Mr. Awesome

Engineering Mathematics - III

Semester III – Mechanical Engineering / Computer Engineering / Electronics & Telecommunication Engineering / Electronics Engineering

(Code : BT-BSC 301)

(Dr. Babasaheb Ambedkar Technological University)

Strictly as per the New Syllabus of Dr. Babasaheb Ambedkar Technological University w.e.f. academic year 2018-2019

Santosh R. Mitkari
M.sc Mathematics
Assistant Professor
F.E. Coordinator
Department of Engineering Sciences and Allied Engineering,
Bharati Vidyapeeth's College of Engineering for Women,
Pune – 411 043. Maharashtra, India

Balasaheb B. Gadekar
M.sc Mathematics
Assistant Professor
Department of Engineering Sciences
JSPM'S Rajarshi Shahu College of Engg.
Tathawade, Pune - 33

Mahesh V. Ghotkar
M.sc Mathematics
Assistant Professor
Department of Engineering Sciences
JSPM'S Rajarshi Shahu College of Engg.
Tathawade, Pune – 33

Dr. Dhondge Satyawan Laxman
M.Sc., P.hd. in Mathematics
Head, Basic Science & Humanities Department
BOS Member in Mechanical Engineering
Deogiri Institute of Engineering & Management Studies
Aurangabad.

Reviewed by ...

Prof. Uttara R. Borsarkar
M. Phil (Maths), M.Sc., (Maths, Computer Science) B. Ed. MIT(E),
Assistant Professor in Mathematics,
Marathwada Institute Of Technology, Aurangabad
Maharashtra

Tech-Max Publications, Pune
Innovation Throughout Engineering Division

व्यंकटेश बुक सेंटर
परभणी फोन.(02452)229892
BATU13A

Engineering Mathematics - III

Santosh R. Mitkari, Balasaheb B. Gadekar , Mahesh V. Ghotkar, Dr. Dhondge Satyawan Laxman

(Semester III – Mechanical Engineering / Computer Engineering / Electronics & Telecommunication Engineerign / Electronics Engineering,)

Dr. Babasaheb Ambedkar Technological University

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First Edition : August 2018

This edition is for sale in India, Bangladesh, Bhutan, Maldives, Nepal, Pakistan, Sri Lanka and designated countries in South-East Asia. Sale and purchase of this book outside of these countries is unauthorized by the publisher.

Printed at : Image Offset, Dugane Ind. Area Survey No. 28/25, Dhayari Near Pari Company, Pune - 41, Maharashtra State, India. E-mail : rahulshahimage@gmail.com

ISBN : 978-93-88200-50-9

Published by
Tech-Max Publications

Head Office : B/5, First floor, Maniratna Complex, Taware Colony, Aranyeshwar Corner, Pune - 411 009. Maharashtra State, India
Ph : 91-20-24225065, 91-20-24217965. Fax 020-24228978.
Email : info@techmaxbooks.com,
Website : www.techmaxbooks.com

[BT-BSC 301] (FID : BATU13/388) (Book Code : BATU13A)

Preface

"Our Passion is for Lucidity. We don't mean simple mindness.
If people can't understand it, why write it."

Dear students,

We are extremely happy to come out with this edition of **"Engineering Mathematics - III"** for you, the Second year engineering students of Dr. BATU. We are very glad for your overwhelming response of our Engineering Mathematics – I / II.books. The purpose of this book is to introduce Engineering Mathematics- III in a simple and easy way. This book has been designed for the use of second year engineering students to remove the fear of the subject. This text book has been written strictly as per the New Syllabus prescribed by University of Dr. Babasaheb Ambedkar Technological University. During our twenty years teaching experience ,we had fully understood the need of the students and hence we had taken a great care to present the subject in the most clear, interesting and complete form ,from the students point of view.

A large number of examples have been solved in simple and honest reasoning explained step by step which anybody can understand easily with concentration. Generally, a student has difficulty to understand the mathematical concepts only because, the author skips steps which he assumes the students to be familiar with. We avoided such gaps and all the necessary and useful part has been arranged in a proper sequence.. Reading this book will give you a great satisfaction to understand the concepts of the subject.The example in each article has been carefully graded. Attention is called to examples occurring where a slight variation in statement changes the nature of problem. We hope that this book will serve as great introductions for engineering students.. We discussed all kinds of problems. We had shown that how to solve problems in a simple way through this book.

We sincerely thank to Prof. S.L. Bhilare, Assistant Executive Director (JSPM'sTathawade campus), Dr. R. K. Jain, Principal RSCOE, Pune, Dr. H. V. Vankudre,, Principal BVCOEW, Pune , Prof. A.S. Devasthali, Vice-Principal RSCOE, Pune. and Prof. A.M. Pawar, HOD, for continuous support in all respect. We are thankful to Mr. Sachin Shah for the encouragement and support that they have extended to us. We are also thankful to staff members of Tech-Max Publications for their efforts to make this book as good it is.

The suggestions for improvement of the text will be thankfully welcome.

- Authors

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SYLLABUS

Semester III

Engineering Mathematics – III

BT-BSC 301	BSC 7	Engineering Mathematics-III	3-1-0	4 Credits

Teaching Scheme:	Examination Scheme:
Lecture: 3 hrs/week Tutorial: 1 hr/week	Continuous Assessment: 20 Marks Mid Semester Exam: 20 Marks End Semester Exam: 60 Marks(Duration 03 hrs)

Pre-Requisites : None

Course Outcomes : At the end of the course, students will be able to:

CO1	
CO2	
CO3	
CO4	
CO5	
CO6	
CO7	
CO8	

Mapping of course outcomes with program outcomes

Course Outcomes	Program Outcomes											
	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
CO1												
CO2												
CO3												
CO4												
CO5												
CO6												
CO7												
CO8												

Course Contents

Unit 1 : Laplace Transform [07 Hours]

Definition – conditions for existence ; Transforms of elementary functions ; Properties of Laplace transforms - Linearity property, first shifting property, second shifting property, transforms of functions multiplied by t^n, scale change property, transforms of functions divided by t, transforms of integral of functions, transforms of derivatives ; Evaluation of integrals by using Laplace transform ; Transforms of some special functions- periodic function, Heaviside-unit step function, Dirac delta function.

(Refer chapter 1)

Unit 2 : Inverse Laplace Transform [07 Hours]

Introductory remarks ; Inverse transforms of some elementary functions ; General methods of finding inverse transforms ; Partial fraction method and Convolution Theorem for finding inverse Laplace transforms ; Applications to find the solutions of linear differential equations and simultaneous linear differential equations with constant coefficients.

(Refer chapter 2)

Unit 3 : Fourier Transform [07 Hours]

Definitions – integral transforms ; Fourier integral theorem (without proof) ; Fourier sine and cosine integrals ; Complex form of Fourier integrals ; Fourier sine and cosine transforms ; Properties of Fourier transforms ; Parseval's identity for Fourier Transforms.

(Refer chapter 3)

Unit 4 : Partial Differential Equations and Their Applications [07 Hours]

Formation of Partial differential equations by eliminating arbitrary constants and functions; Equations solvable by direct integration; Linear equations of first order (Lagrange's linear equations); Method of separation of variables – applications to find solutions of one dimensional heat flow equation $\left(\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}\right)$, and two dimensional heat flow equation (i.e. Laplace equation : $\frac{\partial^2 u}{\partial x^2} + c^2 \frac{\partial^2 u}{\partial y^2} = 0$).

(Refer chapter 4)

Unit 5 : Functions of Complex Variables (Differential calculus) [07 Hours]

Limit and continuity of f(z); Derivative of f(z) ; Analytic functions; Cauchy- Riemann equations in Cartesian and polar forms; Harmonic functions in Cartesian form; Mapping: Translation, magnification and rotation, inversion and reflection , bilinear transformation; Conformal mapping.

(Refer chapter 5)

Unit 6 : Functions of Complex Variables (Integral calculus) [07 Hours]

Cauchy's integral theorem; Cauchy's integral formula; Residues; Cauchy's residue theorem (All theorems without proofs).

(Refer chapter 6)

□□□

UNIT I

Chapter 1 : Laplace Transform 1-1 to 1-64

Syllabus : Definition – conditions for existence ; Transforms of elementary functions ; Properties of Laplace transforms - Linearity property, first shifting property, second shifting property, transforms of functions multiplied by t^n, scale change property, transforms of functions divided by t, transforms of integral of functions, transforms of derivatives ; Evaluation of integrals by using Laplace transform ; Transforms of some special functions- periodic function, Heaviside-unit step function, Dirac delta function.

1.1 Prerequisite 1-1
✓ **Syllabus Topic :** Definition of Laplace Transform 1-1
1.2 Definition of Laplace Transform 1-1
✓ **Syllabus Topic :** Conditions For Existence 1-2
1.3 Existence Condition 1-2
1.4 Examples using Definition of Laplace Transform 1-2
✓ **Syllabus Topic :** Laplace Transforms of elementary functions 1-4
1.5 Laplace Transform of Elementary Functions 1-4
1.6 Examples using Standard Functions of Laplace Transform 1-7
✓ **Syllabus Topic :** Properties of Laplace transforms 1-9
1.7 Properties of Laplace Transform 1-9
✓ **Syllabus Topic :** Linearity property 1-9
1.7.1 Linearity property : If a, b, c be any Constants and f, g, h any Functions of t, then 1-9
✓ **Syllabus Topic :** First Shifting Property 1-9
1.7.2 First Shifting Theorem **(May 17)** 1-9
✓ **Syllabus Topic :** Scale Change Property 1-9
1.7.3 Change of Scale Property 1-9
✓ **Syllabus Topic :** Transforms of Functions Multiplied by t^n 1-10
1.7.4 Multiplication by t^n 1-10
✓ **Syllabus Topic :** Transforms of Functions divided by t 1-10
1.7.5 Division by t 1-10
✓ **Syllabus Topic :** Transforms of Derivatives 1-11
1.7.6 Laplace Transform of Derivatives 1-11
1.7.7 The Convolution Theorem 1-11
✓ **Syllabus Topic :** Transforms of Integral of Functions 1-12
1.7.8 Laplace Transform of Intergration : If L[f (t)] = f (s) then **(Nov. 16)** 1-12
1.7.9 Second Shifting Theorem 1-12
1.7.10 Initial Value Theorem 1-12
1.7.11 Final Value Theorem 1-12
1.8 Examples Using Properties of Laplace Transform 1-13
1.9 Examples using Trigonometric Formula 1-15
1.10 Examples on First Shifting Theorem 1-20
1.11 Examples on Change of Scale Property 1-24
1.12 Miscelleneous Examples 1-25
✓ **Syllabus Topic :** Second Shifting Property 1-43
1.13 Examples on Second Shifting Theorem 1-43
✓ **Syllabus Topic :** Evaluation of integrals by using Laplace transform 1-46
1.14 Evaluation of Integrals by Using Laplace Transform 1-46
1.15 Laplace Transform by Series Method 1-54
✓ **Syllabus Topic :** Transforms of some special functions 1-56
1.16 Laplace Transform of Special Functions 1-56
✓ **Syllabus Topic :** Transforms of Periodic Function 1-56
1.17 Periodic Function 1-56
1.17.1 Examples on Periodic Function 1-56
✓ **Syllabus Topic :** Heaviside-Unit Step Function 1-58
1.18 Unit Step Function (Heviside Unit Step Function) 1-58
1.18.1 Properties of Laplace Transform of Unit Step Function 1-59
1.18.2 Second Shifting Theorem 1-60
1.18.3 Examples on Unit Step Function (Heviside Unit Step Function) 1-60
✓ **Syllabus Topic :** Dirac delta function 1-61
1.19 Dirac-delta Function (Unit Impulse Function) 1-61
1.19.1 Examples : Using Unit Step Function / Dirac-delta Function 1-62
1.20 Bessel Function 1-63
1.20.1 Examples on Based Function 1-63
• Chapter Ends 1-64

Multiple Choice Questions with Answers M1-1 to M1-10

UNIT II

Chapter 2 : Inverse Laplace Transform 2-1 to 2-43

Syllabus : Introductory remarks ; Inverse transforms of some elementary functions ; General methods of finding inverse transforms ; Partial fraction method and Convolution Theorem for finding inverse Laplace transforms ; Applications to find the solutions of linear differential equations and simultaneous linear differential equations with constant coefficients.

✓ **Syllabus Topic :** Introductory Remarks 2-1
2.1 Inverse Laplace Transforms 2-1
✓ **Syllabus Topic :** Inverse transforms of some elementary functions 2-1
✓ **Syllabus Topic :** General methods of finding inverse transforms 2-1
2.1.1 Examples on Inverse Laplace Transform 2-1
2.2 Properties of Inverse Laplace Transform 2-2
2.3 Examples using First Shifting Property of Inverse Laplace Transform 2-3

2.4 Examples of LT using Property 2-7
2.5 Examples Using Second Shifting Theorem 2-9
✓ **Syllabus Topic :** Inverse Laplace Transforms Using Partial fraction method 2-11
2.6 Partial Fraction of Proper Fraction 2-11
2.6.1 For Linear Factors 2-11
2.6.2 Working Rule to Find Partial Fraction 2-11
2.6.3 Examples using Partial Fraction 2-12
2.7 Inverse Laplace Transform of Inverse trigonometric, inverse hyperbolic and Logarithmic Function 2-21
✓ **Syllabus Topic :** Convolution Theorem for finding inverse Laplace transforms 2-29
2.8 Convolution Theorem 2-29
✓ **Syllabus Topic :** Applications to find the solutions of linear differential equations 2-31
2.9 Laplace Transform of Derivative 2-31
2.10 Application of Laplace Transform to the Differential Equation 2-32
✓ **Syllabus Topic :** Applications to Find the Solutions of Simultaneous Linear Differential Equations with Constant Coefficients 2-39
2.11 Simultaneous Differential Equations by Laplace Transform 2-39
• Chapter Ends 2-43

Multiple Choice Questions with Answers M2-1 to M2-8

UNIT III

Chapter 3 : Fourier Transform 3-1 to 3-53

Syllabus : Definitions – integral transforms ; Fourier integral theorem (without proof) ; Fourier sine and cosine integrals ; Complex form of Fourier integrals ; Fourier sine and cosine transforms ; Properties of Fourier transforms ; Parseval's identity for Fourier Transforms.

3.1 Introduction 3-1
✓ **Syllabus Topic :** Integral transforms 3-1
3.1.1 Integral Transforms 3-1
✓ **Syllabus Topic :** Fourier's Integral Theorem 3-1
3.1.2 Fourier's Integral Theorem 3-1
✓ **Syllabus Topic :** Complex form of Fourier Integrals 3-2
3.2 Complex Exponential form of Fourier Series 3-2
3.3 Fourier Integral 3-3
3.4 Fourier Transform 3-3
3.4.1 Definition 3-3
3.4.2 Even Function 3-3
3.4.3 Odd Function 3-4
3.4.4 Fourier transform for Even and Odd Function 3-4
✓ **Syllabus Topic :** Fourier Sine and Cosine Transforms 3-7
3.5 Fourier Sine and Cosine Transforms 3-7
3.5.1 Fourier Cosine Transform (FCT) 3-7
3.5.2 Fourier Sine Transform (FST) 3-7
✓ **Syllabus Topic :** Properties of Fourier Transforms 3-9
3.5.3 Type - I Examples on Fourier integral representation 3-9
3.5.4 Type - II Examples on Fourier Sine/Cosine Transform 3-20
3.5.5 Examples on Use of Differentiation Under Integral Sign(DUIS) Rule 3-31
3.5.6 Type - IV Examples on to Establish the given Integral 3-33
3.5.7 Type-V Examples on Solution of Integral Equation 3-43
✓ **Syllabus Topic :** Parseval's identity 3-52
3.5.8 Parseval's Identity 3-52
• Chapter Ends 3-53

Multiple Choice Questions with Answers M3-1 to M3-9

UNIT IV

Chapter 4 : Partial Differential Equations and Their Applications 4-1 to 4-88

Syllabus :

Formation of Partial differential equations by eliminating arbitrary constants and functions; Equations solvable by direct integration ; Linear equations of first order (Lagrange's linear equations); Method of separation of variables – applications to find solutions of one dimensional heat flow equation $\left(\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}\right)$ and two dimensional heat flow equation (i.e. Laplace equation : $\frac{\partial^2 u}{\partial x^2} + c^2 \frac{\partial^2 u}{\partial y^2} = 0$).

4.1 Introduction 4-1
✓ **Syllabus Topic :** Formation of Partial Differential Equations by Eliminating Arbitrary Constants and Functions 4-1
4.2 Formation of Partial Differential Equations 4-1
4.2.1 Formation of Partial differential Equation by Eliminating an Arbitrary Constants 4-1
4.2.2 Formation of Partial Differential Equation by Eliminating an Arbitrary Function 4-9
4.3 Solutions of Partial Differential Equations 4-17
✓ **Syllabus Topic :** Equations Solvable by Direct Integration 4-18
4.3.1 Solution of Partial Differential Equation - By Direct Integration 4-18
✓ **Syllabus Topic :** Linear Equations of First Order (Lagrange's Linear Equations) 4-23
4.4 Linear Partial Differential Equations of First Order (Lagrange's Linear Equations) 4-23
4.5 Solution of Partial Differential Equations 4-41
✓ **Syllabus Topic :** Method of Separation of Variables 4-41
4.5.1 Method of Separation of Variables 4-41
4.5.2 Working Rule 4-41
4.6 Applications - Partial Differential Equations 4-46
4.6.1 Introduction 4-46

4.6.2 Prerequisite 4-47
4.7 Modelling of Vibrating String / Wave Equation 4-47
4.7.1 Solution of One-dimensional Wave Equation 4-49
4.7.1.1 Procedure to Solve One Dimensional Wave Equation 4-51
✓ **Syllabus Topic :** One Dimensional Heat Flow Equations 4-53
4.8 One Dimensional Heat Flow Equation 4-53
4.8.1 Solution of One Dimensional Heat Equation 4-54
4.8.1.1 Procedure to Solve One Dimensional Heat Flow Equation 4-56
✓ **Syllabus Topic :** Two Dimensional Heat Flow Equations 4-71
4.9 Two Dimensional Heat Flow Equations (Laplace Equations in two Dimensions) 4-71
4.9.1 Solution of Laplace Equation in Two Dimensions 4-72
4.9.1.1 Procedure to Solve Two Dimensional Heat Flow Equation 4-73
• Chapter Ends 4-88

Multiple Choice Questions with Answers M4-1 to M4-2

UNIT V

Chapter 5 : Functions of Complex Variables (Differential calculus) 5-1 to 5-51

Syllabus : Limit and continuity of f(z); Derivative of f(z) ; Analytic functions; Cauchy- Riemann equations in Cartesian and polar forms; Harmonic functions in Cartesian form;Mapping: Translation, magnification and rotation, inversion and reflection , bilinear transformation; Conformal mapping.

5.1 Introduction 5-1
5.2 Pre-requisite 5-1
5.2.1 Some Important Curves 5-1
5.3 Functions of Complex Variable 5-2
5.3.1 Definition 5-2
5.3.2 Single Valued Function 5-2
✓ **Syllabus Topic :** Limit and continuity of f (z) 5-3
5.3.3 Limit of f (z) 5-3
5.3.4 Continuity of f (z) 5-3
✓ **Syllabus Topic :** Derivative of f (z) 5-3
5.3.5 Derivative of f (z) 5-3
✓ **Syllabus Topic :** Analytic Functions 5-3
5.3.6 Analytic Function 5-3
5.3.7 Singular Point 5-4
✓ **Syllabus Topic :** Cauchy-Riemann Equations in Cartesian form 5-4
5.4 Cauchy-Riemann Equations 5-4
5.4.1 Cauchy-Riemann Equation in Cartesian Form 5-4
✓ **Syllabus Topic :** Cauchy-Riemann Equation in Polar Form 5-5
5.4.2 Cauchy-Riemann Equation in Polar Form 5-5
✓ **Syllabus Topic :** Harmonic Function in Cartesian Form 5-5
5.5 Harmonic Function in Cartesian Form 5-5
5.5.1 Milne-Thomson Method 5-5
5.5.2 Orthogonal System 5-5
5.6 The Laplacian Operator 5-31
✓ **Syllabus Topic :** Mapping : Conformal Mapping, Translation, Magnification and rotation, Inversion and Reflection 5-34
5.7 Conformal Mapping 5-34
5.7.1 Definition of Conformal Mapping 5-35
✓ **Syllabus Topic :** Bilinear Transformation 5-45
5.7.2 Bilinear Transformation 5-45
5.7.2.1 Cross ratio of Four Points 5-46
• Chapter Ends 5-51

Multiple Choice Questions with Answers M5-1 to M5-2

UNIT VI

Chapter 6 : Function of Complex Variable (Integral Calculus) 6-1 to 6-31

Syllabus : Cauchy's integral theorem; Cauchy's integral formula; Residues; Cauchy's residue theorem (All theorems without proofs).

6.1 Complex Integration (Line Integral) 6-1
6.2 Contour Integral 6-3
6.2.1 Simple Closed Curve 6-3
6.2.2 Connected Region 6-3
6.2.3 Simply Connected 6-3
✓ **Syllabus Topic :** Cauchy's Integral Theorem 6-4
6.3 Cauchy's Integral Theorem 6-4
✓ **Syllabus Topic :** Cauchy's Integral Formula 6-5
6.4 Cauchy's Integral Formula 6-5
6.5 Taylors Series 6-19
6.6 Laurent's Series 6-19
✓ **Syllabus Topic :** Residues 6-21
6.7 Residues 6-21
6.7.1 Calculation of Residues 6-22
✓ **Syllabus Topic :** Cauchy's Residue Theorem 6-22
6.8 Cauchy's Residue Theorem 6-22
• Chapter Ends 6-31

Multiple Choice Questions with Answers M6-1 to M6-2

□□□

CHAPTER 1

UNIT 1

Laplace Transform

Syllabus

Definition – conditions for existence ; Transforms of elementary functions ; Properties of Laplace transforms - Linearity property, first shifting property, second shifting property, transforms of functions multiplied by t^n, scale change property, transforms of functions divided by t, transforms of integral of functions, transforms of derivatives ; Evaluation of integrals by using Laplace transform ; Transforms of some special functions- periodic function, Heaviside-unit step function, Dirac delta function.

1.1 Prerequisite

1. Gamma function : $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx \ (n > 0)$

2. $\Gamma(n+1) = \begin{cases} n! & ; \text{ if n is poitive integer} \\ n\Gamma n & ; \text{ if otherwise} \end{cases}$

3. $\Gamma 1 = 1$

4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

5. $\Gamma m \cdot \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma 2m$

6. $\Gamma P \cdot \Gamma(1-P) = \frac{\pi}{\sin P\pi}$ if $0 < P < 1$

7. **Error functions :**

(a) $\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2}\, du$

(b) $\text{erf}_c(f) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-u^2}\, du$

(c) $\text{erf}(t) + \text{erf}_c(t) = 1$

8. $\int_0^\infty e^{-at} \sin(bt + c)\, dt = \frac{b}{a^2 + b^2}$

9. $\int_0^\infty e^{-at} \cos(bt + c)\, dt = \frac{a}{a^2 + b^2}$

10. DUIS [Rule I] : If $I(\alpha) = \int_a^b f(x, \alpha)\, dx$

a, b are constants

then, $\frac{dI}{d\alpha} = \int_a^b \frac{\partial}{\partial\alpha}[f(x, \alpha)]\, dx$

Syllabus Topic : Definition of Laplace Transform

1.2 Definition of Laplace Transform

Definition : If f (t) is a function of t defined for all positive values of t, then the Laplace transform of f (t) is defined as

$$\int_0^\infty e^{-st} f(t)dt$$

It is defined as L [f(t)]

$$\therefore \ L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

Provided that integral exist. 's' is a parameter which may be a real or complex.

L[f (t)] is a function of s, which is denote by f (s)

$$\therefore \ L[f(t)] = \int_0^\infty e^{-st} f(t)dt = f(s)$$

There is one to one correspondence between f (t) and f (s).

f (t) is called object function which is defined for $t \geq 0$, f(s) is the resultant function or image function.

Syllabus Topic : Conditions For Existence

1.3 Existence Condition

The Laplace transform of function f (t) exist if it satisfies following conditions :

(i) f (t) is piecewise continuous on every finite interval.

(ii) f (t) satisfies the following equality :

$$|f(t)| \le b e^{at} \quad \text{for all } t \ge 0$$

Where a, b are constants.

Condition (ii) is known as exponential order.

1.4 Examples using Definition of Laplace Transform

Ex. 1.4.1 : Find Laplace transform of

$$f(t) = \begin{cases} t & 0 < t < 4 \\ 5 & t > 4 \end{cases}$$

Soln. :

We have, definition of Laplace transform

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

Given, $f(t) = \begin{cases} t & 0 < t < 4 \\ 5 & t > 4 \end{cases}$

Here f(t) is defined in the interval 0 to 4 and 4 to ∞.

So split integral with limit 0 to 4 and 4 to ∞.

$$\therefore L[f(t)] = \int_0^4 e^{-st} f(t)\,dt + \int_4^\infty e^{-st} f(t)\,dt$$

$$\therefore L[f(t)] = \int_0^4 e^{-st}(t)\,dt + \int_4^\infty e^{-st}(5)\,dt$$

➔ **Using product rule of integration**

$$\left[\int uv\,dx = u v_1 - u' v_2 + u'' v_3 - u''' v_4 + \ldots \text{ where dashes means derivative and suffix means integration}\right]$$

$$= \int_0^4 \underset{u}{t}\,\underset{v}{e^{-st}}\,dt + 5\int_4^\infty e^{-st}\,dt$$

$$= \left[(t)\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^4 + \left[(5)\left(\frac{e^{-st}}{-s}\right)\right]_4^\infty$$

Substitute t as : (upper limit) – (lower limit)

$$= \left\{\left[-4\frac{e^{-4s}}{s} - \frac{e^{-4s}}{s^2}\right] - \left[0 - \frac{1}{s^2}\right]\right\} + \left\{[0] - \left[-\frac{5e^{-4s}}{s}\right]\right\}$$

$$\ldots\left[\because e^{-\infty} = 0 \text{ and } e^0 = 1\right]$$

$$\therefore L[f(t)] = \frac{-4e^{-4s}}{s} - \frac{e^{-4s}}{s^2} + \frac{1}{s^2} + \frac{5e^{-4s}}{s}$$

$$= \left(\frac{-4e^{-4s}}{s} + \frac{5e^{-4s}}{s}\right) - \left(\frac{e^{-4s}}{s^2}\right) + \frac{1}{s^2}$$

$$= \frac{e^{-4s}}{s} - \frac{e^{-4s}}{s^2} + \frac{1}{s^2}$$

$$\therefore L[f(t)] = \frac{e^{-4s}}{s^2}[s-1] + \frac{1}{s^2} \checkmark \qquad \ldots\text{Ans.}$$

Ex. 1.4.2 : Find Laplace transform of

$$f(t) = \begin{cases} a & 0 < t < b \\ 0 & t > b \end{cases}$$

Soln. :

We have, definition of Laplace transform

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

Given, $f(t) = \begin{cases} a & 0 < t < b \\ 0 & t > b \end{cases}$

Here f(t) is defined in the interval 0 to b and b to ∞.

So split integral with limit 0 to b and b to ∞

$$\therefore L[f(t)] = \int_0^b e^{-st} f(t)\,dt + \int_b^\infty e^{-st} f(t)\,dt$$

$$\therefore L[f(t)] = \int_0^b e^{-st}(a)\,dt + \int_b^\infty e^{-st}(0)\,dt$$

$$= \int_0^b a e^{-st}\,dt = a\int_0^b e^{-st}\,dt$$

➔ **Using standard result of integration :** $\ldots\left[\int e^{ax}\,dx = \frac{e^{ax}}{a}\right]$

$$= a\left[\frac{e^{-st}}{-s}\right]_0^b$$

Substitute t as : (upper limit) – (lower limit)

$$= a\left[\left(\frac{e^{-bs}}{-s}\right) - \left(\frac{1}{-s}\right)\right] = a\left[\frac{e^{-bs}}{s} + \frac{1}{s}\right]$$

$$\therefore L[f(t)] = \frac{a}{s}(1 - e^{-bs}) \checkmark \qquad \ldots\text{Ans.}$$

Ex. 1.4.3 : Find Laplace transform of

$$f(t) = \begin{cases} \cos t & 0 < t < 2\pi \\ 0 & t > 2\pi \end{cases}$$

Soln. :

Given : $f(t) = \begin{cases} \cos t & 0 < t < 2\pi \\ 0 & t > 2\pi \end{cases}$

By definition, of Laplace transform,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

Here f (t) defined in the interval 0 to 2π and 2π onward. So split integral with limit 0 to ∞ into two parts as integral with 0 to 2π and 2π to ∞.

$$\therefore L[f(t)] = \int_0^{2\pi} e^{-st} \underbrace{f(t)}_{f(t)=\cos t}\,dt + \int_{2\pi}^\infty e^{-st} \underbrace{f(t)}_{f(t)=0}\,dt$$

$$L[f(t)] = \int_0^{2\pi} e^{-st}\cos t\,dt + \int_{2\pi}^\infty e^{-st}(0)\,dt$$

$$= \int_0^{2\pi} e^{-st}\cos t\,dt + 0$$

➔ **Using the integration formula**

$$\ldots\left[\begin{array}{l} \int e^{ax}\cos(bx+c)\,dx = \dfrac{e^{ax}}{a^2+b^2} \\ \left[a\cos(bx+c) + b\sin(bx+c)\right] \end{array}\right]$$

Here a = –s and b = 1 c = 0

$$= \left[\frac{e^{-st}}{s^2+1}(-s\cos t + \sin t)\right]_0^{2\pi}$$

Substitute t as : $\underbrace{\left(\text{Upper limit}\right)}_{2\pi} - \underbrace{\left(\text{lower limit}\right)}_{0}$

$$\therefore L[f(t)] = \left[\frac{e^{-2\pi s}}{s^2+1}[-s\cos(2\pi) + \sin 2\pi]\right] - \left[\frac{e^0}{s^2+1}[-s\cos 0 + \sin 0]\right]$$

$$= \left[\frac{e^{-2\pi s}}{s^2+1}(-s)\right] - \left[\frac{1}{s^2+1}(-s)\right]$$

$$\ldots[\because \cos 2\pi = 1, \sin 2\pi = 0, \sin 0 = 0, \cos 0 = 1]$$

Common out $\left(\frac{s}{s^2+1}\right)$,

$$\therefore L[f(t)] = \frac{s}{s^2+1}[-e^{-2\pi s} + 1] = \frac{s(1-e^{-2\pi s})}{s^2+1}$$

$$\therefore L[f(t)] = \frac{s(1-e^{-2\pi s})}{s^2+1} \checkmark \qquad \ldots\text{Ans.}$$

Ex. 1.4.4 : Find Laplace transform of

$$f(t) = \begin{cases} 0; & 0 \le t < 1 \\ t; & 1 < t < 2 \\ 0; & t > 2 \end{cases}$$

Soln. :

Given, $f(t) = \begin{cases} 0; & 0 \le t < 1 \\ t; & 1 < t < 2 \\ 0; & t > 2 \end{cases}$

We have,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt \qquad \ldots\text{(by definition)}$$

Here f(t) defined in the interval 0 to 1, 1 to 2 and 2 onwards.

So split integral with limit 0 to 1, 1 to 2 and 2 to ∞.

$$\therefore L[f(t)] = \int_0^1 e^{-st}\underset{0}{f(t)}\,dt + \int_1^2 e^{-st}\underset{t}{f(t)}\,dt + \int_0^\infty e^{-st}\underset{0}{f(t)}\,dt$$

$$\therefore L[f(t)] = \int_0^1 e^{-st}(0)\,dt + \int_1^2 e^{-st}t\,dt + \int_2^\infty e^{-st}(0)\,dt$$

$$= 0 + \int_1^2 e^{-st}t\,dt + 0 = \int_1^2 e^{-st}\cdot(t)\,dt$$

➔ **Using integration by parts**

$$\ldots\left[\int u\cdot v\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\cdot\int v\,dx\right]dx\right.$$

$$\left.\text{Here, } u = t \text{ and } v = e^{-st}\right]$$

$$\therefore L[f(t)] = \left[(t)\int e^{-st}\,dt\right]_1^2 - \int_1^2\left[\frac{d}{dt}(t)\cdot\int e^{-st}\,dt\right]dt$$

➔ **Using standard result of integration and derivative**

$$\ldots\left[\int e^{-st}\,dt = \frac{e^{-st}}{-s} \text{ and } \frac{d}{dt}(t) = 1\right]$$

$$\therefore L[f(t)] = \left[t\cdot\left(\frac{e^{-st}}{-s}\right)\right]_1^2 - \int_1^2(1)\cdot\left(\frac{e^{-st}}{-s}\right)dt$$

$$= \left[t\left(\frac{e^{-st}}{-s}\right)\right]_1^2 - \left(\frac{1}{-s}\right)\int_1^2 e^{-st}\,dt$$

$$= \left[t\cdot\left(\frac{e^{-st}}{-s}\right)\right]_1^2 + \left[\frac{1}{s}\left(\frac{e^{-st}}{-s}\right)\right]_1^2$$

$$= \left[(t)\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_1^2$$

Substitute t as : $\left(\text{Upper limit}\right) - \left(\text{lower limit}\right)$

$$\therefore L[f(t)] = \left[(2)\left(\frac{e^{-2s}}{-s}\right) - (1)\left(\frac{e^{-2s}}{s^2}\right)\right]$$

$$-\left[(1)\left(\frac{e^{-s}}{-s}\right) - (1)\left(\frac{e^{-s}}{s^2}\right)\right]$$

$$= \frac{-2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s} + \frac{e^{-s}}{s^2}$$

$$= \frac{\overbrace{-2se^{-2s} - e^{-2s}}^{-e^{-2s}(2s+1)} + \overbrace{se^{-s} + e^{-s}}^{e^{-s}(s+1)}}{s^2}$$

... by simplification

$$= \frac{-e^{-2s}(2s+1) + e^{-s}(s+1)}{s^2}$$

$$\therefore \; L[f(t)] = \frac{e^{-s}(s+1) - e^{-2s}(2s+1)}{s^2} \checkmark \qquad \text{...Ans.}$$

Ex. 1.4.5 : Find Laplace transform of

$$f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$$

Soln. :

Given, $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$

We have,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

Here f (t) defined in the interval 0 to π and π to ∞

So split integral with limit 0 to π and π to ∞.

$$\therefore \; L[f(t)] = \int_0^\pi e^{-st} \underset{\cos t}{\underset{\uparrow}{f(t)}}\, dt + \int_\pi^\infty e^{-st} \underset{\sin t}{\underset{\uparrow}{f(t)}}\, dt$$

$$\therefore \; L[f(t)] = \int_0^\pi e^{-st} (\cos t)\, dt + \int_\pi^\infty e^{-st} (\sin t)\, dt$$

➔ **Using the integration formula**

$$\left[\int e^{ax} \cos(bx+c)\, dx = \frac{e^{ax}}{a^2+b^2} [a\cos(bx+c) + b\sin(bx+c)] \right. \text{ and } \left. \int e^{ax} \sin(bx+c)\, dx = \frac{e^{ax}}{a^2+b^2} [a\sin(bx+c) - b\cos(bx+c)] \right]$$

Here $a = -s$ and $b = 1, c = 0$

$$\therefore \; L[f(t)] = \left[\frac{e^{-st}}{s^2+1} [-s\cos t + \sin t] \right]_0^\pi + \left[\frac{e^{-st}}{s^2+1} [-s\sin t - \cos t] \right]_\pi^\infty$$

Substitute t as : $\left(\begin{matrix}\text{Upper}\\ \text{limit}\end{matrix}\right) - \left(\begin{matrix}\text{lower}\\ \text{limit}\end{matrix}\right)$

$$= \left[\frac{e^{-\pi s}}{s^2+1} (-s\cos\pi + \sin\pi) \right] - \left[\frac{e^0}{s^2+1} (-s\cos 0 + \sin 0) \right] + \left[\frac{e^{-\infty}}{s^2+1} (-s\sin\infty - \cos\infty) \right] - \left[\frac{e^{-\pi s}}{s^2+1} (-s\sin\pi - \cos\pi) \right]$$

➔ **Using standard trigonometric values**

...[$\sin 0 = 0, \cos 0 = 1, \sin\pi = 0, \cos\pi = -1, e^{-\infty} = 0$]

$$\therefore L[f(t)] = \left[\frac{e^{-\pi s}}{s^2+1} (s) \right] - \left[\frac{1}{s^2+1} (-s) \right] + [0] - \left[\frac{e^{-\pi s}}{s^2+1} (1) \right]$$

$$= \frac{se^{-\pi s}}{s^2+1} + \frac{s}{s^2+1} - \frac{e^{-\pi s}}{s^2+1}$$

$$= \underbrace{\left(\frac{se^{-\pi s}}{s^2+1} - \frac{e^{-\pi s}}{s^2+1} \right)}_{\text{Common out } \frac{e^{-\pi s}}{s^2+1}} + \frac{s}{s^2+1}$$

...Rearrange the terms

$$= \frac{e^{-\pi s}}{s^2+1}(s-1) + \frac{s}{s^2+1}$$

$$= \frac{1}{s^2+1} [e^{-\pi s}(s-1) + s]$$

$$\therefore \; L[f(t)] = \frac{[e^{-\pi s}(s-1) + s]}{s^2+1} \checkmark \qquad \text{...Ans.}$$

Syllabus Topic : Laplace Transforms of elementary functions

1.5 Laplace Transform of Elementary Functions

(I) f (t) = 1

We have by definition of Laplace Transform,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt \qquad \text{...(definition)}$$

$$\therefore \; L[1] = \int_0^\infty e^{-st} (1)\, dt \qquad \text{[Here, } f(t) = 1]$$

$$= \int_0^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s} \right]_0^\infty$$

...(by integration of standard function)

$$= \underbrace{\left(\frac{e^{-\infty}}{-s} \right)}_{\text{Upper limit}} - \underbrace{\left(\frac{e^0}{-s} \right)}_{\text{lower limit}}$$

$$= \left[0 - \left(\frac{-1}{s} \right) \right] \qquad [\because e^{-\infty} = 0, e^0 = 1]$$

$$\therefore \; L[1] = \frac{1}{s} \qquad (s > 0)$$

$$\therefore \; L[1] = \frac{1}{s} \checkmark \qquad \text{...Ans.}$$

(II) f (t) = e^{at}

We know by definition of Laplace transform,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[e^{at}] = \int_0^\infty e^{-st} e^{at}\, dt \qquad \text{...(by definition)}$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)}}{-(s-a)} \right]_0^\infty$$

$$= \underbrace{\left[\frac{e^{-\infty}}{-(s-a)} \right]}_{\text{Upper limit}} - \underbrace{\left[\frac{e^0}{-(s-a)} \right]}_{\text{lower limit}}$$

$$= 0 - \left[\frac{-1}{s-a} \right]$$

$$= 0 + \frac{1}{s-a} \qquad \text{...} [\because e^{-\infty} = 0, \; e^0 = 1]$$

[if $s - a > 0$]

$$\therefore \; L[e^{at}] = \frac{1}{s-a}; \quad \text{if } s > a$$

(III) f (t) = sin at

We know by definition of Laplace transform,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[\sin at] = \int_0^\infty e^{-st} \sin at\, dt \qquad \text{...(by definition)}$$

➔ **Using standard integration formula**

$$\left[\int_0^\infty e^{-at} \sin(bt+c)\, dt = \frac{b}{a^2+b^2} \right]$$

$$\therefore \; L[\sin at] = \frac{a}{s^2+a^2}; \quad \text{if } s > a$$

(IV) f (t) = cos at

We know by definition of Laplace transform,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[\cos at] = \int_0^\infty e^{-st} \cos at\, dt \qquad \text{...(by definition)}$$

➔ **Using standard formula**

$$\left[\int_0^\infty e^{-at} \cos(bt+c)\, dt = \frac{a}{a^2+b^2} \right]$$

$$\therefore \; L[\cos at] = \frac{s}{s^2+a^2}; \quad \text{if } s > a \checkmark \qquad \text{...Ans.}$$

(V) f (t) = sinh at

We know by definition of Laplace transform,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[\sinh at] = \int_0^\infty e^{-st} \sinh at\, dt$$

➔ **Using definition of hyperbolic function**

$$\left[\sinh at = \frac{e^{at} - e^{-at}}{2} \right]$$

$$= \int_0^\infty e^{-st} \left[\frac{e^{at} - e^{-at}}{2} \right] dt$$

$$= \frac{1}{2} \left[\int_0^\infty (e^{-st} e^{at} - e^{-st} e^{-at})\, dt \right]$$

$$= \frac{1}{2} \left[\int_0^\infty e^{-st} e^{at}\, dt - \int_0^\infty e^{-st} e^{-at}\, dt \right]$$

$$= \frac{1}{2} \left[\int_0^\infty e^{-(s-a)t}\, dt - \int_0^\infty e^{-(s+a)t}\, dt \right]$$

$$= \frac{1}{2} \left\{ \left[\frac{e^{-(s-a)}}{-(s-a)} \right]_0^\infty - \left[\frac{e^{-(s+a)}}{-(s+a)} \right]_0^\infty \right\}$$

...by standard integration

if $s - a > 0$ and $s + a > 0$

$$= \frac{1}{2} \left\{ \left[0 + \frac{1}{s-a} \right] - \left[0 + \frac{1}{s+a} \right] \right\}$$

if $s > a$ and $s > -a$

$$L[\sinh at] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{(s-a)(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right]^{*} \quad \text{if } s > |a|$$

$$\therefore \quad L[\sinh at] = \frac{a}{s^2-a^2} \quad \text{if } s > |a| \checkmark \quad \text{...Ans.}$$

(VI) $f(t) = \cosh at$

We know by definition of Laplace transform,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[\cosh at] = \int_0^\infty e^{-st} \cosh at\,dt \quad \text{...(by definition)}$$

➔ **Using definition of hyperbolic function**

$$\ldots\left[\cosh at = \frac{e^{at}+e^{-at}}{2}\right]$$

$$\therefore \quad L[\cosh at] = \int_0^\infty e^{-st}\left[\frac{e^{at}+e^{-at}}{2}\right]dt$$

$$= \frac{1}{2}\int_0^\infty [e^{-st}e^{at} + e^{-st}e^{-at}]\,dt$$

$$= \frac{1}{2}\int_0^\infty [e^{-(s-a)t} + e^{-(s+a)t}]\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)}}{-(s-a)} + \frac{e^{-(s+a)}}{-(s+a)}\right]_0^\infty$$

... by standard integration

$$= \frac{1}{2}\left\{\underbrace{\left[\frac{e^{-\infty}}{-(s-a)} + \frac{e^{-\infty}}{-(s+a)}\right]}_{\text{Upper limit}} - \underbrace{\left[\frac{e^{0}}{-(s-a)} + \frac{e^{0}}{-(s+a)}\right]}_{\text{lower limit}}\right\}$$

$$= \frac{1}{2}\left\{[0+0] - \left[\frac{1}{-(s-a)} + \frac{1}{-(s+a)}\right]\right\}$$

$$= \frac{1}{2}\left[0 + \frac{1}{s-a} + \frac{1}{s+a}\right] \quad \text{...by simplification}$$

if $s-a>0$ and $s+a>0$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{(s-a)(s+a)}\right]$$

$$= {}^{*}\frac{1}{2}\left[\frac{2s}{s^2-a^2}\right] = \frac{s}{s^2-a^2} \quad \text{if } s > a \text{ and } s > -a$$

$$\therefore \quad L[\cosh at] = \frac{s}{s^2-a^2} \quad \text{if } s > |a| \checkmark \quad \text{...Ans.}$$

(VII) $f(t) = t^n$

We know by definition of Laplace transform,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[t^n] = \int_0^\infty e^{-st} t^n\,dt \quad \text{...(by definition)}$$

Put $st = u \Rightarrow t = \frac{u}{s}$

t	0	∞
u	0	∞

$$\therefore \quad dt = \frac{1}{s}du$$

$$= \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^n \frac{1}{s}du$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-u}u^n\,du$$

➔ **Using definition of Gamma function**

$$\ldots\left[\Gamma n = \int_0^\infty e^{-x}x^{n-1}\,dx, (n>0)\right]$$

$$L[t^n] = \frac{1}{s^{n+1}}\Gamma(n+1) \quad \text{...(by Gamma function)}$$

$$= \frac{\Gamma(n+1)}{s^{n+1}} \text{ OR } = \frac{n!}{s^{n+1}} \quad \text{...if } n \text{ is integer}$$

Table 1.5.1 : Laplace Transform of Elementary function

f (t)	L[f (t)] = f (s)	f (t)	L[f (t)] = f (s)
1	$\frac{1}{s}$; $s>0$	e^{at}	$\frac{1}{s-a}$; $s>a$
sin at	$\frac{a}{s^2+a^2}$; $s>0$	cos at	$\frac{s}{s^2+a^2}$; $s>0$
sinh at	$\frac{a}{s^2-a^2}$; $s>\lvert a\rvert$	cosh at	$\frac{s}{s^2-a^2}$; $s>\lvert a\rvert$

f (t)	L[f (t)] = f (s)	f (t)	L[f (t)] = f (s)
$\frac{t}{2a}\sin at$	$\frac{s}{(s^2-a^2)^2}$; $s>\lvert a\rvert$	$\frac{1}{2a^3}(\sin at - at\cos at)$	$\frac{1}{(s^2+a^2)^2}$; $s>0$
t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$ or $\frac{n!}{s^{n+1}}$ when $n = 0, 1, 2, 3, \ldots$; $s>0$		

1.6 Examples using Standard Functions of Laplace Transform

Ex. 1.6.1 : Find Laplace transform of e^{5t}.

Soln. :

Let, $f(t) = e^{5t}$

➔ **Using standard result :** $\ldots\left[L(e^{at}) = \frac{1}{s-a}, \text{ Here } a=5\right]$

$$\therefore \quad L[f(t)] = \frac{1}{s-5}$$

$$\therefore \quad L[e^{5t}] = \frac{1}{s-5} \checkmark \quad \text{...Ans.}$$

Ex. 1.6.2 : Find Laplace transform of e^{-3t}.

Soln. :

Let, $f(t) = e^{-3t}$

➔ **Using standard result**

$$\ldots\left[L(e^{at}) = \frac{1}{s-a}; \text{ Here, } a=-3\right]$$

$$\therefore \quad L[f(t)] = \frac{1}{s-(-3)}$$

$$\therefore \quad L[e^{-3t}] = \frac{1}{s+3} \checkmark \quad \text{...Ans.}$$

Ex. 1.6.3 : Find Laplace transform of $e^{\frac{t}{2}}$

Soln. :

Let, $f(t) = e^{\frac{t}{2}} = e^{\left(\frac{1}{2}\right)t}$

➔ **Using standard result of Laplace Transform**

$$\ldots\left[L(e^{at}) = \frac{1}{s-a}; \text{ Here, } a=\frac{1}{2}\right]$$

$$\therefore \quad L[f(t)] = \frac{1}{s-\frac{1}{2}}$$

Multiple and divide by 2

$$= \frac{1}{s-\frac{1}{2}} \times \frac{2}{2}$$

$$= \frac{2}{2s-1}$$

$$\therefore \quad L\left[e^{\frac{t}{2}}\right] = \frac{2}{2s-1} \checkmark \quad \text{...Ans.}$$

Ex. 1.6.4 : Find Laplace transform of sin 3t.

Soln. :

Let, $f(t) = \sin 3t$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L(\sin at) = \frac{a}{s^2+a^2}; \text{ Here, } a=3\right]$$

$$\therefore \quad L[f(t)] = L[\sin 3t]$$

$$= \frac{3}{s^2+(3)^2} \quad \{\text{compare : sin at and sin } \underset{\substack{\uparrow \\ a}}{③}t\}$$

$$= \frac{3}{s^2+9}$$

$$\therefore \quad L[\sin 3t] = \frac{3}{s^2+9} \checkmark \quad \text{...Ans.}$$

Ex. 1.6.5 : Find Laplace transform of cos 2t.

Soln. :

Let, $f(t) = \cos 2t$

$\therefore L[f(t)] = L[\cos 2t]$ $\{\text{compare : cos at and cos } \underset{\substack{\uparrow \\ a}}{②}t\}$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L(\cos at) = \frac{s}{s^2+a^2}; \text{ Here, } a=2\right]$$

$$= \frac{s}{s^2+(2)^2}$$

$$= \frac{s}{s^2+4}$$

$$\therefore \quad L[\cos 2t] = \frac{s}{s^2+4} \checkmark \quad \text{...Ans.}$$

Ex. 1.6.6 : Find the Laplace transform of cos 3t.

Soln. :

Let, $f(t) = \cos 3t$

$\therefore L[f(t)] = L[\cos 3t]$ $\left\{\text{compare cos at and cos } \underset{\substack{\uparrow \\ a}}{③}t\right\}$

➔ **Using the result of Laplace transform**

$$\ldots\left[L(\cos at)] = \frac{s}{s^2+a^2}; \text{here, } a=3\right]$$

$$= \frac{s}{s^2+(3)^2} = \frac{s}{s^2+9}$$

$$\therefore \quad L[\cos 3t] = \frac{s}{s^2+9} \checkmark \qquad \text{...Ans.}$$

Ex. 1.6.7 : Find the Laplace transform of sin 5t.

Soln. :

Let $f(t) = \sin 5t$

$\therefore L[f(t)] = L[\sin 5t]$

{Compare sin at and sin ⑤t}
↑
a

➔ **Using the result of Laplace transform**

$$\ldots\left[L[\sin at] = \frac{a}{s^2+a^2}; \text{Here } a = 5\right]$$

$$= \frac{5}{s^2+(5)^2} = \frac{5}{s^2+25}$$

$$\therefore L[\sin 5t] = \frac{5}{s^2+25} \checkmark \qquad \text{...Ans.}$$

Ex. 1.6.8 : Find the Laplace transform of cos 4t.

Soln. :

Let, $f(t) = \cos 4t$

$\therefore L[f(t)] = L[\cos 4t]$

...{Compare : cos at and cos ④t}
↑
a

➔ **Using result of Laplace transform**

$$\ldots\left[L(\cos at) = \frac{s}{s^2+a^2}; \text{ Here, } a = 4\right]$$

$$= \frac{s}{s^2+(4)^2} = \frac{s}{s^2+16}$$

$$\therefore L[\cos 4t] = \frac{s}{s^2+16} \checkmark \qquad \text{...Ans.}$$

Ex. 1.6.9 : Find Laplace transform of sinh 2t.

Soln. :

Let, $f(t) = \sinh 2t$

$\therefore L[f(t)] = L[\sinh 2t]$

...{compare : sinh ②t with sinh at}
↑
a

➔ **Using standard result of Laplace transform**

$$\ldots\left[L(\sinh at) = \frac{a}{s^2-a^2}; \text{ Here, } a = 2\right]$$

$$= \frac{2}{s^2-(2)^2}$$

$$\therefore L[f(t)] = \frac{2}{s^2-4}$$

$$\therefore L[\sinh 2t] = \frac{2}{s^2-4} \checkmark \qquad \text{...Ans.}$$

Ex. 1.6.10 : Find the Laplace transform of cosh t.

Soln. :

Let, $f(t) = \cosh t = \cosh 1t$

$\therefore L[f(t)] = L[\cosh 1t]$

...{compare : cosh ①t with cosh at}
↑
a

➔ **Using standard result of Laplace transform**

$$\ldots\left[L(\cosh at) = \frac{s}{s^2-a^2}; \text{ Here, } a = 1\right]$$

$$= \frac{s}{s^2-(1)^2}$$

$$\therefore L[f(t)] = \frac{s}{s^2-1}$$

$$\therefore L[\cosh t] = \frac{s}{s^2-1} \checkmark \qquad \text{...Ans.}$$

Ex. 1.6.11 : Find Laplace transform of f(t) = t.

Soln. :

Given, $f(t) = t$

$\therefore L[f(t)] = L[t] = L[t^1]$ {Compare : t ①↑ with t^n}
n

➔ **Using the standard result of Laplace transform,**

$$\ldots\left[L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} \text{ or } \frac{n!}{s^{n+1}} \text{ if n is an integer}\right]$$

$$= \frac{1!}{s^{1+1}}$$

$$\therefore L[f(t)] = \frac{1}{s^2}$$

$$\therefore L[t] = \frac{1}{s^2} \checkmark \qquad \text{...Ans.}$$

Ex. 1.6.12 : Find Laplace transform of $f(t) = t^2$.

Soln. :

Given, $f(t) = t^2$

$\therefore L[f(t)] = L[t^2]$ {Compare : t ②↑ with t^n}
n

➔ **Using the standard result of Laplace transform,**

$$\ldots\left[L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} \text{ or } \frac{n!}{s^{n+1}} \text{ if n is an integer}\right]$$

$$= \frac{2!}{s^{2+1}}$$

$$\therefore L[f(t)] = \frac{2}{s^3}$$

$$\therefore L[t^2] = \frac{2}{s^3} \checkmark \qquad \text{...Ans.}$$

Ex. 1.6.13 : Find Laplace transform of t^3.

Soln. :

Given, $f(t) = t^3$

$\therefore L[f(t)] = L[t^3]$ {Compare : t ③↑ with t^n}
n

➔ **Using the standard result of Laplace transform,**

$$\ldots\left[L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} \text{ OR } \frac{n!}{s^{n+1}} \text{ if n is an integer}\right]$$

$$\therefore L[f(t)] = \frac{3!}{s^{3+1}} \qquad \{\because 3! = 1\times2\times3 = 6\}$$

$$\therefore L[f(t)] = \frac{6}{s^4}$$

$$\therefore L[t^3] = \frac{6}{s^4} \checkmark \qquad \text{...Ans.}$$

Syllabus Topic : Properties of Laplace transforms

1.7 Properties of Laplace Transform

Syllabus Topic : Linearity property

1.7.1 Linearity property : If a, b, c be any Constants and f, g, h any Functions of t, then

$$L\{a f(t) + b g(t) - c h(t)\} = a L[f(t)] + b L[g(t)] - c L[h(t)]$$

Proof :

$$L\{a f(t) + b g(t) - c h(t)\} = \int_0^\infty e^{-st}[a f(t) + b g(t) - c h(t)]\, dt$$

...(by definition)

$$= \int_0^\infty e^{-st} a f(t)\, dt + \int_0^\infty e^{-st} b g(t)\, dt - \int_0^\infty e^{-st} c h(t)\, dt$$

$$= a\int_0^\infty e^{-st} f(t)\, dt + b\int_0^\infty e^{-st} g(t)\, dt - c\int_0^\infty e^{-st} h(t)\, dt$$

$$L\{a f(t) + b g(t) - c h(t)\} = a L[f(t)] + b L[g(t)] - c L[h(t)]$$

Syllabus Topic : First Shifting Property

1.7.2 First Shifting Theorem

➔ **(Dr. BATU : May 17, 5 Marks)**

If $L[f(t)] = f(s)$ then $L[e^{-at} f(t)] = f(s+a)$

Proof :

$$L\{e^{-at} f(t)\} = f(s+a)$$

$$L[e^{-at} f(t)] = \int_0^\infty e^{-st} e^{-at} f(t)\, dt = \int_0^\infty e^{-(s+a)t} f(t)\, dt$$

...(by definition)

Put $s + a = s_1$

$$= \int_0^\infty e^{-s_1 t} f(t)\, dt = f(s_1)$$

$$L[e^{-at} f(t)] = f(s+a)$$

Also, $L[e^{at} f(t)] = f(s-a)$

Note : $L[e^{-at} f(t)] = [f(s)]_{s \to s+a} = f(s+a)$

Syllabus Topic : Scale Change Property

1.7.3 Change of Scale Property

If $L[f(t)] = f(s)$ then $L\{f(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$

Proof :

By definition of Laplace transform,

$$L[f(at)] = \int_0^\infty e^{-st} f(at)\, dt \qquad \ldots(1)$$

Put $at = u \Rightarrow t = \frac{u}{a}$

Differentiate both sides,

$$\therefore dt = \frac{1}{a} du \qquad \begin{array}{|c|c|c|} \hline t & 0 & \infty \\ \hline u & 0 & \infty \\ \hline \end{array}$$

Using these substitution in Equation (1), we get,

$$= \int_0^\infty e^{-s\left(\frac{u}{a}\right)} f(u) \frac{1}{a} du$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)u} f(u)\, du$$

$$= \frac{1}{a}\underbrace{\int_0^\infty e^{-s_1} f(u)\, du}_{L[f(u)] = f(s_1)} \qquad \ldots\left(\text{put } s_1 = \frac{s}{a}\right)$$

...by definition of L. T.

$$= \frac{1}{a} f(s_1) \quad \text{since } s_1 = \frac{s}{a}$$

$$\therefore \quad L[f(at)] = \frac{1}{a} f\left(\frac{s}{a}\right) \checkmark^{*} \qquad \text{...Ans.}$$

Syllabus Topic : Transforms of Functions Multiplied by t^n

1.7.4 Multiplication by t^n

If $L[f(t)] = f(s)$ then

(i) $L\{t f(t)\} = -\frac{d}{ds} f(s)$

(ii) $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$,

Where $n = 1, 2, 3, 4, \ldots$

Proof :

We have, definition of Laplace transform

$$L[F(t)] = \int_0^\infty e^{-st} f(t)\, dt = f(s)$$

$$\therefore \int_0^\infty e^{-st} f(t)\, dt = f(s)$$

Differentiate both sides w.r.t. 's'

$$\frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)\, dt\right] = \frac{d}{ds} f(s)$$

➔ **Using DUIS rule I for L.H.S.**

$$\left[\text{If } I(\alpha) = \int_a^b f(x, \alpha)\, dx \text{ then } \frac{dI}{d\alpha} = \int_a^b \frac{\partial}{\partial \alpha} f(x, \alpha)\, dx\right]$$

$$\int_0^\infty \frac{\partial}{\partial s}[e^{-st} f(t)]\, dt = \frac{d}{ds} f(s)$$

$$\int_0^\infty (-t)\, e^{-st} f(t)\, dt = \frac{d}{ds}[f(s)]$$

$$\int_0^\infty e^{-st}[t f(t)]\, dt = -\frac{d}{ds}[f(s)]$$

By definition of Laplace transform on L.H.S., we get,

$$\therefore \quad L[t f(t)] = -\frac{d}{ds}[f(s)]$$

To generalize the rule

We can write, $L[t^2 f(t)] = L[t \cdot t f(t)]$

$$= -\frac{d}{ds}[L(t f(t))]$$

$$= -\frac{d}{ds}\left[-\frac{d}{ds} f(s)\right]$$

$$L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}[f(s)]$$

In general,

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[f(s)] \qquad n = 1, 2, 3, \ldots$$

Syllabus Topic : Transforms of Functions divided by t

1.7.5 Division by t

If $L[f(t)] = f(s)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(s)\, ds$

Proof : We have

$$\int_0^\infty e^{-st} f(t)\, dt = f(s)$$

i.e. $$f(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Integrating from s to ∞ on both sides

$$\int_s^\infty f(u)\, du = \int_s^\infty \left[\int_0^\infty e^{-ut} f(t)\, dt\right] du$$

$$= \int_0^\infty \left[\int_s^\infty e^{-ut}\, du\right] f(t)\, dt$$

...[Checking the order of integretion]

$$= \int_0^\infty \left[\frac{e^{-ut}}{-t}\right]_s^\infty f(t)\, dt = \int_0^\infty \left[0 + \frac{e^{-st}}{t}\right] f(t)\, dt$$

$$\int_s^\infty f(u)\, du = \int_0^\infty e^{-st} \frac{f(t)}{t}\, dt = L\left[\frac{f(t)}{t}\right]$$

$$\therefore \quad L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(u)\, du$$

Syllabus Topic : Transforms of Derivatives

1.7.6 Laplace Transform of Derivatives

If $L[f(t)] = f(s)$, then $L\{f'(t)\} = s f(s) - f(0)$

Also, $L\{f^{(n)}(t)\} = s^n f(s) - s^{n-1} f(0) - s^{n-2} f'(0) \ldots\ldots - s f^{(n-2)}(0) - f^{(n-1)}(0)$

Proof

By definition of Laplace Transform

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt \qquad \text{...(by definition)}$$

➔ **Using Integration by parts**

$$\left[\int uv\, dx = u \int v\, dx - \int\left[\frac{du}{dx}\int v\, dx\right] dx \quad \text{with } u = e^{-st};\ v = f'(t)\right]$$

$$\therefore L[f'(t)] = \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s)\, e^{-st} f(t)\, dt$$

$$= 0 - f(0) + s \underbrace{\int_0^\infty e^{-st} f(t)\, dt}_{L[f(t)]} \qquad \left[\lim_{t \to \infty} e^{-st} f(t) = 0\right]$$

...by definition of Laplace transform

$$= -f(0) + sL[f(t)]$$

i.e. $L[f'(t)] = -f(0) + s f(s)$

$\therefore L[f'(t)] = s f(s) - f(0) \checkmark$ ...Ans.

