap T$.

4.16 The proof is not correct. This is because the author needs to construct an integer k for which $n = 2k$. Indeed, the author shows that, for the integer p, the value of $k = \sqrt{2p}/2$ satisfies $n = 2k$. However, the author has not shown that $k = \sqrt{2p}/2$ is an integer.

4.17 The mistake is that the author has shown that 2 divides $m^2 + n^2 - 1$ when the proposition requires the author to show that 4 divides $m^2 + n^2 - 1$.

4.18 The mistake is that the author has claimed that there exists an integer n such that $m < n < p$ but has failed to show that such an integer exists. For example, if $m = 1$ and $p = 2$, then there is no integer n between m and p.

4.19 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show that an integer (namely, a) divides another integer (namely, c)?" By the definition, one answer is to show that

B1: There is an integer k such that $c = ak$.

The appearance of the quantifier "there is" in $B1$ suggests turning to the forward process to construct the desired integer k.

From the hypothesis that $a|b$ and $b|c$, by definition,

A1: There are integers p and q such that $b = ap$ and $c = bq$.

Therefore, it follows that

A2: $c = bq = (ap)q = a(pq)$,

and so the desired integer k is $k = pq$.

Proof. Because $a|b$ and $b|c$, by definition, there are integers p and q for which $b = ap$ and $c = bq$. But then $c = bq = (ap)q = a(pq)$ and so $a|c$. □

4.20 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I to show that an integer (namely, a) divides another integer (namely, b)?" One answer, by definition, is to show that

B1: There is an integer m such that $b = ma$.

Recognizing the keywords "there is" in the statement $B1$ means you should now use the construction method to produce the desired integer m.

Working forward from the hypothesis that the integer x satisfies the equation $ax^2 + bx + b - a = 0$ and $a \neq 0$, from the quadratic formula,

A1: $x = \frac{-b\pm\sqrt{b^2-4a(b-a)}}{2a} = \frac{-b\pm(b-2a)}{2a} = -1$ or $\frac{-b+a}{a}$.

Now because it is stated in the hypothesis that $x > 0$, it must be that

A2: $x = \frac{-b+a}{a}$.

Looking at $B1$, you can solve the equation in $A2$ for b to obtain

A3: $b = a(1-x)$.

It is easy to see from $A3$ that the desired value for the integer m in $B1$ is $1-x$ because $b = (1-x)a = ma$. This completes the construction method and the proof.

Proof. Solving the quadratic equation $ax^2 + bx + b - a = 0$ for x yields

$$x = \frac{-b \pm \sqrt{b^2 - 4a(b-a)}}{2a} = -1 \text{ or } \frac{-b+a}{a}.$$

Now because x is a positive integer, it must be that

$$x = \frac{-b+a}{a}.$$

Solving the foregoing equation for b yields $b = (1-x)a$, which means that $a|b$ and so the proof is complete. □

4.21 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show that an integer (namely, a) divides another integer (namely, c)?" By the definition, one answer is to show that

B1: There is an integer k such that $c = ak$.

The appearance of the quantifier "there is" in $B1$ suggests turning to the forward process to construct the desired integer k.

From the hypothesis that $a|b$ and $a|(b+c)$, by definition,

A1: There are integers p and q such that $b = ap$ and $b+c = aq$.

Solving for c in the second equality in $A1$ and substituting $b = ap$ yields

A2: $c = aq - b = aq - ap = a(q-p)$.

From $A2$ it is now easy to see that the desired integer k in $B1$ is $k = q - p$.

Proof. Because $a|b$ and $a|(b+c)$, by definition, there are integers p and q for which $b = ap$ and $b + c = aq$. But then it follows that $c = aq - b = aq - ap = a(q-p)$ and so $a|c$, thus completing the proof. □

4.22 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show that a real number (namely, s/t) is rational?" By the definition, one answer is to show that

B1: There are integers p and q with $q \neq 0$ such that $s/t = p/q$.

The appearance of the quantifier "there are" in the backward statement $B1$ suggests turning to the forward process to construct the desired p and q.

From the hypothesis that s and t are rational numbers, by the definition,

A1: There are integers a, b, c, and d with $b \neq 0$ and $d \neq 0$ such that $s = a/b$ and $t = c/d$.

Because $t \neq 0$, $c \neq 0$, and thus $bc \neq 0$. Hence,

A2: $s/t = (a/b)/(c/d) = (ad)/(bc)$.

So the desired integers p and q are $p = ad$ and $q = bc$. Observe that because $b \neq 0$ and $c \neq 0$, $q \neq 0$; also, $s/t = p/q$, thus completing the proof.

Proof. Because s and t are rational, there are integers a, b, c, and d with $b \neq 0$ and $d \neq 0$ such that $s = a/b$ and $t = c/d$.

Because $t \neq 0$, $c \neq 0$. Constructing $p = ad$ and $q = bc$, and noting that $q \neq 0$, one has $s/t = (a/b)/(c/d) = (ad)/(bc) = p/q$, and hence s/t is rational, thus completing the proof. □

4.23 **Analysis of Proof.** The keywords "there is" in the conclusion suggest using the construction method to produce real numbers c and d such that $(a + bi)(c + di) = 1$. The approach taken here is to find conditions on a and b so that, whatever values are constructed for c and d, it will be possible to show they satisfy the property that

B1: $(a + bi)(c + di) = 1$.

Performing the multiplication of the two complex numbers in $B1$, you will need to show that

B2: $1 = (a + bi)(c + di) = ac - bd + (bc + ad)i$.

In order for the complex number on the right of the equality in $B2$ to be equal to $1 = 1 + 0i$, you will need to show that

$B3:$ $ac - bd = 1$ and $bc + ad = 0$.

To solve the two linear equations in $B3$ for the two unknowns c and d, you need the following property on a and b to hold (see Proposition 4 on page 43):

$$a^2 + b^2 \neq 0.$$

While the foregoing condition on a and b is certainly correct, observe that this condition is true if and only if the following simpler condition holds:

Property $P:$ a and b are not both zero.

As long as Property P holds, you can solve the two linear equations in $B3$ to construct c and d as follows:

A1: $c = \frac{a}{a^2+b^2}$ and $d = \frac{-b}{a^2+b^2}$.

It remains to show that, for the values of c and d in $A1$, $B3$ is true. Indeed,

$$ac - bd = a\left(\frac{a}{a^2 + b^2}\right) - b\left(\frac{-b}{a^2 + b^2}\right) = 1.$$

and

$$cb + ad = \left(\frac{a}{a^2 + b^2}\right) b + a\left(\frac{-b}{a^2 + b^2}\right) = 0.$$

The proof is now complete because the constructed values of c and d in $A1$ satisfy $(a + bi)(c + di) = 1$.

Proof. Assuming that a and b are not both zero (Property P), let $c = \frac{a}{a^2+b^2}$ and $d = \frac{-b}{a^2+b^2}$. It is easy to verify that $(a + bi)(c + di) = 1$ and so the proof is complete. $\square$

5

Solutions to Exercises

5.1 a. Object: real number x.
Certain property: none.
Something happens: $f(x) \le f(x^*)$.
b. Object: element x.
Certain property: $x \in S$.
Something happens: $g(x) \ge f(x)$.
c. Object: element x.
Certain property: $x \in S$.
Something happens: $x \le u$.

5.2 a. Object: real numbers x and y.
Certain property: $x < y$.
Something happens: $f(x) < f(y)$.
b. Object: elements x and y, and real numbers t.
Certain property: x and y in C, and $0 \le t \le 1$.
Something happens: $tx + (1-t)y$ is an element of C.
c. Object: real numbers x, y, and t.
Certain property: $0 \le t \le 1$.
Something happens: $f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$.

5.3 a. $\exists$ a mountain $\ni$ $\forall$ other mountains, this one is taller than the others.
b. $\forall$ angle t, $\sin(2t) = 2\sin(t)\cos(t)$.
c. $\forall$ nonnegative real numbers p and q, $\sqrt{pq} \ge (p+q)/2$.
d. $\forall$ real numbers x and y with $x < y$, $\exists$ a rational number $r \ni x < r < y$.

5.4 a If p is a prime number, then $p + 7$ is composite.
b. If A, B, and C are sets with the property that $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
c. If p and q are integers with $q \neq 0$, then p/q is rational.

5.5 a. Choose a real number x'.
It will be shown that $f(x') \leq f(x^*)$.
b. Choose an element $x' \in S$.
It will be shown that $g(x') \geq f(x')$.
c. Choose an element $x' \in S$.
It will be shown that $x' \leq u$.

5.6 a. Choose real numbers x', y' with $x' < y'$.
It will be shown that $f(x') < f(y')$.
b. Choose elements x', $y' \in C$ and a real number t' with $0 \leq t' \leq 1$.
It will be shown that $t'x' + (1 - t')y' \in C$.
c. Choose real numbers x', y', t' with $0 \leq t' \leq 1$.
It will be shown that $f(t'x' + (1 - t')y') \leq t'f(x') + (1 - t')f(y')$.

5.7 a. You could not use the choose method because this statement contains the keywords "there is" instead of "for all" (so the construction method is an appropriate technique to consider).
b. You could use the choose method because the backward statement contains the keywords "for all." In this case you would choose an integer $n \geq 4$ for which you must show that $n! \geq 2^n$.
c. You could not use the choose method because the conclusion of this implication contains the keywords "there is" instead of "for all" (so the construction method is an appropriate technique to consider).
d. You could use the choose method because the conclusion of the implication contains the keywords "for all." In this case you would choose an element $x \in S$ for which you must show that $|x| < 20$.
e. You could use the choose method because the backward statement contains the keywords "for all." In this case you would choose real numbers a, b, and c for which you must show that if $4ac \leq b^2$, then $ax^2 + bx + c$ has real roots.

5.8 Key Question: How can I show that a set (namely, R) is a subset of another set (namely, T)?
Key Answer: Show that every element of the first set is in the second set and so it must be shown that
B1: For every element $r \in R$, $r \in T$.
A1: Choose an element $r' \in R$ for which it must be shown that
B2: $r' \in T$.

5.9 Key Question: How can I show that a real number (namely, v) is an upper bound for a set of real numbers (namely, S)?

Key Answer: Show that every element in the set is $\leq$ the number and so it must be shown that

B1: For every element $s \in S$, $s \leq v$.

A1: Choose an element $s' \in S$ for which it must be shown that

B2: $s' \leq v$.

5.10 Key Question: How can I show that a function (namely, f) is strictly increasing?

Key Answer: Show that for each pair of real numbers where the first number is less than the second, the value of the function at the first number is less than the value of the function at the second.

B1: For all real numbers x and y with $x < y$, $f(x) < f(y)$.

A1: Choose real numbers x' and y' with $x' < y'$ for which it must be shown that

B2: $f(x') < f(y')$.

5.11 Key Question: How can I show that a set of real numbers (namely, C) is convex?

Key Answer: Show that for any two elements in the set and any real number between 0 and 1, the real number times the first element plus one minus the real number times the second element is also in the set.

B1: For all elements $x, y \in C$ and real numbers t with $0 \leq t \leq 1$, $tx + (1-t)y \in C$.

A1: Choose two elements $x', y' \in C$ and a real number t' with $0 \leq t' \leq 1$ for which it must be shown that

B2: $t'x' + (1-t')y' \in C$.

5.12 Key Question: How can I show that a function (namely, $f+g$) is convex?

Key Answer: Show that for each pair of real numbers and another special real number between 0 and 1, the value of the function at the special real number times the first number plus one minus the special real number times the second number is less than or equal to the special number times the value of the function at the first number plus one minus the special number times the value of the function at the second number.

B1: For all real numbers x, y, and t with $0 \leq t \leq 1$, $(f+g)(tx+(1-t)y) \leq t(f+g)(x) + (1-t)(f+g)(y)$.

A1: Choose real numbers x', y', and t' with $0 \leq t' \leq 1$ for which it must be shown that

B2: $(f+g)(t'x'+(1-t')y') \leq t'(f+g)(x') + (1-t')(f+g)(y')$.

5.13 a. Incorrect because (x', y') should be chosen in S.
b. Correct.
c. Incorrect because (x', y') should be chosen in S (not in T). Also, you should show that (x', y') is in T (not in S).
d. Incorrect because you should not choose specific values for x and y but rather general values for x and y for which $(x, y) \in S$.
e. Correct. Note that this is simply a notational modification of part (b).

5.14 a. Incorrect because you should be applying the choose method to the for-all statement in the backward process (not in the forward process).
b. Incorrect because you should choose objects with the certain property (that is, you should choose real numbers x and y with $0 < x < y$) for which you should then show that the something happens [that is, you should show that $p(x) < p(y)$].
c. Incorrect because you should choose objects with the certain property, that is, you should choose real numbers x and y with $0 < x < y$ (not $x < y$).
d. Incorrect because you should not choose specific values for x and y but rather general objects with the certain property.
e. Correct.

5.15 The choose method is used in the first sentence of the proof where it says, "Let x be a real number." More specifically, the author asked the key question, "How can I show that a real number (namely, x^*) is a maximizer of a function (namely, $f(x) = ax^2 + bx + c$)?" Using the definition, one answer is to show that

B1: For all real numbers x, $f(x^*) \geq f(x)$.

Recognizing the quantifier "for all" in $B1$, the author uses the choose method to choose

A1: A real number x,

for which it must be shown that

B2: $f(x^*) \geq f(x)$, that is, that $a(x^*)^2 + bx^* + c \geq ax^2 + bx + c$.

The author then reaches $B2$ by considering the following two separate cases.

Case 1. $x^* \geq x$. In this case, the author correctly notes that $x^* - x \geq 0$ and also that $a(x^* + x) + b \geq 0$ because $x^* = -b/(2a)$ and so

A2: $a(x^* + x) + b = -b/2 + ax + b = (2ax + b)/2$.

However, because $x^* = -b/(2a) \geq x$, multiplying the inequality through by $2a < 0$ and adding b to both sides yields

A3: $2ax + b \geq 0$.

Thus, from $A2$ and $A3$, the author correctly concludes that

A4: $a(x^* + x) + b \geq 0$.

Multiplying $A4$ through by $x^* - x \geq 0$ and rewriting yields

A5: $a(x^*)^2 - ax^2 + bx^* - bx \geq 0$.

Bringing all x-terms to the right and adding c to both sides yields $B2$, thus completing this case.

Case 2. $x^* < x$. In this case, the author leaves the following steps for you to create. Now $x^* - x < 0$ and also $a(x^* + x) + b \leq 0$ because

A2: $a(x^* + x) + b = -b/2 + ax + b = (2ax + b)/2$.

However, because $x^* = -b/(2a) < x$ and $a < 0$, it follows that

A3: $2ax + b < 0$.

Thus, from $A2$ and $A3$,

A4: $a(x^* + x) + b < 0$.

Multiplying $A4$ through by $x^* - x < 0$ and rewriting yields

A5: $a(x^*)^2 - ax^2 + bx^* - bx > 0$.

Bringing all x-terms to the right and adding c to both sides yields $B2$, thus completing this case and the proof.

5.16 The choose method is used in the second sentence where the author says, "To that end, let $x \in R \cap S$." The choose method is used because, as stated in the first sentence, the author wants to show that

B1: For all $x \in R \cap S$, $x \in T$.

The author then recognizes the keywords "for all" in the backward statement $B1$ and uses the choose method to choose

A1: An element $x \in R \cap S$,

for which it must be shown that

B2: $x \in T$.

The author then correctly establishes $B2$ in the remainder of the proof.

5.17 **Analysis of Proof.** Because the conclusion contains the keywords "for all," the choose method is used to choose

A1: Real numbers x and y with $x < y$,

for which it must be shown that

B1: $f(x) < f(y)$, that is, $mx + b < my + b$.

Work forward from the hypothesis that $m > 0$ to multiply both sides of the inequality $x < y$ in $A1$ by m yielding

A2: $mx < my$.

Adding b to both sides gives precisely $B2$.

5.18 **Analysis of Proof.** The appearance of the quantifier "for every" in the conclusion suggests using the choose method, whereby one chooses

A1: An element $t \in T$,

for which it must be shown that

B1: t is an upper bound for the set S.

A key question associated with $B1$ is, "How can I show that a real number (namely, t) is an upper bound for a set (namely, S)?" By definition, one must show that

B2: For every element $x \in S$, $x \leq t$.

The appearance of the quantifier "for every" in the backward statement $B2$ suggests using the choose method, whereby one chooses

A2: An element $x \in S$,

for which it must be shown that

B3: $x \leq t$.

To do so, work forward from $A2$ and the definition of the set S in the hypothesis to obtain

A3: $x(x - 3) \leq 0$.

From $A3$, either $x \geq 0$ and $x - 3 \leq 0$, or, $x \leq 0$ and $x - 3 \geq 0$. But the latter cannot happen, so

A4: $x \geq 0$ and $x - 3 \leq 0$.

From $A4$,

A5: $x \leq 3$.

But, from $A1$ and the definition of the set T in the hypothesis

A6: $t \geq 3$.

Combining $A5$ and $A6$ yields $B3$, thus completing the proof.

5.19 **Analysis of Proof.** The author recognizes the keywords "for all" in the conclusion and, without telling you, uses the choose method to choose

A1: Nonzero integers $q < r$ having the same sign,

for which it must be shown that

B1: $p/q > p/r$.

The author then works forward from the hypothesis that q and r have the same sign (so $qr > 0$) and the hypothesis that $p > 0$ to state that

A2: $p/(qr) > 0$.

The author continues to work forward using the hypothesis that $r > q$ by multiplying both sides of $r > q$ by $p/(qr) > 0$ (see $A2$) to state that

A3: $rp/(qr) > qp/(qr)$.

Simple algebra applied to $A2$ results in the desired conclusion that

$p/q > p/r$.

The proof is now complete because $A4$ is the same as $B1$.

5.20 a. The key question is, "How can I show that a number (namely, $\sqrt{2}-x$) is an upper bound for a set (namely, S)?" The key answer, by definition, is to show that each element of the set is $\leq$ the number.

b. The author used the choose method in the first sentence of the proof. Specifically, the author chooses an element $\frac{1}{n} \in S$ for which it must be shown that $\frac{1}{n} \leq \sqrt{2} - x$.

c. The hypothesis states that $0 < \frac{2-x^2}{2x+1}$ and so if n is any integer with $n > \frac{2x+1}{2-x^2} > 0$, it will follow that $0 < \frac{1}{n} < \frac{2-x^2}{2x+1}$ and so $\frac{1}{n} \in S$.

d. $n \geq 1$, which is equivalent to $0 < \frac{1}{n} \leq 1$, is used to obtain the first inequality to claim that $2x + \frac{1}{n} \leq 2x + 1$. The fact that $\frac{1}{n} < \frac{2-x^2}{2x+1}$, which implies $\frac{1}{n}(2x+1) < 2 - x^2$ (because $x > 0$), is used to obtain the second inequality.

5.21 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show that a function (namely, $x+1$) is greater than or equal to another function (namely, $(x-1)^2$) on a set (namely, S)?" The definition provides the answer that one must show that

B1: For all $x \in S$, $x + 1 \geq (x-1)^2$.

The appearance of the quantifier "for all" in the backward process suggests using the choose method to choose

A1: An element $x \in S$,

for which it must be shown that

B2: $x + 1 \geq (x - 1)^2$.

Bringing the term $x+1$ to the right and performing algebraic manipulations, it must be shown that

B3: $x(x - 3) \leq 0$.

However, working forward from $A1$ using the definition of S yields

A2: $0 \leq x \leq 3$.

But then $x \geq 0$ and $x - 3 \leq 0$ so $B3$ is true and the proof is complete.

Proof. Let $x \in S$, so $0 \leq x \leq 3$. It will be shown that $x + 1 \geq (x - 1)^2$. However, because $x \geq 0$ and $x \leq 3$, it follows that $x(x - 3) = x^2 - 3x \leq 0$. Adding $x + 1$ to both sides and factoring leads to the desired conclusion that $x + 1 \geq (x - 1)^2$. □

5.22 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show that a set (namely, C) is convex?" One answer is by the definition, whereby it must be shown that

B1: For all elements x and y in C, and for all real numbers t with $0 \leq t \leq 1, tx + (1 - t)y \in C$.

The appearance of the quantifiers "for all" in the backward statement $B1$ suggests using the choose method to choose

A1: Elements x and y in C, and a real number t with $0 \leq t \leq 1$,

for which it must be shown that

B2: $tx + (1 - t)y \in C$, that is, $a(tx + (1 - t)y) \leq b$.

Turning to the forward process, because x and y are in C (see $A1$),

A2: $ax \leq b$ and $ay \leq b$.

Multiplying both sides of the two inequalities in $A2$, respectively, by the nonnegative numbers t and $1 - t$ (see $A1$) and adding the inequalities yields:

A3: $tax + (1 - t)ay \leq tb + (1 - t)b$.

Performing algebra on $A3$ yields $B2$, and so the proof is complete.

Proof. Let t be a real number with $0 \leq t \leq 1$, and let x and y be in C. Then $ax \leq b$ and $ay \leq b$. Multiplying both sides of these inequalities by $t \geq 0$ and $1 - t \geq 0$, respectively, and adding yields $a[tx + (1 - t)y] \leq b$. Hence, $tx + (1 - t)y \in C$. Therefore, C is a convex set and the proof is complete. □

5.23 **Analysis of Proof.** The keywords "for all" in the conclusion suggest using the choose method to choose

A1: Real numbers x and y,

for which it must be shown that

B1: $f(x) \geq f(y) + (2ay + b)(x - y)$.

Bringing the right side of the inequality in $B1$ to the left side means you must show that

B2: $f(x) - f(y) - (2ay + b)(x - y) \geq 0$.

Using the definition of the function f, this means it must be shown that

B3: $ax^2 + bx + c - ay^2 - by - c - 2ayx + 2ay^2 - bx + by =$
$ax^2 - 2axy + ay^2 \geq 0$.

Rewriting using algebra means you must show that

B4: $a(x - y)^2 \geq 0$.

However, it is given in the hypothesis that $a > 0$ and so $B4$ is true because $(x - y)^2 \geq 0$, thus completing the proof.

Proof. Let x and y be real numbers, for which it will be shown that $f(x) \geq f(y) + (2ay + b)(x - y)$, that is, $f(x) - f(y) - (2ay + b)(x - y) \geq 0$. However, from algebra, $f(x) - f(y) - (2ay + b)(x - y) = a(x - y)^2$. Because $a > 0$ from the hypothesis and $(x - y)^2 \geq 0$, $a(x - y)^2 \geq 0$ and so the proof is complete. □

5.24 **Analysis of Proof.** The keywords "for all" in the conclusion suggest using the choose method to choose

A1: Real numbers a and b, at least one of which is not 0,

for which it must be shown that

B1: $a^2 + ab + b^2 > 0$.

Working forward from the left side of $B1$ by algebra you have:

A2:
$$\begin{aligned} a^2 + ab + b^2 &> \tfrac{a^2+b^2}{2} + ab && \text{(from the hint)} \\ &= \tfrac{a^2+2ab+b^2}{2} && \text{(algebra)} \\ &= \tfrac{(a+b)^2}{2} && \text{(factor).} \end{aligned}$$

The last term on the right side in $A2$ is nonnegative, thus establishing $B1$ and completing the proof.

Proof. Let a and b be real numbers, at least one of which is not zero. It then follows that

$$\begin{array}{rcll} a^2+ab+b^2 & > & [(a^2+b^2)/2]+ab & (\text{because } a^2+b^2 > (a^2+b^2)/2) \\ & = & (a+b)^2/2 & (\text{algebra}). \end{array}$$

The desired conclusion follows by noting that $(a+b)^2 \geq 0$, thus completing the proof. □

5.25 **Analysis of Proof.** A key question associated with the conclusion is, "How can I show that a function (namely, $f(x) = x^3$) is strictly increasing?" Using the definition, the answer is to show that

B1: For all real numbers x and y with $x < y$, $x^3 < y^3$.

Recognizing the keywords "for all" in the backward statement $B1$, the choose method is used to choose

A1: Real numbers x and y with $x < y$,

for which it must be shown that

B2: $x^3 < y^3$.

Subtracting x^3 from both sides of $B2$ and factoring, you must show that

B3: $(y-x)(x^2+xy+y^2) > 0$.

Now the term $y - x$ is positive because $x < y$ from $A1$. Also, the term $x^2 + xy + y^2 > 0$ from the result in Exercise 5.24. Thus the product of these two terms is positive, hence establishing $B3$ and completing the proof.

Proof. To reach the conclusion that $f(x) = x^3$ is strictly increasing, let x and y be real numbers with $x < y$. Thus $y - x > 0$ and also $x^2 + xy + y^2 > 0$ from the result in Exercise 5.24, so $y^3 - x^3 = (y-x)(x^2+xy+y^2) > 0$. This means that $y^3 > x^3$ and so $f(x) = x^3$ is strictly increasing. □

5.26 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show a function (namely, $f(x) = mx+b$) is convex?" Using the definition in Exercise 5.2(c), one must show that

B1: For all real numbers x and y, and for all t with $0 \leq t \leq 1$,
$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$, that is,
$m(tx+(1-t)y)+b \leq t(mx+b)+(1-t)(my+b)$.

The appearance of the quantifier "for all" in the backward statement $B1$ suggests using the choose method, whereby one chooses

A1: Real numbers x' and y', and a real number t' that satisfies
$0 \leq t' \leq 1$,

for which it must be shown that

B2: $m(t'x' + (1-t')y') + b \le t'(mx' + b) + (1-t')(my' + b)$.

Applying algebra to the right side of the inequality in $B2$ results in

$$\textbf{A2:} \quad \begin{aligned} m(t'x' + (1-t')y') + b &= mt'x' + my' - mt'y' + b \\ &= mt'x' + bt' + my' - mt'y' + b - bt' \\ &= t'(mx' + b) + (1-t')(my' + b). \end{aligned}$$

The proof is now complete because $A2$ establishes the inequality in $B2$.

Proof. To show that $f(x) = mx + b$ is convex, it will be shown that for all real numbers x and y, and for all t satisfying $0 \le t \le 1$,

$$m(tx + (1-t)y) + b \le t(mx + b) + (1-t)(my + b).$$

To that end, let x', y' and t' be real numbers with $0 \le t' \le 1$. Then

$$\begin{aligned} m(t'x' + (1-t')y') + b &= mt'x' + my' - mt'y' + b \\ &= t'(mx' + b) + (1-t')(my' + b). \end{aligned}$$

Thus the desired inequality holds and so the proof is complete. □

6

Solutions to Exercises

6.1 The reason you need to show that Y has the certain property is because you only know that the something happens for objects with the certain property. You do not know that the something happens for objects that do not satisfy the certain property. Therefore, if you want to use specialization to claim that the something happens for this particular object Y, you must be sure that Y has the certain property.

6.2 You can apply specialization to the statement,

A1: For every object X with the property Q, T happens,

to reach the conclusion that S happens for the particular object Y if, when you replace X with Y in $A1$, the something that happens (namely, T) leads to the desired conclusion that S happens. To apply specialization, you must show that the particular object Y has the property Q in $A1$, most likely by using the fact that Y has property P.

To use specialization in this problem you need to show two things: (1) that you can specialize $A1$ to the object Y (which will be possible if whenever property P holds, so does property Q) and (2) that if T happens (plus any other information you can assume), then S happens.

6.3 a. (1) Look for a specific real number, say y, with which to apply specialization and (2) conclude that $f(y) \leq f(x^*)$ as a new statement in the forward process.

b. (1) Look for a specific element, say y, with which to apply specialization, (2) show that $y \in S$, and (3) conclude that $g(y) \geq f(y)$ as a new statement in the forward process.

c. (1) Look for a specific element, say y, with which to apply specialization, (2) show that $y \in S$, and (3) conclude that $y \leq u$ as a new statement in the forward process.

6.4 a. (1) Look for specific real numbers, say x' and y', (2) show that $x' < y'$, and (3) conclude that $f(x') < f(y')$ as a new statement in the forward process.

b. (1) Look for specific elements, say x' and y', and a real number t' with which to apply specialization, (2) show that x', $y' \in C$ and that $0 \leq t' \leq 1$, and (3) conclude that $t'x' + (1 - t')y' \in C$ as a new statement in the forward process.

c. (1) Look for specific real numbers, say x', y', and t' with which to apply specialization, (2) show that $0 \leq t' \leq 1$, and (3) conclude that $f(t'x' + (1-t')y') \leq t'f(x') + (1-t')f(y')$ as a new statement in the forward process.

6.5 a. You would not use specialization because the quantifier "for all" does not appear in the hypothesis.

b. You would not use specialization because the quantifier "for all" appears in the conclusion rather than the hypothesis.

c. You would use specialization because the quantifier "for all" appears in the hypothesis.

d. You would use specialization because the quantifier "for all" appears in the hypothesis.

6.6 a. You must verify that the integer m is a prime number. Given that this is so, you can conclude that $m + 7$ is composite.

b. You must verify that the real number $y > 0$ and in the set S. Given that this is so, you can conclude that $p(y) = 0$.

c. You do not need to verify anything. You can, however, conclude that the length c of the hypotenuse of the triangle ABC satisfies the property that $c^2 = m^2 + m^2 - 2m^2 \cos(\pi/2) = 2m^2$.

d. You must verify that the triangles CDE and FDA are equilateral. Given that this is so, you can conclude that, if one side of triangle CDE is parallel to one side of triangle FDA, then the remaining two sides of these triangles are parallel. Further, because you know that DE is parallel to DA, you can also conclude that the remaining two sides of these triangles are parallel.

6.7 a. You can reach the desired conclusion that $\sin(2X) = 2\sin(X)\cos(X)$ by specializing the statement, "For all angles α and β, $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$" to $\alpha = X$ and $\beta = X$, which you can do because X is an angle. The result of this specialization is that $\sin(X+X) = \sin(X)\cos(X) + \cos(X)\sin(X)$, that is, $\sin(2X) = 2\sin(X)\cos(X)$.

b. You can reach the desired conclusion that $(A \cap B)^c = A^c \cup B^c$ by specializing the statement, "For any sets S and T, $(S \cup T)^c = S^c \cap T^c$" to $S = A^c$ and $T = B^c$, which you can do because A^c and B^c are sets. The result of this specialization is that $(A^c \cup B^c)^c = (A^c)^c \cap (B^c)^c = A \cap B$. Applying the complement to both sides now leads to the desired conclusion that $A^c \cup B^c = (A \cap B)^c$.

6.8 a. You can reach the desired conclusion that $f(1/2) \leq (f(0) + f(1))/2$ by specializing the statement, "For all real numbers x, y, and t with $0 \leq t \leq 1$, $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$" to $x = 0$, $y = 1$, and $t = 1/2$ because doing so results in $f(1/2) \leq (f(0) + f(1))/2$. You can do this because this value of $t = 1/2$ satisfies the certain property that $0 \leq t \leq 1$.

b. You can obtain the desired result by specializing the statement, "for all real numbers c and d with $c^2 \geq d^2$, $\sqrt{c^2 - d^2} \leq c$," to $c = (a+b)/2$ and $d = (a-b)/2$ because doing so results in $\sqrt{[(a+b)/2]^2 - [(a-b)/2]^2} = \sqrt{ab} \leq (a+b)/2$. You can do this specialization because the values $c = (a+b)/2$ and $d = (a-b)/2$, together with a, $b \geq 0$, satisfy

$$c^2 = \left(\frac{a+b}{2}\right)^2 = \frac{a^2 + 2ab + b^2}{4} \geq \frac{a^2 - 2ab + b^2}{4} = \left(\frac{a-b}{2}\right)^2 = d^2.$$

6.9 a. The author is using the choose method. The choose method is used because the author has asked the key question, "How can I show that a set (namely, C) is convex?" Using the definition, the answer is to show that

B1: For all elements $x, y \in C$ and for all real numbers t with $0 \leq t \leq 1$, $tx + (1-t)y \in C$.

Recognizing the keywords "for all" in the backward statement $B1$, the author now uses the choose method to choose

A1: Elements $a, b \in C$ and a real number t with $0 \leq t \leq 1$,

for which it must be shown that

B2: $ta + (1-t)b \in C$.

b. Specialization is used in the second sentence. Specifically, the author works forward by definition from the hypothesis that f is convex to claim that

A2: For all real numbers x, y, and t with $0 \leq t \leq 1$, $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$.

Recognizing the keywords "for all" in the forward process, the author specializes $A2$ with $t = t$, $x = a$, and $y = b$ to conclude that

A3: $f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$.

c. The author is justified in saying that $f(a) \leq y$ and $f(b) \leq y$ because this is the defining property of the set C and a and b are in C [see $A1$ in part (a)].

6.10 The author makes a mistake when saying that, "In particular, for the specific elements x and y, and for the real number t, it follows that $tx + (1-t)y \in R$." In this sentence, the author is applying specialization to the statement

A1: For any two elements u and v in R, and for any real number s with $0 \leq s \leq 1$, $su + (1-s)v \in R$.

Specifically, the author is specializing $A1$ with $u = x$, $v = y$, and $s = t$. However, the author has failed to verify that these specific objects satisfy the certain properties in $A1$. In particular, to specialize $A1$ to $u = x$ and $v = y$, it is necessary to verify that $x \in R$ and $y \in R$. The author fails to do so (in fact it is not necessarily true that $x \in R$ and $y \in R$). Thus, the author is not justified in applying this specialization.

6.11 The author makes a mistake in the last sentence because, when dividing

$$a\epsilon \leq b + 2ax^*$$

by $a < 0$, the inequality must reverse, resulting in the following (undesirable) conclusion:

$$\epsilon \geq \frac{b + 2ax^*}{a}.$$

6.12 The author makes the first mistake when saying that, "In particular, for $t = 1 - a_3 > 0$, $x = \frac{a_1}{1-a_3}x_1 + \frac{a_2}{1-a_3}x_2$ and $y = x_3$, you have

$$\begin{aligned} f(tx + (1-t)y) &= f\left((1-a_3)\left(\tfrac{a_1}{1-a_3}x_1 + \tfrac{a_2}{1-a_3}x_2\right) + a_3x_3\right) \\ &\leq (1-a_3)f\left(\tfrac{a_1}{1-a_3}x_1 + \tfrac{a_2}{1-a_3}x_2\right) + a_3f(x_3). \end{aligned}$$

The author is specializing the definition of a convex function to particular values of t, x, and y; however, in order to do so, the author must verify that these objects satisfy the necessary property that $0 \leq t \leq 1$. In particular, the author failed to verify that this value of $t = 1 - a_3 \leq 1$ (which, in fact, is not true if $a_3 = -1$).

The author makes the same mistake in the second specialization of a convex function to particular values of t, x, and y. In order to do so, the author must verify that these objects satisfy the necessary property that $0 \leq t \leq 1$. In particular, the author failed to verify this property for $t = \frac{a_1}{1-a_3}$ (which, in fact, is not true if $a_1 = -1$ and $a_3 = -1$ as $t = \frac{a_1}{1-a_3} < 0$).

6.13 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show that a set (namely, R) is a subset of another set (namely, T)?" One answer is by the definition, so one must show that

B1: For all elements $r \in R$, $r \in T$.

The appearance of the quantifier "for all" in the backward process suggests using the choose method. So choose

A1: An element $r' \in R$,

for which it must be shown that

B2: $r' \in T$.

Turning to the forward process, the hypothesis says that R is a subset of S and S is a subset of T. By definition, this means, respectively, that

A2: For all elements $r \in R$, $r \in S$, and
A3: For all elements $s \in S$, $s \in T$.

Specializing $A2$ to $r = r'$ (which is in R from $A1$), one has that

A4: $r' \in S$.

Specializing $A3$ to $r = r'$ (which is in S from $A4$), one has that

A5: $r' \in T$.

The proof is now complete because $A5$ is the same as $B2$.

Proof. To show that $R \subseteq T$, it must be shown that for all $r \in R$, $r \in T$. Let $r' \in R$. By hypothesis, $R \subseteq S$, so $r' \in S$. Also, by hypothesis, $S \subseteq T$, so $r' \in T$. □

6.14 **Analysis of Proof.** Recognizing the keywords "for all" in the conclusion, you should use the choose method to choose

A1: A real number $\epsilon' > 0$,

for which it must be shown that

B1: $a \le b + \epsilon'$.

The inequality in $B1$ is established by working forward from the hypothesis:

A: For every integer $n > 0$, $a \le b + \frac{1}{n}$.

Recognizing the keywords "for every" in the forward process, you should now specialize A to one particular value of the integer n, the question being what value? Whatever value is used for n, the result of specialization will be

A2: $a \le b + \frac{1}{n}$.

You can see that $B1$ will follow from $A2$ provided that $\frac{1}{n} \le \epsilon'$. Solving the foregoing inequality for n, you want $n \ge \frac{1}{\epsilon'}$. In summary, if you specialize A to any integer $n \ge \frac{1}{\epsilon'}$ (which is > 0 because $\epsilon' > 0$ from $A1$), the result is $A2$. Statement $B2$ is obtained from $A2$ by noting that because, $n \ge \frac{1}{\epsilon'}$, it follows that $\frac{1}{n} \le \epsilon'$ and so the proof is complete.

Proof. Let $\epsilon' > 0$. Let n be an integer with $n > \frac{1}{\epsilon'} > 0$. As such $\frac{1}{n} < \epsilon'$ and so, from the hypothesis, $a \le b + \frac{1}{n} \le b + \epsilon'$, completing the proof. $\square$

6.15 **Analysis of Proof.** Recognizing the keywords "for all" in the conclusion, the choose method is used to choose

A1: Real numbers x, y and z,

for which it will be shown that

B1: $|x - z| \le |x - y| + |y - z|$.

The forward process is used to establish $B1$. Specifically, recognizing the keywords "for all" in the hypothesis, specialization is used. However, to avoid overlapping notation, the hypothesis of this problem is rewritten using a change in symbols, as follows:

A1: For all real numbers a and b, $|a + b| \le |a| + |b|$.

The key is to identify appropriate values for a and b so that the result of specialization is $B1$. To that end, specializing $A1$ with $a = x - y$ and $b = y - z$ yields

A2: $|(x - y) + (y - z)| = |x - z| \le |x - y| + |y - z|$.

The proof is now complete because $A2$ is the same as $B1$.

Proof. From the hypothesis, for all real numbers a and b, $|a + b| \le |a| + |b|$. In particular, for $a = x - y$ and $b = y - z$, it follows that $|(x - y) + (y - z)| = |x - z| \le |x - y| + |y - z|$. $\square$

6.16 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show that a function (namely, f) is $\geq$ another function (namely, h) on a set (namely, S)?" According to the definition, it is necessary to show that

B1: For every element $x \in S$, $f(x) \geq h(x)$.

Recognizing the keywords "for all" in the backward statement $B1$, the choose method is used to choose

A1: An element $x \in S$,

for which it must be shown that

B2: $f(x) \geq h(x)$.

Turning now to the forward process, because $f \geq g$ on S, by definition,

A2: For every element $y \in S$, $f(y) \geq g(y)$.

(Note the use of the symbol y so as not to overlap with the symbol x in $A1$.) Likewise, because $g \geq h$ on S, by definition,

A3: For every element $z \in S$, $g(z) \geq h(z)$.

Recognizing the keywords "for every" in the forward statements $A2$ and $A3$, the desired conclusion in $B2$ is obtained by specialization. Specifically, specializing $A2$ with $y = x$ (noting that $x \in S$ from $A1$) yields:

A4: $f(x) \geq g(x)$.

Likewise, specializing $A3$ with $z = x$ (noting that $x \in S$ from $A1$) yields:

A5: $g(x) \geq h(x)$.

Combining $A4$ and $A5$, you have

A6: $f(x) \geq g(x) \geq h(x)$.

The proof is now complete because $A6$ is the same as $B2$.

Proof. Let $x \in S$. From the hypothesis that $f \geq g$ on S, it follows that $f(x) \geq g(x)$. Likewise, because $g \geq h$ on S, you have $g(x) \geq h(x)$. Combining these two means that $f(x) \geq g(x) \geq h(x)$ and so $f \geq h$ on S. $\square$

6.17 **Analysis of Proof.** Recognizing the keywords "for all" in the conclusion, the choose method is used to choose

A1: An element $x \in S$,

for which it must be shown that

B1: $|x| \leq v$.

A key question associated with $B1$ is, "How can I show that the absolute value of a number (namely, x) is $\leq$ another number (namely, v)?" By definition of absolute value, you must show that

B2: $x \leq v$ and $-x \leq v$.

Turning to the forward process, because u is an upper bound for S, by definition,

A2: For all elements $y \in S$, $y \leq u$.

Recognizing the keywords "for all" in the forward statement $A2$, you can specialize $A2$ with $y = x$ (noting that $x \in S$ from $A1$) to claim that

A3: $x \leq u$.

Furthermore from the hypothesis that $u \leq v$, from $A3$, you have

A4: $x \leq u \leq v$.

Looking at $B2$ it remains to show that $-x \leq v$. To that end, working forward from the hypothesis that v is an upper bound for the set $-S$, by definition,

A5: For all elements $z \in -S$, $z \leq v$.

Recognizing the keywords "for all" in the forward statement $A5$, you can specialize $A5$ with $z = -x$ (noting that $x \in S$ from $A1$ and so $-x \in -S$) to claim that

A6: $-x \leq v$.

The proof is now complete because $A4$ together with $A6$ yields $B2$.

Proof. Let $x \in S$. From the hypothesis that u is an upper bound for S and $u \leq v$, you have $x \leq u \leq v$. Also, because $x \in S$, $-x \in -S$. It then follows from the hypothesis that v is an upper bound for $-S$ that $-x \leq v$. But then $|x| \leq v$, and so the proof is complete. $\square$

6.18 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show that a real number (namely, v) is an upper bound for a set of real numbers (namely, S)? Using the definition, the answer is to show that

B1: For all elements $x \in S$, $x \leq v$.

Recognizing the keywords "for all" in the backward statement $B1$, the choose method is used to choose

A1: An element $x \in S$,

for which it must be shown that

B2: $x \leq v$.

Turning to the forward process, from the hypothesis that u is an upper bound for S, by definition,

A2: For all elements $y \in S$, $y \leq u$.

Recognizing the keywords "for all" in the forward process, specialize $A2$ with $y = x$ which is in S (see $A1$) and so

A3: $x \leq u$.

The desired conclusion in $B2$ now follows from $A3$ using the hypothesis that $u \leq v$, so $x \leq u \leq v$. Thus the proof is complete.

Proof. To show that v is an upper bound for S, let $x \in S$. It will be shown that $x \leq v$. To that end, from the hypothesis that u is an upper bound for S, it follows that for every element $y \in S$, $y \leq u$. In particular, for $x \in S$, $x \leq u$. The proof is completed by noting from the hypothesis that $u \leq v$. $\square$

6.19 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show that a set (namely, $S \cap T$) is convex?" One answer is by the definition, whereby it must be shown that

B1: For all $x, y \in S \cap T$, and for all t with $0 \leq t \leq 1$,
$tx + (1-t)y \in S \cap T$.

The appearance of the quantifier "for all" in the backward process suggests using the choose method to choose

A1: $x', y' \in S \cap T$, and t' with $0 \leq t' \leq 1$,

for which it must be shown that

B2: $t'x' + (1-t')y' \in S \cap T$.

Working forward from the hypothesis and $A1$, $B2$ will be established by showing that

B3: $t'x' + (1-t')y'$ is in both S and T.

Specifically, from the hypothesis that S is convex, by definition, it follows that

A2: For all $x, y \in S$, and for all $0 \leq t \leq 1$, $tx + (1-t)y \in S$.

Specializing $A2$ to $x = x'$, $y = y'$, and $t = t'$ (noting from $A1$ that $0 \leq t' \leq 1$) yields that

A3: $t'x' + (1-t')y' \in S$.

A similar argument shows that $t'x' + (1-t')y' \in T$, thus completing the proof.

Proof. To see that $S \cap T$ is convex, let $x', y' \in S \cap T$, and let t' with $0 \le t' \le 1$. It will be established that $t'x' + (1-t')y' \in S \cap T$. From the hypothesis that S is convex, one has that $t'x' + (1-t')y' \in S$. Similarly, $t'x' + (1-t')y' \in T$. Thus it follows that $t'x' + (1-t')y' \in S \cap T$, and so $S \cap T$ is convex. $\square$

6.20 **Analysis of Proof.** The appearance of the quantifier "for all" in the conclusion indicates that you should use the choose method to choose

A1: A real number $s' \ge 0$,

for which it must be shown that

B1: The function $s'f$ is convex.

An associated key question is, "How can I show that a function (namely, $s'f$) is convex?" Using the definition in Exercise 5.2(c), one answer is to show that

B2: For all real numbers x and y, and for all t with $0 \le t \le 1$, $s'f(tx + (1-t)y) \le ts'f(x) + (1-t)s'f(y)$.

The appearance of the quantifier "for all" in the backward process suggests using the choose method to choose

A2: Real numbers x' and y', and $0 \le t' \le 1$,

for which it must be shown that

B3: $s'f(t'x' + (1-t')y') \le t's'f(x') + (1-t')s'f(y')$.

The desired result is obtained by working forward from the hypothesis that f is a convex function. By the definition in Exercise 5.2(c), you have that

A3: For all real numbers x and y, and for all t with $0 \le t \le 1$, $f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$.

Specializing the statement in $A3$ to $x = x'$, $y = y'$, and $t = t'$ (noting that $0 \le t' \le 1$) yields

A4: $f(t'x' + (1-t')y') \le t'f(x') + (1-t')f(y')$.

The desired statement $B3$ is obtained by multiplying both sides of the inequality in $A4$ by the nonnegative number s', thus completing the proof.

Proof. Let $s' \ge 0$. To show that $s'f$ is convex, let x' and y' be real numbers, and let t' with $0 \le t' \le 1$. It will be shown that $s'f(t'x' + (1-t')y') \le t's'f(x') + (1-t')s'f(y')$.

Because f is a convex function by hypothesis, it follows from the definition that $f(t'x' + (1-t')y') \le t'f(x') + (1-t')f(y')$. The desired result is obtained by multiplying both sides of this inequality by the nonnegative number s'. $\square$

6.21 **Analysis of Proof.** Recognizing the keywords "for all" in the conclusion, the choose method is used to choose

A1: A real number x,

for which it must be shown that

B1: $f(x) \leq g(x^*)$.

Turning to the forward process, from the hypothesis that $g \geq f$ on the set of real numbers, by definition,

A2: For all real numbers y, $g(y) \geq f(y)$.

