ENM 360: Introduction to Data-driven Modeling

Lecture #5: Probability and Statistics primer

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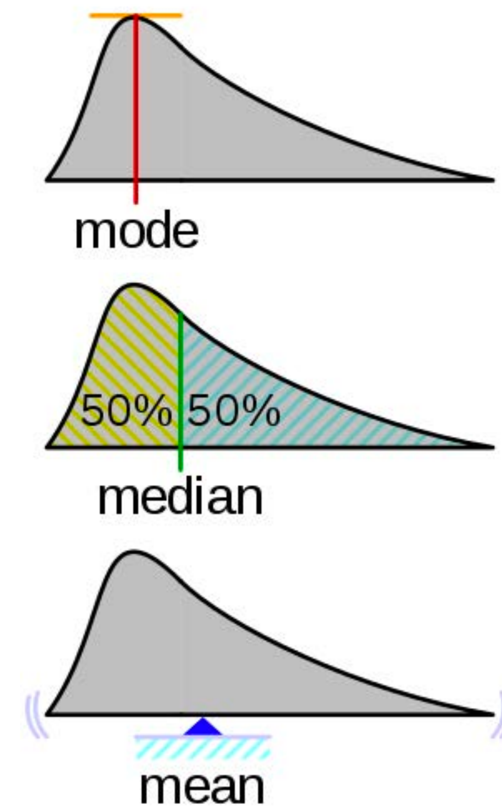
Recap: Continuous random variables

- A ***continuous random variable*** is one which takes an infinite number of possible values. Continuous random variables are usually measurements. Examples include height, weight, the amount of sugar in an orange, the time required to run a mile.
- A continuous random variable is not defined at specific values. Instead, it is defined over an *interval* of values, and is represented by the ***area under a curve*** (in advanced mathematics, this is known as an *integral*). The probability of observing any single value is equal to 0, since the number of values which may be assumed by the random variable is infinite.

- Suppose a random variable X may take all values over an interval of real numbers. Then the probability that X is in the set of outcomes A, $P(A)$, is defined to be the area above A and under a curve. The curve, which represents a function $p(x)$, must satisfy the following:
- ***1:*** *The curve has no negative values ($p(x) \geq 0$ for all x)*
- ***2:*** *The total area under the curve is equal to 1.*
- A curve meeting these requirements is known as a ***density curve***.

Probability density functions



$$\Pr[a \le X \le b] = \int_a^b f_X(x)\,dx.$$

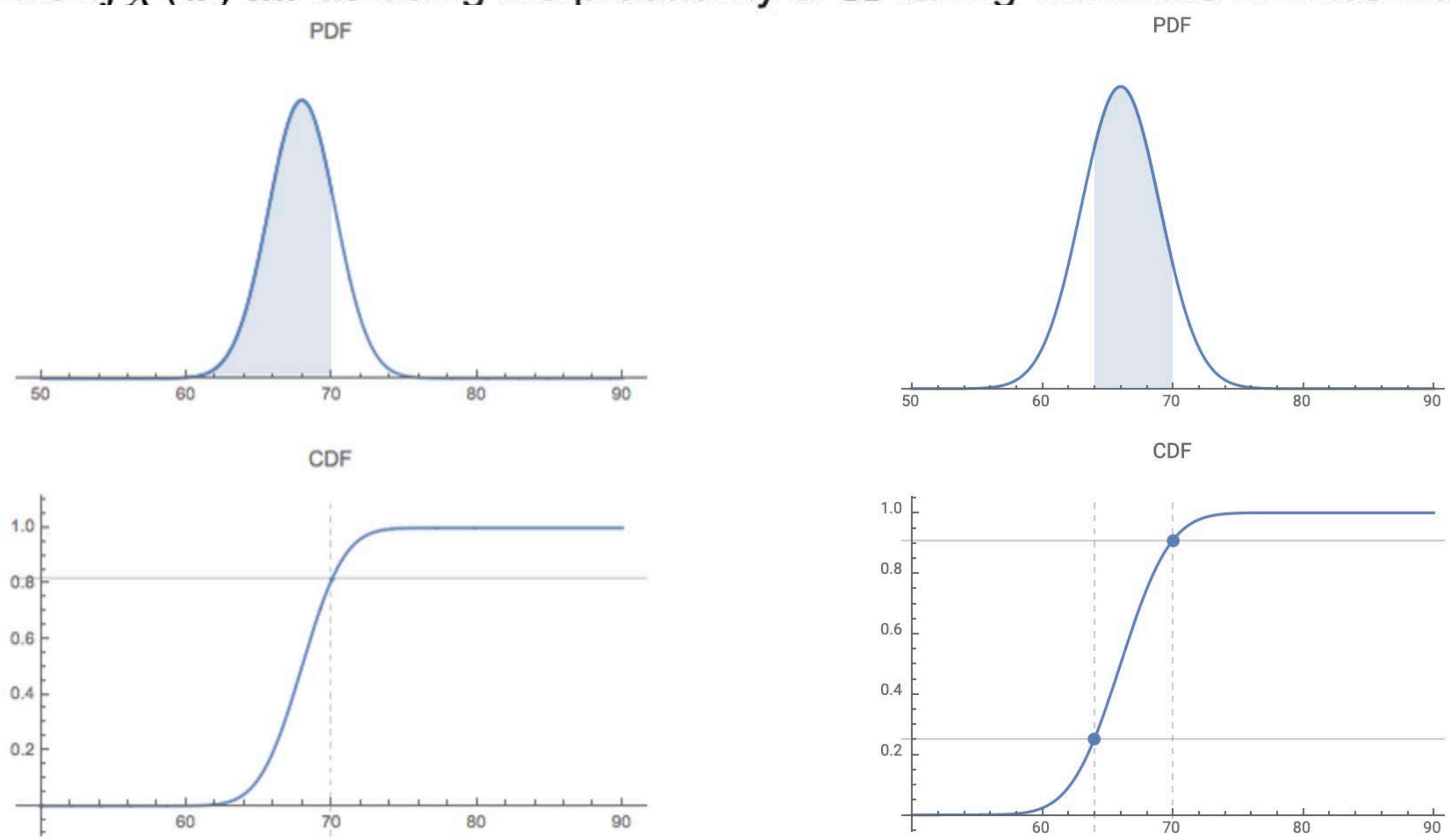
Hence, if F_X is the cumulative distribution function of X, then:

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du,$$

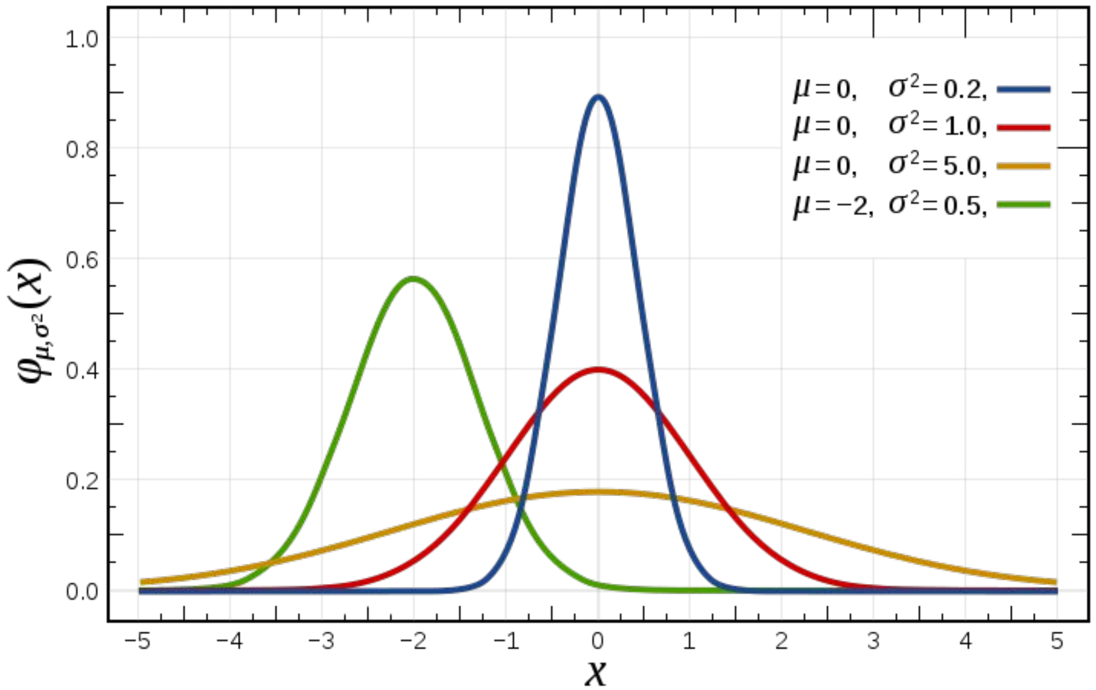
and (if f_X is continuous at x)