B : If the function f(t) is piecewise continuous [at a point of discontinuty]

$$L[f'(t)] = sf(s) - f(0) - [f(a+0) - f(a-0)]\, e^{-as}$$

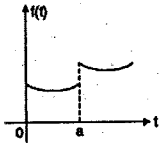

Fig. 1.7.1

C : If f(t) and its first (n – 1) derivatives are continuous then

$$L[f^n(t)] = s^n f(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots\ldots - f^{n-1}(0)$$

1.7.7 The Convolution Theorem

Convolution of two functions f(t) and g(t), t > 0 is defined as,

$$\int_0^t f(u)\, g(t-u)\, du.$$

It is denoted as $f(t)*g(t)$

$$\therefore f(t)*g(t) = \int_0^t f(u)\, g(t-u)\, du$$

Note that:

$$\therefore g(t)*f(t) = \int_0^t g(u)\, f(t-u)\, du$$

If $L[f(t)] = f(s)$ and $L[g(t)] = g(s)$, then

$$L\left[\int_0^t f(u)\, g(t-u)\, du\right] = f(s)*g(s) = L\left[\int_0^t f(t-u)\, g(u)\, du\right]$$

Proof : Consider $\phi(t) = \int_0^t f(u)\, g(t-u)\, du$

$$\therefore L[\phi(t)] = \int_0^\infty e^{-st}\left[\int_0^t f(u)\, g(t-u)\, du\right] dt \qquad \text{...by definition}$$

Since here limits are

$u = 0$ to $u = t$ and $t = 0$ to $t = \infty$

∴ Region of intergration is as shown in Fig. 1.7.2 change the order of intergration, take horizontal strip

∴ limits are

$t = u$ to $t = \infty$ and $u = 0$ to $u = \infty$

$$\therefore L[\phi(t)] = \int_0^\infty \int_u^\infty e^{-st} f(u)\, g(t-u)\, dt\, du$$

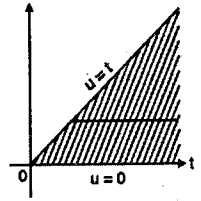

Fig. 1.7.2

$$\therefore L[\phi(t)] = \int_0^\infty \int_u^\infty e^{-st}(e^{su} e^{-su}) f(u)\, g(t-u)\, dt\, du$$

...[Note the adjustment]

$$= \int_0^\infty e^{-su} f(u)\left[\int_u^\infty e^{-s(t-u)} g(t-u)\, dt\right] du$$

Put $t - u = p \Rightarrow dt = dp$

t	u	∞
p	0	∞

$$L[\phi(t)] = \int_0^\infty e^{-su} f(u)\left[\int_0^\infty e^{-sp} g(p)\, dp\right] du$$

$$= \int_0^\infty e^{-su} f(u)\,[g(s)]\,du = \left[\int_0^\infty e^{-su} f(u)\,du\right]\cdot g(s)$$

$$L[\phi(t)] = f(s)\cdot g(s)$$

i.e. $$L\left[\int_0^t f(u)\,g(t-u)\,du\right] = f(s)\,g(s)$$

Convolution is denoted as *

$$\therefore \quad L\left[\int_0^t f(u)\,g(t-u)\,du\right] = f(s) * g(s)$$

Syllabus Topic : Transforms of Integral of Functions

1.7.8 Laplace Transform of Intergration : If L[f (t)] = f (s) then

➔ **(Dr. BATU : Nov. 16, 4 Marks)**

$$L\left[\int_0^t f(u)du\right] = \frac{1}{s} f(s)$$

Proof :

If we put, $g(t) = 1 \therefore g(t-u) = 1$

[in above convolution theorem]

and $$L[g(t)] = \frac{1}{s}$$

$$\therefore \quad L\left[\int_0^t f(u)\,(1)\,du\right] = f(s)\cdot\frac{1}{s}$$

i.e. $$L\left[\int_0^t f(u)\,du\right] = \frac{1}{s} f(s)$$

1.7.9 Second Shifting Theorem

If L[f (t)] = f (s) then

$$L\{F(t)\} = e^{-as} f(s)$$

where, $$F(t) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t < a \end{cases}$$

Proof :

We have, definition of Laplace transform,

$$L[F(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

Given function,

$$F(t) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t < a \end{cases}$$

Split limits of integration 0 to ∞ into two parts as 0 to a and a to ∞,

$$\therefore L[F(t)] = \int_0^a e^{-st} F(t)\,dt + \int_a^\infty e^{-st} F(t)\,dt$$

$$= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

Put, $t - a = u$

differentiate both sides,

$\therefore dt = du$

Limits are;

t	a	∞
u	0	∞

$$\therefore L[F(t)] = \int_0^\infty e^{-s(u+a)} f(u)\,du = \int_0^\infty e^{-su} e^{-sa} f(u)\,du$$

$$= e^{-sa}\int_0^\infty e^{-su} f(u)\,du = e^{-as} L[f(t)]$$

$$\therefore L[F(t)] = e^{-as} f(s) ✓$$ ...Ans.

1.7.10 Initial Value Theorem

If $L[f(t)] = f(s)$ then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sf(s)$

Provided limits exist.

E.g. $$L[4e^{-3t}] = \frac{4}{s+3} = f(s)$$

$$\therefore \lim_{t\to 0} f(t) = \lim_{t\to 0} 4e^{-3t} = 4$$

and $$\lim_{t\to 0} s\,f(s) = \lim_{s\to\infty}\frac{4s}{s+3} = \lim_{s\to\infty}\frac{4}{1+\frac{3}{s}} = 4$$

$$\therefore \lim_{t\to 0} f(t) = \lim_{s\to\infty} s\,f(s)$$

1.7.11 Final Value Theorem

If $L[f(t)] = f(s)$ then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sf(s)$

Provided limit exist

E.g. $$L[4e^{-3t}] = \frac{4}{s+3} = f(s)$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} 4e^{-3t} = 0$$

and $$\lim_{s\to 0} s\,f(s) = \lim_{s\to 0}\frac{4s}{s+3} = 0$$

$$\therefore \lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,f(s)$$

Table 1.7.1

f (t)	L [f (t)] = f (s)
First shifting property : $e^{-at} f(t)$	$f(s+a)$
Second shifting property : $F(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$	$e^{-as} f(s)$
Chang of scale : f (at)	$\frac{1}{a} f\left(\frac{s}{a}\right)$
Multiplication of t : $t^n f(t)$	$(-1)^n \frac{d^n}{ds^n} f(s)$
In particular : $t\,f(t)$	$-\frac{d}{ds} f(s)$
Division by t : $\frac{f(t)}{t}$	$\int_s^\infty f(s)\,ds$
For Derivative : $f^{(n)}(t)$	$s^n f(s) - s^{n-1} f(0) - s^{n-2} f'(0) \ldots\ldots\ldots s f^{(n-2)}(0) - f^{(n-1)}(0)$
In particular : $f'(t)$	$s\,f(s) - f(0)$
For Integration : $\int_0^t f(u)du$	$\frac{1}{s} f(s)$
Convolution Theorem : $\int_0^t f(u)g(t-u)\,du = \int_0^t f(t-u)\,g(u)\,du$	$f(s) * g(s)$

1.8 Examples Using Properties of Laplace Transform

Ex. 1.8.1 : Find the Laplace transform of $f(t) = \sinh 4t + \sin 3t$

Soln. :

Given

$$f(t) = \sinh 4t + \sin 3t$$

$$L[f(t)] = L[\sinh 4t + \sin 3t]$$

➔ **Using linearity property of Laplace transform**

$$\ldots [L[a\,f(t) + b\,g(t) - c\,h(t)] = a\,L[f(t)] + b\,L[g(t)] - c\,L[h(t)]]$$

$$\therefore L[f(t)] = L[\sinh 4t] + L[\sin 3t]$$

➔ **Using the standard result of Laplace transform**

$$\ldots\left[L[\sinh at] = \frac{a}{s^2 - a^2} \text{ and } L[\sin at] = \frac{a}{s^2 + a^2}\right]$$

[compare sinh ④t with sinh at and sin ③t with sin at]

$$L[f(t)] = \underbrace{L(\sinh 4t)}_{\frac{4}{s^2-(4)^2}} + \underbrace{L(\sin 3t)}_{\frac{3}{s^2+(3)^2}}$$

$$L[f(t)] = \frac{4}{s^2-(4)^2} + \frac{3}{s^2+(3)^2}$$

$$L[f(t)] = \frac{4}{s^2-16} + \frac{3}{s^2+9}$$

$$\therefore L[\sinh 4t + \sin 3t] = \frac{4}{s^2-(16)} + \frac{3}{s^2+(9)} ✓$$ ...Ans.

Note : We can further simplify it as,

$$\frac{4}{s^2-16} + \frac{3}{s^2+9} = \frac{4(s^2+9) + 3(s^2-16)}{(s^2-16)(s^2+9)} \text{ ...(by simplification)}$$

$$= \frac{4s^2 + 36 + 3s^2 - 48}{(s^2-16)(s^2+9)}$$

$$= \frac{(4s^2+3s^2) + (36-48)}{(s^2-16)(s^2+9)} \text{ ...(Grouping the terms)}$$

$$L[\sinh 4t + \sin 3t] = \frac{7s^2-12}{(s^2-16)(s^2+9)} ✓$$ ...Ans.

Ex. 1.8.2 : Find the Laplace transform of $f(t) = \cos 4t + \sinh 2t$

Soln. :

Given

$$f(t) = \cos 4t + \sinh 2t$$

$$L[f(t)] = L[\cos 4t + \sinh 2t]$$

➔ **Using linearity property of Laplace transform**

$$\ldots\left[L[a\,f(t) + b\,g(t) - c\,h(t)] = aL[f(t)] + bL[g(t)] - c\,L[h(t)]\right]$$

$$L[f(t)] = L[\cos 4t] + L[\sinh 2t]$$

➔ **Using the standard result of Laplace transform**

$$\ldots\left[L[\cos at] = \frac{s}{s^2+a^2} \text{ and } L[\sinh at] = \frac{a}{s^2-a^2}\right]$$

[compare cos ④t with cos at sinh ②t with sinh at]

$$\therefore L[f(t)] = \underbrace{L[\cos 4t]}_{\frac{s}{s^2+(4)^2}} + \underbrace{L[\sinh 2t]}_{\frac{2}{s^2-(2)^2}}$$

$$L[f(t)] = \frac{s}{s^2+(4)^2} + \frac{2}{s^2-(2)^2}$$

$$L[f(t)] = \frac{s}{s^2+16} + \frac{2}{s^2-4}$$

$$\therefore L[\cos 4t + \sinh 2t] = \frac{s}{s^2+16} + \frac{2}{s^2-4} \checkmark \quad \text{...Ans.}$$

Ex. 1.8.3 : Find the Laplace transform of

$$f(t) = 5e^{-\frac{t}{2}} + 7\sin\left(\frac{t}{2}\right)$$

Soln. : Given

$$f(t) = 5e^{-\frac{t}{2}} + 7\sin\left(\frac{t}{2}\right)$$

$$L[f(t)] = L\left[5e^{-\frac{t}{2}} + 7\sin\left(\frac{t}{2}\right)\right]$$

➔ **Using linearity property of Laplace transform**

$$\ldots\left[L[a\,f(t) + b\,g(t) - c\,h(t)] = aL[f(t)] + bL[g(t)] - c\,L[h(t)]\right]$$

$$\therefore L[f(t)] = 5L\left[e^{-\frac{t}{2}}\right] + 7L\left[\sin\left(\frac{t}{2}\right)\right]$$

➔ **Using the standard result of Laplace transform**

$$\ldots\left[L[e^{at}] = \frac{1}{s-a} \text{ and } L[\sin at] = \frac{a}{s^2+a^2}\right]$$

$$\left[\text{Compare : } e^{\underset{a}{\uparrow}t} \text{ with } e^{at} \quad \sin\left(\underset{a}{\uparrow}t\right) \text{ with } \sin at\right]$$

$$L[f(t)] = 5\,\underbrace{L\left[e^{-\frac{1}{2}t}\right]}_{\frac{1}{s-\left(-\frac{1}{2}\right)}} + 7\,\underbrace{L\left[\sin\left(\frac{1}{2}t\right)\right]}_{\frac{\frac{1}{2}}{s^2+\left(\frac{1}{2}\right)^2}}$$

$$L[f(t)] = 5\,\frac{1}{s-\left(-\frac{1}{2}\right)} + 7\,\frac{\frac{1}{2}}{s^2+\left(\frac{1}{2}\right)^2}$$

$$L[f(t)] = 5\,\frac{1}{s+\frac{1}{2}} + \frac{7}{2}\cdot\frac{1}{s^2+\frac{1}{4}}$$

$$= \underbrace{\left(\frac{5}{s+\frac{1}{2}}\right)\times\frac{2}{2}}_{\text{multiple and divide by 2}} + \frac{7}{\cancel{2}}\left(\frac{1}{s^2+\frac{1}{4}}\right)\times\underbrace{\frac{\cancel{4}^{2}}{4}}_{\text{multiple and divide by 4}}$$

$$= \frac{10}{2s+1} + \frac{14}{4s^2+1} \quad \text{...by simplification}$$

$$\therefore L\left[5e^{-\frac{t}{2}} + 7\sin\left(\frac{t}{2}\right)\right] = \frac{10}{2s+1} + \frac{14}{4s^2+1} \checkmark \quad \text{...Ans.}$$

Ex. 1.8.4 : Find the Laplace transform of $f(t) = \cos 2t + \sin 3t$

Soln. :

Given

$$f(t) = \cos 2t + \sin 3t$$

$$L[f(t)] = L[\cos 2t + \sin 3t]$$

➔ **Using linearity property of Laplace transform**

$$\ldots\left[L[a\,f(t) + b\,g(t) - c\,h(t)] = aL[f(t)] + bL[g(t)] - c\,L[h(t)]\right]$$

$$\therefore L[f(t)] = L[\cos 2t] + L[\sin 3t]$$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L[\cos at] = \frac{s}{s^2+a^2} \text{ and } L[\sin at] = \frac{a}{s^2+a^2}\right]$$

$$\ldots\left[\text{compare } \cos\underset{a}{\uparrow}2t \text{ with } \cos at \text{ and } \sin\underset{a}{\uparrow}3t \text{ with } \sin at\right]$$

$$\therefore L[f(t)] = \underbrace{L[\cos 2t]}_{\frac{s}{s^2+(2)^2}} + \underbrace{L[\sin 3t]}_{\frac{3}{s^2+(3)^2}}$$

$$L[f(t)] = \frac{s}{s^2+(2)^2} + \frac{3}{s^2+(3)^2}$$

$$L[f(t)] = \frac{s}{s^2+4} + \frac{3}{s^2+9}$$

$$\therefore L[\cos 4t + \sinh 2t] = \frac{s}{s^2+4} + \frac{3}{s^2+9} \checkmark \quad \text{...Ans.}$$

Ex. 1.8.5 : Find Laplace transform of $f(t) = e^{2t} - \cos 3t$

Soln. :

Given

$$f(t) = e^{2t} - \cos 3t$$

$$\therefore L[f(t)] = L[e^{2t} - \cos 3t]$$

➔ **Using the linearity property of Laplace transform**

$$\ldots\left[L[a\,f(t) + b\,g(t) - c\,h(t)] = aL[f(t)] + bL[g(t)] - c\,L[h(t)]\right]$$

$$L[f(t)] = L[e^{2t} - \cos 3t]$$

$$= L[e^{2t}] - L[\cos 3t]$$

➔ **Using the standard result of Laplace transform**

$$\ldots\left[L[e^{at}] = \frac{1}{s-a} \text{ and } L[\cos at] = \frac{s}{s^2+a^2}\right]$$

$$\left[\text{compare } e^{\underset{a}{\uparrow}2t} \text{ with } e^{at} \text{ and } \cos\underset{a}{\uparrow}3t \text{ with } \cos at\right]$$

$$L[f(t)] = \underbrace{L[e^{2t}]}_{\frac{1}{s-2}} - \underbrace{L[\cos 3t]}_{\frac{s}{s^2+(3)^2}}$$

$$L[f(t)] = \frac{1}{s-2} - \frac{s}{s^2+(3)^2}$$

$$L[f(t)] = \frac{1}{s-2} - \frac{s}{s^2+9}$$

$$\therefore L[e^{2t} - \cos 3t] = \frac{1}{s-2} - \frac{s}{s^2+9} \checkmark \quad \text{...Ans.}$$

Ex. 1.8.6 : Find Laplace transform of $f(t) = t^3 + 5t - 2$.

Soln. :

Given

$$f(t) = t^3 + 5t - 2$$

$$\therefore L[f(t)] = L[t^3 + 5t - 2] = L[t^3 + 5t - 2(1)]$$

➔ **Using linearity property of Laplace transform**

$$\ldots\left[L[a\,f(t) + b\,g(t) - c\,h(t)] = aL[f(t)] + bL[g(t)] - c\,L[h(t)]\right]$$

$$L[f(t)] = L[t^3] + 5L[t] - 2L[1]$$

➔ **Using the standard result of Laplace transform**

$$\ldots\left[L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} \text{ OR } \frac{n!}{s^{n+1}} \text{ and } L(1) = \frac{1}{s}\right]$$

$$\left[\text{Compare : } t^{\underset{n}{\uparrow}3} \text{ with and } t^{\underset{n}{\uparrow}1} \text{ with } t^n\right]$$

$$\therefore L[f(t)] = \underbrace{L[t^3]}_{\frac{3!}{s^{3+1}}} + 5\underbrace{L[t]}_{\frac{1!}{s^{1+1}}} - 2\underbrace{L[1]}_{\frac{1}{s}}$$

$$L[f(t)] = \frac{3!}{s^4} + 5\frac{1!}{s^2} - 2\left(\frac{1}{s}\right)$$

$$L[f(t)] = \frac{6}{s^4} + \frac{5}{s^2} - \frac{2}{s} \quad \{\because 3! = 1\times2\times3 = 6\}$$

$$\therefore L[t^3 + 5t - 2] = \frac{6}{s^4} + \frac{5}{s^2} - \frac{2}{s} \checkmark \quad \text{...Ans.}$$

Ex. 1.8.7 : Find Laplace transform of $f(t) = t^4 - 3t^2 + 4t$.

Soln. :

Given $f(t) = t^4 - 3t^2 + 4t$

$$\therefore L[f(t)] = [t^4 - 3t^2 + 4t]$$

➔ **Using Linearly property of Laplace transform**

$$\ldots\left[L[a\,f(t) + b\,g(t) - c\,h(t)] = aL[f(t)] + bL[g(t)] - cL[h(t)]\right]$$

$$\therefore L[f(t)] = L[t^4] - 3L[t^2] + 4[t]$$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} \text{ OR } \frac{n!}{s^{n+1}} \ldots \text{If n is integer}\right]$$

$$\left\{\text{compare } \left(t^{\underset{n}{\uparrow}4}\right) \text{ with } t^n \text{ and } \left(t^{\underset{n}{\uparrow}2}\right) \text{ with } t^n\right\}$$

$$\therefore L[f(t)] = \underbrace{L[t^4]}_{\frac{4!}{s^{4+1}}} - 3\underbrace{L[t^2]}_{\frac{2!}{s^{2+1}}} + 4\underbrace{L[t]}_{\frac{1!}{s^{1+1}}}$$

$$L[f(t)] = \frac{4!}{s^5} - 3\frac{2!}{s^3} + 4\left(\frac{1}{s^2}\right)$$

$$= \frac{24}{s^5} - 3\frac{2}{s^3} + \frac{4}{s^2} \quad \{\because 4! = 1\times2\times3\times4 = 24\}$$

$$L[t^4 - 3t^2 + 4t] = \frac{24}{s^5} - \frac{6}{s^3} + \frac{4}{s^2} \checkmark \quad \text{...Ans.}$$

1.9 Examples using Trigonometric Formula

Double angle formulae

1. $\cos 2x = 2\cos^2 x - 1 \Rightarrow \cos^2 x = \frac{1+\cos 2x}{2}$
2. $\cos 2x = 1 - 2\sin^2 x \Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}$

Triple angle formulae

1. $\sin 3x = 3\sin x - 4\sin^3 x$
2. $\cos 3x = 4\cos^3 x - 3\cos x$

Product to sum formulae

1. $2\sin A\cdot\cos B = \sin(A+B) + \sin(A-B)$
2. $2\cos A\cdot\sin B = \sin(A+B) - \sin(A-B)$
3. $2\cos A\cdot\cos B = \cos(A+B) + \cos(A-B)$
4. $2\sin A\cdot\sin B = \cos(A-B) - \cos(A+B)$

Ex. 1.9.1 : Find Laplace transform of $\sin^2 t$.

Soln. :

Let, $f(t) = \sin^2 t$

$$f(t) = \frac{1-\cos 2t}{2} \quad \text{...(standard formula)}$$

$$\therefore L[f(t)] = L\left[\frac{1-\cos 2t}{2}\right]$$

$$= L\left[\frac{1}{2}(1-\cos 2t)\right]$$

➔ **Using Linearity property**

$$\ldots\left[L[a\,f(t)+b\,g(t)-c\,h(t)] = a\,L[f(t)]+b\,L[g(t)]-c\,L[h(t)]\right]$$

$$\therefore\ L[f(t)] = \frac{1}{2}L(1-\cos 2t)$$

$$L[f(t)] = \frac{1}{2}[L(1)-L(\cos 2t)]$$

➔ **Using the standard result of Laplace transform,**

$$\ldots\left[L(1)=\frac{1}{s}\ \text{and}\ L[\cos at]=\frac{s}{s^2+a^2}\right]$$

....(compare : cos ②t with cos at)

$$\therefore\ L[f(t)] = \frac{1}{2}\{\underbrace{L(1)}_{\frac{1}{s}}-\underbrace{L[\cos 2t]}_{\frac{s}{s^2+(2)^2}}\}$$

$$\therefore\ L[f(t)] = \frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+(2)^2}\right]$$

$$\therefore\ L[f(t)] = \frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4}\right]$$

$$\therefore\ L[\sin^2 t] = \frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4}\right]\checkmark \qquad \ldots\text{Ans.}$$

Note : We can further simplify it as,

$$\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4}\right]=\frac{1}{2}\left[\frac{s^2+4-s^2}{s(s^2+4)}\right]=\frac{1}{2}\left[\frac{4}{s(s^2+4)}\right]$$

$$=\frac{2}{s(s^2+4)}$$

Ex. 1.9.2 : Find Laplace transform of $\cos^2 t$.

Soln. :

Let, $f(t) = \cos^2 t$

$$f(t) = \frac{1+\cos 2t}{2} \qquad \ldots\text{(standard formula)}$$

$$\therefore\ L[f(t)] = L\left\{\frac{1}{2}(1+\cos 2t)\right\}$$

➔ **Using Linearity property of Laplace transform**

$$\ldots\left[L[a\,f(t)+b\,g(t)-ch(t)] = a\,L[f(t)]+b\,L[g(t)]-cL[h(t)]\right]$$

$$\therefore\ L[f(t)] = \frac{1}{2}L(1+\cos 2t)$$

$$= \frac{1}{2}[L(1)+L(\cos 2t)]$$

➔ **Using the standard result of Laplace transform**

$$\ldots\left[L(1)=\frac{1}{s}\ \text{and}\ L[\cos at]=\frac{s}{s^2+a^2}\right]$$

....(compare : cos ②t with cos at)

$$\therefore\ L[f(t)] = \frac{1}{2}\{\underbrace{L(1)}_{\frac{1}{s}}+\underbrace{L(\cos 2t)}_{\frac{s}{s^2+(2)^2}}\}$$

$$\therefore\ L[f(t)] = \frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+(2)^2}\right]$$

$$\therefore\ L[f(t)] = \frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+4}\right]\checkmark \qquad \ldots\text{Ans.}$$

Ex. 1.9.3 : Find Laplace transform of $\sin^3 t$.

Soln. :

Let, $f(t) = \sin^3 t$ ...(1)

➔ **Using standard trigonometric formula**

$$\ldots[\sin 3x = 3\sin x - 4\sin^3 x]$$

$$\sin 3x = 3\sin x - 4\sin^3 x$$

$$\therefore 4\sin^3 x = 3\sin x - \sin 3x$$

$$\therefore\ \sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x \quad \ldots\text{(Divide throughout by 4)}$$

$\therefore$ Equation (1) we can write as,

$$\therefore\ f(t) = \sin^3 t = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t$$

Taking Laplace transform of both sides, We get,

$$\therefore\ L[f(t)] = L\left[\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\right]$$

➔ **Using the linearity property of Laplace transform,**

$$\ldots[L[a\,f(t)+b\,g(t)-ch(t)] = a\,L[f(t)]+b\,L[g(t)]-c\,L[h(t)]]$$

$$= \frac{3}{4}L(\sin t) - \frac{1}{4}L(\sin 3t) \qquad \ldots(1)$$

$$= \frac{3}{4}L(\sin 1t) - \frac{1}{4}L(\sin 3t) \qquad \ldots(2)$$

➔ **Using the standard result of Laplace transform,**

$$\ldots\left[L(\sin at) = \frac{a}{s^2+a^2}\right]$$

$$\left\{\text{Compare : } \sin \underset{\substack{\uparrow\\ a}}{①}t \text{ with } \sin at,\ \sin \underset{\substack{\uparrow\\ a}}{③}t \text{ with } \sin at\right\}$$

$$L[f(t)] = \frac{3}{4}\underbrace{L(\sin 1t)}_{\frac{1}{s^2+(1)^2}} - \frac{1}{4}\underbrace{L(\sin 3t)}_{\frac{3}{s^2+(3)^2}}$$

$$= \frac{3}{4}\frac{1}{s^2+(1)^2} - \frac{1}{4}\frac{3}{s^2+(3)^2}$$

$$L[f(t)] = \frac{3}{4}\cdot\frac{1}{s^2+1} - \frac{1}{4}\frac{3}{s^2+9}$$

$$\therefore\ L[\sin^3 t] = \frac{3}{4}\left(\frac{1}{s^2+1}\right) - \frac{3}{4}\left(\frac{1}{s^2+9}\right)$$

Common out $\frac{3}{4}$ from R.H.S., we get,

$$\therefore\ L[\sin^3 t] = \frac{3}{4}\left[\frac{1}{s^2+1}-\frac{1}{s^2+9}\right]\checkmark \qquad \ldots\text{Ans.}$$

Ex. 1.9.4 : Find Laplace transform of $\cos^3 t$

Soln. :

Let $f(t) = \cos^3 t$ ...(1)

➔ **Using the trigonometric formula**

$$\ldots[\cos 3x = 4\cos^3 x - 3\cos x]$$

$$\therefore \cos 3x = 4\cos^3 x - 3\cos x$$

$$\cos 3x + 3\cos x = 4\cos^3 x \quad \ldots\text{(Shifting 3 cos x on L.H.S.)}$$

$$\therefore 4\cos^3 x = \cos 3x + 3\cos x$$

Divide throughout by 4

$$\cos^3 x = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x$$

Using this formula, Equation (1) becomes,

$$\therefore\ f(t) = \cos^3 t = \frac{1}{4}\cos 3t + \frac{3}{4}\cos t$$

$$\therefore\ L[f(t)] = L\left[\frac{1}{4}\cos 3t + \frac{3}{4}\cos t\right]$$

➔ **Using linearity property of Laplace transform**

$$\ldots\left[L[a\,f(t)+b\,g(t)-c\,h(t)] = a\,L[f(t)]+b\,L[g(t)]-c\,L[h(t)]\right]$$

$$\therefore\ L[f(t)] = \frac{1}{4}L(\cos 3t) + \frac{3}{4}L(\cos t)$$

➔ **Using the standard result of Laplace transform**

$$\ldots\left[L(\cos at) = \frac{s}{s^2+a^2}\right]$$

{compare cos ③t with cos at and cos ①t with cos at}

$$L[f(t)] = \frac{1}{4}\underbrace{L(\cos 3t)}_{\frac{s}{s^2+(3)^2}} + \frac{3}{4}\underbrace{L(\cos t)}_{\frac{s}{s^2+(1)^2}}$$

$$= \frac{1}{4}\frac{s}{s^2+(3)^2} + \frac{3}{4}\frac{s}{s^2+(1)^2}$$

$$L[f(t)] = \frac{1}{4}\frac{s}{s^2+9} + \frac{3}{4}\frac{s}{s^2+1}$$

Common out $\frac{1}{4}$, we get,

$$\therefore\ L[\cos^3 t] = \frac{1}{4}\left[\frac{s}{s^2+9}+\frac{3s}{s^2+1}\right]\checkmark \qquad \ldots\text{Ans.}$$

Ex. 1.9.5 : Find Laplace Transform of $(\sin t - \cos t)^2$

Soln. :

Let $f(t) = (\sin t - \cos t)^2$

➔ **Using the algebraic formula :** $\ldots[(a-b)^2 = a^2 - 2ab + b^2]$

$$\therefore\ f(t) = (\sin t)^2 - 2(\sin t)(\cos t) + (\cos t)^2$$

$$f(t) = \sin^2 t - 2\sin t\cos t + \cos^2 t$$

$$= (\sin^2 t + \cos^2 t) - (2\sin t\cos t)$$

➔ **Using standard trigonometric formulae**

$$\ldots[\sin^2\theta + \cos^2\theta = 1 \text{ and } \sin 2\theta = 2\sin\theta\cos\theta]$$

$$\therefore\ f(t) = 1 - \sin 2t$$

$$\therefore\ L[f(t)] = L[1 - \sin 2t]$$

➔ **Using linearity property of Laplace Transform**

$$\ldots\left[L[a\,f(t)+bg(t)-c\,h(t)] = a\,L[f(t)]+b\,L[g(t)]-cL[h(t)]\right]$$

$$\therefore\ L[f(t)] = L(1) - L(\sin 2t)$$

➔ **Using standard result of Laplace transform**

$$\ldots\left\{L(\sin at) = \frac{a}{s^2+a^2}\right\}$$

$$\therefore\ L[f(t)] = \underbrace{L(1)}_{\frac{1}{s}} - \underbrace{L(\sin 2t)}_{\frac{2}{s^2+(2)^2}}$$

$$L[f(t)] = \frac{1}{s} - \frac{2}{s^2+(2)^2}$$

$$L[f(t)] = \frac{1}{s} - \frac{2}{s^2+4}\checkmark \qquad \ldots\text{Ans.}$$

Ex. 1.9.6 : Find Laplace transform of $\sin 3t \cdot \cos 2t$

Soln. :

Let $f(t) = \sin 3t \cdot \cos 2t$

➔ **Using standard trigonometric formula**

$$\ldots\left[\sin A\cdot\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]\right]$$

$$f(t) = \sin \underset{\substack{\uparrow\\ A}}{3t}\cdot\cos \underset{\substack{\uparrow\\ B}}{2t}$$

$$= \frac{1}{2}[\sin(3t+2t)+\sin(3t-2t)]$$

$$f(t) = \frac{1}{2}[\sin 5t + \sin t]$$

$$\therefore\ L\,f(t) = L\left\{\frac{1}{2}(\sin 5t + \sin t)\right\}$$

➔ **Using linearity property of Laplace transform**

$$\ldots\left[L[a\,f(t)+b\,g(t)-c\,h(t)] = a\,L[f(t)]+bL[g(t)] - c\,L[h(t)]\right]$$

$$L[f(t)] = \frac{1}{2}\{L(\sin 5t) + L(\sin 1t)\}$$

➔ **Using the standard result of Laplace transform**

$$\ldots\left\{L[\sin at] = \frac{a}{s^2+a^2}\right\}$$

{compare sin ⑤t with sin at and sin ①t with sin at}

$$L[f(t)] = \frac{1}{2}\{\underbrace{L(\sin 5t)}_{\frac{5}{s^2+(5)^2}} + \underbrace{L(\sin 1t)}_{\frac{1}{s^2+(1)^2}}\}$$

$$L[f(t)] = \frac{1}{2}\left\{\frac{5}{s^2+(5^2)} + \frac{1}{s^2+(1^2)}\right\}$$

$$L[f(t)] = \frac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$$

$$\therefore L[\sin 3t \cos 2t] = \frac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right] \checkmark \quad \ldots\text{Ans.}$$

Ex. 1.9.7 : Find the Laplace transform of sin 2t · cos 3t

Soln. :

Let $f(t) = \sin 2t \cdot \cos 3t$

since 3t > 2t therefore rearrange it as,

$f(t) = \cos 3t \cdot \sin 2t$

➔ **Using the trigonometric formula**

$$\ldots\left[\cos A \cdot \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]\right]$$

$$f(t) = \cos(\overset{A}{3t}) \cdot \sin(\overset{B}{2t})$$

$$= \frac{1}{2}[\sin \underbrace{(3t+2t)}_{5t} - \sin \underbrace{(3t-2t)}_{1t}]$$

$$f(t) = \frac{1}{2}[\sin 5t - \sin t]$$

$$\therefore L[f(t)] = L\left\{\frac{1}{2}(\sin 5t - \sin 1t)\right\}$$

➔ **Using linearity property of Laplace transform**

$$\ldots\left[L[a f(t) + b g(t) - c h(t)] = a L[f(t)] + bL[g(t)] - c L[h(t)]\right]$$

$$L[f(t)] = \frac{1}{2}\{L(\sin 5t) - L(\sin 1t)\}$$

➔ **Using the standard result of Laplace transform**

$$\ldots\left[L[\sin at] = \frac{a}{s^2+a^2}\right]$$

...[compare sin ⑤t with sin at and sin ①t with sin at]

$$L[f(t)] = \frac{1}{2}\{\underbrace{L(\sin 5t)}_{\frac{5}{s^2+(5)^2}} - \underbrace{L(\sin 1t)}_{\frac{1}{s^2+(1)^2}}\}$$

$$L[f(t)] = \frac{1}{2}\left\{\frac{5}{s^2+(5)^2} - \frac{1}{s^2+(1)^2}\right\}$$

$$L[f(t)] = \frac{1}{2}\left[\frac{5}{s^2+25} - \frac{1}{s^2+1}\right]$$

$$\therefore L[\sin 2t \cos 3t] = \frac{1}{2}\left[\frac{5}{s^2+25} - \frac{1}{s^2+1}\right] \checkmark \quad \ldots\text{Ans.}$$

Ex. 1.9.8 : Find Laplace transform of sin 5t · cos 3t

Soln. :

Let $f(t) = \sin 5t \cdot \cos 3t$

➔ Using standard result of trigonometry

$$\ldots\left[\sin A \cdot \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]\right]$$

$$f(t) = \sin(\overset{A}{5t}) \cdot \cos(\overset{B}{3t})$$

$$= \frac{1}{2}[\sin \underbrace{(5t+3t)}_{8t} + \sin \underbrace{(5t-3t)}_{2t}]$$

$$f(t) = \frac{1}{2}[\sin 8t + \sin 2t]$$

$$L[f(t)] = L\left\{\frac{1}{2}(\sin 8t + \sin 2t)\right\}$$

➔ **Using linearity property of Laplace transform**

$$\ldots L\left[[a f(t) + b g(t) - c h(t)] = a L[f(t)] + b L[g(t)] - c L[h(t)]\right]$$

$$L[f(t)] = \frac{1}{2}\{L(\sin 8t) + L(\sin 2t)\}$$

➔ **Using the standard result of Laplace transform**

$$\ldots\left[L[\sin at] = \frac{a}{s^2+a^2}\right]$$

...[compare sin ⑧t with sin at and sin ②t with sin at]

$$L[f(t)] = \frac{1}{2}\{\underbrace{L(\sin 8t)}_{\frac{8}{s^2+(8)^2}} + \underbrace{L(\sin 2t)}_{\frac{2}{s^2+(2)^2}}\}$$

$$L[f(t)] = \frac{1}{2}\left\{\frac{8}{s^2+(8)^2} + \frac{2}{s^2+(2)^2}\right\}$$

$$L[f(t)] = \frac{1}{2}\left\{\frac{8}{s^2+64} + \frac{2}{s^2+4}\right\}$$

$$\therefore L[\sin 5t \cdot \cos 3t] = \frac{1}{2}\left[\frac{8}{s^2+64} + \frac{2}{s^2+4}\right] \checkmark \quad \ldots\text{Ans.}$$

Ex. 1.9.9 : Find Laplace transform of f (t) = sin 2t cos t

Soln. : Given, $f(t) = \sin 2t \cos t$

➔ **Using standard trigonometric formula**

$$\ldots\left[\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]\text{; with } A = 2t, B = t\right]$$

$$\therefore f(t) = \frac{1}{2}[\underbrace{\sin(2t+t)}_{\sin 3t} + \underbrace{\sin(2t-t)}_{\sin t}]$$

$$f(t) = \frac{1}{2}[\sin 3t + \sin t]$$

Taking Laplace transform of both sides,

$$\therefore L[f(t)] = L\left\{\frac{1}{2}[\sin 3t + \sin t]\right\}$$

➔ **Using linearity property of Laplace transform**

$$\left[L[a f(t) + b g(t) - c h(t)] = a L[f(t)] + b L[g(t)] - c L[h(t)]\right]$$

$$\therefore L[f(t)] = \frac{1}{2}[L(\sin 3t) + L[\sin t]]$$

➔ **Using Laplace Transform of standard function**

$$\ldots\left[L[\sin at] = \frac{a}{s^2+a^2}\right]$$

$$L[f(t)] = \frac{1}{2}[\underbrace{L(\sin 3t)}_{\frac{3}{s^2+(3)^2}} + \underbrace{L(\sin 1t)}_{\frac{1}{s^2+(1)^2}}]$$

$$\therefore L\{f(t)\} = \frac{1}{2}\left[\frac{3}{s^2+9} + \frac{1}{s^2+1}\right] \quad \ldots\text{(by table)}$$

$$= \frac{1}{2}\left[\frac{3s^2+3+s^2+9}{(s^2+9)(s^2+1)}\right] \quad \ldots\text{(By simplification)}$$

$$= \frac{1}{2}\left[\frac{4s^2+12}{(s^2+1)(s^2+9)}\right] = \frac{1}{2}\left[\frac{4(s^2+3)}{(s^2+1)(s^2+9)}\right]$$

$$\therefore L\{f(t)\} = 2\left[\frac{s^2+3}{(s^2+1)(s^2+9)}\right] \checkmark \quad \ldots\text{Ans.}$$

Ex. 1.9.10 : Find Laplace transform of f (t) = sin t cos 3t.

Soln. :

Given : $f(t) = \cos 3t \sin t$ [Rearrange : as 3t > t]

➔ **Using standard formula**

$$[\cos A \cdot \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$$

with A = 3t, B = t]

$$\therefore f(t) = \frac{1}{2}[\sin(3t+t) - \sin(3t-t)]$$

$$= \frac{1}{2}[\sin 4t - \sin 2t]$$

Taking Laplace transform of both sides

$$\therefore L[f(t)] = L\left\{\frac{1}{2}[\sin 4t - \sin 2t]\right\}$$

➔ **Using Linearity property of Laplace transform**

$$\ldots\left[L[a f(t) + b g(t) - c h(t)] = a L[f(t)] + b L[g(t)] - c L[h(t)]\right]$$

$$\therefore L[f(t)] = \frac{1}{2}\{L[\sin 4t] - L[\sin 2t]\}$$

➔ **Using Laplace Transform of standard function**

$$\ldots\left[L[\sin at] = \frac{a}{s^2+a^2}\right]$$

$$= \frac{1}{2}\{\underbrace{L(\sin 4t)}_{\frac{4}{s^2+(4)^2}} - \underbrace{L(\sin 2t)}_{\frac{2}{s^2+(2)^2}}\}$$

$$= \frac{1}{2}\left[\frac{4}{s^2+16} - \frac{2}{s^2+4}\right] \quad \ldots\text{(by table)}$$

$$= \frac{2s^2+8-s^2-16}{(s^2+16)(s^2+4)} \quad \ldots\text{(By simplification)}$$

$$L[f(t)] = \frac{s^2-8}{(s^2+4)(s^2+16)} \checkmark \quad \ldots\text{Ans.}$$

Ex. 1.9.11 : Find Laplace transform of f (t) = sin 2t sin 3t.

Soln. :

Given, $f(t) = \sin 3t \sin 2t$ ...[Rearrange : As 3t > 2t]

➔ **Using standard formula**

$$\ldots\left[\sin A \cdot \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]\right.$$

$$\left.\text{with } A = 3t,\ B = 2t\right]$$

$$\therefore f(t) = -\frac{1}{2}[\underbrace{\cos(3t+2t)}_{\cos 5t} - \underbrace{\cos(3t-2t)}_{\cos t}]$$

$$= -\frac{1}{2}[\cos 5t - \cos t]$$

Taking transform of both sides,

$$\therefore L[f(t)] = L\left\{\frac{1}{2}[\cos 5t - \cos t]\right\}$$

➔ **Using Linearity property of Laplace transform**

$$\ldots\left[L[a f(t) + b g(t) - c h(t)] = a L[f(t)] + b L[g(t)] - c L[h(t)]\right]$$

$$\therefore L[f(t)] = \frac{1}{2}\{L(\cos t) - L(\cos 5t)\}$$

➔ **Using Laplace transform of standard function**

$$\ldots\left[L(\cos at) = \frac{s}{s^2+a^2}\right]$$

$$= \frac{1}{2}\{\underbrace{L(\cos 1t)}_{\frac{s}{s^2+(1)^2}} - \underbrace{L(\cos 5t)}_{\frac{s}{s^2+(5)^2}}\}$$

$$= \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+25}\right]$$

$$= \frac{s}{2}\left[\frac{\cancel{s^2}+25-\cancel{s^2}-1}{(s^2+1)(s^2+25)}\right] \text{ ..By simplification}$$

$$= \frac{s}{\cancel{2}}\left[\frac{\cancel{24}^{12}}{(s^2+1)(s^2+25)}\right]$$

$$L[f(t)] = \frac{12s}{(s^2+1)(s^2+25)} \checkmark \quad \text{...Ans.}$$

Ex. 1.9.12 : Find Laplace transform of

$$f(t) = \cos t \cos 2t \cos 3t.$$

Soln. :

$$\text{Let } f(t) = \cos t \cos 2t \cos 3t$$

$$f(t) = \cos 3t \cdot \cos 2t \cdot \cos t$$

[Rearrange as $3t > 72t > t$]

➔ **Using standard trigonometric formula**

$$\ldots\left[\cos A \cdot \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]\right]$$

$$f(t) = [\cos \underset{A}{3t} \cdot \cos \underset{B}{2t}] \cdot \cos t$$

$$f(t) = \frac{1}{2}[\underbrace{\cos(3t+2t)}_{\cos 5t} + \underbrace{\cos(3t-2t)}_{\cos t}] \cdot \cos t$$

$$= \frac{1}{2}[\cos 5t + \cos t]\cos t$$

$$= \frac{1}{2}[\cos 5t \cdot \cos t + \cos^2 t]$$

➔ **Using standard trigonometric formula**

$$\ldots\left[\cos A \cdot \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)] \quad \cos^2 x = \frac{1+\cos 2x}{2}\right]$$

$$f(t) = \frac{1}{2}\left\{[\cos \underset{A}{5t} \cdot \cos \underset{B}{1t}] + [\cos^2 t]\right\}$$

$$= \left\{\frac{1}{2}[\underbrace{\cos(5t+t)}_{\cos 6t} + \underbrace{\cos(5t-t)}_{\cos 4t}] + \left[\frac{1+\cos 2t}{2}\right]\right\}$$

$$= \frac{1}{2}\times\frac{1}{2}[\cos 6t + \cos 4t + 1 + \cos 2t]$$

$$f(t) = \frac{1}{4}[\cos 6t + \cos 4t + \cos 2t + 1]$$

$$\therefore L[f(t)] = L\left\{\frac{1}{4}[\cos 6t + \cos 4t + \cos 2t + 1]\right\}$$

➔ **Using linearity property of Laplace transforms**

$$\ldots\{L[a f(t) + b g(t) - c h(t)] = a L[f(t)] + b L[g(t)] - c L[h(t)]\}$$

$$\therefore L[f(t)] = \frac{1}{4}\{L[\cos 6t] + L[\cos 4t] + L[\cos 2t] + L(1)\}$$

➔ **Using the standard result of Laplace transform**

$$\ldots\left[L[\cos at] = \frac{s}{s^2+a^2} \text{ and } L(1) = \frac{1}{s}\right]$$

$$\therefore L[f(t)] = \frac{1}{4}\left\{\frac{s}{s^2+(6)^2} + \frac{s}{s^2+(4)^2} + \frac{s}{s^2+(2)^2} + \frac{1}{s}\right\}$$

$$L[f(t)] = \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{s}{s^2+4} + \frac{1}{s}\right] \checkmark \quad \text{...Ans.}$$

1.10 Examples on First Shifting Theorem

Statement : First shifting Theorem

If $L[f(t)] = f(s)$ then

$$L[e^{at} f(t)] = [L[f(t)]]_{s \to s-a} = [f(s)]_{s \to s-a}$$

Similarly,

$$L[e^{-at} f(t)] = f(s+a)$$

$s \to s - a$ means, Every s replace by $(s-a)$

Ex. 1.10.1 : Find the Laplace transform of $e^{at}\sin bt$.

Soln. :

We know, by standard result of Laplace transform

$$L[\sin bt] = \frac{b}{s^2+b^2} \quad \ldots(1)$$

➔ **Using first shifting property of Laplace transform**

$$\ldots\left[L[e^{at} f(t)] = [L[f(t)]]_{s \to s-a} = [f(s)]_{s \to s-a}\right]$$

$$L[e^{at}\sin bt] = [L(\sin bt)]_{s \to s-a} \left(\begin{array}{l} s \to s-a \text{ means} \\ \text{Every s replace by } s-a \end{array}\right)$$

Using result from Equation (1),

$$L[e^{at}\sin bt] = \left[\frac{b}{s^2+b^2}\right]_{s \to s-a}$$

$$L[e^{at}\sin bt] = \frac{b}{(s-a)^2+b^2} \checkmark \quad \text{...Ans.}$$

Ex. 1.10.2 : Find Laplace transform of $e^{at}\cos bt$.

Soln. :

We know, by standard result of Laplace transform

$$L[\cos bt] = \frac{s}{s^2+b^2} \quad \ldots(1)$$

➔ **Using first shifting property of Laplace transform**

$$\ldots\left[L[e^{at} f(t)] = [L[f(t)]]_{s \to s-a} = [f(s)]_{s \to s-a}\right]$$

$$L[e^{at}\cos bt] = [L(\cos bt)]_{s \to s-a} \left(\begin{array}{l} s \to s-a \text{ means} \\ \text{every s replace by } s-a \end{array}\right)$$

Using result from Equation (1),

$$L[e^{at}\cos bt] = \left[\frac{s}{s^2+b^2}\right]_{s \to s-a}$$

$$L[e^{at}\cos bt] = \frac{s-a}{(s-a)^2+b^2} \checkmark \quad \text{...Ans.}$$

Ex. 1.10.3 : Find Laplace transform of $e^{at}\sinh bt$.

Soln. :

We know by standard result of Laplace transform

$$L[\sinh bt] = \frac{b}{s^2-b^2} \quad \ldots(1)$$

➔ **Using first shifting property of Laplace transform**

$$\ldots\left[L[e^{at} f(t)] = [L f(t)]_{s \to s-a} = [f(s)]_{s \to s-a}\right]$$

$$\therefore L[e^{at}\sinh bt] = [L(\sinh bt)]_{s \to s-a} \left(\begin{array}{l} s \to s-a \text{ means} \\ \text{every s replace by} \\ s-a \end{array}\right)$$

Using result from Equation (1),

$$L[e^{at}\sinh bt] = \left[\frac{b}{s^2-b^2}\right]_{s \to s-a}$$

$$L[e^{at}\sinh bt] = \frac{b}{(s-a)^2-b^2} \checkmark \quad \text{...Ans.}$$

Ex. 1.10.4 : Find Laplace transform of $e^{at}\cosh bt$.

Soln. :

We know, by standard result of Laplace transform

$$L[\cosh bt] = \frac{s}{s^2-b^2} \quad \ldots(1)$$

➔ **Using first shifting property of Laplace transform**

$$\ldots\left[L[e^{at} f(t)] = [L f(t)]_{s \to s-a} = [f(s)]_{s \to s-a}\right]$$

$$L[e^{at}\cosh bt] = [L(\cosh bt)]_{s \to s-a} \left(\begin{array}{l} s \to s-a \text{ means} \\ \text{every s replace by } s-a \end{array}\right)$$

Using result from Equation (1),

$$\therefore L[e^{at}\cosh bt] = \left[\frac{s}{s^2-b^2}\right]_{s \to s-a}$$

$$\therefore L[e^{at}\cosh bt] = \frac{s-a}{(s-a)^2-b^2} \checkmark \quad \text{...Ans.}$$

Ex. 1.10.5 : Find Laplace transform of $e^{at} t^n$.

Soln. :

We know, by standard result of Laplace transform

$$L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} \text{ OR } \frac{n!}{s^{n+1}} \text{ if n is integer} \quad \ldots(1)$$

➔ **Using first shifting property of Laplace transform**

$$\ldots\left[L[e^{at} f(t)] = L[f(t)]_{s \to s-a} = [f(s)]_{s \to s-a}\right]$$

$$\therefore L[e^{at} t^n] = [L(t^n)]_{s \to s-a} \left(\begin{array}{l} s \to s-a \text{ means} \\ \text{every s replace by } s-a \end{array}\right)$$

➔ **Using result from Equation (1),**

$$L[e^{at} t^n] = \left[\frac{\Gamma(n+1)}{s^{n+1}}\right]_{s \to s-a} \text{ OR } \left[\frac{n!}{s^{n+1}}\right]_{s \to s-a} \text{ If n is integer}$$

$$L[e^{at} t^n] = \frac{\Gamma(n+1)}{(s-a)^{n+1}} \text{ OR } \frac{n!}{(s-a)^{n+1}} \text{ If n is integer. ...Ans.}$$

Ex. 1.10.6 : Find the Laplace transform of $e^{-t}\cos 2t$

Soln. :

To find $L[e^{-t}\cos 2t]$

➔ **Use the first shifting property of Laplace transform**

$$\ldots\left[\begin{array}{l} L[e^{at} f(t)] = [L(f(t))]_{s \to s-a} = [f(s)]_{s \to s-a} \\ \text{where } L[f(t)] = f(s) \text{ and} \\ s \to s-a \text{ means every s replace by } (s-a) \end{array}\right]$$

$$\left\{\begin{array}{l} \text{Compare : } e^{-t} \cos 2t \text{ with } e^{at} f(t) \\ \qquad\qquad \downarrow \qquad\qquad \uparrow \\ \qquad\qquad e^{at} \qquad f(t) \\ \text{with } a = -1 \end{array}\right\}$$

$$\therefore L[e^{-t}\cos 2t] = [L(\cos 2t)]_{s \to s-(-1)}$$

$$= [L(\cos 2t)]_{s \to s+1} \quad \ldots(1)$$

Now find $L[f(t)] = L[\cos 2t]$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L(\cos at) = \frac{s}{s^2+a^2};\ \text{Here, } a = 2\right]$$

$$\therefore L[f(t)] = L[\cos 2t] \{\text{compare : } \cos at \text{ and } \cos \underset{\uparrow a}{2}t\}$$

$$= \frac{s}{s^2+(2)^2} = \frac{s}{s^2+4}$$

$$\therefore L[\cos 2t] = \frac{s}{s^2+4}$$

Using result $L[\cos 2t] = \dfrac{s}{s^2+4}$ in Equation (1),

$$\therefore L[e^{-t}\cos 2t] = \left[\frac{s}{s^2+4}\right]_{s \to s+1}$$

$$= \frac{(s+1)}{(s+1)^2+4} \quad \text{[every s replace by (s + 1)]}$$

$$= \frac{s+1}{(s^2+2s+1)+4}$$

[Use : $(a+b)^2 = a^2+2ab+b^2$]

$$= \frac{s+1}{s^2+2s+5}$$

$$\therefore L[e^{-t}\cos 2t] = \frac{s+1}{s^2+2s+5} \checkmark \quad \text{...Ans.}$$

Ex. 1.10.7 : Find Laplace transform of $f(t) = e^{-4t}\cos 3t$

Soln. :

To find $L[e^{-4t}\cos 3t]$

➔ **Use the first shifting property of Laplace transform**

$$\text{...}\left[\begin{array}{l} L[e^{at}f(t)] = [L(f(t))]_{s\to s-a} = [f(s)]_{s\to s-a} \\ \text{where } L[f(t)] = f(s) \text{ and} \\ s\to s-a \text{ means every s replace by } (s-a) \end{array}\right]$$

Compare

$$\left\{\begin{array}{ll} e^{-4t} & \cos 3t \text{ with } e^{at}f(t) \\ \downarrow & \uparrow \\ e^{at} & f(t) \\ \textit{with } a=-4 & \end{array}\right\}$$

$$\therefore L[e^{-4t}\cos 3t] = [L(\cos 3t)]_{s\to s-(-4)}$$

$$= [L(\cos 3t)]_{s\to s+4} \quad \text{...(1)}$$

Now find $L[f(t)] = L[\cos 3t]$

➔ **Using standard result of Laplace transform**

$$\text{...}\left[L(\cos at) = \frac{s}{s^2+a^2};\ \text{Here, } a=3\right]$$

$$\therefore L[f(t)] = L[\cos 3t] \quad \text{{compare : cos at and cos } } \underset{a}{\underset{\uparrow}{③}}\text{t}\}$$

$$= \frac{s}{s^2+(3)^2} = \frac{s}{s^2+9}$$

$$\therefore L[\cos 3t] = \frac{s}{s^2+9}$$

Using result $L[\cos 3t] = \frac{s}{s^2+9}$ in Equation (1),

$$\therefore L[e^{-4t}\cos 3t] = \left[\frac{s}{s^2+9}\right]_{s\to s+4}$$

$$= \frac{(s+4)}{(s+4)^2+9} \quad \text{[every s replace by (s + 4)]}$$

$$= \frac{s+4}{(s^2+8s+16)+9}$$

[Use : $(a+b)^2 = a^2+2ab+b^2$]

$$= \frac{s+4}{s^2+8s+25}$$

$$\therefore L[e^{-4t}\cos 3t] = \frac{s+4}{s^2+8s+25} \checkmark \quad \text{...Ans.}$$

Ex. 1.10.8 : Find Laplace transform of $e^{-t}\sin 3t$

Soln. :

To find $L[e^{-t}\sin 3t]$

➔ **Use the first shifting property of Laplace transform**

$$\text{...}\left[L[e^{at}f(t)] = [L(f(t))]_{s\to s-a} = [f(s)]_{s\to s-a}\right]$$

$$\left\{\begin{array}{l} \text{where } L[f(t)] = f(s) \text{ and } s\to s-a \text{ means} \\ \text{every s replace by } (s-a) \end{array}\right\}$$

$$\left[\begin{array}{l} \text{Compare : } \underset{\substack{\uparrow \\ e^{at}}}{e^{-t}}\ \underbrace{\sin 3t}_{\substack{\uparrow \\ f(t)}} \text{ with } e^{at}f(t) \\ \text{with } a=-1 \end{array}\right]$$

$$\therefore L[e^{-t}\sin 3t] = [L(\sin 3t)]_{s\to s-(-1)}$$

$$= [L(\sin 3t)]_{s\to s+1} \quad \text{...(1)}$$

Now find $L[f(t)] = L[\sin 3t]$

➔ **Using standard result of Laplace transform**

$$\text{...}\left[L(\sin at) = \frac{a}{s^2+a^2};\ \text{Here, } a=3\right]$$

$$\therefore L[f(t)] = L[\sin 3t]$$

$$= \frac{3}{s^2+(3)^2} \quad \text{{compare : sin at and sin } } \underset{a}{\underset{\uparrow}{③}}\text{t}\}$$

$$= \frac{3}{s^2+9}$$

$$\therefore L[\sin 3t] = \frac{3}{s^2+9} \quad \text{...Ans.}$$

Using result : $L[\sin 3t] = \frac{3}{s^2+9}$ in Equation (1)

$$\therefore L[e^{-t}\sin 3t] = \left[\frac{3}{s^2+9}\right]_{s\to s+1}$$

$$= \frac{3}{(s+1)^2+9} \quad \text{...(every s replace by s + 1)}$$

$$= \frac{3}{(s^2+2s+1)+9} \quad \left[\begin{array}{l}\text{Use : } (a+b)^2 \\ = a^2+2ab+b^2\end{array}\right]$$

$$\therefore L[e^{-t}\sin 3t] = \frac{3}{s^2+2s+10} \checkmark \quad \text{...Ans.}$$

Ex. 1.10.9 : Find Laplace transform of $f(t) = e^{-2t}\sin 3t$

Soln. :

To find $L[e^{-2t}\sin 3t]$

➔ **Use the first shifting property of Laplace transform**

$$\text{...}\left[L[e^{at}f(t)] = [L(f(t))]_{s\to s-a} = [f(s)]_{s\to s-a}\right]$$

$$\left\{\begin{array}{l} \text{where } L[f(t)] = f(s) \text{ and } s\to s-a \text{ means} \\ \text{every s replace by } (s-a) \end{array}\right\}$$

$$\left[\begin{array}{l} \text{Compare : } \underset{\substack{\uparrow \\ e^{at}}}{e^{-2t}}\ \underset{\substack{\uparrow \\ f(t)}}{\sin 3t} \text{ with } e^{at}f(t) \\ \text{with } a=-2 \end{array}\right]$$

$$\therefore L[e^{-2t}\sin 3t] = [L(\sin 3t)]_{s\to s-(-2)}$$

$$= [L(\sin 3t)]_{s\to s+2} \quad \text{...(1)}$$

Now find $L[f(t)] = L[\sin 3t]$

➔ **Using standard result of Laplace transform**

$$\text{...}\left[L(\sin at) = \frac{a}{s^2+a^2};\ \text{Here, } a=3\right]$$

$$\therefore L[f(t)] = L[\sin 3t]$$

$$= \frac{3}{s^2+(3)^2} \quad \text{{compare : sin at and sin } } \underset{a}{\underset{\uparrow}{③}}\text{t}\}$$

$$= \frac{3}{s^2+9}$$

$$\therefore L[\sin 3t] = \frac{3}{s^2+9}$$

➔ **Using result : $L[\sin 3t] = \frac{3}{s^2+9}$ in Equation (1)**

$$\therefore L[e^{-2t}\sin 3t] = \left[\frac{3}{s^2+9}\right]_{s\to s+2}$$

$$= \frac{3}{(s+2)^2+9} \quad \text{...(every s replace by s + 2)}$$

$$= \frac{3}{(s^2+4s+4)+9} \quad \left[\begin{array}{l}\text{Use : } (a+b)^2 \\ = a^2+2ab+b^2\end{array}\right]$$

$$\therefore L[e^{-t}\sin 3t] = \frac{3}{s^2+4s+13} \checkmark \quad \text{...Ans.}$$

Ex. 1.10.10 : Find Laplace transform of $t\,e^{6t}$

Soln. :

Let, $f(t) = t\,e^{6t} = e^{6t}\,t$

First find $L[t]$

➔ **Using the standard result of Laplace transform**

$$\text{...}\left[L[t^n] = \frac{\overline{|n+1}}{s^{n+1}} \text{ OR } \frac{n!}{s^{n+1}} \text{ if n is an integer}\right]$$

$$\therefore L[t] = L[t^1] \quad \left\{\begin{array}{l}\text{compare :} \\ t^{①} \text{ with } t^n \\ \uparrow \\ n\end{array}\right\}$$

$$= \frac{1!}{s^{1+1}}$$

$$L[t] = \frac{1}{s^2} \quad \text{...(1)}$$

Now to find : $L[e^{6t}\,t]$

➔ **Use the first shifting property of Laplace transform**

$$\text{...}\left[\begin{array}{l} L[e^{at}f(t)] = [L(f(t))]_{s\to s-a} = [f(s)]_{s\to s-a} \\ \text{wher } L[f(t)] = f(s) \text{ and } s\to s-a \text{ means} \\ \text{every s replace by } (s-a) \end{array}\right]$$

$$\left\{\text{compare : } e^{⑥t}\ t \text{ with } e^{at}f(t)\right\}$$

$$\uparrow \quad \uparrow$$

$$a \quad f(t)$$

$$\therefore L[e^{6t}\,t] = [L(t)]_{s\to s-6}$$

$$= \left[\frac{1}{s^2}\right]_{s\to s-6} \quad \text{...[using result of Equation (1)]}$$

$s\to s-a$ means, every s replace by $(s-a)$

$$\therefore L[e^{6t}\,t] = \left[\frac{1}{(s-6)^2}\right] \checkmark \quad \text{...Ans.}$$

Ex. 1.10.11 : Find Laplace transform of $f(t) = t^2 e^{3t}$

Soln. :

Let $f(t) = t^2 e^{3t} = e^{3t}\,t^2$ ...(Rearrange)

First find $L[t^2]$

➔ **Using the standard result of Laplace transform**

$$\text{...}\left[L[t^n] = \frac{\overline{|n+1}}{s^{n+1}} \text{ OR } \frac{n!}{s^{n+1}} \text{ if n is an integer}\right]$$

$$\therefore L[t] = L[t^2] = \frac{2!}{s^{2+1}} \quad \left\{\begin{array}{l}\text{compare :} \\ t^{②} \text{ with } t^n \\ \uparrow \\ n\end{array}\right\}$$

$$L[t] = \frac{2}{s^{-3}} \quad \text{...(1)}$$

Now to find $L[e^{3t}\,t^2]$

➔ **Use the first shifting property of Laplace transform**

$$\text{...}\left[\begin{array}{l} L[e^{at}f(t)] = [L(f(t))]_{s\to s-a} = [f(s)]_{s\to s-a} \\ \text{wher } L[f(t)] = f(s) \text{ and} \\ s\to s-a \text{ means every s replace by } (s-a) \end{array}\right]$$

$$\left[\text{Compare : } e^{③t}\ \underset{\substack{\uparrow \\ f(t)}}{t^2} \text{ with } e^{at}f(t)\right]$$

$$\uparrow$$

$$a$$

$$\therefore L[e^{3t}\,t^2] = [L(t^2)]_{s\to s-(-3)}$$

$$= \left[\frac{2}{s^3}\right]_{s\to s-3} \quad \text{[using Equation of (1)]}$$

$s\to s-a$ means, every s replace by $(s-a)$

$$= \frac{2}{(s-3)^3}$$

$$\therefore L[e^{-2t}\,t^3] = \frac{2}{(s-3)^3} \checkmark \quad \text{...Ans.}$$

Ex. 1.10.12 : Find Laplace transform of $t^3 e^{-2t}$

Soln. :

Let $f(t) = t^3 e^{-2t} = e^{-2t}\,t^3$ ... (Rearrange)

First find $L[t^3]$

➔ **Using the standard result of Laplace transform**

$$\left[L[t^n] = \frac{\overline{|n+1}}{s^{n+1}} \text{ OR } \frac{n!}{s^{n+1}} \text{ if n is an integer}\right]$$

$$\therefore L[t] = L[t^3] = \frac{3!}{s^{3+1}} \qquad \left\{\begin{array}{l}\text{compare :}\\ t^{3} \\ \uparrow \\ n\end{array} \text{ with } t^n\right\}$$

$$L[t] = \frac{6}{s^4} \qquad ...(1)$$

Now to find $L[e^{-2t}t^3]$

➔ **Use the first shifting property of Laplace transform**

$$\left[\begin{array}{l} L[e^{at}f(t)] = [L(f(t))]_{s\to s-a} = [f(s)]_{s\to s-a} \\ \text{wher } L[f(t)] = f(s) \text{ and} \\ s\to s-a \text{ means every s replace by } (s-a)\end{array}\right]$$

$$\left[\text{Compare : } e^{\underset{a}{-2}t}\ \underset{f(t)}{t^3} \text{ with } e^{at}f(t)\right]$$

$$\therefore \quad L[e^{-2t}t^3] = [L(t^3)]_{s\to s-(-2)}$$

$$= \left[\frac{6}{s^4}\right]_{s\to s+2} \qquad \text{[using equation of (1)]}$$

$s\to s-a$ means, every s replace by $(s-a)$

$$= \frac{6}{(s+2)^4}$$

$$\therefore L[e^{-2t}t^3] = \frac{6}{(s+2)^4} \checkmark \qquad ...\text{Ans.}$$

Ex. 1.10.13 : Find Laplace transform of

$$f(t) = 2t^2e^{-2t} - 3e^{-2t}.$$

Soln. :