Recognizing the keywords "for all" in the forward process, specialize $A2$ to the real number $y = x$ chosen in $A1$. The result is

A3: $f(x) \leq g(x)$.

Also, from the hypothesis that x^* is a maximizer of g, by definition,

A4: For all real numbers y, $g(y) \leq g(x^*)$.

Recognizing the keywords "for all" in the forward process, specialize $A4$ to the real number $y = x$ chosen in $A1$. The result is

A5: $g(x) \leq g(x^*)$.

The desired result in $B1$ follows from $A3$ and $A5$ because $f(x) \leq g(x) \leq g(x^*)$, thus completing the proof.

Proof. To reach the conclusion, let x be a real number, for which it will be shown that $f(x) \leq g(x^*)$. To that end, the hypothesis that $g \geq f$ means that for every real number y, $f(y) \leq g(y)$. In particular, $f(x) \leq g(x)$. Also, because x^* is a maximizer of g, it follows that $g(x) \leq g(x^*)$. You now have that $f(x) \leq g(x) \leq g(x^*)$, and so the proof is complete. $\square$

6.22 **Analysis of Proof.** To reach the conclusion that

B: $a < 0$,

work forward from the hypothesis that $x^* = -b/(2a)$ is a maximizer of $ax^2 + bx + c$, which, by definition, means that

A1: For all real numbers x, $ax^2 + bx + c \leq a(x^*)^2 + bx^* + c$.

Recognizing the keywords "for all" in the forward statement $A1$, specialization is used. Specifically, as suggested in the hint, choose any value of $\epsilon > 0$ and then specialize $A1$ with $x = x^* + \epsilon > 0$ to obtain

A2: $a(x^* + \epsilon)^2 + b(x^* + \epsilon) + c \leq a(x^*)^2 + bx^* + c$.

Expanding the left side and subtracting $a(x^*)^2+bx^*+c$ from both sides yields

A3: $2ax^*\epsilon + a\epsilon^2 + b\epsilon \le 0$.

Dividing $A3$ through by $\epsilon > 0$ results in

A4: $2ax^* + a\epsilon + b \le 0$.

Substituting $x^* = -b/(2a)$ in $A4$ and dividing both sides by $\epsilon > 0$ results in

A5: $a \le 0$.

The proof is complete on noting that $a \neq 0$ is given and so it must be that $a < 0$, which is precisely $B1$.

Proof. From the hypothesis that x^* is a maximizer of ax^2+bx+c, it follows that for all real numbers x, $ax^2+bx+c \le a(x^*)^2+bx^*+c$. In particular, for $x = x^* + \epsilon$ (where ϵ is any chosen positive number), you have that

$$a(x^*+\epsilon)^2 + b(x^*+\epsilon) + c \le a(x^*)^2 + bx^* + c.$$

Expanding the left side of the foregoing inequality, canceling common terms, and then dividing through by $\epsilon > 0$, yields

$$2ax^* + a\epsilon + b \le 0.$$

Substituting $x^* = -b/(2a)$ from the hypothesis and dividing by $\epsilon > 0$, it follows that $a \le 0$. The proof is complete on noting that $a \neq 0$ and so it must be that $a < 0$. $\square$

6.23 **Analysis of Proof.** The author has used the forward-backward method to ask the key question, "How can I show that a function (namely, g) is greater than or equal to another function (namely, f) on a set (namely, R)?" The definition provides the answer that it is necessary to show that

B1: For every element $r \in R$, $g(r) \ge f(r)$.

Recognizing the quantifier "for every" in $B1$, the author uses the choose method to choose

A1: An element $x \in R$,

for which it must be shown that

B2: $g(x) \ge f(x)$.

To reach $B2$, the author turns to the forward process and works forward from the hypothesis that R is a subset of S by definition to claim that

A2: For every element $r \in R$, $r \in S$.

Recognizing the quantifier "for every" in $A2$, the author specializes $A2$ to the element $r = x \in R$ chosen in $A1$. The result of this specialization is

A3: $x \in S$.

Then the author works forward from the hypothesis that $g \geq f$ on S, which, by definition, means that

A4: For every element $s \in S$, $g(s) \geq f(s)$.

Recognizing the quantifier "for every" in $A4$, the author specializes $A4$ to the element $s = x \in S$ (see $A3$). The result of this specialization is

A5: $g(x) \geq f(x)$.

The proof is now complete because $A5$ is the same as $B2$. □

6.24 **Analysis of Proof.** The author has asked the key question, "How can I show that a function (namely, $f + g$) is convex?" Using the definition, it must be shown that

B1: For all real numbers x, y, and t with $0 \leq t \leq 1$,
$f(tx + (1-t)y) + g(tx + (1-t)y) \leq t[f(x) + g(x)] +$
$(1-t)[f(y) + g(y)]$.

Recognizing the keywords "for all" in the backward statement $B1$, the choose method is used to choose

A1: Real numbers x, y, and t with $0 \leq t \leq 1$,

for which it must be shown that

B2: $f(tx + (1-t)y) + g(tx + (1-t)y) \leq$
$t[f(x) + g(x)] + (1-t)[f(y) + g(y)]$.

The author then works forward from the hypothesis that f is convex which, by definition, means that

A2: For all real numbers x', y', and t' with $0 \leq t' \leq 1$,
$f(t'x' + (1-t')y') \leq t'f(x') + (1-t')f(y')$.

Recognizing the keywords "for all" in the forward statement $A2$, the author applies specialization with $x' = x$, $y' = y$, and $t' = t$ (noting, from $A1$, that $0 \leq t \leq 1$), resulting in

A3: $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$.

The author then applies the same reasoning to the hypothesis that g is convex to conclude that

A4: $g(tx + (1-t)y) \leq tg(x) + (1-t)g(y)$.

The desired result in $B2$ is then obtained by adding the two inequalities in $A3$ and $A4$, thus completing the proof.

6.25 **Analysis of Proof.** Recognizing the keywords "for all" in the conclusion $B1$, the author uses the choose method to choose

A1: Real numbers $x_1, x_2, x_3, a_1, a_2, a_3 \geq 0$ with $a_1+a_2+a_3 = 1$,

for which it must be shown that

B1: $f(a_1x_1 + a_2x_2 + a_3x_3) \leq a_1f(x_1) + a_2f(x_2) + a_3f(x_3)$.

The author notes that there are two possible values for a_3, namely, $a_3 = 1$ and $a_3 \neq 1$ and proves that $B1$ is true in each of these case, as follows.

Case 1: $a_3 = 1$. In this case, because a_1 and $a_2 \geq 0$ and $a_1 + a_2 + a_3 = 1$, it must be that

A2: $a_1 = a_2 = 0$.

But then $B1$ is true because

A3: $f(a_1x_1 + a_2x_2 + a_3x_3) = f(x_3) \leq f(x_3) = a_1f(x_1) + a_2f(x_2) + a_3f(x_3)$.

Case 2: $a_3 \neq 1$. In this case, the author observes that

A4: $1 - a_3 > 0$.

Note that $A4$ is true because $a_1 + a_2 + a_3 = 1$ and $a_1, a_2 \geq 0$.

The author now works forward from the hypothesis that f is a convex function, which, by definition, means that

A5: For all real numbers x, y, and t with $0 \leq t \leq 1$, $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$.

Recognizing the keywords "for all" in the forward process, the author specializes $A5$ with $t = 1 - a_3 > 0$ (see $A4$), $x = \frac{a_1}{1-a_3}x_1 + \frac{a_2}{1-a_3}x_2$ and $y = x_3$. The result of this second specialization is

A6: $f(tx+(1-t)y) = f\left((1-a_3)\left(\frac{a_1}{1-a_3}x_1 + \frac{a_2}{1-a_3}x_2\right) + a_3x_3\right)$
$\leq (1-a_3)f\left(\frac{a_1}{1-a_3}x_1 + \frac{a_2}{1-a_3}x_2\right) + a_3f(x_3)$.

The author then specializes $A5$ again, but this time with $x = x_1$, $y = x_2$, and $t = \frac{a_1}{1-a_3}$. To apply specialization, the author notes that $t \geq 0$ (because $a_1 \geq 0$ and $1-a_3 > 0$ from $A4$) and also that $t \leq 1$. The latter is true because $a_1 + a_2 + a_3 = 1$ and $1 - t = 1 - \frac{a_1}{1-a_3} = \frac{1-a_3-a_1}{1-a_3} = \frac{a_2}{1-a_3}$ and $a_2 \geq 0$ and $1 - a_3 > 0$ (see $A4$). The result of this specialization is

A7: $f\left(\frac{a_1}{1-a_3}x_1 + \frac{a_2}{1-a_3}x_2\right) \leq \frac{a_1}{1-a_3}f(x_1) + \frac{a_2}{1-a_3}f(x_2)$.

On multiplying $A7$ through by $1 - a_3 > 0$ and combining the result with $A6$ you obtain $B1$, and so the proof is complete.

7

Solutions to Exercises

7.1 a. For the quantifier "there is":
- Object: real number y.
- Certain property: none.
- Something happens: for every real number x, $f(x) \leq y$.

For the quantifier "for every":
- Object: real number x.
- Certain property: none.
- Something happens: $f(x) \leq y$.

b. For the quantifier "there is":
- Object: real number M.
- Certain property: $M > 0$.
- Something happens: $\forall$ element $x \in S$, $|x| < M$.

For the quantifier "for all":
- Object: element x.
- Certain property: $x \in S$.
- Something happens: $|x| < M$.

c. For the first quantifier "for all":
- Object: real number ϵ.
- Certain property: $\epsilon > 0$.
- Something happens: there is a real number $\delta > 0$ such that, for all real numbers y with $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$.

For the quantifier "there is":
- Object: real number δ.
- Certain property: $\delta > 0$.
- Something happens: for all real numbers y with $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$.

For the second quantifier "for all":
- Object: real number y.
- Certain property: $|x - y| < \delta$.
- Something happens: $|f(x) - f(y)| < \epsilon$.

d. For the first quantifier "for all":
- Object: real number ϵ.
- Certain property: $\epsilon > 0$.
- Something happens: $\exists$ an integer $j \geq 1 \ni \forall$ integer k with $k > j$, $|x_k - x| < \epsilon$.

For the quantifier "there is":
- Object: integer j.
- Certain property: $j \geq 1$.
- Something happens: $\forall$ integer k with $k > j$, $|x_k - x| < \epsilon$.

For the second quantifier "for all":
- Object: integer k.
- Certain property: $k > j$.
- Something happens: $|x_k - x| < \epsilon$.

7.2 a. A set of real numbers S has the property that, for every element $x \in S$, there is an element $y \in S$ such that $y > x$.

b. A function f of a single variable has the property that there is a real number y such that for every real number x, $|f(x)| \leq y$.

7.3 a. Both $S1$ and $S2$ are true. This is because, when you apply the choose method to each statement, in either case you will choose real numbers x and y with $0 \leq x \leq 1$ and $0 \leq y \leq 2$ for which you can then show that $2x^2 + y^2 \leq 6$.

b. $S1$ and $S2$ are different—$S1$ is true and $S2$ is false. You can use the choose method to show that $S1$ is true. To see that $S2$ is false, consider $y = 1$ and $x = 2$. For these real numbers, $2x^2 + y^2 = 2(4) + 1 = 9 > 6$.

c. These two statements are the same when the properties P and Q do not depend on the objects X and Y. This is the case in part (a) but not in part (b).

7.4 a. $S1$ and $S2$ are both true. This is because you can construct values for x and y, say $x = 2$ and $y = 1$, that make both $S1$ and $S2$ true.

b. $S1$ and $S2$ are different—$S1$ is false and $S2$ is true. In particular, $S1$ is false because there are no values of $0 \le x \le 1$ and $0 \le y \le 2x$ such that $2x^2 + y^2 > 6$. This is because when $0 \le x \le 1$ and $0 \le y \le 2$, $2x^2 + y^2 \le 6$. In contrast, $S2$ is true because you can construct values for x and y, say $x = 2$ and $y = 1$, that make $S2$ true.

c. These two statements are the same when the properties P and Q do not depend on the objects X and Y, as in part (a) but not part (b).

7.5 a. Recognizing the keywords "for all" as the first quantifier from the left, the first step in the backward process is to choose an object X with a certain property P, for which it must be shown that there is an object Y with property Q such that something happens. Recognizing the keywords "there is," one then needs to construct an object Y with property Q such that something happens.

b. Recognizing the keywords "there is" as the first quantifier from the left, the first step in the backward process is to construct an object X with property P. After constructing X, the choose method is used to show that, for the object X you constructed, it is true that for all objects Y with property Q that something happens. Recognizing the keywords "for all," you would choose an object Y with property Q and show that something happens.

7.6 a. Recognizing the keywords "for all" as the first quantifier from the left, you should apply specialization to a particular object, say $X = Z$. To do so, you need to be sure that Z satisfies the property P. If so, then you can conclude, as a new statement in the forward process, that

A1: There is an object Y with the certain property Q such that something happens.

You can work forward from the object Y and its certain property Q.

b. You know that there is an object X such that

A1: For every object Y with the certain property Q, something happens.

Recognizing the keywords "for all" in the forward statement $A1$, you should apply specialization to a particular object, say $Y = Z$. To do so, you need to be sure that Z satisfies the property Q. If so, then you can conclude, as a new statement in the forward process, that

A2: The something happens for Z.

7.7 a. The construction method is used first to construct a real number $M > 0$. The choose method is used next to show that, for the value of M you constructed, it is true that for all elements $x \in T$, $|x| \leq M$. In so doing, you would choose an element $x \in T$, for which you must then show that $|x| \leq M$.

b. The choose method is used first to choose a real number $M > 0$, for which it must be shown that there is an element $x \in T$ such that $|x| > M$. To show this, the construction method is used next whereby you must construct an element $x \in T$ and then show that this element x satisfies $|x| > M$.

c. The choose method is used first to choose a real number $\epsilon > 0$ for which it must be shown that there is a real number $\delta > 0$ such that for all real numbers x and y with $|x-y| < \delta$, $|f(x)-f(y)| < \epsilon$. To show this, the construction method is used next to construct a real number $\delta > 0$. You must then show that this value of δ satisfies the property that for all real numbers x and y with $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$. To do this, use the choose method next to choose real numbers x and y with $|x - y| < \delta$, for which you must show that $|f(x) - f(y)| < \epsilon$.

7.8 a. You know that there is an object M such that

A1: For every element x in the set T, $|x| \leq M$.

Recognizing the keywords “for every” in the forward statement $A1$, you should apply specialization to a particular element, say x'. To do so, you need to be sure that the element x' is in T. If so, then you can conclude, as a new statement in the forward process, that

A2: $|x'| \leq M$.

b. Recognizing the keywords “for all” as the first quantifier from the left, you should apply specialization to some real number, say M'. To do so, you need to be sure that $M' > 0$. Then you can conclude, as a new statement in the forward process, that

A1: There is an element $x \in T$ such that $|x| > M'$.

You can now work forward from the fact that this element $x \in T$ satisfies $|x| > M'$.

c. Recognizing the keywords “for all” as the first quantifier from the left, you should apply specialization to some real number, say ϵ'. To do so, you need to be sure that $\epsilon' > 0$. Then you can conclude, as a new statement in the forward process, that there is a real number $\delta' > 0$ such that

A1: $\forall$ real numbers x and y with $|x-y| < \delta'$, $|f(x)-f(y)| < \epsilon'$.

Recognizing the keywords "for all" in the forward statement $A1$, you should specialize this statement to two real numbers, say x' and y'. To do so, you need to be sure that $|x' - y'| < \delta'$. You can then conclude, as a new statement in the forward process, that

A2: $|f(x') - f(y')| < \epsilon'$

.

7.9 a. A key question is, "How can I show that a set (namely S) is bounded?" By definition, the answer is to show that

B1: There is a real number $M > 0$ such that for every element $s \in S$, $|s| < M$.

Recognizing the keywords "there is" in the backward statement $B1$, you should use the construction method to produce a real number $M > 0$ for which you must then show that

B2: For every element $s \in S, |s| < M$.

Recognizing the keywords "for every" in the backward statement $B2$, you should then use the choose method to choose

A1: An element $s \in S$,

for which you must show that

B3: $|s| < M$.

b. A key question is, "How can I show that a function (namely $f(g(x))$) is onto?" By definition, you must show that

B1: For every real number y, there is a real number x such that $f(g(x)) = y$.

Recognizing the keywords "for every" in the backward statement $B1$, you should now use the choose method to choose

A1: A real number y,

for which you must show that

B2: There is a real number x such that $f(g(x)) = y$.

Recognizing the keywords "there is" in the backward statement $B2$, you should now use the construction method to produce a real number x for which you must then show that $f(g(x)) = y$.

c. A key question is, "How can I show that a function (namely f) is bounded above?" By definition, you must show that

B1: There is a real number y such that for every real number x, $f(x) < y$.

Recognizing the keywords "there is" in the backward statement $B1$, you should now use the construction method to produce a real number y for which you must then show that

B2: For every real number x, $f(x) < y$.

Recognizing the keywords "for every" in the backward statement $B2$, you should now use the choose method to choose

A1: A real number x,

for which you must show that

B3: $f(x) < y$.

7.10 a. By definition of a bounded set, for the set T you know that

> **A1:** There is a real number $M' > 0$ such that for every element $t \in T$, $|t| < M'$.

From $A1$, you know that $M' > 0$ satisfies

> **A2:** For every element $t \in T$, $|t| < M'$.

Recognizing the keywords "for every" in the forward statement $A2$, you should now use specialization. To do so, find one particular element, say x, for which you must verify that $x \in T$. On doing so you can conclude, as a new statement in the forward process, that

> **A3:** $|x| < M'$.

b. By definition of onto, for the function f you know that

> **A1:** For every real number y, there is a real number x such that $f(x) = y$.

Recognizing the keywords "for every" in the forward statement $A1$, you should now use the specialization. To do so, find one particular real number, say y'. On doing so you can conclude, as a new statement in the forward process, that

> **A2:** There is a real number x such that $f(x) = y$.

Recognizing the keywords "there is" in the forward statement $A2$, you can now work forward from the fact that $f(x) = y$. (Fro the hypothesis that g is onto, you can work forward by definition to obtain the same foregoing statements $A1$ and $A2$ for the function g.)

c. By definition of a function bounded above, for the function g you know that

> **A1:** There is a real number y' such that for every real number x, $g(x) < y'$.

Recognizing the keywords "there is" in the forward statement $A1$, you know that the real number y' satisfies the property that

> **A2:** For every real number $x, g(x) < y'$.

Recognizing the keywords "for every" in the forward statement $A2$, you should now use the specialization. To do so, find one particular real number, say x'. On doing so you can conclude, as a new statement in the forward process, that

> **A3:** $g(x') < y'$.

7.11 **Analysis of Proof.** Recognizing the keywords "for every" in the conclusion, you should use the choose method to choose

A1: Real numbers $\epsilon > 0$ and $a > 0$,

for which you must show that

B1: There is an integer $n > 0$ such that $\frac{a}{n} < \epsilon$.

Recognizing the keywords "there is" in the backward statement $B1$, you should now use the construction method to produce an integer $n > 0$ for which you must then show that

B2: $\frac{a}{n} < \epsilon$.

To construct this integer n, work backward from the fact that you want $\frac{a}{n} < \epsilon$. Multiplying both sides of $B2$ by $n > 0$ and then dividing both sides by $\epsilon > 0$ (see $A1$), you can see that you need $\frac{a}{\epsilon} < n$. In other words, constructing n to be any integer $> \frac{a}{\epsilon} > 0$, it follows that $B2$ is true and so the proof is complete.

Proof. Choose a real number $\epsilon > 0$. Now let n be a positive integer for which $n > \frac{a}{\epsilon}$. It then follows that $\frac{a}{n} < \epsilon$ and so the proof is complete. □

7.12 a. Recognizing the keywords "for all" as the first quantifier from the left, the author uses the choose method (as indicated by the words, "Let x and y be real numbers with $x < y$."). Accordingly, the author must now show that

B1: There is a rational number r such that $x < r < y$.

b. The author is using previous knowledge of the statement in Exercise 7.11. Specifically, recognizing the keywords "for all" in the forward process, the author is specializing that statement to the particular value $\epsilon = y - x > 0$ and $a = 2 > 0$. The result of that specialization is that there is an integer $n > 0$ such that $\frac{2}{n} < \epsilon$, that is, $n\epsilon > 2$, as the author states in the second sentence of the proof.

c. The construction method is used because the author recognized the keywords "there is" in the backward statement $B1$ in the answer to part (a). Specifically, the author constructs the rational number $r = \frac{m}{n}$ and claims that this value satisfies the property that $x < r < y$ in $B1$.

d. The author is justified in claiming that the proof is complete because the author has constructed the rational number $r = \frac{m}{n}$ and, because n is a positive integer [see the answer to part (b)] and $nx < m < ny$, it follows that $x < \frac{m}{n} = r < y$. This means that r satisfies the certain property in $B1$ and so the proof is complete.

7.13 a. Recognizing that the first quantifier from the left in the conclusion is "for all," the author uses the choose method to choose a prime number n, for which it must be shown that

B1: There is a prime number p such that $p > n$.

The author then recognizes the quantifier "there is" in the backward statement $B1$ and so constructs p as any prime number that divides $n! + 1$. From $B1$, it remains to show that

B2: $p > n$.

b. Recognizing the keyword "every," you can use the choose method to choose an integer k with the property that $2 \leq k \leq n$, for which it must be shown that $k|n!$. An associated key question is, "How can I show that an integer (namely, k) divides another integer (namely, $n!$)?" Using the definition leads you to show that there is an integer a such that $n! = ak$. To construct this integer a, note that, by definition, $n! = 1 * 2 * \cdots * k * \cdots * n$ and so it is not hard to verify that $a = 1 * 2 * \cdots * (k-1) * (k+1) * \cdots * n$ is the desired integer for which $n! = ak$.

c. From part (b), $k|n!$ but k does not divide 1 because $k > 1$. So $\frac{n!+1}{k} = \frac{n!}{k} + \frac{1}{k}$, is an integer plus some number less than 1 and so not an integer $p > 1$.

d. The author is specializing the following statement in the previous sentence:

A1: For every integer k with $2 \leq k \leq n$, k does not divide $n! + 1$.

If, in fact, $2 \leq p \leq n$, then it would be possible to specialize $A1$ to $k = p$ and conclude that

A2: p does not divide $n! + 1$.

7.14 a. The author uses the choose method in the first sentence. This is because the author first uses the forward-backward method and asks the key question,"How can I show that a function (namely, f) is convex?" Using the definition, the answer is to show that

B1: For all real numbers x, y and t with $0 \le t \le 1$, $f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$.

Recognizing the quantifier "for all" in the backward statement $B1$, the author now uses the choose method (as indicated in the first sentence of the proof) to choose

A1: Real numbers x, y and t with $0 \le t \le 1$,

for which it must be shown that

B1: $f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$.

b. The author is defining the variable z to simplify the notation.

c. The author is specializing the hypothesis that the real number m satisfies the property that for all real numbers x and y, $f(y) \ge f(x) + m(y-x)$ two separate times. Specializing this for-all statement with $y = x$ and $x = z$ results in

A2: $f(x) \ge f(z) + m(x - z)$ [which is inequality (a)].

Likewise, specializing the for-all statement with $y = y$ and $x = z$ results in

A3: $f(y) \ge f(z) + m(y - z)$ [which is inequality (b)].

d. The fact that $0 \le t \le 1$ is used in the fourth sentence when inequalities (a) and (b) are multiplied by t and $1-t$, respectively, without changing the direction of the inequality.

e. To see $f(z)+m(tx+(1-t)y-z) = f(z)$ note that, from the definition of z, $tx + (1-t)y - z = tx + (1-t)y - (tx + (1-t)y) = 0$. Therefore, $f(z) + m(tx + (1-t)y - z) = f(z) + 0 = f(z)$.

f. Recognizing that $z = tx + (1-t)y$, the inequality in (c) becomes precisely $B1$ [see the answer to part (a)] and so f is convex and the author is justified in claiming that the proof is complete.

7.15 In the last sentence, the author incorrectly assumes that $x > 0$, which is what the author needs to show, along with $x(ax + b - m) \ge 0$. From what the author has written, you can see that if $x \ge \frac{m-b}{a}$ and $x \ge 0$, then $x(ax + b - m) \ge 0$. Thus, you should construct x as a real number such that $x \ge \frac{m-b}{a}$ and $x \ge 0$ [for example, $x = \max\left\{\frac{m-b}{a}, 0\right\}$].

7.16 The error occurs when the author claims in the last sentence that $v - \epsilon$ is an upper bound for the set S. This is because, by definition, the author must show that

B1: For every element $x \in S$, $x \leq v - \epsilon$.

Recognizing the keywords "for every" in the backward statement $B1$, the author (correctly) uses the choose method to choose

A1: An element $x \in S$,

for which it must be shown that

B2: $x \leq v - \epsilon$.

Unfortunately, the author shows, in the next-to-last sentence of the proof, that $x - \epsilon \leq v - \epsilon$ (instead of showing that $x \leq v - \epsilon$) and so the proof is not correct.

7.17 **Analysis of Proof.** The keywords "for every" in the conclusion suggest using the choose method to choose

A1: A real number $x' > 2$,

for which it must be shown that

B1: There is a real number $y < 0$ such that $x' = 2y/(1 + y)$.

The keywords "there is" in $B1$ suggest using the construction method to construct the desired y. Working backward form the fact that y must satisfy $x' = 2y/(1 + y)$, it follows that y must be constructed so that

B2: $x' + x'y = 2y$, or
B3: $y(2 - x') = x'$, or
B4: $y = x'/(2 - x')$.

To see that the value of y in $B4$ is correct, it is easily seen that $x' = 2y/(1+y)$. However, it must also be shown that $y < 0$, which it is, because $x' > 2$.

Proof. Let $x' > 2$ and construct $y = x'/(2 - x')$. Because $x' > 2$, $y < 0$. It is also easy to verify that $x' = 2y/(1 + y)$. $\square$

7.18 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show that a function (namely, $f \circ g$) is onto?" By definition, it must be shown that

B1: For every real number y, there is a real number x such that $(f \circ g)(x) = y$, that is, $f(g(x)) = y$.

Recognizing "for every" as the first keywords from the left in the backward statement $B1$, the choose method is used to choose

A1: A real number y,

for which it must be shown that

B2: There is a real number x such that $f(g(x)) = y$.

Recognizing the keywords “there is” in the backward statement $B2$, the construction method is used to produce the desired value for x.

Turning to the forward process, from the hypothesis that f is onto, by definition, you have that

A2: For every real number v, there is a real number u such that $f(u) = v$.

Recognizing the keywords “for every” in the forward statement $A2$, you can specialize $A2$ with $v = y$ from $A1$ to conclude that

A3: There is a real number u such that $f(u) = y$.

Now, from the hypothesis that g is onto, by definition, you have that

A4: For every real number b, there is a real number a such that $g(a) = b$.

Recognizing the keywords “for every” in the forward statement $A4$, you can specialize $A4$ with $b = u$ from $A3$ to conclude that

A5: There is a real number a such that $g(a) = u$.

You can now construct $x = a$ from $A5$. This value is correct because, from $A5$ and $A3$, it follows that

$$f(g(x)) = f(g(a)) = f(u) = y.$$

This establishes $B2$, thus completing the proof.

Proof. To see that $f \circ g$ is onto, let y be a real number. It will be shown that there is a real number x such that $f(g(x)) = y$. To that end, from the hypothesis that f is onto, there is a real number u such that $f(u) = y$. Similarly, because g is onto, there is a real number a such that $g(a) = u$. It then follows that for $x = a$,

$$f(g(x)) = f(g(a)) = f(u) = y.$$

The proof is now complete. $\square$

7.19 **Analysis of Proof.** Recognizing the keywords “for every” as the first quantifier from the left in the conclusion, the choose method is used to choose

A1: A real number $\epsilon > 0$,

for which it must be shown that

B1: There is an element $x \in S$ such that $x^2 > 2 - \epsilon$.

Recognizing the keywords "there is" in $B1$, the construction method is used to produce the desired value of x, which depends on whether $2 > \epsilon$ or $2 \leq \epsilon$.

Case 1: $2 > \epsilon$. In this case, $2 - \epsilon > 0$, so you can construct the real number x as any value satisfying

A2: $\sqrt{2-\epsilon} < x < \sqrt{2}$.

To see that this value for x satisfies the properties in $B1$, you must show that

B2: $x \in S$ and $x^2 > 2 - \epsilon$.

Now $x > 0$ from $A2$ because $\sqrt{2-\epsilon} > 0$. Also, from $A2$, $x^2 < 2$. Thus $x \in S$ because x satisfies the defining property of S. Finally, squaring both sides of the first inequality in $A2$ yields that $2 - \epsilon < x^2$. Thus $B2$ is true, completing this case.

Case 2: $2 \leq \epsilon$. In this case, you can construct x as any value satisfying

A3: $x \in S$.

To see that this value for x satisfies the properties in $B1$, you must show that

B3: $x \in S$ and $x^2 > 2 - \epsilon$.

Now $x \in S$ from $A3$. Also, $x^2 > 2 - \epsilon$ because $x^2 > 0$ and $0 \geq 2 - \epsilon$. Thus, $B3$ is true, completing this case and the proof.

Proof. Let $\epsilon > 0$ be a real number. It will be shown that there is a real number $x \in S$ such that $x^2 > 2 - \epsilon$. In the event that $2 > \epsilon$, construct x as any value for which

$$\sqrt{2-\epsilon} < x < \sqrt{2}.$$

It is easy to see that $x \in S$ because $x > \sqrt{2-\epsilon} > 0$ and also $x^2 < 2$. Furthermore, because $\sqrt{2-\epsilon} < x$, it follows that $2 - \epsilon < x^2$.

Alternatively, if $2 \leq \epsilon$, then any $x \in S$ satisfies the property that $x^2 > 2-\epsilon$ because $x^2 > 0$ and $0 \geq 2 - \epsilon$. The proof is now complete. □

7.20 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show that a function (namely, f) is bounded above?" One answer is by the definition, whereby one must show that

B1: There is a real number y such that for every real number x, $-x^2 + 2x \leq y$.

The appearance of the quantifier "there is" in the backward statement $B1$ suggests using the construction method to produce the desired value for y. Trial and error might lead you to construct $y = 1$ (any value of $y \geq 1$ will also work). Now it must be shown that this value of $y = 1$ is correct, that is:

B2: For every real number x, $-x^2 + 2x \leq 1$.

The appearance of the quantifier "for all" in the backward statement $B2$ suggests using the choose method to choose

A1: A real number x,

for which it must be shown that

B3: $-x^2 + 2x \leq 1$, that is, $x^2 - 2x + 1 \geq 0$.

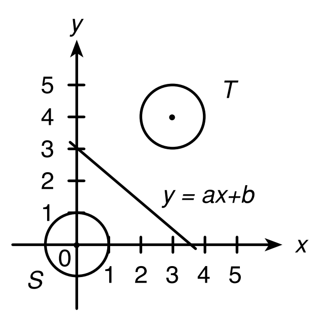
But because $x^2 - 2x + 1 = (x-1)^2$, this number is always ≥ 0. Thus $B3$ is true, completing the proof.

Proof. To see that the function $f(x) = -x^2 + 2x$ is bounded above, it will be shown that for all real numbers x, $-x^2 + 2x \leq 1$. To that end, let x be any real number. Then $x^2 - 2x + 1 = (x-1)^2 \geq 0$, thus completing the proof. □

7.21 **Analysis of Proof.** The keywords "there are" in the conclusion suggest using the construction method to construct

B1: Real numbers a and b such that for every $(x, y) \in S$, $y \leq ax + b$ and for every $(x, y) \in T$, $y \geq ax + b$.

To construct the values of a and b, observe that S is the circle of radius 1 centered at the origin and T is the circle of radius 1 centered at the point $(3, 4)$. As shown in the following figure, any line $y = ax + b$ for which S is on one side and T is on the other provides the desired values for a and b.



For example, construct

A1: $a = -1$ and $b = 2$.

To see that these values are correct, from $B1$, it must be shown that

B2: For every $(x, y) \in S$, $y \leq -x + 2$ and for every $(x, y) \in T$, $y \geq -x + 2$.

Recognizing the first keywords "for every" in the backward statement $B2$, the choose method is used to choose

A2: $(x, y) \in S$,

for which it must be shown that

B3: $y \leq -x + 2$, that is, $x + y \leq 2$.

Working forward from $A2$, from the definition of S, you have

A3: $x^2 + y^2 \leq 1$, so $x \leq 1$ and $y \leq 1$, and thus $x + y \leq 2$.

Turning now to the second keywords "for every" in $B2$, the choose method is used to choose

A4: $(x, y) \in T$,

for which it must be shown that

B4: $y \geq -x + 2$, that is, $x + y \geq 2$.

Working forward from $A4$, from the definition of T, you have

A5: $(x - 3)^2 + (y - 4)^2 \leq 1$, so $x \geq 2$ and $y \geq 3$, and thus $x + y \geq 5 \geq 2$.

$A5$ establishes $B2$, and so the proof is complete.

Proof. The desired values are $a = -1$ and $b = 2$. To see that these values are correct, first let $(x, y) \in S$. It then follows from the definition of S that $x \leq 1$ and $y \leq 1$, so $x + y \leq 2$, that is, $y \leq -x + 2$. Likewise, let $(x, y) \in T$. It then follows from the definition of T that $x \geq 2$ and $y \geq 3$, so $x + y \geq 5 \geq 2$, that is, $y \geq -x + 2$. It has now been shown that, for every $(x, y) \in S$, $y \leq -x + 2$ and for every $(x, y) \in T$, $y \geq -x + 2$ and so the proof is complete. □

7.22 **Analysis of Proof.** The first keywords in the conclusion from the left are "for every," so the choose method is used to choose

A1: A real number $\epsilon > 0$,

for which it must be shown that

B1: There is an element $x \in S$ such that $x > 1 - \epsilon$.

Recognizing the keywords "there is" in the backward statement $B1$, the construction method is used to produce the desired element in S. To that end, from the hint, you can write S as follows:

$$S = \{\text{real numbers } x : \text{there is an integer } n \geq 2 \text{ with } x = 1 - 1/n\}.$$

Thus, you can construct $x = 1 - 1/n$, for an appropriate choice of the integer $n \geq 2$. To find the value for n, from $B1$, you want

$$x = 1 - 1/n > 1 - \epsilon, \quad \text{that is,} \quad 1/n < \epsilon, \quad \text{that is,} \quad n > 1/\epsilon.$$

In summary, noting that $\epsilon > 0$ from $A1$, if you let $n \geq 2$ be any integer $> 1/\epsilon$, then $x = 1 - 1/n \in S$ satisfies the desired property in $B1$, namely, that $x = 1 - 1/n > 1 - \epsilon$. The proof is now complete.

Proof. Let $\epsilon > 0$. To see that there is an element $x \in S$ such that $x > 1 - \epsilon$, let $n \geq 2$ be an integer for which $n > 1/\epsilon$. It then follows from the defining property that $x = 1 - 1/n \in S$, and, by the choice of n, that $x = 1 - 1/n > 1 - \epsilon$. It has thus been shown that, for every real number $\epsilon > 0$, there is an element $x \in S$ such that $x > 1 - \epsilon$, thus completing the proof. □

7.23 **Analysis of Proof.** The author is working forward by definition from the hypothesis that f and g are linear functions to state that

> **A1:** There are real numbers m_1, m_2, b_1, b_2 such that for every real number x, $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$.

Working backward from the conclusion, an appropriate key question is, "How can I show that a function (namely, $f + g$) is linear?" Using the definition, the answer is to show that

> **B1:** There are real numbers m and b such that, for all real numbers x, $(f + g)(x) = mx + b$.

Recognizing the quantifier "there is" as the first keywords in the backward statement $B1$, the construction method is used to produce the desired real numbers m and b. Indeed, the author constructs

> **A2:** $m = m_1 + m_2$ and $b = b_1 + b_2$.

To see that this construction is correct, it must be shown that

> **B2:** For all real numbers x, $(f + g)(x) = mx + b$.

Recognizing the keywords "for all" in the backward statement $B2$, the choose method is used to choose

> **A3:** A real number x,

for which it must be shown that

> **B3:** $(f + g)(x) = mx + b$.

The result in $B3$ is obtained by working forward from the for-all statement in $A1$ and using $A2$, as follows:

$$
\begin{array}{rcll}
(f+g)(x) & = & f(x) + g(x) & \text{(definition of } (f+g)(x)\text{)} \\
& = & (m_1x + b_1) + (m_2x + b_2) & \text{(specializing } A1\text{)} \\
& = & (m_1 + m_2)x + (b_1 + b_2) & \text{(algebra)} \\
& = & mx + b & \text{(from } A2\text{)}
\end{array}
$$

The proof is now complete because $B3$ is true.

8

Solutions to Exercises

8.1 a. The real number x^* is not a maximum of the function f means that there is a real number x such that $f(x) > f(x^*)$.

b. Suppose that f and g are functions of one variable. Then g is not $\geq f$ on the set S of real numbers means that there exists an element $x \in S$ such that $g(x) < f(x)$.

c. The real number u is not an upper bound for a set S of real numbers means that there exists an element $x \in S$ such that $x > u$.

8.2 a. The function f is not strictly increasing means that there exist real numbers x and y with $x < y$ and $f(x) \geq f(y)$.

b. The set C of real numbers is not a convex set means that there exist elements $x, y \in C$ and there exists a real number t with $0 \leq t \leq 1$ such that $tx + (1-t)y \notin C$.

c. The function f of one real variable is not convex means that there exist real numbers x and y and t with $0 \leq t \leq 1$ such that $f(tx + (1-t)y) > tf(x) + (1-t)f(y)$.

8.3 a. A positive integer $p > 1$ is **not prime** if and only if there is an integer n with $2 \leq n < p$ such that n divides p.

b. A sequence $x_1, x_2, \ldots$, of real numbers is **not increasing** if and only if there is an integer $k \geq 1$ such that $x_k \geq x_{k+1}$.

c. A sequence $x_1, x_2, \ldots$, of real numbers is **not decreasing** if and only if there is an integer $k \geq 1$ such that $x_k \leq x_{k+1}$.

d. A sequence $x_1, x_2, \ldots$, of real numbers is **not strictly monotone** if and only if the sequence is neither increasing nor decreasing, that is, if there is an integer $k \geq 1$ such that $x_k \geq x_{k+1}$ and there is an integer $j \geq 1$ such that $x_j \leq x_{j+1}$.

e. An integer d is **not the greatest common divisor** of the integers a and b if and only if, either (i) the remainder on dividing a or b by d is positive or (ii) there is an integer c for which $c|a$ and $c|b$ such that the remainder on dividing d by c is positive.

f. The real number $f'(\bar{x})$ is **not the derivative of the function f at the point** $\bar{x}$ if and only if $\exists$ a real number $\epsilon > 0$ such that $\forall$ real number $\delta > 0$, $\exists$ a real number x with $0 < |x - \bar{x}| < \delta$ such that $|\frac{f(x)-f(\bar{x})}{x-\bar{x}} - f'(\bar{x})| \geq \epsilon$.

8.4 a. There does not exist an element $x \in S$ such that $x \notin T$.

b. It is not true that, for every angle t between 0 and $\pi/2$, $\sin(t) > \cos(t)$ or $\sin(t) < \cos(t)$.

c. There does not exist an object with the certain property such that the something does not happen.

d. It is not true that, for every object with the certain property, the something does not happen.

8.5 a. For every real number x, $x > a^{-x}$ or $x < a^{-x}$.

b. B.

c. B is true and C is false (this is when "B implies C" is false).

d. There is a real number $\epsilon > 0$ such that $\forall$ elements $x \in S$, $x \leq u - \epsilon$.

8.6 a. Work forward from $NOT\ B$.
Work backward from $(NOT\ C)$ *OR* $(NOT\ D)$.

b. Work forward from $NOT\ B$.
Work backward from $(NOT\ C)$ *AND* $(NOT\ D)$.

c. Work forward from $(NOT\ C)$ *OR* $(NOT\ D)$.
Work backward from $NOT\ A$.

d. Work forward from $(NOT\ C)$ *AND* $(NOT\ D)$.
Work backward from $NOT\ A$.

8.7 a. Work forward from integer k divides integer $n + 1$.
Work backward from k does not divide n.

b. Work forward from (mn is not divisible by 4) *AND* (n is divisible by 4).
Work backward from (n is an odd integer) *OR* (m is an even integer).

c. Work forward from n is an odd integer or m is an even integer.
Work backward from (mn is not divisible by 4) AND (n is divisible by 4).

8.8 a. *NOT B* implies *NOT A* is: If there exists a real number M such that, for every real number x, $ax^2 + bx + c \leq M$, then $a \leq 0$. From the hypothesis, you know that there is a real number M such that

A1: For every real number x, $ax^2 + bx + c \leq M$.

Recognizing the keywords "for every" in the forward process, you should specialize $A1$ to one particular value of the real number x.

b. *NOT B* implies *NOT A* is: If $a > 0$, then, for every real number M, there is a real number x such that $ax^2 + bx + c > M$. Recognizing the keywords "for every" as the first quantifier from the left in the backward process, you should use the choose method to choose

A1: A real number M,

for which you must show that

B1: There is a real number x such that $ax^2 + bx + c > M$.

Recognizing the keywords "there is" in the backward statement $B1$, you should now use the construction method to produce a real number x for which $ax^2 + bx + c > M$.

c. *NOT B* implies *NOT A* is: If p is not prime, then there is an integer m with $1 < m \leq \sqrt{p}$ such that $m|p$. Recognizing the keywords "there is" in the backward process, you should use the construction method to produce an integer m with $1 < m \leq \sqrt{p}$ such that $m|p$.

d. *NOT B* implies *NOT A* is: If for every number x, $x \neq a^{-x}$, then $a \leq 0$. Recognizing the keywords "for every" in the forward process, you should specialize the hypothesis to one particular value of the real number x.

e. *NOT B* implies *NOT A* is: If $n = m!$, then, for every integer k with $1 < k < m$, k divides n. Recognizing the keywords "for every" in the backward process, you should use the choose method to choose

A1: An integer k with $1 < k < m$,

for which you must show that

B1: k divides n.

f. *NOT B* implies *NOT A* is: If there is a real number $N > 0$ such that, for every element $y \in T$, $|y| < N$, then there is a real number $M > 0$ such that for every element $x \in S$, $|x| < M$. Recognizing the keywords "there is" as the first quantifier from the left in the backward process, you should use the construction method to produce a real number $M > 0$, for which you will then have to show that

B1: For every element $x \in S$, $|x| < M$.

Recognizing the keywords "for every" in the backward statement $B1$, you should then use the choose method to choose

A1: An element $x \in S$,

for which you must then show that

B2: $|x| < M$.

Also, from the hypothesis, you know that there is a real number $N > 0$ such that

A2: For every element $y \in T$, $|y| < N$.

Recognizing the keywords "for every" in the forward process, you should specialize $A2$ to one particular element $y \in T$.

8.9 a. $x = 3$ is a counterexample because $x^2 = 9 > 3 = x$. (Any value of a real number x for which $x^2 > x$ provides a counterexample.)

b. $n = 4$ is a counterexample because $n^2 = 16 < 24 = n!$. (Any value of an integer n for which $n^2 < n!$ provides a counterexample.)

c. $a = 2$, $b = 3$, and $c = 5$ constitute a counterexample because $a|(b+c)$ [that is, $2|(3+5)$] and yet $a = 2$ does not divide $b = 3$ (nor does $a = 2$ divide $c = 5$). (Any value of the integers a, b, and c for which $a|(b+c)$ and either a does not divide b or a does not divide c provides a counterexample.)

8.10 a. $x = \frac{1}{4}$ is a counterexample because $\sqrt{x} = \frac{1}{2} > \frac{1}{4} = x$. (Any value of a real number x for which $\sqrt{x} > x$ provides a counterexample.)

b. $n = 41$ is a counterexample because $41^2 + 41 + 41 = 41(43)$ is not prime. (Any value of a positive integer n for which $n^2 + n + 41$ is not prime provides a counterexample.)

c. $p = 4 > 0$ and $m = 2$ is a counterexample because p is not prime (as $p = 2 * 2$) and $m = 2$ satisfies $1 < m \leq \sqrt{p} = 2$ and m divides p. (Any value of a positive integer p that is not prime and an integer m for which $1 < m \leq \sqrt{p}$ and $m|p$ provides a counterexample.)

9

Solutions to Exercises

9.1 a. Assume that l, m, and n are three consecutive integers and that 24 divides $l^2 + m^2 + n^2 + 1$.

b. Assume that $n > 2$ is an integer and $x^n + y^n = z^n$ has an integer solution for x, y, and z.

c. Assume that f and g are two functions such that $g \geq f$, f is unbounded above, and g is not unbounded above.

9.2 a. Assume that $n > 2$ is an integer and there exist positive integers x, y, and z such that $x^n + y^n = z^n$.

b. Assume that a is a positive real number and that there are real numbers b, c, and M with $M > 0$ such that, for every real number x, $ax^2 + bx + c \leq M$.

c. Assume that matrix M is not singular and the rows of M are linearly dependent.

9.3 a. With the contradiction method, it is not possible to work backward from any statement because you do not know, beforehand, what the contradiction is going to be.

b. Yes, the student has completed the proof because the fact that b is an even integer contradicts the hypothesis that b is odd.

9.4 a. There are not a finite number of primes.
b. The positive integer p cannot be divided by any positive integer other than 1 and p.
c. The lines l and l' do not intersect.

9.5 a. Either the real number $ad - bc > 0$ or $ad - bc < 0$.
b. At least one side of triangle ABC is different in length from the other two sides. (Or, at least one angle of triangle ABC is not 60 degrees.)
c. Every root of the polynomial $a_0 + a_1x + \cdots + a_nx^n$ is a complex number of the form $c + di$ in which $d > 0$ or $d < 0$.

9.6 a. Recognizing the keywords "there is," you should use the construction method to construct an element $s \in S$ for which you must show that s is also in T.
b. Recognizing the quantifier "for all" as the first keywords from the left, you should use the choose method to choose

A1: An element $s \in S$,

for which you must show that

B1: There is no element $t \in T$ such that $s > t$.

Recognizing the keyword "no" in the backward statement $B1$, you should proceed with the contradiction method and assume that

A2: There is an element $t \in T$ such that $s > t$.