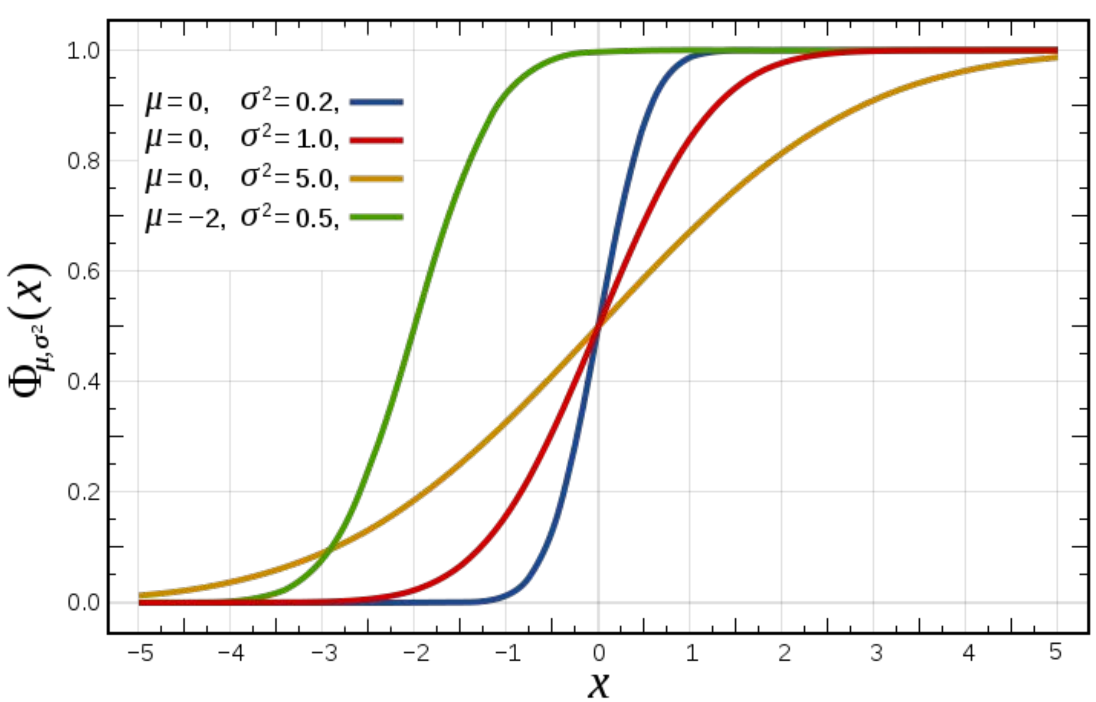
$$f_X(x) = \frac{d}{dx}F_X(x).$$

Intuitively, one can think of $f_X(x)\,dx$ as being the probability of X falling within the infinitesimal interval $[x, x + dx]$.