Given,

$$f(t) = 2t^2e^{-2t} - 3e^{-2t}$$

$$f(t) = e^{-2t}(2t^2 - 3) \qquad ...(\text{common out } e^{-2t})$$

$$\therefore L[f(t)] = L[e^{-2t}(2t^2-3)]$$

➔ **Using the first shifting property of Laplace transform**

$$\left\{\begin{array}{l} L[e^{at}f(t)] = [L(f(t))]_{s\to s-a} = [f(s)]_{s\to s-a} \\ \text{where } L[f(t)] = f(s) \text{ and } s\to s-a \text{ means} \\ \text{every s replace by } (s-a)\end{array}\right\}$$

$$\left[\text{Compare : } \underset{e^{at}}{e^{-2t}}\ \underbrace{(2t^2-3)}_{f(t)} \text{ with } e^{at}f(t) \quad \text{with } a = -2\right]$$

$$\therefore L[e^{-2t}(2t^2-3)] = [L(2t^2-3)]_{s\to s-(-2)}$$

$$= [L(2t^2-3)]_{s\to s+2} \qquad ...(1)$$

Now find $L[f(t)] = L[2t^2-3] = L[2t^2-3(1)]$

➔ **Using linearity property of Laplace transform**

$$\left[L[a\,f(t) + b\,g(t) - c\,h(t)] = a\,L[f(t)] + bL[g(t)] - cL[h(t)]\right]$$

$$\therefore L[f(t)] = 2L[t^2] - 3L[1]$$

➔ **Using standard result of Laplace transform**

$$\left[L[t^n] = \frac{\overline{|n+1}}{s^{n+1}} \text{ OR } \frac{n!}{s^{n+1}} \text{ if n is an integer and } L[1] = \frac{1}{s}\right]$$

$$\therefore L[f(t)] = 2\underbrace{L[t^2]}_{\frac{2!}{s^{2+1}}} - 3\underbrace{L[1]}_{\frac{1}{s}} \qquad \left\{\begin{array}{l}\text{compare :}\\ t^{2} \\ \uparrow \\ n\end{array} \text{ with } t^n\right\}$$

$$L[f(t)] = 2\left(\frac{2!}{s^3}\right) - 3\left(\frac{1}{s}\right) \qquad (\because 2! = 1\times 2 = 2)$$

$$\therefore L[2t^2-3] = 2\left(\frac{2}{s^3}\right) - \frac{3}{s} = \frac{4}{s^3} - \frac{3}{s}$$

Using result $L[2t^2-3] = \frac{4}{s^3} - \frac{3}{s}$ in Equation (1),

$$\therefore \quad L[e^{-2t}(2t^2-3)] = \left[\frac{4}{s^3} - \frac{3}{s}\right]_{s\to s+2}$$

$s\to s+2$ means every s replace by $s+2$

$$\therefore L[e^{-2t}(2t^2-3)] = \frac{4}{(s+2)^3} - \frac{3}{(s+2)} \checkmark \qquad ...\text{Ans.}$$

1.11 Examples on Change of Scale Property

Ex. 1.11.1 : If Laplace transform of f(t) is $\frac{4s}{(s^2+4)^2}$ Then find Laplace transform of f(4t)

Soln. :

Given :

$$L[f(t)] = \frac{4s}{(s^2+4)^2}$$

➔ **Using change of scale property of Laplace transform**

$$\left[L[f(at)] = \frac{1}{a}f\left(\frac{s}{a}\right) \text{ where } L[f(t)] = f(s)\right]$$

$$\left[f\left(\frac{s}{a}\right) \text{ mean } [f(s)]_{s\to\frac{s}{a}}\right]$$

$$L[f(4t)] = \frac{1}{4}[L[f(t)]]_{s\to\frac{s}{4}}$$

$$= \frac{1}{4}\left[\frac{4s}{(s^2+4)^2}\right]_{s\to\frac{s}{4}}$$

$$= \frac{1}{4}\left[\frac{4\left(\frac{s}{4}\right)}{\left[\left(\frac{s}{4}\right)^2+4\right]^2}\right] \qquad \left(\begin{array}{l} s\to\frac{s}{4} \text{ means} \\ \text{every s relace by } \frac{s}{4}\end{array}\right)$$

$$= \frac{1}{4}\left[\frac{s}{\left(\frac{s^2}{16}+4\right)^2}\right]$$

$$= \frac{1}{4}\left[\frac{s}{\left(\frac{s^2+64}{16}\right)^2}\right] = \frac{1}{4}\left[\frac{s}{\left(\frac{1}{16}\right)^2(s^2+64)^2}\right]$$

$$= \frac{1}{4}\times\frac{1}{\left(\frac{1}{16}\right)^2}\frac{s}{(s^2+64)^2}$$

$$= \frac{1}{4}\times\frac{16\times 16}{1}\left[\frac{s}{(s^2+64)^2}\right]$$

$$L[f(4t)] = \frac{64\,s}{(s^2+64)^2} \checkmark \qquad ...\text{Ans.}$$

Ex. 1.11.2 : If $L[f(t)] = \frac{4s}{s^2+2s+3}$ then find L [f(2t)]

Soln. :

Given :

$$L[f(t)] = \frac{4s}{s^2+2s+3}$$

➔ **Using change of scale property of Laplace transform**

$$\left[L[f(at)] = \frac{1}{a}f\left(\frac{s}{a}\right) \text{ where } L[f(t)] = f(s)\right]$$

$$\left[f\left(\frac{s}{a}\right) \text{ mean } [f(s)]_{s\to\frac{s}{a}}\right]$$

$$L[f(2t)] = \frac{1}{2}[L[f(t)]]_{s\to\frac{s}{2}}$$

$$= \frac{1}{2}\left[\frac{4s}{s^2+2s+3}\right]_{s\to\frac{s}{2}}$$

$$= \frac{1}{2}\left[\frac{4\left(\frac{s}{2}\right)}{\left(\frac{s}{2}\right)^2+2\left(\frac{s}{2}\right)+3}\right] = \frac{1}{2}\left[\frac{2s}{\frac{s^2}{4}+s+3}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{\frac{1}{4}(s^2+4s+12)}\right]$$

$$= \frac{1}{\cancel{2}}\times\frac{\cancel{2}}{\left(\frac{1}{4}\right)}\left[\frac{s}{s^2+4s+12}\right]$$

$$L[f(2t)] = \frac{4s}{s^2+4s+12} \checkmark \qquad ...\text{Ans.}$$

Ex. 1.11.3 : If $L[f(t)] = \frac{8+12s-2s^2}{(s^2+4)^2}$ then find L [f(2t)]

Soln. :

Given :

$$L[f(t)] = \frac{8+12s-2s^2}{(s^2+4)^2}$$

➔ **Using change of scale property of Laplace transform**

$$\left[L[f(at)] = \frac{1}{a}f\left(\frac{s}{a}\right) \text{ where } L[f(t)] = f(s)\right]$$

$$\left[f\left(\frac{s}{a}\right) \text{ mean } [f(s)]_{s\to\frac{s}{a}}\right]$$

$$\therefore L[f(2t)] = \frac{1}{2}[L[f(t)]]_{s\to\frac{s}{2}}$$

$$= \frac{1}{2}\left[\frac{8+12s-2s^2}{(s^2+4)^2}\right]_{s\to\frac{s}{2}}$$

$$= \frac{1}{2}\left[\frac{8+12\left(\frac{s}{2}\right)-2\left(\frac{s}{2}\right)^2}{\left[\left(\frac{s}{2}\right)^2+4\right]^2}\right]$$

$$\left(\begin{array}{l} s\to\frac{s}{2} \text{ means} \\ \text{every s relace by } \frac{s}{2}\end{array}\right)$$

$$= \frac{1}{2}\left[\frac{8+6s-2\left(\frac{s^2}{4}\right)}{\left[\left(\frac{s}{4}\right)^2+4\right]^2}\right]$$

$$= \frac{1}{2}\left[\frac{8+6s-\frac{s^2}{2}}{\left(\frac{s^2+16}{4}\right)^2}\right]$$

$$= \frac{1}{2}\left[\frac{8+6s-\frac{s^2}{2}}{\frac{1}{16}(s^2+16)^2}\right] = \frac{1}{2}\left[\frac{\frac{1}{2}(16+12s-s^2)}{\frac{1}{16}(s^2+16)^2}\right]$$

$$= \frac{1}{2}\times\frac{1}{2}\times\frac{16}{1}\frac{16+12s-s^2}{(s^2+16)^2}$$

$$L[f(2t)] = 4\left(\frac{16+12s-s^2}{(s^2+16)^2}\right) \checkmark \qquad ...\text{Ans.}$$

1.12 Miscelleneous Examples

Ex. 1.12.1 : Find Laplace transform of f (t) = t cos 3t

Soln. : We have,

$$L[\cos 3t] = \frac{s}{s^2+9} \qquad ...(\text{by table})$$

$$\therefore L[t\cos 3t] = \frac{-d}{ds}\left[\frac{s}{s^2+9}\right] \qquad \left[\because L[t\,f(t)] = -\frac{d}{ds}f(s)\right]$$

$$= -\left[\frac{(s^2+9)-s(2s)}{(s^2+9)^2}\right]$$

$$= \frac{s^2-9}{(s^2+9)^2} \checkmark \qquad ...\text{Ans.}$$

Ex. 1.12.2 : Find Laplace transform of f (t) = t sin 4 t

Soln. :

To find : L[f(t) = L [t sin 4t]

First find L [sin 4t]

➔ **Using the standard result of Laplace transform**

$$\left[L(\sin at) = \frac{a}{s^2+a^2};\ \text{Here } a = 4\right]$$

$$\therefore L[\sin 4t] = \frac{4}{s^2+(4)^2}$$

$$L[\sin 4t] = \frac{4}{s^2+16} \quad \ldots(1)$$

➔ **Using property of Laplace transform**

$$\ldots\left[L[t\,f(t)] = \frac{-d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore L[t\sin 4t] = \frac{-d}{ds}[L(\sin 4t)]$$

$$= \frac{-d}{ds}\left[\frac{4}{s^2+16}\right] \quad \ldots[\text{Using Equation (1)}]$$

➔ **Using quotient rule of derivative**

$$\ldots\left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$L[t\sin 4t] = -\left[\frac{(s^2+16)(0)-(4)(2s)}{(s^2+16)^2}\right]$$

$$= -\left[\frac{0-8s}{(s^2+16)^2}\right]$$

$$\therefore L[t\sin 4t] = \frac{8s}{(s^2+16)^2} \checkmark \quad \ldots\text{Ans.}$$

Ex. 1.12.3 : Find the Laplace transform of $f(t) = t\,e^{3t}\sin 2t$

Soln. :

We have,

First find L [sin 2t]

➔ **Using the standard result of Laplace transform**

$$\ldots\left[L(\sin at) = \frac{a}{s^2+a^2};\ \text{Here } a = 2\right]$$

$$\therefore L[\sin 2t] = \frac{2}{s^2+(2)^2}$$

$$L[\sin 2t] = \frac{2}{s^2+4} \quad \ldots(1)$$

➔ **Using property of Laplace transform**

$$\ldots\left[L[t\,f(t)] = \frac{-d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore L[t\sin 2t] = \frac{-d}{ds}[L(\sin 2t)]$$

$$= \frac{-d}{ds}\left[\frac{2}{s^2+4}\right] \quad \ldots[\text{Using Equation (1)}]$$

➔ **Using quotient rule of derivative**

$$\ldots\left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$L[t\sin 2t] = -\left[\frac{(s^2+4)(0)-(2)(2s)}{(s^2+4)^2}\right]$$

$$= -\left[\frac{0-4s}{(s^2+4)^2}\right]$$

$$\therefore L[t\sin 2t] = \frac{4s}{(s^2+4)^2}$$

➔ **Using first shifting property of Laplace transform**

$$\ldots\left[L(e^{at}f(t)) = [L[f(t)]]_{s\to s-a} = [f(s)]_{s\to s-a}\right]$$

$(s \to s-a$ means every s replace by $s-a)$

$$\therefore L[e^{3t}t\sin t] = [L(t\sin t)]_{s\to s-3}$$

$$L[e^{3t}t\sin t] = \left[\frac{4s}{(s^2+4)^2}\right]_{s\to s-3} = \frac{4(s-3)}{[(s-3)^2+4]^2} \checkmark \ldots\text{Ans.}$$

Ex. 1.12.4 : Find the Laplace transform of $f(t) = \int_0^t \sin 3t\,dt$

Soln. :

Let, $f(t) = \sin 3t$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L(\sin at) = \frac{a}{s^2+a^2};\ \text{Here, } a = 3\right]$$

$$\therefore L[f(t)] = L[\sin 3t]$$

$$= \frac{3}{s^2+(3)^2} \quad \{\text{compare : } \sin at \text{ and } \sin \underset{\substack{\uparrow \\ a}}{3}t\}$$

$$= \frac{3}{s^2+9}$$

$$\therefore L[\sin 3t] = \frac{3}{s^2+9}$$

➔ **Using Laplace transform of integration**

$$\ldots\left[L\left[\int_0^t f(u)\,du\right] = \frac{1}{s}f(s) \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore L\left[\int_0^t \sin 3t\,dt\right] = \frac{1}{s}L[\sin 3t]$$

$$= \frac{1}{s}\left[\frac{3}{s^2+9}\right]$$

$$\therefore L\left[\int_0^t \sin 3t\,dt\right] = \frac{3}{s(s^2+9)} \checkmark \quad \ldots\text{Ans.}$$

Ex. 1.12.5 : Find Laplace transform of $t\int_0^t e^{-2t}\sin t\,dt$.

Soln. :

Step I :

We have,

➔ **Using standard result of Laplace transform**

$$\ldots\left\{L(\sin at) = \frac{a}{s^2+a^2}\right\}$$

$$L[\sin t] = \frac{1}{s^2+(1)^2} = \frac{1}{s^2+1}$$

Step II :

➔ **Using first shifting property of Laplace transform**

$$\ldots\left[L(e^{at}f(t)) = [L[f(t)]]_{s\to s-a} = [f(s)]_{s\to s-a} = f(s-a)\right]$$

$(s \to s-a$ means every s replace by $s-a)$

$$L[e^{-2t}\sin t] = [L(\sin t)]_{s\to s+2}\left[\frac{1}{s^2+1}\right]_{s\to s+2}$$

$$\therefore L[e^{-2t}\sin t] = \frac{1}{(s+2)^2+1} \quad \ldots(1)$$

$\ldots[\because$ every s replace by $(s+2)]$

Step III :

➔ **Using Laplace transform of integration**

$$\ldots\left\{L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}f(s) \text{ where } f(s) = L[f(t)]\right\}$$

$$L\left[\int_0^t e^{-2t}\sin t\,dt\right] = \frac{1}{s}L[e^{-2t}\sin t]$$

$$= \frac{1}{s}\left[\frac{1}{(s+2)^2+1}\right]$$

[From Equation (1)]

$$= \frac{1}{s}\left[\frac{1}{s^2+4s+4+1}\right] = \frac{1}{s^3+4s^2+5s}$$

➔ **Using property of Laplace transform**

$$\ldots\left\{L[t\,f(t)] = -\frac{d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right\}$$

$$\therefore L\left[t\int_0^t e^{-2t}\sin t\,dt\right] = \frac{d}{ds}\left[L\left[\int_0^t e^{-2t}\sin t\,dt\right]\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s^3+4s^2+5s}\right]$$

➔ **Using quotient rule of derivative**

$$\ldots\left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$= -\left[\frac{-1}{(s^3+4s^2+5s)^2}(3s^2+8s+5)\right]$$

$$\therefore L\left[t\int_0^t e^{-2t}\sin t\,dt\right] = \frac{3s^2+8s+5}{(s^3+4s^2+5s)^2} \checkmark \quad \ldots\text{Ans.}$$

Ex. 1.12.6 : Find Laplace transform of $e^{-3t}\int_0^t t\sin 2t\,dt$.

Soln. :

First find L [sin 2t]

➔ **Using the standard result of Laplace transform**

$$\ldots\left[L(\sin at) = \frac{a}{s^2+a^2};\ \text{Here } a = 2\right]$$

$$\therefore L[\sin 2t] = \frac{2}{s^2+(2)^2}$$

$$L[\sin 2t] = \frac{2}{s^2+4} \quad \ldots(1)$$

➔ **Using property of Laplace transform**

$$\ldots\left[L[t\,f(t)] = \frac{-d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore L[t\sin 2t] = \frac{-d}{ds}[L(\sin 2t)]$$

$$= \frac{-d}{ds}\left[\frac{2}{s^2+4}\right] \quad \ldots[\text{Using Equation (1)}]$$

➔ **Using quotient rule of derivative**

$$\ldots\left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$L[t\sin 2t] = -\left[\frac{(s^2+4)(0)-(2)(2s)}{(s^2+4)^2}\right]$$

$$= -\left[\frac{0-4s}{(s^2+4)^2}\right]$$

$$\therefore L[t\sin 2t] = \frac{4s}{(s^2+4)^2}$$

➔ **Using Laplace transform of integration**

$$\ldots\left\{L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}f(s) \text{ where } f(s) = L[f(t)]\right\}$$

$$\therefore L\left[\int_0^t t\sin 2t\,dt\right] = \frac{1}{s}L[t\sin 2t]$$

$$= \frac{1}{\cancel{s}}\frac{4\cancel{s}}{(s^2+4)^2}$$

$$L\left[\int_0^t t\sin 2t\,dt\right] = \frac{4}{(s^2+4)^2}$$

➔ **Using first shifting property of Laplace transform :**

$$\ldots\left[L(e^{at}f(t)) = [L[f(t)]]_{s\to s-a} = [f(s)]_{s\to s-a}\right]$$

$(s \to s-a$ means every s replace by $s-a)$

$$\therefore L\left[e^{-3t}\int_0^t t\sin 2t\,dt\right] = \left[L\left[\int_0^t t\sin 2t\,dt\right]\right]_{s\to s-(-3)}$$

$$= \left[\frac{4}{(s^2+4)^2}\right]_{s\to s+3}$$

$$L\left[e^{-3t}\int_0^t t\sin 2t\,dt\right] = \frac{4}{[(s+3)^2+4]^2} \checkmark \quad \text{...Ans}$$

Ex. 1.12.7 : Find Laplace transform of $e^{-4t}\int_0^t t\sin 3t\,dt$

Soln. :

First find L [sin 3t]

➔ **Using the standard result of Laplace transform**

$$\ldots\left[L(\sin at) = \frac{a}{s^2+a^2};\ \text{Here } a = 3\right]$$

$$\therefore L[\sin 3t] = \frac{3}{s^2+(3)^2}$$

$$L[\sin 3t] = \frac{3}{s^2+9} \quad \ldots(1)$$

➔ **Using property of Laplace transform**

$$\ldots\left[L[t\,f(t)] = \frac{-d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore L[t\sin 3t] = \frac{-d}{ds}[L(\sin 3t)]$$

$$= \frac{-d}{ds}\left[\frac{3}{s^2+9}\right] \quad \ldots\text{[Using Equation (1)]}$$

➔ **Using quotient rule of derivative**

$$\ldots\left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$L[t\sin 3t] = -\left[\frac{(s^2+9)(0)-(3)(2s)}{(s^2+9)^2}\right]$$

$$= -\left[\frac{0-6s}{(s^2+9)^2}\right]$$

$$\therefore L[t\sin 3t] = \frac{6s}{(s^2+9)^2}$$

➔ **Using Laplace transform of integration**

$$\ldots\left\{L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}f(s) \text{ where } f(s) = L[f(t)]\right\}$$

$$\therefore L\left[\int_0^t t\sin 3t\,dt\right] = \frac{1}{s}L[t\sin 3t] = \frac{1}{\cancel{s}}\frac{6\cancel{s}}{(s^2+9)^2}$$

$$L\left[\int_0^t t\sin 3t\,dt\right] = \frac{6}{(s^2+9)^2}$$

➔ **Using first shifting property of Laplace transform**

$$\ldots\left[L(e^{at}f(t)) = [L[f(t)]]_{s\to s-a} = [f(s)]_{s\to s-a}\right]$$

$(s \to s-a$ means every s replace by $s-a)$

$$\therefore L\left[e^{-4t}\int_0^t t\sin 3t\,dt\right] = \left[L\left[\int_0^t t\sin 3t\,dt\right]\right]_{s\to s-(-4)}$$

$$= \left[\frac{6}{(s^2+9)^2}\right]_{s\to s+4}$$

$$L\left[e^{-4t}\int_0^t t\sin 3t\,dt\right] = \frac{6}{[(s+4)^2+9]^2} \checkmark \quad \text{...Ans.}$$

Ex. 1.12.8 : Find Laplace transform of $\int_0^t t\,e^{-t}\sin 2t\,dt$

Soln. :

First find L [sin 2t]

➔ **Using the standard result of Laplace transform**

$$\ldots\left[L(\sin at) = \frac{a}{s^2+a^2};\ \text{Here } a = 2\right]$$

$$\therefore L[\sin 2t] = \frac{2}{s^2+(2)^2}$$

$$L[\sin 2t] = \frac{2}{s^2+4} \quad \ldots(1)$$

➔ **Using property of Laplace transform**

$$\ldots\left[L[t\,f(t)] = \frac{-d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore L[t\sin 2t] = \frac{-d}{ds}[L(\sin 2t)]$$

$$= \frac{-d}{ds}\left[\frac{2}{s^2+4}\right] \quad \ldots\text{[Using Equation (1)]}$$

➔ **Using quotient rule of derivative**

$$\ldots\left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$L[t\sin 2t] = -\left[\frac{(s^2+4)(0)-(2)(2s)}{(s^2+4)^2}\right]$$

$$= -\left[\frac{0-4s}{(s^2+4)^2}\right]$$

$$\therefore L[t\sin 2t] = \frac{4s}{(s^2+4)^2}$$

➔ **Using first shifting property of Laplace transform**

$$\ldots\left[L(e^{at}f(t)) = [L[f(t)]]_{s\to s-a} = [f(s)]_{s\to s-a}\right]$$

$(s \to s-a$ means every s replace by $s-a)$

$$\therefore L[e^{-t}t\sin 2t] = [L(t\sin 2t)]_{s\to s-(-1)}$$

$$L[e^{-t}t\sin 2t] = \left[\frac{4s}{(s^2+4)^2}\right]_{s\to s+1} = \frac{4(s+1)}{[(s+1)^2+4]^2}$$

➔ **Using Laplace transform of integration**

$$\ldots\left\{L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}f(s) \text{ where } f(s) = L[f(t)]\right\}$$

$$\therefore L\left[\int_0^t e^{-t}t\sin 2t\,dt\right] = \frac{1}{s}\left[L[e^{-t}t\sin 2t]\right]$$

$$\therefore L\left[\int_0^t e^{-t}t\sin 2t\,dt\right] = \frac{1}{s}\frac{4(s+1)}{[(s+1)^2+4]^2} \checkmark \quad \text{...Ans.}$$

Ex. 1.12.9 : Find Laplace transform of $\int_0^t t\,e^{-4t}\sin 3t\,dt$

Soln. :

First find L [sin 3t]

➔ **Using the standard result of Laplace transform**

$$\ldots\left[L(\sin at) = \frac{a}{s^2+a^2};\ \text{Here } a = 3\right]$$

$$\therefore L[\sin 3t] = \frac{3}{s^2+(3)^2}$$

$$L[\sin 3t] = \frac{3}{s^2+9} \quad \ldots(1)$$

➔ **Using property of Laplace transform**

$$\ldots\left[L[t\,f(t)] = \frac{-d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore L[t\sin 3t] = \frac{-d}{ds}[L(\sin 3t)]$$

$$= \frac{-d}{ds}\left[\frac{3}{s^2+9}\right] \quad \ldots\text{[Using Equation (1)]}$$

➔ **Using quotient rule of derivative**

$$\ldots\left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$L[t\sin 3t] = -\left[\frac{(s^2+9)(0)-(3)(2s)}{(s^2+9)^2}\right]$$

$$= -\left[\frac{0-6s}{(s^2+9)^2}\right]$$

$$\therefore L[t\sin 3t] = \frac{6s}{(s^2+9)^2}$$

➔ **Using first shifting property of Laplace transform**

$$\ldots\left[L(e^{at}f(t)) = [L[f(t)]]_{s\to s-a} = [f(s)]_{s\to s-a}\right]$$

$(s \to s-a$ means every s replace by $s-a)$

$$\therefore L[e^{-4t}t\sin 3t] = [L(t\sin 3t)]_{s\to s-(-4)}$$

$$L[e^{-4t}t\sin 3t] = \left[\frac{6s}{(s^2+9)^2}\right]_{s\to s+4} = \frac{6(s+4)}{[(s+4)^2+9]^2}$$

➔ **Using Laplace transform of integration**

$$\ldots\left\{L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}f(s) \text{ where } f(s) = L[f(t)]\right\}$$

$$\therefore L\left[\int_0^t e^{-4t}t\sin 3t\,dt\right] = \frac{1}{s}\left[L[e^{-4t}t\sin 3t]\right]$$

$$= \frac{1}{s}\frac{6(s+4)}{[(s+4)^2+9]^2} \checkmark \quad \text{...Ans.}$$

Ex. 1.12.10 : Find Laplace transform of $f(t) = t\int_0^t e^{-t}\cos 2t\,dt$

Soln. :

To find L $[e^{-t}\cos 2t]$

➔ **Use the first shifting property of Laplace transform**

$$\ldots\left[\begin{array}{l} L[e^{at}f(t)] = [L(f(t))]_{s\to s-a} = [f(s)]_{s\to s-a} \\ \text{where } L[f(t)] = f(s) \text{ and} \\ s \to s-a \text{ means every s replace by } (s-a) \end{array}\right]$$

Compare : e^{-t} $\cos 2t$ with $e^{at}f(t)$

$e^{-t} \downarrow e^{at}$, $\cos 2t \uparrow f(t)$

with $a = -1$

$$\therefore L[e^{-t}\cos 2t] = [L(\cos 2t)]_{s\to s-(-1)}$$

$$= [L(\cos 2t)]_{s\to s+1} \quad \ldots(1)$$

Now find $L[f(t)] = L[\cos 2t]$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L(\cos at) = \frac{s}{s^2+a^2};\ \text{Here, } a = 2\right]$$

$\therefore$ $L[f(t)] = L[\cos 2t]$ {compare : cos at and cos ②t}

a

$$= \frac{s}{s^2+(2)^2} = \frac{s}{s^2+4}$$

$$\therefore L[\cos 2t] = \frac{s}{s^2+4}$$

Using result $L[\cos 2t] = \frac{s}{s^2+4}$ in Equation (1),

$$\therefore L[e^{-t}\cos 2t] = \left[\frac{s}{s^2+4}\right]_{s\to s+1}$$

$$= \frac{(s+1)}{(s+1)^2+4} \quad \text{[every s replace by } (s+1)]$$

$$= \frac{s+1}{(s^2+2s+1)+4}$$

[Use : $(a+b)^2 = a^2+2ab+b^2$]

$$= \frac{s+1}{s^2+2s+5}$$

$$\therefore L[e^{-t}\cos 2t] = \frac{s+1}{s^2+2s+5}$$

➔ **Using Laplace transform of integration**

$$\ldots\left[L\left[\int_0^t f(u)\,du\right] = \frac{1}{s}f(s) \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore L\left[\int_0^t e^{-t}\cos 2t\,dt\right] = \frac{1}{s}L[e^{-t}\cos 2t]$$

$$= \frac{1}{s}\left(\frac{s+1}{s^2+2s+5}\right) \text{ [from Equation (1)]}$$

$$= \frac{s+1}{s(s^2+2s+5)}$$

$$\therefore L\left[\int_0^t e^{-t}\cos 2t\,dt\right] = \frac{s+1}{s^3+2s^2+5s}$$

➔ **Using property of Laplace transform**

$$\ldots\left\{L[t\,f(t)] = -\frac{d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right\}$$

$$\therefore L\left[t\int_0^t e^{-t}\cos 2t\,dt\right] = -\frac{d}{ds}\left[\frac{s+1}{s^3+2s^2+5s}\right]$$

➔ **Using quotient rule of derivative**

$$\ldots\left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$\therefore L\left[t\int_0^t e^{-t}\cos 2t\,dt\right] = -\left[\frac{(s^3+2s^2+5s)\frac{d}{ds}(s+1) - (s+1)\cdot\frac{d}{ds}(s^3+2s^2+5s)}{(s^3+2s^2+5s)^2}\right]$$

$$= -\left[\frac{(s^3+2s^2+5s)(1) - (s+1)(3s^2+4s+5)}{[s(s^2+2s+5)]^2}\right]$$

$$= -\left[\frac{(s^3+2s^2+5s) - s(3s^2+4s+5) - (3s^2+4s+5)}{s^2(s^2+2s+5)^2}\right]$$

$$= -\left[\frac{s^3+2s^2+\cancel{5s}-3s^3-4s^2-\cancel{5s}-3s^2-4s-5}{s^2[s^2+2s+5]^2}\right]$$

$$= -\left[\frac{-2s^3-5s^2-4s-5}{s^2(s^2+2s+5)^2}\right]$$

$$\therefore L\left[\int_0^t e^{-t}\cos 2t\,dt\right] = \frac{2s^3+5s^2+4s+5}{s^2(s^2+2s+5)^2} \checkmark \quad \text{...Ans.}$$

Ex. 1.12.11 : Find the Laplace transform of $e^{-3t}\int_0^t t\sin 3t\,dt$.

Soln. :

First find L [sin 3t]

➔ **Using the standard result of Laplace transform**

$$\ldots\left[L(\sin at) = \frac{a}{s^2+a^2};\ \text{Here } a = 3\right]$$

$$\therefore L[\sin 3t] = \frac{3}{s^2+(3)^2}$$

$$L[\sin 3t] = \frac{3}{s^2+9} \quad \ldots(1)$$

➔ **Using property of Laplace transform**

$$\ldots\left[L[t\,f(t)] = \frac{-d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore L[t\sin 3t] = \frac{-d}{ds}[L(\sin 3t)]$$

$$= \frac{-d}{ds}\left[\frac{3}{s^2+9}\right] \quad \text{...[Using Equation (1)]}$$

➔ **Using quotient rule of derivative**

$$\ldots\left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$L[t\sin 3t] = -\left[\frac{(s^2+9)(0) - (3)(2s)}{(s^2+9)^2}\right]$$

$$= -\left[\frac{0-6s}{(s^2+9)^2}\right]$$

$$\therefore L[t\sin 3t] = \frac{6s}{(s^2+9)^2}$$

➔ **Using Laplace transform of integration**

$$\ldots\left\{L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}f(s) \text{ where } f(s) = L[f(t)]\right\}$$

$$\therefore L\left[\int_0^t t\sin 3t\,dt\right] = \frac{1}{s}L[t\sin 3t] = \frac{1}{s}\left[\frac{6s}{(s^2+9)^2}\right]$$

$$\therefore L\left[\int_0^t t\sin 3t\,dt\right] = \frac{6}{(s^2+9)^2}$$

➔ **Using first shifting property of Laplace transform**

$$\ldots\left[L(e^{at}f(t)) = [L[f(t)]]_{s\to s-a} = [f(s)]_{s\to s-a}\right]$$

($s \to s-a$ means every s replace by $s-a$)

$$L\left[e^{-3t}\int_0^t t\sin 3t\,dt\right] = \left[L\left(\int_0^t t\sin 3t\,dt\right)\right]_{s\to s+3} = \left[\frac{6}{(s^2+9)^2}\right]_{s\to s+3}$$

$$\therefore L\left[e^{-3t}\int_0^t t\sin 3t\,dt\right] = \frac{6}{[(s+3)^2+9]^2} \checkmark \quad \text{...Ans.}$$

Ex. 1.12.12 : Find Laplace transform of $f(t) = \left(\frac{\sin 4t}{t}\right)$

Soln. :

➔ **Using the standard result replace transform**

$$\ldots\left[L(\sin at) = \frac{a}{s^2+a^2};\ \text{Here } a = 4\right]$$

$$\therefore L[\sin 4t] = \frac{4}{s^2+(4)^2}$$

$$L[\sin 4t] = \frac{4}{s^2+16} \quad \ldots(1)$$

➔ **Using property of Laplace transform**

$$\ldots\left\{L\left(\frac{f(t)}{t}\right) = \int_s^\infty f(s)\,ds \text{ where } f(s) = L[f(t)]\right\}$$

$$\therefore L\left(\frac{\sin 4t}{t}\right) = \int_s^\infty L(\sin 4t)\,ds$$

$$= \int_s^\infty \frac{4}{s^2+16}\,ds \quad \text{...[by Equations (1)]}$$

$$= 4\int_s^\infty \frac{1}{s^2+(4)^2}\,ds$$

➔ **Using standard result of integration**

$$\ldots\left[\int\frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \text{ with } x = s,\ a = 4\right]$$

$$\therefore L\left(\frac{\sin 4t}{t}\right) = 4\left[\frac{1}{4}\tan^{-1}\left(\frac{s}{4}\right)\right]_s^\infty$$

$$= 4\times\frac{1}{4}\left[\tan^{-1}\left(\frac{s}{4}\right)\right]_s^\infty$$

Substitute s as : (upper limit) – (lower limit)

$$L\left(\frac{\sin 4t}{t}\right) = \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{4}\right)$$

➔ **Using standard trigonometric value**

$$\ldots\left[\tan\left(\frac{\pi}{2}\right) = \infty \Rightarrow \tan^{-1}(\infty) = \frac{\pi}{2}\right]$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{4}\right)$$

since $\frac{\pi}{2} - \tan^{-1}x = \cot^{-1}x$

$$\therefore L\left(\frac{\sin 4t}{t}\right) = \cot^{-1}\left(\frac{s}{4}\right) \checkmark \quad \text{...Ans.}$$

Ex. 1.12.13 : Find Laplace transform of $\int_0^t \frac{\sin t}{t}\,dt$

Dr. BATU - May 2017

Soln. :

➔ **Using the standard result replace transform**

$$\ldots\left[L(\sin at) = \frac{a}{s^2+a^2};\ \text{Here } a = 1\right]$$

$$\therefore L[\sin t] = \frac{1}{s^2+(1)^2}$$

$$L[\sin t] = \frac{1}{s^2+1} \quad \ldots(1)$$

➔ **Using property of Laplace transform**

$$\ldots\left\{L\left(\frac{f(t)}{t}\right) = \int_s^\infty f(s)\,ds \text{ where } f(s) = L[f(t)]\right\}$$

$$\therefore L\left(\frac{\sin 1t}{t}\right) = \int_s^\infty L(\sin 1t)\,ds$$

$$= \int_s^\infty \frac{1}{s^2+1}\,ds \quad \text{...[by Equations (1)]}$$

$$= \int_s^\infty \frac{1}{s^2+(1)^2}\,ds$$

➔ **Using standard result of integration**

$$\ldots\left[\int\frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \text{ with } x = s,\ a = 1\right]$$

$$\therefore L\left(\frac{\sin 1t}{t}\right) = \left[\frac{1}{1}\tan^{-1}\left(\frac{s}{1}\right)\right]_s^\infty$$

$$= [\tan^{-1}(s)]_s^\infty$$

Substitute s as : (upper limit) – (lower limit)

$$L\left(\frac{\sin 1t}{t}\right) = \tan^{-1}(\infty) - \tan^{-1}(s)$$

➔ **Using standard trigonometric value**

$$\ldots\left[\tan\left(\frac{\pi}{2}\right) = \infty \Rightarrow \tan^{-1}(\infty) = \frac{\pi}{2}\right]$$

$$= \frac{\pi}{2} - \tan^{-1}(s)$$

since $\frac{\pi}{2} - \tan^{-1}x = \cot^{-1}x$

$$\therefore L\left(\frac{\sin t}{t}\right) = \cot^{-1}(s)$$

➔ **Using property of Laplace transform**

$$\ldots\left[L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}f(s) \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore L\left[\int_0^t \frac{\sin t}{t}dt\right] = \frac{1}{s}L\left(\frac{\sin t}{t}\right)$$

$$= \frac{1}{s}\cot^{-1}(s) \qquad \text{...[From Equation (1)]}$$

$$\therefore L\left[\int_0^t \frac{\sin t}{t}\,dt\right] = \frac{1}{s}\cot^{-1}(s) \checkmark \qquad \text{...Ans.}$$

Ex. 1.12.14 : Find Laplace transform of $\frac{d}{dt}\left(\frac{\sin t}{t}\right)$

Soln. :

➔ **Using the standard result replace transform**

$$\text{...}\left[L(\sin at) = \frac{a}{s^2+a^2};\ \text{Here } a = 1\right]$$

$$\therefore L[\sin t] = \frac{1}{s^2+(1)^2}$$

$$L[\sin t] = \frac{1}{s^2+1} \qquad \text{...(1)}$$

➔ **Using property of Laplace transform**

$$\text{...}\left\{L\left(\frac{f(t)}{t}\right) = \int_s^\infty f(s)\,ds \ \text{ where } f(s) = L[f(t)]\right\}$$

$$\therefore L\left(\frac{\sin 1t}{t}\right) = \int_s^\infty L(\sin 1t)\,ds$$

$$= \int_s^\infty \frac{1}{s^2+1}\,ds \qquad \text{...[by Equations (1)]}$$

$$= \int_s^\infty \frac{1}{s^2+(1)^2}\,ds$$

➔ **Using standard result of integration**

$$\text{...}\left[\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \text{ with } x = s,\ a = 1\right]$$

$$\therefore L\left(\frac{\sin 1t}{t}\right) = \left[\frac{1}{1}\tan^{-1}\left(\frac{s}{1}\right)\right]_s^\infty$$

$$= [\tan^{-1}(s)]_s^\infty$$

Substitute s as : (upper limit) – (lower limit)

$$L\left(\frac{\sin 1t}{t}\right) = \tan^{-1}(\infty) - \tan^{-1}(s)$$

➔ **Using standard trigonometric value**

$$\text{...}\left[\tan\left(\frac{\pi}{2}\right) = \infty \Rightarrow \tan^{-1}(\infty) = \frac{\pi}{2}\right]$$

$$= \frac{\pi}{2} - \tan^{-1}(s)$$

since $\frac{\pi}{2} - \tan^{-1}x = \cot^{-1}x$

$$\therefore L\left(\frac{\sin t}{t}\right) = \cot^{-1}(s) \qquad \text{...(2)}$$

➔ **Using Laplace transform of derivative**

$$\text{...}\left\{L[f'(t)] = sf(s) - f(o) \ \text{ where } f(s) = L[f(t)]\right\}$$

$$\therefore L\left[\frac{d}{dt}\left(\frac{\sin t}{t}\right)\right] = s\,L\left[\frac{\sin t}{t}\right] - \left[\frac{\sin t}{t}\right]_{t\to 0}$$

$$\text{...}\left[\text{From Equation (2) and } \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$$L\left[\frac{d}{dt}\left(\frac{\sin t}{t}\right)\right] = s\cot^{-1}(s) - 1 \checkmark \qquad \text{...Ans.}$$

Ex. 1.12.15 : Find Laplace transform of $\int_0^t \frac{e^{-t}\sin t}{t}\,dt$

Soln. :

➔ **Using the standard result replace transform**

$$\text{...}\left[L(\sin at) = \frac{a}{s^2+a^2};\ \text{Here } a = 1\right]$$

$$\therefore L[\sin t] = \frac{1}{s^2+(1)^2}$$

$$L[\sin t] = \frac{1}{s^2+1} \qquad \text{...(1)}$$

➔ **Using property of Laplace transform**

$$\text{...}\left\{L\left(\frac{f(t)}{t}\right) = \int_s^\infty f(s)\,ds \ \text{ where } f(s) = L[f(t)]\right\}$$

$$\therefore L\left(\frac{\sin 1t}{t}\right) = \int_s^\infty L(\sin 1t)\,ds$$

$$= \int_s^\infty \frac{1}{s^2+1}\,ds \qquad \text{...[by Equations (1)]}$$

$$= \int_s^\infty \frac{1}{s^2+(1)^2}\,ds$$

➔ **Using standard result of integration**

$$\text{...}\left[\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \text{ with } x = s,\ a = 1\right]$$

$$\therefore L\left(\frac{\sin 1t}{t}\right) = \left[\frac{1}{1}\tan^{-1}\left(\frac{s}{1}\right)\right]_s^\infty$$

$$= [\tan^{-1}(s)]_s^\infty$$

Substitute s as : (upper limit) – (lower limit)

$$L\left(\frac{\sin 1t}{t}\right) = \tan^{-1}(\infty) - \tan^{-1}(s)$$

➔ **Using standard trigonometric value**

$$\text{...}\left[\tan\left(\frac{\pi}{2}\right) = \infty \Rightarrow \tan^{-1}(\infty) = \frac{\pi}{2}\right]$$

$$= \frac{\pi}{2} - \tan^{-1}(s)$$

since $\frac{\pi}{2} - \tan^{-1}x = \cot^{-1}x$

$$\therefore L\left(\frac{\sin t}{t}\right) = \cot^{-1}(s)$$

Step III :

➔ **Using first shifting property of Laplace transform**

$$\text{...}\left[L(e^{at}f(t)) = [L[f(t)]]_{s\to s-a} = [f(s)]_{s\to s-a}\right]$$

$(s \to s - a$ means every s replace by $s - a)$

$$\therefore L\left[\frac{e^{-t}\sin t}{t}\right] = \left[\cot^{-1}(s)\right]_{s\to s+1}$$

$$\therefore L\left[e^{-t}\cdot\frac{\sin t}{t}\right] = \cot^{-1}(s+1)$$

Step IV :

➔ **Using Laplace transform of integration**

$$\text{...}\left\{L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}f(s) \text{ where } f(s) = L[f(t)]\right\}$$

$$\therefore L\left[\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right] = \frac{1}{s}\cot^{-1}(s+1) \checkmark \qquad \text{...Ans.}$$

Ex. 1.12.16 : Find the Laplace transform of $e^{-3t}\int_0^t \frac{\sin t}{t}\,dt$

Soln. :

➔ **Using the standard result replace transform**

$$\text{...}\left[L(\sin at) = \frac{a}{s^2+a^2};\ \text{Here } a = 1\right]$$

$$\therefore L[\sin t] = \frac{1}{s^2+(1)^2}$$

$$L[\sin t] = \frac{1}{s^2+1} \qquad \text{...(1)}$$

➔ **Using property of Laplace transform**

$$\text{...}\left\{L\left(\frac{f(t)}{t}\right) = \int_s^\infty f(s)\,ds \ \text{ where } f(s) = L[f(t)]\right\}$$

$$\therefore L\left(\frac{\sin 1t}{t}\right) = \int_s^\infty L(\sin 1t)\,ds$$

$$= \int_s^\infty \frac{1}{s^2+1}\,ds \qquad \text{...[by Equations (1)]}$$

$$= \int_s^\infty \frac{1}{s^2+(1)^2}\,ds$$

➔ **Using standard result of integration**

$$\text{...}\left[\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \text{ with } x = s,\ a = 1\right]$$

$$\therefore L\left(\frac{\sin 1t}{t}\right) = \left[\frac{1}{1}\tan^{-1}\left(\frac{s}{1}\right)\right]_s^\infty$$

$$= [\tan^{-1}(s)]_s^\infty$$

Substitute s as : (upper limit) – (lower limit)

$$L\left(\frac{\sin 1t}{t}\right) = \tan^{-1}(\infty) - \tan^{-1}(s)$$

➔ **Using standard trigonometric value**

$$\text{...}\left[\tan\left(\frac{\pi}{2}\right) = \infty \Rightarrow \tan^{-1}(\infty) = \frac{\pi}{2}\right]$$

$$= \frac{\pi}{2} - \tan^{-1}(s)$$

since $\frac{\pi}{2} - \tan^{-1}x = \cot^{-1}x$

$$\therefore L\left(\frac{\sin t}{t}\right) = \cot^{-1}(s)$$

➔ **Using property of Laplace transform**

$$\text{...}\left[L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}f(s) \ \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore L\left[\int_0^t \frac{\sin t}{t}\,dt\right] = \frac{1}{s}L\left(\frac{\sin t}{t}\right)$$

$$= \frac{1}{s}\cot^{-1}(s) \qquad \text{...[From Equation (1)]}$$

$$\therefore L\left[\int_0^t \frac{\sin t}{t}\,dt\right] = \frac{1}{s}\cot^{-1}(s)$$

➔ **Using first shifting property of Laplace transform**

$$\text{...}\left[L(e^{at}f(t)) = [L[f(t)]]_{s\to s-a} = [f(s)]_{s\to s-a}\right]$$

$(s \to s - a$ means every s replace by $s - a)$

$$\therefore L\left[e^{-3t}\int_0^t \frac{\sin t}{t}\,dt\right] = \left[L\left[\int_0^t \frac{\sin t}{t}\,dt\right]\right]_{s\to s-(-3)}$$

$$= \left[\frac{1}{s}\cot^{-1}(s)\right]_{s\to s+3}$$

$$\therefore L\left[e^{-3t}\int_0^t \frac{\sin t}{t}\,dt\right] = \frac{1}{s+3}\cot^{-1}(s+3) \checkmark \qquad \text{...Ans.}$$

Ex. 1.12.17 : Find Laplace transform of $\frac{1-\cos t}{t}$

Soln. :

➔ **Using linearity property of Laplace Transform**

$$\ldots\left[L[a f(t) + bg(t) - c h(t)] = a L[f(t)] + b L[g(t)] - cL[h(t)]\right]$$

$$L[1 - \cos t] = L(1) - L[\cos t]$$

➔ **Using Laplace transform of standard functions**

$$\ldots\left[L(1) = \frac{1}{s} \text{ and } L(\cos at) = \frac{s}{s^2 + a^2}\right]$$

$$= \frac{1}{s} - \frac{s}{s^2 + (1)^2}$$

$$L[1 - \cos t] = \frac{1}{s} - \frac{s}{s^2 + 1}$$

➔ **Using property of Laplace transform**

$$\ldots\left[L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\, ds\right]$$

$$\therefore L\left[\frac{1 - \cos t}{t}\right] = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2 + 1}\right) ds$$

➔ **Using linearity property of integration**

$$\ldots\left\{\int [af(x) \pm bg(x)]\, dx = a \int f(x)\, dx \pm b \int g(x)\, dx\right\}$$

$$= \int_s^\infty \frac{1}{s}\, ds - \int_s^\infty \frac{s}{s^2 + 1}\, ds$$

$$= \int_s^\infty \frac{1}{s}\, ds - \frac{1}{2} \int_s^\infty \frac{2s}{s^2 + 1}\, ds \quad \left(\text{Note adjustment}\right)$$

➔ **Using integration of standard functions**

$$\ldots\left[\int \frac{1}{ax + b} dx = \frac{\log(ax + b)}{a} \text{ and } \int \frac{f'(x)}{f(x)} dx = \log f(x)\right]$$

$$\therefore L\left[\frac{1 - \cos t}{t}\right] = [\log s]_s^\infty - \left[\frac{1}{2} \log(s^2 + 1)\right]_s^\infty$$

$$= [\log s]_s^\infty - \left[\log (s^2 + 1)^{\frac{1}{2}}\right]_s^\infty$$

$$= [\log s]_s^\infty - \left[\log \sqrt{s^2 + 1}\right]_s^\infty$$

$$= \left[\log s - \log \sqrt{s^2 + 1}\right]_s^\infty$$

➔ **Using property of logarithm :** $\ldots\left[\log a - \log b = \log\left(\frac{a}{b}\right)\right]$

$$= \left[\log\left(\frac{s}{\sqrt{s^2 + 1}}\right)\right]_s^\infty$$

Substitute s as : (upper limit) – (lower limit)

$$= 0 - \log\left(\frac{s}{\sqrt{s^2 + 1}}\right) = -\log\left(\frac{s}{\sqrt{s^2 + 1}}\right)$$

➔ **Using property of logarithm :** $\ldots[m \log a = \log a^m]$

$$= \log\left(\frac{s}{\sqrt{s^2 + 1}}\right)^{-1} \quad \ldots\text{Since } \left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

$$L\left[\frac{1 - \cos t}{t}\right] = \log\left(\frac{\sqrt{s^2 + 1}}{s}\right) \checkmark \quad \ldots\textbf{Ans.}$$

Ex. 1.12.18 : Find Laplace transform of $\frac{e^{-2t} \sin 4t}{t}$

Soln. :

➔ **Using the standard result replace transform**

$$\ldots\left[L(\sin at) = \frac{a}{s^2 + a^2}; \text{Here } a = 4\right]$$

$$\therefore L[\sin 4t] = \frac{4}{s^2 + (4)^2}$$

$$L[\sin 4t] = \frac{4}{s^2 + 16} \quad \ldots(1)$$

➔ **Using property of Laplace transform**

$$\ldots\left\{L\left(\frac{f(t)}{t}\right) = \int_s^\infty f(s)\, ds \text{ where } f(s) = L[f(t)]\right\}$$

$$\therefore L\left(\frac{\sin 4t}{t}\right) = \int_s^\infty L(\sin 4t)\, ds$$

$$= \int_s^\infty \frac{4}{s^2 + 16}\, ds \quad \ldots[\text{by Equations (1)}]$$

$$= 4 \int_s^\infty \frac{1}{s^2 + (4)^2}\, ds$$

➔ **Using standard result of integration**

$$\ldots\left[\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) \text{ with } x = s, a = 4\right]$$

$$\therefore L\left(\frac{\sin 4t}{t}\right) = 4\left[\frac{1}{4} \tan^{-1}\left(\frac{s}{4}\right)\right]_s^\infty$$

$$= 4 \times \frac{1}{4}\left[\tan^{-1}\left(\frac{s}{4}\right)\right]_s^\infty$$

Substitute s as : (upper limit) – (lower limit)

$$L\left(\frac{\sin 4t}{t}\right) = \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{4}\right)$$

➔ **Using standard trigonometric value**

$$\ldots\left[\tan\left(\frac{\pi}{2}\right) = \infty \Rightarrow \tan^{-1}(\infty) = \frac{\pi}{2}\right]$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{4}\right)$$

since $\frac{\pi}{2} - \tan^{-1} x = \cot^{-1} x$

$$\therefore L\left(\frac{\sin 4t}{t}\right) = \cot^{-1}\left(\frac{s}{4}\right)$$

➔ **Using first shifting property of Laplace transform**

$$\ldots\left[L(e^{at} f(t)) = [L[f(t)]]_{s \to s-a} = [f(s)]_{s \to s-a}\right]$$

$(s \to s - a$ means every s replace by $s - a)$

$$\therefore L\left[e^{-2t} \frac{\sin 4t}{t}\right] = \left[L\left(\frac{\sin 4t}{t}\right)\right]_{s \to s-(-2)}$$

$$= \left[\cot^{-1}\left(\frac{s}{4}\right)\right]_{s \to s+2}$$

$$\therefore L\left[\frac{e^{-2t} \sin 4t}{t}\right] = \cot^{-1}\left(\frac{s+2}{4}\right) \checkmark \quad \ldots\textbf{Ans.}$$

Ex. 1.12.19 : Find Laplace transform of $\frac{e^{-3t} \sin 2t}{t}$

Soln. :

➔ **Using the standard result replace transform**

$$\ldots\left[L(\sin at) = \frac{a}{s^2 + a^2}; \text{Here } a = 2\right]$$

$$\therefore L[\sin 2t] = \frac{2}{s^2 + (2)^2}$$

$$L[\sin 2t] = \frac{2}{s^2 + 4} \quad \ldots(1)$$

➔ **Using property of Laplace transform**

$$\ldots\left\{L\left(\frac{f(t)}{t}\right) = \int_s^\infty f(s)\, ds \text{ where } f(s) = L[f(t)]\right\}$$

$$\therefore L\left(\frac{\sin 2t}{t}\right) = \int_s^\infty L(\sin 2t)\, ds$$

$$= \int_s^\infty \frac{2}{s^2 + 4}\, ds \quad \ldots[\text{by Equations (1)}]$$

$$= 2 \int_s^\infty \frac{1}{s^2 + (2)^2}\, ds$$

➔ **Using standard result of integration**

$$\ldots\left[\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) \text{ with } x = s, a = 2\right]$$

$$\therefore L\left(\frac{\sin 2t}{t}\right) = 2\left[\frac{1}{2} \tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$$

$$= 2 \times \frac{1}{2}\left[\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$$

Substitute s as : (upper limit) – (lower limit)

$$L\left(\frac{\sin 2t}{t}\right) = \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{2}\right)$$

➔ **Using standard trigonometric value**

$$\ldots\left[\tan\left(\frac{\pi}{2}\right) = \infty \Rightarrow \tan^{-1}(\infty) = \frac{\pi}{2}\right]$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$$

since $\frac{\pi}{2} - \tan^{-1} x = \cot^{-1} x$

$$\therefore L\left(\frac{\sin 2t}{t}\right) = \cot^{-1}\left(\frac{s}{2}\right) \quad \ldots(2)$$

➔ **Using first shifting property of Laplace transform**

$$\ldots\left[L(e^{at} f(t)) = [L[f(t)]]_{s \to s-a} = [f(s)]_{s \to s-a}\right]$$

$(s \to s - a$ means every s replace by $s - a)$

$$\therefore L\left[e^{-3t} \frac{\sin 2t}{t}\right] = \left[L\left(\frac{\sin 2t}{t}\right)\right]_{s \to s-(-3)}$$

From Equation (3)

$$= \left[\cot^{-1}\left(\frac{s}{2}\right)\right]_{s \to s+3}$$

$$\therefore L\left[\frac{e^{-3t} \sin 2t}{t}\right] = \cot^{-1}\left(\frac{s+3}{2}\right) \checkmark \quad \ldots\textbf{Ans.}$$

Ex. 1.12.20 : Find Laplace transform of $\frac{e^t - \cos 2t}{t}$

Soln. :

➔ **Using linearity property of Laplace Transform**

$$\ldots\left[L[a f(t) + bg(t) - c h(t)] = a L[f(t)] + b L[g(t)] - cL[h(t)]\right]$$

$$\therefore L[e^t - \cos 2t] = L(e^t) - L(\cos 2t)$$

➔ **Using Laplace transform of standard functions**

$$\ldots\left[L(e^{at}) = \frac{1}{s - a} \text{ and } L(\cos at) = \frac{s}{s^2 + a^2}\right]$$

$$\therefore L[e^t - \cos 2t] = \frac{1}{s - 1} - \frac{s}{s^2 + (2)^2}$$

$$L[e^t - \cos 2t] = \frac{1}{s - 1} - \frac{s}{s^2 + 4}$$

➔ **Using property of Laplace transform**

$$\ldots\left[L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\, ds\right]$$

$$\therefore L\left[\frac{e^t - \cos 2t}{t}\right] = \int_s^\infty \left(\frac{1}{s - 1} - \frac{s}{s^2 + 4}\right) ds$$

$$= \int_s^\infty \frac{1}{s - 1}\, ds - \int_s^\infty \frac{s}{s^2 + 4}\, ds$$

$$= \int_s^\infty \frac{1}{s - 1}\, ds - \frac{1}{2} \int_s^\infty \frac{2s}{s^2 + 4}\, ds$$

➔ **Using integration of standard functions**

$$\ldots\left[\int \frac{1}{ax+b}\,dx = \frac{\log(ax+b)}{a} \text{ and } \int \frac{f'(x)}{f(x)}\,dx = \log f(x)\right]$$

$$\therefore L\left[\frac{e^t - \cos 2t}{t}\right] = \left[\log(s-1)\right]_s^\infty - \left[\frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \left[\log(s-1) - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

➔ **Using property of logarithm :** $\ldots[m \log a = \log a^m]$

$$= \left[\log(s-1) - \log(s^2+4)^{\frac{1}{2}}\right]_s^\infty$$

$$= \left[\log(s-1) - \log\sqrt{s^2+4}\right]_s^\infty$$

➔ **Using property of logarithm :** $\ldots\left[\log a - \log b = \log\left(\frac{a}{b}\right)\right]$

$$\therefore L\left[\frac{e^t - \cos 2t}{t}\right] = \left[\log\left(\frac{s-1}{\sqrt{s^2+4}}\right)\right]_s^\infty$$

Substitute s as : (upper limit) – (lower limit)

$$\therefore L\left[\frac{e^t - \cos 2t}{t}\right] = 0 - \log\left(\frac{s-1}{\sqrt{s^2+4}}\right)$$

➔ **Using property of logarithm :** $\ldots[m \log a = \log a^m]$

$$= \log\left(\frac{s-1}{\sqrt{s^2+4}}\right)^{-1} \qquad \left[\because \left(\frac{a}{b}\right)^{-1} = \frac{b}{a}\right]$$

$$\therefore L\left[\frac{e^t - \cos 2t}{t}\right] = \log\left(\frac{\sqrt{s^2+4}}{s-1}\right) \checkmark \qquad \ldots\text{Ans.}$$

Ex. 1.12.21 : Find the Laplace transform of $\frac{\cos 2t - \cos 3t}{t}$

Soln. :

➔ **Using linearity property of Laplace Transform**

$$\ldots\left[L[a\,f(t) + b g(t) - c\,h(t)] = a\,L[f(t)] + b\,L[g(t)] - cL[h(t)]\right]$$

$$L[\cos t\,2t - \cos 3t] = L[\cos 2t] - L[\cos 3t]$$

➔ **Using the Laplace transform of standard function**

$$\ldots\left[L(\cos at) = \frac{s}{s^2+a^2}\right]$$

$$\therefore L[\cos 2t - \cos 3t] = \frac{s}{s^2+(2)^2} - \frac{s}{s^2+(3)^2}$$

$$L[\cos t\,2t - \cos 3t] = \frac{s}{s^2+4} - \frac{s}{s^2+9}$$

➔ **Using property of Laplace transform**

$$\ldots\left[L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds\right]$$

$$\therefore L\left[\frac{\cos 2t - \cos 3t}{t}\right] = \int_s^\infty \left[\frac{s}{s^2+4} - \frac{s}{s^2+9}\right] ds$$

➔ **Using linearity property of integration**

$$\ldots\left\{\int [af(x) \pm bg(x)]\,dx = a\int f(x)\,dx \pm b\int g(x)\,dx\right\}$$

$$\therefore L\left[\frac{\cos 2t - \cos 3t}{t}\right] = \int_s^\infty \frac{s}{s^2+4}\,ds - \int_s^\infty \frac{s}{s^2+9}\,ds$$

$$= \frac{1}{2}\int_s^\infty \frac{2s}{s^2+4}\,ds - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+9}\,ds$$

..Note the Adjustment

➔ **Using integration of standard function**

$$\ldots\left[\int \frac{f'(x)}{f(x)}\,dx = \log f(x)\right]$$

$$\therefore L\left[\frac{\cos 2t - \cos 3t}{t}\right] = \left[\frac{1}{2}\log(s^2+4)\right]_s^\infty - \left[\frac{1}{2}\log(s^2+9)\right]_s^\infty$$

$$= \left[\frac{1}{2}\log(s^2+4) - \frac{1}{2}\log(s^2+9)\right]_s^\infty$$

➔ **Using property of logarithm :** $\ldots[m \log a = \log a^m]$

$$= \left[\log(s^2+4)^{\frac{1}{2}} - \log(s^2+9)^{\frac{1}{2}}\right]_s^\infty$$

$$= \left[\log\sqrt{(s^2+4)} - \log\sqrt{(s^2+9)}\right]_s^\infty$$

➔ **Using property of logarithm :** $\ldots\left[\log a - \log b = \log\left(\frac{a}{b}\right)\right]$

$$\therefore L\left[\frac{\cos 2t - \cos 3t}{t}\right] = \left[\log\left(\frac{\sqrt{s^2+4}}{\sqrt{s^2+9}}\right)\right]_s^\infty$$

Substitute s as : (upper limit) – (lower limit)

$$= 0 - \log\left(\frac{\sqrt{s^2+4}}{\sqrt{s^2+9}}\right)$$

➔ **Using property of logarithm :** $\ldots[m \log a = \log a^m]$

$$= \log\left(\frac{\sqrt{s^2+4}}{\sqrt{s^2+9}}\right)^{-1}$$

Since $= \left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$

$$\therefore L\left[\frac{\cos 2t - \cos 3t}{t}\right] = \log\frac{\sqrt{s^2+9}}{\sqrt{s^2+4}} \qquad \because \frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$$

$$\therefore L\left[\frac{\cos 2t - \cos 3t}{t}\right] = \log\sqrt{\frac{s^2+9}{s^2+4}} \checkmark \qquad \ldots\text{Ans.}$$

Ex. 1.12.22 : Find the Laplace transform of $\left(\frac{\cos at - \cos bt}{t}\right)$

Dr. BATU - Nov. 2016, 4 Marks

Soln. :

➔ **Using linearity property of Laplace Transform**

$$\ldots\left[L[a\,f(t) + bg(t) - c\,h(t)] = a\,L[f(t)] + b\,L[g(t)] - cL[h(t)]\right]$$

$$L[\cos at - \cos bt] = L[\cos at] - L[\cos bt]$$

➔ **Using the Laplace transform of standard function**

$$\ldots\left[L(\cos at) = \frac{s}{s^2+a^2}\right]$$

$$\therefore L[\cos at - \cos bt] = \frac{s}{s^2+(a)^2} - \frac{s}{s^2+(b)^2}$$

$$L[\cos t\,at - \cos bt] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

➔ **Using property of Laplace transform**

$$\ldots\left[L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds\right]$$

$$\therefore L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right] ds$$

➔ **Using linearity property of integration**

$$\ldots\left\{\int [af(x) \pm bg(x)]\,dx = a\int f(x)\,dx \pm b\int g(x)\,dx\right\}$$

$$\therefore L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty \frac{s}{s^2+a^2}\,ds - \int_s^\infty \frac{s}{s^2+b^2}\,ds$$

$$= \frac{1}{2}\int_s^\infty \frac{2s}{s^2+a^2}\,ds - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+b^2}\,ds$$

➔ **Using integration of standard function**

$$\ldots\left[\int \frac{f'(x)}{f(x)}\,dx = \log f(x)\right]$$

$$\therefore L\left[\frac{\cos at - \cos bt}{t}\right] = \left[\frac{1}{2}\log(s^2+a^2)\right]_s^\infty - \left[\frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

➔ **Using property of logarithm :** $\ldots[m \log a = \log a^m]$

$$= \left[\log(s^2+a^2)^{\frac{1}{2}} - \log(s^2+b^2)^{\frac{1}{2}}\right]_s^\infty$$

$$= \left[\log\sqrt{(s^2+a^2)} - \log\sqrt{(s^2+b^2)}\right]_s^\infty$$

➔ **Using property of logarithm :** $\ldots\left[\log a - \log b = \log\left(\frac{a}{b}\right)\right]$

$$\therefore L\left[\frac{\cos at - \cos bt}{t}\right] = \left[\log\left(\frac{\sqrt{s^2+a^2}}{\sqrt{s^2+b^2}}\right)\right]_s^\infty$$

➔ **Substitute s as : (upper limit) – (lower limit)**

$$= 0 - \log\left(\frac{\sqrt{s^2+a^2}}{\sqrt{s^2+b^2}}\right)$$

➔ **Using property of logarithm :** $\ldots[m \log a = \log a^m]$

$$= \log\left(\frac{\sqrt{s^2+a^2}}{\sqrt{s^2+b^2}}\right)^{-1}$$

since $= \left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$

$$\therefore L\left[\frac{\cos at - \cos bt}{t}\right] = \log\frac{\sqrt{s^2+b^2}}{\sqrt{s^2+a^2}} \qquad \because \frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$$

$$\therefore L\left[\frac{\cos at - \cos bt}{t}\right] = \log\sqrt{\frac{s^2+b^2}{s^2+a^2}} \checkmark \qquad \ldots\text{Ans.}$$

Ex. 1.12.23 : Find the Laplace transform of $\frac{e^{at} - e^{bt}}{t}$

Soln. :