You must then work forward from $A1$ and $A2$ to reach a contradiction.
c. Recognizing the keyword "no," you should first use the contradiction method, whereby you should assume that

A1: There is a real number $M > 0$ such that, for all elements $x \in S$, $|x| < M$.

To reach a contradiction, you will probably have to apply specialization to the forward statement, "for all elements $x \in S$, $|x| < M$."
Alternatively, you can pass the NOT through the nested quantifiers and hence you must show that

B1: $\forall M > 0$, $\exists x \in S$ such that $|x| \geq M$.

Recognizing the quantifier "for all" as the first keywords in the backward statement $B1$, you can now use the choose method to choose

A1: A real number $M > 0$,

for which you must show that

B2: There is an element $x \in S$ such that $|x| \geq M$.

Recognizing the keywords "there is" in the backward statement $B2$, you should then use the construction method to produce the desired element $x \in S$ for which $|x| \geq M$.

9.7 a. Recognizing the keywords "for every" as the first quantifier from the left, you should start with the choose method. Specifically, choose

A1: A real number $\epsilon > 0$,

for which it must be shown that

B1: There is an element $x \in S$ such that $x > u - \epsilon$.

Recognizing the keywords "there is" in $B1$, you should now use the construction method to construct an element $x \in S$ such that $x > u - \epsilon$.

b. Recognizing the keywords "there is" as the first quantifier from the left, you should start with the construction method. Specifically, you should construct a real number y for which you must show that

B1: $y > 0$ and for every element $x \in S$, $f(x) < y$.

After constructing the value for $y > 0$, to establish $B1$, you should recognize the keywords "for every" in the backward process and so should use the choose method to choose

A1: An element $x \in S$,

for which it must be shown that

B2: $f(x) < y$.

c. Recognizing the quantifier "for every" as the first keywords from the left, you should use the choose method to choose a

A1: line l in the plane that is parallel to, but different from, l',

for which it must be shown that

B1: There is no point on l that is also in S.

In $B1$, the keyword "no" takes precedence and so you should proceed by contradiction. Accordingly, you should assume that

A2: There is a point on l that is also in S.

You must now work forward from $A1$ and $A2$ to reach a contradiction.

9.8 **Analysis of Proof.** To use the contradiction method, assume:

A: n is an integer for which n^2 is even.
A1 (NOT B): n is not even, that is, n is odd.

Work forward from these assumptions using the definition of an odd integer to reach the contradiction that n^2 is odd. Applying a definition to work forward from *NOT B* yields

A2: There exists an integer k such that $n = 2k + 1$.

Squaring both sides of the equality in $A2$ and performing simple algebraic manipulations leads to

A3: $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$, so
A4: $n^2 = 2(2k^2 + 2k) + 1$,

which says that $n^2 = 2p + 1$, where $p = 2k^2 + 2k$. Thus n^2 is odd, and this contradiction to the hypothesis that n^2 is even completes the proof.

Proof. Assume, to the contrary, that n is odd and n^2 is even. Hence, there is an integer k such that $n = 2k + 1$. Consequently,

$$n^2 = (2k + 1)^2 = 2(2k^2 + 2k) + 1,$$

and so n^2 is odd, contradicting the initial assumption. □

9.9 **Analysis of Proof.** According to the contradiction method, you can assume the hypothesis, A, and

A1 (NOT B): There are real numbers x and y with $x \neq y$ such that $x^3 = y^3$, that is, $x^3 - y^3 = 0$.

Working forward by factoring, you have that

A2: $(x - y)(x^2 + xy + y^2) = 0$.

Using the fact that $x \neq y$ from $A1$, you can divide both sides of $A2$ by $x - y \neq 0$ to obtain

A3: $x^2 + xy + y^2 = 0$.

Thinking of $A3$ as a quadratic equation of the form $ax^2 + bx + c$, in which $a = 1$, $b = y$, and $c = y^2$, the quadratic formula yields

A4: $x = \frac{-y \pm \sqrt{y^2 - 4(1)(y^2)}}{2} = \frac{-y \pm \sqrt{-3y^2}}{2}$.

The only way the value in $A4$ can result in a real value for x is if

A5: $y = 0$.

But then, from the fact that $x^3 = y^3$ in $A1$, you have that $x = y = 0$, which contradicts the assumption in $A1$ that $x \neq y$, thus completing the proof.

Proof. Assume, to the contrary, that there are real numbers x and y with $x \neq y$ such that $x^3 = y^3$. Then

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2) = 0.$$

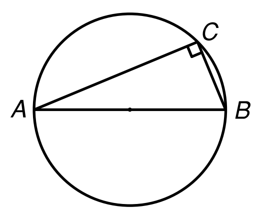
Dividing both sides of the foregoing equality by $x - y \neq 0$ results in $x^2 + xy + y^2 = 0$. Thinking of this as a quadratic equation of the form $ax^2 + bx + c = 0$, in which $a = 1$, $b = y$, and $c = y^2$, the quadratic formula yields

$$x = \frac{-y \pm \sqrt{y^2 - 4(1)(y^2)}}{2} = \frac{-y \pm \sqrt{-3y^2}}{2}.$$

The only way this can result in a real value for x is if $y = 0$, but then $x = 0$ and so $y = x$. This contradicts the assumption that $x \neq y$, thus completing the proof. □

9.10 **Analysis of Proof.** Proceed by assuming that there is a chord of a circle that is longer than its diameter. Using this assumption and the properties of a circle, you must arrive at a contradiction.

Let AC be the chord of the circle that is longer than the diameter of the circle (see the figure below). Let AB be a diameter of the circle. This construction is valid because, by definition, a diameter is a line passing through the center terminating at the perimeter of the circle.



It follows that angle ACB has 90 degrees because ACB is an angle inscribed in a semi-circle. Hence the triangle ABC is a right triangle in which AB is the hypotenuse. Then the desired contradiction is that the hypotenuse of a right triangle is shorter than one side of the triangle, which is impossible.

Proof. Assume that there does exist a chord, say AC, of a circle that is longer than a diameter. Construct a diameter that has one of its ends coinciding with one end of the chord AC. Joining the other ends produces a right triangle in which the diameter AB is the hypotenuse. But then the hypotenuse is shorter than one leg of the right triangle, which is a contradiction. □

9.11 **Analysis of Proof.** According to the contradiction method, you can assume the hypothesis, A, and

A1 (NOT B): There is a rational number x such that $ax^2 + bx + c = 0$.

By definition of a rational number, this means that

A2: There are integers p and q with $q \neq 0$ such that $a(p/q)^2 + b(p/q) + c = 0$.

A contradiction to the hypothesis is reached by showing that

B1: There is a rational number y such that $cy^2 + by + a = 0$.

Recognizing the keywords "there is" in $B1$, the construction method is used to produce the desired value for y. To that end, multiply both sides of the equality in $A2$ by q^2/p^2 (noting that $p \neq 0$, for otherwise, from $A2$, $c = 0$ which, according to the hypothesis, is not the case) to obtain

A3: $a + b(q/p) + c(q^2/p^2) = 0$.

From $A3$, you can see that the desired value for y in $B1$ is $y = q/p$. This number is rational because p and q are integers and, as stated previously, $p \neq 0$. From $A3$ you can see that $cy^2 + by + a = 0$, thus completing the proof.

Proof. Assume, to the contrary, that there is a rational number $x = p/q$ (where p and q are integers with $q \neq 0$) such that

$$a\left(\frac{p}{q}\right)^2 + b\left(\frac{p}{q}\right) + c = 0.$$

Note that $p \neq 0$, for otherwise $c = 0$ which, according to the hypothesis, is not the case. Thus, you can multiply both sides of the foregoing equality by q^2/p^2, resulting in

$$a + b\left(\frac{q}{p}\right) + c\left(\frac{q}{p}\right)^2 = 0.$$

But then q/p is a rational root of the equation $cx^2 + bx + a$, which contradicts the hypothesis and completes the proof. □

9.12 **Analysis of Proof.** By contradiction, assume A and $NOT\ B$, that is:

A: $n - 1$, n, and $n + 1$ are consecutive positive integers.
A1 (NOT B): $(n + 1)^3 = n^3 + (n - 1)^3$.

A contradiction is reached by showing that

B1: $n^2(n - 6) = 2$ and $n^2(n - 6) \geq 49$.

To that end, rewriting $A1$ by algebra, you have:

A2: $n^3 + 3n^2 + 3n + 1 = n^3 + n^3 - 3n^2 + 3n - 1$, or
A3: $n^3 - 6n^2 = 2$, or
A4: $n^2(n - 6) = 2$.

From $A4$, because $n^2 > 0$, it must be that

A5: $n - 6 > 0$, that is, $n \geq 7$.

But when $n \geq 7$, $n - 6 \geq 1$, and $n^2 \geq 49$, so

A6: $n^2(n-6) \geq n^2 \geq 49$.

Now $A6$ contradicts $A4$ because $A6$ states that $n^2(n-6) \geq 49$ while $A4$ states that $n^2(n-6) = 2$. This contradiction completes the proof.

Proof. Assume, to the contrary, that the three consecutive integers $n-1$, n, and $n+1$ satisfy

$$(n+1)^3 = n^3 + (n-1)^3.$$

Expanding these expressions and rewriting yields

$$n^2(n-6) = 2.$$

Because $n^2 > 0$, $n - 6 > 0$, that is, $n \geq 7$. But then $n^2(n-6) \geq n^2 \geq 49$. This contradicts the fact that $n^2(n-6) = 2$, thus completing the proof. $\square$

9.13 **Analysis of Proof.** By the contradiction method, you can assume the hypothesis, A, and

$A1$ (NOT B): q is prime.

By definition, this means that

A2: The only integers that divide q are 1 and q.

A contradiction is reached by showing that

B1: There is an integer a with $a \neq 1$ and $a \neq q$ such that a divides q.

To that end, working forward from the hypothesis that p divides q, by definition,

A3: There is an integer b such that $q = bp$.

You can see that the desired value for a in $B1$ is b. To see that this value is correct, it must be shown that

B2: $b \neq 1$, $b \neq q$, and b divides q.

Because the hypothesis states that $q \neq p$, it follows from $A3$ that $b \neq 1$. Also, $b \neq q$, for otherwise, from $A3$, it would follow that $p = 1$, which cannot happen because the hypothesis states that p is prime. Finally, the fact that $q = bp$ in $A3$ means that b divides q. So b is an integer other than 1 and q that divides q, thus contradicting the assumption that q is prime and completing the proof.

Proof. Assume, to the contrary, that q is prime. A contradiction is reached by showing that there is an integer other than 1 and q that divides q. To that end, from the hypothesis that p divides q, there is an integer b such that

$$q = bp.$$

Now b is the desired integer that leads to the contradiction. To see that this is so, note first from the hypothesis that because $p \neq q$, $b \neq 1$. Furthermore, $b \neq q$, for otherwise, $p = 1$, which cannot happen because p is prime. Finally, because $q = bp$ it follows by definition that b divides q. It has thus been shown that b is an integer other than 1 and q that divides q. This contradiction establishes the proof. □

9.14 **Analysis of Proof.** To use contradiction, assume that no two people have the same number of friends; that is, everybody has a different number of friends. Because there are n people, each of whom has a different number of friends, you can number the people in an increasing sequence according to the number of friends each person has. In other words,

person number 1 has no friends,
person number 2 has 1 friend,
person number 3 has 2 friends,
⋮
person number n has $n - 1$ friends.

By doing so, there is a contradiction that the last person is friends with all the other $n - 1$ people, including the first one, who has no friends.

Proof. Assume, to the contrary, that no two people have the same number of friends. Number the people at the party in such a way that, for each $i = 1, \ldots, n$, person i has $i - 1$ friends. It then follows that the person with $n - 1$ friends is a friend of the person who has no friends, which is a contradiction. □

9.15 **Analysis of Proof.** To use contradiction, assume that no two people are born on the same second of the same hour of the same day in the twentieth century. That is, everybody in the twentieth century is born at a different time. You can therefore number the people born in the twentieth century in an increasing sequence according to the order in which they were born in the twentieth century. In other words,

People Born in the Twentieth Century
person number 1 is born first in the twentieth century,
person number 2 is born second in the twentieth century,
person number 3 is born third in the twentieth century,
⋮

Noting that there are $60 \times 60 \times 24 \times 366 \times 100 = 3,162,240,000$ seconds in the twentieth century, you can see that person number 4,000,000,000 is born at least that many seconds after person number 1. But this means that person number 4,000,000,000 is not born in the twentieth century. This contradiction establishes the claim.

Proof. By contradiction, assume that no two people are born on the same second of the same hour of the same day in the twentieth century. That is, everybody in the twentieth century is born at a different time. You can therefore number the people born in the twentieth century in an increasing sequence according to the order in which they were born in the twentieth century. In other words, person number 1 is born first in the twentieth century, person number 2 is born second in the twentieth century, and so on. Noting that there are $60 \times 60 \times 24 \times 366 \times 100 = 3,162,240,000$ seconds in the twentieth century, you can see that person number 4,000,000,000 is born at least that many seconds after person number 1. But this means that person number 4,000,000,000 is not born in the twentieth century. This contradiction establishes the claim. □

9.16 **Analysis of Proof.** By the contradiction method, you can assume the hypothesis and

A1 (NOT B): There is a positive integer m with $m \neq n$ such that $m^3 - m - 6 = 0$.

A contradiction to $A1$ is reached by showing that $m = n$. To reach this contradiction, work forward by algebra from the hypothesis that $n^3 - n - 6 = 0$ to state that

A2: $n(n+1)(n-1) = 6$.

The only integer that satisfies $A2$ is $n = 2$. This is because it is clear that $n = 1$ does not satisfy $A2$ while $n = 2$ does, and for any value of $n > 2$, the left side of $A2$ is > 6. Likewise, working forward from $A1$ by algebra, you have that

A3: $m(m+1)(m-1) = 6$.

Similarly, the only integer that satisfies $A4$ is $m = 2$. But then $m = n = 2$ and this contradiction completes the proof.

Proof. Assume, to the contrary, that there is a positive integer m with $m \neq n$ such that $m^3 - m - 6 = 0$. It then follows by factoring that $m(m+1)(m-1) = 6$. The only such integer is $m = 2$ because $m = 1$ does not satisfy the equation and for $m > 2$, $m(m+1)(m-1) > 6$. Likewise, from the hypothesis that $n^3 - n - 6 = 0$, you have that $n(n+1)(n-1) = 6$. Once again, the only such integer is $n = 2$. It now follows that $m = n = 2$, which contradicts the assumption that $m \neq n$, thus completing the proof. □

9.17 **Analysis of Proof.** When using the contradiction method, you can assume the hypothesis that

A: $x \geq 0$, $y \geq 0$, $x + y = 0$,

and also that

A1 (NOT B): Either $x \neq 0$ or $y \neq 0$.

From $A1$, suppose first that

A2: $x \neq 0$.

Because $x \geq 0$ from A, it must be that

A3: $x > 0$.

A contradiction to the fact that $y \geq 0$ is reached by showing that

B1: $y < 0$.

Specifically, because $x + y = 0$ from A,

A4: $y = -x$.

Because $-x < 0$ from $A3$, a contradiction has been reached. A similar argument applies for the case where $y \neq 0$ (see $A1$).

Proof. Assume that $x \geq 0$, $y \geq 0$, $x + y = 0$, and that either $x \neq 0$ or $y \neq 0$. If $x \neq 0$, then $x > 0$ and $y = -x < 0$, but this contradicts the fact that $y \geq 0$. Similarly, if $y \neq 0$, then $y > 0$, and $x = -y < 0$, but this contradicts the fact that $x \geq 0$. □

9.18 **Analysis of Proof.** Not recognizing any keywords, the forward-backward method gives rise to the key question, "How can I show that a set (namely, $(S^c)^c$) is equal to another set (namely, S)?" By definition, it must be shown that

B1: $(S^c)^c \subseteq S$ and $S \subseteq (S^c)^c$.

A key question associated with the first statement in $B1$ is, "How can I show that a set (namely, $(S^c)^c$) is a subset of another set (namely, S)?" By the definition of a subset, you must show that

B2: For every element $x \in (S^c)^c$, $x \in S$.

Recognizing the keywords "for every" in the backward statement $B2$, you should use the choose method to choose

A1: An element $x \in (S^c)^c$,

for which you must show that

B3: $x \in S$.

Working forward from $A1$, because $x \in (S^c)^c$, by definition of the complement, this means that

A2: $x \notin S^c$.

Again, by definition of the complement, because $x \notin S^c$, it must be that

A3: $x \in S$.

This part of the proof is now complete because $A3$ is the same as $B3$. From $B1$, it still remains to show that

B4: $S \subseteq (S^c)^c$.

An associated key question is, "How can I show that a set (namely, S) is a subset of another set (namely, $(S^c)^c$)?" By the definition of a subset, you must show that

B5: For every element $x \in S$, $x \in (S^c)^c$.

Recognizing the keywords "for every" in the backward statement $B5$, you should use the choose method to choose

A4: An element $x \in S$,

for which you must show that

B6: $x \in (S^c)^c$.

A key question associated with $B6$ is, "How can I show that an element (namely, x) is in the complement of a set (namely, S^c)?" By definition of the complement of a set, you must show that

B7: $x \notin S^c$.

Recognizing the keyword "not" in the backward statement $B7$, you should proceed by contradiction and thus assume that

A5: $x \in S^c$.

But this means that $x \notin S$, which contradicts $A4$ and completes the proof.

Proof. It will be shown that $(S^c)^c \subseteq S$ and $S \subseteq (S^c)^c$. So first, let $x \in (S^c)^c$. This means that $x \notin S^c$ and so, in fact, $x \in S$. To see that $S \subseteq (S^c)^c$, let $x \in S$. It will be shown that $x \in (S^c)^c$, that is, that $x \notin S^c$. To that end, assume, to the contrary, that $x \in S^c$. But then $x \notin S$, which contradicts the fact that $x \in S$ and completes the proof. $\square$

9.19 a. The author is using the forward-backward and choose methods. This is because the author first asks the key question, "How can I show that a function (namely, a^x) is one-to-one?" By definition, the answer is to show that

B1: For all real numbers x and y with $x \neq y$, $a^x \neq a^y$.

Recognizing the keywords "for all" in the backward statement $B1$, the author now uses the choose method to choose

A1: Real numbers x and y with $x \neq y$,

for which it must be shown that

B2: $a^x \neq a^y$.

b. The author is using contradiction because the keyword "not" appears in the backward statement $B2$. Thus, the author assumes that

A2: $a^x = a^y$.

c. The author uses the hypothesis that $a > 0$ when taking the log of a^x and a^y (because the log is only defined for numbers greater than 0).

d. The author is justified in claiming the proof is complete because the author has correctly shown that $x = y$, which contradicts $A1$ and completes the contradiction method.

9.20 **Analysis of Proof.** Recognizing the keyword "not" in the conclusion, the contradiction method is used, whereby you can assume that

A1 (NOT B): The function $ax^2 + bx + c$ is one-to-one.

By definition, this means that

A2: For all real numbers x and y with $x \neq y$, $ax^2 + bx + c \neq ay^2 + by + c$.

A contradiction is reached by showing that $A2$ is not true, that is, by showing

B1: There are real numbers x and y with $x \neq y$ such that $ax^2 + bx + c = ay^2 + by + c$.

Recognizing the keywords "there are" in the backward statement $B1$, the construction method is used to produce the values of x and y. Specifically, using the hypothesis that $a \neq 0$, you can construct the following real numbers:

A3: $x = \frac{-b}{2a} + 1$ and $y = \frac{-b}{2a} - 1$.

To complete the construction method, it remains to show that, for the values of x and y in $A3$,

B2: $x \neq y$ and $ax^2 + bx + c = ay^2 + by + c$.

For the first part of $B2$, you should recognize the keyword "not" and so proceed by contradiction and assume that

A4: $x = y$, that is, $\frac{-b}{2a} + 1 = \frac{-b}{2a} - 1$.

Subtracting $\frac{-b}{2a}$ from both sides of $A4$ results in the contradiction that $1 = -1$ and so it must be that $x = y$. To complete the proof of $B2$, you can see that

$$\begin{aligned} ax^2 + bx + c &= a\left(\tfrac{-b}{2a} + 1\right)^2 + b\left(\tfrac{-b}{2a} + 1\right) + c \\ &= \tfrac{b^2}{4a} + a - \tfrac{b^2}{2a} + c \\ &= a\left(\tfrac{-b}{2a} - 1\right)^2 + b\left(\tfrac{-b}{2a} - 1\right) + c \\ &= ay^2 + by + c. \end{aligned}$$

The proof is now complete.

Proof. Given a, b, and c are real numbers with $a \neq 0$ and the function $f(x) = ax^2 + bx + c$, it will be shown that for $x = \frac{-b}{2a} + 1$ and $y = \frac{-b}{2a} - 1$, $x \neq y$ and $f(x) = f(y)$. You can see that $x \neq y$ for otherwise, $1 = -1$. To see that $f(x) = f(y)$ note that

$$\begin{aligned} f(x = \tfrac{-b}{2a} + 1) &= \tfrac{b^2}{4a} + a + \tfrac{-b^2}{2a} + c \text{ and} \\ f(y = \tfrac{-b}{2a} - 1) &= \tfrac{b^2}{4a} + a + \tfrac{-b^2}{2a} + c. \end{aligned}$$

Therefore f is not one-to-one. $\square$

9.21 The first error in this proof is the assumption that the $\log(x^a)$ is defined. For example, if $x = -1$ and $a = 3$ then $\log(x^a) = \log(-1)$ is undefined. Even if you were to assume that $x^a > 0$ for all real numbers x, $\log(x^a) = \log(y^a)$ only implies that $a\log(|x|) = a\log(|y|)$. With the assumption that x and y are greater than 0, the function would be one-to-one.

9.22 The contradiction is that b is both odd (as stated in the hypothesis) and even (as has been shown because $b = -2a$ is even).

9.23 The author has first shown that $1 \cdots k \cdots (n-1)n = ck - 1$, and, to reach a contradiction, is claiming that k divides the integer on the left side of the foregoing equality but not the integer on the right side. It is correct that k divides the integer on the left side because $1 \cdots k \cdots (n-1)n = k(1 \cdots (k-1)(k+1) \cdots (n-1)n)$. It is also correct that k does not divide $ck - 1$. To see that this is true, by contradiction (because of the keyword "not"), suppose that k does divide $ck - 1$. By definition, this means that there is an integer b such that $ck - 1 = bk$. But then $(c - b)k = 1$. The only way the product of the two integers $c - b$ and k can be 1 is if $c - b = \pm 1$ and $k = \pm 1$. But $k > 1$ and so this contradiction means that k does not divide $ck - 1$.

9.24 **Analysis of Proof.** By the contradiction method, assume that

A1: The number of primes is finite.

From $A1$, there will be a prime number that is larger than all the other prime numbers. So,

A2: Let n be the largest prime number.

Consider the number $n! + 1$ and let

A3: p be any prime number that divides $n! + 1$.

A contradiction is reached by showing that

B1: $p > n$.

To see that $p > n$, by contradiction, suppose that $1 < p \leq n$. From the proposition in Exercise 9.23, it follows that p does not divide $n! + 1$, which contradicts $A3$ and completes the proof.

Proof. Assume, to the contrary, that there are a finite number of primes. Let n be the largest prime and let p be any prime divisor of $n! + 1$. Now, if $1 < p \leq n$, then, by the proposition in Exercise 9.23, it follows that p does not divide $n! + 1$. This contradiction completes the proof. $\square$

9.25 The contradiction is that, if $p = \pm q$, then $p/q = \pm 1$ are rational roots of $ax^4 + bx^2 + a$. However, it is stated earlier in the proof that ± 1 are not roots of $ax^4 + bx^2 + a$.

9.26 **Analysis of Proof.** The proof is done by contradiction because the author is assuming

NOT B: $xz - y^2 > 0$.

A contradiction is reached by showing that the square of a number is negative. Specifically, the author shows that

B1: $(az - cx)^2 < 0$.

The contradiction in $B1$ is obtained by working forward from $NOT\ B$ and the hypothesis. Specifically, from $NOT\ B$ and the hypothesis that $ac - b^2 > 0$, it follows that

A1: $xz > y^2$,
A2: $ac > b^2$.

Multiplying corresponding sides of the inequalities in $A1$ and $A2$ yields:

A3: $(ac)(xz) > b^2y^2$.

Then, adding $2by$ to both sides of the hypothesis $az - 2by + cx = 0$ and squaring both sides results in:

A4: $(az + cx)^2 = 4b^2y^2$.

From $A3$, it follows that $4b^2y^2 < 4(ac)(xz)$ and so, from $A4$,

A5: $(az + cx)^2 < 4(ac)(xz)$.

Expanding the left side of $A5$, subtracting $4(ac)(xz)$ from both sides, and then factoring yields the contradiction that $(az - cx)^2 < 0$. □

9.27 **Analysis of Proof.** The proof is by contradiction, so the author assumes that

$A1$ (NOT B): The polynomial x^4+2x^2+2x+2 can be expressed as the product of the two polynomials x^2+ax+b and x^2+cx+d in which a, b, c, and d are integers.

Working forward by multiplying the two polynomials, you have that

A2: $x^4 + 2x^2 + 2x + 2 = (x^2 + ax + b)(x^2 + cx + d) =$
$x^4 + (a + c)x^3 + (b + ac + d)x^2 + (bc + ad)x + bd$.

Equating coefficients of like powers of x on both sides, it follows that

A3: $a + c = 0$.
A4: $b + ac + d = 2$.
A5: $bc + ad = 2$.
A6: $bd = 2$.

From $A6$, b is odd (± 1) and d is even (± 2) or vice versa. Suppose, first, that

A7: Case 1: b is odd and d is even.

(Subsequently, the case when b is even and d is odd is considered.) It now follows that, because the right side of $A5$ is even, the left side is also. Because d is even, so is ad. It must therefore be the case that

A8: bc is even.

However, from $A7$, b is odd, so it must be that

A9: c is even.

But then, from the left side of $A4$, you have

A10: $b + ac + d$ is odd + even + even, which is odd.

This is because b is odd (see $A7$), ac is even (from $A9$), and d is even (see $A7$). However, $A10$ is a contradiction because the right side of $A4$, namely, 2, is even. This establishes a contradiction for Case 1 in $A7$. A similar contradiction is reached in Case 2, when b is even and d is odd, thus completing the proof.

9.28 **Analysis of Proof.** The keyword "not" in the conclusion suggests using the contradiction method, whereby you can assume that

A1: k divides $n! + 1$.

Working forward from $A1$ by definition, it follows that

A2: There is an integer c such that $n! + 1 = ck$, that is, $n! = ck - 1$.

By definition of $n!$, this means that

A3: $n(n-1)\cdots k \cdots 1 = ck - 1$.

A contradiction is reached by showing that k divides the integer on the left side of $A3$ but k does not divide the integer on the right side (as it should). You can see that k divides the integer on the left side of $A3$ because $1 \cdots k \cdots (n-1)n = k(1 \cdots (k-1)(k+1) \cdots (n-1)n)$. To establish the contradiction, it remains to show that

B1: k does not divide $ck - 1$.

Recognizing the keyword "not" in the backward statement $B1$, contradiction is used. Thus, suppose

A4: k does divide $ck - 1$.

By definition, this means that

A5: There is an integer b such that $ck - 1 = bk$, that is, $(c - b)k = 1$.

The only way the product of the two integers $c - b$ and k can be 1 is if

A6: $c - b = \pm 1$ and $k = \pm 1$.

But $k > 1$ from the hypothesis and so this contradiction means that $B1$ is true and completes the proof.

9.29 **Analysis of Proof.** The contradiction method is used, whereby you should assume that

A1: b is odd.

A contradiction is reached by showing that $x = \pm 1$ are not roots of ax^4+bx^2+a and yet $x = \pm 1$ are roots of this equation. The proposition in Exercise 9.22 establishes that $x = \pm 1$ are not roots of $ax^4 + bx^2 + a$ and so it remains to show that

B1: $x = \pm 1$ are roots of $ax^4 + bx^2 + a$.

Now $B1$ is established by showing that

B2: There is a rational root $x \neq 0$ of $ax^4 + bx^2 + a$ such that $x = 1/x$ and so $x = \pm 1$.

Recognizing the keywords "there is" in the backward statement $B2$, the construction method is used to produce this rational root. To that end, note that

A2: Rational roots of the equation $ax^4 + bx^2 + a$ come in pairs.

This is true because, suppose that p and q are integers with $q \neq 0$ for which $x = p/q$ is a rational root of $ax^4 + bx^2 + a$, that is, $a(p/q)^4 + b(p/q)^2 + a = 0$. Then because $p \neq 0$ (otherwise $a = 0$, which is not the case according to the hypothesis), it follows by algebra that $a(q/p)^4 + b(q/p)^2 + a = 0$ and so q/p is also a rational root $ax^4 + bx^2 + a$. Now because the hypothesis states that $ax^4 + bx^2 + a$ has an odd number of rational roots and $A2$ is true, it must be that

A3: There is a rational root of $ax^4 + bx^2 + a$ that is repeated, that is, there is a rational root $x = p/q$ for which $x = q/p$.

But $A3$ means that $x = 1/x$ and so $B2$ is true and the proof is complete.

9.30 **Analysis of Proof.** Recognizing the first keywords "for every" in the conclusion, the choose method is used to choose

A1: A real number $y < 0$,

for which it must be shown that

B1: The set $C = \{\text{real numbers } x : ax^2 \leq y\}$ is not bounded.

Recognizing the keyword "not" in the backward statement $B1$, the contradiction method is used. Accordingly, you can assume that

A2: The set C is bounded.

Working forward by definition of a bounded set, this means that

A3: There is a real number $M > 0$ such that, for every element $x \in C$, $|x| < M$.

A contradiction is reached by showing that the for-all statement in $A3$ is not true. Thus, it will be shown that

B2: There is an element $x \in C$ such that $|x| \geq M$.

Recognizing the keywords "there is" in the backward statement $B2$, the construction method is used. Specifically, the author constructs

A4: $x = \max\{M, \sqrt{\frac{y}{a}}\}$.

To complete the construction method, it must be shown that

B3: $x \in C$ and $|x| \geq M$.

By the defining property of C in $B1$, it must be shown that

B4: $ax^2 \leq y$.

But $B4$ is true because, from $A4$,

A5: $x \geq \sqrt{\frac{y}{a}}$, that is, $x^2 \geq \frac{y}{a}$.

Using the fact that $a < 0$ in the hypothesis, it follows from $A5$ that $ax^2 \leq y$ and so $B4$ is true. From $B3$, it remains to show that $|x| \geq M$, but this follows immediately from $A4$ and so the proof is complete.

10

Solutions to Exercises

10.1 a. Work forward from: n is an odd integer.
Work backward from: n^2 is an odd integer.
b. Work forward from: S is a subset of T and T is bounded.
Work backward from: S is bounded.
c. Work forward from: The integer $p > 1$ is not prime.
Work backward from: There is an integer $1 < n \leq \sqrt{p}$, such that $n|p$.

10.2 a. Work forward from: $n > 0$, $p > 0$ and $n > p$.
Work backward from: n does not divide p.
b. Work forward from: There exist real numbers b, c, and M with $M > 0$, such that for all x, $ax^2 + bx + c \leq M$.
Work backward from: $a \leq 0$.
c. Work forward from: $f(x)$ is linear.
Work backward from: x_1, x_2, and x_3 are three real numbers with $x_1 < x_2 < x_3$ and either $f(x_1) \leq f(x_2)$ or $f(x_2) \geq f(x_3)$.

10.3 No, Mary's statement is not correct. Thinking of Bob's statement as being of the form, "If A (I study hard enough), then B (I will get at least a B in this course)," Mary stated that because B is true, A must also be true. This need not be the case, that is, B could be true and yet A could be false and Bob's statement, "If A, then B" is still true (see Table 1.1 in Chapter 1).

10.4 Statement (b) is a result of the forward process because you can assume that there is a real number t with $0 < t < \pi/4$ such that $\sin(t) = r\cos(t)$. The answer in part (b) results on squaring both sides and replacing $\cos^2(t)$ with $1 - \sin^2(t)$.

10.5 Statement (c) is a result of the forward process because you can assume that there is a positive integer root for $mx^2+nx+(n-m)$. Using the definition of a root one obtains statement (c).

10.6 Key Question: How can I show that one integer (namely, m) divides another integer (namely n)?

Key Answer: By definition, show that there is an integer a such that $n = am$.

10.7 The correct key question is in (d). To understand why, recall that with the contrapositive method you work forward from $NOT\ B$ and backward from $NOT\ A$. Thus, you should apply the key question to the statement, "The derivative of the function f at the point x is 0."

a. Incorrect because the key question is applied to $NOT\ B$. Also, the question uses symbols and notation from the specific problem.
b. Incorrect because the key question uses symbols and notation from the specific problem.
c. Incorrect because the key question is applied to $NOT\ B$.
d. Correct.

10.8 The correct key question is in (d). To understand why, recall that with the contrapositive method you work forward from $NOT\ B$ and backward from $NOT\ A$. Thus, you should apply the key question to the statement, "There is an element $x \in S$ with the property that $f(x) = 0$."

a. This is incorrect because the key question is applied to the wrong statement.
b. This is incorrect because it includes symbols and specific notation from the statement.
c. This is incorrect because the key question is applied to the wrong statement.
d. Correct.

10.9 The contradiction method is used in this proof because the author works forward from A and $NOT\ B$ to reach the contradiction that the integer $p > p$.

10.10 The contrapositive method is used in the proof in Exercise 9.22 because the author works forward from $NOT\ B$ to show $NOT\ A$.

10.11 The contradiction method is used in the proof in Exercise 9.23. The contradiction is that the integer k divides and does not divide the same integer.

10.12 With the contrapositive method, you will assume

NOT B: p is not prime.

You must then show that

NOT A: There is an integer m with $1 < m \leq \sqrt{p}$ such that $m|p$.

Recognizing the keywords "there is" in the backward statement $NOT\ A$, you should use the construction method to construct an integer m with the property that $1 < m \leq \sqrt{p}$ such that $m|p$.

10.13 With the contrapositive method, you will assume

NOT B: For every real number x, $ax^2 + bx + c < 0$.

You must then show that

NOT A: There is a real number x with $ax^2 + bx + c = 0$.

Recognizing the keywords "for every" in the forward statement $NOT\ B$, you should use specialization, that is, you should find a specific value for the real number x. The result of this specialization will be that, for this specific value of x, $ax^2 + bx + c < 0$.

10.14 With the contrapositive method, you will assume

NOT B: There are real numbers b, c, and M with $M > 0$ such that, for every real number x, $ax^2 + bx + c > M$.

You must then show that

NOT A: $a \leq 0$.

Recognizing the keywords "there are" as the first quantifier in the backward statement $NOT\ B$, you should use the construction method to construct real numbers b, c, and M for which you must then show that $M > 0$ and

B1: For every real number x, $ax^2 + bx + c > M$.

Recognizing the keywords "for every" in the backward statement $B1$, you should then use the choose method to choose

A1: A real number x,

for which you must show that

B2: $ax^2 + bx + c \leq M$.

10.15 **Analysis of Proof.** By the contrapositive method, assume that

NOT B: $p = q$.

It must be shown that

NOT A: $\sqrt{pq} = (p + q)/2$.

However, from $NOT\ B$ and the fact that p, $q > 0$, it follows that

A1: $\sqrt{pq} = \sqrt{p^2} = p = (p + p)/2 = (p + q)/2$.

The proof is now complete because $NOT\ A$ is true.

Proof. By the contrapositive method, assume that $p = q$. It must be shown that $\sqrt{pq} = (p + q)/2$. However, because p, $q > 0$, it follows that

$$\sqrt{pq} = \sqrt{p^2} = p = (p + p)/2 = (p + q)/2.$$

The proof is now complete. □

10.16 **Analysis of Proof.** With the contrapositive method, you can assume

A1 (NOT B): $(a + b)/2 \leq \sqrt{ab}$.

You must show that

B1 (NOT A): $a = b$.

To that end, multiply both sides of $A1$ by 2, square both sides, and then subtract $4ab$ from both sides to obtain

A2: $a^2 - 2ab + b^2 \leq 0$, that is, $(a - b)^2 \leq 0$.

Of course you also know that

A3: $(a - b)^2 \geq 0$.

Combining $A2$ and $A3$, it follows that

A4: $(a - b)^2 = 0$.

From $A4$, it follows that

A5: $a - b = 0$, that is, $a = b$.

The fact that $A5$ is the same as $B1$ completes the proof.

Proof. By the contrapositive method, assume that

$$(a + b)/2 \leq \sqrt{ab}.$$

It must be shown that $a = b$. To that end, multiply both sides of the foregoing inequality by 2, square both sides, and then subtract $4ab$ from both sides to obtain

$$a^2 - 2ab + b^2 \leq 0, \text{that is, } (a-b)^2 \leq 0.$$

Of course $(a-b)^2 \geq 0$, so it must be that $(a-b)^2 = 0$, that is, $a - b = 0$. Thus, $a = b$, and so the proof is complete. □

10.17 **Analysis of Proof.** With the contrapositive method, you assume

A1 (NOT B): There is an integer solution, say m, to the equation $n^2 + n - c = 0$.

You must then show that

B1 (NOT A): c is not odd, that is, c is even.

But from $A1$, you have that

A2: $c = m + m^2$.

Observe that $m + m^2 = m(m+1)$ is the product of two consecutive integers and is therefore even, thus establishing $B1$ and completing the proof.

Proof. Assume that there is an integer solution, say m, to the equation $n^2 + n - c = 0$. It will be shown that c is even. But $c = m + m^2 = m(m+1)$ is even because the product of two consecutive integers is even. □

10.18 **Analysis of Proof.** With the contrapositive method, you assume

A1 (NOT B): $mx^2 + nx + n - m$ has a positive integer root.

You must then show that

B1 (NOT A): m divides n.

A key question associated with $B1$ is, "How can I show that an integer (namely, m) divides another integer (namely, n)?" By definition, one answer is to show that

B2: There is an integer k such that $n = km$.

Recognizing the keywords "there is" in the backward statement $B2$, you should use the construction method to produce the desired integer k. To that end, working forward from $A1$, you know that

A2: There is an integer $x > 0$ such that $mx^2 + nx + n - m = 0$.

Because $m \neq 0$, you can divide the equality in $A2$ through by m and factor to obtain

A3: $(x+1)(x-1) + \frac{n}{m}(x+1) = 0$.

From $A2$, you also know that $x > 0$ and so $x + 1 > 0$ and, on dividing $A3$ through by $x + 1$ you obtain

A4: $x - 1 + \frac{n}{m} = 0$, that is, $n = (1 - x)m$.

From $A4$, you can see that the desired integer k in $B2$ is $k = 1 - x$ because, for this value of k, $n = km$ and so the proof is complete.

Proof. Assume that there is an integer $x > 0$ such that $mx^2 + nx + n - m = 0$. It will be shown that $m|n$. Because $m \neq 0$, you can divide the equation by m and then factor to obtain $(x + 1)(x - 1) + \frac{n}{m}(x + 1) = 0$. Now x is a positive integer so $x + 1 \neq 0$. Dividing by $x + 1$ and solving for n gives $n = (1 - x)m$. This means that $m|n$ and so the proof is complete. □

10.19 **Analysis of Proof.** The definition is used to answer the key question, "How can I show that a function (namely, $f(x) = x^3$) is one to one?" It must therefore be shown that

B1: For all real numbers x and y with $x \neq y$, $x^3 \neq y^3$.

Recognizing the keywords "for all" in the backward statement $B1$, the choose method is used to choose

A1: Real numbers x and y with $x \neq y$,

for which it must be shown that

B2: $x^3 \neq y^3$.

Recognizing the keyword "not" in $B2$, the contrapositive method is used to show that $A1$ implies $B2$. Accordingly, you can assume that

A2 (NOT B2): $x^3 = y^3$, that is, $x^3 - y^3 = 0$.

It must be shown that

B3 (NOT A1): $x = y$.

Note that $B3$ is true when $x = y = 0$, so you can also assume that

A3: Either $x \neq 0$ or $y \neq 0$.

Working forward from $A2$ by factoring, you have that

A4: $(x - y)(x^2 + xy + y^2) = 0$.

The desired result in $B3$ would follow from $A5$ provided that $x^2 + xy + y^2 \neq 0$, so it will now be shown that

B4: $x^2 + xy + y^2 \neq 0$.

Recognizing the keyword "not" in $B4$, the contradiction method is used, whereby, you can assume that

A5 (NOT B4): $x^2 + xy + y^2 = 0$.

Thinking of $A5$ as a quadratic equation $ax^2 + bx + c$, in which $a = 1$, $b = y$, and $c = y^2$, the quadratic formula yields

A6: $x = \frac{-y \pm \sqrt{y^2 - 4y^2}}{2} = \frac{-y \pm \sqrt{-3y^2}}{2}$.

The only way that the value in $A6$ can result in a real value for x is if $y = 0$, in which case, $x = 0$. This, however, contradicts $A3$ and establishes $B4$, thus completing the proof.

Proof. By definition, it must be shown that for all real numbers x and y with $x \neq y$, $x^3 \neq y^3$. To that end, let x and y be real numbers for which $x^3 = y^3$, that is, $x^3 - y^3 = 0$. It will be shown that $x = y$. Note first that $x = y$ when $x = y = 0$, so you can also assume that either $x \neq 0$ or $y \neq 0$. Then, factoring $x^3 - y^3 = 0$, you have that $(x - y)(x^2 + xy + y^2) = 0$. The desired result that $x = y$ would follow provided that $x^2 + xy + y^2 \neq 0$. By contradiction, assume that $x^2 + xy + y^2 = 0$. Thinking of this as a quadratic equation $ax^2 + bx + c$, in which $a = 1$, $b = y$, and $c = y^2$, the quadratic formula yields

$$x = \frac{-y \pm \sqrt{y^2 - 4y^2}}{2} = \frac{-y \pm \sqrt{-3y^2}}{2}.$$

The only way that the foregoing expression can result in a real value for x is if $y = 0$ and hence $x = 0$. This, however, contradicts the fact that either $x \neq 0$ or $y \neq 0$ and so the proof is complete. □

10.20 **Analysis of Proof.** By the contrapositive method, you can assume

A1 (NOT B): The quadrilateral $RSTU$ is not a rectangle.

You must then show that

B1 (NOT A): There is an obtuse angle.

The appearance of the quantifier "there is" in the backward statement $B1$ suggests turning to the forward process to construct the obtuse angle.

Working forward from $A1$, you can conclude that

A2: At least one angle of the quadrilateral is not 90 degrees, say angle R.

If angle R has more than 90 degrees, then R is the desired angle and the proof is complete. Otherwise,

A3: Angle R has less than 90 degrees.

Because the sum of all the angles in $RSTU$ is 360 degrees, $A3$ means that

A4: The remaining angles of the quadrilateral must add up to more than 270 degrees.

Among these three angles that add up to more than 270 degrees, one of them must be greater than 90 degrees, and that is the desired obtuse angle. The proof is now complete.

Proof. Assume that the quadrilateral $RSTU$ is not a rectangle, and hence, one of its angles, say R, is not 90 degrees. An obtuse angle will be found. If angle R has more than 90 degrees, then R is the desired obtuse angle. Otherwise the remaining three angles add up to more than 270 degrees. Thus one of the remaining three angles is obtuse, and so the proof is complete. □

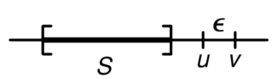
10.21 **Analysis of Proof.** With the contrapositive method, you can assume

> **A1 (NOT B):** There is a real number $u < v$ such that u is an upper bound for S.

It must be shown that

> **B1 (NOT A):** There is a real number $\epsilon > 0$ such that, for every element $x \in S$, $x \le v - \epsilon$.

Recognizing the keywords "there is" in the backward statement $B1$, the construction method is used to produce the desired value of ϵ. To that end, note that, according to $B1$, you want ϵ to satisfy the property that all of S is to the left of $v - \epsilon$. The following figure illustrates the relative positions of S, u, and v and also indicates how to construct ϵ:



From the foregoing figure, you can see that $\epsilon > 0$ should have the property that $v - \epsilon \ge u$. In particular, you can construct

> **A2:** $\epsilon = v - u$.

You can see that $\epsilon = v - u > 0$ because, from $A1$, $v > u$. According to $B1$, it remains to show that

> **B2:** For every element $x \in S$, $x \le v - \epsilon$.

You can see that this is the case in the foregoing figure. To make this formal, recognize the keywords "for every" in the backward statement $B2$ and use the choose method to choose

> **A3:** An element $x \in S$,

for which it must be shown that

> **B3:** $x \le v - \epsilon$.

To that end, work forward from the fact that u is an upper bound for S (see $A1$). Thus, by definition,

A4: For every element $s \in S$, $s \leq u$.

Recognizing the keywords "for every" in the forward statement $A4$, you can specialize this statement to $s = x \in S$ (see $A3$) and so

A5: $x \leq u$.

The desired conclusion in $B3$ follows from $A5$ by noting that $u = v - \epsilon$ (see $A2$), thus completing the proof.

Proof. According to the contrapositive method, assume that there is a real number $u < v$ such that u is an upper bound for S. It will be shown that there is a real number $\epsilon > 0$ such that for every element $x \in S$, $x \leq v - \epsilon$. Specifically, let $\epsilon = v - u > 0$. It must be shown that, for every element $x \in S$, $x \leq v - \epsilon$. To that end, let $x \in S$. However, from the fact that u is an upper bound for S, it follows that $x \leq u$. The fact that $x \leq v - \epsilon$ follows on noting that, by construction, $u = v - \epsilon$. The proof is now complete. □

10.22 **Analysis of Proof.** By the contrapositive method, you can assume

A1 (NOT B): $x < 0$.

It must be shown that

B1 (NOT A): There is a real number $\epsilon > 0$ such that $x < -\epsilon$.

Recognizing the keywords "there is" in the backward statement $B1$, the construction method is used to produce the desired $\epsilon > 0$. Turning to the forward process to do so, from $A1$, because $x < 0$, construct ϵ as any value with $0 < \epsilon < -x$. (Note that this construction is possible because $-x > 0$.) By design, $\epsilon > 0$ and, because $\epsilon < -x$, it follows that $x < -\epsilon$. Thus ϵ has all the needed properties in $B1$, and the proof is complete.