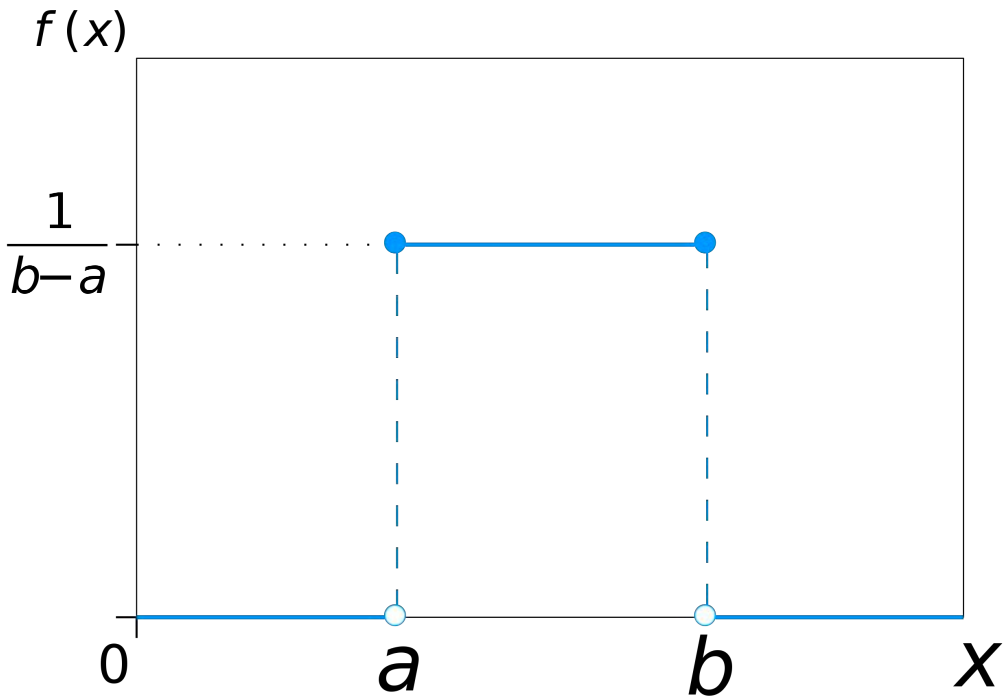
Recap: Univariate Gaussian distribution

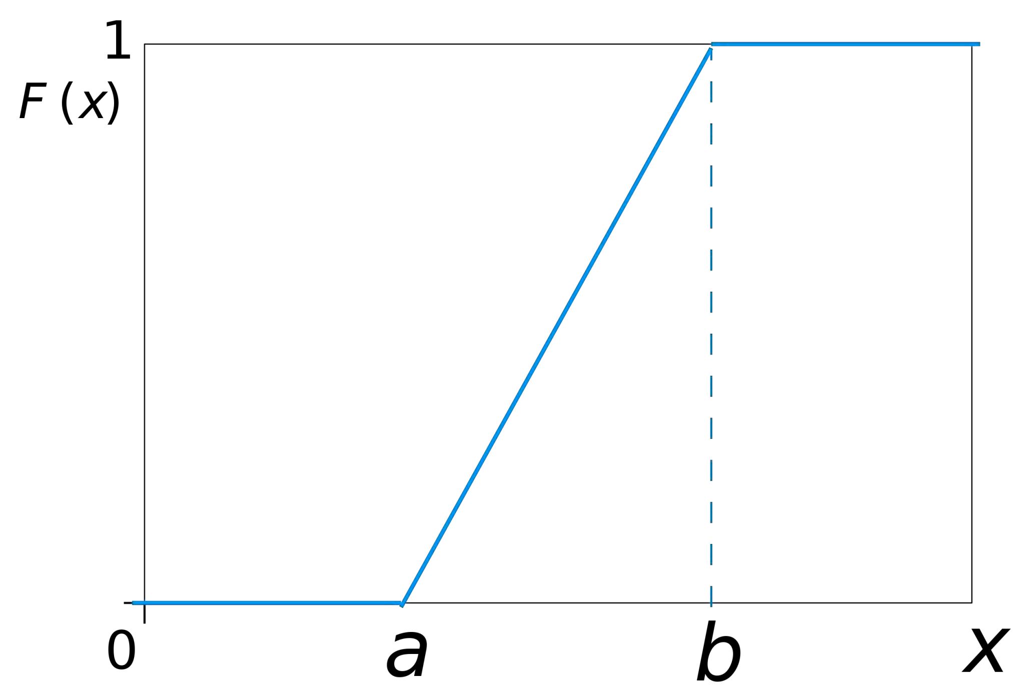




Notation	$\mathcal{N}(\mu,\sigma^2)$
Parameters	$\mu \in \mathbb{R}$ = mean (location) $\sigma^2 > 0$ = variance (squared scale)
Support	$x \in \mathbb{R}$
PDF	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
CDF	$\frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$
Quantile	$\mu+\sigma\sqrt{2}\operatorname{erf}^{-1}(2F-1)$
Mean	μ
Median	μ
Mode	μ
Variance	σ^2
Skewness	0
Ex. kurtosis	0
Entropy	$\frac{1}{2}\log(2\pi e\sigma^2)$
MGF	$\exp(\mu t+\sigma^2 t^2/2)$
CF	$\exp(i\mu t-\sigma^2 t^2/2)$
Fisher information	$\mathcal{I}(\mu,\sigma)=\begin{pmatrix}1/\sigma^2 & 0\\ 0 & 2/\sigma^2\end{pmatrix}$ $\mathcal{I}(\mu,\sigma^2)=\begin{pmatrix}1/\sigma^2 & 0\\ 0 & 1/(2\sigma^4)\end{pmatrix}$
Kullback-Leibler divergence	$D_{\mathrm{KL}}(\mathcal{N}_0\Vert\mathcal{N}_1)=\frac{1}{2}\{(\sigma_0/\sigma_1)^2+\frac{(\mu_1-\mu_0)^2}{\sigma_1^2}-1+2\ln\frac{\sigma_1}{\sigma_0}\}$