➔ **Using linearity property of Laplace Transform**

$$\ldots\left[L[a\,f(t) + bg(t) - c\,h(t)] = a\,L[f(t)] + b\,L[g(t)] - cL[h(t)]\right]$$

$$\therefore L[e^{at} - e^{bt}] = L[e^{at}] - L[e^{bt}]$$

➔ **Using Laplace transform of standard functions**

$$\ldots\left[L[e^{at}] = \frac{1}{s-a}\right]$$

$$\therefore L[e^{at} - e^{bt}] = \frac{1}{s-a} - \frac{1}{s-b}$$

➔ **Using property of Laplace transform**

$$\ldots\left[L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds\right]$$

$$L\left[\frac{e^{at} - e^{bt}}{t}\right] = \int_s^\infty \left(\frac{1}{s-a} - \frac{1}{s-b}\right) ds$$

➔ **Using integration of standard function**

$$\ldots\left[\int \frac{1}{ax+b}\,dx = \frac{\log(ax+b)}{a}\right]$$

$$L\left[\frac{e^{at} - e^{bt}}{t}\right] = \left[\log(s-a) - \log(s-b)\right]_s^\infty$$

➔ **Using property of logarithm :** $\ldots\left[\log a - \log b = \log\left(\frac{a}{b}\right)\right]$

$$= \left[\log\left(\frac{s-a}{s-b}\right)\right]_s^\infty$$

Substitute s as : (upper limit) – (lower limit)

$$L\left[\frac{e^{at}-e^{bt}}{t}\right] = 0 - \log\left(\frac{s-a}{s-b}\right)$$

$$= -\log\left(\frac{s-a}{s-b}\right)$$

➔ **Using property of logarithm :** ...[m log a = log a^m]

$$= \log\left(\frac{s-a}{s-b}\right)^{-1} \quad \text{...since } \left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

$$L\left[\frac{e^{at}-e^{bt}}{t}\right] = \log\left(\frac{s-b}{s-a}\right) \checkmark \quad \text{...Ans.}$$

Ex. 1.12.24 : Find Laplace transform $\frac{e^{-at}-e^{-bt}}{t}$

Soln. :

➔ **Using linearity property of Laplace Transform**

$$\text{...}\left[L[a\,f(t) + bg(t) - c\,h(t)] = a\,L[f(t)] + b\,L[g(t)] - cL[h(t)]\right]$$

$$\therefore L[e^{-at} - e^{-bt}] = L[e^{-at}] - L[e^{-bt}]$$

➔ **Using Laplace transform of standard functions**

$$\text{...}\left[L[e^{-at}] = \frac{1}{s+a}\right]$$

$$\therefore L[e^{-at} - e^{-bt}] = \frac{1}{s+a} - \frac{1}{s+b}$$

➔ **Using property of Laplace transform**

$$\text{...}\left[L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds\right]$$

$$L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

➔ **Using integration of standard function**

$$\text{...}\left[\int \frac{1}{ax+b}\,dx = \frac{\log(ax+b)}{a}\right]$$

$$L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = [\log(s+a) - \log(s+b)]_s^\infty$$

➔ **Using property of logarithm :** $\text{...}\left[\log a - \log b = \log\left(\frac{a}{b}\right)\right]$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

Substitute s as : (upper limit) – (lower limit)

$$L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = 0 - \log\left(\frac{s+a}{s+b}\right)$$

$$= -\log\left(\frac{s+a}{s+b}\right)$$

➔ **Using property of logarithm :** ...[m log a = log a^m]

$$= \log\left(\frac{s+a}{s+b}\right)^{-1} \quad \text{...since } \left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

$$L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \log\left(\frac{s+b}{s+a}\right) \checkmark \quad \text{...Ans.}$$

Ex. 1.12.25 : Find Laplace transform of $e^{4t}\int_0^t \frac{1-\cos 2t}{t}\,dt$

Soln. :

➔ **Using linearity property of Laplace Transform**

$$\text{...}\left[L[a\,f(t) + bg(t) - c\,h(t)] = a\,L[f(t)] + b\,L[g(t)] - cL[h(t)]\right]$$

$$L[1 - \cos 2t] = L(1) - L[\cos 2t]$$

➔ **Using Laplace transform of standard functions**

$$\text{...}\left[L(1) = \frac{1}{s} \text{ and } L(\cos at) = \frac{s}{s^2+a^2}\right]$$

$$= \frac{1}{s} - \frac{s}{s^2+(2)^2}$$

$$L[1 - \text{cost } 2t] = \frac{1}{s} - \frac{s}{s^2+4}$$

➔ **Using property of Laplace transform**

$$\text{...}\left[L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds\right]$$

$$\therefore L\left[\frac{1-\cos 2t}{t}\right] = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$$

➔ **Using linearity property of integration**

$$\text{...}\left\{\int [af(x) \pm bg(x)]\,dx = a\int f(x)\,dx \pm b\int g(x)\,dx\right\}$$

$$= \int_s^\infty \frac{1}{s}\,ds - \int_s^\infty \frac{s}{s^2+4}\,ds$$

$$= \int_s^\infty \frac{1}{s}\,ds - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+4}\,ds$$

➔ **Using integration of standard functions**

$$\text{...}\left[\int \frac{1}{ax+b}\,dx = \frac{\log(ax+b)}{a} \text{ and } \int \frac{f'(x)}{f(x)}\,dx = \log f(x)\right]$$

$$\therefore L\left[\frac{1-\cos 2t}{t}\right] = [\log s]_s^\infty - \left[\frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= [\log s]_s^\infty - \left[\log(s^2+4)^{\frac{1}{2}}\right]_s^\infty$$

$$= [\log s]_s^\infty - \left[\log\sqrt{s^2+4}\right]_s^\infty$$

$$= \left[\log s - \log\sqrt{s^2+4}\right]_s^\infty$$

➔ **Using property of logarithm :** $\text{...}\left[\log a - \log b = \log\left(\frac{a}{b}\right)\right]$

$$= \left[\log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]_s^\infty$$

Substitute s as : (upper limit) – (lower limit)

$$= 0 - \log\left(\frac{s}{\sqrt{s^2+4}}\right) = -\log\left(\frac{s}{\sqrt{s^2+4}}\right)$$

➔ **Using property of logarithm :** ...[m log a = log a^m]

$$= \log\left(\frac{s}{\sqrt{s^2+4}}\right)^{-1} \quad \text{...Since } \left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

$$L\left[\frac{1-\cos 2t}{t}\right] = \log\left(\frac{\sqrt{s^2+4}}{s}\right) \quad \text{...(1)}$$

➔ **Using property of Laplace transform**

$$\text{...}\left[\int_0^t L(f(t)\,dt) = \frac{1}{s}f(s)\right]$$

$$\therefore L\left[\int_0^t \frac{1-\cos 2t}{t}\,dt\right] = \frac{1}{s}L\left[\frac{1-\cos 2t}{t}\right]$$

...[from Equation (1)]

$$= \frac{1}{s}\log\frac{\sqrt{s^2+4}}{s} \quad \text{...(2)}$$

➔ **Using first shifting property of Laplace transform**

$$\text{...}\left[L(e^{at}f(t)) = [L[f(t)]]_{s\to s-a} = [f(s)]_{s\to s-a}\right]$$

($s \to s - a$ means every s replace by $s - a$)

$$\therefore L\left[e^{4t}\int_0^t \frac{1-\cos 2t}{t}\,dt\right] = \left[L\left[\int_0^t \frac{1-\cos 2t}{t}\,dt\right]\right]_{s\to s-4}$$

[from Equation (2)]

$$= \left[\frac{1}{s}\log\frac{\sqrt{s^2+4}}{s}\right]_{s\to s-4}$$

$$\therefore L\left[e^{4t}\int_0^t \frac{1-\cos 2t}{t}\,dt\right] = \frac{1}{s-4}\log\left(\frac{\sqrt{(s-4)^2+4}}{s-4}\right) \checkmark \quad \text{...Ans.}$$

Ex. 1.12.26 : Find Laplace transform of $e^{-2t}\int_0^t \frac{1-\cos 2t}{t}\,dt$

Soln. :

➔ **Using linearity property of Laplace Transform**

$$\text{...}\left[L[a\,f(t) + bg(t) - c\,h(t)] = a\,L[f(t)] + b\,L[g(t)] - cL[h(t)]\right]$$

$$L[1 - \cos 2t] = L(1) - L[\cos 2t]$$

➔ **Using Laplace transform of standard functions**

$$\text{...}\left[L(1) = \frac{1}{s} \text{ and } L(\cos at) = \frac{s}{s^2+a^2}\right]$$

$$= \frac{1}{s} - \frac{s}{s^2+(2)^2}$$

$$L[1 - \text{cost } 2t] = \frac{1}{s} - \frac{s}{s^2+4}$$

➔ **Using property of Laplace transform**

$$\text{...}\left[L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds\right]$$

$$\therefore L\left[\frac{1-\cos 2t}{t}\right] = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$$

➔ **Using linearity property of integration**

$$\text{...}\left\{\int [af(x) \pm bg(x)]\,dx = a\int f(x)\,dx \pm b\int g(x)\,dx\right\}$$

$$= \int_s^\infty \frac{1}{s}\,ds - \int_s^\infty \frac{s}{s^2+4}\,ds$$

$$= \int_s^\infty \frac{1}{s}\,ds - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+4}\,ds$$

➔ **Using integration of standard functions**

$$\text{...}\left[\int \frac{1}{ax+b}\,dx = \frac{\log(ax+b)}{a} \text{ and } \int \frac{f'(x)}{f(x)}\,dx = \log f(x)\right]$$

$$\therefore L\left[\frac{1-\cos 2t}{t}\right] = [\log s]_s^\infty - \left[\frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= [\log s]_s^\infty - \left[\log(s^2+4)^{\frac{1}{2}}\right]_s^\infty$$

$$= [\log s]_s^\infty - \left[\log\sqrt{s^2+4}\right]_s^\infty$$

$$= \left[\log s - \log\sqrt{s^2+4}\right]_s^\infty$$

➔ **Using property of logarithm :** $\text{...}\left[\log a - \log b = \log\left(\frac{a}{b}\right)\right]$

$$= \left[\log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]_s^\infty$$

Substitute s as : (upper limit) – (lower limit)

$$= 0 - \log\left(\frac{s}{\sqrt{s^2+4}}\right) = -\log\left(\frac{s}{\sqrt{s^2+4}}\right)$$

➔ **Using property of logarithm :** ...[m log a = log a^m]

$$= \log\left(\frac{s}{\sqrt{s^2+4}}\right)^{-1} \quad \text{...Since } \left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

$$L\left[\frac{1-\cos 2t}{t}\right] = \log\left(\frac{\sqrt{s^2+4}}{s}\right) \quad ...(1)$$

➔ **Using property of Laplace transform**

$$...\left[\int_0^t L(f(t)\,dt) = \frac{1}{s}f(s)\right]$$

$$\therefore L\left[\int_0^t \frac{1-\cos 2t}{t}dt\right] = \frac{1}{s}L\left[\frac{1-\cos 2t}{t}\right]$$

...[from Equation (1)]

$$= \frac{1}{s}\log\frac{\sqrt{s^2+4}}{s} \quad ...(2)$$

➔ **Using first shifting property of Laplace transform**

$$...\left[L(e^{at}f(t)) = [L[f(t)]]_{s\to s-a} = [f(s)]_{s\to s-a}\right]$$

($s \to s-a$ means every s replace by $s-a$)

$$\therefore L\left[e^{-2t}\int_0^t \frac{1-\cos 2t}{t}dt\right] = \left[L\left[\int_0^t \frac{1-\cos 2t}{t}dt\right]\right]_{s\to s-(-2)}$$

[from Equation (2)]

$$= \left[\frac{1}{s}\log\frac{\sqrt{s^2+4}}{s}\right]_{s\to s+2}$$

$$\therefore L\left[e^{-2t}\int_0^t \frac{1-\cos 2t}{t}dt\right] = \frac{1}{s+2}\log\left(\frac{\sqrt{(s+2)^2+4}}{s+2}\right) \checkmark \text{...Ans.}$$

Ex. 1.12.27 : Find Laplace transform of $\int_0^t \frac{e^{-at}\sin bt}{t}dt$

Soln. :

➔ **Using the standard result replace transform**

$$...\left[L(\sin at) = \frac{a}{s^2+a^2}\right]$$

$$\therefore L[\sin bt] = \frac{b}{s^2+(b)^2}$$

$$\therefore L[\sin bt] = \frac{b}{s^2+b^2} \quad ...(1)$$

➔ **Using property of Laplace transform**

$$...\left\{L\left(\frac{f(t)}{t}\right) = \int_s^\infty f(s)\,ds \text{ where } f(s) = L[f(t)]\right\}$$

$$\therefore L\left(\frac{\sin bt}{t}\right) = \int_s^\infty L(\sin bt)\,ds$$

$$= \int_s^\infty \frac{b}{s^2+b^2}ds \quad ...\text{[by Equations (1)]}$$

$$= b\int_s^\infty \frac{1}{s^2+(b)^2}ds$$

➔ **Using standard result of integration**

$$...\left[\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \text{ with } x = s, a = b\right]$$

$$\therefore L\left(\frac{\sin bt}{t}\right) = b\left[\frac{1}{b}\tan^{-1}\left(\frac{s}{b}\right)\right]_s^\infty$$

$$= b\times\frac{1}{b}\left[\tan^{-1}\left(\frac{s}{b}\right)\right]_s^\infty$$

Substitute s as : (upper limit) – (lower limit)

$$L\left(\frac{\sin bt}{t}\right) = \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{b}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{b}\right)$$

$$\left[\because \tan\left(\frac{\pi}{2}\right) = \infty \Rightarrow \tan^{-1}(\infty) = \frac{\pi}{2}\right]$$

since $\frac{\pi}{2} - \tan^{-1}x = \cot^{-1}x$

$$\therefore L\left(\frac{\sin bt}{t}\right) = \cot^{-1}\left(\frac{s}{b}\right)$$

➔ **Using first shifting property of Laplace transform**

$$...\left[L(e^{at}f(t)) = [L[f(t)]]_{s\to s-a} = [f(s)]_{s\to s-a}\right]$$

($s \to s-a$ means every s replace by $s-a$)

$$\therefore L\left[e^{-at}\frac{\sin bt}{t}\right] = \left[L\left(\frac{\sin bt}{t}\right)\right]_{s\to s-(-a)}$$

$$= \left[\cot^{-1}\left(\frac{s}{b}\right)\right]_{s\to s+a}$$

$$\therefore L\left[\frac{e^{-at}\sin bt}{t}\right] = \cot^{-1}\left(\frac{s+a}{b}\right)$$

➔ **Using Laplace transform of integration**

$$...\left\{L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}f(s) \text{ where } f(s) = L[f(t)]\right\}$$

$$\therefore L\left[\int_0^t \frac{e^{-at}\sin bt}{t}dt\right] = \frac{1}{s}L\left[\frac{e^{-at}\sin bt}{t}\right]$$

...[from Equation (1)]

$$\therefore L\left[\int_0^t \frac{e^{-at}\sin bt}{t}dt\right] = \frac{1}{s}\cot^{-1}\left(\frac{s+a}{b}\right) \checkmark \quad \text{...Ans.}$$

Ex. 1.12.28 : Find Laplace transform of $\int_0^t \frac{1-e^{-x}}{x}dx$

Soln. :

➔ **Using linearity property of Laplace Transform**

$$...\left[L[a f(t) + bg(t) - c h(t)] = a L[f(t)] + b L[g(t)] - cL[h(t)]\right]$$

$$\therefore L[1-e^{-t}] = L[1] - L[e^{-t}]$$

➔ **Using Laplace transform of standard functions**

$$...\left[L[e^{at}] = \frac{1}{s-a}\right]$$

$$\therefore L[1-e^{-t}] = \frac{1}{s} - \frac{1}{s+1}$$

➔ **Using property of Laplace transform**

$$...\left[L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds\right]$$

$$L\left[\frac{1-e^{-t}}{t}\right] = \int_s^\infty\left(\frac{1}{s} - \frac{1}{s+1}\right)ds$$

➔ **Using integration of standard function**

$$...\left[\int \frac{1}{ax+b}dx = \frac{\log(ax+b)}{a}\right]$$

$$L\left[\frac{1-e^{-t}}{t}\right] = [\log(s) - \log(s+1)]_s^\infty$$

➔ **Using property of logarithm :** $...\left[\log a - \log b = \log\left(\frac{a}{b}\right)\right]$

$$= \left[\log\left(\frac{s}{s+1}\right)\right]_s^\infty$$

➔ **Substitute s as : (upper limit) – (lower limit)**

$$L\left[\frac{1-e^{-t}}{t}\right] = 0 - \log\left(\frac{s}{s+1}\right)$$

$$= -\log\left(\frac{s}{s+1}\right)$$

➔ **Using property of logarithm :** ...[$m \log a = \log a^m$]

$$= \log\left(\frac{s}{s+1}\right)^{-1} \quad ...\text{since } \left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

$$L\left[\frac{1-e^{-t}}{t}\right] = \log\left(\frac{s+1}{s}\right)$$

➔ **Using property of Laplace transform**

$$...\left[L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds\right]$$

$$\therefore L\left[\int_0^t \frac{1-e^{-t}}{t}dt\right] = \frac{1}{s}L\left[\frac{1-e^{-t}}{t}\right]$$

From Equation (1), we get,

$$\therefore L\left[\int_0^t \frac{1-e^{-t}}{t}dt\right] = \frac{1}{s}\log\left(\frac{s+1}{s}\right) \checkmark \quad \text{...Ans.}$$

Ex. 1.12.29 : Find Laplace transform of $\cosh t \int_0^t t\cosh at\,dt$.

Soln. :

Step I :

➔ **Using standard result of Laplace transform**

$$...\left[L[\cosh at] = \frac{a}{s^2-a^2}\right]$$

Step II :

➔ **Using property of Laplace transform**

$$...\left\{L[t f(t)] = -\frac{d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right\}$$

$$\therefore L[t\cosh at] = -\frac{d}{ds}[L(\cosh at)] = -\frac{d}{ds}\left[\frac{s}{s^2-a^2}\right]$$

➔ **Using quotient rule of derivative**

$$...\left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$\therefore L[t\cosh at] = -\left[\frac{(s^2-a^2)(1) - s(2s)}{(s^2-a^2)^2}\right]$$

$$= -\left[\frac{s^2-a^2-2s^2}{(s^2-a^2)^2}\right]$$

$$= -\left[\frac{-s^2-a^2}{(s^2-a^2)^2}\right] = \frac{s^2+a^2}{(s^2-a^2)^2}$$

$$\therefore L[t\cosh at] = \frac{s^2+a^2}{(s^2-a^2)^2}$$

Step III :

➔ **Using Laplace transform of integration**

$$...\left\{L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}f(s) \text{ where } f(s) = L[f(t)]\right\}$$

$$\therefore L\left[\int_0^t t\cosh at\,dt\right] = \frac{1}{s}\left(\frac{s^2+a^2}{(s^2-a^2)^2}\right) \quad ...(1)$$

Step IV :

Now,

$$\left[\cosh at \int_0^t t\cosh at\, dt\right] = L\left[\frac{e^t+e^{-t}}{2}\left(\int_0^t t\cosh at\, dt\right)\right]$$

$$= \frac{1}{2} L\left[e^t \int_0^t t\cosh t\, dt + e^{-t}\int_0^t t\cosh t\, dt\right]$$

➔ **Using first shifting property of Laplace transform**

$$\ldots\left[L(e^{at} f(t)) = [L[f(t)]]_{s\to s-a} = [f(s)]_{s\to s-a}\right]$$

($s \to s-a$ means every s replace by $s-a$)

$$= \frac{1}{2}\left\{\left[L\int_0^t t\cosh at\right]_{s\to s-1} + \left[L\int_0^t t\cdot\cosh at\right]_{s\to s+1}\right\}$$

➔ **Using result from Equation (1), we get,**

$$= \frac{1}{2}\left\{\left[\frac{s^2+a^2}{s(s^2-a^2)^2}\right]_{s\to s-1} + \left[\frac{s^2+a^2}{s(s^2-a^2)^2}\right]_{s\to s+1}\right\}$$

$$\left[\cosh t\int_0^t t\cosh t\, dt\right] = \frac{1}{2}\left\{\frac{(s-1)^2+a^2}{(s-1)\left((s-1)^2-a^2\right)^2} + \frac{(s+1)^2+a^2}{(s+1)\left((s+1)^2-a^2\right)^2}\right\} \checkmark$$

...Ans.

Ex. 1.12.30 : If $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$ then find :

(i) $\int_0^\infty J_0(t)\, dt$ (ii) $\int_0^\infty e^{-t} J_0(t)\, dt$

(iii) $L[J_0(at)]$ (iv) $L[t J_0(at)]$

(v) $L[e^{-at} J_0(at)]$

Soln. :

$$L[J_0(t)] = \frac{1}{\sqrt{s^2+1}} \quad \ldots(1)$$

We know definition of Laplace transform,

$$L[F(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

1. $\int_0^\infty J_0(t)\, dt$

$$\int_0^\infty e^{st} J_0(t)\, dt = \int_0^\infty e^{-st} J_0(t)\, dt$$

$$= L[J_0(t)] \quad (s=0) \quad \ldots\text{(by definition)}$$

$$= \frac{1}{\sqrt{s^2+1}} \quad (s=0) \ldots \text{[given Equation (1)]}$$

$$\therefore \int_0^\infty J_0(t)\, dt = 1$$

2. Also, $\int_0^\infty e^{-t} J_0(t)\, dt = \int_0^\infty e^{-st} J_0(t)\, dt \quad (s=1)$

$$= L[J_0(t)] \quad (s=1) \quad \ldots\text{(by definition)}$$

$$= \frac{1}{\sqrt{s^2+1}} \quad (s=1) \ldots \text{[given Equation (1)]}$$

$$= \frac{1}{\sqrt{2}}$$

3. Since $L[f(at)] = \frac{1}{a} f\left(\frac{s}{a}\right)$

$$\therefore \quad L[J_0(at)] = \frac{1}{a}\frac{1}{\sqrt{\left(\frac{s}{a}\right)^2+1}} = \frac{1}{\sqrt{s^2+a^2}}$$

$$\because \quad L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$$

4. $$L[tJ_0(at)] = \frac{-d}{ds}\left[\frac{1}{\sqrt{s^2+a^2}}\right]$$

$$= \frac{s}{(s^2+a^2)^{3/2}} \quad \left[\because L[t f(t)] = \frac{-d}{ds} f(s)\right]$$

5. $$L[e^{-at} J_0(at)] = \frac{1}{\sqrt{(s+a)^2+a^2}} = \frac{1}{\sqrt{s^2+2as+2a^2}}$$

$$\left\{\because L[e^{at} f(t) = f(s-a)\right\}$$

Ex. 1.12.31 : Verify convolution theorem for the functions $f(t) = t$ and $g(t) = e^{at}$

Soln. :

We have,

Convolution theorem,

If $L[f(t)] = f(s)$ and $L[g(t)] = g(s)$

Then,

$$L\left[\int_0^t f(u)\, g(t-u)\, du\right] = f(s) * g(s)$$

$$= L\left[\int_0^t f(t-u)\, g(u)\, du\right] \quad \ldots(1)$$

Step I : Consider, $f(t) = t$ and $g(t) = e^{at}$

By Laplace transform standard functions

$$L[f(t)] = L[t] = \frac{1}{s^2} = f(s)$$

and $$L[g(t)] = L[e^{at}] = \frac{1}{s-a} = g(s)$$

$$\therefore f(s)\, g(s) = \frac{1}{s^2}\frac{1}{s-a} = \frac{1}{s^2(s-a)} \quad \ldots(2)$$

Step II : Now,

$$\therefore \int_0^t f(u)\, g(t-u)\, du = \int_0^t u\cdot e^{a(t-u)}\, du = \int_0^t u\, e^{at} e^{-au}\, du$$

$$= e^{at}\int_0^t u\, e^{-au}\, du$$

➔ **Using Integration Formula**

$$\ldots\left[\int u\, v\, dx - \int\left[\frac{du}{dx}\right]\cdot\int v\, dx\right] dx$$

$$= e^{at}\left[(u)\left(\frac{e^{-au}}{-a}\right) - (1)\left(\frac{e^{-au}}{a^2}\right)\right]_0^t$$

Substitute t as : (upper limit) – (lower limit)

$$= e^{at}\left\{\left[-\frac{t e^{-at}}{a} - \frac{e^{-at}}{a^2}\right] - \left[0 - \frac{1}{a^2}\right]\right\}$$

$$= -\frac{t}{a} - \frac{1}{a^2} + \frac{e^{at}}{a^2}$$

$$\therefore \int_0^t f(u)\, g(t-u)\, du = \frac{e^{at}}{a^2} - \frac{1}{a^2} - \frac{t}{a}$$

$$\therefore L\left[\int_0^t f(u)\, g(t-u)\, du\right] = L\left[\frac{e^{at}}{a^2} - \frac{1}{a^2} - \frac{t}{a}\right]$$

➔ **Using linearity property of Laplace Transform**

$$\ldots\left[L[a f(t) + bg(t) - c h(t)] = a L[f(t)] + b L[g(t)] - cL[h(t)]\right]$$

$$= \frac{1}{a^2} L[e^{at}] - \frac{1}{a^2} L[1] - \frac{1}{a} L[t]$$

➔ **Using Laplace transform standard functions**

$$\ldots\left[L[e^{at}] = \frac{1}{s-a};\ L[t^n] = \frac{n!}{s^{n+1}}\right]$$

$$= \frac{1}{a^2}\left(\frac{1}{s-a}\right) - \left(\frac{1}{a^2}\right)\left(\frac{1}{s}\right) - \frac{1}{a}\left(\frac{1}{s^2}\right)$$

By simplification, we get

$$= \frac{s^2 - s(s-a) - a(s-a)}{a^2 s^2 (s-a)}$$

$$= \frac{s^2 - s^2 + as - as + a^2}{a^2 s^2 (s-a)} = \frac{a^2}{a^2 s^2 (s-a)}$$

$$\therefore \quad L\left[\int_0^t f(u)\, g(t-u)\, du\right] = \frac{1}{s^2(s-a)} \quad \ldots(3)$$

From Equations (1), (2) and (3) convolution theorem vrified.

Syllabus Topic : Second Shifting Property

1.13 Examples on Second Shifting Theorem

Statement : Second shifting Theorem :

If $L[f(t)] = f(s)$ then

$$L\{F(t)\} = e^{-as} f(s)$$

$$\text{where, } F(t) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t < a \end{cases}$$

Ex. 1.13.1 : Find Laplace transform of

$$F(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & t > \frac{2\pi}{3} \\ 0 & t < \frac{2\pi}{3} \end{cases}$$

Soln. :

We have,

Second shifting theorem,

$$L[F(t)] = e^{-as} f(s)$$

$$\text{where } F(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases} \quad \ldots(1)$$

$$\text{Given,} \quad F(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & t > \frac{2\pi}{3} \\ 0 & t < \frac{2\pi}{3} \end{cases}$$

Comparing with

$$F(t) = \begin{cases} f(t-a); & t > a \\ 0; & t < a \end{cases}$$

$$\therefore \text{ Here } a = \frac{2\pi}{3}$$

$$f(t-a) = \cos\left(t - \frac{2\pi}{3}\right)$$

$$\therefore f(t) = \cos t$$

$$\therefore L[F(t)] = e^{-\frac{2\pi s}{3}}\cdot f(s) \quad \text{where } f(s) = L[f(t)] \ldots(2)$$

$$\therefore f(s) = L[f(t)] = L[\cos t]$$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L(\cos at) = \frac{s}{s^2+a^2}\right]$$

$$= \frac{s}{s^2+1} \quad \ldots(3)$$

[From Equations (1), (2) and (3)], we get

$$\therefore L[F(t)] = e^{-\frac{2\pi s}{3}}\cdot\frac{s}{s^2+1} \checkmark$$

...Ans.

Ex. 1.13.2 : Find the Laplace transform of

$$F(t) = \begin{cases} 0 & ; 0 < t < \pi \\ \sin^2(t-\pi) & ; t > \pi \end{cases}$$

Soln. :

We know,

Second shifting theorem of Laplace transform,

$$L[f(t)] = e^{-as} f(s)$$

where $F(t) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t > a \end{cases}$; Note that, $f(s) = L[f(t)]$

Given,

$$F(t) = \begin{cases} 0 & ; 0 < t < \pi \\ \sin^2(t-\pi) & ; t > \pi \end{cases}$$

Rewrite given function as

$$F(t) = \begin{cases} \sin^2(t-\pi) & ; t > \pi \\ 0 & ; 0 < t < \pi \end{cases}$$

Compare with

$$F(t) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t < a \end{cases}$$

$\therefore f(t-a) = \sin^2(t-\pi)$ and $a = \pi$

$\therefore f(t) = \sin^2 t$

$\therefore L[F(t)] = e^{-\pi s} f(s)$ ...(1)

...Where, $f(s) = L[f(t)] = L[\sin^2 t]$

Now find $L[f(t)] = L[\sin^2 t]$

$f(t) = \sin^2 t$

$f(t) = \dfrac{1-\cos 2t}{2}$ ...(Standard formula)

$\therefore L[f(t)] = L\left[\dfrac{1-\cos 2t}{2}\right]$

➔ **Using linearity property of Laplace transform**

...[$L[a f(t) + b g(t) - c h(t)] = a L[f(t)] + b L[g(t)] - c L[h(t)]$]

$$= L\left[\frac{1}{2}(1-\cos 2t)\right]$$

$$= \frac{1}{2} L(1-\cos 2t)$$

$$L[f(t)] = \frac{1}{2}\{L(1) - L(\cos 2t)\}$$

➔ **Using the standard result of Laplace transform**

...$\left[L(1) = \dfrac{1}{s} \text{ and } L[\cos at] = \dfrac{s}{s^2+a^2}\right]$

$\left\{\text{compare } \cos ②t \text{ with } \cos at \ (\uparrow a)\right\}$

$$L[f(t)] = \frac{1}{2}\{\underbrace{L(1)}_{\frac{1}{s}} - \underbrace{L[\cos 2t]}_{\frac{s}{s^2+(2)^2}}\}$$

$$L[f(t)] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+(2)^2}\right]$$

$$\therefore L[f(t)] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$\therefore L[\sin^2 t] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

Substitute the value $L[\sin^2 t]$ in Equation (1),

$$\therefore L[F(t)] = e^{-\pi s}\frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}$$

$$= e^{-\pi s}\frac{1}{2}\left\{\frac{s^2+4-s^2}{s(s^2+4)}\right\} \quad \ldots\left\{\text{by simplification : } \frac{a}{b} - \frac{c}{d} = \frac{ab-bc}{bd}\right\}$$

$$= \frac{e^{-\pi s}}{2}\left[\frac{4}{s(s^2+4)}\right]$$

$$\therefore L[F(t)] = \frac{2e^{-\pi s}}{s(s^2+4)} \checkmark \quad \text{...Ans.}$$

Ex. 1.13.3 : Find the Laplace transform of

$$F(t) = \begin{cases} (t-1)^2 & ; t > 1 \\ 0 & ; 0 < t < 1 \end{cases}$$

Soln. :

We know,

Second shifting theorem of Laplace transform,

$$L[F(t)] = e^{-as} f(s)$$

where

$$F(t) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t < a \end{cases} \quad \text{Note that, } f(s) = L[F(t)]$$

Given,

$$F(t) = \begin{cases} (t-1)^2 & ; t > 1 \\ 0 & ; 0 < t < 1 \end{cases} \quad \ldots(1)$$

Compare with

$$F(t) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t < a \end{cases}$$

$\therefore f(t-a) = (t-1)^2$ and $a = 1$

$\therefore L[F(t)] = e^{-s} f(s)$; where $f(s) = L[f(t)]$ ...(2)

Here, $f(t) = t^2$

Now find

$$f(s) = L[f(t)] = L[t^2]$$

➔ **Using the standard result of Laplace transform**

$$\ldots\left[L(t^n) = \frac{\sqrt{n+1}}{s^{n+1}} \text{ OR } \frac{n!}{s^{n+1}} \text{ if n is an integer}\right]$$

$$\therefore L[f(t)] = \underbrace{L[t^2]}_{\frac{2!}{s^2+1}} \quad \left\{\text{compare : } t^{②} \text{ with } t^n \ (\uparrow n)\right\}$$

$$= \frac{2!}{s^{2+1}}$$

$$\therefore L[f(t)] = \frac{2}{s^3}$$

$$\therefore L[t^2] = \frac{2}{s^3}$$

Substitute the value $L[t^2] = \dfrac{2}{s^3}$ in Equation (2)

$$\therefore L[F(t)] = e^{-s}\left(\frac{2}{s^3}\right) \checkmark \quad \text{...Ans.}$$

Ex. 1.13.4 : Find the Laplace transform of

$$F(t) = \begin{cases} \cos\frac{1}{3}(2P-3t) & ; t > \frac{2P}{3} \\ 0 & ; 0 < t < \frac{2P}{3} \end{cases}$$

Soln. : We know,

Second shifting theorem of Laplace transform,

$L[F(t)] = e^{-as} f(s)$ where $F(t) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t < a \end{cases}$

Note that, $f(s) = L[F(t)]$

Given :

$$F(t) = \begin{cases} \cos\frac{1}{3}(2P-3t) & ; t > \frac{2P}{3} \\ 0 & ; 0 < t < \frac{2P}{3} \end{cases}$$

Compare with,

$$F(t) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t < a \end{cases}$$

$$\therefore f(t-a) = \cos\frac{1}{3}(2P-3t)$$

$$\therefore f(t-a) = \cos\left(\frac{2P}{3} - t\right)$$

$$f(t-a) = \cos\left[\underbrace{-\left(t - \frac{2P}{3}\right)}_{\theta}\right]$$

$$\therefore f(t-a) = \cos\left(t - \frac{2P}{3}\right) \quad \ldots \text{Since, } \cos(-\theta) = \cos\theta$$

$\therefore f(t) = \cos t$ and $a = \dfrac{2P}{3}$

We know,

$$L[F(t)] = e^{-as} f(s) = e^{-\frac{2P}{3}s} f(s) \quad \ldots(1)$$

Here, $f(s) = L[f(t)] = L[\cos t]$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L[\cos at] = \frac{s}{s^2+a^2}\right]$$

$$\therefore L[f(t)] = L[\cos 1t]$$

$$\left\{\text{compare : } \cos ①t \text{ with } \cos at \ (\uparrow a)\right\}$$

$$L[f(t)] = \frac{s}{s^2+(1)^2}$$

$$L[f(t)] = \frac{s}{s^2+1}$$

$$\therefore f(s) = L[f(t)] = L[\cos t] = \frac{s}{s^2+1}$$

$\therefore$ Substitute the value $f(s) = \dfrac{s}{s^2+1}$ in Equation (1),

$$\therefore L[F(t)] = e^{-\frac{2Ps}{3}} \frac{s}{s^2+1}$$

$$L[F(t)] = \frac{se^{-\frac{2Ps}{3}}}{s^2+1} \checkmark \quad \text{...Ans.}$$

Ex. 1.13.5 : Find the Laplace transform of

$$F(t) = \begin{cases} e^{-4(t-3)}\sin 3(t-3) & t > 3 \\ 0 & t < 3 \end{cases}$$

Soln. :

We have, Second shifting theorem,

$$L[F(t)] = e^{-as} f(s)$$

where $F(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$ ...(1)

Given, $F(t) = \begin{cases} e^{-4(t-3)}\sin 3(t-3) & t > 3 \\ 0 & t < 3 \end{cases}$

Comparing with

$$F(t) = \begin{cases} f(t-a); & t > a \\ 0; & t < a \end{cases}$$

$\therefore$ Here $a = 3, f(t-a) = e^{-4(t-3)} \cdot \sin 3(t-3)$

$\therefore f(t) = e^{-4t} \sin 3t$ and $a = 3$

$\therefore L[F(t)] = e^{-3s} \cdot f(s)$,

where $f(s) = L[f(t)] = L[e^{-t} \sin 3t]$ ...(2)

To find $L[e^{-4t} \sin 3t]$

➔ **Use the first shifting property of Laplace transform**

$$\ldots\left[L[e^{at} f(t)] = [L(f(t))]_{s \to s-a} = [f(s)]_{s \to s-a}\right]$$

$$\left\{\begin{array}{l}\text{where } L[f(t)] = f(s) \text{ and } s \to s-a \text{ means} \\ \text{every s replace by } (s-a)\end{array}\right\}$$

$$\left[\begin{array}{c}\text{Compare : } \underset{\substack{\uparrow \\ e^{at}}}{e^{-4t}} \quad \underset{\substack{\uparrow \\ f(t)}}{\sin 3t} \text{ with } e^{at} f(t) \\ \text{with } a = -4\end{array}\right]$$

$\therefore L[e^{-t} \sin 3t] = [L(\sin 3t)]_{s \to s-(-4)}$

$= [L(\sin 3t)]_{s \to s+4}$ ...(3)

Now find $L[f(t)] = L[\sin 3t]$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L(\sin at) = \frac{a}{s^2 + a^2}; \text{ Here, } a = 3\right]$$

$\therefore L[f(t)] = L[\sin 3t]$

$= \dfrac{3}{s^2 + (3)^2}$ {compare : sin at and sin ③t}

$\uparrow$ a

$= \dfrac{3}{s^2 + 9}$

$\therefore L[\sin 3t] = \dfrac{3}{s^2 + 9}$

➔ **Using result : $L[\sin 3t] = \dfrac{3}{s^2 + 9}$ in Equation (3)**

$\therefore L[e^{-t} \sin 3t] = \left[\dfrac{3}{s^2 + 9}\right]_{s \to s+4}$

$= \dfrac{3}{(s+4)^2 + 9}$ ...(every s replace by s + 4)

$= \dfrac{3}{(s^2 + 8s + 16) + 9}$ $\left[\begin{array}{l}\text{Use : } (a+b)^2 \\ = a^2 + 2ab + b^2\end{array}\right]$

$\therefore L[e^{-t} \sin 3t] = \dfrac{3}{s^2 + 8s + 25}$ ...(4)

From Equations (1), (2) and (4), we get

$\therefore L[F(t)] = \dfrac{3e^{-3s}}{s^2 + 8s + 25}$ ✓ ...Ans.

Syllabus Topic : Evaluation of integrals by using Laplace transform

1.14 Evaluation of Integrals by Using Laplace Transform

We can use Laplace transform to evaluate some definite intergrads such as,

$$\int_0^\infty \frac{\sin t}{t} dt; \quad \int_0^\infty \frac{1 - e^{-t}}{t} dt \text{ etc.}$$

Each integer if limit from 0 to ∞ write as $\int_0^\infty e^{-st} f(t)$ form and compare.

Ex. 1.14.1 : Evaluate by using Laplace transform

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt$$

Soln. :

Let $I = \displaystyle\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt$

We have, definition of Laplace transform

$$L[f(t)] = \int_0^\infty e^{-st} f(t) dt$$

Compare : $\displaystyle\int_0^\infty e^{-st} f(t) dt$ and $\displaystyle\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt$

$\therefore \displaystyle\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \int_0^\infty e^{0t} \underbrace{\left[\frac{e^{-at} - e^{-bt}}{t}\right]}_{f(t)} dt$ $(\because e^{0t} = 1)$

$= L\left[\dfrac{e^{-at} - e^{-bt}}{t}\right]$ (s = 0)...1

First Find $L\left[\dfrac{e^{-at} - e^{-bt}}{t}\right]$

➔ **Using linearity property of Laplace Transform**

$$\ldots\left[L[a f(t) + b g(t) - c h(t)] = a L[f(t)] + b L[g(t)] - c L[h(t)]\right]$$

$\therefore L[e^{at} - e^{bt}] = L[e^{at}] - L[e^{bt}]$

➔ **Using Laplace transform of standard functions**

$$\ldots\left[L[e^{at}] = \frac{1}{s-a}\right]$$

$\therefore L[e^{at} - e^{bt}] = \dfrac{1}{s-a} - \dfrac{1}{s-b}$

➔ **Using property of Laplace transform**

$$\ldots\left[L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s) ds\right]$$

$$L\left[\frac{e^{at} - e^{bt}}{t}\right] = \int_s^\infty \left(\frac{1}{s-a} - \frac{1}{s-b}\right) ds$$

➔ **Using integration of standard function**

$$\ldots\left[\int \frac{1}{ax+b} dx = \frac{\log(ax+b)}{a}\right]$$

$$L\left[\frac{e^{at} - e^{bt}}{t}\right] = [\log(s-a) - \log(s-b)]_s^\infty$$

➔ **Using property of logarithm :** $\ldots\left[\log a - \log b = \log\left(\dfrac{a}{b}\right)\right]$

$$= \left[\log\left(\frac{s-a}{s-b}\right)\right]_s^\infty$$

Substitute s as : (upper limit) – (lower limit)

$$L\left[\frac{e^{at} - e^{bt}}{t}\right] = 0 - \log\left(\frac{s-a}{s-b}\right)$$

$$= -\log\left(\frac{s-a}{s-b}\right)$$

➔ **Using property of logarithm :** ...[m log a = log a^m]

$$= \log\left(\frac{s-a}{s-b}\right)^{-1} \quad \ldots\text{since } \left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

$$L\left[\frac{e^{at} - e^{bt}}{t}\right] = \log\left(\frac{s-b}{s-a}\right)$$

Substitute these in Equation (1)

$$\therefore \int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \log\left(\frac{s-b}{s-a}\right) \quad \ldots\text{since } s = 0$$

$$= \log\left(\frac{0-b}{0-a}\right)$$

$$\therefore \int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \log\left(\frac{b}{a}\right) ✓ \quad \ldots\text{Ans.}$$

Ex. 1.14.2 : Evaluate by using Laplace transform

$$f(t) = \int_0^\infty e^{-t} \sin t dt.$$

Soln. :

We know, definition of Laplace transform,

$$L[f(t)] = \int_0^\infty e^{-st} f(t) dt$$

$$\therefore \int_0^\infty e^{-t} \underbrace{\sin(t)}_{f(t)} dt = L[\sin t] \quad (\text{with } s = 1) \quad \ldots(1)$$

➔ **Using standard result of Laplace transform**

$$\ldots\left\{L(\sin at) = \frac{a}{s^2 + a^2}\right\}$$

$$= \frac{1}{(1)^2 + 1} \quad (\because \text{ since } s = 1 \text{ here})$$

$$= \frac{1}{1+1}$$

$$\int_0^\infty e^{-t} \sin t dt = \frac{1}{s^2 + 1} = \frac{1}{2} ✓ \quad \ldots\text{Ans.}$$

Ex. 1.14.3 : Evaluate by using Laplace transform

$$f(t) = \int_0^\infty e^{-2t} \sinh t dt.$$

Soln. :

We know, definition of Laplace transform,

$$L[f(t)] = \int_0^\infty e^{-st} f(t) dt$$

$$\int_0^\infty e^{-2t} \underbrace{\sinh 4t}_{f(t)} dt = L[\sinh t] \quad (\text{with } s = 2)$$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L(\sin h\, at) = \frac{a}{s^2 - a^2}\right]$$

$$\int_0^\infty e^{-2t} \sin h\, 4t dt = \frac{1}{s^2 - 1} = \frac{1}{(2)^2 - 1} \quad (\because s = 2 \text{ here})$$

$$\int_0^\infty e^{-2t} \sin h\, 4t dt = \frac{1}{4-1} = \frac{1}{3} ✓ \quad \ldots\text{Ans.}$$

Ex. 1.14.4 : Evaluate by using Laplace transform

$$f(t) = \int_0^\infty e^{-3t} \sin 3t dt$$

Soln. :

We know, definition of Laplace transform,

$$L[f(t)] = \int_0^\infty e^{-st} f(t) dt$$

$$\int_0^\infty e^{-3t}\underbrace{\sin 3t}_{f(t)}\,dt = L[\sin 3t] \quad (\text{with } s = 3)$$

➔ **Using standard result of Laplace transform**

$$\ldots\left\{L(\sin at) = \frac{a}{s^2+a^2}\right\}$$

$$\int_0^\infty e^{-3t}\sin 3t\,dt = \frac{3}{s^2+9} = \frac{3}{(3)^2+9} \quad \ldots(\because s = 3 \text{ here})$$

$$= \frac{3}{9+9} = \frac{3}{18}$$

$$\int_0^\infty e^{-3t}\sin 3t\,dt = \frac{1}{6} \checkmark \qquad \ldots\text{Ans.}$$

Ex. 1.14.5 : Using the properties of the Laplace transform, find the value of $\int_0^\infty e^{-3t}\sin^2 t\,dt$

Dr. BATU – Dec. 2016, 4 Marks

Soln. :

We know definition of Laplace transform,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore \int_0^\infty e^{-3t}\underbrace{\sin^2 t}_{f(t)}\,dt = L[\sin^2 t] \qquad \text{Here, } s = 3$$

Let, $f(t) = \sin^2 t$

$$f(t) = \frac{1-\cos 2t}{2} \qquad \ldots(\text{standard formula})$$

$$\therefore L[f(t)] = L\left[\frac{1-\cos 2t}{2}\right]$$

$$= L\left[\frac{1}{2}(1-\cos 2t)\right]$$

➔ **Using Linearity property**

$$\ldots\left[L[a f(t) + b g(t) - c h(t)] = a L[f(t)] + b L[g(t)] - c L[h(t)]\right]$$

$$\therefore L[f(t)] = \frac{1}{2} L(1-\cos 2t)$$

$$L[f(t)] = \frac{1}{2}[L(1) - L(\cos 2t)]$$

➔ **Using the standard result of Laplace transform,**

$$\ldots\left[L(1) = \frac{1}{s} \text{ and } L[\cos at] = \frac{s}{s^2+a^2}\right]$$

$$\ldots(\text{compare : } \cos 2t \text{ with } \cos at)$$

$$\therefore L[f(t)] = \frac{1}{2}\{\underbrace{L(1)}_{\frac{1}{s}} - \underbrace{L[\cos 2t]}_{\frac{s}{s^2+(2)^2}}\}$$

$$\therefore L[f(t)] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+(2)^2}\right]$$

$$\therefore L[f(t)] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$\therefore L[\sin^2 t] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] \checkmark \qquad \ldots\text{Ans.}$$

Note : We can further simplify it as,

$$\frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] = \frac{1}{2}\left[\frac{s^2+4-s^2}{s(s^2+4)}\right] = \frac{1}{2}\left[\frac{4}{s(s^2+4)}\right] = \frac{2}{s(s^2+4)}$$

Since $s = 3$

$$\therefore \int_0^\infty e^{-3t}\sin^2 t\,dt = \frac{1}{2}\left[\frac{1}{3} - \frac{3}{(3)^2+4}\right]$$

$$= \frac{1}{2}\left[\frac{1}{3} - \frac{3}{9+4}\right] = \frac{1}{2}\left[\frac{1}{3} - \frac{3}{13}\right]$$

$$= \frac{1}{2}\left[\frac{13-9}{39}\right] = \frac{1}{2}\left[\frac{4}{39}\right]$$

$$\therefore \int_0^\infty e^{-3t}\sin^2 t\,dt = \frac{2}{39} \checkmark \qquad \ldots\text{Ans.}$$

Ex. 1.14.6 : Evaluate the integral $\int_0^\infty e^{-2t}\frac{\sinh t}{t}\,dt$ by Laplace transform.

BATU - May 2017, 5 Marks

Soln. :

We know definition of Laplace transform

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

Given :

$$\therefore \int_0^\infty e^{-2t}\frac{\sinh t}{t}\,dt = L\left[\frac{\sinh t}{t}\right] \qquad \text{with } s = 2$$

➔ **Using property of Laplace transform**

$$\ldots\left[L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds\right]$$

$$\therefore \int_0^\infty e^{-2t}\frac{\sinh t}{t}\,dt = \int_s^\infty L[\sinh t]\,ds$$

➔ **Using standard result of Laplace transform**

$$\ldots\left\{L[\sinh at] = \frac{a}{s^2-a^2}\right\}$$

$$\therefore \int_0^\infty e^{-2t}\frac{\sinh t}{t}\,dt = \int_s^\infty \frac{1}{s^2-(1)^2}\,ds$$

➔ **Using standard formula of integration**

$$\ldots\left[\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)\right]$$

$$\therefore \int_0^\infty e^{-2t}\frac{\sinh t}{t}\,dt = \left[\frac{1}{2}\log\left(\frac{s-1}{s+1}\right)\right]_s^\infty$$

Substitute s as : (upper limit) – (lower limit)

$$= \frac{1}{2}\left\{[0] - \left[\log\left(\frac{s-1}{s+1}\right)\right]\right\}$$

$$= \frac{1}{2}\left[-\log\left(\frac{s-1}{s+1}\right)\right]$$

➔ **Using property of logarithm :** $\ldots[m \log a = \log a^m]$

$$\therefore \int_0^\infty e^{-2t}\frac{\sinh t}{t}\,dt = \frac{1}{2}\left[\log\left(\frac{s-1}{s+1}\right)^{-1}\right]$$

$$= \frac{1}{2}\log\left(\frac{s+1}{s-1}\right) \qquad \because \left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

Since $s = 2$

$$\therefore \int_0^\infty e^{-2t}\frac{\sinh t}{t}\,dt = \frac{1}{2}\log\left(\frac{2+1}{2-1}\right) = \frac{1}{2}\log\left(\frac{3}{1}\right)$$

$$\therefore \int_0^\infty e^{-2t}\frac{\sinh t}{t}\,dt = \frac{1}{2}\log(3) \checkmark \qquad \ldots\text{Ans.}$$

Ex. 1.14.7 : Evaluate by using Laplace transform

$$\int_0^\infty t^2 e^{-t}\sin t\,dt$$

Soln. :

$$\text{Let } I = \int_0^\infty t^2 e^{-t}\sin t\,dt$$

We have, definition of Laplace transform

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

Compare : $\int_0^\infty e^{-st} f(t)\,dt$ and $\int_0^\infty t^2 e^{-t}\sin t\,dt$

$$\therefore \int_0^\infty t^2 e^{-t}\sin t\,dt = \int_0^\infty e^{-t}\underbrace{[t^2\sin t]}_{f(t)}\,dt = L[t^2\sin t]$$

$$\text{with } s = 1 \qquad \ldots(1)$$

➔ **Using property of Laplace transform**

$$\ldots\left\{L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} f(s) \text{ where } f(s) = L[f(t)]\right\}$$

$$= (-1)^2 \frac{d^2}{ds^2}[L(\sin t)]$$

$$= \frac{d^2}{ds^2}\left[\frac{1}{s^2+1}\right] = \frac{d}{ds}\left[\frac{-1}{(s^2+1)^2}(2s)\right]$$

$$= \frac{d}{ds}\left[\frac{-2s}{(s^2+1)^2}\right] \qquad (s = 1)$$

$$\therefore \int_0^\infty t^2 e^{-t}\sin t\,dt = \left[\frac{(s^2+1)^2(-2) - (-2s)\,2(s^2+1)(2s)}{(s^2+1)^4}\right] \qquad (s = 1)$$

$$= \frac{2(s^2+1)[-(s^2+1)+4s^2]}{(s^2+1)^4} = \frac{2(3s^2-1)}{(s^2+1)^3} \qquad (s = 1)$$

Since $s = 1$

$$\therefore \int_0^\infty t^2 e^{-t}\sin t\,dt = \frac{2(3-1)}{(1+1)^3} = \frac{1}{2} \qquad (s = 1)$$

$$\int_0^\infty t^2 e^{-t}\sin t\,dt = \frac{1}{2} \checkmark \qquad \ldots\text{Ans.}$$

Ex. 1.14.8 : Evaluate by using Laplace transform of

$$\int_0^\infty e^{-2t} t^2 \sin 3t\,dt$$

Soln. :

$$\text{Let } I = \int_0^\infty e^{-2t} t^2 \sin 3t\,dt$$

We have, definition of Laplace transform

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

Compare : $\int_0^\infty e^{-st} f(t)\,dt$ and $\int_0^\infty e^{-2t} t^2 \sin 3t\,dt$

$$\therefore \int_0^\infty e^{-2t}\underbrace{[t^2 \sin 3t]}_{f(t)}\, dt = L[t^2 \sin 3t] \quad \text{with } s = 2$$

➔ **Using property of Laplace transform**

$$\ldots\left\{L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} f(s) \text{ where } f(s) = L[f(t)]\right\}$$

$$= (-1)^2 \frac{d^2}{ds^2} L[\sin 3t]$$

$$= \frac{d^2}{ds^2}\left[\frac{3}{s^2+9}\right] = \frac{d}{ds}\left[\frac{-3}{(s^2+9)^2}(2s)\right] \quad (s = 2)$$

$$= \frac{d}{ds}\left[\frac{-6s}{(s^2+9)^2}\right] = \frac{(s^2+9)^2(-6) - (-6s)(2)(s^2+9)(2s)}{(s^2+9)^4}$$

$$= \frac{(-6)(s^2+9)[s^2+9-4s^2]}{(s^2+9)^4} \quad (s = 2)$$

$$\therefore \int_0^\infty e^{-2t} t^2 \sin 3t\, dt = \frac{-6(9-3s^2)}{(s^2+9)^4} = \frac{(-6)(-3)}{13^4} = \frac{18}{13^4} \quad (s = 2)$$

$$\therefore \int_0^\infty e^{-2t} t^2 \sin 3t\, dt = \frac{18}{28561} \checkmark \quad \ldots\text{Ans.}$$

Ex. 1.14.9 : Evaluate by using Laplace transform

$$\int_0^\infty t e^{-2t} \cos t\, dt$$

Soln. :

$$\text{Let} \quad I = \int_0^\infty t e^{-2t} \cos t\, dt$$

We have, definition of Laplace transform

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$\text{Compare : } \int_0^\infty e^{-st} f(t)\, dt \text{ and } \int_0^\infty t e^{-2t} \cos t\, dt$$

$$\therefore \int_0^\infty t e^{-2t} \cos t\, dt = \int_0^\infty e^{-2t}\underbrace{(t \cos t)}_{f(t)}\, dt = L[t \cos t] \quad (\text{with } s = 2)$$

➔ **Using property of Laplace transform**

$$\ldots\left\{L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} f(s) \text{ where } f(s) = L[f(t)]\right\}$$

$$= \frac{-d}{ds} L(\cos t) = \frac{-d}{ds}\left[\frac{s}{s^2+1}\right] \quad (s = 2)$$

➔ **Using quotient rule of derivative**

$$\ldots\left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$= -\left[\frac{(s^2+1)(1) - s(2s)}{(s^2+1)^2}\right] = \frac{s^2-1}{(s^2+1)^2} \quad (s = 2)$$

$$\therefore \int_0^\infty t e^{-2t} \cos t\, dt = \frac{3}{(5)^2} = \frac{3}{25}$$

$$\therefore \int_0^\infty t e^{-2t} \cos t\, dt = \frac{3}{25} \checkmark \quad \ldots\text{Ans.}$$

Ex. 1.14.10 : Evaluate $\int_0^\infty \frac{e^{-t} - e^{-4t}}{t}\, dt$.

Soln. :

$$\text{Let} \quad I = \int_0^\infty \frac{e^{-t} - e^{-4t}}{t}\, dt$$

We have, definition of Laplace transform

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$\text{Compare : } \int_0^\infty e^{-st} f(t)\, dt \text{ and } \int_0^\infty \frac{e^{-t} - e^{-4t}}{t}\, dt.$$

$$\therefore \int_0^\infty \frac{e^{-t} - e^{-4t}}{t}\, dt = \int_0^\infty e^{0t}\left[\frac{e^{-t} - e^{-4t}}{t}\right] dt$$

$$= L\left[\frac{e^{-t} - e^{-4t}}{t}\right] \quad (\text{with } s = 0) \quad \ldots(1)$$

First find $L\left[\frac{e^{-t} - e^{-4t}}{t}\right]$

➔ **Using linearity property of Laplace Transform**

$$\ldots\left[L[a f(t) + b g(t) - c h(t)] = a L[f(t)] + b L[g(t)] - c L[h(t)]\right]$$

$$\therefore L[e^{-t} - e^{-4t}] = L[e^{-t}] - L[e^{-4t}]$$

➔ **Using Laplace transform of standard functions**

$$\ldots\left[L[e^{at}] = \frac{1}{s-a}\right]$$

$$\therefore L[e^{-t} - e^{-4t}] = \frac{1}{s-(-1)} - \frac{1}{s-(-4)} = \frac{1}{s+1} - \frac{1}{s+4}$$

➔ **Using property of Laplace transform**

$$\ldots\left[L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\, ds\right]$$

$$L\left[\frac{e^{-t} - e^{-4t}}{t}\right] = \int_s^\infty \left(\frac{1}{s+1} - \frac{1}{s+4}\right) ds$$

➔ **Using integration of standard function**

$$\ldots\left[\int \frac{1}{ax+b}\, dx = \frac{\log(ax+b)}{a}\right]$$

$$L\left[\frac{e^{at} - e^{bt}}{t}\right] = [\log(s+1) - \log(s+4)]_s^\infty$$

➔ **Using property of logarithm :** $\ldots\left[\log a - \log b = \log\left(\frac{a}{b}\right)\right]$

$$= \left[\log\left(\frac{s+1}{s+4}\right)\right]_s^\infty$$

➔ **Substitute s as : (upper limit) – (lower limit)**

$$L\left[\frac{e^{-t} - e^{-4t}}{t}\right] = 0 - \log\left(\frac{s+1}{s+4}\right)$$

$$= -\log\left(\frac{s+1}{s+4}\right)$$

➔ **Using property of logarithm :** $\ldots[m \log a = \log a^m]$

$$= \log\left(\frac{s+1}{s+4}\right)^{-1} \quad \ldots\text{since } \left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

$$L\left[\frac{e^{-t} - e^{-4t}}{t}\right] = \log\left(\frac{s+4}{s+1}\right)$$

Substitute these in Equation (1), we get,

$$\therefore \int_0^\infty \frac{e^{-t} - e^{-4t}}{t}\, dt = \log\left(\frac{s+4}{s+1}\right) \quad (s = 0)$$

$$\int_0^\infty \frac{e^{-t} - e^{-4t}}{t}\, dt = \log\left(\frac{4}{1}\right) = \log 4 \checkmark \quad \ldots\text{Ans.}$$

Ex. 1.14.11 : Evaluate by using Laplace transform

$$\int_0^\infty e^{-2t} \sin^3 t\, dt$$

Soln. :

$$\text{Let} \quad I = \int_0^\infty e^{-2t} \sin^3 t\, dt$$

We have, definition of Laplace transform

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$\therefore \int_0^\infty e^{-2t} \sin^3 t\, dt = \int_0^\infty e^{-2t}\underbrace{\sin^3 t}_{f(t)}\, dt = L[\sin^3 t] \text{ with } s = 2 \ldots(1)$$

First find $L[\sin^3 t]$

$$\text{Let,} \quad f(t) = \sin^3 t \quad \ldots(1)$$

➔ **Using standard trigonometric formula**

$$\ldots[\sin 3x = 3 \sin x - 4 \sin^3 x]$$

$$\sin 3x = 3 \sin x - 4 \sin^3 x$$

$$\therefore 4 \sin^3 x = 3 \sin x - \sin 3x$$

$$\therefore \sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x \quad \ldots\text{(Divide throughout by 4)}$$

$\therefore$ Equation (1) we can write as,

$$\therefore f(t) = \sin^3 t = \frac{3}{4} \sin t - \frac{1}{4} \sin 3t$$

Taking Laplace transform of both sides, We get,

$$\therefore L[f(t)] = L\left[\frac{3}{4} \sin t - \frac{1}{4} \sin 3t\right]$$

$$= \frac{3}{4} L(\sin t) - \frac{1}{4} L(\sin 3t)$$

➔ **Using the linearity property of Laplace transform,**

$$\left[L[a f(t) + b g(t) - c h(t)] = a L[f(t)] + b L[g(t)] - c L[h(t)]\right]$$

➔ **Using the standard result of Laplace transform,**

$$\ldots\left[L(\sin at) = \frac{a}{s^2 + a^2}\right]$$

$$\therefore L[f(t)] = \frac{3}{4} L(\sin 1t) - \frac{1}{4} L(\sin 3t)$$

$$\left\{\text{Compare : } \sin \underset{\substack{\uparrow \\ a}}{1} t \text{ with } \sin at,\ \sin \underset{\substack{\uparrow \\ a}}{3} t \text{ with } \sin at\right\}$$

$$L[f(t)] = \frac{3}{4} \underbrace{L(\sin 1t)}_{\frac{1}{s^2+(1)^2}} - \frac{1}{4} \underbrace{L(\sin 3t)}_{\frac{3}{s^2+(3)^2}}$$

$$= \frac{3}{4} \frac{1}{s^2+(1)^2} - \frac{1}{4} \frac{3}{s^2+(3)^2}$$

$$L[f(t)] = \frac{3}{4} \cdot \frac{1}{s^2+1} - \frac{1}{4} \frac{3}{s^2+9}$$

$$\therefore L[\sin^3 t] = \frac{3}{4}\left(\frac{1}{s^2+1}\right) - \frac{3}{4}\left(\frac{1}{s^2+9}\right)$$

Common out $\frac{3}{4}$ from R.H.S., we get,

$$\therefore L[\sin^3 t] = \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

Substitute these in Equation (1),

$$\therefore \int_0^\infty e^{-2t} \sin^3 t\, dt = \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

Since s = 1

$$= \frac{3}{4}\left[\frac{1}{(2)^2+1} - \frac{1}{(2)^2+9}\right]$$

$$= \frac{3}{4}\left[\frac{1}{4+1} - \frac{1}{4+9}\right]$$

$$\therefore \int_0^\infty e^{-2t}\sin^3 t\,dt = \frac{3}{4}\left(\frac{1}{5}\right) - \frac{3}{4}\left(\frac{1}{13}\right) = \frac{3}{20} - \frac{3}{52} = \frac{6}{65}$$

$$\int_0^\infty e^{-2t}\sin^3 t\,dt = \frac{6}{65} \checkmark \qquad \text{...Ans.}$$

Ex. 1.14.12 : Evaluate by using Laplace transform

$$\int_0^\infty e^{-3t}\sin^3 t\,dt$$

Soln. :

$$\text{Let } I = \int_0^\infty e^{-3t}\sin^3 t\,dt$$

We have definition of Laplace transform,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt \qquad \text{...(definition)}$$

$$\therefore \int_0^\infty \underbrace{e^{-2t}\sin^3 t}_{f(t)}\,dt = L[\sin^3 t] \qquad \text{with } s = 3 \qquad \text{...(1)}$$

First find $L[\sin^3 t]$

Let, $f(t) = \sin^3 t$

➔ **Using standard trigonometric formula**

$$\text{...}[\sin 3x = 3\sin x - 4\sin^3 x]$$

$$\sin 3x = 3\sin x - 4\sin^3 x$$

$$\therefore 4\sin^3 x = 3\sin x - \sin 3x$$

$$\therefore \sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x \quad \text{....(Divide throughout by 4)}$$

$\therefore$ Equation (1) we can write as,

$$\therefore f(t) = \sin^3 t = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t$$

Taking Laplace transform of both sides, We get,

$$\therefore L[f(t)] = L\left[\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\right]$$

$$= \frac{3}{4}L(\sin t) - \frac{1}{4}L(\sin 3t)$$

➔ **Using the linearity property of Laplace transform,**

$$[L[a f(t) + b g(t) - c h(t)] = a L[f(t)] + b L[g(t)] - c L[h(t)]]$$

➔ **Using the standard result of Laplace transform,**

$$\text{...}\left[L(\sin at) = \frac{a}{s^2 + a^2}\right]$$

$$\therefore L[f(t)] = \frac{3}{4}L(\sin t) - \frac{1}{4}L(\sin 3t)$$

$$\left\{\text{Compare : } \sin \underset{\substack{\uparrow \\ a}}{①}t \text{ with } \sin at,\ \sin \underset{\substack{\uparrow \\ a}}{③}t \text{ with } \sin at\right\}$$

$$L[f(t)] = \frac{3}{4}\underbrace{L(\sin t)}_{\frac{1}{s^2+(1)^2}} - \frac{1}{4}\underbrace{L(\sin 3t)}_{\frac{3}{s^2+(3)^2}}$$

$$= \frac{3}{4}\frac{1}{s^2+(1)^2} - \frac{1}{4}\frac{3}{s^2+(3)^2}$$

$$L[f(t)] = \frac{3}{4}\cdot\frac{1}{s^2+1} - \frac{1}{4}\frac{3}{s^2+9}$$

$$\therefore L[\sin^3 t] = \frac{3}{4}\left(\frac{1}{s^2+1}\right) - \frac{3}{4}\left(\frac{1}{s^2+9}\right)$$

Common out $\frac{3}{4}$ from R.H.S., we get,

$$\therefore L[\sin^3 t] = \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

Substitute these in Equation (1)

$$\therefore \int_0^\infty e^{-3t}\sin^3 t\,dt = \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right] = \frac{3}{4}\left[\frac{1}{(3)^2+1} - \frac{1}{(3)^2+9}\right]$$

$$\int_0^\infty e^{-3t}\sin^3 t\,dt = \frac{3}{4}\left(\frac{1}{10}\right) - \frac{3}{4}\left(\frac{1}{18}\right) = \frac{3}{40} - \frac{3}{72} = \frac{1}{30}$$

$$\int_0^\infty e^{-3t}\sin^3 t\,dt = \frac{1}{30} \checkmark \qquad \text{...Ans.}$$

Ex. 1.14.13 : Evaluate by using Laplace transform

$$\int_0^\infty t^2 e^{-3t}\sinh 2t\,dt$$

Soln. :

$$\text{Let } I = \int_0^\infty t^2 e^{-3t}\sinh 2t\,dt$$

We have, definition of Laplace transform

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore \int_0^\infty t^2 e^{-3t}\sinh 2t\,dt = \int_0^\infty e^{-3t}\underbrace{(t^2\sinh 2t)}_{f(t)}\,dt = L[t^2 \sinh 2t]$$

(with $s = 3$)

$$= (-1)^2\frac{d^2}{ds^2}L(\sinh 2t) \text{ ...}$$

$$\left[\because L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} f(s)\right] \quad (s = 3)$$

$$= \frac{d^2}{ds^2}\left[\frac{2}{s^2-2}\right]$$

➔ **Using quotient rule of derivative**

$$\text{...}\left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-2}{(s^2-2)^2}(2s)\right] \qquad (s = 3)$$

$$= \frac{d}{ds}\left[\frac{-4s}{(s^2-2)^2}\right] \qquad (s = 3)$$

➔ **Using quotient rule of derivative**

$$\text{...}\left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$= \frac{(s^2-2)^2(-4) - (-4s)(2)(s^2-2)(2s)}{(s^2-2)^4} \qquad (s = 3)$$

$$= \frac{4(s^2-2)[-s^2+2+4s^2]}{(s^2-2)^4} \qquad (s = 3)$$

$$\int_0^\infty t^2 e^{-3t}\sinh 2t\,dt = \frac{4(3s^2+2)}{(s^2-2)^3} \qquad \text{with } s = 3$$

$$= \frac{4[3(3)^2+2]}{[(3)^2-2]^3} = \frac{4[27+2]}{(9-2)^3} = \frac{4(29)}{7^3} = \frac{116}{343}$$

$$\therefore \int_0^\infty t^2 e^{-3t}\sinh 2t\,dt = \frac{116}{343} \checkmark \qquad \text{...Ans.}$$

Ex. 1.14.14 : Show that : $\int_0^\infty \left(\frac{\sin 2t + \sin 3t}{t e^t}\right)dt = \frac{3\pi}{4}$

Soln. :

$$\text{Let } I = \int_0^\infty \left(\frac{\sin 2t + \sin 3t}{t e^t}\right)dt = \frac{3\pi}{4}$$

We have, definition of Laplace transform

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore \int_0^\infty \left(\frac{\sin 2t + \sin 3t}{te^t}\right)dt$$

$$= \int_0^\infty e^{-t}\underbrace{\left(\frac{\sin 2t + \sin 3t}{t}\right)}_{f(t)}dt = L\left[\frac{\sin 2t + \sin 3t}{t}\right]$$

(with $s = 1$)

➔ **Using property of Laplace transform**

$$\text{...}\left[L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds\right]$$

$$= \int_s^\infty L(\sin 2t + \sin 3t)\,ds \qquad \left[\because L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds\right]$$

➔ **Using standard result of Laplace transform**

$$\text{...}\left\{L(\sin at) = \frac{a}{s^2+a^2}\right\}$$

$$= \int_s^\infty \left(\frac{2}{s^2+4} + \frac{3}{s^2+9}\right)ds \qquad \text{(by table)}$$

➔ **Using standard result of integration**

$$\text{...}\left[\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \left[2\cdot\frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right) + 3\frac{1}{3}\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty$$

$$= \left[\tan^{-1}\left(\frac{s}{2}\right) + \tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty \qquad (s = 1)$$

Substitute s as : (upper limit) – (lower limit)

$$= \left(\frac{\pi}{2} + \frac{\pi}{2}\right) - \left[\tan^{-1}\left(\frac{s}{2}\right) + \tan^{-1}\left(\frac{s}{3}\right)\right] \qquad (s = 1)$$

$$= \pi - \left[\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)\right]$$

➔ **Using trigonometric formula**

$$\text{...}\left[\tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)\right]$$

$$= \pi - \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}\right)$$

$$= \pi - \tan^{-1}(1)$$

➔ **Using standard value :** $\text{...}\left[\tan\frac{\pi}{4} = 1 \Rightarrow \tan^{-1}(1) = \frac{\pi}{4}\right]$

$$= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\therefore \int_0^\infty \left(\frac{\sin 2t + \sin 3t}{t e^t}\right)dt = \frac{3\pi}{4} \text{ Hence proved}$$

Ex. 1.14.15 : Evaluale by using Laplace transform :

$$\int_0^\infty e^{-2t}\sin 3t\,dt$$

Soln. :

$$\text{Let} \quad I = \int_0^\infty e^{-2t} \sin 3t \, dt$$

We have, definition of Laplace transform

$$L[f(t)] = \int_0^\infty e^{-st} f(t) \, dt$$

$$\therefore \int_0^\infty e^{-2t} \sin 3t \, dt = L[\sin 3t] \qquad \because s = 2$$

Let, $f(t) = \sin 3t$

➔ Using standard result of Laplace transform

$$\ldots\left[L(\sin at) = \frac{a}{s^2 + a^2}\ ;\ \text{Here, } a = 3\right]$$

$$\therefore L[f(t)] = L[\sin 3t]$$

$$= \frac{3}{s^2 + (3)^2} \quad \{\text{compare : sin at and sin } ③t\}$$

$$\uparrow$$
$$a$$

$$= \frac{3}{s^2 + 9}$$

$$\therefore L[\sin 3t] = \frac{3}{s^2 + 9} \checkmark \qquad \ldots\text{Ans.}$$

$$= \left[\frac{3}{s^2 + 9}\right] \quad s = 2$$

$$= \frac{3}{(2)^2 + 9} = \frac{3}{4 + 9} = \frac{3}{13}$$

$$\therefore \int_0^\infty e^{-2t} \sin 3t \, dt = \frac{3}{13} \checkmark \qquad \ldots\text{Ans.}$$

1.15 Laplace Transform by Series Method

We know, Maclaurin's series,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots + \frac{x^n}{n!} f^{n}(0) + \ldots$$

Also we have some standard series as :

(i) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

(ii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

(iii) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots$

We can use these series to find Laplace transform of functions. In this section we shall see how to use series to find Laplace transform of functions.