Proof. Assume, to the contrary, that $x < 0$. It will be shown that there is a real number $\epsilon > 0$ such that $x < -\epsilon$. To that end, construct ϵ as any value with $0 < \epsilon < -x$ (noting that this is possible because $-x > 0$). Clearly $\epsilon > 0$, and, because $\epsilon < -x$, $x < -\epsilon$, thus completing the proof. □

10.23 **Analysis of Proof.** With the contrapositive method, you can assume

A1 (NOT B): $\exists$ a real number $\epsilon > 0$ such that, $\forall\, x \in S$, $x \leq u - \epsilon$.

It must be shown that

B1 (NOT A): u is not a least upper bound for S.

Applying the word "not" to the definition of a least upper bound, it must be shown that

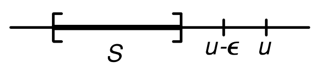
B2: Either u is not an upper bound for S or else $\exists$ an upper bound t for S such that $t < u$.

The latter is shown, that is,

B3: $\exists$ an upper bound t for S such that $t < u$.

Recognizing the keywords "there is" in the backward statement $B3$, the construction method is used to produce the value for t.

To construct this value for t, work forward from $A1$, which is represented in the following figure:



Because you want $t < u$ to be an upper bound for S, construct

A2: $t = u - \epsilon < u$ (because $\epsilon > 0$ from $A1$).

It remains to show that

B4: t is an upper bound for S.

According to the definition, it must therefore be shown that

B5: $\forall$ elements $x \in S$, $x \leq t$.

Recognizing the keywords "for all" in the backward statement $B5$, the choose method is used to choose

A3: An element $y \in S$,

for which it must be shown that

B6: $y \leq t$.

You can establish $B6$ by specializing the for-all statement in $A1$ to $x = y$, which is in S (see $A3$). The result is that

A4: $y \leq u - \epsilon$.

The result in $B6$ follows from $A4$ by noting that $u - \epsilon = t$ (see $A2$), thus completing the proof.

Proof. By the contrapositive method, you can assume that $\exists$ a real number $\epsilon > 0$ such that, $\forall$ elements $x \in S$, $x \leq u - \epsilon$. It will be shown that u is not a least upper bound for S. Specifically, it will be shown that $\exists$ an upper bound t for S such that $t < u$. To that end, let $t = u - \epsilon < u$. It remains to show that t is an upper bound for S. To see this, let $y \in S$. According to the definition, it must be shown that $y \leq t$. However, from the fact that $\forall$ elements $x \in S$, $x \leq u - \epsilon$, it follows that for the element $y \in S$,

$$y \leq u - \epsilon = t.$$

It has now been shown that u is not a least upper bound for S, thus completing the proof. $\square$

10.24 **Analysis of Proof.** This is a proof by the contrapositive method because the author assumes that

NOT B: p is not prime

and eventually shows that

NOT A: There is an integer m with $1 < m \leq \sqrt{p}$ such that $m|p$.

Recognizing the keywords "there is" in $NOT\ A$, the construction method is used to produce the value for the integer m. To do so, the author works forward from $NOT\ B$ to claim that, because p is not prime,

A1: There is an integer n with $1 < n < p$ such that $n|p$.

The author now considers two possibilities: $n \leq \sqrt{p}$ and $n > \sqrt{p}$. In the former case,

A2: $n \leq \sqrt{p}$,

and the author constructs

A3: $m = n$.

This value for m in $A3$ is correct because, from $A1$ and $A2$, $1 < n = m$ and $m = n \leq \sqrt{p}$. Also, from $A1$, $n|p$ and $m = n$, so $m|p$.

In the latter case,

A4: $n > \sqrt{p}$.

To construct the value for m, the author works forward from $A1$ and the definition of $n|p$ to state that

A5: There is an integer k such that $p = nk$.

The author then constructs

A6: $m = k$.

The author shows that this value for m is correct by establishing that

A7: $1 < k \leq \sqrt{p}$.

To do so, the author argues by contradiction that $1 < k$. For otherwise, from $A5$, it would follow that $p = nk \leq n$, which cannot happen because, from $A1$, $n < p$. Finally, $k \leq \sqrt{p}$ because otherwise, $k > \sqrt{p}$ and then, since $n > \sqrt{p}$ from $A4$, it would follow that $nk > \sqrt{p}\sqrt{p} = p$, which cannot happen because, from $A5$, $p = nk$. Observe that the author omits noting that $k|p$, which is true because, from $A5$, $p = nk$. The proof is now complete. $\square$

10.25 **Analysis of Proof.** This is a contrapositive proof because the author assumes that

A1 (NOT B): T is bounded,

and states that it will be shown that

B1 (NOT A): S is bounded.

To understand the rest of the proof, you must realize that the author has worked backward from $B1$ by asking the key question, "How can I show that a set (namely, S) is bounded?" The author answers this question using the definition, so, it must be shown that

B2: There is a real number $M > 0$ such that, for all elements $x \in S$, $|x| < M$.

Recognizing the keywords "there is" in $B2$, the author uses the construction method to produce the desired value of M.

Turning to the forward process, from $A1$, by definition, the author states that

A2: There is a real number $M' > 0$ such that, for all elements $t \in T$, $|t| < M'$.

The desired value for M in $B2$ is $M' > 0$ from $A2$ (note that the author does not state this explicitly). According to the construction method, from $B2$ it remains to show that

B3: For all elements $x \in S$, $|x| < M$.

Recognizing the keywords "for all" in the backward statement $B3$, the author uses the choose method to choose

A3: An element $x \in S$,

for which it must be shown that

B4: $|x| < M$.

To that end, from the fact that $S \subseteq T$ (as stated in the problem), the author uses the definition of subset to claim that

A4: For every element $s \in S$, $s \in T$.

Recognizing the keywords "for every" in the forward statement $A4$, the author specializes $A4$ to $s = x \in S$ (see $A3$) to conclude that

A5: $x \in T$.

The statement in $B4$ results on specializing the for-all statement in $A2$ to $t = x \in T$ (see $A5$ and note that $M = M'$), thus completing the proof.

10.26 a. The author is assuming $NOT\ B$ and thus could be using the contradiction or the contrapositive method.

b. The author is working forward using the definition of a rational number to state that there are integers p and q with $q \neq 0$ such that $\sqrt{a} = p/q$.

c. Multiplying both sides of $\sqrt{a} = p/q$ by q and then squaring both sides leads to the equation $p^2 = aq^2$. To justify the statement that $q^2|p^2$, first note that p and q are integers and so p^2 and q^2 are also integers. Furthermore, a is an integer. Using the definition of "divides", $q^2|p^2$ if there is an integer k, such that $p^2 = kq^2$. Because $p^2 = aq^2$, you can see that $k = a$ is such an integer.

d. If x and y are integers for which $x^2|y^2$, then $x|y$.

e. By showing that $a = b^2$, the author has shown, by definition, that a is a square, which is $NOT\ A$. In other words, the author has assumed $NOT\ B$ and demonstrated $NOT\ A$, thus completing the contrapositive method.

11

Solutions to Exercises

11.1 a. i) Show that the lines $y = mx + b$ and $y = cx + d$ both pass through the points (x_1, y_1) and (x_2, y_2).
ii) Conclude that $y = mx + b$ and $y = cx + d$ are the same line.
b. i) Show that (x_1, y_1) and (x_2, y_2) are solutions to the equations $ax + by = 0$ and $cx + dy = 0$.
ii) Conclude that $(x_1, y_1) = (x_2, y_2)$.
c. i) Show that $(a + bi)(r + si) = (a + bi)(t + ui) = 1$.
ii) Conclude that $r + si = t + ui$.

11.2 a. i) Show that x^* and y^* are both maximizers of $ax^2 + bx + c$, that is, for all real numbers z, $az^2 + bz + c \le a(x^*)^2 + bx^* + c$ and $az^2 + bz + c \le a(y^*)^2 + by^* + c$.
ii) Conclude that $x^* = y^*$.
b. i) Show that, for every real number x, $f(F(x)) = F(f(x)) = x$ and $f(G(x)) = G(f(x)) = x$.
ii) Conclude that $F = G$.
c. i) Show that $p > 0$, $p|n$, $q > 0$, $q|n$, and for every integer $b > 0$ such that $b|n$, $b|p$ and $b|q$.
ii) Conclude that $p = q$.

11.3 a. i) First, construct a line, say $y = mx + b$, that goes through the two given points. Then assume that $y = cx + d$ also passes through those two points. You must now work forward to show that these two lines are the same—that is, that $m = c$ and $b = d$.

ii) First, construct a line, say $y = mx + b$, that goes through the two given points. Then assume that a different line, say $y = cx + d$, also passes through those two points. You must now work forward to reach a contradiction.

b. i) First, construct a solution, say (x_1, y_1), to the system of equations $ax + by = 0$ and $cx + dy = 0$. Then assume that you have another solution to the equations, say (x_2, y_2). You must now work forward to show that $(x_1, y_1) = (x_2, y_2)$.

ii) First, construct a solution, say (x_1, y_1), to the two equations $ax + by = 0$ and $cx + dy = 0$. Then assume that you have a different solution, say $(x_2, y_2) \neq (x_1, y_1)$, to the two equations. You must now work forward to reach a contradiction.

c. i) First, construct a complex number, say $r + si$, that satisfies $(a+bi)(r+si) = 1$. Then assume that you have another complex number, say $t + ui$, that also satisfies $(a + bi)(t + ui) = 1$. You must now work forward to show that $r + si$ and $t + ui$ are the same—that is, that $r + si = t + ui$.

ii) First, construct a complex number, say $r+si$, that satisfies $(a+bi)(r + si) = 1$. Then assume that there is a different complex number, say $t+ui \neq r+si$, that also satisfies $(a+bi)(t+ui) = 1$. You must now work forward to reach a contradiction.

11.4 a. i) First, construct a maximizer, say x^*, of $ax^2 + bx + c$. Then assume that you have another maximizer, say y^*, of ax^2+bx+c. You must now work forward to show that $x^* = y^*$.

ii) First, construct a maximizer, say x^*, of $ax^2 + bx + c$. Then assume that you have a different maximizer, say $y^* \neq x^*$, of $ax^2 + bx + c$. You must now work forward to reach a contradiction.

b. i) First, construct a function, say G, such that $f(G(x)) = G(f(x)) = x$. Then assume that there is a second function, say F, such that $f(F(x)) = F(f(x)) = x$. You must now work forward to show that $F = G$.

ii) First, construct a function, say G, such that $f(G(x)) = G(f(x)) = x$. Then assume that there is a different function, say $F \neq G$, such that $f(F(x)) = F(f(x)) = x$. You must now work forward to reach a contradiction.

c. i) First, construct an integer, say $p > 0$, such that $p|n$ and for every integer $b > 0$ for which $b|n$, $p|b$. Then assume that there is another integer, say $q > 0$, that also satisfies $q|n$ and for every integer $b > 0$ for which $b|n$, $q|b$. You must then show that $p = q$.

ii) First, construct an integer, say $p > 0$, such that $p|n$ and for every integer $b > 0$ for which $b|n$, $p|b$. Then assume that there is a different integer, say $q \neq p$, with $q > 0$ that also satisfies $q|n$ and for every integer $b > 0$ for which $b|n$, $q|b$. You must then work forward to reach a contradiction.

11.5 a. The reason that the author does not first construct an object with the certain property is because this object (namely, u) is already given in the hypothesis of the proposition.

b. The proof uses the direct uniqueness method as the author demonstrates that $u = v$ rather than assuming $u \neq v$ and working forward to a contradiction.

c. The author is specializing the forward statement in the first sentence, namely, "for all $x \in S$, $x \leq v$," to the value of $x = u \in S$ and hence concludes that $u \leq v$.

d. The author has successfully completed the direct uniqueness method by showing correctly that $v = u$. Specifically, given that u is one element of S such that for all $x \in S$, $x \leq u$, the author has correctly assumed that v is also an element of S such that for all $x \in S$, $x \leq v$. The author has then reached the conclusion that $u = v$ by showing that $u \leq v$ and $v \leq u$.

11.6 **Analysis of Proof.** Recognizing the keyword "unique" in the conclusion, the direct uniqueness method is used. Accordingly, it is first necessary to construct a maximizer of the function $ax^2 + c$. This has already been done in Exercise 5.15, so you know that

A1: $x = 0$ is a real number such that for every real number z, $a(0)^2 + c \geq az^2 + c$.

To complete the direct uniqueness method, assume that

A2: y is also a maximizer of the function, that is, for every real number z, $ay^2 + c \geq az^2 + c$.

It must be shown that

B1: $y = x = 0$.

Specializing the for-all statement in $A2$ to the value of $z = 0$, it follows that

A3: $ay^2 + c \geq a(0)^2 + c = c$.

Subtracting c from both sides of $A3$ and then dividing by $a < 0$ leads to

A4: $y^2 \leq 0$.

Now the only way that $A4$ can be true is if

A5: $y = 0$.

The proof is now complete because $A5$ is the same as $B1$.

11.7 **Analysis of Proof.** Recognizing the keyword "not" in the conclusion, the contradiction method is used, so you can assume the hypothesis and

A1 (NOT B): $ad - bc = 0$.

To reach a contradiction, you can work forward from the hypothesis that the equations $ax + by = 0$ and $cx + dy = 0$ have a unique solution to state that

A2: Either $c \neq 0$ or $d \neq 0$.

A contradiction to $A2$ is now established by showing $c = 0$ and $d = 0$. To that end, it is easy to verify from $A1$ that

A3: $x = d$ and $y = -c$ is a solution to the equations $ax+by = 0$ and $cx + dy = 0$, as is $x = 0$ and $y = 0$.

By the forward uniqueness method, it must be that these two solutions are the same, that is,

A4: $c = 0$ and $d = 0$.

Now $A4$ contradicts $A2$ and thus the proof is complete.

11.8 **Analysis of Proof.** The direct uniqueness method is used, whereby one must first construct the real number y. This is done in Exercise 7.17, so

A1: $y < 0$ and $x = 2y/(1 + y)$.

It remains to show the uniqueness by assuming that z is also a number with

A2: $z < 0$ and $x = 2z/(1 + z)$.

Working forward via algebraic manipulations, it will be shown that

B1: $y = z$.

Specifically, combining $A1$ and $A2$ yields

A3: $x = 2y/(1 + y) = 2z/(1 + z)$.

Dividing both sides of $A3$ by 2 and clearing the denominators, you have that

A4: $y + yz = z + yz$.

Subtracting yz from both sides of $A4$ yields the desired conclusion that $y = z$.

Proof. The existence of the real number y is established in Exercise 7.17. To show that y is unique, suppose that y and z satisfy $y < 0$, $z < 0$, $x = 2y/(1+y)$, and also $x = 2z/(1+z)$. But then $2y/(1+y) = 2z/(1+z)$ and so $y + yz = z + yz$, or, $y = z$, as desired. □

11.9 **Analysis of Proof.** According to the direct uniqueness method, the first step is to construct an integer k such that $b = ka$. This integer k comes from the hypothesis that $a|b$, which, by definition, means that

A1: There is an integer k such that $b = ka$.

To show the uniqueness by the direct uniqueness method, you should now assume that

A2: There is an integer m such that $b = ma$.

It must be shown that

B1: $k = m$.

To that end, from $A1$ and $A2$, you have that

A3: $ka = ma$.

$B1$ follows on dividing both sides of $A3$ by a (note that $a \neq 0$ is given in the hypothesis), thus completing the proof.

Proof. The fact that there is an integer k such that $b = ka$ follows by definition from the hypothesis that $a|b$. To see that k is the unique such integer, suppose that the integer m also satisfies $b = ma$. But then, $ka = ma$ and, dividing both sides by $a \neq 0$ yields the desired conclusion that $k = m$, thus establishing the uniqueness and completing the proof. □

11.10 **Analysis of Proof.** According to the indirect uniqueness method, one must first construct a real number x for which $mx + b = 0$. But because the hypothesis states the $m \neq 0$, you can construct

A1: $x = -b/m$.

This value is correct because

A2: $mx + b = m(-b/m) + b = -b + b = 0$.

To establish the uniqueness by the indirect uniqueness method, suppose that

A3: y is a real number with $y \neq x$ such that $my + b = 0$.

A contradiction to the hypothesis that $m \neq 0$ is reached by showing that

B1: $m = 0$.

Specifically, from $A2$ and $A3$,

A4: $mx + b = my + b$.

Subtracting the right side of the equality in $A4$ from the left side and rewriting yields

A5: $m(x - y) = 0$.

On dividing both sides of the equality in $A5$ by the nonzero number $x - y$ (see $A3$), it follows that $m = 0$. This contradiction establishes the uniqueness.

Proof. To construct the number x for which $mx + b = 0$, let $x = -b/m$ (because $m \neq 0$). Then $mx + b = m(-b/m) + b = 0$.

Now suppose that $y \neq x$ and also that $my + b = 0$. Then $mx + b = my + b$, and so $m(x - y) = 0$. But because $x - y \neq 0$, it must be that $m = 0$. This contradicts the hypothesis that $m \neq 0$ and completes the proof. □

11.11 **Analysis of Proof.** The first step is to construct an integer n such that $2n^2 - 3n - 2 = 0$. By factoring, you have $2n^2 - 3n - 2 = (2n - 4)(n + \frac{1}{2})$, so

A1: $n = 2$ is an integer for which $2n^2 - 3n - 2 = 0$.

According to the indirect uniqueness method, you should now assume that

A2: $m \neq n$ is also an integer for which $2m^2 - 3m - 2 = 0$.

Combining $A1$ and $A2$, it follows that

A3: $2m^2 - 3m - 2 = 2n^2 - 3n - 2$, or, $2(m^2 - n^2) - 3(m - n) = 0$.

Factoring out the term $m - n$, you have

A4: $(m - n)(2(m + n) - 3) = 0$.

Dividing both sides of $A4$ by $m - n \neq 0$ (see $A2$), you have

A5: $2(m + n) - 3 = 0$, that is, $3 = 2(m + n)$.

But $A5$ provides the contradiction that 3 is both odd and even. This completes the indirect uniqueness method and hence the proof.

Proof. It is easy to see that $n = 2$ satisfies $2n^2 - 3n - 2 = 0$. To see that this is the unique such value, suppose that m is an integer with $m \neq n$ such that $2m^2 - 3m - 2 = 0$. It then follows that $2m^2 - 3m - 2 = 2n^2 - 3n - 2$. Applying algebra yields $(m - n)(2(m + n) - 3) = 0$. Dividing both sides of this equality by $m - n \neq 0$ results in $3 = 2(m + n)$, which says that 3 is both odd and even. This contradiction completes the proof. □

11.12 **Analysis of Proof.** The issue of existence is addressed first. To construct the complex number $c + di$ that satisfies $(a + bi)(c + di) = 1$, multiply the two terms using complex arithmetic to obtain

$$\begin{aligned} ac - bd &= 1, \quad \text{and} \\ bc + ad &= 0. \end{aligned}$$

Solving these two equations for the two unknowns c and d in terms of a and b leads you to construct $c = a/(a^2 + b^2)$ and $d = -b/(a^2 + b^2)$ (noting that the denominator is not 0 because, by the hypothesis, at least one of a and b is not 0). To see that this construction is correct, note that

$$\begin{aligned} \textbf{A1:}\ (a+bi)(c+di) &= (ac-bd)+(bc+ad)i \\ &= [(a^2+b^2)/(a^2+b^2)]+0i \\ &= 1. \end{aligned}$$

To establish the uniqueness, suppose that $e + fi$ is also a complex number that satisfies

A2: $(e + fi)(a + bi) = 1$.

It will be shown that

B1: $c + di = e + fi$.

Working forward by multiplying both sides of the equality in $A1$ by $e + fi$ and using associativity yields

A3: $[(e + fi)(a + bi)](c + di) = (e + fi)$.

Because $(e + fi)(a + bi) = 1$ from $A2$, it follows from $A3$ that $c + di = e + fi$, and so $B1$ is true, completing the proof.

Proof. Because either $a \neq 0$ or $b \neq 0$, $a^2 + b^2 \neq 0$, and so it is possible to construct the complex number $c + di$ in which $c = a/(a^2 + b^2)$ and $d = -b/(a^2 + b^2)$, for then

$$(a + bi)(c + di) = (ac - bd) + (bc + ad)i = 1.$$

To see the uniqueness, assume that $e + fi$ also satisfies $(a + bi)(e + fi) = 1$. Multiplying the foregoing displayed equality through by $e + fi$ yields

$$[(e + fi)(a + bi)](c + di) = e + fi.$$

Using the fact that $(a + bi)(e + fi) = 1$, it follows that $c + di = e + fi$ and so the uniqueness is established. □

11.13 **Analysis of Proof.** Not recognizing any keywords in the hypothesis or the conclusion, the forward-backward method is used to begin the proof. A key question associated with the conclusion is, "How can I show that a function (namely, f) is one-to-one?" Using the definition, one answer is to show that

B1: For all real numbers x and y with $x \neq y$, $f(x) \neq f(y)$.

Recognizing the keywords "for all" in the backward statement $B1$, the choose method is used to choose

A1: Real numbers s and t with $s \neq t$,

for which it must be shown that

B2: $f(s) \neq f(t)$.

Recognizing the keyword "not" in the backward statement $B2$, the contrapositive method is used to prove that "$A1$ implies $B2$." Accordingly, you can assume that

A2 (NOT B2): $f(s) = f(t)$.

You must show that

B3 (NOT A1): $s = t$.

Working forward, you can specialize the for-all statement in the hypothesis to $y = f(s)$ to claim that

A3: There is a unique value of x such that $f(x) = f(s)$.

You can see that both $x = s$ and $x = t$ satisfy $f(x) = f(s)$ (see $A2$) and so, by the forward uniqueness method, it must be that

A4: $s = t$.

The proof is now complete because $A4$ is the same as $B3$.

Proof. Let s and t be two real numbers with $f(s) = f(t)$. It will be shown that $s = t$. However, both $x = s$ and $x = t$ satisfy $f(x) = f(s)$. Thus, from the hypothesis that for each real number y there is a unique real number x such that $f(x) = y$, it follows that $s = t$ and so the proof is complete. $\square$

12

Solutions to Exercises

12.1 a. Applicable.

b. Not applicable because the statement contains the quantifier "there is" instead of "for all."

c. Applicable.

d. Applicable.

e. Not applicable because in this statement, n is a real number, and induction is applicable only to integers.

12.2 The reason you can assume that $P(n)$ is true is because, in Step 2 of induction, you must show that, "If $P(n)$ is true, then $P(n+1)$ is true." When proving that, "If A is true, then B is true," you can assume that A is true. Thus, for Step 2 of induction, you can assume that $P(n)$ is true.

12.3 a. The time to use induction instead of the choose method to show that, "For every integer $n \geq n_0$, $P(n)$ is true" is when you can relate $P(n)$ to $P(n-1)$, for then you can use the induction hypothesis that $P(n-1)$ is true, and this should help you establish that $P(n)$ is true. If you were to use the choose method, you would choose

A1: An integer $n \geq n_0$,

for which it must be shown that

B1: $P(n)$ is true.

With the choose method, you cannot use the assumption that $P(n-1)$ is true to do so.

b. It is not possible to use induction when the object is a real number because showing that $P(n)$ implies $P(n{+}1)$ "skips over" many values of the object. As a result, the statement will not have been proved for such values.

12.4 For Step 1 of induction, you must show that, "If $A(1)$ is true, then $B(1)$ is true." To do so, you can assume that $A(1)$ is true and must show that $B(1)$ is true (perhaps by using the forward-backward method).

For Step 2 of induction, you can assume that

P(n): If $A(n)$ is true, then $B(n)$ is true.

You must show that

P(n+1): If $A(n+1)$ is true, then $B(n+1)$ is true.

Looking at the form of $P(n+1)$, you should proceed by assuming that

C: $A(n+1)$ is true.

You must then show that

D: $B(n+1)$ is true.

The idea now is to relate $A(n+1)$ to $A(n)$ so that you can use $P(n)$ to conclude that $B(n)$ is true. That, in turn, should help you to reach the desired conclusion in D that $B(n+1)$ is true.

12.5 a. Verify that $P(n)$ is true for the initial value of $n = n_0$. Then, assuming that $P(n)$ true, prove that $P(n-1)$ is true.

b. Verify that $P(n)$ is true for some integer n_0. Assuming that $P(n)$ is true for n, prove that $P(n+1)$ is true and that $P(n-1)$ is also true.

12.6 The approach is to prove by induction that, for every integer $k \geq 1$, $P(2k-1)$ is true. Specifically, you must first show that for $k = 1$, $P(1)$ is true. You would then assume that $P(2k-1)$ is true and show that $P(2k+1)$ is true.

12.7 **Proof.** First it must be shown that $P(n)$ is true for $n = 1$. Replacing n by 1, it must be shown that $1(1!) = (1+1)! - 1$. But this is clear because $1(1!) = 1 = (1+1)! - 1$.

Now assume that $P(n)$ is true and use that fact to show that $P(n+1)$ is true. So assume

P(n): $1(1!) + \cdots + n(n!) = (n+1)! - 1.$

It must be shown that

P(n + 1): $1(1!) + \cdots + (n+1)(n+1)! = (n+2)! - 1.$

Starting with the left side of $P(n+1)$ and using $P(n)$:

$$\begin{aligned}
&1(1!) + \cdots + n(n!) + (n+1)(n+1)! \\
&\qquad = [1(1!) + \cdots + n(n!)] + (n+1)(n+1)! \\
&\qquad = [(n+1)! - 1] + (n+1)(n+1)! \qquad (P(n) \text{ is used here.}) \\
&\qquad = (n+1)![1 + (n+1)] - 1 \\
&\qquad = (n+1)!(n+2) - 1 \\
&\qquad = (n+2)! - 1. \quad \square
\end{aligned}$$

12.8 **Proof.** It must first be shown that $P(1)$ is true, that is, $2^{1-1} = 2^1 - 1$. This is clearly true because both sides evaluate to 1 and are therefore equal. To complete the proof, assume that

P(k): $1 + 2^1 + \cdots + 2^{k-1} = 2^k - 1$.

It must be shown that

P(k+1): $1 + 2^1 + \cdots + 2^k = 2^{k+1} - 1$.

Starting with the left side of $P(k+1)$, you have that

$$\begin{aligned}
1 + 2^1 + \cdots + 2^k &= (1 + 2^1 + \cdots + 2^{k-1}) + 2^k && \text{(grouping terms)} \\
&= (2^k - 1) + 2^k && \text{(induction hypothesis)} \\
&= 2^{k+1} - 1 && \text{(algebra).}
\end{aligned}$$

This completes the proof. $\square$

12.9 **Proof.** First it is shown that $P(n)$ is true for $n = 5$. But $2^5 = 32$ and $5^2 = 25$, so $2^5 > 5^2$ and so $P(n)$ is true for $n = 5$. Assuming that $P(n)$ is true, you must then prove that $P(n+1)$ is true. So assume

P(n): $2^n > n^2$.

It must be shown that

P(n + 1): $2^{n+1} > (n+1)^2$.

Starting with the left side of $P(n+1)$ and using the fact that $P(n)$ is true, you have:

$$2^{n+1} = 2(2^n) > 2(n^2).$$

To obtain $P(n+1)$, it must still be shown that for $n > 5$, $2n^2 > (n+1)^2 = n^2 + 2n + 1$, or, by subtracting $n^2 + 2n - 1$ from both sides and factoring, that $(n-1)^2 > 2$. This last statement is true because, for $n > 5$, $(n-1)^2 \geq 4^2 = 16 > 2$. $\square$

12.10 It must first be shown that $P(1)$ is true, that is, $1/1! \leq 1/2^{1-1}$. This is true because both sides evaluate to 1. Assume now that

P(n): $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$.

It must be shown that

P(n+1): $\frac{1}{(n+1)!} \leq \frac{1}{2^n}$.

Starting with the left side of $P(n+1)$, you have that

$$\begin{array}{rcll} \frac{1}{(n+1)!} & = & \left(\frac{1}{n!}\right)\left(\frac{1}{n+1}\right) & \text{(definition of } (n+1)!\text{)} \\ & \leq & \left(\frac{1}{2^{n-1}}\right)\left(\frac{1}{n+1}\right) & \text{(induction hypothesis)} \\ & \leq & \left(\frac{1}{2^{n-1}}\right)\left(\frac{1}{2}\right) & (1/(n+1) \leq 1/2) \\ & = & \frac{1}{2^n} & \text{(algebra).} \end{array}$$

This completes the proof. □

12.11 **Proof.** Let $\$X$ be the principal investment, $\$X_n$ be the total capital after n years of investment at an annual interest rate of $i\%$, and $r = \frac{i}{100}$. It must first be shown that $P(n)$ is true for $n = 0$, but this is true because $\$X_0 = \$X = \$(1+r)^0 X$. For the second step of induction, assume that

P(n): $\$X_n = \$(1+r)^n X$.

It must be shown that

P(n+1): $\$X_{n+1} = \$(1+r)^{n+1} X$.

Starting with the left side of $P(n+1)$, you have that

$$\begin{array}{rcll} \$X_{n+1} & = & \$(1+r)X_n & (X_n + \text{annual interest}) \\ & = & \$(1+r)[(1+r)^n X] & \text{(induction hypothesis)} \\ & = & \$(1+r)^{n+1} X & \text{(algebra).} \end{array}$$

This completes the proof. □

12.12 **Proof.** The statement is true for $n = 1$ because the subsets of a set consisting of one element, say x, are $\{x\}$ and $\emptyset$; that is, there are $2^1 = 2$ subsets. Assume that, for a set with n elements, the number of subsets is 2^n. It will be shown that, for a set with $n+1$ elements, the number of subsets is 2^{n+1}. For a set S with $n+1$ elements, one can construct all the subsets by listing first all those subsets that include the first n elements, and then, to each such subset, one can add the last element of S. By the induction hypothesis, there are 2^n subsets using the first n elements. An additional 2^n subsets are created by adding the last element of S to each of the subsets of n elements. Thus the total number of subsets of S is $2^n + 2^n = 2^{n+1}$, and so the statement is true for $n+1$. □

12.13 It must first be shown that $(1+x)^2 > 1 + 2x$. This is true because the left side expands to $1 + 2x + x^2 > 1 + 2x$ as $x \neq 0$. Assume now that

P(n): $(1+x)^n > 1 + nx$.

It must be shown that

P(n+1): $(1+x)^{n+1} > 1+(n+1)x$.

Starting with the left side of $P(n+1)$, you have that

$$\begin{array}{rcll} (1+x)^{n+1} & = & (1+x)[(1+x)^n] & \text{(factor out } 1+x) \\ & > & (1+x)(1+nx) & \text{(induction hypothesis)} \\ & = & 1+x+nx+nx^2 & \text{(algebra)} \\ & > & 1+(1+n)x & (x \neq 0 \text{ and } n > 1). \end{array}$$

This completes the proof. □

12.14 It must first be shown that $P(2)$ is true, that is, $C_1 \cap C_2$ is a convex set. This is done in Exercise 6.19. For Step 2 of induction, assume that

P(n): $\cap_{i=1}^{n} C_i$ is a convex set.

It must be shown that

P(n+1): $\cap_{i=1}^{n+1} C_i$ is a convex set.

Now $\cap_{i=1}^{n+1} C_i = \cap_{i=1}^{n} C_i \cap C_{n+1}$. Using the induction hypothesis, it follows that $\cap_{i=1}^{n} C_i = S$ is a convex set. As demonstrated in Exercise 6.19, the intersection of two convex sets is a convex set. So, $\cap_{i=1}^{n+1} C_i = S \cap C_{n+1}$ is a convex set. This completes the proof. □

12.15 **Proof.** Let

$$S = 1+2+\cdots+n.$$

Then

$$S = n+(n-1)+\cdots+1.$$

On adding the two foregoing equations, one obtains

$$2S = n(n+1), \text{ that is, } S = n(n+1)/2. \quad \square$$

12.16 **Proof.** The key is to reword the problem so that induction is appropriate. Specifically, you want to prove that for every integer $n \geq 1$,

P(n): A machine that has $2n$ candies consisting of an odd number of caramel candies and an odd number of chocolate candies eventually dispenses a pair consisting of one of each type of candy.

Proceeding by induction, when $n = 1$, it must be that the machine only has 1 caramel candy and 1 chocolate candy. Thus, the machine can only dispense one pair which, of necessity, consists of one type of each and so the statement is true for $n = 1$.

Assume now that $P(n)$ is true. Then, for $n+1$, it must be shown that

P(n+1): A machine that has $2n+2$ candies consisting of an odd number of caramel candies and an odd number of chocolate candies eventually dispenses a pair consisting of one of each type of candy.

To relate $P(n+1)$ to $P(n)$, consider the first pair of candies dispensed. If this pair consists of one of each type, then $P(n+1)$ is true. Otherwise, this pair consists of two candies of the same type. In this case, the machine has $2n$ remaining candies still consisting of an odd number of caramel candies and an odd number of chocolate candies. Hence, the induction hypothesis applies and so the machine eventually dispenses a pair consisting of one of each type of candy. Thus, $P(n+1)$ is true and the proof is complete. □

12.17 **Proof.** For $n = 1$ the statement becomes:

P(1): $[\cos(x) + i\sin(x)]^1 = \cos(1x) + i\sin(1x)$.

Now $P(1)$ is true because both sides evaluate to $\cos(x) + i\sin(x)$.

Now assume the statement is true for $n-1$, that is:

P(n − 1): $[\cos(x) + i\sin(x)]^{n-1} = \cos((n-1)x) + i\sin((n-1)x)$.

It must be shown that $P(n)$ is true, that is:

P(n): $[\cos(x) + i\sin(x)]^n = \cos(nx) + i\sin(nx)$.

Using $P(n-1)$ and the facts that

$$\begin{aligned}\cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b),\\ \sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b),\end{aligned}$$

starting with the left side of $P(n)$, you have:

$$\begin{aligned}[\cos(x) + i\sin(x)]^n &= [\cos(x) + i\sin(x)]^{n-1}[\cos(x) + i\sin(x)]\\ &= [\cos((n-1)x) + i\sin((n-1)x)][\cos(x) + i\sin(x)]\\ &= [\cos((n-1)x)\cos(x) - \sin((n-1)x)\sin(x)]+\\ &\quad\; i[\sin((n-1)x)\cos(x) + \cos((n-1)x)\sin(x)]\\ &= \cos(nx) + i\sin(nx).\end{aligned}$$

This establishes that $P(n)$ is true, thus completing the proof. □

12.18 **Proof.** Let $n \geq 2$ be the number of people in line. If $n = 2$, then the line consists of only two people, the first of which is a woman and the last of which is a man. Thus, there is a man standing behind a woman and so the statement is true for $n = 2$.

Assume now that the statement is true for n, that is,

P(n): In a line of n people in which the first is a woman and the last is a man, there is a man standing directly behind a woman somewhere in the line.

For $n+1$, it must be shown that

P(n+1): In a line of $n+1$ people in which the first is a woman and the last is a man, there is a man standing directly behind a woman somewhere in the line.

Consider, therefore, a line of $n+1$ people in which the first is a woman and the last is a man. To relate $P(n+1)$ to $P(n)$, consider the second person in line. If that person is a man, then that man is standing behind the woman in the front of the line and so $P(n+1)$ is true. If, however, the second person in line is a woman, then consider the line from that second woman to the end. This line then consists of n people, the first of which is a woman and the last of which is a man. In this case, the induction hypothesis applies and so somewhere there is a man standing behind a woman and so $P(n+1)$ is true, thus completing the proof. □

12.19 The author relates $P(n+1)$ to $P(n)$ by expressing the product of the n terms associated with the left side of the equality in $P(n+1)$ to the product of the n terms associated with left side of the equality in $P(n)$ and one additional term—specifically,

$$\underbrace{\prod_{k=2}^{n+1}\left(1-\frac{1}{k^2}\right)}_{\text{left side of } P(n+1)} = \underbrace{\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)}_{\text{left side of } P(n)} \underbrace{\left(1-\frac{1}{(n+1)^2}\right)}_{\text{extra term}}$$

The induction hypothesis is used when the author subsequently replaces $\prod\limits_{k=2}^{n}(1-\frac{1}{k^2})$, which is the left side of $P(n)$, with $\frac{n+1}{2n}$, which is the right side of $P(n)$.

12.20 The author relates $P(n+1)$ to $P(n)$ by using the product rule of differentiation to express

$$[x(x^n)]' = (x)'(x^n) + x(x^n)'.$$

The author then uses the induction hypothesis to replace $(x^n)'$ with nx^{n-1}.

12.21 The author relates $P(n+1)$ to $P(n)$ by expressing the product of $n+1$ terms as the product of the first n terms times the term $(1-x_{n+1})$. Specifically,

$$\underbrace{(1-x_1)(1-x_2)\cdots(1-x_{n+1})}_{\text{left side of } P(n+1)} = \underbrace{[(1-x_1)(1-x_2)\cdots(1-x_n)]}_{\text{left side of } P(n)}\underbrace{(1-x_{n+1})}_{\text{extra term}}$$

The author then uses the induction hypothesis to make the claim that $(1-x_1)\ (1-x_2)\cdots(1-x_n) > 1-x_1-\cdots-x_n$.

12.22 a. $\sqrt{x_1^2+x_2^2}^2 = x_1^2 + x_2^2$ and $(|x_1|+|x_2|)^2 = |x_1|^2 + |x_2|^2 + 2|x_1||x_2| \geq |x_1|^2 + |x_2|^2$ because $2|x_1||x_2| \geq 0$. Therefore $\sqrt{x_1^2+x_2^2} \leq |x_1| + |x_2|$.

b. In part (a), it was shown that for all real numbers x_1 and x_2, $\sqrt{x_1^2+x_2^2} \leq |x_1|+|x_2|$. Specializing this for-all statement with $x_1 = z$ and $x_2 = x_{n+1}$, it follows that $\sqrt{z^2 + x_{n+1}^2} \leq |z| + |x_{n+1}|$.

c. This is true by the induction hypothesis.

12.23 a. Recognizing the keywords "for any" in the conclusion, the author is using the choose method to choose a real number x.

b. Recognizing the keywords "for all" in the hypothesis, the author is applying specialization to the real number x that was chosen in the first sentence and to the point x_*.

c. The inequality results by specializing the statement, "for all real numbers x and y, $|f(x) - f(y)| \leq \alpha|x-y|$" in the hypothesis to the values $x = f^n(x_0)$ and $y = x_*$ to obtain $|f(f^n(x_0)) - f(x_*)| \leq \alpha|f^n(x_0) - x_*|$.

d. This is true by the induction hypothesis.

12.24 The mistake occurs in the last sentence, where it states that, "Then, because all the colored horses in this (second) group are brown, the uncolored horse must also be brown." How do you know that there is a colored horse in the second group? In fact, when the original group of $n+1$ horses consists of exactly 2 horses, the second group of n horses does not contain a colored horse. The entire difficulty is caused by the fact that the statement should have been verified for the initial integer $n = 2$, not $n = 1$. This, of course, you will not be able to do.

12.25 The proof is incorrect because when $n = 1$ and $r = 1$, the right side of $P(1)$ is undefined because you cannot divide by zero. A similar problem arises throughout the rest of the proof.

12.26 An error arises in the proof of the case when $n = 2$. In particular, when

A1: $x, y \in S_1 \cup S_2$ and $0 \leq t \leq 1$,

the author claims that because S_1 is convex, $tx + (1-t)y \in S_1$. This, however, is not correct. To see why, note that you do know that S_1 is convex, so, by definition,

A2: For all real numbers $u, v \in S_1$ and for all real numbers s with $0 \leq s \leq 1$, $su + (1-s)v \in S_1$.

The author has specialized $A2$ to $u = x$, $v = y$, and $s = t$ (and hence reached the conclusion that $tx + (1-t)y \in S_1$, as stated by the author). However, you cannot apply specialization in this case because you do not know that $x, y \in S_1$. That is, just because $x, y \in S_1 \cup S_2$ (from $A1$), it does not follow that $x, y \in S_1$. Thus, the author is not justified in specializing $A2$ and therefore cannot conclude that $tx + (1-t)y \in S_1$. The author makes the same mistake when claiming that $tx + (1-t)y \in S_2$.

12.27 It is straightforward to see that the statement is true for $n = 1$ because for $k = 3$, you have $f^3(1) = f^2(4) = f^1(2) = 1$. A difficulty arises in trying to prove that if $P(n)$ is true, then $P(n+1)$ is true. Specifically, it is challenging to relate $P(n+1)$ to $P(n)$ (or to $P(j)$, for some integer j with $1 \leq j < n$).

13

Solutions to Exercises

13.1 a. $a|b$ and $b|a$ implies either $a = b$ or $a = -b$.
b. Either $x > 3$ or $x < -3$.
c. Either $x = 0$ or $y = 0$.
d. Either $z \in S$ or $z \in T$.

13.2 a. This means that C cannot be true and hence, from Table 1.1, that "C implies B" is true. Thus, to complete the proof, you now need to do the second case by showing that "D implies B."
b. This situation arises when the author recognizes that $x \geq 2$ and $x \leq 1$ is an impossibility and so goes on to finish the proof by assuming the other half of the either/or statement, namely that $x \leq 2$ and $x \geq 1$.

13.3 a. To apply a proof by elimination to the statement, "If A, then C *OR* D *OR* E," you would assume that A is true, C is not true, and D is not true; you must conclude that E is true. (Alternatively, you can assume that A is true and that any two of the three statements C, D, and E are not true; you would then have to conclude that the remaining statement is true.)
b. To apply a proof by cases to the statement, "If C *OR* D *OR* E, then B," you must do all three of the following proofs: (1) "If C, then B," (2) "If D, then B," and (3) "If E, then B."

13.4 a. You should use a proof by cases. This is because, with the contradiction method, you assume A and NOT $(C$ AND $D)$, that is, $(NOT\ C)$ OR $(NOT\ D)$. The appearance of the keyword "or" in the forward process now suggests using a proof by cases.

b. You should use a proof by cases. This is because, with the contradiction method, you assume that A is true and that NOT $[(NOT\ C)$ AND $(NOT\ D)]$, that is, C OR D. The appearance of the keyword "or" in the forward process now suggests using a proof by cases.

13.5 The author uses a proof by cases when the following statement containing the keywords "either/or" is encountered in the forward process:

A1: Either $(x \leq 0$ and $x - 3 \geq 0)$ or $(x \geq 0$ and $x - 3 \leq 0)$.

Accordingly, the author considers the following two cases:

Case 1: $x \leq 0$ and $x - 3 \geq 0$. The author then observes that this cannot happen and so proceeds to Case 2.

Case 2: $x \geq 0$ and $x - 3 \leq 0$. The author then reaches the desired conclusion.

13.6 The author uses a proof by cases when the following statement containing the keywords "either/or" is encountered in the forward process:

A1: Either the factor b is odd or even.

Accordingly, the author considers the following two cases:

Case 1: The factor b is odd. The author then works forward from this information to establish the contradiction that the left side of equation (2) is odd and yet is equal to the even number 2.

Case 2: The factor b is even. The author claims, without providing details, that this case also leads to a contradiction.

13.7 The author uses a proof by cases when the following statement containing the hidden keywords "either/or" is encountered in the forward process:

A1: $x = \frac{-b \pm (b-2a)}{2a}$ (so, either $x = \frac{-b+(b-2a)}{2a}$ or $x = \frac{-b-(b-2a)}{2a}$).

Accordingly, the author considers the following two cases:

Case 1: $x = \frac{-b+(b-2a)}{2a}$. The author then works forward and reaches a contradiction and proceeds to Case 2.

Case 2: $x = \frac{-b-(b-2a)}{2a}$. The author works forward to reach the desired conclusion.

13.8 The author uses a proof by cases when the following statement containing the keywords "either/or" is encountered in the forward process:

A1: Either $n+1$ is prime or $n+1$ is not prime.

Accordingly, the author considers the following two cases:

Case 1: $n+1$ is prime. In this case, $n+1$ is the product of primes, namely, itself and so the proof is complete.

Case 2: $n+1$ is not prime. In this case, the author uses this fact to express $n+1$ as the product of a prime p and an integer q to which the induction hypothesis can be applied to complete the proof.

13.9 The author uses a proof by elimination because the conclusion contains the keywords " either/or," as follows:

B: Either $C : a \neq 0$ or $D : b \neq 0$.

Accordingly, the author assumes the hypothesis A and, in this proof, that

A1 (NOT C): $a = 0$.

This is evidenced in the sentence, "If $a = 0$, then $-bd = 1$." With a proof by elimination, the author must then work forward to reach the statement

B1 (D): $b \neq 0$.

Indeed, in the last sentence of the proof, the author states that $B1$ is true.

13.10 a. If x is a real number that satisfies $x^3 + 3x^2 - 9x - 27 \geq 0$, then $x \leq -3$ or $x \geq 3$.

b. **Analysis of Proof.** With this proof by elimination, you assume that

A: $x^3 + 3x^2 - 9x - 27 \geq 0$ and
A1 (NOT C): $x > -3$.

It must be shown that

B1 (D): $x \geq 3$, that is, $x - 3 \geq 0$.

By factoring A, it follows that

A2: $x^3 + 3x^2 - 9x - 27 = (x-3)(x+3)^2 \geq 0$.

From $A1$, because $x > -3$, $(x+3)^2$ is strictly positive. Thus, dividing both sides of $A2$ by $(x+3)^2$ yields $B1$ and completes the proof.

Proof. Assume that $x^3 + 3x^2 - 9x - 27 \geq 0$ and $x > -3$. Then it follows that $x^3 + 3x^2 - 9x - 27 = (x-3)(x+3)^2 \geq 0$. Because $x > -3$, $(x+3)^2$ is positive, so $x - 3 \geq 0$, or equivalently, $x \geq 3$. □

13.11 **Analysis of Proof.** With this proof by elimination, you assume that

A: $x^3 + 3x^2 - 9x - 27 \geq 0$ and
A1 (NOT D): $x < 3$.

It must be shown that

B1 (C): $x \leq -3$.

By factoring A, it follows that

A2: $x^3 + 3x^2 - 9x - 27 = (x-3)(x+3)^2 \geq 0$.

Dividing both sides of $A2$ by $x - 3 < 0$ (from $A1$) yields

A3: $(x+3)^2 \leq 0$.

Because $(x+3)^2$ is also ≥ 0, from $A3$, it must be that

A4: $(x+3)^2 = 0$, so $x + 3 = 0$, that is, $x = -3$.

Thus $B1$ is true, completing the proof.