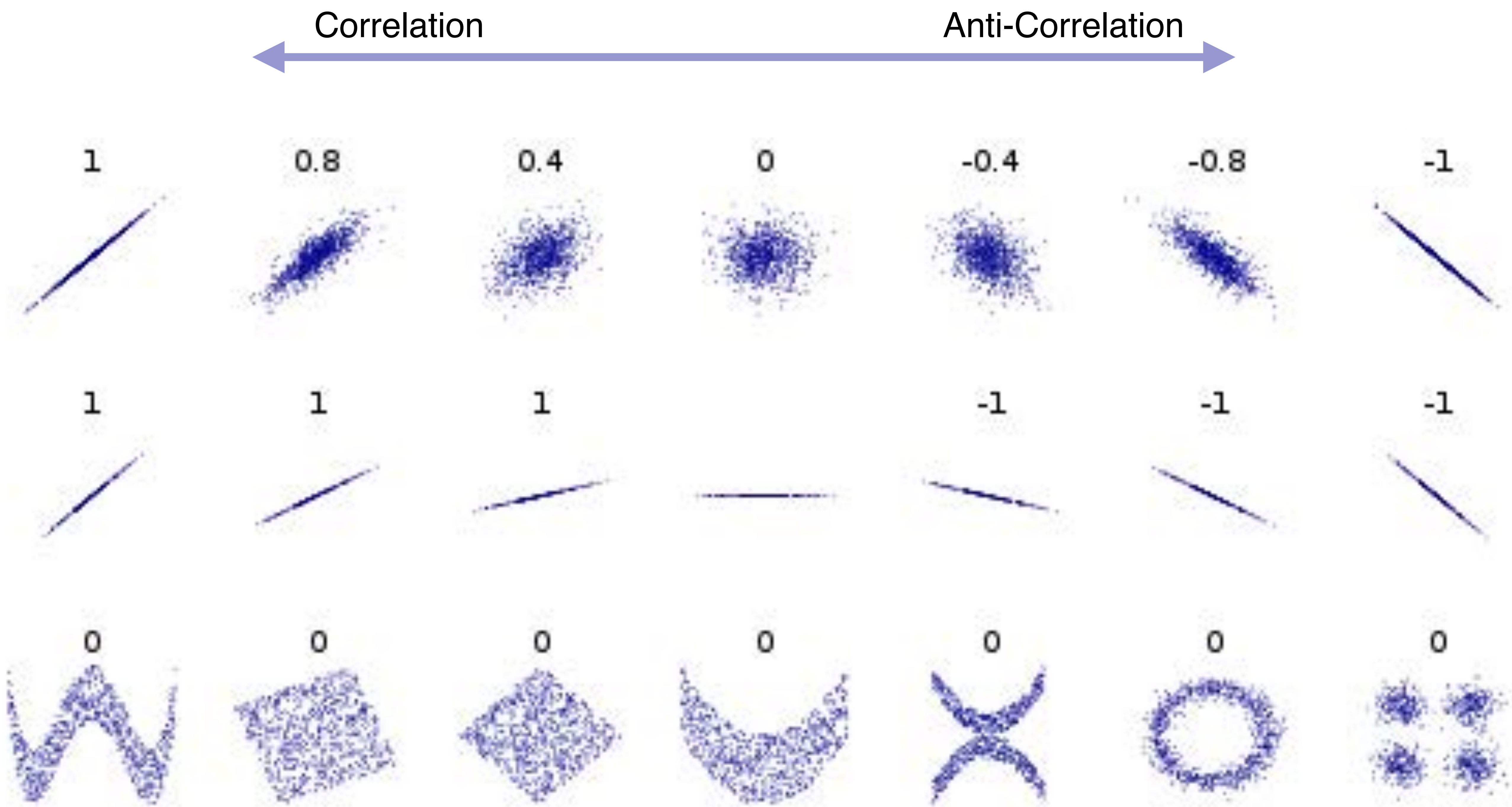
Recap: Uniform distribution



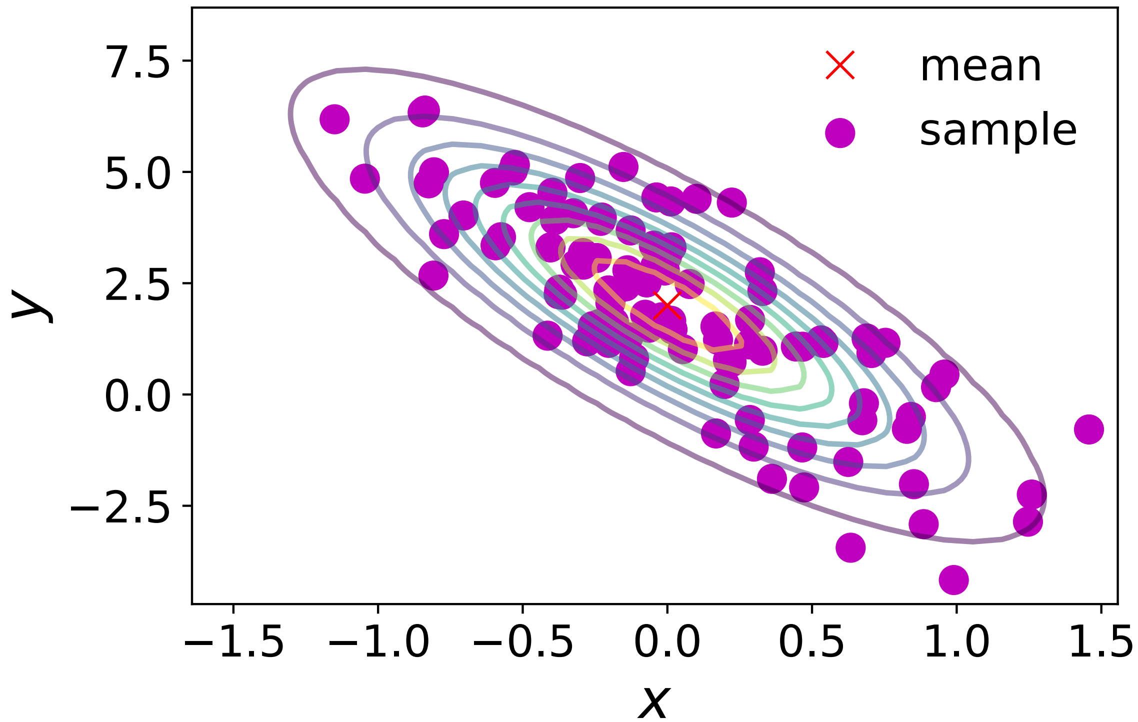


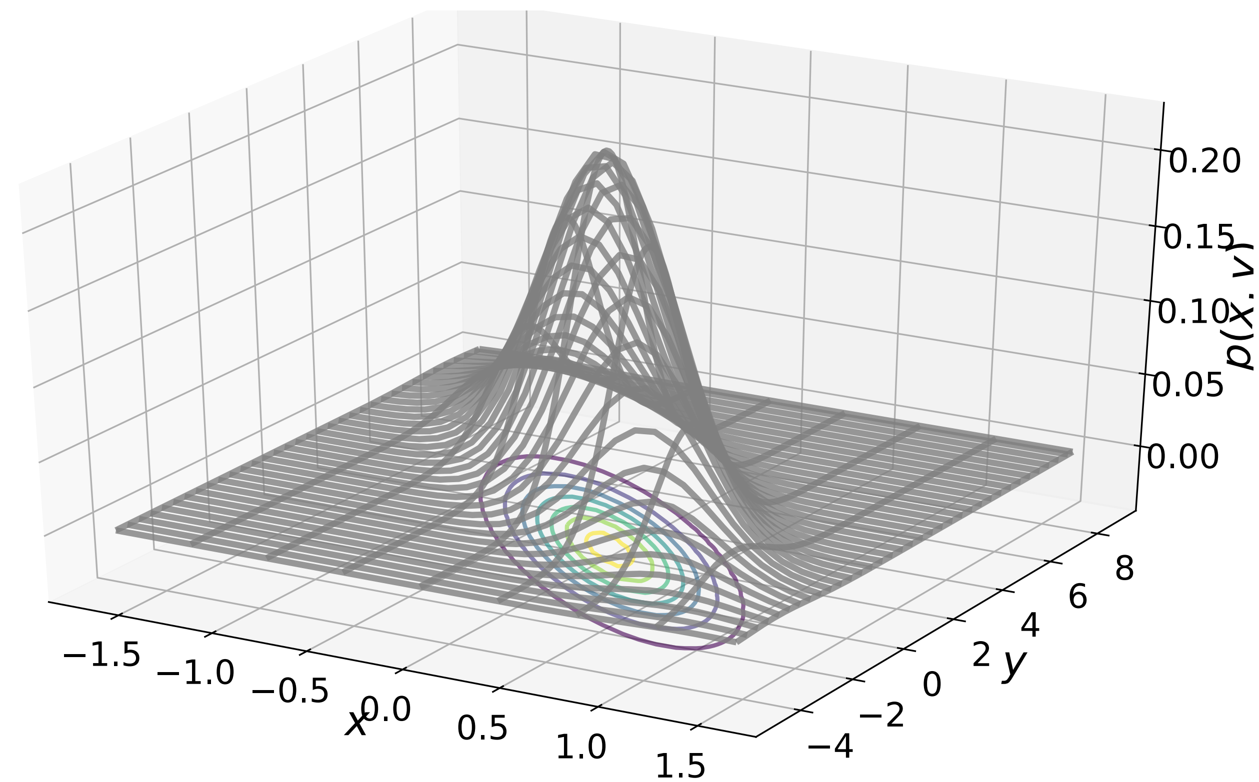
Notation	$\mathcal{U}(a,b)$ or $\mathrm{unif}(a,b)$
Parameters	$-\infty < a < b < \infty$
Support	$x \in [a,b]$
PDF	$\begin{cases} \frac{1}{b-a} & \text{for } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$
CDF	$\begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } x \in [a,b) \\ 1 & \text{for } x \geq b \end{cases}$
Mean	$\frac{1}{2}(a+b)$
Median	$\frac{1}{2}(a+b)$
Mode	any value in (a,b)
Variance	$\frac{1}{12}(b-a)^2$
Skewness	0
Ex. kurtosis	$-\frac{6}{5}$
Entropy	$\ln(b-a)$
MGF	$\begin{cases} \frac{e^{tb}-e^{ta}}{t(b-a)} & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}$
CF	$\begin{cases} \frac{e^{itb}-e^{ita}}{it(b-a)} & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}$