Ex. 1.15.1 : Find the Laplace transform of $\sin\sqrt{t}$

Soln. :

➔ Using standard series of function

$$\ldots \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\sin\sqrt{t} = (\sqrt{t}) - \frac{1}{3!}(\sqrt{t})^3 + \frac{1}{5!}(\sqrt{t})^5 - \frac{1}{7!}(\sqrt{t})^7 + \ldots.$$

...(by Expansion)

$$\sin\sqrt{t} = t^{1/2} - \frac{1}{3!} t^{3/2} + \frac{1}{5!} t^{5/2} - \frac{1}{7!} t^{7/2} + \ldots.$$

Taking Laplace transform of both sides,

$$L[\sin\sqrt{t}] = L\left[t^{1/2} - \frac{1}{3!} t^{3/2} + \frac{1}{5!} t^{5/2} - \frac{1}{7!} t^{7/2} + \ldots\right]$$

➔ Using linearity property of Laplace Transform

$$\ldots\left[L[a f(t) + b g(t) - c h(t)] = a L[f(t)] + b L[g(t)] - c L[h(t)]\right]$$

$$\therefore L[\sin\sqrt{t}] = L\left[t^{1/2}\right] - \frac{1}{3!} L\left[t^{3/2}\right] + \frac{1}{5!} L\left[t^{5/2}\right] - \frac{1}{7!} L\left[t^{7/2}\right] \ldots$$

➔ Using standard result of Laplace transform

$$\ldots\left[L(t^n) \frac{\Gamma(n+1)}{s^{n+1}} \text{ OR } \frac{n!}{s^{n+1}} \text{ if n is integer}\right]$$

$$= \frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}} - \frac{1}{3!} \frac{\Gamma\left(\frac{5}{2}\right)}{s^{5/2}} + \frac{1}{5!} \frac{\Gamma\left(\frac{7}{2}\right)}{s^{7/2}} - \frac{1}{7!} \frac{\Gamma\left(\frac{9}{2}\right)}{s^{9/2}} \ldots.$$

$$\left[\because L(t^n) \frac{\Gamma(n+1)}{s^{n+1}}\right]$$

➔ Using property of Gamma Function

$$\ldots\left\{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \text{ and } \Gamma(n+1) = n\Gamma(n)\right\}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{1}{3!} \frac{\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{s^{5/2}} + \frac{1}{5!} \frac{\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{s^{7/2}} - \frac{1}{7!} \frac{\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{s^{9/2}} + \ldots$$

Common out $\frac{\sqrt{\pi}}{2s^{3/2}}$ Throughout R.H.S., we get,

$$L[\sin\sqrt{t}] = \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{32s^2} - \frac{1}{384s^3} \ldots\right]$$

➔ Using standard series of function

$$\ldots\left[e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} \underbrace{\left[1 - \left(\frac{1}{4s}\right) + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \frac{1}{3!}\left(\frac{1}{4s}\right)^3 + \ldots.\right]}_{\text{Rearrange as a series of } e^{-x}}$$

$$\therefore L[\sin\sqrt{t}] = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}} \checkmark \qquad \ldots\text{Ans.}$$

Ex. 1.15.2 : Find Laplace transform of $\frac{\cos\sqrt{t}}{\sqrt{t}}$

Soln. :

➔ Using standard series of function

$$\ldots\left[\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right]$$

$$\therefore \cos\sqrt{t} = 1 - \frac{(\sqrt{t})^2}{2!} + \frac{(\sqrt{t})^4}{4!} - \frac{(\sqrt{t})^6}{6!} + \ldots.$$

Divide throughout by $\sqrt{t}$, we get,

$$\therefore \frac{\cos\sqrt{t}}{\sqrt{t}} = \frac{1}{\sqrt{t}} - \frac{\sqrt{t}}{2!} + \frac{(\sqrt{t})^3}{4!} - \frac{(\sqrt{t})^5}{6!} + \ldots$$

Taking Laplace transform of both sides,

$$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = L\left[t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} \ldots.\right]$$

➔ Using linearity property of Laplace Transform

$$\ldots\left[L[a f(t) + b g(t) - c h(t)] = a L[f(t)] + b L[g(t)] - c L[h(t)]\right]$$

$$= L\left[t^{-1/2}\right] - \frac{1}{2!} L\left[t^{\frac{1}{2}}\right] + \frac{1}{4!} L\left[t^{\frac{3}{2}}\right] - \frac{1}{6!} L\left[t^{\frac{3}{2}}\right] + \ldots.$$

➔ Using standard result of Laplace transform

$$\ldots\left\{L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} \text{ OR } \frac{n!}{s^{n+1}} \text{ if n is integer}\right\}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} - \frac{1}{2!} \frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}} + \frac{1}{4!} \frac{\Gamma\left(\frac{5}{2}\right)}{s^{5/2}} - \frac{1}{6!} \frac{\Gamma\left(\frac{7}{2}\right)}{s^{7/2}} \ldots.$$

$$\ldots\left\{\because L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}\right\}$$

➔ Using property of gamma function

$$\ldots\left[\Gamma(n+1) = n\Gamma(n) \text{ and } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{\sqrt{\pi}}{s^{1/2}} - \frac{1}{2!} \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} + \frac{1}{4!} \frac{\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{s^{5/2}} - \frac{1}{6!} \frac{\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{s^{7/2}} + \ldots$$

$$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \left[\frac{\sqrt{\pi}}{s^{1/2}}\right]\left[1 - \frac{1}{4s} + \frac{1}{32s^2} - \frac{1}{384s^3} \ldots.\right]$$

➔ Using standard series of function

$$\ldots\left[e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots\right]$$

$$= \frac{\sqrt{\pi}}{s^{1/2}} \underbrace{\left[1 - \left(\frac{1}{4s}\right) + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \frac{1}{3!}\left(\frac{1}{4s}\right)^3 \ldots.\right]}_{\text{Rearrange as series of } e^{-x}}$$

$$\therefore L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}} e^{-\frac{1}{4s}} \checkmark \qquad \ldots\text{Ans.}$$

Exercise

Find Laplace transform of

Ex. 1 : $t + t^2 + t^3$

Ans. : $\left[\frac{1}{s^2} + \frac{2}{s^3} + \frac{6}{s^4}\right]$

Ex. 2 : $\sin t \cos t$

Ans. : $\left[\frac{1}{s^2 + 4}\right]$

Ex. 3 : $e^{-t} \cos^2 t$

Ans. : $\left[\frac{1}{2s + 2} + \frac{s + 1}{2s^2 + 4s + 10}\right]$

Ex. 4 : $\sinh^3 t$

Ans. : $\left[\frac{6}{(s^2 - 1)(s^2 - 9)}\right]$

Ex. 5 : $\cosh at \sin at$

Ans. : $\left[\frac{a(s^2 + 2a^2)}{s^4 + 4a^4}\right]$

Ex. 6 :

$$f(t) = \begin{cases} t^2 & 0 < t < 2 \\ t - 1 & 2 < t < 3 \\ 7 & t > 3 \end{cases}$$

Ans. : $\left[\frac{2}{s^3} - \frac{e^{-2s}}{s^3}(2 + 3s + 3s^2) + \frac{e^{-3s}}{s^2}(5s - 1)\right]$

Ex. 7 : $t \cosh at$

Ans. : $\left[\frac{s^2 + a^2}{(s^2 - a^2)^2}\right]$

Ex. 8 : $t^2 \sin t$

Ans. : $\left[\frac{2(3s^2 - 1)}{(s^2 + 1)^3}\right]$

Ex. 9 : $t \sin^2 3t$

Ans. : $\left[\frac{1}{2}\left[\frac{1}{s^2} - \frac{s^2 - 36}{(s^2 + 36)^2}\right]\right]$

Ex. 10 : $t e^{at} \sin at$

Ans. : $\left[\frac{2a(s - a)}{(s^2 - 2as + 2a^2)^2}\right]$

Ex. 11 : $\int_0^t e^{-2t} t \sin^3 t \, dt$

Ans. : $\left[\frac{3(s + 2)}{2s}\left(\frac{1}{(s + 2)^2 + 9} - \frac{1}{[(s + 2)^2 + 1]^2}\right)\right]$

Ex. 12 : $t^2 e^{-2t} \cos t$

Ans. : $\left[\dfrac{2(s^3+\cos^2 t+25s+22)}{(s^2+4s+5)^3}\right]$

Ex. 13 : $t e^{-t} \cosh 2t$

Ans. : $\left[\dfrac{s^2+2s+5}{(s^2+2s-3)^2}\right]$

Ex. 14 : $\dfrac{\sin^2 t}{t}$

Ans. : $\left[\dfrac{1}{4}\log\left(\dfrac{s^2+4}{s^2}\right)\right]$

Ex. 15 : $\dfrac{\sinh t}{t}$

Ans. : $\left[\dfrac{-1}{2}\log\left(\dfrac{s-1}{s+1}\right)\right]$

Ex. 16 : $\dfrac{e^{-t}\sin t}{t}$

Ans. : $[\cot^{-1}(s+1)]$

Evaluate using Lapalce transform

Ex. 1 : $\displaystyle\int_0^t t e^{-2t} \sin t \, dt$

Ans. : $\left[\dfrac{4}{25}\right]$

Syllabus Topic : Transforms of some special functions

1.16 Laplace Transform of Special Functions

Following are the special functions which are the necessary to discuss :

1. Periodic Function
2. Unit Step Function
3. Dirac-delta Function (Unit Impulse Function)
4. Bessel Function

Syllabus Topic : Transforms of Periodic Function

1.17 Periodic Function

The Laplace transform of the periodic function f (t) with period T i.e. f (t + T) = f (t) is,

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-su} f(u)\, du$$

where T is period of the function f(t).

1.17.1 Examples on Periodic Function

Ex. 1.17.1 : Find the Laplace transform of periodic function defined as,

$$f(t) = \begin{cases} t & ; \quad 0 < t < \pi \\ \pi - t & ; \quad \pi < t < 2\pi \end{cases} \quad \text{and } f(t+2\pi) = f(t)$$

Soln. : We know,

Laplace transform of periodic function, f(t) defined in the interval (0, T) is,

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-su} f(u)\, du$$

Where T is period of the function

Given, $f(t) = \begin{cases} t & 0 < t < \pi \\ \pi - t & \pi < t < 2\pi \end{cases}$ and $f(t+2\pi) = f(t)$

i.e. $T = 2\pi$

$$\therefore L[f(t)] = \frac{1}{1-e^{-s(2\pi)}} \int_0^{2\pi} e^{-su} f(u)\, du$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \int_0^{\pi} e^{-su} u\, du + \int_{\pi}^{2\pi} e^{-su} (\pi - u)\, du \right\}$$

➔ **Using product rule of integration**

$$\cdots \left[\begin{array}{l} \int uv\, dx = u v_1 - u' v_2 + u'' v_3 - u''' v_4 + \ldots \\ \text{where dashes mean derivative and} \\ \text{suffix mean integration} \end{array} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \left[(u)\left(\frac{e^{-su}}{-s}\right) - (1)\left(\frac{e^{-su}}{s^2}\right) \right]_0^{\pi} + \left[(\pi - u)\left(\frac{e^{-su}}{-s}\right) - (-1)\left(\frac{e^{-su}}{s^2}\right) \right]_{\pi}^{2\pi} \right\}$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \left[-\frac{\pi e^{-s\pi}}{s} - \frac{e^{-s\pi}}{s^2} \right] - \left[0 - \frac{1}{s^2} \right] + \left[\frac{\pi e^{-2\pi s}}{s} + \frac{e^{-2\pi s}}{s^2} \right] - \left[0 + \frac{e^{-s\pi}}{s^2} \right] \right\}$$

$$\therefore L[f(t)] = \frac{1}{1-e^{-2\pi s}} \left\{ -\frac{\pi e^{-s\pi}}{s} - \frac{e^{-s\pi}}{s^2} + \frac{1}{s^2} + \frac{\pi e^{-2\pi s}}{s} + \frac{e^{-2\pi s}}{s^2} - \frac{e^{-s\pi}}{s^2} \right\}$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ e^{-s\pi}\left(-\frac{\pi}{s} - \frac{1}{s^2}\right) + \frac{\pi e^{-\pi s}}{s} + \frac{e^{-\pi s}}{s^2} - \frac{1}{s^2} + \frac{1}{s^2} \right\}$$

...(by simplification)

$$= \frac{e^{-s\pi}}{1-e^{-2\pi s}} \left[-\frac{\pi}{s}\left(1-e^{-\pi s}\right) - \frac{1}{s^2}\left(1-e^{-\pi s}\right) - \frac{1}{s^2} \right] + \frac{1}{s^2\left(1-e^{-2\pi s}\right)}$$

$$= \frac{e^{-s\pi}}{1-e^{-2\pi s}} \left(1-e^{-\pi s}\right)\left(\frac{-\pi}{s} - \frac{1}{s^2}\right) - \frac{e^{-s\pi}}{s^2\left(1-e^{-2\pi s}\right)} + \frac{1}{s^2\left(1-e^{-2\pi s}\right)}$$

$$= \frac{-e^{-s\pi}\left(1-e^{-\pi s}\right)}{\left(1-e^{-2\pi s}\right)} \cdot \left(\frac{\pi s + 1}{s^2}\right) + \frac{1}{s^2\left(1-e^{-2\pi s}\right)}\left(1-e^{-s\pi}\right)$$

$$= \frac{\left(1-e^{-\pi s}\right)\left[1-e^{-s\pi}(\pi s+1)\right]}{s^2\left(1-e^{-2\pi s}\right)}$$

$$= \frac{\left(1-e^{-\pi s}\right)\left(1-e^{-s\pi}(\pi s+1)\right)}{s^2\left(1-e^{-\pi s}\right)\left(1+e^{-\pi s}\right)}$$

$$\therefore L[f(t)] = \frac{1-e^{-\pi s}(\pi s+1)}{s^2\left(1+e^{-\pi s}\right)} \checkmark \qquad \text{...Ans.}$$

Ex. 1.17.2 : Find the Laplace transform of the period function,

$$f(t) = \begin{cases} 0 & ; \quad 0 < t < \dfrac{\pi}{\omega} \\ -\sin \omega t & ; \quad \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega} \end{cases}$$

and $f(t) = f\left(t + \dfrac{2\pi}{\omega}\right)$

Soln. :

Given, $f(t) = \begin{cases} 0 & 0 < t < \dfrac{\pi}{\omega} \\ -\sin \omega t & \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega} \end{cases}$

and $f(t) = f\left(t + \dfrac{2\pi}{\omega}\right)$

➔ **Using Laplace transform of periodic function**

$$\cdots \left[L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-su} f(u)\, du \right]$$

T-is period of the function.

$$\therefore L[f(t)] = \frac{1}{1-e^{-s\left(\frac{2\pi}{\omega}\right)}} \left[\int_0^{\pi/\omega} 0 + \int_{\pi/\omega}^{2\pi/\omega} e^{-su}(-\sin \omega u)\, du \right]$$

$$= \frac{-1}{1-e^{-2\pi s/\omega}} \int_{\pi/\omega}^{2\pi/\omega} e^{-su} \sin \omega u\, du$$

➔ **Using standard formula of integram**

$$\cdots \left[\int e^{ax} \sin(bx+c)\, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin(bx+c) - b\cos(bx+c)\right] \right]$$

$$= \frac{-1}{1-e^{-2\pi s/\omega}} \left[\frac{e^{-su}}{s^2+\omega^2}\left[-s \sin \omega u - \omega \cos \omega u\right] \right]_{\pi/\omega}^{2\pi/\omega}$$

$$= \frac{-1}{1-e^{-2\pi s/\omega}} \left\{ \frac{e^{-2\pi s/\omega}}{s^2+\omega^2}[0-\omega] - \frac{e^{-\pi s/\omega}}{s^2+\omega^2}[0-\omega(-1)] \right\}$$

$$= \frac{\omega}{1-e^{-2\pi s/\omega}} \cdot \frac{e^{-\pi s/\omega}}{s^2+\omega^2}\left(1+e^{-\pi s/\omega}\right)$$

$$= \frac{\omega e^{-\frac{\pi s}{\omega}}\left(1+e^{-\frac{\pi s}{\omega}}\right)}{(s^2+\omega^2)\left[(1)^2-(e^{-\frac{\pi s}{\omega}})^2\right]}$$

$$\therefore L\{f(t)\} = \frac{\omega e^{-\pi s/\omega}\cancel{(1+e^{\pi s/\omega})}}{(s^2+\omega^2)(1-e^{-\pi s/\omega})\cancel{(1+e^{-\pi s/\omega})}}$$

$\because (a^2-b^2) = (a-b)(a+b)$

$$\therefore L\{f(t)\} = \frac{\omega e^{-\pi s/\omega}}{(s^2+\omega^2)(1-e^{-\pi s/\omega})} \checkmark \qquad \text{...Ans.}$$

Ex. 1.17.3 : Find the Laplace transform of

$$f(t) = \begin{cases} \sin \omega t & ; \quad 0 < t < \dfrac{\pi}{\omega} \\ 0 & ; \quad \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega} \end{cases}$$

and $f(t) = f\left(t + \dfrac{2\pi}{\omega}\right)$

Soln. :

Given, $f(t) = \begin{cases} \sin \omega t & 0 < t < \dfrac{\pi}{\omega} \\ 0 & \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega} \end{cases}$

and $f(t) = f\left(t + \dfrac{2\pi}{\omega}\right)$

This is periodic function with period $T = \dfrac{2\pi}{\omega}$

➔ **Using Laplace transform of periodic function,**

$$\cdots \left[L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-su} f(u)\, du \right]$$

$$\therefore L[f(t)] = \frac{1}{1-e^{-s\left(\frac{2\pi}{\omega}\right)}}\left\{\int_0^{\pi/\omega} e^{-su}(\sin\omega u)\,du + \int_{\pi/\omega}^{2\pi/\omega} 0\right\}$$

➔ **Using standard integration formula**

$$\ldots\left\{\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right]\right\}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left\{\frac{e^{-su}}{s^2+\omega^2}\left[-s\sin\omega u - \omega\cos\omega u\right]\right\}_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left\{\frac{e^{-s\pi/\omega}}{s^2+\omega^2}[0-\omega(-1)] - \frac{1}{s^2+\omega^2}[-\omega]\right\}$$

$$= \frac{1}{(1-e^{-2\pi s/\omega})}\,\frac{\omega}{s^2+\omega^2}(1+e^{-s\pi/\omega})$$

$$= \frac{\omega}{s^2+\omega^2}\,\frac{(1+e^{-s\pi/\omega})}{(1+e^{-s\pi/\omega})(1-e^{-s\pi/\omega})}$$

$$\therefore L[f(t)] = \frac{\omega}{(s^2+\omega^2)(1-e^{-s\pi/\omega})} \checkmark \quad \ldots\text{Ans.}$$

Ex. 1.17.4 : Find the Laplace transform of

$$f(t) = \begin{cases} 1 & 0<t<a \\ -1 & a<t<2a \end{cases}$$

and $f(t) = f(t+2a)$

Soln. :

Given, $f(t) = \begin{cases} 1 & 0<t<a \\ -1 & a<t<2a \end{cases}$

and $f(t) = f(t+2a)$

This is periodic function with period $T = 2a$

We have,

➔ **Using Laplace transform of periodic function,**

$$\ldots\left[L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-su} f(u)\,du\right]$$

$$\therefore L[f(t)] = \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-su}(1)\,du + \int_a^{2a} e^{-su}(-1)\,du\right]$$

$$= \frac{1}{1-e^{-2as}}\left\{\left[\frac{e^{-su}}{-s}\right]_0^a - \left[\frac{e^{-su}}{-s}\right]_a^{2a}\right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{\left[\frac{e^{-as}}{-s} - \frac{1}{-s}\right] - \left[\frac{e^{-2as}}{-s} - \frac{e^{-as}}{-s}\right]\right\}$$

$$= \frac{1}{(-s)(1-e^{-2as})}\left[e^{-as} - 1 - e^{-2as} + e^{-as}\right]$$

$$= \frac{-e^{-2as} + 2e^{-as} - 1}{(-s)(1-e^{-2as})} = \frac{1-2e^{-as}+e^{-2as}}{s(1-e^{-2as})}$$

$$= \frac{(1-e^{-as})^2}{s(1-e^{-as})(1+e^{-as})} = \frac{(1-e^{-as})}{s(1+e^{-as})}$$

$$\therefore L[f(t)] = \frac{(1-e^{-as})}{s(1+e^{-as})} \checkmark \quad \ldots\text{Ans.}$$

Exercise

Ex. 1 : Find the Laplace transform of

$$f(t) = \begin{cases} \sin t & ; \quad 0<t<\pi \\ 0 & ; \quad \pi<t<2\pi \end{cases} \quad \text{and} \quad f(t+2\pi) = f(t)$$

Ans. : $L[f(t)] = \dfrac{1}{(s^2+1)(1-e^{-\pi s})}$

Ex. 2 : Saw toothed wave function defined as

$f(t) = \dfrac{kt}{T}$ where $0<t<T$ and $f(t) = f(t+T)$.

Find Laplace transform of $f(t)$.

Ans. : $L[f(t)] = k\left\{\dfrac{1}{Ts^2} - \dfrac{e^{-sT}}{s(1-e^{-sT})}\right\}$

Ex. 3 : Find the Laplace transform of the waveform

$f(t) = \dfrac{2t}{3},\ 0 \le t \le 3$

Ans. : $\dfrac{2e^{-3s}}{-s(1-e^{-3s})} + \dfrac{2}{3s^2}$

Ex. 4 : Obtain Laplace transform of the periodic function

$f(t) = e^t$ for $0<t<2\pi$

Ans. : $\dfrac{e^{-2\pi(s-1)}-1}{(1-s)(1-e^{-2\pi s})}$

Ex. 5 : $f(t) = \begin{cases} \cos\omega t & \text{for } 0<t<\dfrac{\pi}{\omega} \\ 0 & \text{for } \dfrac{\pi}{\omega}<t<\dfrac{2\pi}{\omega} \end{cases}$

Ans. : $\dfrac{s}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$

Syllabus Topic : Heaviside-Unit Step Function

1.18 Unit Step Function (Heviside Unit Step Function)

By using unit step function, we can find inverse transform of functions which can not be determined with previous method.

Unit step function is defined as, $U(t) = \begin{cases} 0 & t<0 \\ 1 & t \ge 0 \end{cases}$

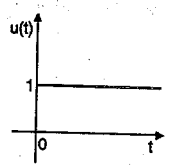

Fig. 1.18.1

Also Displaced Unit Step Function:

$$U(t-a) = \begin{cases} 0, & t<a \\ 1, & t \ge a \end{cases}$$

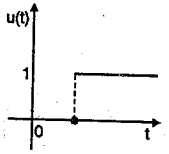

Fig. 1.18.2

Note :

1. $f(t-a)\,U(t-b) = \begin{cases} 0 & t<b \\ f(t-a) & t \ge b \end{cases}$

2. $f(t)\,[U(t-a) - U(t-b)] = \begin{cases} 0 & t<a \\ f(t) & a \le t \le b \\ 0 & t>b \end{cases}$

1.18.1 Properties of Laplace Transform of Unit Step Function

(i) $L[U(t)] = \dfrac{1}{s}$

Proof

We know, definition of Laplace transform,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

(i) Since $U(t) = \begin{cases} 0 & ; \quad t<0 \\ 1 & ; \quad t \ge 0 \end{cases}$

$$L[U(t)] = \int_0^\infty e^{-st}U(t)\,dt = \int_0^\infty e^{-st}(1)\,dt$$

$$= \int_0^\infty e^{-st}\,dt$$

➔ **Using standard result of integration**

$$\ldots\left[\int e^{at}\,dt = \frac{e^{at}}{a}\right]$$

$$L[U(t)] = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{1}{s}\left[e^{-st}\right]_0^\infty$$

➔ **Using linearity property of Laplace Transform**

$$\ldots\left[L[a\,f(t) + bg(t) - c\,h(t)] = a\,L[f(t)] + b\,L[g(t)] - cL[h(t)]\right]$$

$$L[U(t)] = -\frac{1}{s}\left[e^{-\infty} - e^0\right]$$

$$= -\frac{1}{s}[0-1] \quad (\because e^{-\infty} = 0,\ e^0 = 1)$$

$$L[U(t)] = \frac{1}{s}$$

(ii) $L[U(t-a)] = \dfrac{e^{-as}}{s}$

Proof

Since by definition,

$$U(t-a) = \begin{cases} 0 & ; \quad t<a \\ 1 & ; \quad t \ge a \end{cases}$$

$$\therefore L[U(t-a)] = \int_0^\infty e^{-st}\cdot U(t-a)\,dt$$

$$\int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}(1)\,dt$$

$$= 0 + \int_a^\infty e^{-st}\,dt$$

$$= \int_a^\infty e^{-st}\,dt$$

➔ **Using standard result of integration :** $\ldots\left[\int e^{at}\,dt = \dfrac{e^{at}}{a}\right]$

$$L[U(t-a)] = \left[\frac{e^{-st}}{-s}\right]_a^\infty = -\frac{1}{s}\left[e^{-st}\right]_0^\infty$$

➔ **Using linearity property of Laplace Transform**

$$\ldots\left[L[a\,f(t) + bg(t) - c\,h(t)] = a\,L[f(t)] + b\,L[g(t)] - cL[h(t)]\right]$$

$$L[U(t-a)] = -\frac{1}{s}\left[e^{-\infty} - e^{-as}\right]$$

$$= -\frac{1}{s}[0 - e^{-as}] = \frac{1}{s}e^{-as} \quad (\because e^{-\infty} = 0)$$

$$L[U(t-a)] = \frac{1}{s}e^{-as}$$

1.18.2 Second Shifting Theorem

If $L[f(t)] = f(s)$ then $L[f(t-a)\,U(t-a)] = e^{-as} f(s)$

Proof :

We know, definition of Laplace transform,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore L[f(t-a)\cdot U(t-a)] = \int_0^\infty e^{-st} f(t-a)\,U(t-a)\,dt$$

Since,

$$U(t-a) = \begin{cases} 0 \;; & t < a \\ 1 \;; & t \ge a \end{cases}$$

$$\therefore L[f(t-a)\,U(t-a)] = \int_0^a e^{-st} f(t-a)\cdot \underbrace{U(t-a)}_{0}\,dt + \int_0^\infty e^{-st} f(t-a)\cdot \underbrace{U(t-a)}_{1}$$

$$= \int_0^a 0 + \int_a^\infty e^{-st} f(t-a)\,(1)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

Put $t-a = u$

t	a	∞
u	0	∞

$\therefore dt = du$

$$\therefore L[f(t-a)\,U(t-a)] = \int_0^\infty e^{-s(u+a)} f(u)\,du$$

$$= \int_0^\infty e^{-su}\, e^{-as} f(u)\,du$$

$$= e^{-as} \int_0^\infty e^{-su} f(u)\,du$$

By definition of laplace transform

$$= e^{-as} L[f(t)]$$

$L[f(t-a)\,U(t-a)] = e^{-as} f(s)$ ✓ **Hence proved**

Note :

1. $L\{U(t)\} = \frac{1}{s}$
2. $L\{U(t-a)\} = \frac{1}{s} e^{-as}$
3. $L\{f(t-a)\,U(t-a)\} = e^{-as} f(s)$
4. $L\{t(t)\,U(t)\} = f(s)$
5. $L\{f(t)\,U(t-a)\} = e^{-as} L\{f(t+a)\}$

   (This formula student can use to find LT of any function containing Unit step function.)
6. If $f(t) = \begin{cases} f_1(t) & 0 < t < a \\ f_2(t) & a < t < b \\ f_3(t) & b < t < c \\ f_4(t) & t \ge c \end{cases}$

   Then we can express f (t) in Heviside Unit step function as,

   $f(t) = f_1(t)\,U(t) + [f_2(t) - f_1(t)]\,U(t-a) + [f_3(t) - f_2(t)]\,U(t-b)$
7. $L^{-1}[e^{-as} f(s)] = U(t-a)\{L^{-1}(f(s))\}_{t\to t-a}$

1.18.3 Examples on Unit Step Function (Heviside Unit Step Function)

Ex. 1.18.1 : Find Laplace transform of $f(t) = (t-\pi)\;U(t-\pi)$.

Soln. :

➔ **Using property of unit step function**

$$\ldots\left[L[f(t-a)\,U(t-a)] = e^{-as} f(s)\right]$$

$$\therefore L[(t-\pi)\,U(t-\pi)] = e^{-\pi s}\,L[t]$$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} \text{ OR } \frac{n!}{s^{n+1}} \text{ if n is integer}\right]$$

$$= \frac{e^{-\pi s}}{s^2}$$

$$\therefore L[(t-\pi)\,U(t-\pi)] = \frac{e^{-\pi s}}{s^2} ✓ \qquad \ldots\text{Ans.}$$

Ex. 1.18.2 : Find Laplace transform of $f(t) = (t-1)^2\,U(t-1)$

Soln. :

➔ **Using standard result of Laplace transform**

$$\ldots\left[L(t^n) \frac{\Gamma(n+1)}{s^{n+1}} \text{ OR } \frac{n!}{s^{n+1}} \text{ if n is integer}\right]$$

$$L[t^2] = \frac{\Gamma(2+1)}{s^{n+1}}\;\frac{\Gamma(3)}{s^3} \qquad \ldots(1)$$

➔ **By property of Gamma function**

$$\ldots\left[\Gamma(n+1) = n! \text{ if n is integer}\right]$$

$$L[t^2] = \frac{2!}{s^3} = \frac{2}{s^3}$$

➔ **Using property of unit step function**

$$\ldots\left[L[f(t)\,U(t-a)] = e^{-as} L[f(t+a)]\right]$$

$$(\because a = 1 \text{ and } t \to t+1)$$

$$\therefore L[(t-1)^2\,U(t-1)] = e^{-s} L[(t+1-1)^2]$$

$$= e^{-s} L[t^2]$$

Using value from Equation (1), we get,

$$= e^{-s}\frac{2}{s^3} = \frac{2e^{-s}}{s^3}$$

$$\therefore L[(t-1)^2\,U(t-1)] = \frac{2e^{-s}}{s^3} ✓ \qquad \ldots\text{Ans.}$$

Ex. 1.18.3 : Find Laplace transform of $f(t) = e^{-t}\sin t\,U(t-\pi)$

Soln. :

Step I :

➔ **Using property of unit step function**

$$\ldots\left[L[f(t)\,U(t-a)] = e^{-as} L[f(t+a)]\right]$$

$$\therefore L[\sin t\cdot U(t-\pi)] = e^{-\pi s}\,L[\sin(t+\pi)]$$

$$(\because a = \pi \text{ and } t \to t+\pi)$$

➔ **Using trigonometric transformation formula :**

$$\ldots[\sin(\pi+\theta) = -\sin\theta]$$

$$= e^{-\pi s} L[-\sin t]$$

➔ **Using standard result of Laplace transform**

$$\ldots\left\{L(\sin at) = \frac{a}{s^2+a^2}\right\}$$

$$= -e^{-\pi s}\frac{1}{s^2+1}$$

$$\therefore L[\sin t\cdot U(t-\pi)] = -\frac{e^{-\pi s}}{s^2+1}$$

Step II :

➔ **Using first shifting property of Laplace transform**

$$\ldots\left[L[e^{at} f(t)] = [L[f(t)]]_{s\to s-a} = [f(s)]_{s\to s-a}\right]$$

(s → s – a means every s replace by s – a)

$$L[e^{-t}\sin t\cdot U(t-\pi)] = \left[-\frac{e^{-\pi s}}{s^2+1}\right]_{s\to s+1}$$

$$L[e^{-t}\sin t\cdot U(t-\pi)] = -\frac{e^{-\pi(s+1)}}{(s+1)^2+1} ✓ \qquad \ldots\text{Ans.}$$

Ex. 1.18.4 : Find Laplace transform of $f(t) = e^{-3t}\,U(t-2)$

Soln. :

➔ **Using property of unit step function**

$$\ldots\left[L[f(t)\,U(t-a)] = e^{-as} L[f(t+a)]\right]$$

$$\therefore L[e^{-3t}U(t-2)] = e^{-2s} L\left[e^{-3(t+2)}\right]$$

$$(\because a = 2 \text{ and } t \to t+2)$$

$$= e^{-2s} L\left[e^{-3t}\cdot e^{-6}\right]$$

$$= e^{-2s}\cdot e^{-6} L[e^{-3t}]$$

➔ **Using Laplace transform of standard functions**

$$\ldots\left[L[e^{at}] = \frac{1}{s-a}\right]$$

$$= e^{-2s}\cdot e^{-6}\,\frac{1}{s+3} = \frac{e^{-2(s+3)}}{s+3}$$

$$\therefore L[e^{-3t}U(t-2)] = \frac{e^{-2(s+3)}}{s+3} ✓ \qquad \ldots\text{Ans.}$$

Ex. 1.18.5 : Find Laplace transform of $f(t) = e^{t-2}\cdot U(t-2)$.

Soln. :

➔ **Using property of unit step function**

$$\ldots\left[L[f(t)\,U(t-a)] = e^{-as} L[f(t+a)]\right]$$

$$\therefore L[e^{t-2}U(t-2)] = e^{-2s} L\left[e^{(t+2)-2}\right]$$

$$(\because a = 2 \text{ and } t \to t+2)$$

$$= e^{-2s} L[e^t]$$

➔ **Using Laplace transform of standard functions**

$$\ldots\left[L[e^{at}] = \frac{1}{s-a}\right]$$

$$= e^{-2s}\,\frac{1}{s-1}$$

$$\therefore L[e^{t-2}U(t-2)] = \frac{e^{-2s}}{s-1} ✓ \qquad \ldots\text{Ans.}$$

Syllabus Topic : Dirac delta function

1.19 Dirac-delta Function (Unit Impulse Function)

In applied mechanics, Impulse function is defined as it is a product of the force and time when a large force acts for a short time. The unit impulse function (or dirac delta) is the limiting function.

The dirac-delta function is defined as,

$$\delta(t-a) = \begin{cases} 0, & t < a \\ \frac{1}{\varepsilon}, & a \le t \le a+\varepsilon \quad \text{where } \varepsilon > 0 \\ 0, & t > a+\varepsilon \end{cases}$$

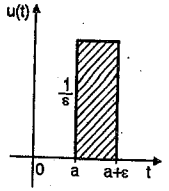

Fig. 1.19.1

Note :

1. $L\{\delta(t-a)\} = e^{-as}$
2. $L\{\delta(t)\} = 1$
3. $L\{f(t)\,\delta(t-a)\} = e^{-as} f(a)$
4. $L\{f(t)\,\delta(t)\} = f(0)$
5. $\int_{-\infty}^{\infty} L\{f(t)\,\delta(t-a)\} = f(a)$
6. $\int_{-\infty}^{\infty} L\{f(t)\,\delta(t)\} = f(0)$
7. $\delta(t-a) = \frac{d}{dt} U(t-a) = U'(t-a)$

1.19.1 Examples : Using Unit Step Function / Dirac-delta Function

Ex. 1.19.1 : Solve : $L[t\,U(t-4) - t^3\,\delta(t-2)]$

Soln. :

➔ **Using linearity property of Laplace Transform**

$$\ldots\left[L[a\,f(t) + b\,g(t) - c\,h(t)] = a\,L[f(t)] + b\,L[g(t)] - c\,L[h(t)]\right]$$

$$L[t\,U(t-4) - t^3\,\delta(t-2)] = L[t\,U(t-4)] - L[t^3\,\delta(t-2)]$$

➔ **Using property of unit step function and Dirac-delta function**

$$\ldots\left[L[f(t)\,U(t-a)] = e^{-as} L[f(t+a)] \text{ and } L[f(t)\,\delta(t-a)] = e^{-as} f(a)\right]$$

$$L[t\,U(t-4) - t^3\delta(t-2)] = e^{-4s} L[t+4] - e^{-2s}(2)^3$$

$$= e^{-4s}\{L(t) + 4\,L(1)\} - e^{-2s}\,8$$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L[t^n] = \frac{\sqrt{n+1}}{s^{n+1}} \text{ OR } \frac{n!}{s^{n+1}} \text{ if n is integer}\right]$$

$$L[t\,U(t-4)] - t^3\delta(t-2)] = e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right) - 8\,e^{-2s} \checkmark \quad \ldots\text{Ans.}$$

(by standard L.T.)

Ex. 1.19.2 : Find the laplace transform of $t^3\,\delta(t-4)$

Soln. :

➔ **Using property of direct delta function**

$$\ldots[L[f(t)\cdot\delta(t-a)] = e^{-as} f(a)]$$

$$\therefore L[t^3\cdot\delta(t-4)] = L[\underset{\underset{f(t)}{\uparrow}}{t^3}\cdot\delta(\underset{\underset{a}{\uparrow}}{t-4})]$$

$$= e^{-4s}(4)^3 = e^{-4s}\,64$$

$$L[t^3\cdot\delta(t-4)] = 64\,e^{-4s} \checkmark \quad \ldots\text{Ans.}$$

Ex. 1.19.3 : Find Laplace transform of $f(t) = \sin t\cdot\delta(t-\pi)$.

Soln. :

➔ **Using property of dirac-delta function**

$$\ldots[L[f(t)\,\delta(t-a)] = e^{-as} f(a)]$$

$$\therefore L[\sin t\,\delta(t-\pi)] = e^{-\pi s}\sin\pi$$

➔ **Using standard trigonometric value :** $\ldots[\sin\pi = 0]$

$$= e^{-\pi s}(0)$$

$$= 0$$

$$\therefore L[\sin t\,\delta(t-\pi)] = 0 \checkmark \quad \ldots\text{Ans.}$$

Ex. 1.19.4 : Find Laplace transform of $t\,e^{-2t}\cdot\delta(t-2)$

Soln. :

We know, property of dirac-delta function,

$$L[f(t)\cdot\delta(t-a)] = e^{-as} f(a)$$

Given, $f(t)\cdot\delta(t-a) = t\,e^{-2t}\cdot\delta(t-2)$

Here $f(t) = t\,e^{-2t}$

and $a = 2$

$$\therefore L[t\,e^{-2t}\cdot\delta(t-2)] = e^{-2s} f(2)$$

$$= e^{-2s}[t\,e^{-2t}]_{t=2} \quad \because f(t) = t\,e^{-2t}$$

$$= e^{-2s}[2\cdot e^{-2(2)}]$$

$$= e^{-2s}\cdot 2\,e^{-4}$$

$$L[t\,e^{-2t}\,\delta(t-2)] = 2\,e^{-2s-4} \checkmark \quad \ldots\text{Ans.}$$

Ex. 1.19.5 : Find Laplace transform of $\sin 2t\cdot\delta\left(t-\frac{\pi}{4}\right)$

Soln. :

We know property of dirac-delta function,

$$L[f(t)\cdot\delta(t-a)] = e^{-as} f(a)$$

Given, $f(t)\cdot\delta(t-a) = \sin 2t\cdot\delta\left(t-\frac{\pi}{4}\right)$

Here $f(t) = \sin 2t$

and $a = \frac{\pi}{4}$

$$\therefore L\left[\sin 2t\cdot\delta\left(t-\frac{\pi}{4}\right)\right] = e^{-\frac{\pi}{4}s}\cdot f\left(\frac{\pi}{4}\right)$$

$$= e^{-\frac{\pi s}{4}}[\sin 2t]_{t=\frac{\pi}{4}} \quad (\because f(t) = \sin 2t)$$

$$= e^{-\frac{\pi s}{4}}\left[\sin\left(2\cdot\frac{\pi}{4}\right)\right] = e^{-\frac{\pi s}{4}}\cdot\left[\sin\frac{\pi}{2}\right]$$

➔ **Using standard trigonometric value :** $\ldots\left[\sin\frac{\pi}{2} = 1\right]$

$$= e^{-\frac{\pi s}{4}}(1) = e^{-\frac{\pi s}{4}}$$

$$L\left[\sin 2t\cdot\delta\left(t-\frac{\pi}{4}\right)\right] = e^{-\frac{\pi s}{4}} \checkmark \quad \ldots\text{Ans.}$$

Ex. 1.19.6 : Evaluate $\int_{-\infty}^{\infty} e^{-3t}\,\delta(t-4)\,dt$

Soln. :

We know, property of dirac-delta function,

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = f(a)$$

Given, $\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = \int_{-\infty}^{\infty} e^{-3t}\,\delta(t-4)\,dt$

Here $f(t) = e^{-3t}$

and $a = 4$

$$\therefore \int_{-\infty}^{\infty} e^{-3t}\,\delta(t-4)\,dt = f(4)$$

$$= [e^{-3t}]_{a=4} \quad [\because f(t) = e^{-3t}]$$

$$= e^{-3(4)} = e^{-12}$$

$$\therefore \int_{-\infty}^{\infty} e^{-3t}\,\delta(t-4)\,dt = e^{-12} \checkmark \quad \ldots\text{Ans.}$$

1.20 Bessel Function

(i) $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2 4^2 6^2}\ldots\ldots$

(ii) $L[J_0(x)] = \frac{1}{\sqrt{s^2+1}}$

(iii) $J_1(x) = -\frac{d}{dx}[J_0(x)]$

1.20.1 Examples on Based Function

Ex. 1.20.1 : If $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$ then find $\int_0^{\infty} J_0(t)\,dt$.

Soln. :

Given, $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$

➔ **Using definition of Laplace transform,**

$$\ldots\left[L[f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt\right]$$

$$\int_0^{\infty} J_0(t)\,dt = \int_0^{\infty} e^{-st} J_0(t)\,dt \quad (\text{here } s = 0)$$

$$= L[J_0(t)] \quad (\text{here } s = 0) \quad \ldots\text{(by definition)}$$

$$= \frac{1}{\sqrt{s^2+1}} \quad (\text{here } s = 0)$$

$$\therefore \int_0^{\infty} J_0(t)\,dt = 1 \checkmark \quad \ldots\text{Ans.}$$

Ex. 1.20.2 : If $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$ then

$$\int_0^{\infty} e^{-t} J_0(t)\,dt = _____.$$

Soln. :

Given : $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$

➔ **Using definition of Laplace transform**

$$\ldots\left[L[f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt\right]$$

$$\therefore \int_0^{\infty} e^{-t} J_0(t)\,dt = \int_0^{\infty} e^{-1t} J_0(t)\,dt = \int_0^{\infty} e^{-st} J_0(t)\,dt \quad (\text{Here } s = 1)$$

$$= L[J_0(t)] \quad (\text{Here } s = 1)$$

$$\ldots\text{(by definition)}$$

$$= \frac{1}{\sqrt{s^2+1}} \quad (\text{Here } s = 1)$$

$$\therefore \int_0^{\infty} e^{-t} J_0(t)\,dt = \frac{1}{\sqrt{2}} \checkmark \quad \ldots\text{Ans.}$$

Ex. 1.20.3 : If $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$ then $L[J_0(at)] = _____$.

Soln. :

Given : $L[J_0(at)] = \frac{1}{\sqrt{s^2+1}}$

➔ **Using change of scale property of Laplace transform**

$$\ldots\left[L[f(at)] = \frac{1}{a} f\left(\frac{s}{a}\right) \text{ where } L[f(t)] = f(s)\right]$$

$$\left[f\left(\frac{s}{a}\right) \text{ mean } [f(s)]_{s\to\frac{s}{a}}\right].$$

$$\therefore L[J_0(at)] = \frac{1}{a}\frac{1}{\sqrt{\left(\frac{s}{a}\right)^2+1}}$$

$$\therefore L[J_0(at)] = \frac{1}{\sqrt{s^2+a^2}} \checkmark \quad \ldots\text{Ans.}$$

Ex. 1.20.4 : If $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$ then

$$L[t\,J_0(at)] = _____.$$

Soln. :

Given : $L[J_0(t)] = \dfrac{1}{\sqrt{s^2+1}}$

➔ **Using change of scale property of Laplace transform**

$$\ldots\left[L[f(at)] = \frac{1}{a} f\left(\frac{s}{a}\right) \text{ where } L[f(t)] = f(s)\right]$$

$$\ldots\left[f\left(\frac{s}{a}\right) \text{ mean } [f(s)]_{s \to \frac{s}{a}}\right]$$

$$\therefore\ L[J_0(at)] = \frac{1}{a}\frac{1}{\sqrt{\left(\frac{s}{a}\right)^2+1}}$$

$$= \frac{1}{\sqrt{s^2+a^2}}$$

➔ **Using property of Laplace transform**

$$\ldots\left[L[t\,f(t)] = -\frac{d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore\ L[tJ_0(at)] = \frac{-d}{ds}\left[\frac{1}{\sqrt{s^2+a^2}}\right]$$

➔ **Using quotient rule of derivative**

$$\ldots\left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$\therefore\ L[t\,J_0(at)] = \frac{s}{(s^2+a^2)^{3/2}} \checkmark \qquad \ldots\text{Ans.}$$

Ex. 1.20.5 : If $L[J_0(t)] = \dfrac{1}{\sqrt{s^2+1}}$ then

$L[e^{-at}J_0(at)] =$ _____.

Soln. :

Given : $L[J_0(t)] = \dfrac{1}{\sqrt{s^2+1}}$

➔ **Using change of scale property of Laplace transform**

$$\ldots\left[L[f(at)] = \frac{1}{a} f\left(\frac{s}{a}\right) \text{ where } L[f(t)] = f(s)\right]$$

$$\ldots\left[f\left(\frac{s}{a}\right) \text{ mean } [f(s)]_{s \to \frac{s}{a}}\right]$$

$$L[J_0(at)] = \frac{1}{a}\frac{1}{\sqrt{\left(\frac{s}{a}\right)^2+1}}$$

$$= \frac{a}{\sqrt{s^2+a^2}}$$

➔ **Using first shifting property of Laplace transform**

$L[(e^{at} f(t)) = [L(f(t))]_{s \to s-a}$ where $L[f(t)] = f(s)$ and

... $s \to s-a$ means every s replace by $(s-a)$

$$\therefore\ L[e^{-at}J_0(at)] = \frac{1}{\sqrt{(s+a)^2+a^2}}$$

$$\therefore\ L[e^{-at}J_0(at)] = \frac{1}{\sqrt{s^2+2as+2a^2}} \checkmark \qquad \ldots\text{Ans.}$$

Chapter Ends...

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UNIT 1

Laplace Transform

Syllabus

Definition – conditions for existence ; Transforms of elementary functions ; Properties of Laplace transforms - Linearity property, first shifting property, second shifting property, transforms of functions multiplied by t^n, scale change property, transforms of functions divided by t, transforms of integral of functions, transforms of derivatives ; Evaluation of integrals by using Laplace transform ; Transforms of some special functions- periodic function, Heaviside-unit step function, Dirac delta function.