Proof. Assume that $x^3 + 3x^2 - 9x - 27 \geq 0$ and $x < 3$. Then it follows that $x^3 + 3x^2 - 9x - 27 = (x-3)(x+3)^2 \geq 0$. Because $x < 3$, $(x+3)^2$ must be 0, so $x + 3 = 0$, that is, $x = -3$. Thus, $x \leq -3$, completing the proof. □

13.12 **Analysis of Proof.** Observe that the conclusion can be written as

B: Either $a = b$ or $a = -b$.

The keywords "either/or" in the backward process now suggest proceeding with a proof by elimination, in which you can assume the hypothesis and

A1: $a \neq b$.

It must be shown that

B1: $a = -b$.

Working forward from the hypotheses that $a|b$ and $b|a$, by definition:

A2: There is an integer k such that $b = ka$.
A3: There is an integer m such that $a = mb$.

Substituting $a = mb$ in the equality in $A2$ yields:

A4: $b = kmb$.

If $b = 0$, then, from $A3$, $a = 0$ and so $B1$ is clearly true and the proof is complete. Thus, you can assume that $b \neq 0$. Therefore, on dividing both sides of the equality in $A4$ by b you obtain:

A5: $km = 1$.

From $A5$ and the fact that k and m are integers (see $A2$ and $A3$), it must be that

A6: Either ($k = 1$ and $m = 1$) or ($k = -1$ and $m = -1$).

Recognizing the keywords "either/or" in the forward statement $A6$, a proof by cases is used.

Case 1: $k = 1$ and $m = 1$. In this case, $A2$ leads to $a = b$, which cannot happen according to $A1$.

Case 2: $k = -1$ and $m = -1$. In this case, from $A2$, it follows that $a = -b$, which is precisely $B1$, thus completing the proof.

Proof. To see that $a = \pm b$, assume that $a|b$, $b|a$, and $a \neq b$. It will be shown that $a = -b$. By definition, it follows that there are integers k and m such that $b = ka$ and $a = mb$. Consequently, $b = kmb$. If $b = 0$, then $a = mb = 0$ and so $a = -b$. Thus, assume that $b \neq 0$. It then follows that $km = 1$. Because k and m are integers, it must be that $k = m = 1$ or $k = m = -1$. However, because $a \neq b$, it must be that $k = m = -1$. From this it follows that $a = mb = -b$, and so the proof is complete. □

13.13 **Analysis of Proof.** Working forward from the hypothesis you have

A1: $n = 1$ or $n = 2$ or $n > 2$.

Recognizing the keywords "either/or" in the forward statement $A1$, a proof by cases is used.

Case 1: $n = 1$. In this case, $n = 1^2$, so n is a square and hence the conclusion is true.

Case 2: $n = 2$. In this case, n is prime and again the conclusion is true.

Case 3: $n > 2$. For this case, it must be shown that

B1: n is prime, or n is a square, or n divides $(n-1)!$.

Recognizing the keywords "either/or" in the backward statement $B1$, a proof by elimination is appropriate, so, assume that

A2: n is not a prime and n is not a square.

It must be shown that

B2: n divides $(n-1)!$.

Working backward from $B2$ by definition, it must be shown that

B3: There is an integer k such that $(n-1)! = kn$.

Recognizing the keywords "there is" in $B3$, the construction method is used to produce the integer k.

Turning to the forward process, from $A2$, because $n > 2$ is not prime,

A3: There are integers a and b with $1 < a < n$ and $1 < b < n$ such that $n = ab$.

Also, working forward from $A2$, because n is not a square,

A4: $a \neq b$.

Combining $A3$ and $A4$, it follows that

A5: $2 \leq a \neq b \leq n - 1$.

Indeed, this means that

A6: a and b are two different terms of $(n-1)(n-2)\cdots 1 = (n-1)!$.

Thus, from $A6$ and $A3$,

A7: $(n-1)! = ab \times$ (the remaining terms of $(n-1)!$) $=$ $n \times$ (the remaining terms of $(n-1)!$).

From $A7$, you can see that the desired value for the integer k in $B3$ is $k =$ the remaining terms of $(n-1)!$. The result in $A7$ establishes that this is the correct value of k and thus completes the proof.

13.14 **Analysis of Proof.** The appearance of the keywords either/or in the hypothesis suggest proceeding with a proof by cases.

Case 1. Assume that

A1: $a|b$.

It must be shown that

B1: $a|(bc)$.

$B1$ gives rise to the key question, "How can I show that an integer (namely, a) divides another integer (namely, bc)?" Applying the definition means you must show that

B2: There is an integer k such that $bc = ka$.

Recognizing the keywords "there is" in $B2$, you should use the construction method to produce the desired integer k. Working forward from $A1$ by definition, you know that

A2: There is an integer p such that $b = pa$.

Multiplying both sides of the equality in $A2$ by c yields

A3: $bc = cpa$.

From $A3$, the desired value for k in $B2$ is cp, thus completing this case.

Case 2. In this case, you should assume that

A1: $a|c$.

You must show that

B1: $a|(bc)$.

The remainder of the proof in this case is similar to that in Case 1 and is not repeated.

Proof. Assume, without loss of generality, that $a|b$. By definition, there is an integer p such that $b = pa$. But then, $bc = (cp)a$, and so $a|(bc)$. □

13.15 **Analysis of Proof.** Recognizing the keywords "either/or" in the conclusion, a proof by elimination is used. Accordingly, you can assume that

A1: $4|n$.

It will be shown that

B1: $4|(mn)$.

$B1$ gives rise to the key question, "How can I show that an integer (namely, 4) divides another integer (namely, mn)?" By definition, the answer is to show that

B2: There is an integer k such that $mn = 4k$.

Recognizing the keywords "there is" in the backward statement $B2$, the construction method is used to produce the desired integer k.

Working forward from $A1$, by definition,

A2: There is an integer p such that $n = 4p$.

Multiplying both sides of the equality in $A2$ by the integer m, you have that

A3: $mn = 4mp$.

It is easy to see from $A3$ that the desired integer k in $B2$ is $k = mp$ and so the proof is complete.

Proof. Suppose that the integer n satisfies $4|n$. By definition, there is an integer p such that $n = 4p$. Thus, $mn = 4mp$ and so $4|(mn)$, completing the proof. □

13.16 **Analysis of Proof.** A key question associated with the conclusion is, "How can I show that a set (namely, $(S \cap T)^c$) is equal to another set (namely, $S^c \cup T^c$)?" Answering this question by definition means you must show that

B1: $(S \cap T)^c \subseteq S^c \cup T^c$ and $S^c \cup T^c \subseteq (S \cap T)^c$.

A key question associated with the first part of $B1$ is, "How can I show that a set (namely, $(S \cap T)^c$) is a subset of another set (namely, $S^c \cup T^c$)?" Answering this question by the definition means you must show that

B2: For all elements $x \in (S \cap T)^c$, $x \in S^c \cup T^c$.

Recognizing the keywords "for all" in the backward statement $B2$, the choose method is used to choose

A1: An element $x \in (S \cap T)^c$,

for which it must be shown that

B3: $x \in S^c \cup T^c$.

A key question associated with $B3$ is, "How can I show that an element belongs to the union of two sets (namely, S^c and T^c)?" By definition of union, you must show that

B4: Either $x \in S^c$ or $x \in T^c$.

Recognizing the keywords "either/or" in the backward statement $B4$, a proof by elimination is appropriate. Accordingly, you should assume that

A2: $x \notin S^c$, that is, $x \in S$.

Working forward from $A1$ and $A2$, you must show that

B5: $x \in T^c$, that is, $x \notin T$.

Working forward from $A1$, by definition of the complement, you have that

A3: $x \notin S \cap T$, that is, $x \notin S$ or $x \notin T$.

Recognizing the keywords "either/or" in the forward statement $A3$, a proof by cases is appropriate. Accordingly,

Case 1: $x \notin S$. This case, however, cannot happen because, from $A2$, you know that $x \in S$.

Case 2: $x \notin T$. In this case, $B5$ is true and this part of the proof is now complete.

From $B1$, it remains to show that

B6: $S^c \cup T^c \subseteq (S \cap T)^c$.

A key question associated with $B6$ is, "How can I show that a set (namely, $S^c \cup T^c$) is a subset of another set (namely, $(S \cap T)^c$)?" Answering this question by the definition means you must show that

B7: For all elements $x \in S^c \cup T^c$, $x \in (S \cap T)^c$.

Recognizing the keywords "for all" in the backward statement $B7$, the choose method is used to choose

A4: An element $x \in S^c \cup T^c$,

for which it must be shown that

B8: $x \in (S \cap T)^c$.

A key question associated with $B8$ is, "How can I show that an element belongs to the complement of a set (namely, $(S \cap T)^c$)?" By definition of complement, you must show that

B9: $x \notin S \cap T$, that is, $x \notin S$ or $x \notin T$.

Recognizing the keywords "either/or" in the backward statement $B9$, a proof by elimination is appropriate. Accordingly, you should assume that

A5: $x \in S$.

Working forward from $A4$ and $A5$, you must show that

B10: $x \notin T$.

Working forward from $A4$, by definition of the union, you have that

A6: $x \in S^c$ or $x \in T^c$.

Recognizing the keywords "either/or" in the forward statement $A6$, a proof by cases is appropriate. Accordingly,

Case 1: $x \in S^c$, that is, $x \notin S$. This, however, cannot happen because from $A5$, you know that $x \in S$.

Case 2: $x \in T^c$, that is, $x \notin T$. In this case, $B10$ is true.

The proof is now complete because it has been shown that $(S \cap T)^c \subseteq S^c \cup T^c$ and $S^c \cup T^c \subseteq (S \cap T)^c$ and so $(S \cap T)^c = S^c \cup T^c$.

Proof. To show that $(S \cap T)^c = S^c \cup T^c$, it will be shown that $(S \cap T)^c \subseteq S^c \cup T^c$ and $S^c \cup T^c \subseteq (S \cap T)^c$. For the first part, let $x \in (S \cap T)^c$, that is, $x \notin S \cap T$. Thus, $x \notin S$ or $x \notin T$. In the former case, $x \in S^c$ and so $x \in S^c \cup T^c$. In the latter case, $x \in T^c$ and so $x \in S^c \cup T^c$. Thus, in either case, $x \in S^c \cup T^c$ and so $(S \cap T)^c \subseteq S^c \cup T^c$.

It remains to show that $S^c \cup T^c \subseteq (S \cap T)^c$. To that end, let $x \in S^c \cup T^c$, that is, $x \in S^c$ or $x \in T^c$. In the former case, $x \notin S$, so, $x \notin S \cap T$, that is,

$x \in (S \cap T)^c$. In the latter case, $x \notin T$, so, $x \notin S \cap T$, that is, $x \in (S \cap T)^c$. Thus, in either case, $x \in (S \cap T)^c$ and so $S^c \cup T^c \subseteq (S \cap T)^c$. The proof is now complete because it has been shown that $(S \cap T)^c = S^c \cup T^c$. □

13.17 a. Recognizing the keywords "either/or" in the conclusion, the author is using a proof by elimination.

b. Because $c|n$, by definition, there is an integer p such that $n = cp$. It is given that $c, n > 0$, so $p > 0$. Furthermore, because $c < n$, it must be that $p \geq 2$ (for if $p = 1$, $c = n$). This means that $2c \leq pc = n$. Dividing the left and right sides of the foregoing inequality by 2 results in the desired conclusion that $c \leq n/2$.

c. The hypothesis states $n \leq 2m$. Squaring both sides and dividing by four gives the desired result, $n^2/4 \leq m^2$.

d. In the previous sentence, the author has shown that $n \leq m^2$. Upon taking the positive square root of both sides of this inequality, it follows that $\sqrt{n} \leq m$. This means that, in trying to prove that "A implies (C or D)," the author has assumed A and NOT C and then demonstrated that D is true. The proof by elimination is thus complete.

13.18 a. Recognizing the keywords "either/or" in the conclusion, the author is using a proof by elimination. Accordingly, the author assumes that p does not divide a and must then show that $p|b$. Indeed, the author claims to have done so in the last sentence of the proof.

b. If q and b are integers for which q is prime and q does not divide b, then there are integers x and y such that $xq + yb = 1$.

c. It has already been shown that $ab = cp$ and $mpb + nab = b$. Substituting cp for ab in the equation $b = mpb + nab$ results in the equality $b = mpb + ncp$.

d. In the previous sentence, the author has shown that $b = (mb + nc)p$. This means that the author has constructed an integer k such that $b = kp$ (namely, $k = mb+nc$) and hence, by definition, has shown that $p|b$. This means that, in trying to prove that "A implies (C or D)," the author has assumed A and NOT C and then demonstrated that D is true. The proof by elimination is thus complete.

14

Solutions to Exercises

14.1 a. For all elements $s \in S$, $s \leq z$.
b. There is an element $s \in S$ such that $s \geq z$.

14.2 a. There is an x with $ax \leq b$ and $x \geq 0$ such that $cx \leq u$.
b. There is an x with $ax \leq b$ and $x \geq 0$ such that $cx \geq u$.
c. For all x with $b \leq x \leq c$, $ax \geq u$.
d. For all x with $b \leq x \leq c$, $ax \leq u$.

14.3 a. For all real numbers x with $0 \leq x \leq 1$, $f(x) \leq y$.
b. There is a real number x with $0 \leq x \leq 1$ such that $f(x) \leq y$.

14.4 **Analysis of Proof.** Recognizing the keyword "min" in the backward process, you must show that the following quantified statement is true:

B1: For all numbers x, $x(x-2) \geq -1$, that is, $x^2 - 2x + 1 \geq 0$.

Recognizing the keywords "for all" in the backward statement $B1$, the choose method is used to choose

A1: A real number y,

for which it must be shown that

B2: $y^2 - 2y + 1 \geq 0$.

Now $B2$ is true because $y^2 - 2y + 1 = (y-1)^2 \geq 0$ and so the proof is complete.

Proof. To show that, for all real numbers x, $x(x-2) \geq -1$, one only needs to show that, for all x, $x^2 - 2x + 1 \geq 0$. To that end, choose a real number y and note that $y^2 - 2y + 1 = (y-1)^2 \geq 0$. □

14.5 **Analysis of Proof.** Recognizing the keyword "max" in the backward process, you must show that the following quantified statement is true:

B1: There is a real number x with $x \leq 2^{-x}$ such that $x \geq 0.5$.

Recognizing the keywords "there is" in the backward statement $B1$, the construction method is used. There are many values of x that satisfy the properties in $B1$. For example, for $x = 0.5$, you have $2^{-x} = 2^{-0.5} \approx 0.7071 > x$ and $x \geq 0.5$, so the proof is complete.

Proof. To show that there is a real number x with $x \leq 2^{-x}$ such that $x \geq 0.5$, one only needs to find a value for x that satisfies the foregoing properties. Indeed, for $x = 0.5$, you have $2^{-x} = 2^{-0.5} \approx 0.7071 > x$ and $x \geq 0.5$. □

14.6 **Analysis of Proof.** The max/min method is used to convert the conclusion to the equivalent statement

B1: For all elements $s \in S$, $s \geq t^*$.

The appearance of the quantifier "for all" in the backward statement $B1$ suggests using the choose method to choose

A1: An element $s' \in S$,

for which it must be shown that

B2: $s' \geq t^*$.

The desired conclusion is obtained by working forward. Specifically, because S is a subset of T, it follows by definition that

A2: For all elements $s \in S$, $s \in T$.

Specializing $A2$ to $s = s'$, which is in S (see $A1$), it follows that

A3: $s' \in T$.

Also, the hypothesis states that

A4: For all elements $t \in T$, $t \geq t^*$.

Specializing $A4$ with $t = s'$, which is in T (see A3), it follows that

A5: $s' \geq t^*$.

Now $A5$ is precisely $B2$, and so the proof is complete.

Proof. To reach the conclusion, let $s' \in S$. It will be shown that $s' \geq t^*$. By the hypothesis that $S \subseteq T$, it follows that $s' \in T$. But then the hypothesis that for all elements $t \in T$, $t \geq t^*$ ensures that $s' \geq t^*$. $\square$

14.7 **Analysis of Proof.** Recognizing the keyword "min" in the conclusion and letting $z = \max\{ub : ua \leq c, u \geq 0\}$, the max/min methods results in the following equivalent quantified statement:

> **B1:** For every real number x with $ax \geq b$ and $x \geq 0$, it follows that $cx \geq z = \max\{ub : ua \leq c, u \geq 0\}$.

The keywords "for every" in the backward statement $B1$ suggest using the choose method to choose

> **A1:** A real number x with $ax \geq b$ and $x \geq 0$,

for which it must be shown that

> **B2:** $cx \geq \max\{ub : ua \leq c, u \geq 0\}$.

Recognizing the keyword "max" in $B2$, the max/min methods lead to the need to prove the following equivalent quantified statement:

> **B3:** For every real number u with $ua \leq c$ and $u \geq 0$, $cx \geq ub$.

The keywords "for every" in the backward statement $B3$ suggest using the choose method to choose

> **A2:** A real number u with $ua \leq c$ and $u \geq 0$,

for which it must be shown that

> **B4:** $cx \geq ub$.

To reach $B4$, multiply both sides of $ax \geq b$ in $A1$ by $u \geq 0$ to obtain

> **A3:** $uax \geq ub$.

Likewise, multiply both sides of $ua \leq c$ in $A2$ by $x \geq 0$ to obtain

> **A4:** $uax \leq cx$.

The desired conclusion in $B4$ that $cx \geq ub$ follows by combining $A3$ and $A4$.

Proof. To reach the desired conclusion that $cx \geq ub$, let x and u be real numbers with $ax \geq b$, $x \geq 0$, $ua \leq c$, and $u \geq 0$. Multiplying $ax \geq b$ through by $u \geq 0$ and $ua \leq c$ through by $x \geq 0$, it follows that $ub \leq uax \leq cx$. $\square$

14.8 **Analysis of Proof.** Recognizing the keyword "max" in the conclusion and letting $z = \min\{s \in S\}$, the max/min methods result in the following equivalent quantified statement that you must show is true:

B1: There is an element $t \in T$ such that $t \geq z$.

The keywords "there is" in the backward statement $B1$ suggest using the construction method to produce the desired element $t \in T$.

Turning to the forward process, from the hypothesis that $S \cap T \neq \emptyset$,

A1: There is an element $x \in S$ such that $x \in T$.

Constructing $t = x \in T$, from $B1$, it remains to show that

B2: $x \geq z = \min\{s \in S\}$.

Now because $z = \min\{s \in S\}$, by the definition of the minimum of a set, you know that

A2: For every element $s \in S$, $s \geq z$.

The desired conclusion in $B2$ follows on specializing $A2$ with $s = x$ (noting from $A1$ that $x \in S$) and so the proof is complete.

Proof. From the hypothesis that $S \cap T \neq \emptyset$, there is an element, say x, that is in both S and T. Now, because $x \in S$, it follows that $x \geq \min\{s \in S\}$. Furthermore, because $x \in T$, $x \leq \max\{t \in T\}$. It therefore follows that $\max\{t \in T\} \geq x \geq \min\{s \in S\}$ and so the proof is complete. $\square$

14.9 **Analysis of Proof.** For notational purposes, let $z^- = \min\{s : s \in S\}$ and $z^+ = \max\{-s : s \in S\}$ and so it must be shown that

B1: $z^- = -z^+$.

A key question associated with $B1$ is, "How can I show that two real numbers (namely, z^- and z^+) are equal?" From the hint, the answer is to show that

B2: $z^- \leq -z^+$ and $z^- \geq -z^+$.

For the first inequality, you must show that

B3: $z^- = \min\{s : s \in S\} \leq -z^+$.

However, working forward from the fact that $z^- = \min\{s : s \in S\}$, you know by the max/min method that

A1: For every element $s \in S$, $z^- \leq s$.

Specializing the for-all statement in $A1$ to $s = -z^+$, noting that $-z^+ \in S$, results in $B3$. It remains from $B2$ to show that

B4: $z^- \geq -z^+ = -\max\{-s : s \in S\}$.

However, working forward from the fact that $z^+ = \max\{-s : s \in S\}$, you know by the max/min method that

A2: For every element $s \in S$, $z^+ \geq -s$.

Specializing the for-all statement in $A2$ to $s = z^-$, noting that $z^- \in S$, results in

A3: $z^+ \geq -z^-$, that is, $z^- \geq -z^+$.

Thus, $B2$ is true and so the proof is complete.

Proof. Let $z^- = \min\{s : s \in S\}$ and $z^+ = \max\{-s : s \in S\}$. By definition of the minimum of a set, for all $s \in S$, $z^- \leq s$, that is, $-z^- \geq -s$. In particular, $-z^+ \in S$, so $-z^- \geq z^+$ or equivalently $z^- \leq -z^+$. Similarly, by definition of the maximum of a set, for all $s \in S$, $z^+ \geq -s$, that is, $-z^+ \leq s$. In particular, $z^- \in S$, so $-z^+ \leq z^-$. Thus $z^- = -z^+$ and so the proof is complete. □

14.10 a. Recognizing the keyword "max" in the conclusion, the author uses the min/max method to convert the conclusion to the equivalent statement: For all real numbers x, $ax^2 + bx + c \leq (4ac - b^2)/(4a)$. Then recognizing the keywords "for all" in the backward process, the author uses the choose method to choose a real number x.

b. $a\left(x + \frac{b}{2a}\right)^2 \leq 0$ because $a < 0$ from the hypothesis. Adding $\frac{4ac-b^2}{4a}$ to both sides results in the inequality in the proof.

14.11 a. The contradiction is that the smallest positive integer for which $P(n)$ is false is not the smallest positive integer for which $P(n)$ is false.

b. Having assumed that there is a positive integer for which $P(n)$ false, the set of integers for which $P(n)$ is false must contain at least that integer.

c. $k \geq 2$ because in the hypothesis, it is given that $P(1)$ is true. So the smallest positive integer for which $P(n)$ is false has to be greater than 1.

d. The author is claiming in the fourth sentence that $P(k-1)$ is false from the hypothesis. However, the hypothesis only ensures that $P(k-1)$ is false when $k - 1 \geq 1$, that is, when $k \geq 2$ and this is why the author needs $k \geq 2$.

e. The author is working forward from the hypothesis that, for all integers $n \geq 1$, $P(n)$ implies $P(n+1)$ and is specializing the statement to the integer $n = k - 1 \geq 1$ to claim that $P(k-1)$ implies $P(k)$. However, because $P(k)$ is false, by the contrapositive method, it must be that $P(k-1)$ is false, which is what the author states in this sentence.

15

Solutions to Exercises

15.1 a. Contrapositive or contradiction method because the keyword "no" appears in the conclusion.
b. Induction method because the conclusion is true for every integer $n \geq 4$.
c. Forward-Backward method because there are no keywords in the hypothesis or conclusion.
d. Max/Min method because the conclusion contains the keyword "maximum."
e. Uniqueness method because the conclusion contains the keywords "one and only one."

15.2 a. Construction method because the first quantifier from the left in the conclusion is "there is."
b. Forward-Backward method because there are no keywords in the hypothesis or conclusion.
c. Either/Or method because the conclusion contains the hidden keywords "either/or" ($a = \pm b$ is the same as "either $a = +b$ or $a = -b$).
d. Contradiction or contrapositive method because the conclusion contains the keyword "no." You might also use specialization because the hypothesis contains the keywords "for all."
e. Choose method because the first quantifier from the left in the conclusion is "for all."

15.3 a. With the contradiction method, you would assume that p and q are odd integers and that x is a rational number that satisfies the equation $x^2 + 2px + 2q = 0$. You would then work forward from this information to reach a contradiction.

With the contrapositive method, you assume that x is a rational number for which $x^2+2px+2q = 0$ and show that either p is even or that q is even (for which a proof by elimination would be appropriate because of the keywords "either/or" in the backward process).

b. To use induction, first show that the statement is true for $n = 4$, that is, that $4! > 4^2$. For the second part of induction, you would assume that $n! > n^2$ and try to prove that $(n+1)! > (n+1)^2$. To do so, you should relate $(n+1)!$ to $n!$ so that you can use the induction hypothesis. You can also use the fact that $n \geq 4$, if necessary.

c. Working backward, you are led to the key question, "How can I show that a function (namely, $f+g$) is convex?" You can use the definition to answer this question. To show that $f + g$ is convex, you should also work forward from the hypothesis that f and g are convex by using the definition of a convex function.

d. According to the max/min method, you should convert the conclusion to the following equivalent quantified statement:

B1: For all real numbers a, b, and c with $a^2 + b^2 + c^2 = 1$, $ab + bc + ac \leq 1$.

Recognizing the keywords "for all" in the backward statement $B1$, the choose method should be used next.

e. To apply the direct uniqueness method, you must first show that there is a line M perpendicular to the given line L through the point P. Next, you should assume that N is also a line perpendicular to L through the point P. You should then work forward to show that M and N are the same.

To apply the indirect uniqueness method, you must first show that there is a line M perpendicular to the given line L through the point P. Next, you should assume that N is a line, different from M, perpendicular to L through the point P. You should then work forward to reach a contradiction.

15.4 a. Working forward from the fact p is not prime, you should construct an integer m and then show that $1 < m \leq \sqrt{p}$ and $m|p$.

b. Working backward, you are led to the key question, "How can I show that a function (namely, $f+g$) is continuous at a point (namely, x)?" You can use the definition to answer this question. You should also work forward from the hypothesis that f and g are continuous at x by using the definition of a continuous function.

c. To use a proof by elimination, you assume the hypothesis that a and b are integers for which $a|b$ and $b|a$. You can also assume that $a \neq +b$ and work forward to show that $a = -b$. Alternatively, in addition to the hypothesis, you can also assume that $a \neq -b$ and work forward to show that $a = +b$.

d. To use contradiction, assume the hypothesis that f and g are two functions such that (1) for all real numbers x, $f(x) \leq g(x)$ and (2) there is no real number M such that, for all x, $f(x) \leq M$. You should also assume that the conclusion is not true, that is, that there is a real number $M > 0$ such that for all real numbers x, $g(x) \leq M$. You must then work forward to reach a contradiction.

To use the contrapositive method, you should assume that the conclusion is not true, that is, that there is a real number $M > 0$ such that for all real numbers x, $g(x) \leq M$. You must then work forward from this information to show that the hypothesis is not true, that is, that either (1) there is a real number x such that $f(x) > g(x)$ or (2) there is a real number M such that for all real numbers x, $f(x) \leq M$.

e. To apply the choose method, you would choose

A1: A real number $\epsilon > 0$,

for which you would then have to show that

B1: There is a real number $\delta > 0$ such that, for all real numbers y with $|x - y| < \delta$, $|f(x) + g(x) - (f(y) + g(y))| < \epsilon$.

Recognizing the keywords "there is" in the backward statement $B1$, you should proceed with the construction method.

15.5 a. Using induction, you would first have to show that $4! > 4^2$. Then you would assume that $n! > n^2$ and $n \geq 4$, and show that $(n+1)! > (n+1)^2$.

b. Using the choose method, you would choose an integer m for which $m \geq 4$. You would then try to show that $m! > m^2$.

c. Converting the statement to the form, "If ... then ..." you obtain, "If n is an integer with $n \geq 4$, then $n! > n^2$." With the forward-backward method, you would then assume that n is an integer with $n \geq 4$ and work forward to show that $n! > n^2$.

d. Using contradiction, you would assume that there is an integer $n \geq 4$ such that $n! \leq n^2$ and then work forward to reach a contradiction.

15.6 a. With the choose method, you would choose an element $x \in S$ for which you must show that $x \leq \epsilon$. To do so, you can also work forward from the hypothesis.

b. With specialization, you would look for one particular element, say $y \in T$. The result of specialization is that $y \leq \epsilon$. This fact should help you to reach the conclusion.

c. With contradiction, you would assume that there is an element $x \in S$ with $x > \epsilon$, $S \subseteq T$, and $\epsilon > 0$ is a real number such that, for every element $x \in T$, $x \leq \epsilon$. You would then work forward to reach a contradiction.

d. With the contrapositive method, you would assume that there is an element $x \in S$ with $x > \epsilon$, $\epsilon > 0$, and $S \subseteq T$. You would then have to conclude that there is an element $t \in T$ such that $t > \epsilon$.

15.7 a. If $X = \{(1+\frac{1}{n})^n : n > 0 \text{ is an integer}\}$, then $\max X \leq 3$.

b. If $X = \{(1+\frac{1}{n})^n : n > 0 \text{ is an integer}\}$, then there is no element $x \in X$ such that $x > 3$.

c. For every integer $n > 0$, $(1+\frac{1}{n})^n \leq 3$.

15.8 a. A proof by elimination is likely to be used because the keywords "either/or" appear in the conclusion.

b. A proof by cases is likely to be used because the keywords "either/or" appear in the hypothesis.

c. The construction method is likely to be used because the keywords "there is" appear in the conclusion. Also, specialization is likely to be used because the keywords "for all" appear in the hypothesis.

d. The construction method is likely to be used because the keywords "there is" appear in the conclusion. To show that the constructed object is correct, the choose method is likely to be used because the keywords "for all" appear in the backward process. Also, specialization is likely to be used because the keywords "for all" appear in the hypothesis.

15.9 a. A proof by elimination is likely to be used because the keywords "either/or" appear in the conclusion.

b. A proof by cases is likely to be used because the keywords "either/or" appear in the hypothesis.

c. The construction method is likely to be used because the keywords "there is" appear in the conclusion. Also, specialization is likely to be used because the keywords "for all" appear in the hypothesis.

d. The construction method is likely to be used because the keywords "there is" appear in the conclusion. Also, specialization is likely to be used because the keywords "for all" appear in the hypothesis.

15.10 The author first uses a proof by cases. Then, recognizing the keywords "either/or" in the conclusion the author uses a proof by elimination.

15.11 The author uses the contrapositive method and assumes $NOT\ B$. When trying to prove $NOT\ A$, the author recognizes the keywords "there is" and hence uses the construction method. To produce the desired object, the author works forward and uses a proof by cases. To show that the constructed object has the desired properties, the contradiction method is used.

15.12 The author recognizes the keywords "for all" in the conclusion and therefore starts with the choose method. A proof by cases is used next because, "either $a_3 = 1$ or else $1 - a_3 > 0$." For the latter case, the author works forward from the hypothesis that f is a convex function by definition. On so doing, the author recognizes the keywords "for all" in the forward process and uses specialization two times to complete this case and the proof.

15.13 **Analysis of Proof.** Recognizing the hidden keywords "there is" in the conclusion, the construction method is used to produce the desired rational root of $\frac{1}{2}ax^2 + cx + b$. To that end, noting that $a > 0$, you can use the quadratic formula to obtain the following two roots:

A1: $x = \frac{-c \pm \sqrt{c^2 - 2ab}}{a}$.

Working forward from the hypothesis that ABC is a right triangle, using previous knowledge of the Pythagorean theorem, you know that

A2: $c^2 = a^2 + b^2$.

Substituting the expression for c^2 from $A2$ in $A1$ yields

A3: $x = \frac{-c \pm \sqrt{a^2 + b^2 - 2ab}}{a} = \frac{-c \pm \sqrt{(a-b)^2}}{a} = \frac{-c \pm (a-b)}{a}$.

Indeed, the positive square root in $A3$ provides the desired rational root of $\frac{1}{2}ax^2 + cx + b$, that is, you can construct

A4: $x = \frac{-c+a-b}{a}$.

To see that this value of x is correct, it must be shown that

B1: $x = \frac{-c+a-b}{a}$ is rational.

A key question associated with $B1$ is, "How can I show that a real number (namely, x) is rational?" Using the definition, it must be shown that

B2: There are integers p and q with $q \neq 0$ such that $x = \frac{p}{q}$.

Recognizing the keywords "there are" in the backward statement $B2$, the construction method is used to produce the integers p and q. Indeed, looking at the values of x in $A4$, it is not hard to construct $p = -c + a - b$ and $q = a$. Because a, b, and c are integers, it follows that p and q are integers. Finally, $q = a \neq 0$ because a is the length of a side of a right triangle.

Proof. Noting that $a > 0$, the roots of $\frac{1}{2}ax^2 + cx + b$ are $x = \frac{-c \pm \sqrt{c^2 - 2ab}}{a}$. It will now be shown that $x = \frac{-c+\sqrt{c^2-2ab}}{a}$ is rational. Because ABC is a right triangle, from the Pythagorean theorem, $x = \frac{-c+\sqrt{c^2-2ab}}{a} = \frac{-c+\sqrt{a^2+b^2-2ab}}{a}$ $= \frac{-c+a-b}{a}$. Now $-c + a - b$ is an integer because a, b, and c are integers and $a \neq 0$ is an integer because it is the length of the leg of a right triangle. Thus, this value of x is the desired rational root. □

15.14 **Analysis of Proof.** Recognizing the keyword "only" in the conclusion, the direct uniqueness method is used. Accordingly, it is first necessary to construct an element $x \in S$ such that for every element $s \in S$, $s \leq x$. Indeed, from the hypothesis, you can construct

A1: $x = \max\{s : s \in S\}$.

By the max/min method, this means that

A2: For every element $s \in S$, $s \leq x$.

To see that x is the only element of S that satisfies the property in $A2$, by the direct uniqueness method, you can assume that y is another such element of S, that is,

A3: $y \in S$ also satisfies the property that for every element $s \in S$, $s \leq y$.

It must be shown that

B1: $x = y$.

A key question associated with $B1$ is, "How can I show that two real numbers (namely, x and y) are equal?" One answer is to show that

B2: $x \leq y$ and $y \leq x$.

To see that $x \leq y$, you can specialize the for-all statement in $A3$ to $s = x \in S$ (from the hypothesis) and conclude that

A4: $x \leq y$.

To see that $y \leq x$, you can specialize the for-all statement in $A2$ to $s = y \in S$ (see $A3$) and conclude that

A5: $y \leq x$.

The proof is now complete because $A4$ and $A5$ mean that $B2$ is true.

Proof. It is given that $x = \max\{s : s \in S\}$ so, by definition, this means that for every element $s \in S$, $s \leq x$. To see that x is the only such element of S, suppose that for $y \in S$, for every element $s \in S$, $s \leq y$. Now $x \in S$, so $x \leq y$. Also $y \in S$ and x is the maximum element of S, so $y \leq x$. As such $x = y$. □

15.15 **Analysis of Proof.** The forward-backward method is used to begin the proof because there are no keywords in the hypothesis or conclusion. An associated key question is, "How can I show that a function (namely, $p + q$) is a polynomial?" By definition, the answer is to show that

B1: There is an integer $k \geq 0$ and real numbers $c_0, c_1, \ldots, c_k$ such that for all numbers x, $(p + q)(x) = c_0 + c_1x^1 + \cdots + c_kx^k$.

Recognizing the keywords "there is" in the backward statement $B1$, the construction method is used to produce the desired integer k and real numbers $c_0, c_1, \ldots, c_k$. Turning to the forward process, from the hypothesis that p is a polynomial, by definition, you know that

A1: There is an integer $m \geq 0$ and real numbers $a_0, a_1, \ldots, a_m$ such that for all numbers x, $p(x) = a_0 + a_1x^1 + \cdots + a_mx^m$.

Likewise, because q is a polynomial, you know that

A2: There is an integer $n \geq 0$ and real numbers $b_0, b_1, \ldots, b_n$ such that for all numbers x, $q(x) = b_0 + b_1x^1 + \cdots + b_nx^n$.

Now you can construct

A3: The integer $k = \max\{m, n\} \geq 0$.

You would also like to construct the numbers $c_0 = a_0 + b_0, c_1 = a_1 + b_1, \ldots$, $c_k = a_k + b_k$, which you can do if $m = n$. However, if $m < n$, then a_j does not exist for $j > m$. Likewise, if $n < m$, then b_j does not exist for $j > n$. In either case, you can proceed by extending the needed real numbers with values of 0. More formally, proceed with a proof by cases, as follows.

Case 1: $m \leq n$. In this case, for each integer $j = m + 1, \ldots, n$, define $a_j = 0$ and construct

A4: The numbers $c_0 = a_0 + b_0, c_1 = a_1 + b_1, \ldots, c_k = a_k + b_k$.

To see that the construction in $A3$ and $A4$ is correct, it remains from $B1$ to show that

B2: For every real number x, $(p+q)(x) = c_0 + c_1x^1 + \cdots + c_kx^k$.

Recognizing the keywords "for every" in the backward statement $B2$, you should use the choose method to choose

A5: A real number x,

for which it must be shown that

B3: $(p + q)(x) = c_0 + c_1x^1 + \cdots + c_kx^k$.

However, specializing the for-all statements in $A1$ and $A2$, you know that

$$\begin{array}{rcll} (p+q)(x) & = & p(x)+q(x) & \text{(definition of } p+q\text{)} \\ & = & (a_0+\cdots+a_m x^m)+(b_0+\cdots+b_n x^n) & \text{(specialize } A1 \text{ and } A2\text{)} \\ & = & (a_0+\cdots+a_n x^n)+(b_0+\cdots+b_n x^n) & (a_j=0 \text{ for } j>m) \\ & = & (a_0+b_0)\cdots+(a_k+b_k)x^k & (k=\max\{m,n\}=n) \\ & = & c_0+\cdots+c_k x^k & \text{(from } A4\text{).} \end{array}$$

The proof of this case is now complete.

Case 2: $n \le m$. In this case, for each integer $j = n+1, \ldots, m$, define $b_j = 0$ and construct

A6: The numbers $c_0 = a_0 + b_0, c_1 = a_1 + b_1, \ldots, c_k = a_k + b_k$.

To see that the construction in $A3$ and $A6$ is correct, it remains from $B1$ to show that

B4: For every real number x, $(p+q)(x) = c_0 + c_1x^1 + \cdots + c_kx^k$.

Recognizing the keywords "for every" in the backward statement $B3$, you should use the choose method to choose

A7: A real number x,

for which it must be shown that

B5: $(p+q)(x) = c_0 + c_1x^1 + \cdots + c_kx^k$.

However, specializing the for-all statements in $A1$ and $A2$, you know that

$$\begin{array}{rcll} (p+q)(x) & = & p(x)+q(x) & \text{(definition of } p+q\text{)} \\ & = & (a_0+\cdots+a_m x^m)+(b_0+\cdots+b_n x^n) & \text{(specialize } A1 \text{ and } A2\text{)} \\ & = & (a_0+\cdots+a_m x^m)+(b_0+\cdots+b_m x^m) & (b_j=0 \text{ for } j>n) \\ & = & (a_0+b_0)\cdots+(a_k+b_k)x^k & (k=\max\{m,n\}=m) \\ & = & c_0+\cdots+c_k x^k & \text{(from } A4\text{).} \end{array}$$

The proof for this case, and hence the whole proof, is now complete.

Proof. Because p is a polynomial, there exists an integer $m \ge 0$ and coefficients $a_0, \cdots, a_n$ such that $p(x) = \sum_{i=0}^{m} a_i x^i$. Similarly for q, there exists an integer $n \ge 0$ and coefficients $b_0, \cdots, b_n$ such that $q(x) = \sum_{i=0}^{n} b_i x^i$. Let $k = \max\{m, n\} \ge 0$ and assume, without loss of generality, that $m \le n$. Defining $a_j = 0$ for each $j = m+1, \ldots, n$, you have $(p+q)(x) = p(x)+q(x) = \sum_{i=0}^{k} a_i x^i + \sum_{i=0}^{k} b_i x^i = \sum_{i=0}^{k}(a_i + b_i)x^i = \sum_{i=0}^{k} c_i x^i$. □

15.16 **Analysis of Proof.** The keyword "not" in the conclusion suggests using the contradiction or contrapositive method. Here, the contradiction method is used, whereby you can assume that

A1 (NOT B): f is linear.

By definition, this means that there are real numbers m and b such that

A2: For every real number x, $f(x) = mx + b$.

A contradiction is now reached by showing that the slope $m > 0$ and $m < 0$. To that end, specialize the for-all statement in $A2$ three separate times to the three real numbers $x = x_1$, $x = x_2$, and $x = x_3$, which results in

A3: $f(x_1) = mx_1 + b$.
A4: $f(x_2) = mx_2 + b$.
A5: $f(x_3) = mx_3 + b$.

Subtracting $A3$ from $A4$ yields

A6: $f(x_2) - f(x_1) = mx_2 + b - (mx_1 + b) = m(x_2 - x_1)$.

From the hypothesis that $x_1 < x_2$, you can divide $A6$ through by $x_2 - x_1 \neq 0$ to obtain

A7: $m = \frac{f(x_2)-f(x_1)}{x_2-x_1}$.

From the hypothesis that $x_1 < x_2$ and $f(x_1) < f(x_2)$, it follows that both the denominator and numerator in $A7$ are positive and so

A8: $m > 0$.

A similar set of algebraic steps shows that $m < 0$. Specifically, subtracting $A4$ from $A5$ yields

A9: $f(x_3) - f(x_2) = mx_3 + b - (mx_2 + b) = m(x_3 - x_2)$.

From the hypothesis that $x_2 < x_3$, you can divide $A9$ through by $x_3 - x_2 \neq 0$ to obtain

A10: $m = \frac{f(x_3)-f(x_2)}{x_3-x_2}$.

From the hypothesis that $x_2 < x_3$ and $f(x_3) < f(x_2)$, it follows that the denominator in $A10$ is positive while the numerator is negative and so

A11: $m > 0$.

The contradiction that $m > 0$ in $A8$ and $m < 0$ in $A11$ completes the proof.

Proof. Assume, to the contrary, that f is linear and so there are real numbers m and b such that for every real number x, $f(x) = mx + b$. In particular, for the given real numbers x_1, x_2, and x_3,

$$\begin{aligned} f(x_1) &= mx_1 + b, \quad (1) \\ f(x_2) &= mx_2 + b, \quad (2) \\ f(x_3) &= mx_3 + b. \quad (3) \end{aligned}$$

A contradiction is reached by showing that $m > 0$ and $m < 0$. To that end, subtracting equation (1) from equation (2) and using the hypothesis that

$x_1 < x_2$ and $f(x_1) < f(x_2)$, it follows that

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0.$$

Likewise, subtracting equation (2) from equation (3) and using the hypothesis that $x_2 < x_3$ and $f(x_3) < f(x_2)$, it follows that

$$m = \frac{f(x_3) - f(x_2)}{x_3 - x_2} < 0.$$

The contradiction that $m > 0$ and $m < 0$ completes the proof. □

15.17 a. The contradiction is that $f(x_2) < f(x_2)$.

b. The author creates a value for t to use when specializing the following for-all statement in the definition of the function f being convex:

A1: For all real numbers x, y, and t with $0 \leq t \leq 1$, $f(tx + (1-t)y \leq tf(x) + (1-t)f(y)$.

c. The value of $t = \frac{x_3 - x_2}{x_3 - x_1} > 0$ because both $x_3 - x_2 > 0$ and $x_3 - x_1 > 0$ from the hypothesis $x_1 < x_2 < x_3$. Similarly, using this hypothesis, $x_2 - x_1 > 0$ and $x_3 - x_1 > 0$, and so $t < 1$ because

$$1 - t = 1 - \frac{x_3 - x_2}{x_3 - x_1} = \frac{x_3 - x_1 - (x_3 - x_2)}{x_3 - x_1} = \frac{x_2 - x_1}{x_3 - x_1} > 0.$$

d. From part (c) it has been shown that $t > 0$ and $1 - t > 0$. From the hypothesis, you have

$$\begin{aligned} f(x_1) &< f(x_2), \quad (1) \\ f(x_3) &< f(x_2). \quad (2) \end{aligned}$$

Multiplying inequality (1) through by $t > 0$ and inequality (2) through by $1 - t > 0$ and adding corresponding sides of the two inequalities yields the given strict inequality:

$$tf(x_1) + (1-t)f(x_3) < tf(x_2) + (1-t)f(x_2).$$

16

Solutions to Exercises

16.1 From the special cases of the triangle, quadrilateral, and pentagon you can see that the sum of interior angles increases by 180° for each additional side of the polygon. Letting n be the number of sides in the polygon, the equation $A(n) = (n-2) * 180°$ gives the correct sum of the interior angles. For the three special cases, you have

Polygon	n	$A(n)$
triangle	3	$(3-2) * 180 = 180°$
quadrilateral	4	$(4-2) * 180 = 360°$
pentagon	5	$(5-2) * 180 = 540°$

16.2 The linear function $ax + b$ is a special case of the polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ in which $n = 1$, $a_0 = b$, and $a_1 = a$. The quadratic function $ax^2 + bx + c$ is a special case of the polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ in which $n = 2$, $a_0 = c$, $a_1 = b$, and $a_2 = a$.

16.3 To explain that the point-to-set map $F : R \to R^*$ is a generalization of the function $F : R \to R$, you need to explain how f is a special case of F. For each real number $x \in R$ the function f maps x to $f(x) \in R$ while F maps x to a set of real numbers. If you let $F(x)$ be the set consisting of the single value $f(x)$, that is, $F(x) = \{f(x)\}$, then f is a special case of F.

16.4 If you substitute the set R of real numbers for the set S in Definition 2, then Definition 2 becomes, "A function f of one real variable is **strictly increasing on the set** R **of real numbers** if and only if for all real numbers $x, y \in R$ with $x < y$, $f(x) < f(y)$," which is precisely Definition 1.

16.5 You need to substitute values for the complex numbers $a+bi$ and $c+di$ so that Proposition 2 becomes Proposition 1. Because Proposition 1 deals with the real numbers x and y, there are no imaginary parts, you should substitute $a = x$, $b = 0$, $c = y$, and $d = 0$. On so doing, Proposition 2 becomes: If $x + 0i$ is a complex number with $x^2 + 0^2 \neq 0$, then there is a complex number $y+0i$ such that $(x+0i)(y+0i) = 1$. You can rewrite this to become Proposition 1 as follows: If x is a real number with $x \neq 0$, then there is a real number y such that $xy = 1$.

16.6 Note that the symbol n is used to describe both the matrix and the vector. To avoid this overlapping notation, suppose the vector has p components. You can see that a p-vector is a special case of a matrix by substituting $m = p$ and $n = 1$ or by substituting $m = 1$ and $n = p$.

16.7 Doing these generalizations requires a new definition as well as verifying the original special case.

a. For a positive integer j, a j-dimensional matrix is a table of real numbers with $m_1 \times m_2 \times \ldots \times m_j$ entries. Setting $j = 2$, $m_1 = m$, and $m_2 = n$ creates a two-dimensional matrix with m rows and n columns.

b. A complex two-dimensional matrix is a table of complex numbers of the form $a+bi$, where a and b are real numbers, organized in m rows and n columns. If the imaginary part of each entry is 0, then every entry in the complex two-dimensional matrix is a real number and hence a two-dimensional matrix of real numbers.