Correlation and linear dependence



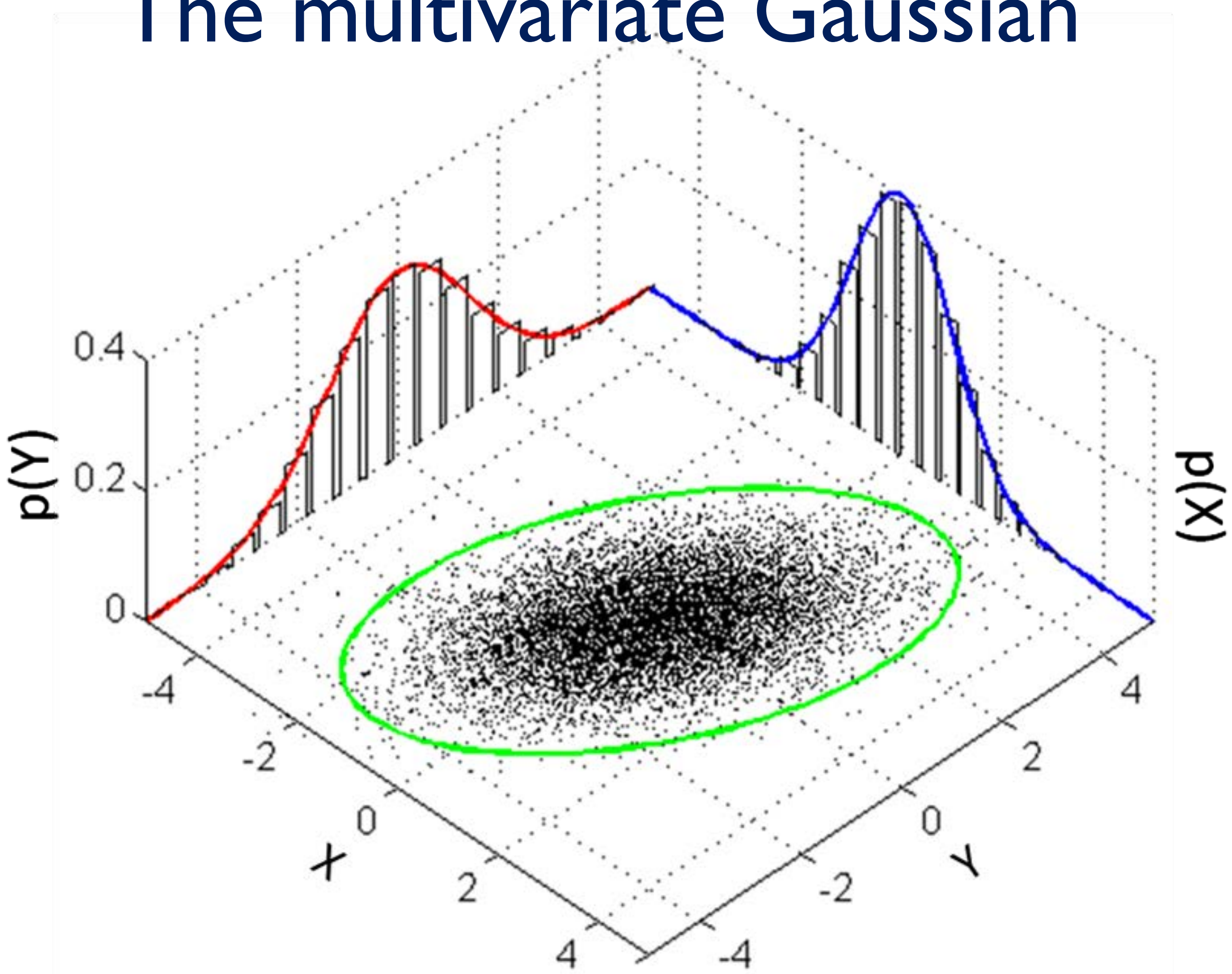
The multivariate Gaussian





$$p(\boldsymbol{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-\frac{D}{2}} |\boldsymbol{\Sigma}|^{-\frac{1}{2}} \exp\big(-\tfrac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\big)$$

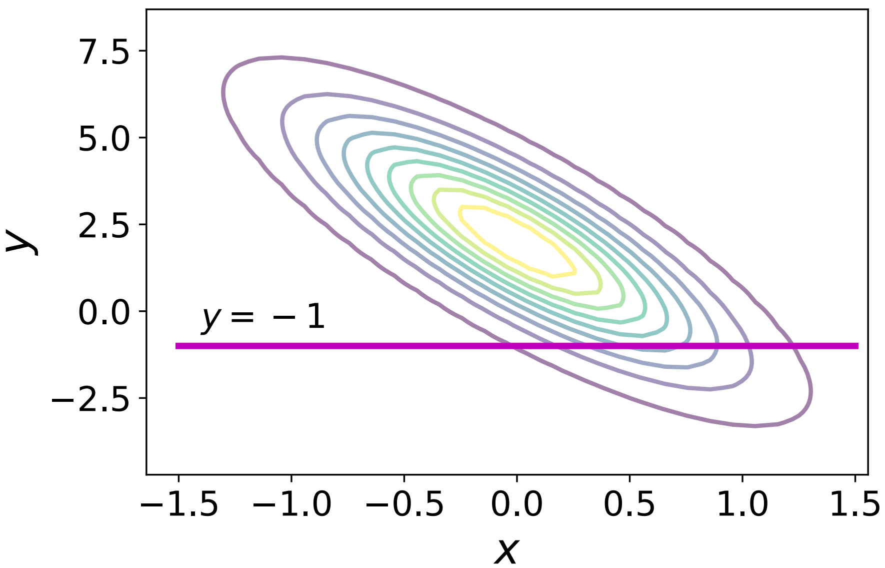
The multivariate Gaussian



Notation	$\mathcal{N}(\boldsymbol{\mu},\, \boldsymbol{\Sigma})$
Parameters	$\boldsymbol{\mu} \in \mathbf{R}^k$ — location $\boldsymbol{\Sigma} \in \mathbf{R}^{k \times k}$ — covariance (positive semi-definite matrix)
Support	$\boldsymbol{x} \in \boldsymbol{\mu} + \mathrm{span}(\boldsymbol{\Sigma}) \subseteq \mathbf{R}^k$
PDF	$\det(2\pi\boldsymbol{\Sigma})^{-\frac{1}{2}}\, e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}$, exists only when $\boldsymbol{\Sigma}$ is positive-definite
Mean	$\boldsymbol{\mu}$
Mode	$\boldsymbol{\mu}$
Variance	$\boldsymbol{\Sigma}$