➢ Chapter 1 : Laplace Transform

CHAPTER 1

UNIT I

Laplace Transform

Multiple Choice Questions

Q. 1 If f(t) is a function of t, (t > 0) then Laplace transform of the function f(t) is _____

(a) $\int_0^\infty e^{-st} f(t)\,dt$ (b) $\int_0^\infty e^{-t} t^{n-1}\,dt$

(c) $\int_0^\infty e^{-st} f'(t)\,dt$ (d) $\int_0^1 e^{-t} f(t)\,dt$

Ans. : (a)

Q. 2 If f(t) = 1, then L [f(t)] is _____

(a) 1 0 (b) $\frac{1}{s}$ s > 0

(c) ∞ (d) s s > 0

Ans. : (b)

Q. 3 If f(t) = sin at then Laplace transform of f(t), L [f(t)] = _____

(a) $\frac{a}{s^2-a^2}$ s > |a| (b) $\frac{s}{s^2-a^2}$ s > |a|

(c) $\frac{s}{s^2+a^2}$ s > a (d) $\frac{a}{s^2+a^2}$ s > a

Ans. : (d)

Q. 4 If f(t) = cos at then Laplace transform L [f(t)] = _____

(a) $\frac{a}{s^2-a^2}$ s > |a| (b) $\frac{s}{s^2-a^2}$ s > |a|

(c) $\frac{s}{s^2+a^2}$ s > a (d) $\frac{a}{s^2+a^2}$ s > a

Ans. : (c)

Q. 5 If f(t) = sinh at then Laplace transform of f(t) is _____

(a) $\frac{a}{s^2-a^2}$ s > |a| (b) $\frac{s}{s^2-a^2}$ s > |a|

(c) $\frac{s}{s^2+a^2}$ s > a (d) $\frac{a}{s^2+a^2}$ s > a

Ans. : (a)

Q. 6 If f(t) = cosh at then Laplace transform of

(a) $\frac{a}{s^2-a^2}$ s > |a|

(b) $\frac{s}{s^2-a^2}$ s > |a|

(c) $\frac{s}{s^2+a^2}$ s > a

(d) $\frac{a}{s^2+a^2}$ s > a

Ans. : (b)

Q. 7 If f(t) = e^{at} then the Laplace transform of f(t) is _____

(a) $\frac{1}{s}$ s > 0 (b) $\frac{1}{s-a}$ s > a

(c) $\frac{1}{s+a}$ s < a (d) $\frac{s}{s+a}$ s < a

Ans. : (b)

Q. 8 If f(t) = e^{-at} then Laplace transform of f(t) is _____

(a) $\frac{1}{s}$ s > 0 (b) $\frac{1}{s-a}$ s > a

(c) $\frac{1}{s+a}$ s > −a (d) $\frac{s}{s+a}$ s < a

Ans. : (c)

Q. 9 If f(t) = t then Laplace transform of f(t) is _____

(a) $\frac{1}{s^2}$ (b) $\frac{1}{s}$

(c) s (d) s^2

Ans. : (a)

Q. 10 If f(t) = t^n, n > 0, n is an integer then Laplace transform of f(t) = _____

(a) $\frac{n!}{s^{n+1}}$ (b) $\frac{1}{s^{n+1}}$

(c) $\frac{(n+1)!}{s^n}$ (d) $\frac{1}{s^n}$

Ans. : (a)

Q. 11 If f(t) = t^n. n > 0 n is not an integer then Laplace transform of f(t) is _____

(a) $\frac{n!}{s^{n+1}}$ (b) $\frac{(n+1)!}{s^n}$

(c) $\frac{\Gamma(n+1)}{s^{n+1}}$ (d) $\frac{\Gamma(n)}{s^n}$

Ans. : (c)

Q. 12 If $f(t) = \sin 4t$ then $L[f(t)] =$ _____

(a) $\frac{4}{s^2+16}$ $s>0$ (b) $\frac{4}{s^2-16}$ $s>4$

(c) $\frac{s}{s^2+16}$ $s>0$ (d) $\frac{s}{s^2-16}$ $s>4$

Ans. : (a)

Q. 13 If $f(t) = \cos 3t$ then $L[f(t)] =$ _____

(a) $\frac{3}{s^2+9}$ $s>0$ (b) $\frac{3}{s^2-9}$ $s>3$

(c) $\frac{s}{s^2+9}$ $s>0$ (d) $\frac{s}{s^2-9}$ $s>3$

Ans. : (c)

Q. 14 If $f(t) = e^{-2t}$ then Laplace transform of $f(t) =$ _____

(a) $\frac{1}{s+2}$ $s>-2$ (b) $\frac{1}{s-2}$ $s>2$

(c) $\frac{s}{s+2}$ $s>-2$ (d) $\frac{s}{s-2}$ $s>2$

Ans. : (a)

Q. 15 If $f(t) = e^{6t}$ then Laplace transform of $f(t) =$ _____

(a) $\frac{1}{s+6}$ $s>-6$ (b) $\frac{1}{s-6}$ $s>6$

(c) $\frac{s}{s+6}$ $s>-6$ (d) $\frac{s}{s-6}$ $s>6$

Ans. : (b)

Q. 16 If $f(t) = \sin^2 t$ then Laplace transform of $f(t) =$ _____

(a) $\frac{1}{s(s^2+4)}$ (b) $\frac{2}{s(s^2+4)}$

(c) $\frac{s^2+2}{s(s^2+4)}$ (d) $\frac{4}{s(s^2+4)}$

Ans. : (b)

Q. 17 If $f(t) = \cos^2 t$ then Laplace transform of $f(t) =$ _____

(a) $\frac{1}{s(s^2+4)}$ (b) $\frac{2}{s(s^2+4)}$

(c) $\frac{s^2+2}{s(s^2+4)}$ (d) $\frac{4}{s(s^2+4)}$

Ans. : (c)

Q. 18 If $f(t) = \sin^3 t$ then Laplace transform of $f(t) =$ _____

(a) $\frac{3}{4}\left[\frac{1}{(s^2+1)(s^2+9)}\right]$ (b) $\frac{3}{s^2+1}$

(c) $\frac{6}{(s^2+1)(s^2+9)}$ (d) $\frac{3}{s^2+9}$

Ans. : (c)

Q. 19 If $f(t) = \sin 2t \cos t$ then Laplace transform of $f(t)$ is _____

(a) $\frac{3s}{(s^2+4)(s^2+1)}$ (b) $\frac{s}{(s^2+4)(s^2+1)}$

(c) $2\left[\frac{s^2+3}{(s^2+1)(s^2+9)}\right]$ (d) $\frac{4s^2+12}{(s^2+1)(s^2+9)}$

Ans. : (c)

Q. 20 The Laplace transform of $\sin t \cos 3t$ is _____

(a) $\frac{s^2-8}{(s^2+4)(s^2+16)}$ (b) $\frac{s^2+8}{(s^2+4)(s^2+16)}$

(c) $\frac{s^2-8}{(s^2-4)(s^2-16)}$ (d) $\frac{s^2+8}{(s^2-4)(s^2-16)}$

Ans. : (a)

Q. 21 The Laplace transform of t^3 is _____

(a) $\frac{1}{s^4}$ (b) $\frac{6}{s^3}$

(c) $\frac{6}{s^4}$ (d) $\frac{1}{s^3}$

Ans. : (c)

Explanation : Use $L\{t^n\} = \frac{n!}{s^{n+1}}$

Q. 22 The Laplace transform of t^2 is _____.

(a) $\frac{2}{s^3}$ (b) $\frac{1}{s^3}$ (c) $\frac{1}{s^2}$ (d) $\frac{2}{s^2}$

Ans. : (a)

Explanation : Use $L\{t^n\} = \frac{n!}{s^{n+1}}$

Q. 23 The Laplace transform of $t^{-1/2}$ is _____.

(a) $\frac{1}{\sqrt{s}}$ (b) $\frac{\pi}{s}$

(c) $\frac{1}{s}$ (d) $\sqrt{\frac{\pi}{s}}$

Ans. : (d)

Explanation : Use $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

$$\therefore \quad L[t^{-1/2}] = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$$

Q. 24 The Laplace transform of $\sin 2t \sin 3t$ is _____

(a) $\frac{12}{(s^2+1)(s^2+25)}$ (b) $\frac{1}{(s^2+1)(s^2+25)}$

(c) $\frac{s^2+13}{(s^2+1)(s^2+25)}$ (d) $\frac{12s}{(s^2+1)(s^2+25)}$

Ans. : (d)

Q. 25 The Laplace transform of $\cos 2t \cos 4t$ is _____

(a) $\frac{20}{(s^2+36)(s^2+4)}$ (b) $\frac{s(s^2+20)}{(s^2+4)(s^2+36)}$

(c) $\frac{s}{(s^2+4)(s^2+36)}$ (d) $\frac{s^2+20}{(s^2+4)(s^2+36)}$

Ans. : (b)

Q. 26 The Laplace transform of $\cos^3 t$ is _____.

(a) $\frac{s^2-7}{(s^2+9)(s^2+1)}$ (b) $\frac{s(s^2+7)}{(s^2+9)(s^2+1)}$

(c) $\frac{s}{(s^2-9)(s^2-1)}$ (d) $\frac{s^2+7}{(s^2+9)(s^2+1)}$

Ans. : (b)

Q. 27 Laplace transform of $f(t) = \sinh 4t + \sin 3t$ is _____.

(a) $\frac{5s^2+12}{(s^2+16)(s^2-9)}$ (b) $\frac{s}{(s^2-16)(s^2+9)}$

(c) $\frac{4s+5}{(s^2-16)(s^2+9)}$ (d) $\frac{7s^2-12}{(s^2-16)(s^2+9)}$

Ans. : (d)

Q. 28 If $f(t) = \cos 4t + \sinh 2t$ then $L[f(t)]$ is _____.

(a) $\frac{s^3+2s^2-4s+32}{(s^2+16)(s^2-4)}$ (b) $\frac{s^3-2s^2+4s-32}{(s^2+16)(s^2-4)}$

(c) $\frac{s^3-2s^2+4s-32}{(s^2-16)(s^2+4)}$ (d) $\frac{s^3+2s^2-4s+32}{(s^2-16)(s^2+4)}$

Ans. : (a)

Q. 29 The Laplace transform of $\sin 2t + \cosh t$ is _____.

(a) $\frac{s^3+2s^2+4s-2}{(s^2-4)(s^2+1)}$ (b) $\frac{s^3-2s^2-4s+2}{(s^2-4)(s^2+1)}$

(c) $\frac{s^3+2s^2+4s-2}{(s^2+4)(s^2-1)}$ (d) $\frac{s^3-2s^2-4s+2}{(s^2+4)(s^2-1)}$

Ans. : (c)

Q. 30 The Laplace transform of $\sinh 2t + \cosh t$ is _____.

(a) $\frac{s^3+2s^2-4s-2}{(s^2-4)(s^2-1)}$ (b) $\frac{s^3+2s^2+4s+2}{(s^2+4)(s^2+1)}$

(c) $\frac{s^3+2s^2+4s+2}{(s^2-4)(s^2-1)}$ (d) $\frac{s^3+2s^2+4s-2}{(s^2+4)(s^2+1)}$

Ans. : (a)

Q. 31 The Laplace transform of $\cosh t + \cos 3t$ is _____.

(a) $\frac{s(s^2+1)}{(s^2+1)(s^2-9)}$ (b) $\frac{2s(s^2+4)}{(s^2-1)(s^2+9)}$

(c) $\frac{s^3+2s+1}{(s^2-1)(s^2-9)}$ (d) $\frac{s(s^2-4)}{(s^2+1)(s^2-9)}$

Ans. : (b)

Q. 32 If $L[f(t)] = f(s)$ then Laplace transform of $e^{at} f(t)$ is ____

(a) $f(s+a)$ (b) $f(s-a)$

(c) $f(as)$ (d) $f\left(\frac{s}{a}\right)$

Ans. : (b)

Q. 33 If $L[f(t)] = f(s)$ then Laplace transform of $e^{-at} f(t)$ is, ____

(a) $f(s+a)$ (b) $f(s-a)$

(c) $f(as)$ (d) $f\left(\frac{s}{a}\right)$

Ans. : (a)

Q. 34 If $L[f(t)] = f(s)$ then $L[f(at)] =$ ____

(a) $f(s/a)$ (b) $af(s/a)$

(c) $a + f(s/a)$ (d) $\frac{1}{a} f(s/a)$

Ans. : (d)

Q. 35 If Laplace transform of $L[f(t)] = f(s)$ then $L[tf(t)] =$ ____

(a) $\int_s^\infty f(s)\,ds$ (b) $\frac{d}{ds}[f(s)]$

(c) $-\frac{d}{ds}[f(s)]$ (d) $-\int_s^\infty f(s)\,ds$

Ans. : (c)

Q. 36 The Laplace transform of $t^2 f(t)$ is ____.

(a) $\frac{d}{ds}[f(s)]$ (b) $-\frac{d^2}{ds^2}[f(s)]$

(c) $\frac{d}{ds}[f(s)]^2$ (d) $\frac{d^2}{ds^2}[f(s)]$

Ans. : (d)

Q. 37 If $L[f(t)] = f(s)$ then $L[t^n f(s)] =$ ____

(a) $(-1)^n \frac{d^n}{ds^n}[f(s)]$ (b) $\frac{d^n}{ds^n}[f(s)]$

(c) $\int_s^\infty\int_s^\infty \ldots$ n times $\int_s^\infty f(s)\,ds$ (d) $\frac{1}{s}\int_s^\infty f(s)\,ds$

Ans. : (a)

Q. 38 If $L[f(t)] = f(s)$ then $L\left[\frac{f(t)}{t}\right]$ is ____

(a) $\frac{1}{s}\int_s^\infty f(s)\,ds$ (b) $\int_s^\infty f(s)\,ds$

(c) $\frac{d}{ds}[f(s)]$ (d) $-\frac{d}{ds}[f(s)]$

Ans. : (b)

Explanation : Please refer Section 3.6

Q. 39 If Laplace transform of $f(t)$ is $f(s)$ [i.e. $L[f(t)] = f(s)$] then Laplace transform of $\int_0^t f(u)\,du$ is ____

(a) $\frac{1}{s} f(s)$ (b) $s f(s)$

(c) $s + f(s)$ (d) $s - f(s)$

Ans. : (a)

Q. 40 If Laplace transform of $f(t)$ is $f(s)$ [i.e. $L[f(t)] = f(s)$] then Laplace transform of $f'(t)$ is

(a) $sf(s)$ (b) $\frac{1}{s} f(s)$

(c) $f(s) - f(0)$ (d) $sf(s) - f(0)$

Ans. : (d)

Q. 41 If L [f(t)] = f(s) then L [f″(t)] is

(a) sf(s) (b) $s^2f(s) - sf(0) - f'(0)$

(c) sf(s) – f(0) (d) $\frac{1}{s}f(s)$

Ans. : (b)

Q. 42 If L [f(t)] = f(s) and $F(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$ the L [F(t)] is _____

(a) $e^{as} f(s)$ (b) $e^{-as} f(s)$

(c) f(s) (d) $e^{as} sf(s)$

Ans. : (b)

Q. 43 If Laplace transform of f(t) is f(s) [i.e. L [f(t)] = f(s)] then the Laplace transform of $f^{(n)}(t)$ = ... (where (n) denotes n^{th} derivative)

(a) $s^n f(s) - s^{n-1} f(0) - s^{n-2} f'(0) .. sf^{(n-2)}(0) - f^{(n-1)}(0)$

(b) $s^n f(s) - s^{n-1} f'(0) - s^{n-2} f''(0) ..... sf^{n-1}(0)$

(c) $s^{n-1} f(s) - s^{n-2} f(0) - s^{n-3} f'(0) .... sf^{n-2}(0) - f^{n-1}(0)$

(d) $s^{n-1} f(s) - s^{n-2} f''(0) - s^{n-3} f''(0) ... sf^{n-2}(0) - f^{n-1}(0)$

Ans. : (a)

Q. 44 Which of the following is correct form of convolution theorem

(a) $L\left\{\int_0^t F(t-u)\, G(u)\, du\right\} = F(s) + G(s)$

(b) $L\left[\int_0^t F(u)\, G(u)\, du\right] = F(s) + G(s)$

(c) $L\left[\int_0^t F(t-u)\, G(u)\, du\right] = F(s) * G(s)$

(d) $L\left[\int_0^t F(u)\, G(u)\, du\right] = F(s) * G(s)$

Ans. : (c)

Q. 45 If L [F(t)] = F(s), L [G(t)] = G (s) then L $\left\{\int_0^t F(t-u)\, G(u)\, du\right\}$ = ____.

(a) F(s) * G(s) (b) F(s) + G(s)

(c) F(t – s) G(s) (d) $\frac{F(s)}{G(s)}$

Ans. : (a)

Q. 46 If L [f(t)] = f(s) then L [t^3 f(t)] is _____.

(a) $-\frac{d^3}{ds^3}[f(s)]$ (b) $\frac{d}{ds}[f(s)]^3$

(c) $-\frac{d}{ds}[f(s)]^3$ (d) $\frac{d^3}{ds^3}[f(s)]$

Ans. : (a)

Q. 47 If L [f(t)] = f(s) then $L\left[\frac{f(t)}{t^2}\right]$ is _____.

(a) $\frac{1}{s^2} f(s)$ (b) $(-1)^2 \frac{d^2}{ds^2}[f(s)]$

(c) $\int_0^\infty \int_0^\infty f(s)\, ds\, ds$ (d) $\frac{d}{ds}[f(s)]^2$

Ans. : (c)

Q. 48 If L[f(t)] = f(s) then $L\left[\frac{f(t)}{t^n}\right]$ is _____.

(a) $\frac{1}{s^n} f(s)$

(b) $\int_0^\infty \int_0^\infty$ .....n times $\int_0^\infty f(s)\, ds\, ds$ ..... ds

(c) $(-1)^n \frac{d^n}{ds^n}[f(s)]$

(d) $\frac{d}{ds}[f(s)]^n$

Ans. : (b)

Q. 49 If * denote convolution and L[f(t)] = f(s) and L[g(t)] = g(s) then f(s) * g(s) is _____.

(a) $L\left[\int_0^t f(u)\, g(t-u)\, du\right]$

(b) $L\left[\int_0^t f(u)\, g(u)\, du\right]$

(c) $L\left[\int_0^s f(s)\, g(s)\, ds\right]$

(d) $L\left[\int_0^t f(t-u)\, du\right]$

Ans. : (a)

Q. 50 The Laplace transform of (e^{-4t} cos 3t) is _____

(a) $\frac{s-4}{(s+4)^2+9}$ (b) $\frac{3}{(s+4)^2+9}$

(c) $\frac{s+4}{(s+4)^2+9}$ (d) $\frac{3}{(s-4)^2+9}$

Ans. : (c)

Q. 51 The Laplace transform of the function t^3e^{2t} is _____

(a) $\frac{6}{s^4}$ (b) $\frac{6}{(s-2)^4}$

(c) $\frac{6}{(s+2)^4}$ (d) $\frac{1}{(s+2)^4}$

Ans. : (b)

Q. 52 The Laplace transform of the function (t sin 2t) is _____

(a) $\frac{s^2-4}{(s^2+4)^2}$ (b) $\frac{4s}{(s^2+4)^2}$

(c) $\frac{s}{(s^2+4)}$ (d) $\frac{s-4}{(s^2+4)}$

Ans. : (b)

Q. 53 The Laplace transform of the function

$f(t) = 5e^{-t/2} + 7\sin\left(\frac{t}{2}\right)$ is _____

(a) $\frac{1}{2s+1} + \frac{1}{4s^2+1}$ (b) $\frac{5}{s+1} + \frac{7}{s^2+1}$

(c) $\frac{5}{2s+1} + \frac{7}{4s^2+1}$ (d) $\frac{10}{2s+1} + \frac{14}{4s^2+1}$

Ans. : (d)

Q. 54 The Laplace transform of (t sin 4t) is _____

(a) $\frac{s}{(s^2+16)}$ (b) $\frac{8s}{(s^2+16)^2}$

(c) $\frac{4}{(s^2+16)}$ (d) $\tan^{-1}(s/4)$

Ans. : (b)

Q. 55 The Laplace transform of the function t cos 3t is _____

(a) $\frac{s^2-9}{(s^2+9)^2}$ (b) $\frac{1}{2}\log(s^2+9)$

(c) $\frac{s}{s^2+9}$ (d) $\frac{3}{s^2+9}$

Ans. : (a)

Q. 56 The Laplace transform of the function t^2e^{3t} is _____

(a) $\left(\frac{1}{s-3}\right)^2$ (b) $1/s^3$

(c) $e^{3s}s^3$ (d) $\frac{2}{(s-3)^3}$

Ans. : (d)

Q. 57 The Laplace transform of the function

$f(t) = \begin{cases} \cos t & 0 < t < 2\pi \\ 0 & t > 2\pi \end{cases}$ is _____

(a) $\frac{s}{s^2+1}$ (b) $\frac{1}{s^2+1}$

(c) $e^{3s}\frac{s}{s^2+1}$ (d) $\frac{s(1-e^{-2\pi s})}{s^2+1}$

Ans. : (d)

Q. 58 The Laplace transform of (e^{2t} – cos 3t) is _____

(a) $\frac{s}{(s+2)(s^2+9)}$ (b) $\frac{2s+9}{(s-2)(s^2+9)}$

(c) $\frac{2s^2-2s+9}{(s+2)(s^2+9)}$ (d) $\frac{1}{(s-2)(s^2+9)}$

Ans. : (b)

Q. 59 The Laplace transform of $\int_0^t \sin 3t\, dt$ is _____

(a) $\frac{3}{s(s^2+9)}$ (b) $\frac{3}{s^2+9}$

(c) $\frac{1}{s} + \frac{1}{s^2+9}$ (d) $\frac{1}{s} - \frac{1}{s^2+9}$

Ans. : (a)

Q. 60 The Laplace transform of the function f (t) = te^{-4t} sin 3t is _____

(a) $\frac{6s}{(s^2+9)^2}$ (b) $\frac{6(s+4)}{[(s+4)^2+9]^2}$

(c) $\frac{s^2+2s+9}{(s^2+9)^2}$ (d) $\frac{s-9}{(s^2+9)^2}$

Ans. : (b)

Q. 61 The Laplace transform of the function $f(t) = \int_0^t e^{-2t}\cos t\, dt$ is _____

(a) $\frac{1}{s}\left(\frac{s+2}{(s+2)^2+1}\right)$ (b) $\frac{(s+2)}{(s+2)^2+1}$

(c) $\frac{3(s+2)}{(s+2)^2+1}$ (d) $\frac{(s-2)}{(s+2)^2+1}$

Ans. : (a)

Q. 62 The Laplace transform of the function

f(t) = cos t cos 2t cos 3t is _____

(a) $\left(\frac{s}{s^2+1}\right)\left(\frac{s}{s^2+4}\right)\left(\frac{s}{s^2+9}\right)$

(b) $\frac{6}{(s^2+1)(s^2+4)(s^2+9)}$

(c) $\frac{1}{4}\left\{\frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{s}{s^2+4} + \frac{1}{s}\right\}$

(d) $\frac{s^3}{(s^2+1)(s^2+4)(s^2+9)}$

Ans. : (c)

Q. 63 The Laplace transform of the function f (t) = e^{-2t} sin 3t is _____

(a) $\frac{(s+2)}{(s+2)^2+9}$ (b) $\frac{s}{s^2+9}$

(c) $\frac{3}{(s+2)^2+9}$ (d) $\frac{3}{s^2+9}$

Ans. : (c)

Q. 64 The Laplace transform of the function $\left(\frac{\sin 4t}{t}\right)$ is _____

(a) $\tan^{-1}(s/4)$ (b) $\frac{1}{4}\tan^{-1}(s/4)$

(c) $\frac{1}{4}\cot^{-1}(s/4)$ (d) $\cot^{-1}(s/4)$

Ans. : (d)

Q. 65 The Laplace transform of $\left(\frac{\sin 2t}{t}\right)$ is f(s) = _____.

(a) $\tan^{-1}(s)$ (b) $\cot^{-1}\left(\frac{s}{2}\right)$

(c) $\tan^{-1}\left(\frac{s}{2}\right)$ (d) $\cot^{-1}(s)$

Ans. : (b)

Q. 66 The Laplace transform of the function $f(t) = \frac{e^{-at}\sin bt}{t}$ is _____

(a) $\cot^{-1}\left(\frac{s+a}{b}\right)$ (b) $\cot^{-1}\left(\frac{s-b}{a}\right)$

(c) $\frac{1}{a}\tan^{-1}\left(\frac{s}{b}\right)$ (d) $\frac{1}{b}\tan^{-1}\left(\frac{s}{a}\right)$

Ans. : (a)

Q. 67 If $L\{f(t)\} = \frac{4s}{s^2+2s+3}$ then $L\{f(2t)\}$ = _____

(a) $\frac{8s}{2s^2+4s+9}$ (b) $\frac{8s}{s^2+2s+3}$

(c) $\frac{4s}{s^2+4s+12}$ (d) $2\left[\frac{4s}{4s^2+4s+9}\right]$

Ans. : (c)

Q. 68 If Laplace transform of f(t) is $\frac{4s}{(s^2+4)^2}$ then Laplace transform of f (4t) is

(a) $\frac{16s}{[(s+4)^2+4]^2}$

(b) $\frac{64s}{(s^2+64)^2}$

(c) $\frac{16s}{(4s^2+4)^2}$

(d) $\frac{64s}{(16s^2+4)^2}$

Ans. : (b)

Q. 69 The Laplace transform of the function $f(t) = e^{3t} + \sin 4t$ is _____

(a) $\frac{12s}{(s^2+16)}$ (b) $\frac{s^2}{(s-3)(s^2+16)}$

(c) $\frac{s^2+4s-8}{(s-3)(s^2+4)}$ (d) $\frac{s^2+4s}{(s+3)(s^2+4)}$

Ans. : (c)

Q. 70 The Laplace transform of the function

$$f(t) = \begin{cases} 0 & 1 \le t < 2 \\ t & 1 < t < 2 \\ 0 & t > 2 \end{cases}$$ is _____.

(a) $\frac{e^{-s}(s+1) - e^{-2s}(2s+1)}{s^2}$ (b) $\frac{(s+1)(e^{-2s}+1)}{s^2}$

(c) $\frac{(s+1)(e^{-2s}-1)}{s^2}$ (d) $\frac{e^{-s}}{s} + \frac{e^{-2s}}{s^2}$

Ans. : (a)

Q. 71 The Laplace transform of the function $(\sin t - \cos t)^2$ is _____

(a) $\frac{1}{s^2+1} - \frac{s}{s^2+1}$ (b) $\frac{s}{(s^2+1)(s^2+1)}$

(c) $\frac{s^2}{(s^2+1)^2}$ (d) $\frac{s^2-2s+4}{s(s^2+4)}$

Ans. : (d)

Q. 72 The Laplace transform of the function f(t) = sinh t cos t is _____.

(a) $\frac{1}{2}\left\{\frac{s-1}{(s-1)^2+1} - \frac{s+1}{(s+1)^2+1}\right\}$

(b) $\frac{1}{2}\left\{\frac{s+1}{(s+1)^2-1} - \frac{s-1}{(s-1)^2-1}\right\}$

(c) $\frac{1}{2}\left\{\frac{1}{(s-1)^2+1} + \frac{1}{(s+1)^2+1}\right\}$

(d) $\frac{1}{2}\left\{\frac{s}{(s-1)^2+1} + \frac{s}{(s+1)^2+1}\right\}$

Ans. : (a)

Q. 73 The Laplace of transform of $f(t) = e^{2t}\sinh 3t$ is _____.

(a) $\frac{3(s-2)}{s^2-4s+4}$ (b) $\frac{(s-2)}{s^2-4s-9}$

(c) $\frac{3(s-2)}{(s^2-4s-5)}$ (d) $\frac{3}{s^2-4s-5}$

Ans. : (d)

Q. 74 The Laplace transform of (t cos ht) is _____.

(a) $\frac{s^2-a^2}{(s^2+a^2)^2}$ (b) $\frac{s^2+a^2}{(s^2-a^2)^2}$

(c) $\frac{s+a}{(s^2+a^2)^2}$ (d) $\frac{s-a}{s^2+a^2}$

Ans. : (b)

Q. 75 If $L\left[\frac{\sin t}{t}\right] = \cot^{-1}(s)$ then $L\left[\frac{d}{dt}\left(\frac{\sin t}{t}\right)\right]$ is _____.

(a) $\tan^{-1}(s)$ (b) $\frac{1}{s^2+1}$

(c) $s\cot^{-1}(s) - 1$ (d) $\frac{-1}{s^2+1}$

Ans. : (c)

Q. 76 If $L\left[\frac{\sin t}{t}\right] = \cot^{-1}(s)$ then $L\left[\int_0^t \frac{\sin t}{t}\,du\right]$ is _____.

(a) $\frac{1}{s}\cot^{-1}(s)$ (b) $s\cot^{-1}(s)$

(c) $\tan^{-1}(s)$ (d) $\frac{1}{s}\tan^{-1}(s)$

Ans. : (a)

Q. 77 If $L\left[\frac{1-\cos 3t}{t}\right] = \log\frac{\sqrt{s^2+9}}{s}$ then

$L\left[\int_0^t \frac{1-\cos 3t}{t}\,dt\right]$ is _____.

(a) $s\log\frac{\sqrt{s^2+9}}{s}$ (b) $\frac{1}{s}\log\frac{\sqrt{s^2+9}}{s}$

(c) $\cot^{-1}\sqrt{s^2+9} - \cot^{-1}s$ (d) $\int_0^s \log\frac{\sqrt{s^2+9}}{s}\,ds$

Ans. : (b)

Q. 78 If $L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \log\left(\frac{s+b}{s+a}\right)$ then $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt$ is equal to _____.

(a) log a + log b (b) log a − log b

(c) $\log\left(\frac{b}{a}\right)$ (d) $\frac{1}{s}\log\left(\frac{s+b}{s+a}\right)$

Ans. : (c)

Q. 79 If $L[t\cos t] = \frac{s^2-1}{(s^2+1)^2}$ then $\int_0^\infty e^{-2t}t\cos t\,dt$ is equal to _____.

(a) $\frac{3}{25}$ (b) $\frac{1}{s}\left[\frac{s^2-1}{(s^2+1)^2}\right]$

(c) $\frac{s+2}{(s^2+2)^2}$ (d) $\frac{6}{25}$

Ans. : (a)

Q. 80 If $L\left[\frac{e^{-t}-e^{-4t}}{t}\right] = \log\left(\frac{s+4}{s+1}\right)$ then $\int_0^\infty \frac{e^{-t}-e^{-4t}}{t}\,dt$ is equal to _____.

(a) $\frac{1}{s}\log\left(\frac{s+4}{s+1}\right)$ (b) log 4

(c) $-\log\left(\frac{s^2}{2}\right)$ (d) $\log\frac{1}{4}$

Ans. : (b)

Q. 81 If $L[\sin^3 t] = \frac{6}{(s^2+1)(s^2+9)}$ then $\int_0^\infty e^{-2t}\sin^3 t\,dt$ is equal to _____.

(a) $\frac{1}{65}$ (b) $\frac{2}{65}$

(c) $\frac{1}{s}\left[\frac{6}{(s^2+1)(s^2+9)}\right]$ (d) $\frac{6}{65}$

Ans. : (d)

Q. 82 If $L[\sin^2 t] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$ and

$$F(t) = \begin{cases} 0 & 0 < t < \pi \\ \sin^2(t-\pi) & t > \pi \end{cases}$$ then

L [F(t)] = _____.

(a) $\pi\left[\frac{2}{s(s^2+4)}\right]$ (b) $\frac{2}{s(s^2+4)}$

(c) $\frac{2e^{-\pi s}}{s(s^2+4)}$ (d) $\frac{1}{\pi}\left[\frac{2}{s(s^2+4)}\right]$

Ans. : (c)

Q. 83 If $L[t^2] = \frac{2}{s^3}$ and $F(t) = \begin{cases} (t-1)^2 & t > 1 \\ 0 & 0 < t < 1 \end{cases}$

then L [F(t)] = _____.

(a) $\frac{2}{s^3}$ (b) $\frac{2e^{-s}}{s^3}$ (c) $\left(\frac{2}{s^3}\right)^2$ (d) 0

Ans. : (b)

Q. 84 If $F(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & t > \frac{2\pi}{3} \\ 0 & t < 0 \end{cases}$

then L [F(t)] = _____.

(a) $e^{-\frac{2\pi s}{3}}\left(\frac{s}{s^2+1}\right)$ (b) $\frac{3}{s^2+1}$

(c) $\frac{2\pi}{3}\left(\frac{s}{s^2+1}\right)$ (d) $\frac{s^{-2\pi/3}}{s^2+1}$

Ans. : (a)

Q. 85 If $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$ then $\int_0^\infty J_0(t)\,dt$ = _____

(a) 2 (b) $\frac{1}{\sqrt{2}}$ (c) 0 (d) 1

Ans. : (d)

Q. 86 If $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$ then $\int_0^\infty e^{-t}J_0(t)\,dt$ = _____

(a) $\frac{1}{\sqrt{2}}$ (b) 1 (c) $\sqrt{2}$ (d) 0

Ans. : (a)

Q. 87 If $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$ then $L[J_0(at)] =$ _____

(a) $\frac{as}{\sqrt{s^2+1}}$ (b) $\frac{1}{\sqrt{s^2+a^2}}$

(c) $\frac{a}{\sqrt{s^2+1}}$ (d) $\frac{s}{\sqrt{s^2+a^2}}$

Ans. : (b)

Q. 88 If $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$ then $L[t\,J_0(at)] =$ _____

(a) $\frac{s}{(s^2+a^2)^{3/2}}$ (b) $\frac{as}{\sqrt{s^2+a^2}}$

(c) $\frac{s}{\sqrt{s^2+a^2}}$ (d) $\frac{a}{(s^2+a^2)^{3/2}}$

Ans. : (a)

Q. 89 If $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$ then $L[e^{-at} J_0(at)] =$ _____

(a) $\frac{a}{\sqrt{s^2+a^2}}$ (b) $\frac{s+a}{\sqrt{s^2+2as+2a^2}}$

(c) $\frac{1}{\sqrt{s^2+2as+2a^2}}$ (d) $\frac{s}{\sqrt{s^2+a^2}}$

Ans. : (c)

Q. 90 By Laplace transform the value of $\int_0^\infty e^{-t}\sin t\,dt$ is

_____

(a) 1 (b) $\frac{1}{2}$

(c) 0 (d) 2

Ans. : (b)

Explanation :

$$\int_0^\infty e^{-t}\sin t\,dt = L[\sin t] \qquad (s=1)$$

(By definition of Laplace transform)

$$= \frac{1}{s^2+1} \qquad (s=1)$$

$$= \frac{1}{2}$$

Q. 91 By Laplace transform the value of $\int_0^\infty e^{-2t}\sinh t\,dt$ is

_____

(a) 3 (b) 2 (c) $\frac{1}{3}$ (d) $\frac{1}{2}$

Ans. : (c)

Explanation :

$$\int_0^\infty e^{-2t}\sinh 4t\,dt = L[\sinh t]$$

(By definition of Laplace transform) (s = 2)

$$= \frac{1}{s^2-1} \qquad (\text{put } s=2)$$

$$= \frac{1}{3}$$

Q. 92 By Laplace transform the value of $\int_0^\infty e^{-3t}\sin 3t\,dt$ is

_____

(a) 1 (b) $\frac{-1}{3}$ (c) $\frac{1}{3}$ (d) $\frac{1}{6}$

Ans. : (b)

Explanation :

$$\int_0^\infty e^{-3t}\sin 3t\,dt = L[\sin 3t] \qquad (s=3)$$

(By definition of Laplace transform)

$$= \frac{3}{s^2+9} = \frac{3}{18} = \frac{1}{6}$$

Q. 93 The Laplace transform of the periodic function f (t) with period T i.e. f (t + T) = f (t) is _____

(a) $\int_0^T e^{-su} f(u)\,du$

(b) $\frac{1}{1-e^{-s}}\int_0^T e^{-s} f(u)\,du$

(c) $\frac{1}{1-e^{-su}}\int_0^T e^{-su} f(t)\,du$

(d) $\frac{1}{1-e^{-sT}}\int_0^T e^{-su} f(u)\,du$

Ans. : (d)

Q. 94 Unit step function U(t) is defined as _____

(a) $\begin{cases} -1 & t<0 \\ 1 & t\ge 0 \end{cases}$ (b) $\begin{cases} 0 & t<0 \\ 1 & t\ge 0 \end{cases}$

(c) $\begin{cases} 1 & t<0 \\ -1 & t\ge 0 \end{cases}$ (d) $\begin{cases} 1 & t<0 \\ 0 & t\ge 0 \end{cases}$

Ans. : (b)

Q. 95 The value of U(t – a) is _____

(a) $\begin{cases} -1 & t<0 \\ 1 & t\ge 0 \end{cases}$ (b) $\begin{cases} 1 & t<0 \\ 0 & t\ge 0 \end{cases}$

(c) $\begin{cases} 0, & t<a \\ 1, & t\ge a \end{cases}$ (d) $\begin{cases} 1, & t<a \\ 0, & t\ge a \end{cases}$

Ans. : (c)

Q. 96 The Laplace transform of unit step function U (t) is _____

(a) $\frac{1}{s}$ (b) $\int_0^T e^{-s} f(u)\,du$

(c) $\frac{1}{s^2}$ (d) $\frac{1}{1-e^{-s}}$

Ans. : (a)

Q. 97 The Laplace transform of U (t – a) is _____

(a) $\frac{1}{s}$ (b) e^{-st}

(c) $\frac{1}{s}e^{-a.s}$ (d) $\frac{1}{1-e^{-s}}$

Ans. : (c)

Q. 98 The dirac-delta function δ (t – a) is defined as _____

(a) $\begin{cases} 0 & t<0 \\ 1 & t\ge 0 \end{cases}$

(b) $\begin{cases} 0, & t<a \\ 1, & t\ge a \end{cases}$

(c) $\begin{cases} f_1(t) & 0<t<a \\ f_2(t) & a<t<b \\ f_3(t) & b<t<c \\ f_4(t) & t\ge c \end{cases}$

(d) $\begin{cases} 0, & t<a \\ \frac{1}{\varepsilon}, & a\le t\le a+\varepsilon \quad \text{where } \varepsilon>0 \\ 0, & t>a+\varepsilon \end{cases}$

Ans. : (d)

Q. 99 The Laplace transform of δ (t – a) is _____

(a) $\frac{1}{s}$ (b) $\frac{1}{1-e^{-st}}$

(c) e^{-as} (d) $\frac{1}{1-e^{-s}}$

Ans. : (c)

Q. 100 The value of $\frac{d}{dt}$ U(t – a)is _____

(a) f (a) (b) δ (t – a)

(c) e^{-as} f (a) (d) δ (t + a)

Ans. : (b)

Q. 101 If Laplace transform of f(t) = f(s) then Laplace transform of f(t) U (t – a) is _____

(a) L[f(t + a)] (b) e^{-as}L[f(t + a)]

(c) e^{-as}L[f(t - a)] (d) e^{as}L[f(t + a)]

Ans. : (b)

Q. 102 If Laplace transform of f(t) = f(s) then Laplace transform of L[f(t) δ(t – a)] is _____

(a) e^{-as}f(a) (b) e^{as}f(t)

(c) e^{-as}L[f(t - a)] (d) e^{-as}L[f(t + a)]

Ans. : (a)

Q. 103 The Laplace transform of $f(t) = (t-1)^2 U(t-1)$is _____

(a) $\frac{2e^{-s}}{s^3}$ (b) $\frac{e^{-2(s+3)}}{s+3}$

(c) $\frac{2e^{-s}}{s^2}$ (d) $\frac{2e^{-(s+1)}}{s^2}$

Ans. : (a)

Explanation :

We have, $L[t^2] = \frac{2}{s^3}$ ...(by table)

Since $L[f(t)U(t-a)] = e^{-as}L[f(t+a)]$

$(\because a = 1 \text{ and } t \to t+1)$

$$\therefore L[(t-1)^2 U(t-1)] = e^{-s}L[(t+1-1)^2]$$

$$= e^{-s}L[t^2] = e^{-s}\frac{2}{s^3} = \frac{2e^{-s}}{s^3}$$

Q. 104 The Laplace transform of $f(t) = e^{t-2}\cdot U(t-2)$ is _____

(a) $\frac{2e^{-s}}{s^3}$ (b) $\frac{e^{-2(s+3)}}{s+3}$

(c) $\frac{e^{-2s}}{s-1}$ (d) $\frac{2e^{-(s+1)}}{s^2}$

Ans. : (c)

Explanation :

$$L[e^{at}] = \frac{1}{s-a}$$

Since, $L[f(t)U(t-a)] = e^{-as}L[f(t+a)]$

$$\therefore L[e^{t-2}U(t-2)] = e^{-2s}L[e^{(t+2)-2}]$$

$(\because a = 2 \text{ and } t \to t+2)$

$$= e^{-2s}L[e^t] = e^{-2s}\frac{1}{s-1}$$

$$\therefore L[e^{t-2}U(t-2)] = \frac{e^{-2s}}{s-1}$$

Q. 105 The Laplace transform of $f(t) = (t-\pi)\,U(t-\pi)$.is _____

(a) $e^{-\pi s}\sin\pi$ (b) $\frac{e^{-2(s+3)}}{s+3}$

(c) $\frac{e^{-2s}}{s-1}$ (d) $\frac{e^{-\pi s}}{s^2}$

Ans. : (d)

Explanation :

$$L[f(t-a)U(t-a)] = e^{-as}f(s)$$

$$\therefore L[(t-\pi)U(t-\pi)] = e^{-\pi s}L[t] = \frac{e^{-\pi s}}{s^2}$$

Q. 106 The Laplace transform of $f(t) = \sin t \cdot \delta(t-\pi)$ is _____

(a) 0 (b) π

(c) $\frac{e^{-2s}}{s-1}$ (d) $\frac{e^{-\pi s}}{s^2}$

Ans. : (a)

Explanation :

$$L[f(t)\,\delta(t-a)] = e^{-as} f(a)$$

$$\therefore \quad L[\sin t\,\delta(t-\pi)] = e^{-\pi s} \sin \pi = 0$$

Q. 107 If $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$ Then the value of $\int_0^\infty J_0(t)\,dt$ is _____

(a) 0 (b) π (c) 1 (d) $\frac{e^{-\pi s}}{s^2}$

Ans. : (c)

Explanation :

$$L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$$

$$\int_0^\infty J_0(t)\,dt = \int_0^\infty e^{-st} J_0(t)\,dt \qquad (s=0)$$

$$= L[J_0(t)] \quad (s=0) \text{ ...(by definition)}$$

$$= \frac{1}{\sqrt{s^2+1}} \qquad (s=0)$$

$$\therefore \int_0^\infty J_0(t)\,dt = 1$$

Q. 108 The Laplace transform of $f(t) = e^{-3t} U(t-2)$ is _____

(a) $\frac{e^{-2(s+3)}}{s+3}$ (b) π

(c) $\frac{e^{-2s}}{s-1}$ (d) $\frac{e^{-2(s+3)}}{s+3}$

Ans. : (d)

Explanation :

We have,

$$L[e^{-at}] = \frac{1}{s+a} \qquad \text{...(by table)}$$

Since, $L[f(t)\,U(t-a)] = e^{-as} L[f(t+a)]$

$$\therefore L[e^{-3t}U(t-2)] = e^{-2s} L[e^{-3(t+2)}]$$

$$(\because a = 2 \text{ and } t \to t+2)$$

$$= e^{-2s} L[e^{-3t} \cdot e^{-6}]$$

$$= e^{-2s} \cdot e^{-6} \frac{1}{s+3} = \frac{e^{-2(s+3)}}{s+3}$$

$$\therefore L[e^{-3t}U(t-2)] = \frac{e^{-2(s+3)}}{s+3}$$

□□□

CHAPTER 2 — UNIT II

Inverse Laplace Transform

Syllabus

Introductory remarks ; Inverse transforms of some elementary functions ; General methods of finding inverse transforms ; Partial fraction method and Convolution Theorem for finding inverse Laplace transforms ; Applications to find the solutions of linear differential equations and simultaneous linear differential equations with constant coefficients.

Syllabus Topic : Introductory Remarks

2.1 Inverse Laplace Transforms

If Laplace transform of the function f (t) is,

$$L[F(t)] = f(s) \quad \text{(if exist)}$$

Then we can find inverse Laplace transform of f (s) is,

$$L^{-1}[f(t)] = f(t)$$

Since there is one to one correspondence between f (t) and f (s)

Syllabus Topic : Inverse transforms of some elementary functions

Note :

(i) $L[1] = \frac{1}{s} \Rightarrow L^{-1}\left[\frac{1}{s}\right] = 1$

(ii) $L[e^{at}] = \frac{1}{s-a} \Rightarrow L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

(iii) $L[\sin at] = \frac{a}{s^2+a^2} \Rightarrow L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sin at$

Similarly we can find others.

Table 2.1.1 : Inverse Laplace Transform

Sr. No.	f (s)	$L^{-1}[f(s)] = f(t)$
1.	$\frac{1}{s}$	1
2.	$\frac{1}{s-a}$ $s > a$	e^{at}
3.	$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$
4.	$\frac{s}{s^2-a^2}$	$\cos at$
5.	$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sinh at$
6.	$\frac{s}{s^2-a^2}$	$\cosh at$
7.	$\frac{s}{(s^2-a^2)^2}$	$\frac{t}{2a}\sin at$
8.	$\frac{1}{(s^2+a^2)^2}$	$\frac{1}{2a^3}(\sin at - at\cos at)$
9.	$\frac{1}{s^{n+1}}$	$\frac{t^n}{\Gamma(n+1)}$ or $\frac{t^n}{n!}$ if n is positive Integer

Syllabus Topic : General methods of finding inverse transforms

2.1.1 Examples on Inverse Laplace Transform

Ex. 2.1.1 : Find inverse Laplace transform of

$$\frac{s}{s^2+9} - \frac{s}{s^2+16}$$

Soln. :

$$\text{Let } f(s) = \frac{s}{s^2+9} - \frac{s}{s^2+16} \qquad \text{...(1)}$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[f(s)] = L^{-1}\left(\frac{s}{s^2+9} - \frac{s}{s^2+16}\right)$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots[L^{-1}[a f(s) + b g(s) - c h(s)] = a L^{-1}[f(s)] + b L^{-1}[g(s)] - c L^{-1}[h(s)]]$$

$$L^{-1}[f(s)] = \underbrace{L^{-1}\left[\frac{s}{s^2+(3)^2}\right]}_{\cos 3t} - \underbrace{L^{-1}\left[\frac{s}{s^2+(4)^2}\right]}_{\cos 4t}$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at\right]$$

$$f(t) = \cos 3t - \cos 4t$$

$$f(t) = \cos 3t - \cos 4t \checkmark \qquad \ldots\text{Ans.}$$

Ex. 2.1.2 : Find inverse Laplace transform of $\frac{1}{s^2+4} + \frac{s}{s^2+9}$

Soln. :

$$\text{Let } f(s) = \frac{1}{s^2+4} + \frac{s}{s^2+9} \qquad \ldots(1)$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[f(s)] = L^{-1}\left(\frac{1}{s^2+4} + \frac{s}{s^2+9}\right)$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots[L^{-1}[a f(s) + b g(s) - c h(s)] = a L^{-1}[f(s)] + b L^{-1}[g(s)] - c L^{-1}[h(s)]]$$

$$L^{-1}[f(s)] = \underbrace{L^{-1}\left[\frac{1}{s^2+(2)^2}\right]}_{\frac{1}{2}\sin 2t} + \underbrace{L^{-1}\left[\frac{s}{s^2+(3)^2}\right]}_{\cos 3t}$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at;\ L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at\right]$$

$$f(t) = \frac{1}{2}\sin 2t + \cos 3t$$

$$f(t) = \frac{1}{2}\sin 2t + \cos 3t \checkmark \qquad \ldots\text{Ans.}$$

Ex. 2.1.3 : Find inverse Laplace transform of $\left[\frac{1}{s-1} - \frac{s}{s^2+2} + \frac{1}{s^2+2}\right]$

Soln. :

$$\text{Let } f(s) = \left[\frac{1}{s-1} - \frac{s}{s^2+2} + \frac{1}{s^2+2}\right] \qquad \ldots(1)$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[f(s)] = L^{-1}\left[\frac{1}{s-1} - \frac{s}{s^2+2} + \frac{1}{s^2+2}\right]$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots[L^{-1}[a f(s) + b g(s) - ch(s)] = a L^{-1}[f(s)] + b L^{-1}[g(s)] - cL^{-1}[h(s)]]$$

$$L^{-1}[f(s)] = L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{s}{s^2+(\sqrt{2})^2}\right] + L^{-1}\left[\frac{1}{s^2+(\sqrt{2})^2}\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s-a}\right] = e^{at};\ L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at \text{ and } \left[\frac{s}{s^2+a^2}\right] = \cos at\right]$$

$$\therefore L^{-1}[f(s)] = \underbrace{L^{-1}\left[\frac{1}{s-1}\right]}_{e^{1t}} - \underbrace{L^{-1}\left[\frac{s}{s^2+(\sqrt{2})^2}\right]}_{\cos(\sqrt{2}\,t)} + \underbrace{L^{-1}\left[\frac{1}{s^2+(\sqrt{2})^2}\right]}_{\frac{1}{\sqrt{2}}\sin(\sqrt{2}\,t)}$$

$$\therefore f(t) = e^{1t} - \cos(\sqrt{2}\,t) + \frac{1}{\sqrt{2}}\sin(\sqrt{2}\,t)$$

$$f(t) = \left[e^{t} - \cos(\sqrt{2}\,t) + \frac{1}{\sqrt{2}}\sin(\sqrt{2}t)\right] \checkmark \qquad \ldots\text{Ans.}$$

2.2 Properties of Inverse Laplace Transform

Property I : First Shifting Property of Inverse Laplace Transform

We know if $L[f(t)] = f(s)$ then $L[e^{-at} f(t)] = f(s+a)$

$$\therefore L^{-1}[f(s-a)] = e^{at} L^{-1}[f(s)]$$

known as first shifting property of inverse Laplace transform.

We can use first shifting property of inverse Laplace transform :

(I) If each term of f(s) is in the form (s – a) or can be written in this form

(II) If f(s) is in the form $\frac{1}{as^2+bs+c}$ where the term $as^2 + bs + c$ can be expressed in perfect square by adjusting constant term.

Procedure : If $f(s) = \frac{N}{as^2+bs+c}$ then to find inverse Laplace transform of f (s).

Express $as^2 + bs + c$ in perfect square say $(s \pm d)^2$

For $as^2 + bs$ find third term as

$$\text{Third term} = \text{T. T.} = \frac{(\text{Second term})^2}{4 \times \text{first term}}$$

Adjust $as^2 + bs$ as,

$$as^2 + bs + (\text{T. T.}) - (\text{T. T.})$$

Property II : Division by s

If $L^{-1}\{f(s)\} = f(t)$ then $L^{-1}\left[\frac{1}{s}f(s)\right] = \int_0^t f(t)$

Property III : Inverse Laplace of derivative

We know, If $L[f(t)] = f(s)$ then $L\{t f(t)\} = -\frac{d}{ds} f(s)$

$$\therefore L^{-1}\left[\frac{d}{ds}f(s)\right] = -t f(t)$$

Note : We apply this property if f (s) is in logarithmic form or inverse trigonometric function.

Property IV : Multiplication by s

We know, If $L[f(t)] = f(s)$, then $L\{f'(t)\} = s f(s) - f(0)$

$$\therefore L^{-1}\{sf(s)\} = f'(t), \text{ if } f(0) = 0$$

Note : We apply this property if f (s) is in the form $\frac{\frac{d}{ds}(as^2+bs+c)}{(as^2+bs+c)^2}$

Property V : Inverse Laplace of Integration

We know, If $L[f(t)] = f(s)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(s)ds$

$$\therefore L^{-1}\left\{\int_s^\infty f(s)\,ds\right\} = \frac{f(t)}{t}$$

Property VI : Convolution Theorem

Convolution of two functions f(t) and g(t), t > 0 is defined as,

$$\int_0^t f(u)\,g(t-u)du.$$

It is denoted as $f(t)*g(t)$

$$\therefore f(t)*g(t) = \int_0^t f(u)\,g(t-u)du$$

Note that :

$$\therefore g(t)*f(t) = \int_0^t g(u)\,f(t-u)du$$

We know, If $L[f(t)] = f(s)$ and $L[g(t)] = g(s)$, then

$$L\left[\int_0^t f(u)\,g(t-u)du\right] = f(s)*g(s) = L\left[\int_0^t f(t-u)\,g(u)du\right]$$

$$\therefore L^{-1}\{f(s)*g(s)\} = \int_0^t f(u)\,g(t-u)\,du$$

Table 2.2.1 : Inverse Laplace Transform using properties

Sr. No.	f (s)	$L^{-1}\{f(s)\} = f(t)$
1.	$f(s-a)$	$e^{at} f(t)$
2.	$e^{-as} f(s)$	$f(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$
3.	$f\left(\frac{s}{a}\right)$	$a f(at)$
4.	$sf(s)$	$f'(t)$ If $f(0) = 0$
5.	$(-1)^n \frac{d^n}{dt^n}\{f(s)\}$	$t^n f(t)$
6.	$\frac{f(s)}{s}$	$\int_0^t f(u)\,du$
7.	$\int_s^\infty f(s)\,ds$	$\frac{f(t)}{s}$
8.	$F(s)\,G(s)$	$\int_0^t f(t-u)\,G(u)\,du$

2.3 Examples using First Shifting Property of Inverse Laplace Transform

$$L^{-1}[f(s-a)] = e^{at} L^{-1}[f(s)]$$

We can use first shifting property of inverse Laplace transform:

(I) If each term of f(s) is in the form (s – a) or can be written in this form

(II) If f(s) is in the form $\frac{1}{as^2+bs+c}$ where the term $as^2 + bs + c$ can be expressed in perfect square by adjusting constant term.

Procedure : If $f(s) = \dfrac{N}{as^2+bs+c}$ then to find inverse Laplace transform of f (s).

Express $as^2 + bs + c$ in perfect square say $(s \pm d)^2$

For $as^2 + bs$ find third term as

$$\text{Third term} = \text{T. T.} = \frac{(\text{Second term})^2}{4 \times \text{first term}}$$

Adjust $as^2 + bs$ as,

$$as^2 + bs + (\text{T. T.}) - (\text{T. T.})$$

Ex. 2.3.1 : Find inverse Laplace transform of $\dfrac{1}{(s+4)^{3/2}}$

Soln. :

Given :

$$f(s) = \frac{1}{(s+4)^{3/2}}$$

$$\therefore \quad L^{-1}[f(s)] = L^{-1}\left[\frac{1}{(s+4)^{3/2}}\right]$$

Here all s are in the form (s + 4)

...{Compare s + 4 with s – a $\therefore -a = 4 \Rightarrow a = -4$}

➔ **Using first shifting property of inverse Laplace transform**

$$\ldots \left\{ L^{-1}[f(s-a)] = e^{at} L^{-1}[f(s)] \right\}$$

$$\therefore \quad L^{-1}[f(s)] = e^{-4t} L^{-1}\left[\frac{1}{s^{3/2}}\right]$$

Here (s – a) replace by s, with e^{-4t} taking outside.

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{\Gamma n} \text{ OR } \frac{t^{n-1}}{(n-1)!} \text{ if n is an integer}\right]$$

$$\therefore L^{-1}[f(s)] = e^{-4t} \underbrace{L^{-1}\left[\frac{1}{s^{3/2}}\right]}_{\frac{t^{\frac{3}{2}-1}}{\Gamma\frac{3}{2}}} \quad \left\{\text{Compare : } \underset{\substack{\uparrow \\ n}}{s^{3/2}} \text{ with } s^n\right\}$$

$$= e^{-4t}\frac{t^{\frac{3}{2}-1}}{\Gamma\frac{3}{2}} = e^{-4t}\frac{t^{\frac{1}{2}}}{\Gamma\frac{3}{2}}$$

➔ **Using property and standard value of Gamma function**

$$\ldots\left[\Gamma(n+1) = n\Gamma n \text{ with } n = \frac{1}{2} \text{ and } \Gamma\frac{1}{2} = \sqrt{\pi}\right]$$

$$f(t) = e^{-4t}\frac{t^{\frac{1}{2}}}{\frac{1}{2}\Gamma\frac{1}{2}} = 2e^{-4t}\frac{t^{\frac{1}{2}}}{\sqrt{\pi}} \quad \left(\because \frac{1}{\frac{a}{b}} = \frac{b}{a}\right)$$

$$\ldots\left(\text{Since standard value of Gamma fucntion : } \Gamma\frac{1}{2} = \sqrt{\pi}\right)$$

$$\therefore f(t) = 2e^{-4t}\frac{\sqrt{t}}{\sqrt{\pi}} \qquad \therefore \left(t^{\frac{1}{2}} \text{ as } \sqrt{t}\right)$$

$$\therefore f(t) = 2e^{-4t}\sqrt{\frac{t}{\pi}} \checkmark \qquad \ldots\text{Ans.}$$

Ex. 2.3.2 : Find inverse Laplace transform of $\dfrac{1}{s+1} - \dfrac{4}{(s+2)^2}$

Soln. :

Let, $$f(s) = \frac{1}{s+1} - \frac{4}{(s+2)^2}$$

Taking inverse Laplace transform both sides,

$$L^{-1}[f(s)] = L^{-1}\left[\frac{1}{s+1} - \frac{4}{(s+2)^2}\right]$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots[L^{-1}[a f(s) + b g(s) - c h(s)] = a L^{-1}[f(s)] + b L^{-1}[g(s)] - c L^{-1}[h(s)]]$$

$$\therefore L^{-1}[f(s)] = L^{-1}\left[\frac{1}{s+1}\right] - 4L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

➔ **Using standard result and first shifting property of inverse Laplace transform**

$$\ldots\left\{L^{-1}\left[\frac{1}{s-a}\right] = e^{at} \text{ and } \underbrace{L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)]}_{\text{Taking } e^{at} \text{ outside}}\right\}$$

every (s – a) replace by s with e^{at} outside

$$\therefore L^{-1}[f(s)] = \underbrace{L^{-1}\left[\frac{1}{s-(-1)}\right]}_{e^{-1t}} - 4\underbrace{L^{-1}\left[\frac{s}{[s-(-2)]^2}\right]}_{e^{-2}L^{-1}\left[\frac{1}{s^2}\right]}$$

$$f(t) = e^{-1t} - 4e^{-2t}L^{-1}\left[\frac{1}{s^2}\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{\Gamma n} \text{ OR } \frac{t^{n-1}}{(n-1)!} \text{ if n is an integer}\right]$$

$$f(t) = e^{-t} - 4e^{-2t}\underbrace{L^{-1}\left[\frac{1}{s^2}\right]}_{\frac{t^{2-1}}{(2-1)!}}$$

$$f(t) = e^{-t} - 4e^{-2t}\frac{t^{2-1}}{(2-1)!}$$

$$f(t) = e^{-t} - 4e^{-2t}\frac{t}{1!}$$

$$f(t) = e^{-t} - 4e^{-2t}.t \checkmark \qquad \ldots\text{Ans.}$$

Ex. 2.3.3 : Find inverse Laplace transform of $\dfrac{1}{s^2+6s+5}$

Soln. :

Let, $$f(s) = \frac{1}{s^2+6s+5} \qquad \ldots(1)$$

For $s^2 + 6s$ find third term (T. T.) as,

$$\text{T. T.} = \frac{(\text{Second term})^2}{4 \times \text{first term}} \qquad \left\{\underset{\substack{\uparrow \\ \text{F.T.}}}{s^2} + \underset{\substack{\uparrow \\ \text{S.T.}}}{6s}\right\}$$

$$= \frac{(6s)^2}{4 \times s^2} = \frac{36s^2}{4 \times s^2} = 9$$

$$\text{T. T.} = 9$$

$$\therefore s^2 + 6s = (s^2 + 6s + 9) - 9$$

Substitute these in Equation (1),

$$\therefore f(s) = \frac{1}{(s^2+6s+9)-9+5}$$

$$f(s) = \frac{1}{(s+3)^2 - 4}$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[f(s)] = L^{-1}\left[\frac{1}{(s+3)^2-4}\right]$$

...(Observe nature of all s as (s – a) i.e. s + 3)

➔ **Using first shifting property of inverse Laplace transform**

$$\ldots[L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)]]$$

$$\therefore L^{-1}[f(s)] = L^{-1}\left[\frac{1}{(s+3)^2-4}\right]$$

$$f(t) = e^{-3t}L^{-1}\left[\frac{1}{s^2-4}\right]$$

...{Replace s + 3 by s with e^{-3t} outside}

$$f(t) = e^{-3t}L^{-1}\left[\frac{1}{s^2-(2)^2}\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at\right]$$

$$\therefore f(t) = e^{-3t}\underbrace{L^{-1}\left[\frac{1}{s^2-(2)^2}\right]}_{\frac{1}{2}\sinh 2t}$$

$$\ldots\left\{\text{compare } s^2 - \underset{\substack{\uparrow \\ a^2}}{(2)^2} \text{ with } s^2 - a^2\right\}$$

$$\therefore f(t) = e^{-3t}.\left(\frac{1}{2}\sinh 2t\right)$$

$$f(t) = \frac{1}{2}e^{-3t}\sinh 2t \checkmark \qquad \ldots\text{Ans.}$$

Ex. 2.3.4 : Find inverse Laplace transform of $\dfrac{2s+5}{s^2+4s+13}$

Soln. :

Let $$f(s) = \frac{2s+5}{s^2+4s+13} \qquad \ldots(1)$$

For $s^2 + 4s$ find third term (T.T.) as

$$\text{T.T.} = \frac{(\text{second term})^2}{4 \times \text{first term}} \qquad \left[\underset{\substack{\uparrow \\ \text{F.T.}}}{s^2} + \underset{\substack{\uparrow \\ \text{S.T.}}}{4s}\right]$$

$$= \frac{(4s)^2}{4 \times s^2} = \frac{16s^2}{4s^2} = 4$$

$$\therefore \text{T.T.} = 4$$

$$\therefore s^2 + 4s = (s^2 + 4s + 4) - 4$$

Substitute these in Equation (1),

$$f(s) = \frac{2s+5}{(s^2+4s+4)-4+13}$$

$$f(s) = \frac{2s+5}{(s+2)^2+9}$$

$$\ldots\left[\because a^2 + 2ab + b^2 = (a+b)^2\right]$$

$$f(s) = \frac{2(s+2-2)+5}{(s+2)^2+9} \quad \left(\text{Try to obtain every s as s + 2}\right)$$

$$= \frac{2(s+2)-4+5}{(s+2)^2+9}$$

$$f(s) = \frac{2(s+2)+1}{(s+2)^2+9}$$

Taking inverse Laplace transform of both sides,

$$\therefore L^{-1}[f(s)] = L^{-1}\left[\frac{2(s+2)+1}{(s+2)^2+9}\right]$$

...[Observe that all s are in the form (s – a) i.e. (s + 2)]

➔ **Using first shifting property of inverse Laplace transform :**

$$\ldots\left\{L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)]\right\}$$

$$\therefore L^{-1}[f(s)] = e^{-2t}L^{-1}\left[\frac{2s+1}{s^2+9}\right]$$

...(Replace (s + 2) by s with e^{-2t} outside)

$$\therefore\ L^{-1}[f(s)] = e^{-2t}L^{-1}\left[\frac{2s}{s^2+9}+\frac{1}{s^2+9}\right]$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[af(s)+bg(s)-ch(s)] = aL^{-1}[f(s)]+bL^{-1}[g(s)]-cL^{-1}[h(s)]\right\}$$

$$\therefore\ L^{-1}[f(s)] = e^{-2t}\left\{2L^{-1}\left[\frac{s}{s^2+9}\right]+L^{-1}\left[\frac{1}{s^2+9}\right]\right\}$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at \text{ and } L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$= e^{-2t}\left\{\underbrace{2L^{-1}\left[\frac{s}{s^2+(3)^2}\right]}_{\cos 3t}+\underbrace{L^{-1}\left[\frac{1}{s^2+(3)^2}\right]}_{\frac{1}{3}\sin 3t}\right\}$$

$$\therefore\ L^{-1}[f(s)] = e^{-2t}\left[2\cos 3t+\frac{1}{3}\sin 3t\right]\checkmark \qquad \ldots\textbf{Ans.}$$

Ex. 2.3.5 : Find inverse Laplace transform of $\frac{s}{s^2+6s+25}$

Soln. :

Let $f(s) = \frac{s}{s^2+6s+25}$

For s^2+6s find third term (T.T.) as

$$T.T. = \frac{(\text{second term})^2}{4\times\text{first term}} \qquad \left[\underset{\text{F.T.}}{s^2}+\underset{\text{S.T.}}{6s}\right]$$

$$= \frac{(6s)^2}{4\times s^2} = \frac{\cancel{36s^2}^{\,9}}{\cancel{4s^2}} = 9$$

$\therefore$ T.T. = 9

$\therefore\ s^2+6s = (s^2+6s+9)-9$

Substitute these in Equation (1),

$$f(s) = \frac{s}{(s^2+6s+9)-9+25}$$

$$f(s) = \frac{s}{(s+3)^2+16}$$

$$\ldots\left[\because\ a^2+2ab+b^2 = (a+b)^2\right]$$

$$f(s) = \frac{(s+3)-3}{(s+3)^2+16} \qquad \ldots\left[\begin{array}{l}\text{Try to obtain every}\\ \text{s as } (s+3)\end{array}\right]$$

Taking inverse Laplace transform of both sides,

$$\therefore\ L^{-1}[f(s)] = L^{-1}\left[\frac{(s+3)-3}{(s+3)^2+16}\right]$$

...{Observe that all s are in the form (s – a) i.e. (s + 3)}

➔ **Using first shifting property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)]\right\}$$

$$\therefore\ L^{-1}[f(s)] = e^{-3t}L^{-1}\left[\frac{s-3}{s^2+16}\right]$$

...(Replace (s + 3) by s with e^{-3t} outside)

$$\therefore\ L^{-1}[f(s)] = e^{-3t}L^{-1}\left[\frac{s}{s^2+16}-\frac{3}{s^2+16}\right]$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[af(s)+bg(s)-ch(s)] = aL^{-1}[f(s)]+bL^{-1}[g(s)]-cL^{-1}[h(s)]\right\}$$

$$\therefore\ L^{-1}[f(s)] = e^{-3t}\left\{L^{-1}\left[\frac{s}{s^2+16}\right]-3L^{-1}\left[\frac{1}{s^2+16}\right]\right\}$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at \text{ and } L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$\therefore\ L^{-1}[f(s)] = e^{-3t}\left\{\underbrace{L^{-1}\left[\frac{s}{s^2+(4)^2}\right]}_{\cos 4t}-3\underbrace{L^{-1}\left[\frac{1}{s^2+(4)^2}\right]}_{\frac{1}{4}\sin 4t}\right\}$$

$$\therefore\ L^{-1}[f(s)] = e^{-3t}\left[\cos 4t-3\cdot\frac{1}{4}\sin 4t\right]$$

$$\therefore\ L^{-1}[f(s)] = e^{-3t}\left[\cos 4t-\frac{3}{4}\sin 4t\right]\checkmark \qquad \ldots\textbf{Ans.}$$

Ex. 2.3.6 : Find $L^{-1}\frac{2(s+1)}{s^2+2s+10}$

Soln. :

Let $f(s) = \frac{2(s+1)}{s^2+2s+10}$ ...(1)

For s^2+2s find third term (T. T.) as,

$$T.T. = \frac{(\text{Second term})^2}{4\times\text{first term}} \qquad \left\{\underset{\text{F.T.}}{s^2}+\underset{\text{S.T.}}{2s}\right\}$$

$$T.T. = \frac{(2s)^2}{4\times s^2} = \frac{4s^2}{4s^2} = 1$$

T. T. = 1

$\therefore\ s^2+2s = (s^2+2s+1)-1$

Substitute these in Equation (1),

$$f(s) = \frac{2(s+1)}{(s^2+2s+1)-1+10}$$

$$f(s) = \frac{2(s+1)}{(s+1)^2+9} \qquad \left[\because\ a^2+2ab+b^2 = (a+b)^2\right]$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[f(s)] = L^{-1}\left[\frac{2(s+1)}{(s^2+1)+9}\right]$$

...{Observe that all s are in the form of (s – a) i.e. (s + 1)}

➔ **Using the first shifting theorem of inverse Laplace transform**

$$\ldots\left[L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)]\right]$$

$$L^{-1}[f(s)] = e^{-1t}L^{-1}\left[\frac{2s}{s^2+9}\right]$$

{Replace (s + 1) by s with e^{-1t} outside}

$$f(t) = 2e^{-t}L^{-1}\left[\frac{s}{s^2+(3)^2}\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at\right]$$

$$\therefore\ f(t) = 2e^{-t}\underbrace{L^{-1}\left[\frac{s}{s^2+(3)^2}\right]}_{\cos 3t}$$

$$\ldots\left\{\text{compare } s^2+\underset{\substack{\uparrow\\ a^2}}{(3)^2} \text{ with } s^2+a^2\right\}$$

$\therefore$ f(t) = $2e^{-t}\cos 3t$✓ ...**Ans.**

2.4 Examples of LT using Property

$$L^{-1}\left[\frac{1}{s}f(s)\right] = \int_0^t f(t)\,dt \text{ Where, } f(t) = L^{-1}[f(s)]$$

Ex. 2.4.1 : Find inverse Laplace transform of $\frac{1}{s\sqrt{s+4}}$

Soln. :

Let $f(s) = \frac{1}{s\sqrt{s+4}} = \frac{1}{s}\cdot\frac{1}{\sqrt{s+4}}$

Taking inverse Laplace transform of both sides,

$$\therefore\ L^{-1}[f(s)] = L^{-1}\left[\frac{1}{s}\cdot\frac{1}{\sqrt{s+4}}\right]$$

➔ **Using property of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s}f(s)\right] = \int_0^t f(t)\,dt \text{ Where } f(t) = L^{-1}[f(s)]\right]$$

$$\therefore L^{-1}[f(s)] = \int_0^t L^{-1}\left[\frac{1}{\sqrt{s+4}}\right]dt$$

➔ **Using first shifting property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)]\right\}$$

$$L^{-1}[f(s)] = \int_0^t e^{-4t}L^{-1}\left[\frac{1}{\sqrt{s}}\right]dt \qquad \left\{\begin{array}{c}\text{Compare } s+4\\ \text{with } s-a\\ \therefore a=-4\end{array}\right\}$$

$$L^{-1}[f(s)] = \int_0^t e^{-4t}L^{-1}\left[\frac{1}{s^{1/2}}\right]dt$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{\Gamma(n+1)}\right]$$

$$= \int_0^t e^{-4t}\frac{t^{\frac{1}{2}-1}}{\Gamma\left(\frac{1}{2}\right)}dt$$

$$= \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t e^{-4t}t^{-1/2}\,dt$$

➔ **Using standard value of Gamma function**

$$\ldots\left[\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$\left(\because\ t^{-1/2} = \frac{1}{t^{1/2}} = \frac{1}{\sqrt{t}}\right)$$

$$\therefore\ L^{-1}\left[\frac{1}{s\sqrt{s+4}}\right] = \frac{1}{\sqrt{\pi}}\int_0^t\frac{e^{-4t}}{\sqrt{t}}\,dt\checkmark \qquad \ldots\textbf{Ans.}$$

Ex. 2.4.2 : Find inverse Laplace transform of $\frac{1}{s(s^2+4)}$

Soln. :