16.8 a. For given real numbers a and b with $a \leq b$, let $Y = \{$real numbers $x : a \leq x \leq b\}$. Setting $a = -2$ and $b = 3$ shows that the interval X is a special case of the interval Y.

b. For given real numbers a_1, b_1, a_2, b_2 with $a_1 \leq b_1$ and $a_2 \leq b_2$, let $Z = \{$real numbers $(x_1, x_2) : a_1 \leq x_1 \leq b_1$ and $a_2 \leq x_2 \leq b_2\}$. Setting $a_1 = a$, $b_1 = b$ and $a_2 = b_2 = 0$ shows that the interval Y is a special case of the rectangle Z.

c. For a given integer $n > 0$ and real numbers c_i, d_i, with $c_i \leq d_i$ for $i = 1, \ldots, n$, let $H = \{$real numbers $(x_1, \ldots, x_n) : c_i \leq x_i \leq d_i$ for each $i = 1, \ldots, n\}$. Setting $n = 2$, $c_1 = a_1$, $c_2 = a_2$, $d_1 = b_1$ and $d_2 = b_2$ shows that the rectangle Z is a special case of the hyperrectangle H.

16.9 Notationally let $B_r(c_1, c_2) = \{(x_1, x_2) : \sqrt{(x_1 - c_1)^2 + (x_2 - c_2)^2} \leq r\}$.

a. Define the distance between two circles of equal radius centered about the points (x_1, x_2) and (y_1, y_2) as follows:

$$\begin{aligned} d(B_r(x_1,x_2), B_r(y_1,y_2)) &= \min\{d[(a_1,a_2),(b_1,b_2)] : \\ &\quad (a_1,a_2) \in B_r(x_1,x_2) \text{ and} \\ &\quad (b_1,b_2) \in B_r(y_1,y_2)\}. \quad (**) \end{aligned}$$

You can see that the distance between two points defined by $d[(x_1, x_2), (y_1, y_2)] = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ is a special case of (**) by substituting $r = 0$ in (**). On so doing and noting that $B_0(x_1, x_2) = \{(x_1, x_2)\}$ and $B_0(y_1, y_2) = \{(y_1, y_2)\}$, (**) becomes:

$$\begin{aligned} d(B_0(x_1,x_2), B_0(y_1,y_2)) &= \min\{d[(a_1,a_2),(b_1,b_2)] : \\ &\quad (a_1,a_2) \in \{(x_1,x_2)\} \text{ and} \\ &\quad (b_1,b_2) \in \{(y_1,y_2)\}\} \\ &= d[(x_1,x_2),(y_1,y_2)]. \end{aligned}$$

b. When the two circles have radii r_1 and r_2, respectively, you can compute the distance between the circles as follows:

$$\begin{aligned} d(B_{r_1}(x_1,x_2), B_{r_2}(y_1,y_2)) &= \min\{d[(a_1,a_2),(b_1,b_2)] : \\ &\quad (a_1,a_2) \in B_{r_1}(x_1,x_2) \text{ and} \\ &\quad (b_1,b_2) \in B_{r_2}(y_1,y_2)\}. \quad (***) \end{aligned}$$

You can see that (**) in part (a) is a special case of (***) by substituting $r_1 = r_2 = r$ in (***). On so doing, you obtain precisely (**).

16.10 A real valued function of one variable, f, can be approximated by a quadratic function $ax^2 + bx + c$ (which reduces to a linear function when $a = 0$), by a cubic function $ax^3 + bx^2 + cx + d$ (which reduces to a linear function when $a = b = 0$), and in general by a polynomial of degree $n \geq 1$ (which reduces to a linear function when $n = 1$).

16.11 a. A set T in the plane is bounded above if and only if there is a real number β such that for all $(x, y) \in T$, $x \leq \beta$ and $y \leq \beta$. To see that Definition 22 is a special case of the foregoing definition, for a set S of real numbers that is bounded above by the real number α, note that substituting $T = \{(x, \alpha) : x \in S\}$ and $\beta = \alpha$ in the foregoing definition results in Definition 22, as follow:

There is a number α such that for all $(x, y) \in T$, $x \leq \alpha$ and $y \leq \alpha$.
There is a number α such that for all $(x, \alpha) \in T$, $x \leq \alpha$ and $\alpha \leq \alpha$.
There is a number α such that for all $x \in S$, $x \leq \alpha$.

b. A set T of n-vectors is bounded above if and only if there is a real number β such that for all $(x_1, \ldots, x_n) \in T$, $x_i \leq \beta$ for $i = 1, \ldots, n$. The definition in part (a) is a special case of the foregoing definition when you substitute $n = 2$.

16.12 A set S of n-vectors is bounded if and only if there is a real number γ such that for any n-vector $\mathbf{x} = (x_1, \ldots, x_n) \in S$, $|x_i| \leq \gamma$, for $i = 1, \ldots, n$. The foregoing definition becomes Definition 24 when you substitute $n = 1$.

16.13 The integer d is the gcd of the n integers $x_1, x_2, \cdots, x_n$ if and only if (1) d divides x_i for all $i = 1, \ldots, n$ and (2) if an integer c divides each x_i for $i = 1, \ldots, n$, then c divides d.

16.14 Syntax errors result because both the $\sqrt{x}$ and $\log(x)$ are undefined when the real number x is replaced with an n-vector $\mathbf{x}$.

16.15 a. One syntax error arises because the operation of dividing one n-vector by another n-vector is undefined.

b. (Answers other than the one presented here are possible.) One generalization that avoids syntax errors in the expression $|\mathbf{x}/\mathbf{y}|$ for n-vectors $\mathbf{x}$ and $\mathbf{y}$ is to define $\mathbf{x}/\mathbf{y}$ to be $\mathbf{x}/||\mathbf{y}||$ and to define the absolute value of a vector $\mathbf{z}$ to be $||\mathbf{z}||$.

c. When you rewrite the given inequality without division you have $|x| < t|y|$. Now replacing absolute value with the norm of a vector, an appropriate generalization without syntax errors is $||\mathbf{x}|| < t||\mathbf{y}||$.

16.16 When the set S of real numbers has an infinite number of values, the operations of adding all the values and then dividing by the number of elements, namely ∞, might not be meaningful.

16.17 a. For the real numbers $p_1, \ldots, p_n$ to be the probabilities that the events $E_1, \ldots, E_n$ occur, you need $p_1, \ldots, p_n \geq 0$ and $\sum_{i=1}^{n} p_i = 1$.

b. For the real numbers $p_1, p_2, \ldots$ to be the probabilities that the events $E_1, E_2, \ldots$ occur, you need $p_i \geq 0$ for all $i = 1, 2, \ldots$ and $\sum_{i=1}^{\infty} p_i = 1$.

c. When there an uncountable number of mutually exclusive events it may not be possible to check that all the probabilities add up to 1.

16.18 An appropriate unification is $\left(\prod_{i=1}^{n} x_i\right)^{1/n} \leq \left(\sum_{i=1}^{n} x_i\right)/n$.

16.19 a. An appropriate unification is $\frac{1}{n} - \frac{1}{n+1} < \left(\frac{1}{n}\right)^2$.

b. For an integer $n \geq 1$, you have $n^2 + n > n^2$ and so

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n^2 + n} < \left(\frac{1}{n}\right)^2.$$

16.20 a. An appropriate unification is $P(n) : \sum_{i=1}^{n}(2i-1) = n^2$.

b. The proof is by induction. Accordingly, the statement is true for $n = 1$ because $\sum_{i=1}^{1}(2i-1) = 1 = 1^2$. Now assume that $P(n)$ is true. Then for $n+1$ you have

$$\sum_{i=1}^{n+1}(2i-1) = \sum_{i=1}^{n}(2i-1) + (2n+1) = n^2 + 2n + 1 = (n+1)^2.$$

The proof is now complete.

16.21 a. An appropriate unification is $P(n) : \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.

b. The proof is by induction. Accordingly, the statement is true for $n = 1$ because $\sum_{i=1}^{1} \frac{i}{i+1} = \frac{1}{1+1}$. Now assume that $P(n)$ is true. Then for $n+1$ you have

$$\begin{aligned} P(n+1) \quad &= \textstyle\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \left(\sum_{i=1}^{n} \frac{1}{i(i+1)}\right) + \frac{1}{(n+1)(n+2)} \\ &= \tfrac{n}{n+1} + \tfrac{1}{(n+1)(n+2)} \\ &= \tfrac{1}{n+1}\left(n + \tfrac{1}{n+2}\right) \\ &= \tfrac{n^2+2n+1}{(n+1)(n+2)} \\ &= \tfrac{n+1}{n+2}. \end{aligned}$$

The proof is now complete.

16.22 The recursive relationship is $a_{n+1} = a_n + n$, for $n = 1, 2, \ldots$. Thus, the next term in the sequence is $a_6 = a_5 + 6 = 15 + 6 = 21$.

16.23 For given real numbers α, β, and γ and a given real-valued function of one variable, $h(t)$, the differential equation $\alpha g''(t) + \beta g'(t) + \gamma g(t) = h(t)$ is a unification of both differential equations with the appropriate substitutions.

16.24 An appropriate unification, called the L_p norm on the n-vector $\mathbf{x} = (x_1, \ldots, x_n)$, is $\left(\sum_i |x_i|^p\right)^{1/p}$. The substitution of $p = 1$ and $p = 2$ verifies the two special cases.

16.25 The proof is by contradiction. Thus, assume that n^m is even and that n is odd. Because n is odd, by definition, there exists an integer k such that $n = 2k+1$. This implies that $n^m = (2k+1)^m = 2k\left(\sum_{i=0}^{n-1}\binom{n}{i}(2k)^{(n-i-1)}\right)+1$, which is odd and hence contradicts that n^m is even.

16.26 The proof is by contradiction. Thus, assume that r is rational and that $r^m = 2^{m-1}$. By definition of a rational number, there exist integers p and q with $q \neq 0$ such that $r = p/q$. It can further be assumed that p and q have no common divisor for, if they did, you could cancel the common divisor from both the numerator p and the denominator q. By substitution,

$r^m = (p/q)^m = 2^{m-1}$. Hence, $p^m = 2^{m-1}q^m$ which implies that p^m is even. Then from Exercise 16.26, it follows that p is even. As such, by definition, there exists an integer k such that $p = 2k$. By substitution, $(2k)^m = 2^{m-1}q^m$ which implies $2k^m = q^m$. This means that q^m is even and so again by Exercise 26, q is also even. It has thus been shown that p and q are even, but this contradicts the assumption that p and q have no common divisor.

16.27 The generalized proposition is that, if $S_1, \ldots, S_n$ are n convex sets, then $C = \cap_{i=1}^n S_i$ is a convex set. Proof. To show that C is convex, it must be shown that for all elements $x, y \in C$ and for all real numbers t with $0 \le t \le 1$, $tx + (1-t)y \in C$. Because of the keywords "for all" in the backward process, the choose method is now used to choose elements $x, y \in C$ and a real number t with $0 \le t \le 1$, for which it must be shown that $tx + (1-t)y \in C$. To that end, from the hypothesis that $S_1, \ldots, S_n$ are convex sets, you know by definition that for each $i = 1, \ldots, n$, for all elements $u, v \in S_i$ and for all real numbers s with $0 \le s \le 1$, $su + (1-s)v \in S_i$. Specializing the foregoing for-all statement with $u = x$, $v = y$, and $s = t$ (noting that $0 \le t \le 1$), it follows that for each $i = 1, \ldots, n$, $tx + (1-t)y \in S_i$. This, in turn, means that $tx + (1-t)y \in \cap_{i=1}^n S_i = C$ and so C is convex and the proof is complete.

16.28 The generalized proposition is that, if $f_1, \ldots, f_n$ are n convex functions, then $F = \sum_{i=1}^n f_i$ is a convex function. Proof. To show that F is a convex function, by definition, you must show that for all real numbers x, y, and t with $0 \le t \le 1$, $F(tx + (1-t)y) \le tF(x) + (1-t)F(y)$. Because of the keywords "for all" in the backward process, the choose method is now used to choose real numbers x, y, and t with $0 \le t \le 1$, for which it must be shown that $F(tx + (1-t)y) \le tF(x) + (1-t)F(y)$. To that end, from the hypothesis that $f_1, \ldots, f_n$ are convex functions, you know by definition that for each $i = 1, \ldots, n$, for all real numbers u, v, and s with $0 \le s \le 1$, $f_i(su + (1-s)v) \le sf_i(u) + (1-s)f_i(v)$. Specializing the foregoing for-all statement with $u = x$, $v = y$, and $s = t$ (noting that $0 \le t \le 1$), it follows that for each $i = 1, \ldots, n$, $f_i(tx + (1-t)y) \le tf_i(x) + (1-t)f_i(y)$. On adding up these inequalities over $i = 1, \ldots, n$, you have

$$\begin{aligned} F(tx + (1-t)y) &= \sum_{i=1}^n f_i(tx + (1-t)y) \\ &\le \sum_{i=1}^n [tf_i(x) + (1-t)f_i(y)] \\ &= t\sum_{i=1}^n f_i(x) + (1-t)\sum_{i=1}^n f_i(y) \\ &= tF(x) + (1-t)F(y). \end{aligned}$$

This means that F is a convex function and so the proof is complete.

17

Solutions to Exercises

17.1 a. (Answers may differ from the one given here.)

Similarities: All values are real numbers; all values are positive.

Differences: The values 0.75, 2/3, and 0.111... are rational numbers while $\sqrt{3}$ and π are irrational.

b. On the basis of the differences, let Group 1 consist of 0.75, 2/3, and 0.111... and Group 2 consist of $\sqrt{3}$ and π.

17.2 a. (Answers may differ from the one given here.)

Similarities: Solutions to all equations are positive real numbers.

Differences: All solutions to the equations in (ii), (iv), and (v) are integers while the solutions to the equations in (i) and (iii) are not integer.

b. On the basis of the differences, let Group 1 consist of the equations in (ii), (iv), and (v) and Group 2 consist of the equations in (i) and (iii).

17.3 a. (Answers may differ from the one given here.)

Similarities: All sets are subsets of real numbers.

Differences: The sets in (i), (iv), and (v) are intervals on the real line while the sets in (ii) and (iii) are not.

b. On the basis of the differences, let Group 1 consist of the sets in (i), (iv), and (v) and Group 2 consist of the sets in (ii) and (iii).

17.4 (Answers may differ from the one given here.)
The functions (i) and (iii) are polynomials while the functions in (ii) and (iv) are not, so let Group 1 consist of the functions in (i) and (iii) and Group 2 consist of the functions in (ii) and (iv).

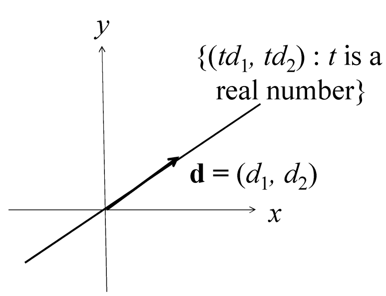
17.5 (Answers may differ from the one given here.)
The result of performing the operations in (i) and (iii) is a vector while the result of performing the operations in (ii) and (iv) is a number, so let Group 1 consist of the operations in (i) and (iii) and Group 2 consist of the operations in (ii) and (iv).

17.6 (Answers may differ from the one given here.)
The operations in (iii) and (v) are closed unary operations; the operations in (i), (iv), and (vii) are closed binary operations; and the operations in (ii) and (vi) are binary relations, so let Group 1 consist of the operations in (iii) and (v), Group 2 consist of the operations in (i), (iv) and (vii), and Group 3 consist of the operations in (ii) and (vi).

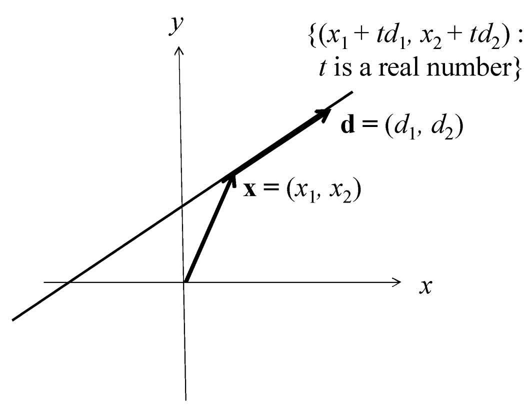
17.7 (Answers may differ from the one given here.)
Similarities: Both contain n values. Both sets and n-vectors can contain positive and negative values.

Differences: The values in n-vector are ordered while those in a set are not. The values in n-vector can be repeated but not in a set. The operations performed on sets are different from those performed on n-vectors.

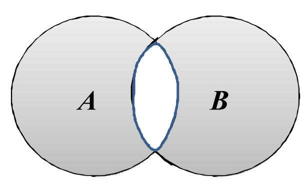
17.8 The following is the figure for $\{(td_1, td_2) : t \text{ is a real number}\}$:



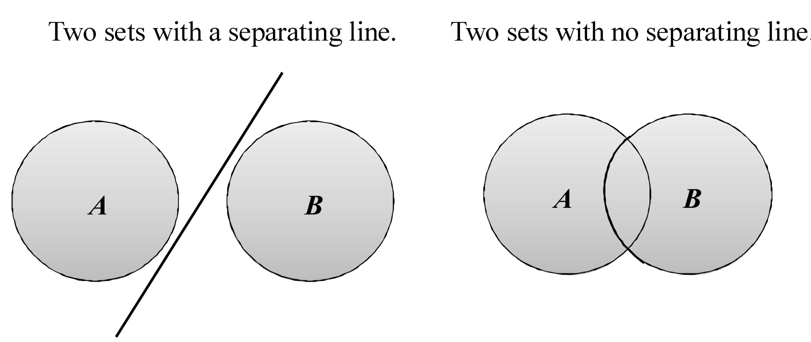
17.9 The following is the figure for $\{(x_1 + td_1, x_2 + td_2) : t \text{ is a real number}\}$:



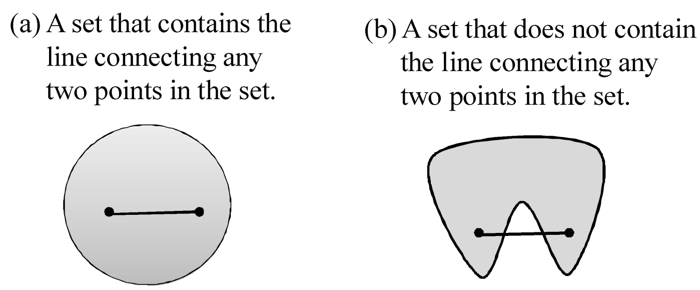
17.10 The shaded portion of the following figure represents the points in the set A that are not in the set B and the points in B that are not in A:



17.11 The two sets on the left side of the following figure have a separating line while the two sets on the right side do not have a separating line:

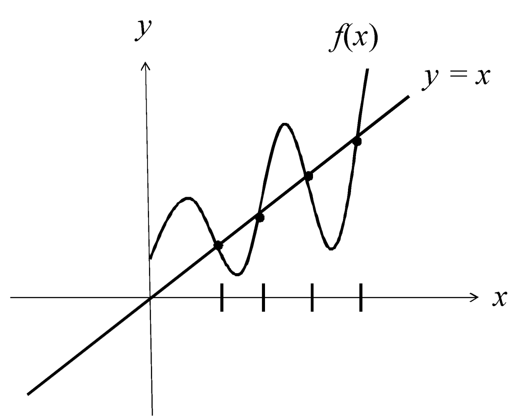


17.12 The solutions to parts (a) and (b) are given in the following figure:

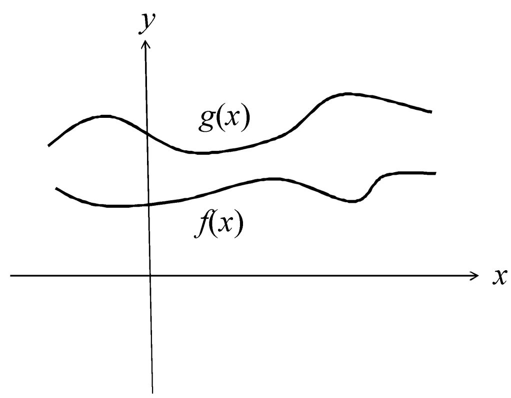


c. The set S in part (a) has the property that for all elements $x, y \in S$, and for all real numbers t with $0 \leq t \leq 1$, $(1-t)x + ty \in S$.

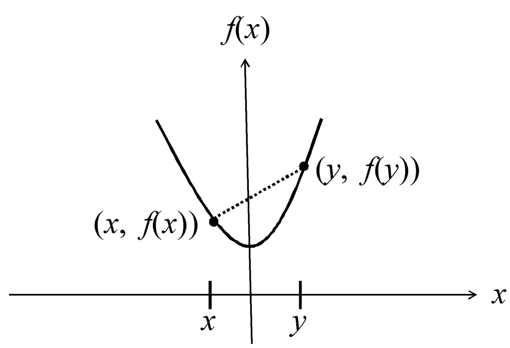
17.13 The vertical bars on the x-axis in the following figure indicate the points x^* for which $f(x^*) = x^*$, that is, where the graph of the function f crosses the line $y = x$:



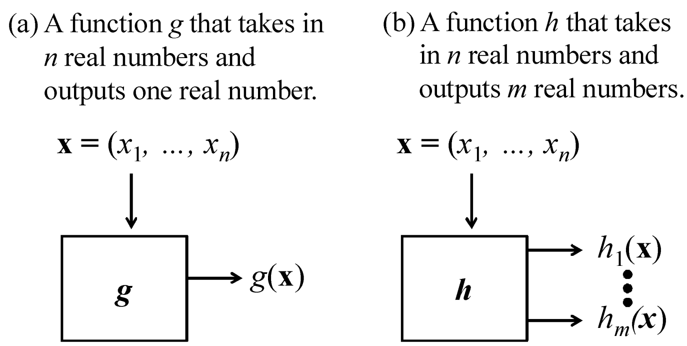
17.14 The following figure indicates real-valued functions f and g of one variable such that for all real numbers x, $g(x) \geq f(x)$:



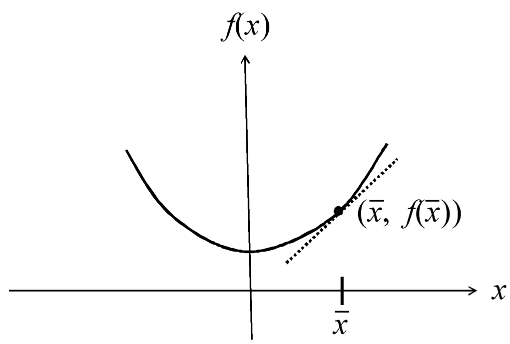
17.15 The following figure is a visual image of a real-valued function f of one variable with the property that for all real numbers x and y with $x < y$, the value of the function at all points between x and y lies below the line segment connecting the points $(x, f(x))$ and $(y, f(y))$:



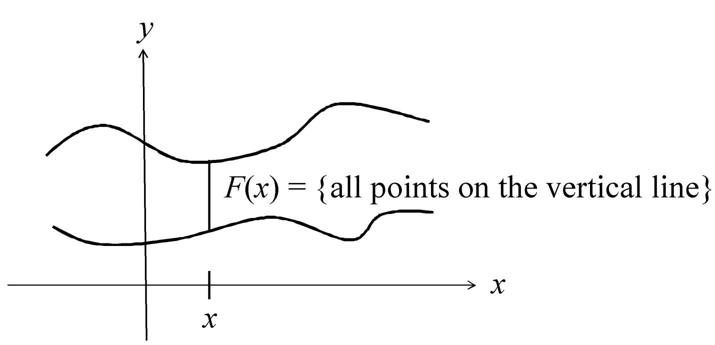
17.16 The solutions to parts (a) and (b) are given in the following figure:



17.17 The following figure is a visual image of a real-valued function f of one variable with the property that for all real numbers $\bar{x}$, the value of the function at all points x always lies above the tangent line to the graph of f at the point $(\bar{x}, f(\bar{x}))$:



17.18 The following figure is an image of a point-to-set map $F(x)$ in which for each real number x, $F(x)$ is an interval:



17.19 If f is the original real-valued function of one variable and c is a positive real number, then the real-valued function g defined by $g(x) = f(x) + c$ is the function f shifted up by the amount c.

17.20 The parabola $ax^2 + bx + c$ will cross the x-axis at two different points provided that $a \neq 0$ and $b^2 - 4ac > 0$.

17.21 The value of $y_i = 1 - x_i$ (when $x_i = 0, y_i = 1 - x_i = 1$ and when $x_i = 1, y_i = 1 - x_i = 0$).

17.22 The inequality $x_1 + x_2 + \cdots + x_n \geq 3$ ensures that at least three projects are funded.

17.23 The statement that for all elements $x \in S$, $x > u$ means that all values in the set S of real numbers are strictly to the right of the real number u.

17.24 The expression $(y_1 - x_1, y_2 - x_2)$ is the vector that points from (x_1, x_2) to (y_1, y_2).

17.25 a. **Verbal form:** The boundary of $B_1^2(c_1, c_2)$ is the set of points in the plane whose distance from (c_1, c_2) is 1.
Mathematical form: The boundary of $B_1^2(c_1, c_2) = \{(x_1, x_2) : (x_1 - c_1)^2 + (x_2 - c_2)^2 = 1\}$.
b. **Verbal form:** The boundary of $B_r^n(c_1, \ldots, c_n)$ is the set of points in the plane whose distance from $(c_1, \ldots, c_n)$ is r.
Mathematical form: The boundary of the set $B_r^n(c_1, \ldots, c_n) = \{(x_1, \ldots, x_n) : (x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 = r^2\}$.

17.26 **Verbal form:** The boundary of a set S in n dimensions is the set of points in n dimensions with the property that for every radius $r > 0$, the ball of radius r in n dimensions contains points in S and not in S.
Mathematical form: The boundary of a set S in n dimensions $= \{\mathbf{x} = (x_1, \ldots, x_n) :$ for every radius $r > 0, B_n^r(\mathbf{x}) \cap S \neq \emptyset$ and $B_n^r(\mathbf{x}) \cap S^c \neq \emptyset\}$.

17.27 **Verbal form:** A line separates the sets A and B in the plane if and only if every point in A is on one side of the line and every point in B is on the opposite side of the line.
Mathematical form: The line $a_1x_1 + a_2x_2 = b$ separates the sets A and B in the plane if and only if for every element $(x_1, x_2) \in A$, $a_1x_1 + a_2x_2 \leq b$ and for every element $(x_1, x_2) \in B$, $a_1x_1 + a_2x_2 > b$.

17.28 **Verbal form:** The sequence $x_1, x_2, \ldots$ is monotonically increasing if each subsequent value is larger than the previous value.
Mathematical form: The sequence $x_1, x_2, \ldots$ is monotonically increasing if $x_1 < x_2 < \cdots$ or equivalently if for every integer $i = 1, 2, \ldots$, it follows that $x_i < x_{i+1}$.

18

Solutions to Exercises

18.1 (Answers other than the ones given here are possible.) The result of performing abstraction is that each component of an n-vector is an object, for which three special cases are: (a) each component is a real number, (b) each component is a function, and (c) each component is a set.

18.2 (Answers other than the ones given here are possible.) The result of performing abstraction is that each of the inputs x and y are objects, as is the output of the function. Three special cases are: (a) the inputs are n-vectors and the output is an n-vector, (b) the inputs are functions and the output is a function, and (c) the inputs are sets and the output is a set.

18.3 When you replace the real number $x > 0$ with an object, the operation of taking the square root of an object is undefined. One way to eliminate this syntax error is to create a meaningful definition of what it means to take the square root of an object. For example, using the binary operator $\odot$, you could define the square root of an object x to be an object y for which $y \odot y = x$.

18.4 When you replace the real numbers x and y with objects, the meaning of "object $x \geq$ object y" is undefined. One way to eliminate this syntax error is to create a binary relation, say $\succeq$, to determine whether the object x is "greater than or equal to" the object y.

18.5 A unary operator v would have to return a positive real number for each object x, for then you could take the log of $v(x)$, that is, $\log(v(x))$ would then be defined.

18.6 a. When x is an object, the operation of raising an object to a positive integer power is undefined. However, using a closed binary operator $\odot$ on the set S of objects, you can create the following meaningful definition of x^j when x is an object and j is a positive integer:

$$x^j = \underbrace{x \odot \cdots \odot x}_{j \text{ times}}$$

b. When j is an object, the operation of raising a real number x to the power j is undefined. However, if a unary operator v associates to each object $j \in S$, a real number $v(j)$, then you can perform the operation $x^{v(j)}$.

18.7 The following tables show the results of performing the respective operations:

	0	**1**
0	0	0
1	0	1

	X	**Y**
X	X	X
Y	X	Y

18.8 (Answers other than the ones given here are possible.)

a. Performing abstraction means that you want to find $1/x$ when x is an object. A syntax error results because you cannot divide 1 by an object. You can eliminate the syntax error by using a unary operator v that associates to each object x in a set S, a nonzero real number, $v(x)$, for then you can compute $1/v(x)$. You therefore have the abstract system (S, v).

b. Performing abstraction means that, given two objects A and B, you want to find $A \cup B$. A syntax error results because the "union" of two objects is undefined. You can eliminate the syntax error by using a closed binary operator, say $\oplus$, that combines two objects A and B in a set S, for then $A \oplus B$. You therefore have the abstract system $(S, \oplus)$.

c. Performing abstraction means that, given an object y, you want to find an object x such that $f(x) = y$. A syntax error results because you cannot put an object x into the function f. You can eliminate the syntax error with a closed unary operator v that associates to each object x in a set S, a real number $v(x)$. In this case, given an object $y \in S$, you want to find an object $x \in S$ such that $f(v(x)) = v(y)$. You therefore have the abstract system (S, v).

18.9 (Answers other than the ones given here are possible.)

a. Performing abstraction means that, given two objects a and b, you want to determine if there is an object c such that $b = ca$. A syntax error results because you cannot multiply two objects c and a. You can eliminate the syntax error by using a closed binary operator, say $\odot$, that combines two objects c and a in a set S, for then $c \odot a$ is valid. You therefore have the abstract system $(S, \odot)$.

b. Performing abstraction means that, given two objects x and y, you want to compute $x \bullet y$. A syntax error results because the dot product of two objects is undefined. You can eliminate the syntax error by using a binary operator, say $\odot$, that combines two objects x and y in a set S and returns a real number, for then $x \odot y$ is valid. You therefore have the abstract system $(S, \odot)$.

c. Performing abstraction means that, given two objects s and t, you want to determine if $s > t$. A syntax error results because you cannot determine if one object is $>$ another object. You can eliminate the syntax error by using a binary relation, say $\succ$, to compare two objects s and t in a set S. You therefore have the abstract system $(S, \succ)$.

18.10 (Answers other than the ones given here are possible.)

a. Performing abstraction means that, for two given objects a and b, you want to find an object x such that $ax = b$. A syntax error results because you cannot multiply two objects a and x. You can eliminate the syntax error by using a closed binary operator, say $\odot$, that combines two objects a and x in a set S. You therefore have the abstract system $(S, \odot)$.

b. Performing abstraction means that you want to find an object x such that $f(x) = x$. A syntax error results because you cannot put an object x into the function f. You can eliminate the syntax error by using a unary operator v that associates to each object x, a real number $v(x)$. For the abstract system (S, v), you then want to find an object $x \in S$ such that $f(v(x)) = v(x)$.

c. Performing abstraction means that, for two objects a and b, you want to compute $\min\{a, b\}$. A syntax error results because you cannot compute the minimum of two objects. You can eliminate the syntax error by using a unary operator, say v, that takes an object $a \in S$ and returns a real number, for then the operation $\min\{v(a), v(b)\}$ is defined. You therefore have the abstract system (S, v).

18.11 (Answers other than the ones given here are possible.)

Axiom 1: This axiom has no syntax errors.

Axiom 2: This axiom has a syntax error in the expression $v(s+t)$ because you cannot add the objects s and t. You can eliminate the syntax error with a closed binary operator, say $\oplus$, on S because you can then rewrite the axiom as follows: For all elements $s, t \in S$, $v(s \oplus t) \leq v(x) + v(t)$.

Axiom 3: This axiom has a syntax error in the expression $v(s)^2 = s^2$ because you cannot square an object s. You can eliminate the syntax error with a binary operator, say $\odot$, that, for two objects $x, y \in S$, results in a real number $x \odot y$. You can then rewrite the axiom as follows: For every element $s \in S$, $v(s)^2 = s \odot s$.

18.12 (Answers other than the ones given here are possible.)

Axiom 1: This axiom has no syntax errors.

Axiom 2: This axiom has a syntax error in the expression $\vee(s,t) \geq s$ because you cannot compare the object $\vee(s,t)$ to the object s with $\geq$. To eliminate the syntax error, you can use a binary relation, say $\succeq$, instead of $\geq$ to compare two objects. You can then rewrite the axiom as follows: For all element $s, t \in S$, $\vee(s,t) \succeq s$.

18.13 (Answers other than the ones given here are possible.)

Axiom 1: This axiom has no syntax errors.

Axiom 2: This axiom has a syntax error in the expression $\emptyset \preceq x$ because you cannot compare the empty set to the object x with the binary relation $\preceq$. To eliminate the syntax error, you need an axiom to create the existence of an object with the same properties as $\emptyset$ and then you can rewrite the axiom as follows: There is an object $z \in S$ such that for every element $x \in S$, $z \preceq x$.

18.14 (Answers other than the one given here are possible.) A special case that satisfies the given axiom is $S =$ the set of all real-valued functions of one variable and $\oplus$ is the operation of adding two functions f and g to obtain the function $f \oplus g$ defined by $(f \oplus g)(x) = f(x) + g(x)$ which satisfies $f \oplus g = g \oplus f$.

18.15 (Answers other than the ones given here are possible). One special case that satisfies the given axiom is $S = \{$ real numbers$\}$ and $-$ is the unary operation of taking the negative of a real number. This special case satisfies the axiom because, for $\forall$ real number y, $\exists$ a real number x such that $y = -x$.

A second special case that satisfies the given axiom is $S =$ the set of all subsets of a universal set U and ! is the unary operation of taking the complement of a set. This special case satisfies the axiom because, for every subset S of U, there is a set $T \subseteq U$ such that $T = S^c$.

18.16 (Answers other than the ones given here are possible). One special case that satisfies the given axioms is $S =$ the set of real numbers and v is the unary operation of taking the absolute value of a number. This special case satisfies Axiom (i) because, for every real number x, $|x| \geq 0$. This special case satisfies Axiom (ii) because, there is a unique real number x such that $|x| = 0$, namely $x = 0$.

A second special case that satisfies the given axioms is $S =$ the set of all finite subsets of a set U and v is the unary operation of counting the number of elements in a finite subset T of U. This special case satisfies Axiom (i) because, for every finite subset T of U, $v(T) =$ the number of elements of T is ≥ 0. This special case satisfies Axiom (ii) because, there is a unique subset T of U for which $v(T) =$ the number of elements of $T = 0$, namely $T = \emptyset$.

18.17 (Answers other than the ones given here are possible). One special case that satisfies the given axioms is $S =$ the set of integers and $\odot$ is the closed binary operation of adding two integers. This special case satisfies Axiom (i) because, there is an integer e such that for all integers x, $x + e = e + x = x$, namely $e = 0$. This special case satisfies Axiom (ii) because, for every integer y, there is an integer x such that $x + y = y + x = 0$, namely $x = -y$.

A second special case that satisfies the given axioms is $S =$ the set of nonzero real numbers and $\odot$ is the closed binary operation of multiplying two nonzero real numbers. This special case satisfies Axiom (i) because, there is a nonzero real number e such that for all real numbers x, $xe = ex = x$, namely $e = 1$. This special case satisfies Axiom (ii) because, for every nonzero real number y, there is a nonzero real number x such that $xy = yx = 1$, namely $x = 1/y$.

18.18 An appropriate axiom is that there is an element $e \in S$ such that for every element $x \in S$, $x \vee e = e \vee x = x$.

18.19 An appropriate axiom is that there is an element $e \in S$ such that for every element $x \in S$, $x \wedge e = e \wedge x = e$.

18.20 An appropriate axiom is that for all elements $x, y, z \in S$, $(x \odot y) \odot z = x \odot (y \odot z)$.

18.21 For two objects $x, y \in S$, define $x|y$ to mean that there is an element $z \in S$ such that $y = z \odot x$.

18.22 The appropriate axiom is that for every element $x \in S$, there is an element $y \in S$ such that $y \odot y = x$.

18.23 (Answers other than the ones given here are possible.)

Axiom 1: For every element $x \in S$, $x \wedge x = x$.

Axiom 2: For all elements $x, y \in S$, $x \wedge y = y \wedge x$.

Axiom 3: There is an element $e \in S$ such that for all elements $x \in S$, $x \wedge e = e \wedge x = e$.

18.24 (Answers other than the ones given here are possible.)

Axiom 1: For every element $x \in S$, $x \vee x = x$.

Axiom 2: For all elements $x, y, z \in S$, $x \vee (y \vee z) = (x \vee y) \vee z$.

Axiom 3: There is an element $e \in S$ such that for all elements $x \in S$, $x \vee e = e \vee x = x$.

18.25 a. For any elements $x, y \in S$, if $x \preceq y$ and $y \preceq x$, then $x = y$.
b. There is an element $e \in S$ such that for every element $x \in S$, $e \preceq x$.
c. There is an element $z \in S$ such that for every element $x \in S$, $x \preceq z$.

18.26 a. For any elements $x_1, x_2 \in S$, either $x_1 \preceq x_2$ or $x_2 \preceq x_1$.
b. There is no element $z \in S$ such that for every element $x \in S$, $x \preceq z$.
c. For any elements $x_1, x_2 \in S$ with $x_1 \preceq x_2$, there is and element $y \in S$ different from x_1 and x_2 such that $x_1 \preceq y \preceq x_2$.

18.27 a. There is an element $z \in S$ such that for every element $x \in S$, $x \oplus z = z \oplus x = x$.
b. For every element $x \in S$, there is an element $-x \in S$ such that $x \oplus (-x) = (-x) \oplus x = z$.
c. A syntax error arises in the definition of $|x|$ because the relation $\geq$ is not defined for the abstract system $(S, \oplus)$ and so the expression $x \geq z$ does not make sense.

Appendix A
Solutions to Exercises

A.1 One common key question is, “How can I show that two sets are equal?” An associated answer to this question—obtained from the definition of two sets being equal—is, “Show that the first set is a subset of the second set and that the second set is a subset of the first set.”

A.2 One different key question is, “How can I show that the intersection of two sets is a subset of another set.” (Other answers are possible.)

A.3 **Analysis of Proof.** The forward-backward method is used to begin the proof because the hypothesis and conclusion do not contain keywords. A key question associated with the conclusion is, “How can I show that a set (namely, $A^c \cap B^c$) is a subset of another set (namely, $(A \cup B)^c$)?” Using the definition of a subset, one answer is to show that

B1: For every element $x \in A^c \cap B^c$, $x \in (A \cup B)^c$.

Recognizing the keywords “for every” in the backward statement $B1$, you should now use the choose method to choose

A1: An element $x \in A^c \cap B^c$,

for which it must be shown that

B1: $x \in (A \cup B)^c$.

A key question associated with $B1$ is, "How can I show that an element (namely, x) is in the complement of a set (namely, $A \cup B$)?" From the definition of the complement of a set, you must show that

B2: $x \notin A \cup B$.

Working forward now from $A1$, by definition of the intersection of two sets, it follows that

A2: $x \in A^c$ AND $x \in B^c$.

By the definition of the complement of a set, $A2$ means that

A3: $x \notin A$ AND $x \notin B$.

Using the rules of negation in Chapter 10, you can rewrite $A3$ as follows:

A4: NOT $[x \in A$ OR $x \in B]$.

According to the definition for the union of two sets, $A4$ means that

A5: NOT $[x \in A \cup B]$, that is, $x \notin A \cup B$.

The proof is now complete because $A5$ is the same as $B2$.

Proof. To see that $A^c \cap B^c \subseteq (A \cup B)^c$, let $x \in (A \cup B)^c$ (the word "let" here indicates that the choose method is used). Because $x \in A^c \cap B^c$, it follows that $x \in A^c$ and $x \in B^c$, that is, $x \notin A$ and $x \notin B$. This, in turn, means that it is not true that $x \in A$ or $x \in B$, that is, $x \notin A \cup B$, or, equivalently, that $x \in (A \cup B)^c$. The proof is now complete. □

A.4 **Analysis of Proof.** The forward-backward method is used to begin the proof because there are no keywords in either the hypothesis or conclusion. A key question associated with the conclusion is, "How can I show that a set (namely, $(A \cap B)^c$) is a subset of another set (namely, $A^c \cup B^c$)?" Using the definition of subset, one answer is to show that

B1: For every element $x \in (A \cap B)^c, x \in A^c \cup B^c$.

Recognizing the keywords "for every" in $B1$, you should now use the choose method to choose

A1: An element $x \in (A \cap B)^c$,

for which you must show that

B2: $x \in A^c \cup B^c$.

Working forward from $A1$, by definition of complement, you can say that

A2: $x \notin A \cap B$.

Now, from the definition for the intersection of two sets, $A2$ is equivalent to the statement NOT $[x \in A$ AND $x \in B]$. Using the techniques in Chapter 8, you can rewrite this statement as follows:

A3: $x \notin A$ OR $x \notin B$.

Again using the definition of the complement of a set, $A3$ becomes

A4: $x \in A^c$ OR $x \in B^c$.

Finally, by the definition for the union of two sets, from $A4$ you have

A5: $x \in A^c \cup B^c$.

The proof is now complete because $A5$ is the same as $B2$.

A.5 **Analysis of Proof.** The forward-backward method is used to begin the proof because there are no keywords in either the hypothesis or conclusion. A key question associated with the conclusion is, "How can I show that a set (namely, $A^c \cup B^c$) is a subset of another set (namely, $(A \cap B)^c$)?" Using the definition of subset, one answer is to show that

B1: For every element $x \in A^c \cup B^c$, $x \in (A \cap B)^c$.

Recognizing the keywords "for every" in $B1$, you should now use the choose method, as the author does, to choose

A1: An element $x \in A^c \cup B^c$,

for which you must show that

B2: $x \in (A \cap B)^c$.

Working forward from $A1$, by definition of the union of two sets, you have

A2: $x \in A^c$ OR $x \in B^c$.

Now, from the definition for the complement of a set, $A2$ is equivalent to

A3: $x \notin A$ OR $x \notin B$.

Using the rules for negation in Chapter 10, you can rewrite $A3$ as follows:

A4: NOT $[x \in A$ AND $x \in B]$.

By the definition for the intersection of two sets, from $A4$ you have

A5: NOT $[x \in A \cap B]$, that is, $x \notin A \cap B$.

Finally, by the definition of the complement of a set, $A5$ means that

A6: $x \in (A \cap B)^c$.

The proof is now complete because $A6$ is the same as $B2$.

A.6 **Analysis of Proof.** The forward-backward method is used to begin the proof because there are no keywords in either the hypothesis or conclusion. A key question associated with the conclusion is, "How can I show that a set [namely, $(A \cup B) \cap (A \cup C)$] is a subset of another set [namely, $A \cup (B \cap C)$]?" Using the definition for subset, one answer is to show that

B1: For every element $x \in (A \cup B) \cap (A \cup C)$, $x \in A \cup (B \cap C)$.

Recognizing the keywords "for every" in the backward statement $B1$, you should use the choose method to choose

A1: An element $x \in (A \cup B) \cap (A \cup C)$,

for which it must be shown that

B3: $x \in A \cup (B \cap C)$.

A key question associated with $B3$ is, "How can I show that an element (namely, x) is in the union of two sets (namely, A and $B \cap C$)?" Answering this question using the definition, you must show that

B4: $x \in A$ OR $x \in B \cap C$.

Turning to the forward process, from $A1$ and the definition for the intersection of two sets, you can say that

A2: $x \in A \cup B$ AND $x \in A \cup C$.

From the first statement in $A2$ and the definition for the union of two sets, you know that

A3: $x \in A$ OR $x \in B$.

Recognizing the keywords "either/or" in the forward statement $A3$, you should now use a proof by cases. Accordingly, you must first assume that $x \in A$ and show that $B4$ is true and then assume that $x \in B$ and show that $B4$ is true, as is done now.

Case 1: Assume that $x \in A$ (for which it will be shown that $B4$ is true).
Because $x \in A$, $x \in A$ or $x \in B \cap C$, so $B4$ is true. This leaves ...

Case 2: Assume that $x \in B$ (for which it will be shown that $B4$ is true).
From the second statement in $A2$, you know that $x \in A \cup C$. By the definition for the union of two sets this means that

A4: $x \in A$ OR $x \in C$.

Proceeding again by cases, if $x \in A$, $B4$ is true, so you can assume that $x \in C$. You now have that $x \in B$ and $x \in C$, which means by definition of the intersection of two sets that $x \in B \cap C$, and so $B4$ is again true. The proof is now complete.

Proof. To see that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$, let $x \in (A \cup B) \cap (A \cup C)$ (the word "let" here indicates that the choose method is used). It will be shown that $x \in A \cup (B \cap C)$. Because $x \in (A \cup B) \cap (A \cup C)$, by definition, $x \in A \cup B$ and $x \in A \cup C$. The first of these statements means that $x \in A$ or $x \in B$. If $x \in A$ (Case 1 in the foregoing Analysis of Proof), then it follows that $x \in A$ or $x \in B \cap C$ and so the proof would be finished. On the other hand, if $x \in B$ (Case 2 in the foregoing Analysis of Proof), then you also know that $x \in A$ or $x \in C$. If $x \in A$, then $x \in A \cup (B \cap C)$ while if $x \in C$, then you have that $x \in B \cap C$ and again $x \in A \cup (B \cap C)$. Thus, in all cases, $x \in A \cup (B \cap C)$, and so the proof is complete. □

A.7 **Analysis of Proof.** The forward-backward method is used to begin the proof because there are no keywords in either the hypothesis or conclusion. A key question associated with the conclusion is, "How can I show that a set [namely, $A \cap (B \cup C)$] is equal to another set [namely, $(A \cap B) \cup (A \cap C)$]?" Using the definition for equality of two sets, one answer is to show that

B1: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and also
$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

A key question associated with the first statement in $B1$ is, "How can I show that a set [namely, $A \cap (B \cup C)$] is a subset of another set [namely, $(A \cap B) \cup (A \cap C)$]?" Using the definition for subset, one answer is to show that

B2: For every element $x \in A \cap (B \cup C)$, $x \in (A \cap B) \cup (A \cap C)$.

