Problem 1. Greedy Densest Subgraphs

This problem is about something something about finding a special case where an algorithm performs poorly. But we will also solve some equations!

Here is an algorithm to consider:

Algorithm 1: `GreedyDensestSubgraph`

```
begin
    Best ← V(G)
    S ← V(G)
    while |S| > 1 do
        u ← vertex of minimum degree in G_S
        S ← S \ {u}
        if d(G_S) > d(G_Best) then
            Best ← S
    return Best
```

Next we have a numbered equation:

$$\sum_i \sum_p \alpha_{ij}\beta_{pq}(x_i \otimes y_p). \tag{1}$$

(*a*) This is the first task to solve.

(*b*) This is the second task.

(*c*) The final task is to compute something.

Solution

Part (a)
Consider the graph constructed below. It is a connected graph with two components, here referred to as the *fan* and *trunk*, where the fan is the topmost component, and the trunk is the bottom 3-regular component. We first note that the fan has density $d(G_{fan}) = \frac{46}{25} \approx 1.84$. Furthermore, note that the fan-subgraph contains two vertices of degree 2, any of which will be removed first by the algorithm. The removal of any of them will result in a total decrease of $|E|$ by 2 and of $|V|$ by 1, while also resulting in a new vertex of first 2, then later 1, appearing. Since the trunk is 3-regular, this will result in the entire fan being removed first before the trunk is processed, and at each step while the fan is being removed the current density is never greater than the initial density observed at the start of the algorithm.

Part (b)
Here we reference the proof by Portugal Renato Portugal[1] where the result is shown.

Part (c)
Regarding the equation 1, we make the following observation…

[1] "Basic Quantum Algorithms," *arXiv:2201.10574 [Quant-Ph]*, 2022, http://arxiv.org/abs/2201.10574.