Let, $f(s) = \frac{1}{s(s^2+4)} = \frac{1}{s}\cdot\frac{1}{s^2+4}$

➔ **Using property of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s}f(s)\right] = \int_0^t f(t)\,dt \text{ Where } L^{-1}[f(s)] = f(t)\right]$$

$$L^{-1}[f(s)] = L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2+4}\right] = \int_0^t L^{-1}\left(\frac{1}{s^2+4}\right)dt$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$L^{-1}[f(s)] = \int_0^t \underbrace{L^{-1}\left[\frac{1}{s^2+(2)^2}\right]}_{\frac{1}{2}\sin 2t}dt$$

$$L^{-1}[f(s)] = \int_0^t\frac{1}{2}\sin 2t\,dt = \frac{1}{2}\int_0^t\sin 2t\,dt$$

➔ **Using standard integration :** $\ldots\left[\int \sin at\, dt = \frac{-\cos at}{a}\right]$

$$\therefore\ L^{-1}[f(s)] = \frac{1}{2}\left[\frac{-\cos 2t}{2}\right]_0^t = \frac{-1}{4}[\cos 2t]_0^t$$

Substitute t as : $\left(\begin{matrix}\text{upper}\\\text{limit}\end{matrix}\right) - \left(\begin{matrix}\text{lower}\\\text{limit}\end{matrix}\right)$

$$\therefore\ L^{-1}[f(s)] = \frac{-1}{4}[\cos 2t - \cos 0]$$

$$= \frac{-1}{4}[\cos 2t - 1] \qquad [\because \cos 0 = 1]$$

$$\therefore\ L^{-1}\left[\frac{1}{s(s^2+4)}\right] = \frac{1-\cos 2t}{4} \checkmark \qquad \ldots\text{Ans.}$$

Ex. 2.4.3 : Find inverse Laplace transform of $\frac{s^2+7}{s(s^2+9)}$

Soln. :

Let, $f(s) = \frac{s^2+7}{s(s^2+9)}$

$$= \frac{s^2+9-2}{s(s^2+9)} \qquad \ldots\text{(Note the adjustment)}$$

$$= \frac{(s^2+9)-2}{s(s^2+9)} = \frac{s^2+9}{s(s^2+9)} - \frac{2}{s(s^2+9)}$$

$$f(s) = \frac{s^2+9}{s(s^2+9)} - \frac{2}{s(s^2+9)}$$

$$\ldots\left[\frac{a \pm b}{c} = \frac{a}{c} \pm \frac{b}{c}\right]$$

$$= \frac{1}{s} - \frac{2}{s(s^2+9)}$$

$$f(s) = \frac{1}{s} - 2 \cdot \frac{1}{s}\,\frac{1}{s^2+9}$$

Taking inverse Laplace transform of both sides,

$$\therefore\ L^{-1}[f(s)] = L^{-1}\left[\frac{1}{s} - 2 \cdot \frac{1}{s} \cdot \frac{1}{s^2+9}\right]$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[af(s) + bg(s) - ch(s)] = a\,L^{-1}[f(s)] + bL^{-1}[g(s)] - cL^{-1}[h(s)]\right\}$$

$$= L^{-1}\left[\frac{1}{s}\right] - 2\,L^{-1}\left[\frac{1}{s} \cdot \frac{1}{s^2+(3)^2}\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left(\frac{1}{s}\right) = 1\right]$$

➔ **Using property of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s}f(s)\right] = \int_0^t f(t)\,dt \text{ Where } f(t) = L^{-1}[f(s)]\right]$$

$$L^{-1}[f(s)] = \underbrace{L^{-1}\left[\frac{1}{5}\right]}_{1} - 2\,\underbrace{L^{-1}\left[\frac{1}{s} \cdot \frac{1}{s^2+9}\right]}_{\int_0^t L^{-1}\left[\frac{1}{s^2+9}\right] dt}$$

$$\therefore\ L^{-1}[f(s)] = 1 - 2\int_0^t L^{-1}\left[\frac{1}{s^2+(3)^2}\right] dt$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$= 1 - 2\int_0^t \underbrace{L^{-1}\left[\frac{1}{s^2+(3)^2}\right]}_{\frac{1}{3}\sin 3t} dt$$

$$\therefore\ L^{-1}[f(s)] = 1 - 2\int_0^t \frac{1}{3}\sin 3t\, dt$$

$$= 1 - \frac{2}{3}\int_0^t \sin 3t\, dt$$

➔ **Using standard integration result**

$$\ldots\left[\int \sin at\, dt = \left(\frac{-\cos at}{a}\right)\right]$$

$$\therefore\ L^{-1}[f(s)] = 1 - \frac{2}{3}\left[\frac{-\cos 3t}{3}\right]_0^t = 1 + \frac{2}{9}[\cos 3t]_0^t$$

Substitute t as : $\left(\begin{matrix}\text{upper}\\\text{limit}\end{matrix}\right) - \left(\begin{matrix}\text{lower}\\\text{limit}\end{matrix}\right)$

$$L^{-1}[f(s)] = 1 + \frac{2}{9}[\cos 3t - \cos 0]$$

$$L^{-1}[f(s)] = 1 + \frac{2}{9}[\cos 3t - 1] \qquad \ldots[\because \cos 0 = 1]$$

$$= 1 + \frac{2}{9}\cos 3t - \frac{2}{9} \qquad \ldots\text{(by simplification)}$$

$$= \left(1 - \frac{2}{9}\right) + \frac{2}{9}\cos 3t \text{ (Grouping the terms)}$$

$$L^{-1}[f(s)] = \frac{7}{9} + \frac{2}{9}\cos 3t$$

$$L^{-1}\left[\frac{s^2+7}{s(s^2+9)}\right] = \frac{1}{9}(7 + 2\cos 3t) \checkmark \qquad \ldots\text{Ans.}$$

Ex. 2.4.4 : Find inverse Laplace transform of $\frac{1}{s^3(s^2+1)}$

Soln. :

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

We get,

$$L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$$

Now,

➔ **Using property of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s}f(s)\right] = \int_0^t f(t)\,dt \text{ Where } f(t) = L^{-1}[f(s)]\right]$$

$$L^{-1}\left[\frac{1}{s} \cdot \frac{1}{s^2+1}\right] = \int_0^t L^{-1}\left(\frac{1}{s^2+1}\right) dt$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$= \int_0^t \underbrace{L^{-1}\left(\frac{1}{s^2+1}\right)}_{\sin t} dt = \int_0^t \sin t\, dt$$

➔ **Using standard integration result**

$$\ldots\left[\int \sin at\, dt = \left(\frac{-\cos at}{a}\right)\right]$$

$$L^{-1}\left[\frac{1}{s} \cdot \frac{1}{s^2+1}\right] = [-\cos t]_0^t = -[\cos t]_0^t$$

Substitute t as : $\left(\begin{matrix}\text{upper}\\\text{limit}\end{matrix}\right) - \left(\begin{matrix}\text{lower}\\\text{limit}\end{matrix}\right)$

$$= -[\cos t - \cos 0]$$

$$= -[\cos t - 1] = -\cos t + 1$$

$$(\because \cos 0 = 1)$$

$$\therefore\ L^{-1}\left[\frac{1}{s \cdot (s^2+1)}\right] = (1 - \cos t) \qquad \ldots(1)$$

Again,

➔ **Using property of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s}f(s)\right] = \int_0^t f(t)\,dt \text{ Where } f(t) = L^{-1}[f(s)]\right]$$

$$L^{-1}\left[\frac{1}{s} \cdot \frac{1}{s(s^2+1)}\right] = \int_0^t L^{-1}\left[\frac{1}{s(s^2+1)}\right] dt$$

$$= \int_0^t (1 - \cos t)\, dt \quad \left(\begin{matrix}\text{From}\\\text{Equation (1)}\end{matrix}\right)$$

By integration of standard functions,

$$\therefore\ L^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = [t - \sin t]_0^t$$

Substitute t as : $\left(\begin{matrix}\text{upper}\\\text{limit}\end{matrix}\right) - \left(\begin{matrix}\text{lower}\\\text{limit}\end{matrix}\right)$

$$= (t - \sin t) - (0 - \sin 0)$$

$$\therefore\ L^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = t - \sin t \qquad \ldots(2) \quad (\because \sin 0 = 0)$$

Again,

➔ **Using property of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s}f(s)\right] = \int_0^t f(t)\,dt \text{ Where } f(t) = L^{-1}[f(s)]\right]$$

$$\therefore\ L^{-1}\left[\frac{1}{s} \cdot \frac{1}{s^2(s^2+1)}\right] = \int_0^t L^{-1}\left[\frac{1}{s^2(s^2+1)}\right] dt$$

$$= \int_0^t (t - \sin t)\, dt$$

$$\ldots \text{(From Equation (2))}$$

By integration of standard functions,

$$\therefore\ L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = \left[\frac{t^2}{2} - (-\cos t)\right]_0^t$$

$$= \left[\frac{t^2}{2} + \cos t\right]_0^t$$

Substitute t as : $\left(\begin{matrix}\text{upper}\\\text{limit}\end{matrix}\right) - \left(\begin{matrix}\text{lower}\\\text{limit}\end{matrix}\right)$

$$= \left[\frac{t^2}{2} + \cos t\right] - [0 + \cos 0]$$

$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = \frac{t^2}{2} + \cos t - 1 \checkmark \quad (\because \cos 0 = 1)\ldots\text{Ans.}$$

2.5 Examples Using Second Shifting Theorem

To find inverse Laplace transform of the function if f(s) is multiplied by e^{-as}

i.e. To find $L^{-1}[e^{-as} f(s)]$ We can use second shifting property

We know, second shifting property,

$$L^{-1}[e^{-as} \cdot f(s)] = F(t)\ ;\ \text{Where } F(t) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t < a \end{cases}$$

Steps to find $L^{-1}[e^{-as} \cdot f(s)]$

(I) Ignore e^{-as} and consider only f(s)

(II) Find $L^{-1}[f(s)] = f(t)$ (say)

(III) Using second shifting property,

$$L^{-1}[e^{-as} \cdot f(s)] = F(t)\ ;\ \text{Where } F(t) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t < a \end{cases}$$

means in f(t) replace every t by (t – a) gives f (t – a)

$$\therefore\ F(t) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t < a \end{cases}$$

which is required inverse Laplace transform

Ex. 2.5.1 : Find inverse Laplace transform of $\frac{e^{-2s}}{\sqrt{s+5}}$

Soln. :

We know, second shifting property,

$$L^{-1}\left[e^{-as}\cdot f(s)\right] = F(t)\,;\ \text{Where } F(t) = \begin{cases} f(t-a) & ;\ t>a \\ 0 & ;\ t<a \end{cases}$$

$$\therefore \text{To find } L^{-1}\left[\underset{\substack{\uparrow \\ e^{-as}}}{e^{-2s}}\cdot\underbrace{\frac{1}{\sqrt{s+5}}}_{f(s)}\right] = F(t) \text{ find } f(t) = L^{-1}[f(s)]$$

$$\text{Here } f(s) = \frac{1}{\sqrt{s+5}}$$

$$\therefore L^{-1}[f(s)] = L^{-1}\left[\frac{1}{\sqrt{s+5}}\right]$$

{Observe all s are in the form (s – a) i.e. (s + 5)}

➔ **Using first shifting property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)]\right\}$$

$$\therefore L^{-1}[f(s)] = e^{-5t}L^{-1}\left[\frac{1}{\sqrt{5}}\right] = e^{-5t}L^{-1}\left[\frac{1}{s^{1/2}}\right]$$

$$L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{\Gamma(n+1)}$$

$$\ldots\ L^{-1}[f(s)] = e^{-5t}\underbrace{L^{-1}\left[\frac{1}{s^{1/2}}\right]}_{\frac{t^{\frac{1}{2}-1}}{\Gamma\left(\frac{1}{2}\right)}} \qquad \left\{\because n+1 = \frac{1}{2}\right\}$$

$$L^{-1}[f(s)] = e^{-5t}\frac{t^{-1/2}}{\Gamma\left(\frac{1}{2}\right)}$$

➔ **Using standard value of Gamma function**

$$\ldots\left[\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= e^{-5t}\frac{t^{-1/2}}{\sqrt{\pi}} = e^{-5t}\frac{1}{\sqrt{\pi}\sqrt{t}}$$

$$\therefore L^{-1}[f(s)] = \frac{e^{-5t}}{\sqrt{\pi}\sqrt{t}} = f(t)$$

We know, second shifting property,

$$L^{-1}\left[e^{-as}\cdot f(s)\right] = F(t)\,;\ \text{Where } F(t) = \begin{cases} f(t-a) & ;\ t>a \\ 0 & ;\ t<a \end{cases}$$

$$\therefore F(t) = \begin{cases} f(t-2) & ;\ t>2 \\ 0 & ;\ t<2 \end{cases}$$

$$\text{i.e. } F(t) = \begin{cases} \dfrac{e^{-5(t-2)}}{\sqrt{\pi(t-2)}} & ;\ t>2 \\ 0 & ;\ t<2 \end{cases} \quad ✓ \quad \ldots\text{Ans.}$$

Ex. 2.5.2 : Find inverse Laplace transform of $\frac{-e^{\pi s}}{s^2+6s+5}$

Soln. :

We know, second shifting property,

$$L^{-1}\left[e^{-as}\cdot f(s)\right] = F(t)\,;\ \text{Where } F(t) = \begin{cases} f(t-a) & ;\ t>a \\ 0 & ;\ t<a \end{cases}$$

$$\textbf{To find : } L^{-1}\left[\underset{\substack{\uparrow \\ e^{-as}}}{e^{-\pi s}}\cdot\underbrace{\frac{1}{s^2+6s+5}}_{f(s)}\right] = F(t) \text{ find } f(t) = L^{-1}[f(s)]$$

$$\text{Here } f(s) = \frac{1}{s^2+6s+5} \qquad \ldots(1)$$

For $s^2 + 6s$ find third term (T. T.) as,

$$\text{T. T.} = \frac{(\text{Second term})^2}{4\times\text{first term}} \qquad \begin{bmatrix} s^2 & + & 6s \\ \uparrow & & \uparrow \\ \text{F.T.} & & \text{S.T.} \end{bmatrix}$$

$$= \frac{(6s)^2}{4\times s^2} = \frac{36s^2}{4\times s^2} = 9$$

$$\text{T. T.} = 9$$

$$\therefore s^2+6s = (s^2+6s+9)-9$$

Substitute these in Equation (1),

$$\therefore f(s) = \frac{1}{(s^2+6s+9)-9+5}$$

$$f(s) = \frac{1}{(s+3)^2-4}$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[f(s)] = L^{-1}\left[\frac{1}{(s+3)^2-4}\right]$$

...(Observe nature of all s as (s – a) i.e. s + 3)

➔ **Using first shifting property of inverse Laplace transform**

$$\ldots\left[L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)]\right]$$

$$\therefore L^{-1}[f(s)] = L^{-1}\left[\frac{1}{(s+3)^2-4}\right]$$

$$f(t) = e^{-3t}L^{-1}\left[\frac{1}{s^2-4}\right]$$

...{Replace s + 3 by s with e^{-3t} outside}

$$f(t) = e^{-3t}L^{-1}\left[\frac{1}{s^2-(2)^2}\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at\right]$$

$$\therefore f(t) = e^{-3t}\underbrace{L^{-1}\left[\frac{1}{s^2-(2)^2}\right]}_{\frac{1}{2}\sinh 2t}$$

$$\ldots\left\{\text{compare } s^2-\underset{\substack{\uparrow \\ a^2}}{(2)^2} \text{ with } s^2-a^2\right\}$$

$$\therefore f(t) = e^{-3t}\cdot\left(\frac{1}{2}\sinh 2t\right)$$

$$f(t) = \frac{1}{2}e^{-3t}\sinh 2t$$

We know, second shifting property,

$$L^{-1}\left[e^{-as}\cdot f(s)\right] = F(t)\,;\ \text{Where } F(t) = \begin{cases} f(t-a) & ;\ t>a \\ 0 & ;\ t<a \end{cases}$$

$$F(t) = \begin{cases} f(t-\pi) & ;\ t>\pi \\ 0 & ;\ t<\pi \end{cases}$$

Here $a = \pi$

$$\therefore F(t) = \begin{cases} \frac{1}{2}e^{-3(t-\pi)}\sinh 2(t-\pi) & ;\ t>\pi \\ 0 & ;\ t<\pi \end{cases} \quad ✓ \quad \ldots\text{Ans.}$$

Ex. 2.5.3 : Find inverse Laplace transform of $\frac{e^{-s}}{(s+5)^{3/2}}$

Soln. :

We know, second shifting property,

$$L^{-1}\left[e^{-as}\cdot f(s)\right] = F(t)\,;\ \text{Where } F(t) = \begin{cases} f(t-a) & ;\ t>a \\ 0 & ;\ t<a \end{cases}$$

$$\text{To find } L^{-1}\left[\underset{\substack{\uparrow \\ e^{-as}}}{e^{-s}}\cdot\underbrace{\frac{1}{(s+5)^{3/2}}}_{f(s)}\right] = F(t) \text{ find } f(t) = L^{-1}[f(s)]$$

$$\text{Here } f(s) = \frac{1}{(s+5)^{3/2}}$$

$$L^{-1}[f(s)] = L^{-1}\left[\frac{1}{(s+5)^{3/2}}\right] \qquad \left\{\begin{array}{c}\text{Observe that all} \\ \text{s are in the form of} \\ \text{(s – a) i.e (s + 4)}\end{array}\right\}$$

➔ **Using first shifting property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)]\right\}$$

$$L^{-1}[f(s)] = e^{-5t}L^{-1}\left[\frac{1}{s^{3/2}}\right]$$

$$L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{\Gamma(n+1)}$$

$$L^{-1}[f(s)] = e^{-5t}\underbrace{L^{-1}\left[\frac{1}{s^{3/2}}\right]}_{\frac{t^{\frac{3}{2}-1}}{\Gamma\left(\frac{3}{2}\right)}} \qquad \left(n+1 = \frac{3}{2}\right)$$

$$f(t) = e^{-5t}\frac{t^{\frac{3}{2}-1}}{\Gamma\left(\frac{3}{2}\right)}$$

➔ **By property of Gamma function**

$$\ldots\left[\Gamma(n+1) = n\Gamma(n) \text{ and } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= e^{-5t}\frac{t^{\frac{1}{2}}}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = e^{-5t}\frac{\sqrt{t}}{\frac{1}{2}\sqrt{\pi}} \qquad \left\{\begin{array}{l}\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) \\ = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)\end{array}\right\}$$

$$f(t) = 2e^{-5t}\sqrt{\frac{t}{\pi}}$$

We know, second shifting property,

$$L^{-1}\left[e^{-as}\cdot f(s)\right] = F(t)\,;\ \text{Where } F(t) = \begin{cases} f(t-a) & ;\ t>a \\ 0 & ;\ t<a \end{cases}$$

$$\therefore F(t) = \begin{cases} f(t-1) & ;\ t>1 \\ 0 & ;\ t<1 \end{cases}$$

$$\therefore F(t) = \begin{cases} 2e^{-5(t-1)}\sqrt{\frac{t-1}{\pi}} & ;\ t>1 \\ 0 & ;\ t<1 \end{cases} \quad ✓ \quad \ldots\text{Ans.}$$

Syllabus Topic : Inverse Laplace Transforms Using Partial fraction method

2.6 Partial Fraction of Proper Fraction

In the proper fraction, the algebraic fraction $\frac{P(x)}{Q(x)}$, $Q(x) \neq 0$ and degree P(x) < degree Q(x).

2.6.1 For Linear Factors

(ax + b), (cx + d) etc. split into addition with numerator as A, B, C etc.

Examples :

(i) $\frac{1}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$

(ii) $\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$

(iii) $\frac{1}{(x-1)(2x+3)(x+5)} = \frac{A}{x-1} + \frac{B}{2x+3} + \frac{C}{x+5}$

2.6.2 Working Rule to Find Partial Fraction

Q. Give the working rule to find partial fraction.

To find partial fraction of $\frac{P(x)}{Q(x)}$ with Q(x) contains only linear non repeated factors.

(i) Find the all factors of Q(x). Suppose all factors are linear and non repeated. In particular these are three linear factors as (a_1x+b_1), (a_2x+b_2), (a_3x+b_3).

$$\therefore Q(x) = (a_1x+b_1)(a_2x+b_2)(a_3x+b_3)$$

(ii) Write in correct form as,

$$\frac{P(x)}{Q(x)} = \frac{A}{a_1x+b_1} + \frac{B}{a_2x+b_2} + \frac{C}{a_3x+b_3} \quad ...(1)$$

(iii) Simplify R.H.S. by taking L.C.M. of R.H.S. denominator.

i.e. $$\frac{P(x)}{Q(x)} = \frac{A(a_2x+b_2)(a_3x+b_3) + B(a_1x+b_1)(a_3x+b_3) + C(a_1x+b_1)(a_2x+b_2)}{(a_1x+b_1)(a_2x+b_2)(a_3x+b_3)}$$

Since denominator of both sides are equal.

Equate numerator of both side. It gives,

$$P(x) = A(a_2x+b_2)(a_3x+b_3) + B(a_1x+b_1)(a_3x+b_3) + C(a_1x+b_1)(a_2x+b_2) \quad ...(2)$$

(iv) **To find A** put $a_1x+b_1=0 \Rightarrow x=\frac{-b_1}{a_1}$ in Equation (2).

To find B put $a_2x+b_2=0 \Rightarrow x=\frac{-b_2}{a_2}$ in Equation (2)

To find C put $a_3x+b_3=0 \Rightarrow x=\frac{-b_3}{a_3}$ in Equation (2)

(v) Substitute these values (A, B, C) in Equation (1). Gives required partial fraction.

Table 2.6.1 : Partial Fraction Decomposition

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots\dots + \frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
(a_1x+b_1) and (ax^2+bx+c)	$\frac{A}{(a_1x+b_1)} + \frac{Bx+C}{(ax^2+bx+c)}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

2.6.3 Examples using Partial Fraction

Ex. 2.6.1 : Find inverse Laplace transform of $\frac{3s+1}{(s-4)(s+3)}$

Soln. :

Let, $$f(s) = \frac{3s+1}{(s-4)(s+3)} \quad ...(1)$$

By partial fraction

$$\frac{3s+1}{(s-4)(s+3)} = \frac{A}{s-4} + \frac{B}{s+3} \quad ...(2)$$

$$\frac{3s+1}{\cancel{(s-4)(s+3)}} = \frac{A(s+3)+B(s-4)}{\cancel{(s-4)(s+3)}}$$

$$\therefore 3s+1 = A(s+3)+B(s-4) \quad ...(3)$$

Put, $s-4 = 0 \Rightarrow s=4$ in Equation (3),

$$3(4)+1 = A(4+3)+B(0)$$
$$12+1 = A(7)+0$$
$$13 = 7A$$
$$\therefore A = \frac{13}{7}$$

Put $s+3 = 0 \Rightarrow s=-3$ in Equation (3),

$$3(-3)+1 = A(0)+B(-3-4)$$
$$-9+1 = 0+B(-7)$$
$$-8 = -7B$$
$$\therefore B = \frac{-8}{-7}\ ;\ B=\frac{8}{7}$$

Substitute these $\left(A=\frac{13}{7},\ B=\frac{8}{7}\right)$ values in Equation (2),

$$\frac{3s+1}{(s-4)(s+3)} = \frac{\frac{13}{7}}{s-4} + \frac{\frac{8}{7}}{s+3}$$

Substitute these in Equation (1),

$$\therefore f(s) = \frac{\frac{13}{7}}{s-4} + \frac{\frac{8}{7}}{s+3}$$

$$f(s) = \frac{13}{7}\cdot\frac{1}{s-4} + \frac{8}{7}\cdot\frac{1}{s+3}$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[f(s)] = L^{-1}\left[\frac{13}{7}\cdot\frac{1}{s-4} + \frac{8}{7}\cdot\frac{1}{s+3}\right]$$

➔ **Using linearity property of inverse Laplace transform**

$$...[L^{-1}[a f(s) + b g(s) - c h(s)] = a L^{-1}[f(s)] + b L^{-1}[g(s)] - c L^{-1}[h(s)]]$$

$$\therefore L^{-1}[f(s)] = L^{-1}\left[\frac{13}{7}\cdot\frac{1}{s-4} + \frac{8}{7}\cdot\frac{1}{s+3}\right]$$

$$f(t) = \frac{13}{7}L^{-1}\left[\frac{1}{s-4}\right] + \frac{8}{7}L^{-1}\left[\frac{1}{s+3}\right]$$

$$f(t) = \frac{13}{7}L^{-1}\left[\frac{1}{s-4}\right] + \frac{8}{7}L^{-1}\left[\frac{1}{s-(-3)}\right]$$

➔ **Using standard result of inverse Laplace transform**

$$...\left[L^{-1}\left[\frac{1}{s-a}\right] = e^{at}\right]$$

$$f(t) = \frac{13}{7}\underbrace{L^{-1}\left[\frac{1}{s-4}\right]}_{e^{4t}} + \frac{8}{7}\underbrace{L^{-1}\left[\frac{1}{s-(-3)}\right]}_{e^{-3t}}$$

$$f(t) = \frac{13}{7}e^{4t} + \frac{8}{7}e^{-3t} \qquad \text{Common out } \frac{1}{7},$$

$$f(t) = \frac{1}{7}(13e^{4t} + 8e^{-3t}) \checkmark \qquad ...\text{Ans.}$$

Ex. 2.6.2 : Find the inverse Laplace transform $\frac{s^2+2}{s(s^2+4)}$

Soln. : Given, $f(s) = \frac{s^2+2}{s(s^2+4)}$ ...(1)

By Partial fraction, $$\frac{s^2+2}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \quad ....(2)$$

$$\frac{s^2+2}{\cancel{s(s^2+4)}} = \frac{A(s^2+4)+s(Bs+C)}{\cancel{s(s^2+4)}}$$

$$\therefore s^2+2 = \underbrace{A(s^2+4)+(Bs+C)(s)}_{As^2+A\cdot 4+sBs+sC}$$

...(by distributive property)

$$s^2+2 = As^2+4A+Bs^2+Cs$$
$$= (As^2+Bs^2)+(Cs)+4A$$

....(Grouping the terms)

$$s^2+2 = (A+B)s^2+Cs+4A$$
$$1s^2+0s+2 = (A+B)s^2+CS+4A$$

Equating coefficient of same powers of s

$$\therefore 1 = A+B\ ;\ 0 = C\ ;\ 2 = 4A$$
$$\therefore \frac{2}{4} = A\ ;\ \therefore A = \frac{1}{2}s$$

Now, $A+B = 1$

$$\frac{1}{2}+B = 1 \qquad ...\left(\because A=\frac{1}{2}\right)$$
$$B = 1-\frac{1}{2}$$
$$B = \frac{1}{2}$$

Substitute these values $\left[A=\frac{1}{2}, B=\frac{1}{2}, C=0\right]$ in Equation (2)

$$\therefore \frac{s^2+2}{s(s^2+4)} = \frac{\frac{1}{2}}{s} + \frac{\frac{1}{2}s+0}{s^2+4}$$

$$f(s) = \frac{\frac{1}{2}}{s} + \frac{\frac{1}{2}s}{s^2+4}$$

$$= \frac{1}{2}\left(\frac{1}{s}\right) + \frac{1}{2}\left(\frac{s}{s^2+4}\right)$$

Taking inverse Laplace transform of both sides,

➔ **Using the linearity property of inverse Laplace transform**

$$...[L^{-1}[a f(s) + b g(s) - c h(s)] = a L^{-1}[f(s)] + b L^{-1}[g(s)] - c L^{-1}[h(s)]]$$

$$\therefore L^{-1}[f(s)] = \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) + \frac{1}{2}L^{-1}\left(\frac{s}{s^2+4}\right)$$

➔ **Using standard result of inverse Laplace transform**

$$...\left[L^{-1}\left[\frac{1}{s}\right] = 1 \text{ and } L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at\right]$$

$$L^{-1}[f(s)] = \frac{1}{2}\underbrace{L^{-1}\left(\frac{1}{s}\right)}_{1} + \frac{1}{2}\underbrace{L^{-1}\left(\frac{s}{s^2+(2)^2}\right)}_{\cos 2t}$$

$$\left\{\text{Compare : } \frac{s}{s^2+4} \text{ with } \frac{s}{s^2+a^2} \therefore a^2 = 4 \Rightarrow a = 2\right\}$$

$$L^{-1}[f(s)] = \frac{1}{2}(1) + \frac{1}{2}(\cos 2t)$$

$$\therefore f(t) = \frac{1}{2} + \frac{1}{2}\cos 2t$$

$$\left[\because L^{-1}[f(s)] = f(t)\right]$$

Common at $\frac{1}{2}$ from RHS,

$$f(t) = \frac{1}{2}(1+\cos 2t) \checkmark \qquad ...\text{Ans.}$$

Ex. 2.6.3 : Find inverse Laplace transform of $\frac{3s+1}{(s-1)(s^2+1)}$

Soln. :

Let, $$f(s) = \frac{3s+1}{(s-1)(s^2+1)} \quad ...(1)$$

By partial fraction

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} \quad ...(2)$$

$$\frac{3s+1}{\cancel{(s-1)(s^2+1)}} = \frac{A(s^2+1)+(Bs+C)(s-1)}{\cancel{(s-1)(s^2+1)}}$$

$$\therefore (3s+1) = A(s^2+1)+(Bs+C)(s-1) \quad ...(3)$$

Put $s-1 = 0 \Rightarrow s=1$ in Equation (3),

$$3(1)+1 = A[(1)^2+1]+[(B(1)+C](0)$$
$$3+1 = A[1+1]+0$$
$$4 = 2A \qquad \frac{4}{2} = A$$
$$\therefore A = \frac{4}{2} \qquad \therefore A = 2$$

Put $s = 0$ in Equation (3),

$$3(0)+1 = A(0+1)+(B(0)+C)(0-1)$$
$$0+1 = A(1)+(0+C)(-1)$$
$$1 = A-C$$

Since, $A = 2$

$$\therefore 1 = 2-C \Rightarrow 1-2 = -C$$
$$\therefore -1 = -C \qquad \therefore C = 1$$

Put $s = -1$ in Equation (3),

$$3(-1)+1 = A[(-1)^2+1]+[B(-1)+C](-1-1)$$
$$-3+1 = A(1+1)+(-B+C)(-2)$$
$$-2 = A(2)+2B-2C$$

Since, $A = 2,\ C = 1$

$$\therefore -2 = (2)(2) + 2B - 2(1)$$
$$-2 = 4 + 2B = 2$$
$$-2 = (4-2) + 2B$$
$$-2 = 2 + 2B$$
$$\therefore -4 = 2B$$
$$\therefore 2B = -\cancel{2} - 4 + \cancel{2}$$
$$2B = -4$$
$$\therefore B = -\frac{4}{2} = -2$$
$$\therefore B = -2$$

Substitute these values (A = 2, B = – 2, C = 1) in Equation (2)

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{2}{s-1} + \underbrace{\frac{-2s+1}{s^2+1}}$$

$$\frac{-2s}{s^2+1} + \frac{1}{s^2+1} \quad \ldots\left\{\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}\right\}$$

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{2}{s-1} + \frac{-2s}{s^2+1} + \frac{1}{s^2+1}$$

Substitute these in Equation (1),

$$f(s) = \frac{2}{s-1} + \frac{-2s}{s^2+1} + \frac{1}{s^2+1}$$
$$\frac{-2s}{s^2+1} + \frac{1}{s^2+1}$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[f(s)] = L^{-1}\left[\frac{2}{s-1} - \frac{-2s}{s^2+1} + \frac{1}{s^2+1}\right]$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots[L^{-1}[a f(s) + b g(s) - c h(s)] = a L^{-1}[f(s)] + b L^{-1}[g(s)] - c L^{-1}[h(s)]]$$

$$L^{-1}[f(s)] = 2L^{-1}\left(\frac{1}{s-1}\right) - 2L^{-1}\left(\frac{s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s^2+1}\right)$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left(\frac{1}{s-a}\right) = e^{at},\ L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at \text{ and } L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at\right]$$

$$\therefore L^{-1}[f(s)] = 2\underbrace{L^{-1}\left(\frac{1}{s-1}\right)}_{e^{1t}} - 2\underbrace{L^{-1}\left(\frac{s}{s^2+(1)^2}\right)}_{\cos 1t} + \underbrace{L^{-1}\left(\frac{1}{s^2+(1)^2}\right)}_{\frac{1}{1}\sin 1t}$$

$$f(t) = 2[e^{1t}] - 2[\cos 1t] + \frac{1}{1}[\sin 1t]$$

$$f(t) = 2e^{t} - 2\cos t + \sin t \checkmark \quad \ldots\text{Ans.}$$

Ex. 2.6.4 : Find inverse Laplace transform of $\dfrac{2s+1}{(s-1)(s^2+1)}$

Soln. :

$$\text{Let } f(s) = \frac{2s+1}{(s-1)(s^2+1)} \quad \ldots(1)$$

By partial fraction

$$\frac{2s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{(s^2+1)} \quad \ldots(2)$$

$$\frac{2s+1}{\cancel{(s-1)(s^2+1)}} = \frac{A(s^2+1) + (Bs+C)(s-1)}{\cancel{(s-1)(s^2+1)}}$$

[by simplification taking L.C.M. of R.H.S.]

$$\therefore 2s+1 = A(s^2+1) + (Bs+C)(s-1) \quad \ldots(3)$$

Put $s - 1 = 0 \Rightarrow s = 1$, in Equation (3),

[To find A from Equation (2), put denominator of A equal to Zero]

$$2(1)+1 = A[(1)^2+1] + [B(1)+C](0)$$
$$2+1 = A(1+1) + 0$$
$$3 = A(2)$$
$$\therefore 3 = 2A$$
$$\frac{3}{2} = A \Rightarrow A = \frac{3}{2}$$

Put, $s = 0$, in Equation (3),

$$2(0)+1 = A[(0)^2+1] + [B(0)+C](0-1)$$
$$0+1 = A(1) + C(-1)$$
$$1 = A - C$$

Since, $A = \frac{3}{2}$

$$1 = \frac{3}{2} - C \qquad \therefore 1 - \frac{3}{2} = -C$$
$$\therefore -1 + \frac{3}{2} = C \qquad \therefore \frac{3}{2} - 1 = C$$
$$\therefore C = \frac{3}{2} - 1 = \frac{3-2}{2}$$
$$\Rightarrow C = \frac{1}{2}$$

Put $s = -1$, in Equation (3),

$$2(-1)+1 = A[(-1)^2+1] + [B(-1)+C](-1-1)$$
$$-2+1 = A(1+1) + (-B+C)(-2)$$
$$-1 = A(2) + 2B - 2C$$

Since, $A = \frac{3}{2}, C = \frac{1}{2}$

$$-1 = \left(\frac{3}{2}\right)(2) + 2B - 2\left(\frac{1}{2}\right)$$
$$-1 = 3 + 2B - 1$$
$$\therefore \cancel{-1} - 3 + \cancel{1} = 2B$$
$$-3 = 2B \Rightarrow \frac{-3}{2} = B, \quad B = \frac{-3}{2}$$

Substitute these values $\left(A = \frac{3}{2}, B = \frac{-3}{2}, C = \frac{1}{2}\right)$ in Equation (2)

Equation (2) becomes,

$$\frac{2s+1}{(s-1)(s^2+1)} = \frac{\frac{3}{2}}{s-1} + \frac{\frac{-3}{2}s + \frac{1}{2}}{s^2+1}$$
$$= \frac{\frac{1}{2}\cdot 3}{s-1} + \frac{\frac{1}{2}(-3)s + \frac{1}{2}}{s^2+1}$$
$$= \frac{\frac{1}{2}\cdot 3}{s-1} + \frac{\frac{1}{2}(-3s+1)}{s^2+1}$$
$$\frac{2s+1}{(s-1)(s^2+1)} = \frac{1}{2}\left[\frac{3}{s-1} + \frac{-3s+1}{s^2+1}\right]$$

Substitute these in Equation (1),

$$f(s) = \frac{1}{2}\left[\frac{3}{s-1} + \underbrace{\frac{-3s+1}{s^2+1}}\right]$$
$$\frac{-\frac{3}{2}s}{s^2+1} + \frac{\frac{1}{2}}{s^2+1} \ldots\left\{\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}\right\}$$

$$f(s) = \frac{1}{2}\left[\frac{3}{s-1} + \frac{-3s}{s^2+1} + \frac{1}{s^2+1}\right]$$
$$f(s) = \frac{3}{2}\cdot\left(\frac{1}{s-1}\right) - \frac{3}{2}\left(\frac{s}{s^2+1}\right) + \frac{1}{2}\left(\frac{1}{s^2+1}\right)$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[f(s)] = L^{-1}\left[\frac{3}{2}\left(\frac{1}{s-1}\right) - \frac{3}{2}\left(\frac{s}{s^2+1}\right) + \frac{1}{2}\left(\frac{1}{s^2+1}\right)\right]$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots[L^{-1}[a f(s) + b g(s) - c h(s)] = a L^{-1}[f(s)] + b L^{-1}[g(s)] - c L^{-1}[h(s)]]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s-a}\right] = e^{at};\ L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at \text{ and } L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$L^{-1}[f(s)] = \frac{3}{2}\underbrace{L^{-1}\left[\frac{1}{s-1}\right]}_{e^{1t}} - \frac{3}{2}\underbrace{L^{-1}\left[\frac{s}{s^2+(1)^2}\right]}_{\cos 1t} + \frac{1}{2}\underbrace{L^{-1}\left[\frac{1}{s^2+(1)^2}\right]}_{\frac{1}{1}\sin 1t}$$

$$f(t) = \frac{3}{2}e^{1t} - \frac{3}{2}\cos 1t + \frac{1}{2}\cdot\left(\frac{1}{1}\right)\sin 1t$$

$$f(t) = \frac{3}{2}e^{t} - \frac{3}{2}\cos t + \frac{1}{2}\sin t \checkmark \quad \ldots\text{Ans.}$$

Ex. 2.6.5 : Find inverse Laplace transform of $\dfrac{2s+3}{s^2(s-1)}$

Soln. :

$$\text{Let, } f(s) = \frac{2s+3}{s^2(s-1)} \quad \ldots(1)$$

By partial fraction

$$\frac{2s+3}{s^2(s-1)} = \frac{A}{s-1} + \frac{B}{s} + \frac{C}{s^2} \quad \ldots(2)$$

$$\frac{2s+3}{\cancel{s^2(s-1)}} = \frac{As^2 + Bs(s-1) + C(s-1)}{\cancel{s^2(s-1)}}$$

[by simplification taking L.C.M. of R.H.S.]

$$2s+3 = As^2 + Bs(s-1) + C(s-1) \quad \ldots(3)$$

Put $s - 1 = 0 \Rightarrow s = 1$, in Equation (3),

[To find A from Equation (2), put denominator of A equal to Zero]

$$(2\times 1)+3 = A(1)^2 + B(1)(0) + C(0)$$
$$2+3 = A(1) + 0 + 0$$
$$5 = A \quad \Rightarrow \quad A = 5$$

Put, $s = 0$, in Equation (3),

[To find C from Equation (2), put denominator of C equal to Zero]

$$2(0)+3 = A(0) + B(0)(0-1) + C(0-1)$$
$$0+3 = 0 + 0 + C(-1)$$
$$3 = -C \quad \Rightarrow \quad C = -3$$

Put $s = -1$, in Equation (3),

$$[2\times(-1)]+3 = A(-1)^2 + B(-1)(-1-1) + C(-1-1)$$
$$-2+3 = A + B(-1)(-2) + C(-2)$$
$$1 = A + B(2) - 2C$$

Since $A = 5$ and $C = -3$

$$\therefore 1 = 5 + 2B - 2(-3) = 5 + 2B + 6$$
$$1 = (5+6) + 2B = 11 + 2B$$
$$\therefore 11 + 2B = 1$$
$$2B = 1 - 11 = -10$$
$$B = \frac{-10}{2} = -5 \quad \Rightarrow \quad B = -5$$

Substitute these values (A = 5, B = –5, C = – 3) in Equation (2)

Equations (2) becomes,

$$\frac{2s+3}{s^2(s-1)} = \frac{5}{s-1} + \frac{-5}{s} + \frac{-3}{s^2}$$
$$\frac{2s+3}{s^2(s-1)} = \frac{5}{s-1} - \frac{5}{s} - \frac{3}{s^2}$$

Substitute these in Equation (1),

$$f(s) = \frac{5}{s-1} - \frac{5}{s} - \frac{3}{s^2}$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[f(s)] = L^{-1}\left[\frac{5}{s-1} - \frac{5}{s} - \frac{3}{s^2}\right]$$

$$f(t) = L^{-1}\left[5\cdot\left(\frac{1}{s-1}\right) - 5\cdot\left(\frac{1}{s}\right) - 3\left(\frac{1}{s^2}\right)\right]$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots[L^{-1}[a f(s) + b g(s) - ch(s)] = a L^{-1}[f(s)] + b L^{-1}[g(s)] - cL^{-1}[h(s)]]$$

$$[f(t)] = 5L^{-1}\left(\frac{1}{s-1}\right) - 5L^{-1}\left(\frac{1}{s}\right) - 3L^{-1}\left(\frac{1}{s^2}\right)$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left(\frac{1}{s-a}\right) = e^{at};\ L^{-1}\left(\frac{1}{s}\right) = 1 \text{ and } L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{\Gamma n} \text{ OR } \frac{t^{n-1}}{(n-1)!} \text{ if n is an integer}\right]$$

$$f(t) = 5\cdot\underbrace{L^{-1}\left(\frac{1}{s-1}\right)}_{e^t} - 5\underbrace{L^{-1}\left(\frac{1}{s}\right)}_{1} - 3\underbrace{L^{-1}\left(\frac{1}{s^2}\right)}_{\frac{t^{2-1}}{(2-1)!}}$$

$$f(t) = 5e^t - 5(1) - 3\left[\frac{t^{2-1}}{(2-1)!}\right] = 5e^t - 5 - 3\frac{t}{1!}$$

$$\therefore f(t) = 5e^t - 5 - 3t \text{ or}$$

$$f(t) = 5(e^t - 1) - 3t \checkmark \qquad \ldots\text{Ans.}$$

Ex. 2.6.6 : Find inverse Laplace transform of $\frac{s+1}{(s+2)(s+3)}$

Soln. :

Let, $f(s) = \frac{s+1}{(s+2)(s+3)}$ ...(1)

By partial fraction

Now, consider, $\frac{s+1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$ ...(2)

$$\frac{s+1}{(s+2)(s+3)} = \frac{A(s+3) + B(s+2)}{(s+2)(s+3)}$$

[by simplification taking L.C.M. of R.H.S.]

$$s+1 = A(s+3) + B(s+2) \qquad \ldots(3)$$

Put $s+2=0 \Rightarrow s=-2$, in Equation (2),

[To find A from Equation (2), put denominator of A equal to Zero]

$$-2+1 = A(-2+3) + B(0)$$

$$-1 = A(1)$$

$$\Rightarrow A = -1$$

Put $s+3=0 \Rightarrow s=-3$, in Equation (3),

[To find B from Equation (2), put denominator of B equal to Zero]

$$-3+1 = A(0) + B(-3+2)$$

$$-2 = 0 + B(-1)$$

$$-2 = -B$$

$$\Rightarrow \mathbf{B = 2}$$

Substitute these values $(A=-1, B=2)$ in Equation (2)

Equation (2) becomes,

$$\therefore \frac{s+1}{(s+2)(s+3)} = \frac{-1}{s+2} + \frac{2}{s+3}$$

$$\frac{s+1}{(s+2)(s+3)} = \frac{2}{s+3} - \frac{1}{s+2} \text{ (Taking positive terms first)}$$

Substitute these in Equation (1),

$$f(s) = \frac{2}{s+3} - \frac{1}{s+2}$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[f(s)] = L^{-1}\left[\frac{2}{s+3} - \frac{1}{s+2}\right]$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots[L^{-1}[a f(s) + b g(s) - ch(s)] = a L^{-1}[f(s)] + b L^{-1}[g(s)] - c L^{-1}[h(s)]]$$

$$L^{-1}[f(s)] = 2L^{-1}\left[\frac{1}{s+3}\right] - L^{-1}\left[\frac{1}{s+2}\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s-a}\right] = e^{at}\right]$$

$$L^{-1}[f(s)] = 2\underbrace{L^{-1}\left[\frac{1}{s-(-3)}\right]}_{e^{-3t}} - \underbrace{L^{-1}\left[\frac{1}{s-(-2)}\right]}_{e^{-2t}}$$

$$f(t) = 2e^{-3t} - e^{-2t} \checkmark \qquad \ldots\text{Ans.}$$

Ex. 2.6.7 : Find inverse Laplace transform of $\frac{1}{(s^2+9)(s^2+16)}$

Soln. :

Let $f(s) = \frac{1}{(s^2+9)(s^2+16)}$

➔ **Using the formula of partial fraction**

$$\ldots\left[\frac{1}{(x-a)(x-b)} = \frac{1}{a-b}\left[\frac{1}{x-a} - \frac{1}{x-b}\right]\right]$$

Compare : $(s^2+9)(s^2+16)$ with $(x-a)(x-b)$

$$x^2 = s,\ -a = 9 \Rightarrow a = -9$$

$$-b = 16 \Rightarrow b = -16$$

$$\therefore f(s) = \frac{1}{(s^2+9)(s^2+16)}$$

$$= \frac{1}{-9-(-16)}\left[\frac{1}{(s^2+9)} - \frac{1}{(s^2+16)}\right]$$

$$f(s) = \frac{1}{7}\left[\frac{1}{s^2+9} - \frac{1}{s^2+16}\right]$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[f(s)] = L^{-1}\left\{\frac{1}{7}\left[\frac{1}{s^2+9} - \frac{1}{s^2+16}\right]\right\}$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots[L^{-1}[a f(s) + b g(s) - ch(s)] = a L^{-1}[f(s)] + b L^{-1}[g(s) - c L^{-1}[h(s)]]$$

$$L^{-1}[f(s)] = L^{-1}\left\{\frac{1}{7}\left[\frac{1}{s^2+9} - \frac{1}{s^2+16}\right]\right\}$$

$$f(t) = \frac{1}{7}\left\{L^{-1}\left[\frac{1}{s^2+9}\right] - L^{-1}\left[\frac{1}{s^2+16}\right]\right\}$$

$$f(t) = \frac{1}{7}\left\{L^{-1}\left[\frac{1}{s^2+(3)^2}\right] - L^{-1}\left[\frac{1}{s^2+(4)^2}\right]\right\}$$

➔ **Using the standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$f(t) = \frac{1}{7}\left\{\underbrace{L^{-1}\left[\frac{1}{s^2+(3)^2}\right]}_{\frac{1}{3}\sin 3t} - \underbrace{L^{-1}\left[\frac{1}{s^2+(4)^2}\right]}_{\frac{1}{4}\sin 4t}\right\}$$

$$f(t) = \frac{1}{7}\left[\frac{1}{3}(\sin 3t) - \frac{1}{4}(\sin 4t)\right] \checkmark \qquad \ldots\text{Ans.}$$

Ex. 2.6.8 : Find inverse Laplace transform of $\frac{s^2}{(s+1)(s+2)(s+3)}$

Soln. :

Let, $f(s) = \frac{s^2}{(s+1)(s+2)(s+3)}$ ...(1)

By partial fraction :

$$\frac{s^2}{(s+1)(s+2)(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} + \frac{C}{(s+3)} \qquad \ldots(2)$$

$$\therefore \frac{s^2}{(s+1)(s+2)(s+3)} = \frac{A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2)}{(s+1)(s+2)(s+3)}$$

[by simplification taking L.C.M. of R.H.S.]

$$\therefore s^2 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2) \qquad \ldots(3)$$

Put $s+1=0 \Rightarrow s=-1$, in Equation (3)

[To find A from Equation (2), put denominator of A equal to Zero]

$$\therefore (-1)^2 = A(-1+2)(-1+3) + B(0)(-1+3) + C(0)(-1+2)$$

$$1 = A(1)(2) + 0 + 0$$

$$1 = 2A \qquad \frac{1}{2} = A$$

$$\therefore A = \frac{1}{2}$$

Put $s+2=0 \Rightarrow s=-2$, in Equation (3),

[To find B from Equation (2), put denominator of B equal to Zero]

$$\therefore (-2)^2 = A(0)(-2+3) + B(-2+1)(-2+3) + C(-2+1)(0)$$

$$4 = 0 + B(-1)(1) + 0$$

$$4 = -B \qquad \Rightarrow B = -4$$

Put $s+3=0 \Rightarrow s=-3$, in Equation (3)

[To find C from Equation (2), put denominator of C equal to Zero]

$$(-3)^2 = A(-3+2)(0) + B(-3+1)(0) + C(-3+1)(-3+2)$$

$$9 = 0 + 0 + C(-2)(-1)$$

$$9 = 2C \qquad \Rightarrow C = \frac{9}{2}$$

Substitute these values $\left(A = \frac{1}{2}, B = -4, C = \frac{9}{2}\right)$ in Equation (2).

$$\therefore \frac{s^2}{(s+1)(s+2)(s+3)} = \frac{\frac{1}{2}}{s+1} + \frac{-4}{s+2} + \frac{\frac{9}{2}}{s+3}$$

$$\frac{s^2}{(s+1)(s+2)(s+3)} = \frac{1}{2}\cdot\frac{1}{s+1} - \frac{4}{s+2} + \frac{9}{2}\cdot\frac{1}{s+3}$$

Substitute these in Equation (1)

$$\therefore f(s) = \frac{1}{2}\cdot\frac{1}{s+1} - \frac{4}{s+2} + \frac{9}{2}\frac{1}{s+3}$$

Taking inverse Laplace transform of both sides

$$L^{-1}[f(s)] = L^{-1}\left[\frac{1}{2}\cdot\frac{1}{s+1} - \frac{4}{s+2} + \frac{9}{2}\cdot\frac{1}{s+3}\right]$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots[L^{-1}[a f(s) + b g(s) - c h(s)] = a L^{-1}[f(s)] + b L^{-1}[g(s) - c L^{-1}[h(s)]]$$

$$\therefore L^{-1}[f(s)] = \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] - 4L^{-1}\left[\frac{1}{s+2}\right] + \frac{9}{2}L^{-1}\left[\frac{1}{s+3}\right]$$

➔ **Using the standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s-a}\right] = e^{at}\right]$$

$$\therefore L^{-1}[f(s)] = \frac{1}{2}\underbrace{L^{-1}\left[\frac{1}{s-(-1)}\right]}_{e^{-1t}} - 4\underbrace{L^{-1}\left[\frac{1}{s-(-2)}\right]}_{e^{-2t}}$$

$$+ \frac{9}{2}\underbrace{L^{-1}\left[\frac{1}{s-(-3)}\right]}_{e^{-3t}}$$

$$f(t) = \frac{1}{2}e^{-1t} - 4e^{-2t} + \frac{9}{2}e^{-3t}$$

$$f(t) = \frac{1}{2}e^{-t} - 4e^{-2t} + \frac{9}{2}e^{-3t} \checkmark \qquad \text{...Ans.}$$

Ex. 2.6.9 : Find inverse Laplace transform of $\dfrac{s^2}{(s+1)(s+2)^2}$

Soln. :

Let, $$f(s) = \frac{s^2}{(s+1)(s+2)^2} \qquad \text{...(1)}$$

By partial fraction

$$\frac{s^2}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} \qquad \text{...(2)}$$

$$\frac{s^2}{\cancel{(s+1)(s+2)^2}} = \frac{A(s+2)^2 + B(s+1)(s+2) + C(s+1)}{\cancel{(s+1)(s+2)^2}}$$

[by simplification taking L.C.M. of R.H.S.]

$$\therefore \quad s^2 = A(s+2)^2 + B(s+1)(s+2) + C(s+1) \qquad \text{...(3)}$$

Put $s+1=0 \Rightarrow s=-1$, in Equation (3),

[To find A from Equation (2), put denominator of A equal to Zero]

$$(-1)^2 = A(-1+2)^2 + B(0)(-1+2) + C(0)$$
$$1 = A(1)^2 + 0 + 0$$
$$1 = A \quad \Rightarrow \quad A = 1$$

Put $s+2=0 \Rightarrow s=-2$, in Equation (3),

[To find C from Equation (2), put denominator of C equal to Zero]

$$\therefore \quad (-2)^2 = A(0) + B(-2+1)(0) + C(-2+1)$$
$$4 = 0 + 0 + C(-1)$$
$$4 = -C \quad \Rightarrow \quad C = -4$$

Put $s=0$, in Equation (2),

[To find B from Equation (1), put denominator of B equal to Zero]

$$(0)^2 = A(0+2)^2 + B(0+1)(0+2) + C(0+1)$$
$$0 = A(4) + B(1)(2) + C(1)$$
$$0 = 4A + 2B + C$$

Since, $A = 1$ and $C = -4$

$$\therefore 0 = (4 \times 1) + 2B - 4 = 4 + 2B - 4$$
$$0 = -2B \quad \Rightarrow \quad B = 0$$

Substitute these values ($A = 1$, $B = 0$, $C = -4$) in Equation (2).

$\therefore$ Equation (2) becomes,

$$\frac{s^2}{(s+1)(s+2)^2} = \frac{1}{s+1} + \frac{0}{s+2} + \frac{-4}{(s+2)^2}$$

$$\frac{s^2}{(s+1)(s+2)^2} = \frac{1}{s+1} - \frac{4}{(s+2)^2}$$

Substitute these in Equation (1),

$$f(s) = \frac{1}{s+1} - \frac{4}{(s+2)^2}$$

Taking inverse Laplace transform both sides,

$$L^{-1}[f(s)] = L^{-1}\left[\frac{1}{s+1} - \frac{4}{(s+2)^2}\right]$$

➔ **Using linearity property of inverse Laplace transform**

$$\text{...}[\ L^{-1}[a f(s) + b g(s) - c h(s)] = a L^{-1}[f(s)] + b L^{-1}[g(s)] - c L^{-1}[h(s)]\ ]$$

$$\therefore L^{-1}[f(s)] = L^{-1}\left[\frac{1}{s+1}\right] - 4L^{-1}\left[\frac{1}{s+2^2}\right]$$

➔ **Using standard result and first shifting property of inverse Laplace transform**

Taking e^{at} outside

↓

$$\text{...}\left\{L^{-1}\left[\frac{1}{s-a}\right] = e^{at} \text{ and } \underbrace{L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)]}\right\}$$

every (s – a) replace by s with e^{at} outside

$$\therefore L^{-1}[f(s)] = \underbrace{L^{-1}\left[\frac{1}{s-(-1)}\right]}_{e^{-1t}} - 4\underbrace{L^{-1}\left[\frac{s}{[s-(-2)]^2}\right]}_{e^{-2}L^{-1}\left[\frac{1}{s^2}\right]}$$

$$f(t) = e^{-1t} - 4e^{-2t}L^{-1}\left[\frac{1}{s^2}\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\text{...}\left[L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{\Gamma n} \text{ OR } \frac{t^{n-1}}{(n-1)!} \text{ if n is an integer}\right]$$

$$f(t) = e^{-t} - 4e^{-2t}\underbrace{L^{-1}\left[\frac{1}{s^2}\right]}_{\frac{t^{2-1}}{(2-1)!}}$$

$$f(t) = e^{-t} - 4e^{-2t}\frac{t^{2-1}}{(2-1)!}$$

$$f(t) = e^{-t} - 4e^{-2t}\frac{t}{1!}$$

$$f(t) = e^{-t} - 4e^{-2t}.t \checkmark \qquad \text{...Ans.}$$

Ex. 2.6.10 : Find inverse Laplace transform of $\dfrac{s+2}{s^2(s+3)}$

Soln. :

Let $$f(s) = \frac{s+2}{s^2(s+3)} \qquad \text{...(1)}$$

By partial fraction

$$\frac{s+2}{s^2(s+3)} = \frac{A}{s+3} + \frac{Bs+C}{s^2} \qquad \text{...(2)}$$

$$\frac{s+2}{s^2(s+3)} = \frac{A(s^2) + (Bs+C)(s+3)}{(s+3)s^2}$$

$$s+2 = As^2 + (Bs+C)(s+3) \qquad \text{...(3)}$$

Put $s+3=0 \Rightarrow s=-3$ in Equation (3),

$$\therefore -3+2 = A(-3)^2 + (B(-3)+C)(0)$$
$$-1 = A(9) + 0$$
$$-1 = 9A \qquad \therefore A = \frac{-1}{9}$$

Put $s=0$ in Equation (3),

$$\therefore 0+2 = A(0) + (B(0)+C)(0+3)$$
$$2 = 0 + C(3)$$
$$2 = 3C \qquad \therefore C = \frac{2}{3}$$

Put $s=1$ in Equation (3),

$$1+2 = A(1)^2 + (B(1)+C)(1+3)$$
$$3 = A + (B+C)(4)$$

since $A = \dfrac{-1}{9}$ and $C = \dfrac{2}{3}$

$$\therefore 3 = \frac{-1}{9} + \left(B + \frac{2}{3}\right)(4)$$
$$3 = \frac{-1}{9} + 4B + \frac{8}{3}$$
$$\therefore 4B = 3 + \frac{1}{9} - \frac{8}{3}$$
$$4B = \frac{27+1-24}{9} = \frac{4}{9}$$
$$\therefore B = \frac{1}{9}$$

Substitute these values in Equation (2)

$$\frac{s+2}{s^2(s+3)} = \frac{\frac{-1}{9}}{s+3} + \frac{\frac{1}{9}s + \frac{2}{3}}{s^2}$$

$$\frac{s+2}{s^2(s+3)} = \frac{-1}{9}\cdot\frac{1}{s+3} + \frac{1}{9}\frac{s}{s^2} + \frac{2}{3}\frac{1}{s^2}$$

Substitute these in Equation (1)

$$\therefore f(s) = \frac{-1}{9}\frac{1}{s+3} + \frac{1}{9}\frac{1}{s} + \frac{2}{3}\frac{1}{s^2}$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[f(s)] = L^{-1}\left[\frac{-1}{9}\frac{1}{s+3} + \frac{1}{9}\cdot\frac{1}{2} + \frac{2}{3}\frac{1}{s^2}\right]$$

➔ **Using linearity property of inverse Laplace transform**

$$\text{...}\{L^{-1}[af(s) + bg(s) - ch(s)] = aL^{-1}[f(s)] + bL^{-1}[g(s)] - cL^{-1}[h(s)]\}$$

$$L^{-1}[f(s)] = \frac{-1}{9}L^{-1}\left(\frac{1}{s+3}\right) + \frac{1}{9}L^{-1}\left(\frac{1}{s}\right) + \frac{2}{3}L^{-1}\left(\frac{1}{s^2}\right)$$

➔ **Using standard result of inverse Laplace transform**

$$\text{...}\left[L^{-1}\left[\frac{1}{s-a}\right] = e^{at};\ L^{-1}\left(\frac{1}{s}\right) = 1;\ \text{and } L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{\Gamma n+1}\right]$$

$$\therefore L^{-1}[f(s)] = \frac{-1}{9}\underbrace{L^{-1}\left(\frac{1}{s+3}\right)}_{e^{-3t}} + \frac{1}{9}\underbrace{L^{-1}\left(\frac{1}{s}\right)}_{1} + \frac{2}{3}\underbrace{L^{-1}\left(\frac{1}{s^2}\right)}_{\frac{t}{12}}$$

$$= \frac{-1}{9}e^{-3t} + \frac{1}{9}(1) + \frac{2}{3}\frac{t}{\Gamma 2}$$

➔ **By property of Gamma function**

$$\text{...}\left[\Gamma n+1 = n! \text{ if n is integer}\right]$$

$$= \frac{1}{9}[-e^{-3t} + 1 + 6t] \qquad \left(\because \Gamma 2 = 1! = 1\right)$$

$$\therefore L^{-1}[f(s)] = \frac{1}{9}(1 + 6t - e^{-3t}) \checkmark \qquad \text{...Ans.}$$

Ex. 2.6.11 : Find inverse Laplace transform of $\dfrac{s^2+36s+6}{(s-1)(s^2+2)}$

Soln. :

Let, $$f(s) = \frac{s^2+36s+6}{(s-1)(s^2+2)} \qquad \text{...(1)}$$

By partial fraction :

$$\frac{s^2+36s+6}{(s-1)(s^2+2)} = \frac{A}{s-1} + \frac{Bs+C}{(s^2+2)} \qquad \text{...(2)}$$

$$\frac{s^2+36s+6}{\cancel{(s-1)(s^2+2)}} = \frac{A(s^2+2) + (Bs+C)(s-1)}{\cancel{(s-1)(s^2+2)}}$$

[by simplification taking L.C.M. of R.H.S.]

$$\therefore s^2+36s+6 = A(s^2+2) + (Bs+C)(s-1) \qquad \text{...(3)}$$

Put $s-1=0 \Rightarrow s=1$, in Equation (3)

[To find A from Equation (2), put denominator of A equal to Zero]

$$\therefore (1)^2 + 36(1) + 6 = A((1)^2+2) + (B(1)+C)(0)$$
$$1 + 36 + 6 = A(1+2) + 0$$
$$43 = 3A \qquad \left[\because A = \frac{43}{3}\right]$$
$$\Rightarrow A = \frac{43}{3}$$

Put $s=0$, in Equation (2)

$$\therefore (0)^2 + 36(0) + 6 = A(0+2) + (B(0)+C)(0-1)$$

$$6 = 2A + C(-1)$$

$$6 = 2\left(\frac{43}{3}\right) - C = \frac{86}{3} - C \qquad \left[\because A = \frac{43}{3}\right]$$

$$\therefore C = \frac{86}{3} - 6 = \frac{86-18}{3}$$

$$C = \frac{83}{3} \qquad \Rightarrow C = \frac{68}{3}$$

Put, $s = -1$, in Equation (3)

$$\therefore (-1)^2 + 36(-1) + 6 = A((-1)^2 + 2) + (B(-1) + C)(-1-1)$$

$$1 - 36 + 6 = A(1+2) + (-B + C)(-2)$$

$$-29 = 3A + 2B - 2C$$

$$-29 = 3\left(\frac{43}{3}\right) + 2B - 2\left(\frac{68}{3}\right)$$

$$\left[\because A = \frac{43}{3},\ C = \frac{68}{3}\right]$$

$$-29 = 43 + 2B - \frac{136}{3}$$

$$2B = -29 - 43 + \frac{136}{3} = \frac{-29(3) - 43(3) + 136}{3}$$

$$2B = \frac{-80}{3} \qquad \Rightarrow B = \frac{-80}{3} \times \frac{1}{2}$$

$$\Rightarrow B = \frac{-40}{3}$$

Substitute these values $\left(A = \frac{43}{3}, B = \frac{-40}{3}, C = \frac{68}{3}\right)$ in Equation (2).