Recognizing the keywords "for every" in the backward statement $B2$, you should use the choose method to choose

A1: An element $x \in A \cap (B \cup C)$,

for which it must be shown that

B3: $x \in (A \cap B) \cup (A \cap C)$.

A key question associated with $B3$ is, "How can I show that an element (namely, x) is in the union of two sets (namely, $A \cap B$ and $A \cap C$)?" Answering this question using the definition, you must show that

B4: $x \in A \cap B$ OR $x \in A \cap C$.

Turning to the forward process, from $A1$ and the definition for the intersection of two sets, you can say that

A2: $x \in A$ AND $x \in B \cup C$.

Because $x \in A$ from $A2$, you can establish $B4$ if you can show that

B5: $x \in B$ OR $x \in C$.

However, $B5$ is true from the second statement in $A2$ according to the definition of the union of two set.

Thus, from $B1$, it remains to show that

B6: $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

A key question associated with $B6$ is, "How can I show that a set [namely, $(A \cap B) \cup (A \cap C)$] is a subset of another set [namely, $A \cap (B \cup C)$]?" Using the definition for subset, one answer is to show that

B7: For every element $x \in (A \cap B) \cup (A \cap C)$, $x \in A \cap (B \cup C)$.

Recognizing the keywords "for every" in the backward statement $B7$, you should use the choose method to choose

A3: An element $x \in (A \cap B) \cup (A \cap C)$,

for which it must be shown that

B8: $x \in A \cap (B \cup C)$.

A key question associated with $B8$ is, "How can I show that an element (namely, x) is in the intersection of two sets (namely, A and $B \cup C$)?" Answering this question using the definition, you must show that

B9: $x \in A$ AND $x \in B \cup C$.

Turning to the forward process, from $A3$ and the definition for the union of two sets, you can say that

A4: $x \in A \cap B$ OR $x \in A \cap C$.

Recognizing the keywords *either/or* in $A4$, you can now use a proof by cases, as follows.

Case 1: Assume that $x \in A \cap B$ (it will be shown that $B9$ is true). Because $x \in A \cap B$, by definition of the intersection of two sets, you have that

A5: $x \in A$ AND $x \in B$.

But this means that $x \in A$ and also that $x \in B$ or $x \in C$, which in turn means that $B9$ is true. This leaves ...

Case 2: Assume that $x \in A \cap C$ (it will be shown that $B9$ is true). Because $x \in A \cap C$, by definition of the intersection of two sets, you have that

A6: $x \in A$ AND $x \in C$.

But this means that $x \in A$ and also that $x \in C$ or $x \in B$, which in turn means that $B9$ is true.

It has now been shown that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and so the proof is complete. □

Proof. To see that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, it is first shown that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and subsequently that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. To that end, let $x \in A \cap (B \cup C)$ (the word "let" here indicates that the choose method is used). But then $x \in A$ and $x \in B \cup C$, that is, $x \in A$ and either $x \in B$ or $x \in C$. In the case that $x \in B$, you have $x \in A$ and $x \in B$, so $x \in A \cap B$. In the case that $x \in C$, you have $x \in A$ and $x \in C$, so $x \in A \cap C$. Thus, in either case, $x \in (A \cap B) \cup (A \cap C)$.

It remains to show that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. To that end, let $x \in (A \cap B) \cup (A \cap C)$ (the word "let" here indicates that the choose method is used). But then $x \in A \cap B$ or $x \in A \cap C$. In the first case, $x \in A$ and $x \in B$, so $x \in A \cap (B \cup C)$. In the second case, $x \in A$ and $x \in C$, so $x \in A \cap (B \cup C)$. Thus, in either case, $x \in A \cap (B \cup C)$ and so $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. It now follows that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and so the proof is complete. □

A.8 **Analysis of Proof.** The first step of induction is to show that the statement is true for $n = 1$ which, for this exercise, means you must show that the following statement is true:

P(1): If the sets S and T each have $n = 1$ element, then $S \times T$ has $1^2 = 1$ element.

Now $P(1)$ is true because if $S = \{s\}$ and $T = \{t\}$, then $S \times T = \{(s,t)\}$, which has exactly one element.

The next step of induction is to assume that the statement is true for n, that is, assume that

P(n): If the sets S and T each have n elements, then $S \times T$ has n^2 elements.

You must then show that the statement is true for $n + 1$, that is, you must show that

P(n+1): If the sets S and T each have $n + 1$ elements, then $S \times T$ has $(n + 1)^2 = n^2 + 2n + 1$ elements.

The key to a proof by induction is to write $P(n + 1)$ in terms of $P(n)$. To that end, suppose that

$$S = \{s_1, \ldots, s_n, s_{n+1}\} \quad \text{and} \quad T = \{t_1, \ldots, t_n, t_{n+1}\}.$$

Letting S' and T' be the sets consisting of the first n elements of S and T, respectively, from the induction hypothesis you know that $S' \times T'$ has n^2

elements. It remains to show that $S \times T$ contains $2n+1$ elements in addition to the n^2 elements of $S' \times T'$. But, the remaining elements of $S \times T$ consist of the following. For the element $s_{n+1} \in S$, you can construct the following additional n elements of $S \times T$: $(s_{n+1}, t_1), \ldots, (s_{n+1}, t_n)$. Likewise, for the element $t_{n+1} \in T$, you can construct the following additional n elements of $S \times T$: $(s_1, t_{n+1}), \ldots, (s_n, t_{n+1})$. The final element of $S \times T$ is (s_{n+1}, t_{n+1}). The proof is now complete.

Proof. If the sets S and T each have $n = 1$ elements, then $S \times T$ has $1^2 = 1$ element because if $S = \{s\}$ and $T = \{t\}$, then $S \times T = \{(s,t)\}$, which has exactly one element.

Assume now that the statement is true for n. It must be shown that if the sets S and T each have $n+1$ elements, then $S \times T$ has $(n+1)^2 = n^2 + 2n + 1$ elements. To that end, letting S' and T' be the sets consisting of the first n elements of S and T, respectively, you can count the following number of elements in $S \times T$:

$S' \times T'$	n^2 elements by the induction hypothesis
$(s_{n+1}, t_1), \ldots, (s_{n+1}, t_n)$	n elements
$(s_1, t_{n+1}), \ldots, (s_n, t_{n+1})$	n elements
(s_{n+1}, t_{n+1})	1 element

Putting together the pieces, $S \times T$ has $n^2 + n + n + 1 = (n+1)^2$ elements, and so the proof is complete. □

A.9 **Analysis of Proof.** The first step of induction is to show that the statement is true for $n = 1$ which, for this exercise, means you must show that the following statement is true:

P(1): If the set T has $n = 1$ element, then 2^T has $2^1 = 2$ elements.

Now $P(1)$ is true because if $T = \{t\}$, then $2^T = \{\{t\}, \emptyset\}$, which has exactly two elements.

The next step of induction is to assume that the statement is true for n, that is, assume that

P(n): If the set T has n elements, then 2^T has 2^n elements.

You must then show that the statement is true for $n+1$, that is, you must show that

P(n+1): If the set T has $n+1$ elements, then 2^T has 2^{n+1} elements.

The key to a proof by induction is to write $P(n+1)$ in terms of $P(n)$. To that end, suppose that

$$T = \{t_1, \ldots, t_n, t_{n+1}\} \quad \text{and} \quad T' = \{t_1, \ldots, t_n\}.$$

The subsets of T consist of those that do not contain the element t_{n+1} and those subsets that do contain the element t_{n+1}. The subsets of T that do not contain the element t_{n+1} are precisely the subsets of the set T'. By the induction hypothesis, there are 2^n such subsets because T' has n elements. The remaining subsets of T are those that contain the element t_{n+1}. Specifically, for each subset S' of T', you can create the subset $S' \cup \{t_{n+1}\}$ of T. Thus, in total, 2^T has 2^n elements from subsets of T' plus another 2^n elements corresponding to those subsets of T that contain t_{n+1}. That is, 2^T has $2^n + 2^n = 2^{n+1}$ elements, and so the proof is now complete.

Proof. If the set T has $n = 1$ element, then 2^T has $2^1 = 2$ elements because if $T = \{t\}$, then $2^T = \{\{t\}, \emptyset\}$, which has exactly two elements.

Assume now that the statement is true for n. It must be shown that if the set T has $n + 1$ elements, say, $T = \{t_1, \ldots, t_n, t_{n+1}\}$, then 2^T has 2^{n+1} elements. Now, some of the subsets of T do not contain the element t_{n+1} while the remaining subsets of T do contain t_{n+1}. Letting $T' = \{t_1, \ldots, t_n\}$, the induction hypothesis leads to 2^n subsets of T', each of which does not contain t_{n+1}. Then, for each of the 2^n subsets S' of T', you can construct the subset $S' \cup \{t_{n+1}\}$ of T that contains t_{n+1}. Putting together the two pieces, 2^T has $2^n + 2^n = 2^{n+1}$ elements, and so the proof is complete. $\square$

A.10 The function $f : A \to B$ is not surjective if and only if there is an element $y \in B$ such that for every element $x \in A$, $f(x) \neq y$. The statement is obtained by using the rules in Chapter 8 to negate the definition of f being surjective, as follows:

> NOT[For every element $y \in B$, there is an element $x \in A$ such that $f(x) = y$].
>
> There is an element $y \in B$ such that NOT[there is an element $x \in A$ such that $f(x) = y$].
>
> There is an element $y \in B$ such that for every element $x \in A$, NOT[$f(x) = y$].
>
> There is an element $y \in B$ such that for every element $x \in A$, $f(x) \neq y$.

A.11 The function $f : A \to B$ is not injective if and only if there are elements $u, v \in A$ with $u \neq v$ such that $f(u) = f(v)$. The statement is obtained by using the rules in Chapter 10 to negate the definition of f being injective, as follows:

> NOT[For all elements $u, v \in A$ with $u \neq v$, $f(u) \neq f(v)$].
>
> There are elements $u, v \in A$ with $u \neq v$ such that NOT [$f(u) \neq f(v)$].
>
> There are elements $u, v \in A$ with $u \neq v$ such that $f(u) = f(v)$.

A.12 Yes, the function $f : R \rightarrow R^+ = \{x \in R : x \geq 0\}$ defined by $f(x) = |x|$ is surjective because of the following proof.

Analysis of Proof. From the backward process, a key question associated with the conclusion is, "How can I show that a function (namely, $f(x) = |x|$) is surjective?" One answer is by the definition, so you must show that

B1: For every element $y \in R^+$, there is an element $x \in R$ such that $f(x) = |x| = y$.

Recognizing the keywords "for every" in $B1$, you should use the choose method to choose

A1: An element $y \in R^+$,

for which you must show that

B2: There is an element $x \in R$ such that $|x| = y$.

Recognizing the keywords "there is" in the backward statement $B2$, you should now use the construction method to construct the value of $x \in R$ such that $|x| = y$. However, because from $A1$ $y \in R^+ = \{x \in R : x \geq 0\}$, you know that $y \geq 0$. Thus, you can

A2: Construct $x = y$.

According to the construction method, you must show that the value of x in $A2$ satisfies the certain property ($x \in R$) and the something that happens ($|x| = y$) in $B2$. Now $x \in R$ because $x = y$ is a real number. Finally, because $x = y \geq 0$, it follows that $|x| = |y| = y$. The proof is now complete.

Proof. To show that $f : R \rightarrow R^+$ defined by $f(x) = |x|$ is surjective, let $y \in R^+$. (The word "let" here indicates that the choose method is used.) It must be shown that there is an element $x \in R$ such that $|x| = y$. To that end, let $x = y$ and y is a real number. (The word "let" here indicates that the construction method is used.) But because $x = y \geq 0$, it follows that $|x| = |y| = y$. □

A.13 The function $f : R \rightarrow S = \{x \in R : x \geq 0\}$ defined by $f(x) = |x|$ is not injective because of the following proof.

Analysis of Proof. From the backward process, a key question associated with the conclusion is, "How can I show that a function (namely, $|x|$) is not surjective?" (An alternative approach is to use the contradiction or contrapositive method). One answer is by writing the negation of the definition of an injective function—as was done in the solution to Exercise A.11—so, you must show that

B1: There are elements $u, v \in R$ with $u \neq v$ such that $|u| = |v|$.

Recognizing the keywords "there are" in the backward statement $B1$, you should use the construction method to construct real numbers u and v with $u \neq v$ such that $|u| = |v|$. Specifically, you can

A1: Construct $u = 1$ and $v = -1$.

It is easy to see that $u \neq v$ and also that $|u| = |1| = 1 = |-1| = |v|$, so the proof is complete.

Proof. To see that the function $f : R \to S = \{x \in R : x \geq 0\}$ defined by $f(x) = |x|$ is not injective, let $u = 1$ and $v = -1$ (the word *let* here indicates that the construction method is being used). Then clearly $u \neq v$ and $|u| = |1| = 1 = |-1| = |v|$, so the proof is complete. $\square$

A.14 The desired condition for $f(x) = ax + b$ to be injective is that $a \neq 0$, as established in the following proof.

Analysis of Proof. A key question associated with the conclusion is, "How can I show that a function (namely, $f(x) = ax + b$) is injective?" Using the definition, one answer is to show that

B1: For all real numbers u and v with $u \neq v$, $au + b \neq av + b$.

Recognizing the keywords "for all" in the backward statement $B1$, you should now use the choose method to choose

A1: Real numbers u and v with $u \neq v$,

for which you must show that

B2: $au + b \neq av + b$.

Recognizing the keyword "not" in $B2$ (and also in $A1$), you should now consider using either the contradiction or contrapositive method. Here, the contrapositive method is used. So, you can assume that

A2 (NOT B2): $au + b = av + b$.

According to the contrapositive method, you must now show that

B3 (NOT A1): $u = v$.

You can obtain $B3$ by working forward from $A2$ and the assumption that $a \neq 0$, as follows:

$$\begin{array}{ll} au + b = av + b & \text{(from } A2\text{)} \\ au = av & \text{(subtract } b \text{ from both sides)} \\ u = v & \text{(divide both sides by } a \neq 0\text{).} \end{array}$$

The proof is now complete.

Proof. To show that the function $f(x) = ax + b$ is injective, assuming that $a \neq 0$, let u and v be real numbers with $f(u) = f(v)$, that is, $au + b = av + b$. It will be shown that $u = v$, but this is true because

$$\begin{aligned} au + b &= av + b && \text{(from } A2\text{)} \\ au &= av && \text{(subtract } b \text{ from both sides)} \\ u &= v && \text{(divide both sides by } a \neq 0\text{).} \end{aligned}$$

The proof is now complete. □

A.15 The desired condition for $f(x) = ax + b$ to be surjective is that $a \neq 0$, as established in the following proof.

Analysis of Proof. A key question associated with the conclusion is, "How can I show that a function (namely, $f(x) = ax + b$) is surjective?" Using the definition, one answer is to show that

B1: For all real numbers y, there is a real number x such that $ax + b = y$.

Recognizing the keywords "for all" in the backward statement $B1$, you should now use the choose method to choose

A1: A real number y,

for which you must show that

B2: There is a real number x such that $ax + b = y$.

Recognizing the keywords "there is" in the backward statement $B2$, you should now use the construction method to construct the desired value for x. To do so, solve the equation in $B2$ for x using the fact that $a \neq 0$:

$$x = \frac{y - b}{a}.$$

It is easy to see that this constructed value of x is a real number that satisfies $ax + b = y$ in $B2$, and so the proof is complete.

Proof. To show that the function $f(x) = ax + b$ is surjective, assuming that $a \neq 0$, let y be a real number. It must be shown that there is a real number x such that $ax + b = y$. However, using the fact that $a \neq 0$, the desired value of x is $x = (y - b)/a$ because

$$ax + b = a\frac{y - b}{a} + b = y - b + b = y.$$

The proof is now complete. □

A.16 **Analysis of Proof.** The keyword "unique" in the conclusion suggests using the backward uniqueness method. Accordingly, it is first necessary to construct a fixed point x_* of f. However, this fixed point is given in the hypothesis. It remains to show that there is at most one fixed point of f. Here, the author uses the indirect uniqueness method to do so and thus assumes that, in addition to the fixed point x_* of f,

A1: $y_* \neq x_*$ is also a fixed point of f.

The author now works forward from $A1$, the hypothesis, and the fact that x_* is a fixed point of f to reach the contradiction that $\alpha \geq 1$. Specifically, the author specializes the hypothesis that for all real numbers x and y, $|f(y) - f(x)| \leq \alpha|y - x|$ to the values $y = y_*$ and $x = x_*$, to obtain

A2: $|f(y_*) - f(x_*)| \leq \alpha|y_* - x_*|$.

The author now works forward from the fact that x_* and y_* are fixed points of f to obtain

$$\begin{array}{rcll} |x_* - y_*| & = & |f(x_*) - f(y_*)| & \text{(because } x_* \text{ and } y_* \text{ are fixed points of } f) \\ & \leq & \alpha|x_* - y_*| & \text{(from } A2). \end{array}$$

The author then divides both sides of the foregoing inequality by the number $|x_* - y_*|$, which is > 0 because $x_* \neq y_*$ (see $A1$), to obtain that $\alpha \geq 1$. But this contradicts the hypothesis that $\alpha < 1$, and so the proof is complete.

A.17 **Analysis of Proof.** The author uses a proof by induction and, accordingly, begins by verifying that the statement is true for the first value of the integer n, namely, $n = 0$. To that end, the author verifies that

$$|x_0 - x_*| = \alpha^0|x_0 - x_*|.$$

The author then correctly assumes the induction hypothesis that

A1: $|x_n - x_*| \leq \alpha^n|x_0 - x_*|$.

According to the induction method, the author must now verify the statement for $n + 1$, that is, that

B1: $|x_{n+1} - x_*| \leq \alpha^{n+1}|x_0 - x_*|$.

The author starts with the left side of the inequality in $B1$ and first states that

A2: $|x_{n+1} - x_*| = |f(x_n) - f(x_*)|$.

Now $A2$ is true because, from the hypothesis, $x_{n+1} = f(x_n)$ and also x_* is a fixed point of f, so $x_* = f(x_*)$. The author then specializes the hypothesis that for all real numbers x and y, $|f(y) - f(x)| \leq \alpha|y - x|$ to the values $y = x_n$ and $x = x_*$, to obtain

A3: $|f(x_n) - f(x_*)| \leq \alpha|x_n - x_*|$.

The author then uses the induction hypothesis to conclude that

A4: $\alpha|x_n - x_*| \leq \alpha(\alpha^n|x_0 - x_*|)$.

Finally, the author uses algebra to rewrite the right side of the last inequality as $\alpha^{n+1}|x_0 - x_*|$. The proof is now complete because the author has shown from $A2$, $A3$, and $A4$, that $|x_{n+1} - x_*| \leq \alpha^{n+1}|x_0 - x_*|$. $\square$

Appendix B
Solutions to Exercises

B.1 a. A key question associated with the properties in (j), (k), and (l) in Table B.1 is, "How can I show that two real numbers are equal?" This is because the dot product of two n-vectors is a real number.

b. Using the definition, one answer to the key question, "How can I show that two n-vectors are equal?" is to show that all n components of the two vectors are equal. Applying this answer to the n-vectors $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ means you must show that for every integer $i = 1, \ldots, n$, $(\mathbf{x}+\mathbf{y})_i = (\mathbf{y}+\mathbf{x})_i$, that is, you must show that

B1: For every integer $i = 1, \ldots, n$, $x_i + y_i = y_i + x_i$.

c. Based on the answer in part (b), the next technique you should use is the choose method because the keywords "for all" appear in the backward statement $B1$. For that statement, you would choose

A1: An integer i with $1 \leq i \leq n$,

for which you must show that

B2: $x_i + y_i = y_i + x_i$.

B.2 **Analysis of Proof.** Following the approach in Exercise B.1, a key question associated with showing that $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ is, "How can I show that two n-vectors are equal?" By definition, one answer is to show that all components are equal; that is,

B1: For every integer $i = 1, \ldots, n$, $(\mathbf{x} + \mathbf{y})_i = (\mathbf{y} + \mathbf{x})_i$; that is, $x_i + y_i = y_i + x_i$.

Recognizing the keywords "for all" in the backward statement $B1$, you should proceed with the choose method and choose

A1: An integer i with $1 \leq i \leq n$,

for which you must show that

B2: $x_i + y_i = y_i + x_i$.

However, $B2$ is true because of the commutative property of addition of real numbers, and so the proof is complete.

Proof. To see that $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$, let i be an integer with $1 \leq i \leq n$. (The word "let" here indicates that the choose method is used.) But then you have that $(\mathbf{x} + \mathbf{y})_i = x_i + y_i = y_i + x_i = (\mathbf{y} + \mathbf{x})_i$ by the commutative property of addition of real numbers, and so the proof is complete. $\square$.

B.3 **Analysis of Proof.** The forward-backward method is used to begin the proof because the hypothesis and conclusion do not contain any keywords. A key question associated with showing that $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ is, "How can I show that two n-vectors are equal?" By definition, one answer is to show that all components are equal, that is,

B1: For every integer $i = 1, \ldots, n$, $[(\mathbf{x}+\mathbf{y})+\mathbf{z}]_i = [\mathbf{x}+(\mathbf{y}+\mathbf{z})]_i$, that is, $(\mathbf{x} + \mathbf{y})_i + \mathbf{z}_i = \mathbf{x}_i + (\mathbf{y} + \mathbf{z})_i$, that is, $(x_i + y_i) + z_i = x_i + (y_i + z_i)$.

Recognizing the keywords "for all" in the backward statement $B1$, you should proceed with the choose method and choose

A1: An integer i with $1 \leq i \leq n$,

for which you must show that

B2: $(x_i + y_i) + z_i = x_i + (y_i + z_i)$.

However, $B2$ is true because of the associative property of addition of real numbers, and so the proof is complete.

Proof. To see that $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$, let i be an integer with $1 \leq i \leq n$ (the word "let" here indicates that the choose method is used). But then you have that $(x_i + y_i) + z_i = x_i + (y_i + z_i)$ by the associative property of addition of real numbers, and so the proof is complete. $\square$.

B.4 **Analysis of Proof.** The forward-backward method is used to begin the proof because the hypothesis and conclusion do not contain any keywords. A key question associated with showing that $t(\mathbf{x} + \mathbf{y}) = t\mathbf{x} + t\mathbf{y}$ is, "How can I show that two n-vectors (namely, $t(\mathbf{x} + \mathbf{y})$ and $t\mathbf{x} + t\mathbf{y}$) are equal?" By definition, one answer is to show that all components are equal, that is,

> **B1:** For every integer $i = 1, \ldots, n$, $[t(\mathbf{x}+\mathbf{y})]_i = (t\mathbf{x} + t\mathbf{y})_i$.

Recognizing the keywords "for all" in the backward statement $B1$, you should proceed with the choose method and choose

> **A1:** An integer i with $1 \leq i \leq n$,

for which you must show that

> **B2:** $[t(\mathbf{x}+\mathbf{y})]_i = (t\mathbf{x} + t\mathbf{y})_i$.

However, $B2$ is true because $[t(\mathbf{x}+\mathbf{y})]_i = t(\mathbf{x}+\mathbf{y})_i = t(x_i + y_i) = tx_i + ty_i = (t\mathbf{x} + t\mathbf{y})_i$ and so the proof is complete.

B.5 Recognizing the keyword *unique* in the conclusion of the proposition, the direct uniqueness method is used. According, in addition to the real numbers $t_1, \ldots, t_k$ in the hypothesis that satisfy

> **A1:** $\mathbf{x} = t_1\mathbf{x}^1 + \cdots + t_k\mathbf{x}^k$,

the author assumes that

> **A2:** There are real numbers $s_1, \ldots, s_k$ that also satisfy
> $\mathbf{x} = s_1\mathbf{x}^1 + \cdots + s_k\mathbf{x}^k$.

According to the direct uniqueness method, it must now be shown that

> **B1:** $s_1 = t_1, \ldots, s_k = t_k$.

The author now works forward to do so. Specifically, from $A1$ and $A2$, the author concludes that

> **A3:** $s_1\mathbf{x}^1 + \cdots + s_k\mathbf{x}^k = t_1\mathbf{x}^1 + \cdots + t_k\mathbf{x}^k$.

Using the properties of vector operations in Table B.1, the author rewrites $A3$ as follows:

> **A4:** $(s_1 - t_1)\mathbf{x}^{\mathbf{1}} + \cdots + (s_k - t_k)\mathbf{x}^{\mathbf{k}} = \mathbf{0}$.

The author then works forward by definition from the hypothesis that the vectors $\mathbf{x}^1, \ldots, \mathbf{x}^k$ are linearly independent to state that

A5: For all real numbers $a_1, \ldots, a_k$ with $a_1\mathbf{x}^1 + \cdots + a_k\mathbf{x}^k = \mathbf{0}$, it follows that $a_1 = \cdots = a_k = 0$.

Recognizing the keywords "for all" in the forward statement $A5$, the author specializes this statement with $a_1 = s_1 - t_1, \ldots, a_k = s_k - t_k$. Noting from $A4$ that these values satisfy the certain property in $A5$ that $a_1\mathbf{x}^1 + \cdots + a_k\mathbf{x}^k = \mathbf{0}$, the result of specialization is

A6: $s_1 - t_1 = 0, \ldots, s_k - t_k = 0$, that is, $s_1 = t_1, \ldots, s_k = t_k$.

The proof is now complete because $A6$ is the same as $B1$.

B.6 **Analysis of Proof.** The forward-backward method is used to begin the proof because the hypothesis and conclusion do not contain any keywords. A key question associated with the conclusion is, "How can I show that two n-vectors (namely, $\mathbf{x}$ and $\mathbf{y}$) are linearly independent?" Using the definition, one answer is to show that

B1: For all real numbers a and b with $a\mathbf{x} + b\mathbf{y} = \mathbf{0}$, $a = b = 0$.

Recognizing the keywords "for all" in the backward statement $B1$, you should now use the choose method to choose

A1: Real numbers a and b with $a\mathbf{x} + b\mathbf{y} = \mathbf{0}$,

for which you must show that

B2: $a = b = 0$.

To that end, work forward from $A1$ by multiplying both sides of the equality in $A1$ by the vector $\mathbf{x}$ to obtain

A2: $(a\mathbf{x} + b\mathbf{y})\bullet \mathbf{x} = \mathbf{0}\bullet \mathbf{x}$.

Using the properties of vector operations in Table B.1, $A2$ becomes

A3: $a(\mathbf{x}\bullet \mathbf{x}) + b(\mathbf{x}\bullet \mathbf{y}) = 0$.

From the hypothesis that $\mathbf{x}\bullet \mathbf{y} = 0$, $A3$ becomes

A4: $a(\mathbf{x}\bullet \mathbf{x}) = 0$.

Finally, because it is stated in the hypothesis the $\mathbf{x} \neq \mathbf{0}$, it follows that $\mathbf{x}\bullet \mathbf{x} = (x_1)x_1 + \cdots + (x_n)x_n \neq 0$. On dividing $A4$ by $\mathbf{x}\bullet \mathbf{x} \neq 0$, you obtain

A5: $a = 0$.

Multiplying both sides of the equality in $A1$ by the vector $\mathbf{y}$ and using forward steps similar to those in $A2$, $A3$, and $A4$ results in

A6: $b = 0$.

The proof is now complete because $A5$ and $A6$ are the same as $B2$.

Proof. To see that the n-vectors $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, let a and b be real numbers with $a\mathbf{x} + b\mathbf{y} = \mathbf{0}$ (the word "let" here indicates that the choose method is used). It will be shown that $a = b = 0$. To see that $a = 0$, you have

$$\begin{array}{rcll} (a\mathbf{x} + b\mathbf{y}) \bullet \mathbf{x} & = & \mathbf{0} \bullet \mathbf{x} & \text{(multiply through by } \mathbf{x}) \\ a(\mathbf{x} \bullet \mathbf{x}) + b(\mathbf{x} \bullet \mathbf{y}) & = & 0 & \text{(properties of vector operations)} \\ a(\mathbf{x} \bullet \mathbf{x}) & = & 0 & \text{(hypothesis that } \mathbf{x} \bullet \mathbf{y} = 0). \end{array}$$

Because the hypothesis states that $\mathbf{x} \neq \mathbf{0}$, you can divide the final of the foregoing equalities through by the nonzero number $\mathbf{x} \bullet \mathbf{x}$ to obtain $a = 0$. A similar set of operations shows that $b = 0$ and completes the proof. □

B.7 The n-vectors $\mathbf{x}^1, \ldots, \mathbf{x}^k$ are **linearly dependent** if and only if there are real numbers $t_1, \ldots, t_k$, not all 0, such that $t_1\mathbf{x}^1 + \cdots + t_k\mathbf{x}^k = \mathbf{0}$. This statement is obtained by using the rules in Chapter 8 to negate the definition of the n-vectors $\mathbf{x}^1, \ldots, \mathbf{x}^k$ being linearly independent, as follows:

> NOT[For all real numbers $t_1, \ldots, t_k$ with $t_1\mathbf{x}^1 + \cdots + t_k\mathbf{x}^k = \mathbf{0}$, it follows that $t_1 = \cdots = t_k = 0$].
>
> There are real numbers $t_1, \ldots, t_k$ with $t_1\mathbf{x}^1 + \cdots + t_k\mathbf{x}^k = \mathbf{0}$ such that NOT[$t_1 = \cdots = t_k = 0$].
>
> There are real numbers $t_1, \ldots, t_k$ not all 0 such that $t_1\mathbf{x}^1 + \cdots + t_k\mathbf{x}^k = \mathbf{0}$.

B.8 **Analysis of Proof.** The forward-backward method is used to begin the proof because the hypothesis and conclusion do not contain keywords. A key question associated with the conclusion is, "How can I show that some n-vectors (namely, $\mathbf{x}^1, \ldots, \mathbf{x}^k, \mathbf{0}$) are linearly dependent?" Using the definition in Exercise B.7, one answer is to show that

> **B1:** There are real numbers $t_1, \ldots, t_k, t_{k+1}$ not all 0 such that $t_1\mathbf{x}^1 + \cdots + t_k\mathbf{x}^k + t_{k+1}\mathbf{0} = \mathbf{0}$.

Recognizing the keywords "there are" in the backward statement $B1$, you should now use the construction method to produce the real numbers $t_1, \ldots, t_k$, t_{k+1} in $B1$. To that end, you can

> **A1:** Construct the real numbers $t_1 = 0, \ldots, t_k = 0, t_{k+1} = 1$.

According to the construction method, you must show that the values constructed in $A1$ satisfy the certain property of not all being 0 as well as the something that happens in $B1$. It is easy to see that not all the values in $A1$ are 0 because $t_{k+1} = 1$. Finally, the constructed values in $A1$ satisfy the something that happens in $B1$ because

$$t_1\mathbf{x}^1 + \cdots + t_k\mathbf{x}^k + t_{k+1}\mathbf{0} = 0\mathbf{x}^1 + \cdots + 0\mathbf{x}^k + 1(\mathbf{0}) = \mathbf{0},$$

the last equality being true from the properties of vector operations in Table B.1. The proof is now complete.

Proof. To see that the n-vectors $\mathbf{x}^1, \ldots, \mathbf{x}^k, \mathbf{0}$ are linearly dependent, by the definition in Exercise B.7, it will be shown that there are real numbers $t_1, \ldots, t_k, t_{k+1}$ not all 0 such that $t_1\mathbf{x}^1 + \cdots + t_k\mathbf{x}^k + t_{k+1}\mathbf{0} = \mathbf{0}$. Specifically, the real numbers $t_1 = 0, \ldots, t_k = 0, t_{k+1} = 1$ are not all 0 and satisfy

$$t_1\mathbf{x}^1 + \cdots + t_k\mathbf{x}^k + t_{k+1}\mathbf{0} = 0\mathbf{x}^1 + \cdots + 0\mathbf{x}^k + 1(\mathbf{0}) = \mathbf{0}.$$

The proof is now complete. $\square$

B.9 **Analysis of Proof.** The forward-backward method is used to begin the proof because the hypothesis and conclusion do not contain keywords. A key question associated with the conclusion is, "How can I show that some n-vectors (namely, $\mathbf{x}^1, \ldots, \mathbf{x}^k, \mathbf{x}$) are linearly dependent?" Using the definition in Exercise B.7, one answer is to show that

B1: There are real numbers $t_1, \ldots, t_k, t_{k+1}$ not all 0 such that $t_1\mathbf{x}^1 + \cdots + t_k\mathbf{x}^k + t_{k+1}\mathbf{x} = \mathbf{0}$.

Recognizing the keywords "there are" in the backward statement $B1$, you should now use the construction method to produce the real numbers $t_1, \ldots, t_k, t_{k+1}$ in $B1$. To that end, you can work forward from the hypothesis that the n-vectors $\mathbf{x}^1, \ldots, \mathbf{x}^k$ are linearly dependent which, by the definition in Exercise B.7, means that

A1: There are real numbers $s_1, \ldots, s_k$ not all zero such that $s_1\mathbf{x}^1 + \cdots + s_k\mathbf{x}^k = \mathbf{0}$.

You can use these values, together with the number 0, to construct the $k+1$ real numbers in $B1$, as follows:

A2: Construct the real numbers $t_1 = s_1, \ldots, t_k = s_k, t_{k+1} = 0$.

According to the construction method, you must show that the values constructed in $A2$ satisfy the certain property of not all being 0 as well as the something that happens in $B1$. It is easy to see that not all the values in $A2$ are 0 because, according to $A1$, not all of the values $s_1, \ldots, s_k$ are 0. Finally, the constructed values in $A1$ satisfy the something that happens in $B1$ because

$$\begin{array}{rcll} t_1\mathbf{x}^1 + \cdots + t_k\mathbf{x}^k + t_{k+1}\mathbf{x} & = & s_1\mathbf{x}^1 + \cdots + s_k\mathbf{x}^k + 0(\mathbf{x}) & (\text{from } A2) \\ & = & s_1\mathbf{x}^1 + \cdots + s_k\mathbf{x}^k & (0\mathbf{x} = \mathbf{0}) \\ & = & \mathbf{0} & (\text{from } A1). \end{array}$$

The proof is now complete.

Proof. To see that the n-vectors $\mathbf{x}^1, \ldots, \mathbf{x}^k, \mathbf{x}$ are linearly dependent, by the definition in Exercise B.7, it will be shown that there are real numbers

$t_1, \ldots, t_k, t_{k+1}$ not all 0 such that $t_1\mathbf{x}^1 + \cdots + t_k\mathbf{x}^k + t_{k+1}\mathbf{x} = \mathbf{0}$. To that end, from the hypothesis that the n-vectors $\mathbf{x}^1, \ldots, \mathbf{x}^k$ are linearly dependent, by the definition in Exercise B.7, there are real numbers $s_1, \ldots, s_k$ not all zero such that $s_1\mathbf{x}^1 + \cdots + s_k\mathbf{x}^k = \mathbf{0}$. Setting $t_1 = s_1, \ldots, t_k = s_k, t_{k+1} = 0$, it is easy to see that not all these values are 0. Furthermore, you have

$$\begin{aligned} t_1\mathbf{x}^1 + \cdots + t_k\mathbf{x}^k + t_{k+1}\mathbf{x} &= s_1\mathbf{x}^1 + \cdots + s_k\mathbf{x}^k + 0(\mathbf{x}) \\ &= s_1\mathbf{x}^1 + \cdots + s_k\mathbf{x}^k \\ &= \mathbf{0}. \end{aligned}$$

The proof is now complete. □

B.10 a. A common key question associated with the properties in Table B.2 is, "How can I show that two matrices are equal?"

b. Using the definition, one answer to the key question in part (a) is to show that the matrices have the same dimensions and that all elements of the two matrices are equal. Applying this answer to the matrices $A+B$ and $B+A$ that both have the dimensions $m \times n$ means you must show that for every integer $i = 1, \ldots, m$ and $j = 1, \ldots, n$, $(A+B)_{ij} = (B+A)_{ij}$, that is, you must show that

B1: For every integer $i = 1, \ldots, m$ and for every integer $j = 1, \ldots, n$, $A_{ij} + B_{ij} = B_{ij} + A_{ij}$.

c. Based on the answer in part (b), the next technique you should use is the choose method because the keywords "for every" appear in the backward statement $B1$. For that statement, you would choose

A1: Integers i and j with $1 \le i \le m$ and $1 \le j \le n$,

for which you must show that

B2: $A_{ij} + B_{ij} = B_{ij} + A_{ij}$.

B.11 **Analysis of Proof.** Following the approach in Exercise B.10, a key question associated with showing that $A + B = B + A$ is, "How can I show that two matrices are equal?" By definition, one answer is to show that they have the same dimensions (which they do because both $A+B$ and $B+A$ are $(m \times n)$ matrices) and that all components are equal; that is,

B1: For every integer $i = 1, \ldots, m$ and for every integer $j = 1, \ldots, n$, $(A+B)_{ij} = (B+A)_{ij}$, that is, $A_{ij} + B_{ij} = B_{ij} + A_{ij}$.

Recognizing the keywords "for every" in the backward statement $B1$, you should proceed with the choose method and choose

A1: Integers i and j with $1 \le i \le m$ and $1 \le j \le n$,

for which you must show that

B2: $A_{ij} + B_{ij} = B_{ij} + A_{ij}$.

However, $B2$ is true because of the commutative property of addition of real numbers, and so the proof is complete.

Proof. To see that $A + B = B + A$, note that the dimensions of both $A + B$ and $B+A$ are $(m \times n)$, so let i and j be integers with $1 \le i \le m$ and $1 \le j \le n$. (The word "let" here indicates that the choose method is used.) But then you have that $(A+B)_{ij} = A_{ij} + B_{ij} = B_{ij} + A_{ij} = (B+A)_{ij}$ by the commutative property of addition of real numbers, and so the proof is complete. □.

B.12 **Analysis of Proof.** The forward-backward method is used to begin the proof because there are no keywords in the hypothesis or conclusion. A key question associated with showing that $(A+B)+C = A+(B+C)$ is, "How can I show that two matrices are equal?" By definition, one answer is to show that the two matrices have the same dimensions (which they do because both $(A+B)+C$ and $A+(B+C)$ are $(m \times n)$ matrices) and that all components are equal, that is,

B1: For every integer $i = 1, \ldots, m$ and for every integer $j = 1, \ldots, n$, $[(A + B) + C]_{ij} = [A + (B + C)]_{ij}$, that is, $(A + B)_{ij} + C_{ij} = A_{ij} + (B + C)_{ij}$, that is, $(A_{ij} + B_{ij}) + C_{ij} = A_{ij} + (B_{ij} + C_{ij})$.

Recognizing the keywords "for every" in the backward statement $B1$, you should proceed with the choose method and choose

A1: Integers i and j with $1 \le i \le m$ and $1 \le j \le n$,

for which you must show that

B2: $(A_{ij} + B_{ij}) + C_{ij} = A_{ij} + (B_{ij} + C_{ij})$.

However, $B2$ is true because of the associative property of addition of real numbers, and so the proof is complete.

Proof. To see that $(A + B) + C = A + (B + C)$, note that the dimensions of both $(A+B)+C$ and $A+(B+C)$ are $(m \times n)$, so let i and j be integers with $1 \le i \le m$ and $1 \le j \le n$ (the word "let" here indicates that the choose method is used). But then you have that $[(A + B) + C]_{ij} = [A + (B + C)]_{ij}$, that is, $(A + B)_{ij} + C_{ij} = A_{ij} + (B + C)_{ij}$, that is, $(A_{ij} + B_{ij}) + C_{ij} = A_{ij} + (B_{ij} + C_{ij})$. The last equality is true by the associative property of addition of real numbers, and so the proof is complete. □.

B.13 **Analysis of Proof.** The keyword "unique" in the conclusion suggests that you should use the backward uniqueness method. Accordingly, it is first necessary to construct a matrix C such that $AC = CA = I$. However, because the hypothesis states that A is invertible, indeed, by definition, you know that

A1: There is an $(n \times n)$ matrix C such that $AC = CA = I$.

The next step is to prove that there is at most one such matrix. To that end, the direct uniqueness method is chosen. Accordingly, assume that there is another such object, so, you should now assume that

A2: There is an $(n \times n)$ matrix D such that $AD = DA = I$,

for which you must show that

B1: $C = D$.

Specifically, you can work forward from $A1$ and $A2$ to establish $B1$, as follows:

$$\begin{array}{rcll} AC & = & I & \text{(from } A1\text{)} \\ D(AC) & = & D & \text{(multiply on the left by } D\text{)} \\ (DA)C & = & D & \text{(property of matrix multiplication)} \\ IC & = & D & \text{(from } A2\text{)} \\ C & = & D & \text{(property of } I\text{).} \end{array}$$

The proof is now complete.

Proof. By definition of an invertible matrix, there is an $(n \times n)$ matrix C such that $AC = CA = I$. To show that C is unique, assume that D is also an $(n \times n)$ matrix that satisfies $AD = DA = I$. But then you have that $C = D$ by the steps in the foregoing analysis of proof, and so the proof is now complete. □

B.14 **Analysis of Proof.** The forward-backward method is used to begin the proof because there are no keywords in the hypothesis or conclusion. A key question associated with the conclusion that $A(B + C) = AB + AC$ is, "How can I show that two matrices are equal?" By definition, one answer is to show that the two matrices have the same dimensions (which they do because both $A(B+C)$ and $AB+AC$ are $(m \times n)$ matrices) and that all components are equal; that is,

B1: For every integer $i = 1, \ldots, m$ and for every integer $j = 1, \ldots, n$, $[A(B + C)]_{ij} = (AB + AC)_{ij}$.

Recognizing the keywords "for every" in the backward statement $B1$, you should proceed with the choose method and choose

A1: Integers i and j with $1 \le i \le m$ and $1 \le j \le n$,

for which you must show that

B2: $[A(B + C)]_{ij} = (AB + AC)_{ij}$.

By definition of matrix multiplication, the element in row i and column j of $A(B+C)$ is $A_{i*} \bullet (B+C)_{*j}$ while the element in row i and column j of $AB + AC$ is $(AB)_{ij} + (AC)_{ij}$. Thus, you must show that

B3: $A_{i*} \bullet (B+C)_{*j} = (AB)_{ij} + (AC)_{ij}$.

However, starting with the left side of $B3$, you have

A2: $A_{i*} \bullet (B+C)_{*j} = A_{i*} \bullet (B_{*j}+C_{*j}) = A_{i*} \bullet B_{*j} + A_{i*} \bullet C_{*j} = (AB)_{ij} + (AC)_{ij}$.

The proof is now complete because $A2$ is the same as $B3$.

Proof. To see that $A(B+C) = AB + AC$, note that both matrices have dimension $(m \times n)$, so, let i and j be integers with $1 \leq i \leq m$ and $1 \leq j \leq n$. (The word "let" here indicates that the choose method is used.) You then have:

$$
\begin{array}{rcll}
[A(B+C)]_{ij} & = & A_{i*} \bullet (B+C)_{*j} & \text{(def. of matrix multiplication)} \\
& = & A_{i*} \bullet (B_{*j} + C_{*j}) & \text{(prop. of matrix addition)} \\
& = & A_{i*} \bullet B_{*j} + A_{i*} \bullet C_{*j} & \text{(prop. of vector dot product)} \\
& = & (AB)_{ij} + (AC)_{ij} & \text{(def. of matrix multiplication)} \\
& = & (AB + AC)_{ij} & \text{(definition of matrix addition).}
\end{array}
$$

The proof is now complete. □

B.15 **Analysis of Proof.** The forward-backward method is used to begin the proof because there are no keywords in the hypothesis or conclusion. In this case, the author works forward by definition from the hypothesis that C is invertible to claim that

A1: There is an $(n \times n)$ matrix D such that $CD = DC = I$.

The author then works forward from $A1$ and the hypothesis that $AC = I$ to claim that all of the following statements are true:

$$
\begin{array}{rcll}
AC & = & I & \text{(from the hypothesis)} \\
(AC)D & = & ID & \text{(multiply on the right by } D\text{)} \\
A(CD) & = & D & \text{(property of matrix multiplication and } I\text{)} \\
AI & = & D & \text{(from } A1\text{)} \\
A & = & D & \text{(property of } I\text{)} \\
CA & = & CD & \text{(multiply on the left by } C\text{)} \\
CA & = & I & \text{(from } A1\text{).}
\end{array}
$$

The proof is now complete because the final equality is the desired conclusion.

B.16 The error is in the author's application of the construction method. Specifically, the author uses the construction method to produce an $(n \times n)$ matrix C such that $AC = CA = I$. The author constructs the matrix C and correctly shows that $AC = I$ but fails to show that $CA = I$, as required.

Appendix C *Solutions to Exercises*

C.1 **Analysis of Proof.** The forward-backward method is used to begin the proof because there are no keywords in the hypothesis or conclusion. A key question associated with the conclusion $a|(-b)$ is, "How can I show that an integer (namely, a) divides another integer (namely, $-b$)?" Using the definition, one answer is to show that

B1: There is an integer d such that $-b = da$.

Recognizing the keywords "there is" in the backward statement $B1$, you should now use the construction method. Specifically, turn to the forward process to construct the desired integer d.

Working forward from the hypothesis that a divides b, by definition, you know that

A1: There is an integer c such that $b = ca$.

Multiplying the equality in $A1$ through by -1 yields that

A2: $-b = (-c)a$.

From $A2$ you can see that the desired value of the integer d in $B2$ is $d = -c$. According to the construction method, you must still show that this value of d satisfies the certain property and the something that happens in $B1$, namely, that $-b = da$, but this is clear from $A2$.

Proof. To show that $a|(-b)$, by definition, it is shown that there is an integer d such that $-b = da$. However, from the hypothesis that $a|b$, there is an integer c such that $b = ca$. Letting $d = -c$, it is easy to see that $-b = (-c)a = da$, thus completing the proof. □

C.2 **Analysis of Proof.** It is important to realize that the conclusion of this proposition contains the hidden keywords "either/or" because you can rewrite the conclusion, as follows:

B1: Either $a = +b$ or $a = -b$.

Recognizing the keywords "either/or" in the backward statement $B1$, you should now use a proof by elimination (see Section 12.1). Accordingly, you should assume the hypothesis, A, and

A1: $a \neq +b$,

and you must show that

B1: $a = -b$.