Marginals and conditionals of a Gaussian

$$p(\boldsymbol{x}, \boldsymbol{y}) = \mathcal{N}\left(\begin{bmatrix}\boldsymbol{\mu}_x \\ \boldsymbol{\mu}_y\end{bmatrix}, \begin{bmatrix}\boldsymbol{\Sigma}_{xx} & \boldsymbol{\Sigma}_{xy} \\ \boldsymbol{\Sigma}_{yx} & \boldsymbol{\Sigma}_{yy}\end{bmatrix}\right)$$



Conditional distribution

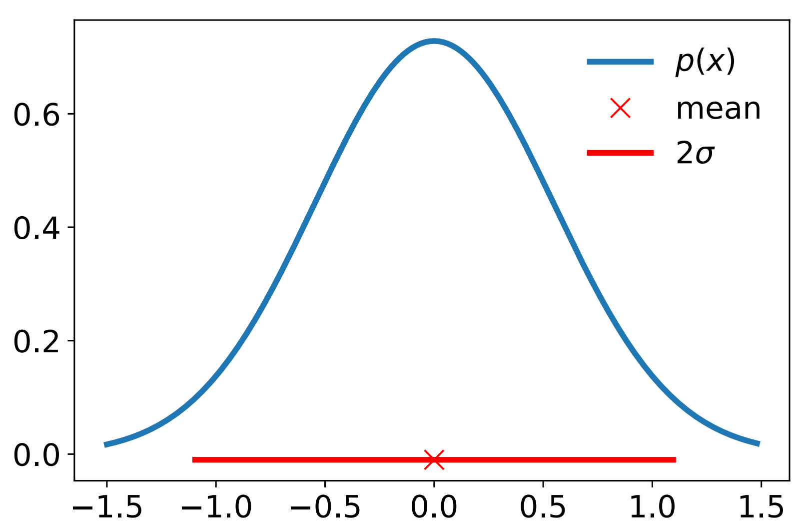
$$p(\boldsymbol{x} \,|\, \boldsymbol{y}) = \mathcal{N}\big(\boldsymbol{\mu}_{x\,|\,y}, \, \boldsymbol{\Sigma}_{x\,|\,y}\big)$$

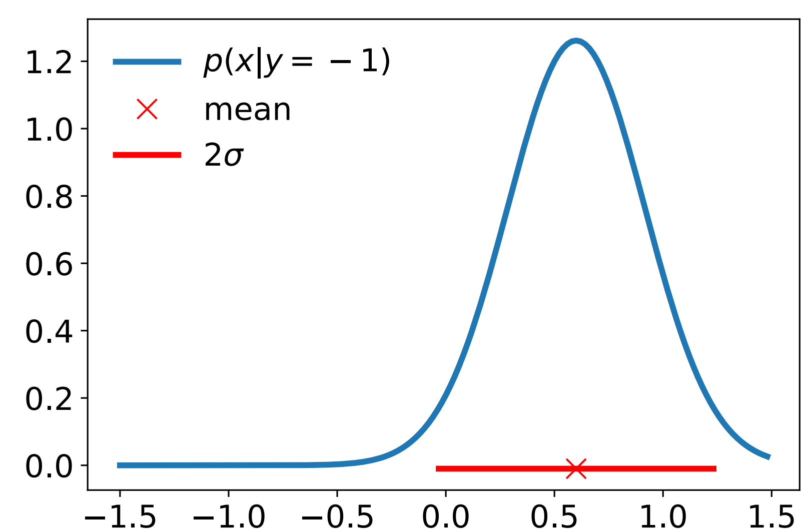
$$\boldsymbol{\mu}_{x\,|\,y} = \boldsymbol{\mu}_x + \boldsymbol{\Sigma}_{xy}\boldsymbol{\Sigma}_{yy}^{-1}(\boldsymbol{y} - \boldsymbol{\mu}_y)$$

$$\boldsymbol{\Sigma}_{x\,|\,y} = \boldsymbol{\Sigma}_{xx} - \boldsymbol{\Sigma}_{xy}\boldsymbol{\Sigma}_{yy}^{-1}\boldsymbol{\Sigma}_{yx} \,.$$

Marginal distribution

$$p(\boldsymbol{x}) = \int p(\boldsymbol{x}, \boldsymbol{y}) d\boldsymbol{y} = \mathcal{N}\big(\boldsymbol{x} \,|\, \boldsymbol{\mu}_x, \, \boldsymbol{\Sigma}_{xx}\big)$$





These are unique properties that make the Gaussian distribution very simple and attractive to compute with! It is essentially our main building block for computing under uncertainty.

Transformations