Turning to the forward process, from the hypothesis that $a|b$, by definition,

A2: There is an integer c such that $b = ca$.

Likewise, from the hypothesis that $b|a$, by definition,

A3: There is an integer d such that $a = db$.

Substituting $A2$ in $A3$, you have

A4: $a = db = (dc)a$.

Observe that if you could divide the equality in $A4$ through by a, it would follow that

A5: $1 = dc$.

The only way the product of the integers c and d can be 1 is if

A6: Either $c = d = 1$ or $c = d = -1$.

Now it cannot be that $c = d = 1$, for otherwise, from $A2$ and $A3$, it would follow that $a = b$, which contradicts $A1$. Thus, it must be that

A7: $c = d = -1$.

Finally, from $A3$ it follows that $a = -b$, and so the proof would be complete. However, recall that statement $A5$ is based on the ability to divide $A4$ through by a. You will only be able to do so when $a \neq 0$, so, you must still handle the case when $a = 0$, as is done in the following condensed proof.

Proof. To see that $a = \pm b$, assume that $a \neq b$. It will be shown that $a = -b$. From the hypothesis that $a|b$, there is an integer c such that $b = ca$. Likewise, from the hypothesis that $b|a$, there is an integer d such that $a = db$. It then follows that $a = db = (dc)a$. Now $a \neq 0$ for otherwise $b = ca = 0$ and so $a = b = 0$, contradicting $a \neq b$. Because $a \neq 0$, dividing the equation $a = (dc)a$ through by a, you have $cd = 1$. The only way this can happen is if $c = d = 1$ or $c = d = -1$. The former cannot happen for otherwise $a = b$. In the latter case, $a = -b$, thus completing the proof. □

C.3 **Analysis of Proof.** The forward-backward method is used to begin the proof because there are no keywords in the hypothesis or conclusion. A key question associated with the conclusion $a|(b+c)$ is, "How can I show that an integer (namely, a) divides another integer (namely, $b + c$)?" Using the definition, one answer is to show that

B1: There is an integer d such that $b + c = da$.

Recognizing the keywords "there is" in the backward statement $B1$, you should now use the construction method. Specifically, turn to the forward process to construct the desired integer d.

Working forward from the hypothesis that a divides b, by definition, you know that

A1: There is an integer e such that $b = ea$.

Likewise, from the hypothesis that a divides c, by definition,

A2: There is an integer f such that $c = fa$.

Adding the equalities in $A1$ and $A2$ results in

A3: $b + c = ea + fa = (e + f)a$.

From $A3$ you can see that the desired value of the integer d in $B1$ is $d = e+f$. According to the construction method, you must still show that this value of d satisfies the certain property and the something that happens in $B1$, namely, that $b + c = da$, but this is clear from $A3$.

Proof. To show that $a|(b + c)$, by definition, it is shown that there is an integer d such that $b + c = da$. However, from the hypothesis that $a|b$, there is an integer e such that $b = ea$. Likewise, from the hypothesis that $a|c$, there is an integer f such that $c = fa$. Letting $d = e + f$, it is easy to see that $b + c = ea + fa = (e + f)a = da$, thus completing the proof. □

C.4 **Analysis of Proof.** The words "Suppose not ..." indicate that the author is using the contradiction method (see Chapter 9). Accordingly, the author assumes that the conclusion is not true; that is, that

A1 (NOT B): There is an integer $a > 1$ such that a cannot be written as a product of primes.

The author must now reach a contradiction (which is identified in the last sentence of the proof). To that end, from $A1$, the author constructs

A2: $b =$ the first integer > 1 such that b cannot be written as a product of primes.

The author is justified in making this statement because the author is specializing the Least Integer Principle to the following set:

$$M = \{c \in Z : c > 1 \text{ and } c \text{ cannot be written as a product of primes}\}.$$

To apply specialization, it is necessary to show that M is a nonempty set of positive integers. However, $A1$ ensures that $M \neq \emptyset$ (because $a \in M$) and the definition of M ensures that all elements are > 0. Hence, the author is justified in making statement $A2$. The remainder of the proof is working forward from $A2$ to show that b can be written as a product of primes, which contradicts $A2$. Specifically, the author first notes from $A2$ that

A3: b is not prime.

Working forward from $A3$ by writing the negation of the definition of a prime, the author states that

A4: There are integers b_1 and b_2 with $1 < b_1, b_2 < b$ such that $b = b_1 b_2$.

The author now uses the fact that b is the smallest integer > 1 that cannot be written as the product of primes (see $A2$). Specifically, without telling you, the author uses a max/min technique (see Section 12.3) to realize that

A5: For every integer a with $1 < a < b$, a can be written as a product of primes.

Recognizing the keywords "for every" in the forward statement $A5$, the author specializes $A5$ to $a = b_1$ and again to $a = b_2$ (which both satisfy $1 < b_1, b_2 < b$ according to $A4$) to obtain

A6: There are primes $q_1, \ldots, q_m$ and $r_1, \ldots, r_n$ such that $b_1 = q_1 q_2 \cdots q_m$ and $b_2 = r_1 r_2 \cdots r_n$.

Multiplying corresponding sides of the equalities in $A6$ results in

A7: $b = b_1 b_2 = (q_1 q_2 \cdots q_m)(r_1 r_2 \cdots r_n)$.

However, $A7$ says that b can be written as a product of primes, which contradicts $A2$ and completes the proof.

C.5 **Analysis of Proof.** The author recognizes the keywords "there is" in the conclusion and so uses the construction method to produce the positive integer d. Specifically, the author constructs the integer d by specializing the Least Integer Principle to the following set:

$$M = \{ax + by > 0 : x, y \in Z\}.$$

To apply specialization, it is necessary to show that M is a nonempty set of positive integers. The author uses the hypothesis that a and b are not both 0 to note that $M \neq \emptyset$ because $0 < a^2 + b^2 = aa + bb \in M$. Also, the definition of M ensures that all elements are > 0. Hence, the author is justified in constructing d as the smallest element of M, that is,

A1: $d = am + bn > 0$ is the smallest element of M.

According to the construction method, it remains to show that the value of d in $A1$ satisfies the something that happens in the conclusion, that is, the author must still show that

B1: $d|a$ and $d|b$.

Working backward by definition, to show that $d|a$, it is necessary to show that

B2: There is an integer q such that $a = dq$.

The author uses the Division Algorithm to do so. Specifically, using Proposition 35, you can say that

A2: There are integers q and r with $0 \leq r < d$ such that $a = dq + r$.

The integer q in $A2$ will satisfy $a = dq$ in $B2$ provided that $r = 0$ in $A2$. Thus, the author must show that

B3: $r = 0$.

This is done by contradiction, so the author assumes that

A3: $r \neq 0$, so, from $A2$, $0 < r < d$.

The author now reaches a contradiction by showing that $r \in M$, for then, because d is the smallest element of M, it must be that $r \geq d$, which contradicts $r < d$ in $A3$. To show that $r \in M$, the author must show that

B4: There are integers x and y such that $r = ax + by > 0$.

Recognizing the keywords "there are" in the backward statement $B4$, the author uses the construction method to produce the integers x and y. Specifically, solving the equality in $A2$ for r and substituting $d = am + bn$ from $A1$ yields

A4: $r = a - dq = a - (am + bn)q = a(1 - mq) + b(-nq)$.

From $A4$, you can see that the desired values of x and y in $B4$ are $x = 1 - mq$ and $y = -nq$. According to the construction method, it is still necessary to show that these values of x and y satisfy the property that $r = ax + by > 0$, but this is given in $A3$.

It has thus been shown that $d|a$. To complete the proof, according to $B1$, it is still necessary to show that $d|b$, but this is true by a similar reasoning used to show that $d|a$, and so the proof is complete.

C.6 **Analysis of Proof.** Recognizing the keyword "only" in the conclusion, you should use the backward uniqueness method (see Section 11.1). Accordingly, you must first construct an identity element of the group G. However, this is given in the hypothesis and is property (2) in the definition of a group, so you already know that

A1: There is an element $e \in G$ such that for all elements $a \in G$, $a \odot e = e \odot a = a$.

To show that there is at most one identity element, the direct uniqueness method is used here. Accordingly, you should now assume that

A2: There is an element $f \in G$ such that for all elements $a \in G$, $a \odot f = f \odot a = a$.

You must show that

B1: $e = f$.

You can obtain $B1$ by specializing the for-all statements in $A1$ and $A2$. Specifically, specializing the for-all statement in $A1$ to $a = f$ (which is in G from $A2$), it follows that

A3: $f \odot e = f$.

Likewise, specializing the for-all statement in $A2$ to $a = e$ (which is in G from $A1$), it follows that

A4: $f \odot e = e$.

The desired conclusion in $B1$ now follows from $A3$ and $A4$ because both e and f are equal to $f \odot e$ and hence $e = f$, thus completing the proof.

Proof. To see that the identity element $e \in G$ is the only element with the property that for all $a \in G$, $a \odot e = e \odot a = a$, suppose that $f \in G$ also satisfies the property that for all $a \in G$, $a \odot f = f \odot a = a$. Then, because e is an identity element of G, $f \odot e = f$. Likewise, because f is an identity element of G, $f \odot e = e$. But then $f \odot e = f = e$ and so $e = f$, completing the proof. □

C.7 **Analysis of Proof.** Recognizing the keyword "only" in the conclusion, you should use the backward uniqueness method (see Chapter 11). Accordingly, you must first construct an inverse element of the group G. However, this is given in the hypothesis and is property (3) in the definition of a group, so, you already know that

> **A1:** There is an $a^{-1} \in G$ such that $a \odot a^{-1} = a^{-1} \odot a = e$.

To show that there is at most one such inverse element, the direct uniqueness method is used. Accordingly, you should now assume that

> **A2:** There is an element $f \in G$ such that for $a \odot f = f \odot a = e$.

You must show that

> **B1:** $f = a^{-1}$.

To obtain $B1$, start by combining both sides of the equality in $A2$ on the left by a^{-1} to obtain

> **A3:** $a^{-1} \odot (a \odot f) = a^{-1} \odot e$.

Specializing the for-all statement in property (1) in the definition of a group, from $A3$ you have

> **A4:** $(a^{-1} \odot a) \odot f = a^{-1} \odot e$.

From $A1$, it follows that

> **A5:** $e \odot f = a^{-1} \odot e$.

Finally, specializing the for-all statement in property (2) of the definition of a group to each side of $A5$ results in

> **A6:** $f = a^{-1}$.

The proof is now complete because $A6$ is the same as $B1$.

Proof. To see that the inverse element $a^{-1} \in G$ is the only element with the property that $a \odot a^{-1} = a^{-1} \odot a = e$, suppose that $f \in G$ also satisfies the property that $a \odot f = f \odot a = e$. Then, using the properties of a group, it follows that

$$
\begin{array}{rcll}
a \odot f & = & e & \text{(property of } f\text{)} \\
a^{-1} \odot (a \odot f) & = & a^{-1} \odot e & \text{(combine both sides with } a^{-1}\text{)} \\
(a^{-1} \odot a) \odot f & = & a^{-1} \odot e & \text{(property (1) of a group)} \\
e \odot f & = & a^{-1} \odot e & \text{(property of } a^{-1}\text{)} \\
f & = & a^{-1} & \text{(property (2) of a group).}
\end{array}
$$

The proof is now complete because it has been shown that the inverse element is unique. □

C.8 **Analysis of Proof.** To reach the conclusion, work forward from the hypothesis that $a \odot b = a \odot c$ by combining both sides on the left with the element a^{-1} (which exists by property (3) of a group) to obtain

A1: $a^{-1} \odot (a \odot b) = a^{-1} \odot (a \odot c)$.

Now specialize the for-all statement in property (1) of a group to both sides of $A1$ to obtain

A2: $(a^{-1} \odot a) \odot b = (a^{-1} \odot a) \odot c$.

Using the fact that $a^{-1} \odot a = e$ from property (3) of a group, $A2$ becomes

A3: $e \odot b = e \odot c$.

Finally, specialize property (2) of a group to both sides of $A3$ to obtain

A4: $b = c$.

The proof is now complete because $A4$ is the same as the conclusion of the proposition. □

Proof. To reach the conclusion that $b = c$, you have that

$$\begin{array}{rcll} a \odot b & = & a \odot c & \text{(hypothesis)} \\ a^{-1} \odot (a \odot b) & = & a^{-1} \odot (a \odot c) & \text{(combine both sides with } a^{-1}\text{)} \\ (a^{-1} \odot a) \odot b & = & (a^{-1} \odot a) \odot c & \text{(property (1) of a group)} \\ e \odot b & = & e \odot c & \text{(property of } a^{-1}\text{)} \\ b & = & c & \text{(property of } e\text{).} \end{array}$$

The proof is now complete. □

C.9 **Analysis of Proof.** To reach the conclusion, work forward from the hypothesis that $b \odot a = c \odot a$ by combining both sides on the right with the element a^{-1} (which exists by property (3) of a group) to obtain

A1: $(b \odot a) \odot a^{-1} = (c \odot a) \odot a^{-1}$.

Now specialize the for-all statement in property (1) of a group to both sides of $A1$ to obtain

A2: $b \odot (a \odot a^{-1}) = c \odot (a \odot a^{-1})$.

Using the fact that $a^{-1} \odot a = e$ from property (3) of a group, $A2$ becomes

A3: $b \odot e = c \odot e$.

Finally, specialize property (2) of a group to both sides of $A3$ to obtain

A4: $b = c$.

The proof is now complete because $A4$ is the same as the conclusion of the proposition. □

Proof. To reach the conclusion that $b = c$, you have that

$$\begin{array}{rcll} b \odot a & = & c \odot a & \text{(hypothesis)} \\ (b \odot a) \odot a^{-1} & = & (c \odot a) \odot a^{-1} & \text{(combine both sides with } a^{-1}) \\ b \odot (a \odot a^{-1}) & = & c \odot (a \odot a^{-1}) & \text{(property (1) of a group)} \\ b \odot e & = & c \odot e & \text{(property of } a^{-1}) \\ b & = & c & \text{(property of } e). \end{array}$$

The proof is now complete. □

C.10 **Analysis of Proof.** The forward-backward method is used to begin the proof because there are no keywords in either the hypothesis or the conclusion. Starting with the forward process, because $a^{-1} \in G$, you can specialize the for-all statement in property (3) of a group to state that

A1: There is an $(a^{-1})^{-1} \in G$ such that $a^{-1} \odot (a^{-1})^{-1} = (a^{-1})^{-1} \odot a^{-1} = e$.

Now, if you can show that $a \in G$ also satisfies the property of $(a^{-1})^{-1}$ in $A1$ then, because the inverse element is unique (see Exercise C.7), by the forward uniqueness method (see Section 11.1), it must be that $(a^{-1})^{-1} = a$. Thus, you must show that

B1: $a^{-1} \odot a = a \odot a^{-1} = e$.

However, $B1$ is true by property (3) in the definition of a group, and so the proof is complete.

Proof. Because $a^{-1} \in G$, by property (3) of a group, there is an element $(a^{-1})^{-1} \in G$ such that $a^{-1} \odot (a^{-1})^{-1} = (a^{-1})^{-1} \odot a^{-1} = e$. Now $a \in G$ also satisfies $a^{-1} \odot a = a \odot a^{-1} = e$. Thus, because the inverse element is unique (see Exercise C.7), it must be that $(a^{-1})^{-1} = a$, completing the proof. □

C.11 **Analysis of Proof.** The forward-backward method is used to begin the proof because there are no keywords in either the hypothesis or the conclusion. Starting with the forward process, because $a \odot b \in G$, you can specialize the for-all statement in property (3) of a group to state that

A1: There is an $(a \odot b)^{-1} \in G$ such that $(a \odot b) \odot (a \odot b)^{-1} = (a \odot b)^{-1} \odot (a \odot b) = e$.

Now, if you can show that $b^{-1} \odot a^{-1} \in G$ satisfies the same property as $(a \odot b)^{-1}$ in $A1$ then, because the inverse element is unique (see Exercise C.7), by the forward uniqueness method (see Section 11.1), it must be that $(a \odot b)^{-1} = b^{-1} \odot a^{-1}$. Thus, you must show that

B1: $(a \odot b) \odot (b^{-1} \odot a^{-1}) = (b^{-1} \odot a^{-1}) \odot (a \odot b) = e$.

However, the second equality in $B1$ is true by specializing the properties of a group, as follows:

$$\begin{array}{rcll} (b^{-1} \odot a^{-1}) \odot (a \odot b) & = & [(b^{-1} \odot a^{-1}) \odot a] \odot b & \text{(property (1) of } G) \\ & = & [b^{-1} \odot (a^{-1} \odot a)] \odot b & \text{(property (1) of } G) \\ & = & (b^{-1} \odot e) \odot b & \text{(property (3) of } G) \\ & = & b^{-1} \odot b & \text{(property (2) of } G) \\ & = & e & \text{(property (3) of } G). \end{array}$$

A similar set of operations shows that $(a \odot b) \odot (b^{-1} \odot a^{-1}) = e$. Thus, $B1$ is true, and so the proof is complete.

Proof. Because $a \odot b \in G$, there is an element $(a \odot b)^{-1} \in G$ such that $(a \odot b) \odot (a \odot b)^{-1} = (a \odot b)^{-1} \odot (a \odot b) = e$. Now you also have that $b^{-1} \odot a^{-1}$ satisfies $(b^{-1} \odot a^{-1}) \odot (a \odot b) = e$ and $(a \odot b) \odot (b^{-1} \odot a^{-1}) = e$ (see the foregoing Analysis of Proof). Because the inverse element is unique (see Exercise C.7), it must be that $(a \odot b)^{-1} = b^{-1} \odot a^{-1}$, thus completing the proof. □

C.12 **Analysis of Proof.** The forward-backward method is used to begin the proof because there are no keywords in either the hypothesis or the conclusion. A key question associated with the conclusion is, "How can I show that a set (namely, H) together with an operation (namely, $\odot$ from G) forms a group?" One answer is to use the definition for a group. Accordingly, you must first show that

B1: The operation $\odot$ satisfies the property that for all elements $a, b \in H$, $a \odot b \in H$.

Recognizing the keywords "for all" in the backward statement $B1$, you should now use the choose method to choose

A1: Elements $a, b \in H$,

for which you must show that

B2: $a \odot b \in H$.

A key question associated with $B2$ is, "How can I show that an element (namely, $a \odot b$) belongs to a set (namely, H)?" Using the defining property of H, you must show that

B3: There is an integer k such that $a \odot b = x^k$.

Recognizing the keywords "there is" in the backward statement $B2$, you should now use the construction method. Accordingly, the author turns to the forward process. Working forward from $A1$ using the defining property of the set H, it follows that

A2: There are integers i, j such that $a = x^i$ and $b = x^j$.

The author now works forward from $A2$ using Proposition 35 to state that

A3: $a \odot b = x^i \odot x^j = x^{i+j}$.

You can now see from $A3$ that the desired value of the integer k in $B2$ is $k = i + j$.

To complete the proof that $(H, \odot)$ is a group, it remains to show that the following three properties hold:

B4:
(1) For all elements $a, b, c \in H$, $(a \odot b) \odot c = a \odot (b \odot c)$.
(2) There is an element $e \in H$ such that for all elements $a \in H$, $a \odot e = e \odot a = a$.
(3) For all elements $a \in H$, there is an element $a^{-1} \in H$ such that $a \odot a^{-1} = a^{-1} \odot a = e$.

The author now uses the following three properties of the group $(G, \odot)$ to establish the corresponding properties in $B4$ for $(H, \odot)$:

A4:
(1) For all elements $a, b, c \in G$, $(a \odot b) \odot c = a \odot (b \odot c)$.
(2) There is an element $e \in G$ such that for all elements $a \in G$, $a \odot e = e \odot a = a$.
(3) For all elements $a \in G$, there is an element $a^{-1} \in G$ such that $a \odot a^{-1} = a^{-1} \odot a = e$.

Recognizing the keywords "for all" in property (1) of the backward statement $B4$, the author uses the choose method to choose

A5: Elements $a, b, c \in H$,

for which it must be shown that

B5: $(a \odot b) \odot c = a \odot (b \odot c)$.

The author states that $B2$ is true by specializing the corresponding for-all statement in property (1) of $A4$ to a, b, c in $A5$. However, to specialize that statement in $A4$, it must be that $a, b, c \in G$. The author states that this is the case without justification. The justification is that, because $a, b, c \in H$, by the defining property of H, there are integers i, j, k such that $a = x^i$, $b = x^j$, $c = x^k$. Each of these elements is in G because $\odot$ combines elements of G and produces an element of G.

Turning now to proving property (2) in $B4$, the author recognizes the keywords "there is" and uses the construction method. Specifically, the author constructs the identity element of H as the identity element e of G. According to the construction method, the author must show that the value of e satisfies the certain property and the something that happens in property (2) in $B4$; that is, the author must show that

B6: $e \in H$ and for all elements $a \in H$, $a \odot e = e \odot a = a$.

The author notes that $e \in H$ because $e = x^0$. Recognizing the keywords "for all" in the backward statement $B6$, the author now uses the choose method to choose

A6: An element $a \in H$,

for which it must be shown that

B7: $a \odot e = e \odot a = a$.

The author states that $B7$ is true, which is obtained by specializing the for-all statement in property (2) of $A4$ to a, noting that $a \in H$ and so $a \in G$.

Turning to proving property (3) in $B4$, the author recognizes the keywords "for all" in the backward statement $B4$ and uses the choose method to choose

A7: An element $x^k \in H$,

for which it must be shown that

B8: There is an element $(x^k)^{-1} \in H$ such that
$x^k \odot (x^k)^{-1} = (x^k)^{-1} \odot x^k = e$.

Recognizing the keywords "there is" in the backward statement $B8$, the author uses the construction method. Specifically, the author constructs $(x^k)^{-1} = x^{-k}$. According to the construction method, the author must show that this value of $(x^k)^{-1} = x^{-k}$ satisfies the certain property and the something that happens in $B8$, namely, that

B9: $(x^k)^{-1} \in H$ and $x^k \odot (x^k)^{-1} = (x^k)^{-1} \odot x^k = e$.

Now $(x^k)^{-1} = x^{-k} \in H$ by the defining property of H. Finally, the author uses Proposition 35 to note that

$$\begin{array}{l} x^k \odot (x^k)^{-1} = x^k \odot x^{-k} = x^0 = e \quad \text{and} \\ (x^k)^{-1} \odot x^k = x^{-k} \odot x^k = x^0 = e. \end{array}$$

The proof is now complete.

C.13 **Analysis of Proof.** Recognizing the keywords "if and only if," two proofs are necessary. Thus, the author supposes first that

A: For all $x, y \in G, x \odot y = y \odot x$,

for which it must be shown that

B: For all $a, b \in G, (a \odot b)^{-1} = a^{-1} \odot b^{-1}$.

Recognizing the keywords "for all" in the forward statement A and in the backward statement B, both specialization and the choose method are used. The author starts with the choose method applied to B and so chooses

A1: Elements $a, b \in G$,

for which it must be shown that

B1: $(a \odot b)^{-1} = a^{-1} \odot b^{-1}$.

The author then works forward from the left side of $B1$ using the result in part (f) of Table C.1 to state that

A2: $(a \odot b)^{-1} = b^{-1} \odot a^{-1}$.

The author then specializes A to $x = b^{-1} \in G$ and $y = a^{-1} \in G$ to claim that

A3: $b^{-1} \odot a^{-1} = a^{-1} \odot b^{-1}$.

The desired statement $B1$ results from combining $A2$ and $A3$.

Turning now to the converse, the author assumes that

A: For all $x, y \in G, (x \odot y)^{-1} = x^{-1} \odot y^{-1}$,

for which it must be shown that

B: For all $a, b \in G, a \odot b = b \odot a$.

Recognizing the keywords "for all" in the forward statement A and in the backward statement B, both specialization and the choose method are used. The author starts with the choose method applied to B and so chooses

A1: Elements $a, b \in G$,

for which it must be shown that

B1: $a \odot b = b \odot a$.

The author obtains $B1$ by performing the following forward steps:

$$\begin{array}{rcll}
(a \odot b)^{-1} & = & a^{-1} \odot b^{-1} & \text{(specialize } A\text{)} \\
(a \odot b)^{-1} & = & b^{-1} \odot a^{-1} & \text{[Table C.1 (f)]} \\
a^{-1} \odot b^{-1} & = & b^{-1} \odot a^{-1} & \text{(from above)} \\
b \odot (a^{-1} \odot b^{-1}) & = & b \odot (b^{-1} \odot a^{-1}) & \text{(combine with } b\text{)} \\
b \odot (a^{-1} \odot b^{-1}) & = & a^{-1} & \text{(properties (1-3) of } G\text{)} \\
a \odot [b \odot (a^{-1} \odot b^{-1})] & = & a \odot a^{-1} & \text{(combine with } a\text{)} \\
(a \odot b) \odot (a^{-1} \odot b^{-1}) & = & e & \text{(properties (1, 3) of } G\text{)} \\
\big[(a \odot b) \odot (a^{-1} \odot b^{-1})\big] \odot b & = & e \odot b & \text{(combine with } b\text{)} \\
(a \odot b) \odot a^{-1} & = & b & \text{(properties (1-3) of } G\text{)} \\
\big[(a \odot b) \odot a^{-1}\big] \odot a & = & b \odot a & \text{(combine with } a\text{)} \\
a \odot b & = & b \odot a & \text{(properties (1-3) of } G\text{).}
\end{array}$$

Appendix D
Solutions to Exercises

D.1 a. A common key question is, "How can I show that a sequence converges to a number?"

b. Using the definition, one answer to the key question in part (a) applied to the specific problem results in having to show that

B1: For every real number $\epsilon > 0$, there is an integer $j \in N$ such that for all $k \in N$ with $k > j$, $|x_k y_k - xy| < \epsilon$.

c. Recognizing "for every" as the first of the nested quantifiers in the backward statement $B1$, you should use the choose method next.

D.2 a. Use the set $T = \{s \in R : s > 0 \text{ and } s^n > 2\}$.

b. **Analysis of Proof.** You can show that the set T in part (a) has an infimum by specializing the Infimum Property of real numbers. To do so, you must show that

B1: $T \neq \emptyset$ and T is bounded below.

You can see that $T \neq \emptyset$ because $2 \in T$ (recall that $n > 1$). Finally, 0 is a lower bound for T. To see that this is so, according to the definition of a lower bound, you must show that

B2: For every element $x \in T, x \geq 0$.

Recognizing the keywords "for every" in the backward statement $B2$, you should now use the choose method to choose

A1: An element $x \in T$,

for which it must be shown that

B3: $x \geq 0$.

However, $B3$ is true because $x \in T$ and by the defining property of T, $x > 0$. The proof is now complete.

Proof. To see that T has an infimum, note that $2 \in T$ and so $T \neq \emptyset$. Also, 0 is a lower bound for T because, for $x \in T$, by the defining property of T, $x > 0$. The fact that T has an infimum now follows from the Infimum Property of real numbers. □

D.3 a. Use the set $T = \{s \in R : s > 0 \text{ and } s^2 > a\}$.

b. **Analysis of Proof.** You can show that the set T in part (a) has an infimum by specializing the Infimum Property of real numbers to the set T. To do so, you must show that

B1: $T \neq \emptyset$ and T is bounded below.

You can see that $T \neq \emptyset$ because $a \in T$ if $a > 1$ while $1 \in T$ if $a \leq 1$. Finally, 0 is a lower bound for T. To see that this is so, according to the definition of a lower bound, you must show that

B2: For every element $x \in T, x \geq 0$.

Recognizing the keywords "for every" in the backward statement $B2$, you should now use the choose method to choose

A1: An element $x \in T$,

for which it must be shown that

B3: $x \geq 0$.

However, $B3$ is true because $x \in T$ and by the defining property of T, $x > 0$. The proof is now complete.

> **Proof.** To see that T has an infimum, note that $a \in T$ if $a > 1$ while $1 \in T$ if $a \leq 1$, so $T \neq \emptyset$. Also, 0 is a lower bound for T because, for $x \in T$, by the defining property of T, $x > 0$. The fact that T has an infimum now follows from the Infimum Property of real numbers. □

D.4 **Analysis of Proof.** Recognizing the keyword "unique" in the conclusion, you should use the backward uniqueness method. Accordingly, the first step is to construct an infimum for the set T. However, this is given in the hypothesis, that is,

A: T has an infimum, say, the real number u.

To complete the backward uniqueness method, you must now show that T has at most one infimum. Here, the direct uniqueness method is used to do so. Accordingly, you should now assume that

A1: The real number v is also an infimum of T.

You must now show that u and v are the same, that is, that

B1: $u = v$.

To that end, working forward from A by definition of an infimum, you have

A2: u is a lower bound for T and for every lower bound w for T, $u \geq w$.

Likewise, from $A1$, because v is a lower bound for T, by definition,

A3: v is a lower bound for T and for every lower bound w for T, $v \geq w$.

Recognizing the keywords "for every" in both forward statements $A2$ and $A3$, you should now use specialization. Specifically, specializing the for-all statement in $A2$ to $w = v$, noting from $A3$ that v is a lower bound for T, the result is that

A4: $u \geq v$.

Likewise, specializing the for-all statement in $A3$ to $w = u$, noting from $A2$ that u is a lower bound for T, the result is that

A5: $v \geq u$.

Statement $B1$ now follows from $A4$ and $A5$, and so the proof is complete.

Proof. The hypothesis ensures the existence of a lower bound, say, u, for the set T. To see that u is unique, assume that v is also a lower bound for T. Then because u is a greatest lower bound for T and v is also a lower bound for T, it follows that $u \geq v$. Likewise, because v is a greatest lower bound for

T and u is also a lower bound for T, it follows that $v \geq u$. This means that $u = v$, and so the proof is complete, □

D.5 **Analysis of Proof.** Not recognizing any keywords in the hypothesis or conclusion, the forward-backward method is used to begin the proof. A key question associated with the conclusion is, "How can I show that a number (namely, 0) is a lower bound for a set (namely, T)?" Using the definition of a lower bound, you must show that

B1: For all elements $x \in T$, $x \geq 0$.

Recognizing the keywords "for all" in the backward statement $B1$, you should now use the choose method to choose

A1: An element $x \in T$,

for which it must be shown that

B2: $x \geq 0$.

Working forward from $A1$ using the defining property of T, you know that

A2: $x > 0$ and $x^2 > 2$.

The fact that $x > 0$ in $A2$ ensures that $B2$ is true, thus completing the proof.

Proof. To see that 0 is a lower bound for T, let $x \in T$ (the word "let" here indicates that the choose method is used). But then, by the defining property of T, $x > 0$. This shows that 0 is a lower bound for T, thus completing the proof. □

D.6 **Analysis of Proof.** Recognizing the keywords "for every" as the first quantifier in the conclusion, the choose method is used to choose

A1: A real number $\epsilon > 0$,

for which it must be shown that

B1: There is an element $x \in T$ such that $x < t + \epsilon$.

Recognizing the keywords "there is" in the backward statement $B1$, the author uses the construction method to produce the element $x \in T$. To do so, the author works forward from the hypothesis that t is the infimum of T which, by definition, means t is a lower bound for T and

A2: For every lower bound s for T, $s \leq t$.

The author then states that

A3: $t + \epsilon$ is not a lower bound for T.

Now $A3$ is true because, if $t + \epsilon$ is a lower bound for T, then you could specialize $A2$ to $s = t + \epsilon$ and hence conclude that $t + \epsilon \leq t$; that is, $\epsilon \leq 0$, which contradicts $A1$. The author then works forward from $A3$ by writing the *NOT* of the definition of a lower bound for a set to obtain

> **A4:** There is an element $x \in T$ such that $x < t + \epsilon$.

The proof is now complete because $A4$ is the same as $B1$.

D.7 **Analysis of Proof.** The author starts with the backward process and asks the key question, "How can I show that a real number (namely, t) is the infimum of a set (namely, T)?" By definition, one answer is to show that

> **B1:** t is a lower bound for T and for every lower bound s for T, $s \leq t$.

The author then notes that the hypothesis states that t is a lower bound for T, so, from $B1$, it remains only to show that

> **B2:** For every lower bound s for T, $s \leq t$.

Recognizing the keywords "for every" the backward statement $B2$, the author uses the choose method to choose

> **A1:** A lower bound s for T,

for which it must be shown that

> **B3:** $s \leq t$.

The author now turns to the contradiction method to show that $B3$ is true and, accordingly, assumes that B is not true, that is, that

> **A2:** $s > t$.

To reach a contradiction, the author first applies specialization to the for-all statement in the hypothesis. Specifically, the author specializes that statement to the real number $\epsilon = s - t$. To apply specialization, this value of ϵ must be > 0, which it is from $A2$. The result of specialization is that

> **A3:** There is an $x \in T$ such that $x < t + \epsilon = t + (s - t) = s$.

Finally, working forward from $A1$ by definition means that

> **A4:** For every element $x \in T$, $x \geq s$.

You can see that $A3$ is the negation of $A4$ and so it is not possible for both $A3$ and $A4$ to be true at the same time. This contradiction completes the proof.

D.8 The sequence $X = (x_1, x_2, \ldots)$ does not converge to the real number x means that there is a real number $\epsilon > 0$ such that for every integer $j \in N$, there is an integer $k \in N$ with $k > j$ such that $|x_k - x| \geq \epsilon$. The statement is obtained by using the rules in Chapter 8 to negate the definition of X converging to x, as follows:

> NOT[for every real number $\epsilon > 0$, there is an integer $j \in N$ such that for all $k \in N$ with $k > j$, $|x_k - x| < \epsilon$].
>
> There is a real number $\epsilon > 0$ such that NOT[there is an integer $j \in N$ such that for all $k \in N$ with $k > j$, $|x_k - x| < \epsilon$].
>
> There is a real number $\epsilon > 0$ such that for every integer $j \in N$, NOT[for all $k \in N$ with $k > j$, $|x_k - x| < \epsilon$].
>
> There is a real number $\epsilon > 0$ such that for every integer $j \in N$, there is an integer $k \in N$ with $k > j$ such that NOT[$|x_k - x| < \epsilon$].
>
> There is a real number $\epsilon > 0$ such that for every integer $j \in N$, there is an integer $k \in N$ with $k > j$ such that $|x_k - x| \geq \epsilon$.

D.9 The sequence $X = (x_1, x_2, \ldots)$ is not monotone increasing means that there is an integer $i \geq 1$ such that $x_i \geq x_{i+1}$. The statement is obtained by using the rules in Chapter 8 to negate the definition of X being monotone increasing, as follows:

> NOT[for each $i = 1, 2, \ldots, x_i < x_{i+1}$].
>
> There is an integer $i \geq 1$ such that NOT[$x_i < x_{i+1}$].
>
> There is an integer $i \geq 1$ such that $x_i \geq x_{i+1}$.

D.10 **Analysis of Proof.** The forward-backward method is used to begin the proof because there are no keywords in the hypothesis or the conclusion. An associated key question is, “How can I show that a sequence [namely, $X = (\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots)$] converges to a real number (namely, 0)?” Using the definition of convergence, one answer is to show that

> **B1:** For every real number $\epsilon > 0$, there is an integer $j \in N$ such that for all $k \in N$ with $k > j$, $|\frac{1}{k} - 0| = \frac{1}{k} < \epsilon$.

Recognizing the keywords “for every” as the first quantifier in the backward statement $B1$, you should use the choose method to choose

> **A1:** A real number $\epsilon > 0$,

for which it must be shown that

> **B2:** There is an integer $j \in N$ such that for all $k \in N$ with $k > j$, $\frac{1}{k} < \epsilon$.

Recognizing the keywords “there is” as the first quantifier in the backward statement $B2$, you should now consider using the construction method. You can turn to the forward process in an attempt to do so. However, another approach is to assume that you have *already* constructed the desired value of $j \in N$. To complete the construction method, you would then have to show that your value of j satisfies the something that happens in $B2$; namely, that

B3: For all $k \in N$ with $k > j$, $\frac{1}{k} < \epsilon$.

The idea now is to try to prove $B3$, and in so doing to discover what value of $j \in N$ allows you to do so. Proceeding with the backward process and recognizing the keywords "for all" in $B3$, you should now use the choose method to choose

A2: An integer $k \in N$ with $k > j$,

for which you must show that

B4: $\frac{1}{k} < \epsilon$.

Now, from $A2$, you know that $k > j$, so

A3: $\frac{1}{k} < \frac{1}{j}$.

You can obtain $B4$ from $A3$ provided that

B5: $\frac{1}{j} < \epsilon$.

Indeed, this tells you that you need to construct j so that $B5$ holds. Solving the inequality in $B5$ for j using the fact that $\epsilon > 0$ leads to

B6: $j > \frac{1}{\epsilon}$.

In other words, constructing j to be any integer satisfying $B6$ enables you to show that $B5$, and hence $B4$, is true, thus completing the proof.

Proof. To see that the sequence $X = (\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots)$ converges to 0, let $\epsilon > 0$. (The word "let" here indicates that the choose method is used.) Now let j be any integer with $j > \frac{1}{\epsilon}$. (The word "let" here indicates that the construction method is used.) Now let $k \in N$ with $k > j$. (The word "let" here indicates that the choose method is used.) You then have that

$$|x_k - 0| = \frac{1}{k} < \frac{1}{j} < \epsilon.$$

The proof is now complete. □

D.11 **Analysis of Proof.** According to Exercise D.8, to show that the sequence $X = (\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots)$ does not converge to 1, you must show that

B1: There is a real number $\epsilon > 0$ such that for every integer $j \in N$, there is an integer $k \in N$ with $k > j$ such that $|\frac{1}{k} - 1| \geq \epsilon$.

Recognizing the keywords "there is" as the first quantifier in the backward statement $B1$, you should use the construction method. You can turn to the forward process in an attempt to do so. However, another approach is to assume that you have *already* constructed the desired value of ϵ. To complete the construction method, you would then have to show that your value of ϵ satisfies the something that happens in $B1$, namely, that

B2: For every integer $j \in N$, there is an integer $k \in N$ with $k > j$ such that $|\frac{1}{k} - 1| \geq \epsilon$.

The idea now is to try to prove $B2$ and, in so doing, to discover what value of ϵ allows you to do so. Proceeding with the backward process and recognizing the keywords "for all" in $B2$, you should use the choose method to choose

A1: An integer $j \in N$,

for which you must show that

B3: There is an integer $k \in N$ with $k > j$ such that $|\frac{1}{k} - 1| \geq \epsilon$.

Recognizing the keywords "there is" in the backward statement $B3$, you should use the construction method to produce an integer $k \in N$ with $k > j$ such that $|\frac{1}{k} - 1| \geq \epsilon$. In other words, you must construct $k > j$ such that

$$\left|\frac{1}{k} - 1\right| = 1 - \frac{1}{k} \geq \epsilon, \quad \text{that is,} \quad \frac{1}{k} \leq 1 - \epsilon. \tag{D.1}$$

Solving equation (D.1) for k means you must construct $k > j$ so that

$$k \geq \frac{1}{1-\epsilon}. \tag{D.2}$$

In summary, if the value of ϵ is, say, $\epsilon = \frac{3}{4}$, then, by constructing the integer $k > j$ with $k \geq 4$ [from (D.2)], this value of ϵ will satisfy the something that happens in $B1$, as shown in the following condensed proof.

Proof. To see that the sequence $X = (\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots)$ does not converge to 1, according to Exercise D.8, it must be shown that there is a real number $\epsilon > 0$ such that for every integer $j \in N$, there is an integer $k \in N$ with $k > j$ such that $|\frac{1}{k} - 1| \geq \epsilon$. To that end, let $\epsilon = \frac{3}{4}$ (the word "let" here indicates that the construction method is used). Now let $j \in N$ (the word "let" here indicates that the choose method is used). Finally, constructing the integer $k > \max\{j, 4\}$, you have that $k > j$ and also

$$\left|\frac{1}{k} - 1\right| = 1 - \frac{1}{k} \geq \frac{3}{4} = \epsilon.$$

The proof is now complete. $\square$

D.12 a. **Analysis of Proof.** The forward-backward method is used to begin the proof because there are no keywords in the hypothesis or the conclusion. A key question associated with the conclusion is, "How can I show that a sequence [namely, $-Y = (-y_1, -y_2, \ldots)$] converges to a real number (namely, $-y$)?" Using the definition of convergence, one answer is to show that

B1: For every real number $\epsilon > 0$, there is an integer $j \in N$ such that for all $k \in N$ with $k > j$, $|-y_k - (-y)| = |y_k - y| < \epsilon$.

Working forward now from the hypothesis that the sequence $Y = (y_1, y_2, \ldots)$ converges to y, by definition you know that

A1: For every real number $\epsilon > 0$, there is an integer $j \in N$ such that for all $k \in N$ with $k > j$, $|y_k - y| < \epsilon$.

The proof is now complete because $A1$ is the same as $B1$.

Proof. To show that the sequence $-Y = (-y_1, -y_2, \ldots)$ converges to y, by definition of convergence, it must be shown that for every real number $\epsilon > 0$, there is an integer $j \in N$ such that for all $k \in N$ with $k > j$, $|-y_k-(-y)| = |y_k-y| < \epsilon$. However, this is true because, from the hypothesis, it is given that the sequence $Y = (y_1, y_2, \ldots)$ converges to y. The proof is now complete. □

b. **Analysis of Proof.** The forward-backward method is used to begin the proof because there are no keywords in the hypothesis or the conclusion. A key question associated with the conclusion is, "How can I show that the sum of two sequences (namely, X and $-Y$) converges to the sum of two real numbers (namely, x and $-y$)?" One answer is to use previous knowledge (see Chapter 3) in the form of Proposition 41, whose conclusion is that the sum of two sequences converges to a sum of two real numbers. Accordingly, you must show that the hypotheses of Proposition 41 are satisfied for the two sequences X and $-Y$, that is, you must show that

B1: The sequence X converges to the real number x and the sequence $-Y$ converges to the real number $-y$.

Now the hypothesis of the proposition in this exercise states that X converges to x. Finally, the fact that the sequence $-Y$ converges to $-y$ is proved in part (a) of this exercise. The proof is now complete because $B1$ is true.

Proof. The hypothesis states that the sequence X converges to the real number x. Also, in part (a), it was proved that the sequence $-Y$ converges to the real number $-y$. Thus, by Proposition 41, it follows that the sequence $X + (-Y) = X - Y$ converges to the real number $x + (-y) = x - y$, and so the proof is complete. □

D.13 **Analysis of Proof.** The words, "Suppose, to the contrary, ..." indicate that the author is using the contradiction method. Accordingly, the author assumes the hypothesis

A: For every real number $\epsilon > 0$, $|x - y| < \epsilon$.

and also

A1 (NOT B): $x \neq y$.

The author then specializes A to the specific value $\epsilon = |x - y|$. To apply specialization, it is necessary to show that this value of ϵ satisfies the certain property in A of being > 0. This, however, is true because $x \neq y$ from $A1$ and so $\epsilon = |x - y| > 0$. The result of specializing $A2$ to $\epsilon = |x - y|$ is that

A2: $|x - y| < |x - y|$.

Now $A2$ is a contradiction because a number (namely, $|x - y|$) cannot be strictly less than itself, thus completing the proof.

D.14 **Analysis of Proof.** Recognizing the keyword "only" in the conclusion, the author uses the backward uniqueness method. The first step therefore is to construct the object which, in this case, is given in the hypothesis, that is,

A: The sequence X converges to the real number x.

To show that x is the only real number to which the sequence X converges, the author then uses the direct uniqueness method. Accordingly, the author assumes that there is another such object, y, that is,

A1: The sequence X also converges to the real number y.

The author must now show that x and y are the same, that is, that

B1: $x = y$.

Working backward from $B1$, the author asks the key question, "How can I show that two real numbers (namely, x and y) are equal?" In this case, the author uses the knowledge in the previous exercise to answer this question. Specifically, $B1$ will follow if the author can show that

B2: For every real number $\epsilon > 0$, $|x - y| < \epsilon$.

Recognizing the keywords "for all" in the backward statement $B2$, the author uses the choose method to choose

A2: A real number $\epsilon > 0$,

for which it must be shown that

B3: $|x - y| < \epsilon$.

The author then turns to the forward process. Specifically, working forward from A by using the definition of convergence, it follows that

> **A3:** For every real number $\bar{\epsilon} > 0$, there is an integer $j \in N$ such that for all $k \in N$ with $k > j$, $|x_k - x| < \bar{\epsilon}$.

Recognizing the keywords "for every" in the forward statement $A3$, the author applies specialization using the value $\bar{\epsilon} = \epsilon/2 > 0$ (from $A2$). Note that, at this point, it is not clear why the value of $\epsilon/2$ is used but should be related to showing that $B3$ is true. Nevertheless, the result of specialization is that

> **A4:** There is an integer $j_1 \in N$ such that for all $k \in N$ with $k > j_1$, $|x_k - x| < \epsilon/2$.

Similarly, working forward from $A1$ using the same reasoning as in $A3$ and $A4$, it follows that

> **A5:** There is an integer $j_2 \in N$ such that for all $k \in N$ with $k > j_2$, $|x_k - y| < \epsilon/2$.

The author then constructs

> **A6:** An integer $k \in N$ with $k > \max\{j_1, j_2\}$.

At this point, it is not clear why the author does so, but the reason should be related to showing that $B3$ is true. Indeed, the author now works forward from the left side of $B3$ to claim that

> **A7:** $|x - y| = |x - x_k + x_k - y| \quad (\text{add } 0 = x_k - x_k)$.

Then, because $|a + b| \leq |a| + |b|$, the author states that

> **A8:** $|x - x_k + x_k - y| \leq |x - x_k| + |x_k - y|$.

The author then notes that each of the terms on the right side of the inequality in $A8$ is $< \epsilon/2$. To see that this is true, specialize the for-all statements in the forward statements $A4$ and $A5$ to the integer $k \in N$ in $A6$. To do so, it must be that $k > j_1$ and $k > j_2$. Indeed k satisfies these properties because k was constructed in $A6$ to satisfy $k > \max\{j_1, j_2\}$. The result of these specializations is that

> **A9:** $|x - x_k| < \epsilon/2$ and $|x_k - y| < \epsilon/2$.

Combining $A7$, $A8$ and $A9$ yields

> **A10:** $|x - y| = |x - x_k + x_k - y| \leq |x - x_k| + |x_k - y| = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

The proof is now complete because $A10$ is the same as $B3$.