$\therefore$ Equation (2) becomes,

$$\frac{s^2 + 36s + 6}{(s-1)(s^2+2)} = \frac{\frac{43}{3}}{s-1} + \frac{\frac{-40}{3}s + \frac{68}{3}}{s^2+2}$$

$$\frac{s^2 + 36s + 6}{(s-1)(s^2+2)} = \frac{1}{3}\left[\frac{43}{s-1} - \underbrace{\frac{40s - 68}{s^2+2}}_{\frac{40s}{s^2+2} - \frac{68}{s^2+2}}\right]$$

$$= \frac{1}{3}\left[\frac{43}{s-1} - \left(\frac{40s}{s^2+1} - \frac{68}{s^2+2}\right)\right]$$

$$\ldots\left\{\text{by } \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}\right\}$$

Substitute these in Equation (1),

$$f(s) = \frac{1}{3}\left[\frac{43}{s-1} - \left(\frac{40s}{s^2+1} - \frac{68}{s^2+2}\right)\right]$$

Substitute these in Equation (1),

$$f(s) = \frac{1}{3}\left[\frac{43}{s-1} - \frac{40s}{s^2+2} + \frac{68}{s^2+2}\right]$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[f(s)] = L^{-1}\left\{\frac{1}{3}\left[\frac{43}{s-1} - \frac{40s}{s^2+2} + \frac{68}{s^2+2}\right]\right\}$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots[L^{-1}[a f(s) + b g(s) - c h(s)] = a L^{-1}[f(s)] + b L^{-1}[g(s)] - c L^{-1}[h(s)]]$$

$$\therefore L^{-1}[f(s)] = \frac{1}{3}\left\{43 L^{-1}\left[\frac{1}{s-1} - 40\frac{40s}{s^2+2} + \frac{68}{s^2+2}\right]\right\}$$

$$L^{-1}[f(s)] = \frac{1}{3}\left\{43 L^{-1}\left[\frac{1}{s-1}\right] - 40 L^{-1}\left[\frac{s}{s^2+2}\right] + 68 L^{-1}\left[\frac{1}{s^2+2}\right]\right\}$$

$$L^{-1}[f(s)] = \frac{43}{3}L^{-1}\left[\frac{1}{s-1}\right] - \frac{40}{3}L^{-1}\left[\frac{s}{s^2+(\sqrt{2})^2}\right] + \frac{68}{3}L^{-1}\left[\frac{1}{s^2+(\sqrt{2})^2}\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s-a}\right] = e^{at};\ L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at \text{ and } \left[\frac{s}{s^2+a^2}\right] = \cos at\right]$$

$$\therefore L^{-1}[f(s)] = \frac{43}{3}\underbrace{L^{-1}\left[\frac{1}{s-1}\right]}_{e^{1t}} - \frac{40}{3}\underbrace{L^{-1}\left[\frac{s}{s^2+(\sqrt{2})^2}\right]}_{\cos(\sqrt{2}\,t)} + \frac{68}{3}\underbrace{L^{-1}\left[\frac{1}{s^2+(\sqrt{2})^2}\right]}_{\frac{1}{\sqrt{2}}\sin(\sqrt{2}\,t)}$$

$$\therefore f(t) = \frac{43}{3}e^{1t} - \frac{43}{3}\cos(\sqrt{2}\,t) + \frac{68}{3}\frac{1}{\sqrt{2}}\sin(\sqrt{2}\,t)$$

$$f(t) = \frac{1}{\sqrt{3}}\left[43e^{t} - 40\cos(\sqrt{2}\,t) + \frac{68}{\sqrt{2}}\sin(\sqrt{2}t)\right] \checkmark \ \ldots\text{Ans.}$$

Ex. 2.6.12 : Find inverse Laplace transform of $\dfrac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}$

Soln. : Given : $f(s) = \dfrac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}$

By Synthetic Division

1	1	-6	11	-6
		1	-5	6
	1	-5	6	$\boxed{0}$

$$s^2 - 5s + 6 = (s-3)(s-2)$$

$$f(s) = \frac{2s^2 - 6s + 5}{(s-1)(s-2)(s-3)} \qquad \ldots(1)$$

$$s^2 - 5s + 6 = (s-3)(s-2)$$

and $s = 1 \Rightarrow (s-1)$

$$\therefore s^3 - 6s^2 + 115 - 6 = (s-1)(s-2)(s-3)$$

By partial fraction,

$$\frac{2s^2 - 6s + 5}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3} \qquad \ldots(2)$$

$$\frac{2s^2 - 6s + 5}{(s-1)(s-2)(s-3)} = \frac{A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)}{(s-1)(s-2)(s-3)}$$

$$\therefore 2s^2 - 6s + 5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2) \qquad \ldots(3)$$

Put $s - 1 = \Rightarrow s = 1$ in Equation (3),

$$\therefore 2(1)^2 - 6(1) + 5 = A(1-2)(1-3) + B(0)(1-3) + C(0)(1-2)$$

$$2 - 6 + 5 = A(-1)(-2) + 0 + 0$$

$$1 = 2A$$

$$\frac{1}{2} = A$$

$$\therefore A = \frac{1}{2}$$

Put $s - 2 = 0 \Rightarrow s = 2$ in Equation (3),

$$2(2)^2 - 6(2) + 5 = A(0)(2-3) + B(2-1)(2-3) + C(2-1)(0)$$

$$2(4) - 12 + 5 = 0 + B(1)(-1) + 0$$

$$8 - 12 + 5 = -B$$

$$1 = -B$$

$$-1 = B$$

$$\therefore B = -1$$

Put $s - 3 = 0 \Rightarrow s = 3$ in Equation (4),

$$\therefore 2(3)^2 - 6(3) + 5 = A(3-2)(0) + B(3-1)(0) + C(3-1)(3-2)$$

$$2(9) - 18 + 5 = 0 + 0 + C(2)(1)$$

$$\cancel{18} - \cancel{18} + 5 = 2C$$

$$5 = 2C$$

$$\frac{5}{2} = C \qquad \therefore C = \frac{5}{2}$$

Substitute these values $\left[A = \frac{1}{2}, B = -1, C = \frac{5}{2}\right]$ in Equation (2),

$$\frac{2s^2 - 6s + 5}{(s-1)(s-2)(s-3)} = \frac{\frac{1}{2}}{s-1} + \frac{-1}{s-2} + \frac{\frac{5}{2}}{s-3}$$

$$\frac{2s^2 - 6s + 5}{(s-1)(s-2)(s-3)} = \frac{1}{2}\left(\frac{1}{s-1}\right) - \left(\frac{1}{s-2}\right) + \frac{5}{2}\left(\frac{1}{s-3}\right)$$

$\therefore$ From Equations (1),

$$f(s) = \frac{1}{2}\left(\frac{1}{s-1}\right) - \left(\frac{1}{s-2}\right) + \frac{5}{2}\left(\frac{1}{s-3}\right)$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[f(s)] = L^{-1}\left[\frac{1}{2}\frac{1}{s-1} - \frac{1}{s-2} + \frac{5}{2}\frac{1}{s-3}\right]$$

➔ **Using the linearity property of inverse Laplace transform**

$$\ldots[L^{-1}[a f(s) + b g(s) - c h(s)] = a L^{-1}[f(s)] + b L^{-1}[g(s)] - c L^{-1}[h(s)]]$$

$$f(t) = \frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s-2}\right) + \frac{5}{2}L^{-1}\left(\frac{1}{s-3}\right)$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left(\frac{1}{s-a}\right) = e^{at}\right]$$

$$f(t) = \frac{1}{2}\underbrace{L^{-1}\left(\frac{1}{s-1}\right)}_{e^{t}} - \underbrace{L^{-1}\left(\frac{1}{s-2}\right)}_{e^{2t}} + \frac{5}{2}\underbrace{L^{-1}\left(\frac{1}{s-3}\right)}_{e^{3t}}$$

$$f(t) = \frac{1}{2}e^{t} - e^{2t} + \frac{5}{2}e^{3t} \checkmark \qquad \ldots\text{Ans.}$$

2.7 Inverse Laplace Transform of Inverse trigonometric, inverse hyperbolic and Logarithmic Function

To find inverse Laplace transform of the function containing inverse trigonometric, inverse hyperbolic and logarithmic function. We use

➔ **Using property of Laplace transform**

$$\ldots\left[L[t f(t)] = -\frac{d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

Use the following steps :

I. Consider

$$L^{-1}[f(s)] = f(t)$$

$$\therefore L[f(t)] = f(s)$$

II.

➔ **Using property of Laplace transform**

$$\ldots\left[L[t f(t)] = -\frac{d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore L[t f(t)] = \frac{-d}{ds}[f(s)]$$

The R.H.S. of above Equation is a standard function of s whose inverse laplace is obtain by using standard table

$$\therefore L[t f(t)] = f(s) \quad \text{(say)}$$

(III) $t f(t) = L^{-1}[f(s)]$

$$f(t) = \frac{\text{Standard inverse Laplace transform}}{t}$$

Ex. 2.7.1 : Find Inverse Laplace Transform of $\log\left(\dfrac{s+b}{s+a}\right)$.

Soln. :

Consider, $L^{-1}\left[\log\left(\dfrac{s+b}{s+a}\right)\right] = f(t)$

$$\therefore L[f(t)] = \log\left(\frac{s+b}{s+a}\right)$$

➔ **Using property of logarithm :** $\ldots\left[\log\left(\frac{a}{b}\right) = \log a - \log b\right]$

$$L[f(t)] = \log(s+b) - \log(s+a)$$

➔ Using property of Laplace transform

$$\ldots\left[L[t\,f(t)] = -\frac{d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore\ L[t\,f(t)] = -\frac{d}{ds}[\log(s+b) - \log(s+a)]$$

$$= \frac{d}{ds}[\log(s+a) - \log(s+b)]$$

$$= \frac{d}{ds}[\log(s+a)] - \frac{d}{ds}[\log(s+b)]$$

➔ Using standard result of derivative : $\ldots\left[\frac{d}{dx}[\log x] = \frac{1}{x}\right]$

$$L[t\,f(t)] = \frac{1}{s+a} - \frac{1}{s+b}$$

$$t\,f(t) = L^{-1}\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

➔ Using linearity property of inverse Laplace transform

$$\ldots\{L^{-1}[af(s) + bg(s) - ch(s)] = a\,L^{-1}[f(s)] + bL^{-1}[g(s)] - cL^{-1}[h(s)]\}$$

$$\therefore\ t\,f(t) = L^{-1}\left[\frac{1}{s+a}\right] - L^{-1}\left[\frac{1}{s+b}\right]$$

➔ Using standard result of inverse Laplace transform

$$\ldots\left[L^{-1}\left[\frac{1}{s-a}\right] = e^{at} \text{ and } L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}\right]$$

$$\therefore\ t\,f(t) = \underbrace{L^{-1}\left[\frac{1}{s+a}\right]}_{e^{-at}} - \underbrace{L^{-1}\left[\frac{1}{s+b}\right]}_{e^{-bt}}$$

$$\therefore\ t\,f(t) = e^{-at} - e^{-bt}$$

$$\therefore\ L^{-1}\left[\log\left(\frac{s+b}{s+a}\right)\right] = \frac{e^{-at} - e^{-bt}}{t} \checkmark \qquad \ldots\text{Ans.}$$

Ex. 2.7.2 : Find inverse Laplace transform of $\log\left(\frac{s+3}{s+2}\right)$

Soln. :

Consider, $L^{-1}\left[\log\left(\frac{s+3}{s+2}\right)\right] = f(t)$

$$\therefore\ L[f(t)] = \log\left(\frac{s+3}{s+2}\right)$$

➔ Using property of logarithm : $\ldots\left[\log\left(\frac{a}{b}\right) = \log a - \log b\right]$

$$L[f(t)] = \log(s+3) - \log(s+2)$$

➔ Using property of Laplace transform

$$\ldots\left[L[t\,f(t)] = -\frac{d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore\ L[t\,f(t)] = -\frac{d}{ds}[\log(s+3) - \log(s+2)]$$

$$= \frac{d}{ds}[\log(s+2) - \log(s+3)]$$

$$= \frac{d}{ds}[\log(s+2)] - \frac{d}{ds}[\log(s+3)]$$

➔ Using standard result of derivative : $\ldots\left[\frac{d}{dx}[\log x] = \frac{1}{x}\right]$

$$L[t\,f(t)] = \frac{1}{s+2} - \frac{1}{s+3}$$

$$t\,f(t) = L^{-1}\left[\frac{1}{s+2} - \frac{1}{s+3}\right]$$

➔ Using linearity property of inverse Laplace transform

$$\ldots\{L^{-1}[af(s) + bg(s) - ch(s)] = a\,L^{-1}[f(s)] + bL^{-1}[g(s)] - cL^{-1}[h(s)]\}$$

$$\therefore\ t\,f(t) = L^{-1}\left[\frac{1}{s+2}\right] - L^{-1}\left[\frac{1}{s+3}\right]$$

➔ Using standard result of inverse Laplace transform

$$\ldots\left[L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}\right]$$

$$\therefore\ t\,f(t) = \underbrace{L^{-1}\left[\frac{1}{s+2}\right]}_{e^{-2t}} - \underbrace{L^{-1}\left[\frac{1}{s+3}\right]}_{e^{-3t}}$$

$$\therefore\ t\,f(t) = e^{-2t} - e^{-3t}$$

$$\therefore\ f(t) = \frac{e^{-2t} - e^{-3t}}{t} \checkmark \qquad \ldots\text{Ans.}$$

Ex. 2.7.3 : Find inverse Laplace transform of $\frac{1}{s}\log\left(\frac{s+3}{s+2}\right)$

Soln. :

Consider, $L^{-1}\left[\log\left(\frac{s+3}{s+2}\right)\right] = f(t)$

$$\therefore\ L[f(t)] = \log\left(\frac{s+3}{s+2}\right)$$

➔ Using property of logarithm

$$\ldots\left[\log\left(\frac{a}{b}\right) = \log a - \log b\right]$$

$$L[f(t)] = \log(s+3) - \log(s+2)$$

➔ Using property of Laplace transform

$$\ldots\left[L[t\,f(t)] = -\frac{d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore\ L[t\,f(t)] = -\frac{d}{ds}[\log(s+3) - \log(s+2)]$$

$$= \frac{d}{ds}[\log(s+2) - \log(s+3)]$$

$$= \frac{d}{ds}[\log(s+2)] - \frac{d}{ds}[\log(s+3)]$$

➔ Using standard result of derivative : $\ldots\left[\frac{d}{dx}[\log x] = \frac{1}{x}\right]$

$$L[t\,f(t)] = \frac{1}{s+2} - \frac{1}{s+3}$$

$$t\,f(t) = L^{-1}\left[\frac{1}{s+2} - \frac{1}{s+3}\right]$$

➔ Using linearity property of inverse Laplace transform

$$\ldots\{L^{-1}[af(s) + bg(s) - ch(s)] = a\,L^{-1}[f(s)] + bL^{-1}[g(s)] - cL^{-1}[h(s)]\}$$

$$\therefore\ t\,f(t) = L^{-1}\left[\frac{1}{s+2}\right] - L^{-1}\left[\frac{1}{s+3}\right]$$

➔ Using standard result of inverse Laplace transform

$$\ldots\left[L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}\right]$$

$$\therefore\ t\,f(t) = \underbrace{L^{-1}\left[\frac{1}{s+2}\right]}_{e^{-2t}} - \underbrace{L^{-1}\left[\frac{1}{s+3}\right]}_{e^{-3t}}$$

$$\therefore\ t\,f(t) = e^{-2t} - e^{-3t}$$

$$\therefore\ f(t) = \frac{e^{-2t} - e^{-3t}}{t}$$

➔ Using property of inverse Laplace transform

$$\ldots\left[L^{-1}\left[\frac{1}{s}f(s)\right] = \int_0^t f(t)\,dt \text{ Where } f(t) = L^{-1}[f(s)]\right]$$

$$\therefore\ L^{-1}\left[\frac{1}{s}\log\left(\frac{s+3}{s+2}\right)\right] = \int_0^t \frac{e^{-2t} - e^{-3t}}{t}\,dt \checkmark \qquad \ldots\text{Ans.}$$

Ex. 2.7.4 : Find inverse Laplace transform of $\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$

Soln. :

Consider $L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = f(t)$ ...(1)

$$\therefore\ L[f(t)] = \log\left(\frac{s^2+a^2}{s^2+b^2}\right)$$

➔ Using property of logarithm : $\ldots\left[\log\left(\frac{a}{b}\right) = \log a - \log b\right]$

$$\therefore\ L[f(t)] = \log(s^2+a^2) - \log(s^2+b^2)$$

➔ Using property of Laplace transform

$$\ldots\left[L[t\,f(t)] = -\frac{d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore\ L[t\,f(t)] = -\frac{d}{ds}[\log(s^2+a^2) - \log(s^2+b^2)]$$

$$= \frac{d}{ds}[\log(s^2+b^2) - \log(s^2+a^2)]$$

$$L[t\,f(t)] = \frac{d}{ds}[\log(s^2+b^2)] - \frac{d}{ds}[\log(s^2+a^2)]$$

➔ Using derivative of composite function

$$\ldots\left[\frac{d}{dx}[\log f(x)] = \frac{1}{f(x)}\frac{d}{dx}[f(x)]\right]$$

$$L[t\,f(t)] = \frac{1}{s^2+b^2}\underbrace{\frac{d}{ds}(s^2+b^2)}_{2s+0} - \frac{1}{s^2+a^2}\underbrace{\frac{d}{ds}(s^2+a^2)}_{2s+0}$$

$$= \frac{1}{s^2+b^2}(2s) - \frac{1}{s^2+a^2}(2s)$$

$$L[t\,f(t)] = 2\left[\frac{s}{s^2+b^2} - \frac{s}{s^2+a^2}\right]$$

$$\therefore\ t\,f(t) = L^{-1}\left\{2\left[\frac{s}{s^2+b^2} - \frac{s}{s^2+a^2}\right]\right\}$$

➔ Using linearity property of inverse Laplace transform

$$\ldots\{L^{-1}[af(s) + bg(s) = ch(s)] = a\,L^{-1}[f(s)] + bL^{-1}[g(s)] = cL^{-1}[h(s)]\}$$

$$t\,f(t) = 2\left[L^{-1}\left(\frac{s}{s^2+b^2}\right) - L^{-1}\left(\frac{s}{s^2+a^2}\right)\right]$$

➔ Using standard result of inverse Laplace transform

$$\ldots\left[L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at\right]$$

$$\therefore\ t\,f(t) = 2\left[\underbrace{L^{-1}\left(\frac{s}{s^2+b^2}\right)}_{\cos bt} - \underbrace{L^{-1}\left(\frac{s}{s^2+a^2}\right)}_{\cos at}\right]$$

$$t\,f(t) = 2(\cos bt - \cos at)$$

$$\therefore\ f(t) = \frac{2(\cos bt - \cos at)}{t} \qquad \ldots(2)$$

From Equations (1) and (2),

$$L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \frac{2(\cos bt - \cos at)}{t} \checkmark \qquad \ldots\text{Ans.}$$

Ex. 2.7.5 : Find inverse Laplace transform of $\frac{1}{s}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$

Soln. :

Consider $L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = f(t)$ ...(1)

$$\therefore\ L[f(t)] = \log\left(\frac{s^2+a^2}{s^2+b^2}\right)$$

➔ Using property of logarithm

$$\ldots\left[\log\left(\frac{a}{b}\right) = \log a - \log b\right]$$

$$\therefore\ L[f(t)] = \log(s^2+a^2) - \log(s^2+b^2)$$

➔ Using property of Laplace transform

$$\ldots\left[L[t\,f(t)] = -\frac{d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore L[t\,f(t)] = -\frac{d}{ds}[\log(s^2+a^2) - \log(s^2+b^2)]$$

$$= \frac{d}{ds}[\log(s^2+a^2) - \log(s^2+a^2)]$$

$$L[t\,f(t)] = \frac{d}{ds}[\log(s^2+b^2)] - \frac{d}{ds}[\log(s^2+a^2)]$$

➔ **Using standard result of derivative**

$$\ldots\left[\frac{d}{dx}[\log f(x)] = \frac{1}{f(x)}\frac{d}{dx}[f(x)]\right]$$

$$L[t\,f(t)] = \frac{1}{s^2+b^2}\underbrace{\frac{d}{ds}(s^2+b^2)}_{2s+0} - \frac{1}{s^2+a^2}\underbrace{\frac{d}{ds}(s^2+a^2)}_{2s+0}$$

$$= \frac{1}{s^2+b^2}(2s) - \frac{1}{s^2+a^2}(2s)$$

$$L[t\,f(t)] = 2\left[\frac{s}{s^2+b^2} - \frac{s}{s^2+a^2}\right]$$

$$\therefore\ t\,f(t) = L^{-1}\left\{2\left[\frac{s}{s^2+b^2} - \frac{s}{s^2+a^2}\right]\right\}$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots\{L^{-1}[af(s)+bg(s)-ch(s)] = a\,L^{-1}[f(s)] + bL^{-1}[g(s)] - cL^{-1}[h(s)]\}$$

$$t\,f(t) = 2\left[L^{-1}\left(\frac{s}{s^2+b^2}\right) - L^{-1}\left(\frac{s}{s^2+a^2}\right)\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at\right]$$

$$\therefore\ t\,f(t) = 2\left[\underbrace{L^{-1}\left(\frac{s}{s^2+b^2}\right)}_{\cos bt} - \underbrace{L^{-1}\left(\frac{s}{s^2+a^2}\right)}_{\cos at}\right]$$

$$t\,f(t) = 2(\cos bt - \cos at)$$

$$\therefore\ f(t) = \frac{2(\cos bt - \cos at)}{t} \qquad \ldots(2)$$

From Equations (1) and (2),

$$L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \frac{2(\cos bt - \cos at)}{t}$$

➔ **Using property of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s}f(s)\right] = \int_0^t f(t)\,dt \text{ Where } f(t) = L^{-1}[f(s)]\right]$$

$$\therefore L^{-1}\left[\frac{1}{s}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \int_0^t \frac{2(\cos bt - \cos at)}{t}\,dt \checkmark \quad \ldots\text{Ans.}$$

Ex. 2.7.6 : Find inverse Laplace transform of $\log\sqrt{\frac{s^2+4}{s^2+9}}$.

Soln. :

Consider $L^{-1}\left[\log\left(\frac{s^2+4}{s^2+9}\right)\right] = f(t)$ ...(1)

$$\therefore\ L[f(t)] = \log\left(\frac{s^2+4}{s^2+9}\right)$$

➔ **Using property of logarithm**

$$\ldots\left[\log\left(\frac{a}{b}\right) = \log a - \log b\right]$$

$$\therefore\ L[f(t)] = \log(s^2+4) - \log(s^2+9)$$

➔ **Using property of Laplace transform**

$$\ldots\left[L[t\,f(t)] = -\frac{d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore L[t\,f(t)] = -\frac{d}{ds}[\log(s^2+4) - \log(s^2+9)]$$

$$= \frac{d}{ds}[\log(s^2+9) - \log(s^2+4)]$$

$$L[t\,f(t)] = \frac{d}{ds}[\log(s^2+9)] - \frac{d}{ds}[\log(s^2+4)]$$

➔ **Using standard result of derivative**

$$\ldots\left[\frac{d}{dx}[\log f(x)] = \frac{1}{f(x)}\frac{d}{dx}[f(x)]\right]$$

$$L[t\,f(t)] = \frac{1}{s^2+9}\underbrace{\frac{d}{ds}(s^2+9)}_{2s+0} - \frac{1}{s^2+4}\underbrace{\frac{d}{ds}(s^2+4)}_{2s+0}$$

$$= \frac{1}{s^2+9}(2s) - \frac{1}{s^2+4}(2s)$$

$$L[t\,f(t)] = 2\left[\frac{s}{s^2+9} - \frac{s}{s^2+4}\right]$$

$$\therefore\ t\,f(t) = L^{-1}\left\{2\left[\frac{s}{s^2+9} - \frac{s}{s^2+4}\right]\right\}$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots\{L^{-1}[af(s)+bg(s)-ch(s)] = a\,L^{-1}[f(s)] + bL^{-1}[g(s)] - cL^{-1}[h(s)]\}$$

$$t\,f(t) = 2\left[L^{-1}\left(\frac{s}{s^2+9}\right) - L^{-1}\left(\frac{s}{s^2+4}\right)\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at\right]$$

$$\therefore\ t\,f(t) = 2\left[\underbrace{L^{-1}\left(\frac{s}{s^2+(3)^2}\right)}_{\cos 3t} - \underbrace{L^{-1}\left(\frac{s}{s^2+(2)^2}\right)}_{\cos 2t}\right]$$

$$t\,f(t) = 2(\cos 3t - \cos 2t)$$

$$\therefore\ f(t) = \frac{2(\cos 3t - \cos 2t)}{t} \qquad \ldots(2)$$

From Equations (1) and (2),

$$L^{-1}\left[\log\left(\frac{s^2+4}{s^2+9}\right)\right] = \frac{2(\cos 3t - \cos 2t)}{t} \checkmark \quad \ldots\text{Ans.}$$

➔ **Using property of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s}f(s)\right] = \int_0^t f(t)\,dt \text{ Where } f(t) = L^{-1}[f(s)]\right]$$

$$\therefore L^{-1}\left[\frac{1}{s}\log\left(\frac{s^2+4}{s^2+9}\right)\right] = \int_0^t \frac{2(\cos 3t - \cos 2t)}{t}\,dt \checkmark \quad \ldots\text{Ans.}$$

Ex. 2.7.7 : Find inverse Laplace transform of $\log\left(\frac{s^2+1}{s^2+4}\right)$

Soln. :

Consider $L^{-1}\left[\log\left(\frac{s^2+1}{s^2+4}\right)\right] = f(t)$ ...(1)

$$\therefore\ L[f(t)] = \log\left(\frac{s^2+1}{s^2+4}\right)$$

➔ **Using property of logarithm**

$$\ldots\left[\log\left(\frac{a}{b}\right) = \log a - \log b\right]$$

$$\therefore\ L[f(t)] = \log(s^2+1) - \log(s^2+4)$$

➔ **Using property of Laplace transform**

$$\ldots\left[L[t\,f(t)] = -\frac{d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore L[t\,f(t)] = -\frac{d}{ds}[\log(s^2+1) - \log(s^2+4)]$$

$$= \frac{d}{ds}[\log(s^2+4) - \log(s^2+1)]$$

$$L[t\,f(t)] = \frac{d}{ds}[\log(s^2+4)] - \frac{d}{ds}[\log(s^2+1)]$$

➔ **Using standard result of derivative**

$$\ldots\left[\frac{d}{dx}[\log f(x)] = \frac{1}{f(x)}\frac{d}{dx}[f(x)]\right]$$

$$L[t\,f(t)] = \frac{1}{s^2+4}\underbrace{\frac{d}{ds}(s^2+4)}_{2s+0} - \frac{1}{s^2+1}\underbrace{\frac{d}{ds}(s^2+1)}_{2s+0}$$

$$= \frac{1}{s^2+4}(2s) - \frac{1}{s^2+1}(2s)$$

$$L[t\,f(t)] = 2\left[\frac{s}{s^2+4} - \frac{s}{s^2+1}\right]$$

$$\therefore\ t\,f(t) = L^{-1}\left\{2\left[\frac{s}{s^2+4^2} - \frac{s}{s^2+1^2}\right]\right\}$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots\{L^{-1}[af(s)+bg(s)-ch(s)] = a\,L^{-1}[f(s)] + bL^{-1}[g(s)] - cL^{-1}[h(s)]\}$$

$$t\,f(t) = 2\left[L^{-1}\left(\frac{s}{s^2+4}\right) - L^{-1}\left(\frac{s}{s^2+1}\right)\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at\right]$$

$$\therefore\ t\,f(t) = 2\left[\underbrace{L^{-1}\left(\frac{s}{s^2+(2)^2}\right)}_{\cos 2t} - \underbrace{L^{-1}\left(\frac{s}{s^2+(1)^2}\right)}_{\cos 1t}\right]$$

$$t\,f(t) = 2(\cos 2t - \cos t)$$

$$\therefore\ f(t) = \frac{2(\cos 2t - \cos t)}{t} \qquad \ldots(2)$$

From Equations (1) and (2),

$$L^{-1}\left[\log\left(\frac{s^2+1^2}{s^2+4^2}\right)\right] = \frac{2(\cos 2t - \cos t)}{t} \checkmark \quad \ldots\text{Ans.}$$

Ex. 2.7.8 : Find inverse Laplace transform of $\cot^{-1}(s)$.

Soln. :

Consider, $L^{-1}[\cot^{-1}(s)] = f(t)$ ...(1)

$$\therefore \cot^{-1}(s) = L[f(t)]$$

i.e. $L[f(t)] = \cot^{-1}(s)$

➔ **Using property of Laplace transform**

$$\ldots\left[L[t\,f(t)] = -\frac{d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore L[t\,f(t)] = -\frac{d}{ds}[\cot^{-1}(s)]$$

➔ **Using standard result of derivative**

$$\ldots\left[\frac{d}{ds}[\cot^{-1}s] = \frac{-1}{1+s^2}\right]$$

$$\therefore\ L[t\,f(t)] = -\left(\frac{-1}{1+s^2}\right)$$

$$= \frac{1}{1+s^2}$$

$$\therefore\ L[t\,f(t)] = \frac{1}{s^2+1}$$

$$\therefore\ t\,f(t) = L^{-1}\left(\frac{1}{s^2+1}\right)$$

➔ **Using inverse Laplace transform of standard function**

$$\ldots\left[L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$\therefore\ t\,f(t) = \underbrace{L^{-1}\left[\frac{1}{s^2+1}\right]}_{\sin t}$$

$$\therefore\ t\,f(t) = \sin t$$

$$\therefore\ f(t) = \frac{\sin t}{t} \qquad \ldots(2)$$

From Equations (1) and (2),

$$\therefore\ L^{-1}[\cot^{-1}(s)] = \frac{\sin t}{t} \checkmark \qquad \text{...Ans.}$$

Ex. 2.7.9 : Find the inverse Laplace transform of $\tan^{-1}(s)$.

Soln. :

Consider, $L^{-1}[\tan^{-1}(s)] = f(t)$ ...(1)

$\therefore\ \tan^{-1}(s) = L[f(t)]$

i.e. $L[f(t)] = \tan^{-1}(s)$

➔ Using property of Laplace transform

$$\ldots\left[L[t\,f(t)] = -\frac{d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore\ L[t\,f(t)] = -\frac{d}{ds}[\tan^{-1}(s)]$$

➔ Using standard result of derivative

$$\ldots\left[\frac{d}{ds}(\tan^{-1}s) = \frac{1}{1+s^2}\right]$$

$$\therefore\ L[t\,f(t)] = -\left[\frac{1}{1+s^2}\right]$$

$$= \frac{-1}{1+s^2}$$

$$L[t\,f(t)] = \frac{-1}{s^2+1}$$

$$\therefore\ t\,f(t) = L^{-1}\left(\frac{-1}{s^2+1}\right) = -L^{-1}\left(\frac{1}{s^2+1}\right)$$

➔ Using inverse Laplace transform of standard function

$$\ldots\left[L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$\therefore\ t\,f(t) = -\underbrace{L^{-1}\left[\frac{1}{s^2+1}\right]}_{\sin t}$$

$$\therefore\ t\,f(t) = -\sin t$$

$$\therefore\ f(t) = -\frac{\sin t}{t} \qquad \ldots(2)$$

From Equations (1) and (2),

$$\therefore\ L^{-1}[\tan^{-1}(s)] = -\frac{\sin t}{t} \checkmark \qquad \text{...Ans.}$$

Ex. 2.7.10 : Find inverse Laplace transform of $\tan^{-1}\frac{1}{s}$.

Soln. :

Consider, $L^{-1}\left[\tan^{-1}\frac{1}{s}\right] = f(t)$ ...(1)

$$\therefore\ \tan^{-1}\left(\frac{1}{s}\right) = L[f(t)]$$

i.e. $L[f(t)] = \tan^{-1}\left(\frac{1}{s}\right)$

➔ Using property of Laplace transform

$$\ldots\left[L[t\,f(t)] = -\frac{d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore\ L(t\,f(t)) = -\frac{d}{ds}\left[\tan^{-1}\left(\frac{1}{s}\right)\right]$$

➔ Using derivative of composite function

$$\ldots\left[\frac{d}{dx}[\tan^{-1}f(x)] = \frac{1}{1+(f(x))^2}\cdot\frac{d}{dx}f(x)\right]$$

$$L[t\,f(t)] = -\left[\frac{1}{1+\left(\frac{1}{s}\right)^2}\cdot\frac{d}{ds}\left(\frac{1}{s}\right)\right]$$

$$= -\left[\frac{1}{1+\frac{1}{s^2}}\cdot\left(\frac{-1}{s^2}\right)\right]$$

$$= -\left[\frac{-1}{\left(\frac{s^2+1}{\cancel{s^2}}\right)\cdot\cancel{s^2}}\right] \quad \text{(by simplification)}$$

$$L[t\,f(t)] = \frac{1}{s^2+1}$$

$$\therefore\ t\,f(t) = L^{-1}\left[\frac{1}{s^2+1}\right]$$

➔ Using standard result of inverse Laplace transform

$$\ldots\left[L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$t\,f(t) = \underbrace{L^{-1}\left[\frac{1}{s^2+1}\right]}_{\sin t}$$

$$\therefore\ t\,f(t) = \sin t$$

$$\therefore\ f(t) = \frac{\sin t}{t} \qquad \ldots(2)$$

From Equations (1) and (2),

$$L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right] = \frac{\sin t}{t} \checkmark \qquad \text{...Ans.}$$

Ex. 2.7.11 : Find inverse Laplace transform of

$$\tan^{-1}\left(\frac{s+1}{2}\right).$$

Soln. :

Consider, $L^{-1}\left[\tan^{-1}\left(\frac{s+1}{2}\right)\right] = f(t)$ ...(1)

$$\therefore\ L[f(t)] = \tan^{-1}\left(\frac{s+1}{2}\right)$$

➔ Using property of Laplace transform

$$\ldots\left[L[t\,f(t)] = -\frac{d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore\ L[t\,f(t)] = -\frac{d}{ds}\left[\tan^{-1}\left(\frac{s+1}{2}\right)\right]$$

➔ Using derivative of composite function

$$\ldots\left[\frac{d}{dx}[\tan^{-1}f(x)] = \frac{1}{1+(f(x))^2}\cdot\frac{d}{dx}f(x)\right]$$

$$L[t\,f(t)] = -\left[\frac{1}{1+\left(\frac{s+1}{2}\right)^2}\frac{d}{ds}\left(\frac{s+1}{2}\right)\right]$$

$$= -\left[\frac{1}{1+\frac{(s+1)^2}{4}}\cdot\frac{1}{2}(1)\right]$$

$$= -\left[\frac{1}{\frac{4+(s+1)^2}{4}}\cdot\frac{1}{2}\right] \quad \text{(by simplification)}$$

$$= -\left[\frac{\cancel{4}^{2}}{4+(s+1)^2}\cdot\frac{1}{\cancel{2}}\right]$$

$$L[t\,f(t)] = -\left[\frac{2}{4+(s+1)^2}\right]$$

$$\therefore\ t\,f(t) = L^{-1}\left\{\frac{-2}{4+(s+1)^2}\right\}$$

{Observe that all s in the form (s + 1)}

➔ Using first shifting property of inverse Laplace transform

$$\ldots\left\{L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)]\right\}$$

$$\therefore\ t\,f(t) = e^{-t}L^{-1}\left[\frac{-2}{4+s^2}\right]$$

...(∵ Replace every (s + 1) by s with taking e^{-t})

Now, $t\,f(t) = e^{-t}(-2)\,L^{-1}\left[\frac{1}{s^2+4}\right]$

➔ Using standard result of inverse Laplace transform

$$\ldots\left[L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$t\,f(t) = (-2)\,e^{-t}\underbrace{L^{-1}\left[\frac{1}{s^2+(2)^2}\right]}_{\frac{1}{2}\sin 2t}$$

$$t\,f(t) = (-\cancel{2})\,e^{-t}\frac{1}{\cancel{2}}\sin 2t$$

$$t\,f(t) = -e^{-t}\sin 2t$$

$$\therefore\ f(t) = \frac{-e^{-t}\sin 2t}{t} \qquad \ldots(2)$$

From Equations (1) and (2),

$$L^{-1}\left[\tan^{-1}\left(\frac{s+1}{2}\right)\right] = -\frac{e^{-t}\sin 2t}{t} \checkmark \qquad \text{...Ans.}$$

Ex. 2.7.12 : Find the inverse Laplace transform of

$$\cot^{-1}\left(\frac{s+1}{2}\right)$$

Soln. :

Consider, $L^{-1}\left[\cot^{-1}\left(\frac{s+1}{2}\right)\right] = f(t)$ ...(1)

$$\therefore\ L[f(t)] = \cot^{-1}\left(\frac{s+1}{2}\right)$$

➔ Using property of Laplace transform

$$\ldots\left[L[t\,f(t)] = -\frac{d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore\ L[t\,f(t)] = -\frac{d}{ds}\left[\cot^{-1}\left(\frac{s+1}{2}\right)\right]$$

➔ Using standard result of derivative

$$\ldots\left[\frac{d}{dx}[\cot^{-1}f(x)] = \frac{-1}{1+(f(x))^2}\cdot\frac{d}{dx}f(x)\right]$$

$$L[t\,f(t)] = -\left[\frac{-1}{1+\left(\frac{s+1}{2}\right)^2}\frac{d}{ds}\left(\frac{s+1}{2}\right)\right]$$

$$= \left[\frac{1}{1+\frac{(s+1)^2}{4}}\cdot\frac{1}{2}(1)\right]$$

$$= \left[\frac{1}{\frac{4+(s+1)^2}{4}}\cdot\frac{1}{2}\right] \quad \text{(by simplification)}$$

$$= \left[\frac{\cancel{4}^{2}}{4+(s+1)^2}\cdot\frac{1}{\cancel{2}}\right]$$

$$L[t\,f(t)] = \left[\frac{2}{4+(s+1)^2}\right]$$

$$\therefore\ t\,f(t) = L^{-1}\left\{\frac{2}{4+(s+1)^2}\right\}$$

{Observe that all s in the form (s + 1)}

➔ Using first shifting property of inverse Laplace transform :

$$\ldots\left\{L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)]\right\}$$

$$\therefore\ t\,f(t) = e^{-t}L^{-1}\left[\frac{2}{4+s^2}\right]$$

Replace every (s + 1) by s with taking e^{-t}

$$t\,f(t) = e^{-t}(2)\,L^{-1}\left[\frac{1}{s^2+4}\right]$$

➔ Using standard result of inverse Laplace transform

$$\ldots\left[L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$t\,f(t) = (2)\,e^{-t}\underbrace{L^{-1}\left[\frac{1}{s^2+(2)^2}\right]}_{\frac{1}{2}\sin 2t}$$

$$t\,f(t) = (2)\,e^{-t}\,\frac{1}{\cancel{2}}\sin 2t$$

$$t\,f(t) = e^{-t}\sin 2t$$

$$\therefore\quad f(t) = \frac{e^{-t}\sin 2t}{t} \qquad \ldots(2)$$

From Equations (1) and (2),

$$L^{-1}\left[\cot^{-1}\left(\frac{s+1}{2}\right)\right] = \frac{e^{-t}\sin 2t}{t}\checkmark \qquad \ldots\text{Ans.}$$

Ex. 2.7.13 : Find the inverse Laplace transform of $\cot^{-1}\left(\frac{s-2}{3}\right)$

Soln. :

Consider, $L^{-1}\left[\cot^{-1}\left(\frac{s-2}{3}\right)\right] = f(t) \qquad \ldots(1)$

$$\therefore\quad L[f(t)] = \cot^{-1}\left(\frac{s-2}{3}\right)$$

➔ **Using property of Laplace transform**

$$\ldots\left[L[t\,f(t)] = -\frac{d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore L[t\,f(t)] = -\frac{d}{ds}\left[\cot^{-1}\left(\frac{s-2}{3}\right)\right]$$

➔ **Using standard result of derivative**

$$\ldots\left[\frac{d}{dx}[\cot^{-1} f(x)] = \frac{-1}{1+(f(x))^2}\cdot\frac{d}{dx}f(x)\right]$$

$$L[t\,f(t)] = -\left[\frac{-1}{1+\left(\frac{s-2}{3}\right)^2}\frac{d}{ds}\left(\frac{s-2}{3}\right)\right]$$

$$= \left[\frac{1}{1+\frac{(s-2)^2}{9}}\cdot\frac{1}{2}(1)\right]$$

$$= \left[\frac{1}{\frac{9+(s-9)^2}{9}}\cdot\frac{1}{2}\right] \qquad \text{(by simplification)}$$

$$= \left[\frac{9}{9+(s-2)^2}\cdot\frac{1}{2}\right]$$

$$L[t\,f(t)] = \frac{9}{2}\left[\frac{1}{9+(s-2)^2}\right]$$

$$\therefore\quad t\,f(t) = \frac{9}{2}L^{-1}\left\{\frac{1}{9+(s-2)^2}\right\}$$

{Observe that all s in the form (s – 2)}

➔ **Using first shifting property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)]\right\}$$

$$\therefore\quad t\,f(t) = \frac{9}{2}e^{2t}L^{-1}\left[\frac{1}{9+s^2}\right]$$

Replace every (s + 1) by s with taking e^{-t}

$$t\,f(t) = \frac{9}{2}e^{2t}\cdot L^{-1}\left[\frac{1}{s^2+(3)^2}\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$t\,f(t) = \frac{9}{2}e^{2t}\underbrace{L^{-1}\left[\frac{1}{s^2+(3)^2}\right]}_{\frac{1}{3}\sin 3t}$$

$$t\,f(t) = \frac{9}{2}e^{2t}\,\frac{1}{3}\sin 3t$$

$$t\,f(t) = \frac{3}{2}e^{2t}\sin 3t$$

$$\therefore\quad f(t) = \frac{\frac{3}{2}e^{2t}\sin 3t}{t} \qquad \ldots(2)$$

From Equations (1) and (2),

$$L^{-1}\left[\cot^{-1}\left(\frac{s-2}{3}\right)\right] = \frac{\frac{3}{2}e^{2t}\sin 3t}{t}\checkmark \qquad \ldots\text{Ans.}$$

Ex. 2.7.14 : Find the inverse Laplace transform of $\tan^{-1}(s+1)$.

Soln. :

Consider, $L^{-1}[\tan^{-1}(s+1)] = f(t) \qquad \ldots(1)$

$$\therefore\quad L[f(t)] = \tan^{-1}(s+1)$$

➔ **Using property of Laplace transform**

$$\ldots\left[L[t\,f(t)] = -\frac{d}{ds}[f(s)] \text{ where } f(s) = L[f(t)]\right]$$

$$\therefore\quad L[t\,f(t)] = -\frac{d}{ds}[\tan^{-1}(s+1)]$$

➔ **Using standard result of derivative**

$$\ldots\left[\frac{d}{dx}[\tan^{-1} f(x)] = \frac{1}{1+(f(x))^2}\cdot\frac{d}{dx}f(x)\right]$$

$$L[t\,f(t)] = -\left[\frac{1}{1+(s+1)^2}\frac{d}{ds}(s+1)\right]$$

$$= -\left[\frac{1}{1+(s+1)^2}\cdot(1)\right]$$

$$L[t\,f(t)] = -\left[\frac{1}{1+(s+1)^2}\right]$$

$$\therefore\quad t\,f(t) = L^{-1}\left\{\frac{-1}{1+(s+1)^2}\right\}$$

{Observe that all s in the form (s + 1)}

➔ **Using first shifting property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)]\right\}$$

$$\therefore\quad t\,f(t) = e^{-t}L^{-1}\left[\frac{1}{1+s^2}\right]$$

Replace every (s + 1) by s with taking e^{-t}

$$t\,f(t) = e^{-t}(-1)\,L^{-1}\left[\frac{1}{s^2+1}\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$t\,f(t) = -e^{-t}\underbrace{L^{-1}\left[\frac{1}{s^2+(1)^2}\right]}_{\frac{1}{1}\sin t}$$

$$t\,f(t) = -e^{-t}\sin t$$

$$t\,f(t) = -e^{-t}\sin t$$

$$\therefore\quad f(t) = \frac{-e^{-t}\sin t}{t} \qquad \ldots(2)$$

From Equations (1) and (2),

$$L^{-1}[\tan^{-1}(s+1)] = -\frac{e^{-t}\sin t}{t}\checkmark \qquad \ldots\text{Ans.}$$

Exercise

Find inverse Laplace transform of :

Ex. 1 : $\frac{2s-5}{s^2+s-6}$

Ans. : $\left[\frac{1}{5}(11\,e^{-3t} - e^{2t})\right]$

Ex. 2 : $\frac{s^2+5}{(s^2+4)(s^2+9)}$

Ans. : $\left[\frac{1}{10}\sin 2t + \frac{4}{15}\sin 3t\right]$

Ex. 3 : $\frac{s}{(s^2+2s+4)(s^2+4s+4)}$

Ans. : $\left[\frac{1}{2}\left[\frac{e^{-t}\sin(\sqrt{3}t)}{\sqrt{3}}\right] - te^{-2t}\right]$

Ex. 4 : $\frac{3s+4}{s^2-3s-4}$

Ans. : $\left[\frac{1}{5}(16e^{4t} - e^{-t})\right]$

Ex. 5 : $\frac{2s-5}{s^2+4s+8}$

Ans. : $f(t) = 2e^{-2t}\cos(2t) - \frac{9}{2}e^{-2t}\sin(2t)$

Ex. 6 : $\frac{s+4}{s-2^3}$

Ans. : $f(t) = t\,e^{2t} + 3t^2 e^{2t}$

Ex. 7 : $\frac{1}{(s+2)(s-1)}$

Ans. : $\frac{1}{3}e^{t} - \frac{1}{3}e^{-2t}$

Ex. 8 : $\frac{4s+2}{s^2+6s+34}$

Ans. : $4\,e^{-3t}\cos 5t - 2\,e^{-3t}\sin 5t$

Ex. 9 : $\frac{1}{(s-2)(s-4)(s-8)}$

Ans. : $\frac{1}{12}e^{2t} - \frac{1}{8}e^{4t} + \frac{1}{24}e^{8t}$

Ex. 10 : $\frac{3s-7}{4s^2+1}$

Ans. : $\frac{3}{4}\cos\frac{t}{2} - \frac{7}{2}\sin\frac{t}{2}$

Syllabus Topic : Convolution Theorem for finding inverse Laplace transforms

2.8 Convolution Theorem

Property VI : Convolution Theorem

Convolution of two functions f(t) and g(t), t > 0 is defined as,

$$\int_0^t f(u)\,g(t-u)\,du.$$

It is denoted as $f(t)*g(t)$

$$\therefore\quad f(t)*g(t) = \int_0^t f(u)\,g(t-u)\,du$$

Note that:

$$\therefore\quad g(t)*f(t) = \int_0^t g(u)\,f(t-u)\,du$$

We know, If $L[f(t)] = f(s)$ and $L[g(t)] = g(s)$, then

$$L\left[\int_0^t f(u)\,g(t-u)\,du\right] = f(s)*g(s) = L\left[\int_0^t f(t-u)\,g(u)\,du\right]$$

$$\therefore\quad L^{-1}\{f(s)*g(s)\} = \int_0^t f(u)\,g(t-u)\,du$$

Ex. 2.8.1 : Find inverse Laplace transform by using convolution theorem $\frac{1}{(s+1)(s^2+1)}$

Soln. :

We know convolution theorem,

$$L^{-1}[f(s)*g(s)] = \int_0^t f(t-u)\cdot g(u)\,du$$

Given function write as $f(s) * g(s)$,

Where, $f(s)$ and $g(s)$ are standard inverse Laplace transform such that,

$$f(s) = L^{-1}[f(t)] \text{ and } g(s) = L^{-1}[g(t)]$$

Given,

$$\frac{1}{(s+1)(s^2+1)} = \frac{1}{s+1} \cdot \frac{1}{s^2+1} = f(s) * g(s) \quad \ldots(1)$$

Where, $f(s) = \frac{1}{s+1}$ and $g(s) = \frac{1}{s^2+1}$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}\right]$$

$$\therefore\ L^{-1}[f(s)] = L^{-1}\left[\frac{1}{s+1}\right]$$

$$= e^{-t} = f(t) \quad \ldots(2)$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$\therefore\ L^{-1}[g(s)] = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t = g(t) \quad \ldots(3)$$

We have, convolution theorem

$$L^{-1}[f(s) * g(s)] = \int_0^t f(t-u) \cdot g(u)\, du$$

From Equations (1), (2) and (3)

$$L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right] = \int_0^t e^{-(t-u)} \sin u\, du$$

$$= \int_0^t e^{-t} \cdot e^{u} \sin u\, du$$

$$= e^{-t} \int_0^t e^{u} \sin u\, du$$

➔ **Using standard integration formula**

$$\ldots\left\{\int e^{ax} \sin(bx+c)\, dx = \frac{e^{ax}}{a^2+b^2}[a \sin(bx+c) - b\cos(bx+c)]\right\}$$

$$= e^{-t}\left[\frac{e^{u}}{(1)^2+(1)^2}(1 \cdot \sin u - 1 \cdot \cos u)\right]_0^t$$

$$\left\{\int e^{u} \sin u\, du \text{ compare with } \int e^{ax} \sin(bx+c)\, dx \quad \therefore\ a = 1, b = 1, c = 0\right\}$$

Substitute t as : $\left(\begin{matrix}\text{upper}\\\text{limit}\end{matrix}\right) - \left(\begin{matrix}\text{lower}\\\text{limit}\end{matrix}\right)$

$$L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right] = e^{-t}\left\{\left[\frac{e^{t}}{2}(\sin t - \cos t)\right] - \left[\frac{e^{0}}{2}(\sin 0 - \cos 0)\right]\right\}$$

$$= e^{-t}\left\{\frac{e^{t}}{2}(\sin t - \cos t) - \frac{1}{2}(0-1)\right\}$$

$$L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right] = e^{-t}\left[\frac{e^{t}}{2}(\sin t - \cos t) + \frac{1}{2}\right]$$

$$L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right] = \frac{1}{2}(\sin t - \cos t) + \frac{1}{2}e^{-t} \quad \ldots\text{Ans.}$$

Ex. 2.8.2 : Find inverse Laplace transform by using convolution theorem $\frac{1}{s^3(s^2+1)}$

Soln. : We know convolution theorem,

$$L^{-1}[f(s) * g(s)] = \int_0^t f(t-u) \cdot g(u)\, du$$

Given function write as $f(s) * g(s)$,

Where, $f(s)$ and $g(s)$ are standard inverse Laplace transform such that,

$$f(s) = L^{-1}[f(t)] \text{ and } g(s) = L^{-1}[g(t)]$$

Given,

$$\frac{1}{s^3(s^2+1)} = \frac{1}{s^3} \cdot \frac{1}{s^2+1} = f(s) * g(s) \quad \ldots(1)$$

Where,

$$f(s) = \frac{1}{s^3} \text{ and } g(s) = \frac{1}{s^2+1}$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{\Gamma(n+1)}\right]$$

$$L^{-1}[f(s)] = L^{-1}\left[\frac{1}{s^3}\right] = \frac{t^2}{\Gamma 3}$$

$$= \frac{t^2}{2!}$$

$$L^{-1}[f(s)] = \frac{t^2}{2} = f(t) \quad \ldots(2)$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$L^{-1}[g(s)] = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t = g(t) \quad \ldots(3)$$

We have, convolution theorem

$$L^{-1}[f(s) * g(s)] = \int_0^t f(t-u) \cdot g(u)\, du$$

From Equations (1), (2) and (3),

$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = \int_0^t \frac{(t-u)^2}{2} \cdot \sin u$$

$$= \frac{1}{2}\int_0^t (t-u)^2 \sin u\, du$$

➔ **Using integration formula**

$$\ldots\left[\int uv\, dx = uv_1 - u'v_2 + u''v_3 - \ldots\ldots\right]$$

$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = \frac{1}{2}\left[(t-u)^2(-\cos u) - (2(t-u)(-1))(-\sin u) + (2)(\cos u)\right]_0^t$$

Substitute t as : $\left(\begin{matrix}\text{upper}\\\text{limit}\end{matrix}\right) - \left(\begin{matrix}\text{lower}\\\text{limit}\end{matrix}\right)$

$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = \frac{1}{2}\left\{[0+0+2\cos t] - [-(t)^2\cos 0 - 2t\sin 0 + 2\cos 0]\right\}$$

$$= \frac{1}{2}[2\cos t + t^2 - 2]$$

$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = \cos t + \frac{1}{2}(t^2 - 2) \checkmark \quad \ldots\text{Ans.}$$

Ex. 2.8.3 : Find inverse Laplace transform by convolution theorem $\frac{1}{(s-2)^4(s+3)}$

Soln. : We know convolution theorem,

$$L^{-1}[f(s) * g(s)] = \int_0^t f(t-u) \cdot g(u)\, du$$

Given function write as $f(s) * g(s)$,

Where, $f(s)$ and $g(s)$ are standard inverse Laplace transform such that,

$$f(s) = L^{-1}[f(t)] \text{ and } g(s) = L^{-1}[g(t)]$$

Given,

$$\frac{1}{(s-2)^4(s+3)} = \frac{1}{s+3} \cdot \frac{1}{(s-2)^4} = f(s) * g(s) \quad \ldots(1)$$

where $f(s) = \frac{1}{s+3}$ and $g(s) = \frac{1}{(s-2)^4}$

➔ **Using standard result of inverse Laplace transform:**

$$\ldots\left[L^{-1}\left[\frac{1}{s-a}\right] = e^{at} \text{ and } L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}\right]$$

$$\therefore\ L^{-1}[f(s)] = L^{-1}\left[\frac{1}{s+3}\right]$$

$$\therefore\ L^{-1}[f(s)] = e^{-3t} = f(t) \quad \ldots(2)$$

➔ **Using first shifting property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)]\right\}$$

$$L^{-1}[g(s)] = L^{-1}\left[\frac{1}{(s-2)^4}\right] = e^{2t}L^{-1}\left[\frac{1}{s^4}\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{\Gamma(n+1)}\right]$$

$$\therefore\ L^{-1}[g(s)] = e^{2t}\left[\frac{t^3}{\Gamma 4}\right]$$

➔ **By property of Gamma function**

$$\ldots\left[\Gamma(n+1) = n! \text{ if n is integer}\right]$$

$$= e^{2t}\frac{t^3}{3!}$$

$$L^{-1}[g(s)] = \frac{e^{2t}t^3}{6} = g(t) \quad \ldots(3)$$

$$(3! = 1 \times 2 \times 3 = 6)$$

We have, convolution theorem.

$$L^{-1}[f(s) * g(s)] = \int_0^t f(t-u) \cdot g(u)\, du$$

From Equations (1), (2) and (3),

$$L^{-1}\left[\frac{1}{(s-2)^4(s+3)}\right] = \int_0^t e^{-3(t-u)} \cdot \frac{e^{2u}u^3}{6}\, du$$

$$= \frac{1}{6}\int_0^t e^{-3t} \cdot e^{3u} e^{2u} u^3\, du$$

$$L^{-1}\left[\frac{1}{(s-2)^4(s+3)}\right] = \frac{1}{6}e^{-3t}\int_0^t e^{5u} u^3\, du$$

➔ **Using integration formula**

$$\ldots\left[\int uv\, dx = uv_1 - u'v_2 + u''v_3 - \ldots\ldots\right]$$

$$= \frac{1}{6}e^{-3t}\left[u^3 \cdot b\left(\frac{e^{5u}}{5}\right) - (3u^2)\left(\frac{e^{5u}}{5\times5}\right) + (6u)\left(\frac{e^{5u}}{5\times5\times5}\right) - (6)\left(\frac{e^{5u}}{5\times5\times5\times5}\right)\right]_0^t$$

$$= \frac{1}{6}e^{-3t}\left[\frac{1}{5}u^3 e^{5u} - \frac{3}{25}u^2 e^{5u} + \frac{6}{125}u e^{5u} - \frac{6}{625}e^{5u}\right]_0^t$$

$$= \frac{1}{6}e^{-3t}\left\{e^{5u}\left[\frac{1}{5}u^3 - \frac{3}{25}u^2 + \frac{6}{125}u - \frac{6}{625}\right]\right\}_0^t$$

Substitute t as : $\left(\begin{matrix}\text{upper}\\\text{limit}\end{matrix}\right) - \left(\begin{matrix}\text{lower}\\\text{limit}\end{matrix}\right)$

$$= \frac{1}{6}e^{-3t}\left\{\left[e^{5t}\left(\frac{t^3}{5} - \frac{3t^2}{25} + \frac{6t}{125} - \frac{6}{625}\right)\right]\left[(1)\left(0 - \frac{6}{625}\right)\right]\right\}$$

$$L^{-1}\left[\frac{1}{(s-2)^4(s+3)}\right] = \frac{e^{-3t}}{6} \cdot e^{5t}\left(\frac{t^3}{5} - \frac{3t^2}{25} + \frac{6t}{125} - \frac{6}{625}\right) + \frac{1}{6}e^{-3t} \cdot \frac{6}{625}$$

$$L^{-1}\left[\frac{1}{(s-2)^4(s+3)}\right] = \frac{e^{2t}}{6}\left[\frac{t^3}{5} - \frac{3t^2}{25} + \frac{6t}{125} - \frac{6}{625}\right] + \frac{e^{-3t}}{625} \checkmark \quad \ldots\text{Ans.}$$

Syllabus Topic : Applications to find the solutions of linear differential equations

2.9 Laplace Transform of Derivative

Laplace transform of derivatives :If $L[f(t)] = f(s)$, then $L\{f'(t)\} = sf(s) - f(0)$

Proof

A :

We know the definition of Laplace transform

$$\therefore\ L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$\therefore\ L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

➔ **Using integration of product rule (integration by parts)**

$$\left[\int f(x)\cdot g(x)\,dx = f(x)\cdot\int g(x)\,dx - \int[f'(x)]\cdot\int g(x)\,dx]\,dx\right]$$

$$= [e^{-st} f(t)]_0^\infty - \int_0^\infty (-s)\,e^{-st} f(t)\,dt$$

$$= 0 - f(0) + s\int_0^\infty e^{-st} f(t)\,dt = -f(0) + sL[f(t)]$$

$$\ldots\left[\lim_{t\to\infty} e^{-st} f(t) = 0\right]$$

$$L[f'(t)] = -f(0) + s\,f(s)$$

$$\therefore\ L[f'(t)] = sf(s) - f(0)\ \text{OR}\ L\left[\frac{df}{dt}\right] = sf(s) - f(0)$$

2.10 Application of Laplace Transform to the Differential Equation

Introduction

Laplace transform is effective tool to solve Linear differential equations whose particular solution can directly be obtained without doing laborious calculation to find values of an arbitrary constants.

We have

$$L[f'(t)] = s\,f(s) - f(0)$$

$$L[f''(t)] = s^2 f(s) - s\,f(0) - f'(0)$$

$$L[f'''(t)] = s^3 f(s) - s^2 f(0) - s\,f'(0) - f''(0)$$

$$L[f^n(t)] = s^n f(s) - s^{n-1} f(0) - s^{n-2} f'(0)\ \ldots\ldots\ldots f^{(n-1)}(0)$$

To solve differential equation, operate Laplace transform on both sides of the given differential equation and use initial conditions. It gives algebraic equation in terms of s with y(s). Then take inverse Laplace transform of both sides gives solution.

e.g. $\dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} + 7y = t \quad y(0) = y'(0) = 1$

Operate Laplace Transform

$$s^2 y(s) - s\,y(0) - y'(0) + 5[s\,y(s) - y(0)] + 7\,y(s) = \frac{1}{s^2}$$

$$(s^2 + 5s + 7)\,y(s) = \frac{1}{s^2}\ \ldots\text{(algebraic equation)(subsidiary equation)}$$

$$\therefore\ y(s) = \frac{1}{s^2(s^2 + 5s + 7)}$$

Take inverse Laplace Transform

$$\therefore\ L^{-1}[y(s)] = L^{-1}\left[\frac{1}{s^2(s^2 + 5s + 7)}\right],$$

$$y(t) = L^{-1}\left[\frac{1}{s^2(s^2 + 5s + 7)}\right]\ \text{Find it.}$$

Ex. 2.10.1 : Solve by Laplace transform method $\dfrac{dy}{dt} + 2y = 4e^{-2t}$; $y(0) = -3$

Soln. : Given differential equation is,

$$\frac{dy}{dt} + 2y = 4e^{-2t}$$

Operate Laplace transform of both sides,

$$L\left[\frac{dy}{dt} + 2y\right] = L[4e^{-2t}]$$

➔ **Using linearity property of Laplace transform**

$$\ldots\left[L[a\,f(t) + b\,g(t) - c\,h(t)] = a\,L\,f[f(t)] + 6\,L[g(t)] - c\,L[h(t)]\right]$$

$$\therefore\ L\left[\frac{dy}{dt}\right] + 2L[y] = 4\,L[e^{-2t}] \qquad \ldots(1)$$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L\left[\frac{df}{dt}\right] = s\,f(s) - f(0);\ L[e^{at}] = \frac{1}{s-a}\right]$$

$$L[y] = y(s)$$

$$L\left[\frac{dy}{dt}\right] = sy(s) - y(0)$$

∴ Equation (1) becomes,

$$sy(s) - y(0) + 2y(s) = 4\left[\frac{1}{s-(-2)}\right]$$

$$\text{since,}\quad y(0) = -3$$

$$sy(s) - 3 + 2y(s) = 4\frac{1}{s+2}$$

grouping the term

$$[sy(s) + 2y(s)] - 3 = \frac{4}{s+2}$$

$$(s+2)\,y(s) - 3 = \frac{4}{s+2}$$

$$(s+2)\,y(s) = \frac{4}{(s+2)} + 3$$

Divide throughout by (s + 2)

$$\cancel{\frac{(s+2)}{(s+2)}}\,y(s) = \frac{4}{(s+2)^2} + \frac{3}{(s+2)}\quad \text{[Algebraic equation]}$$

$$y(s) = \frac{4}{(s+2)^2} + \frac{3}{(s+2)}$$

Now operating inverse Laplace transform on both sides,

$$L^{-1}[y(s)] = L^{-1}\left\{\frac{4}{(s+2)^2} + \frac{3}{(s+2)}\right\}$$

...(observe that each s of the form s – a i.e. s + 2)

➔ **Using first shifting theorem of inverse Laplace transform**

$$\ldots\left[L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)]\right]$$

$$\therefore L^{-1}[y(s)] = L^{-1}\underbrace{\left\{\frac{4}{(s+2)^2} + \frac{3}{(s+2)}\right\}}_{e^{-2t}L^{-1}\left[\frac{4}{s^2} + \frac{3}{s}\right]}\quad \text{\{compare s + 2 with s – a\}}$$

{Taking e^{-2t} out side and L^{-1} [every s + 2 replace by s]}

$$\therefore\ y(t) = e^{-2t}L^{-1}\left[\frac{4}{s^2} + \frac{3}{s}\right]$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots\{L^{-1}[a\,f(s) + b\,g(s) - c\,h(s)] = a\,L^{-1}[f(s)] + b\,L^{-1}[g(s) - cL^{-1}[h(s)]\}$$

$$y(t) = e^{-2t}\left\{4\,L^{-1}\left[\frac{1}{s^2}\right] + 3\,L^{-1}\left[\frac{1}{s}\right]\right\}$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{\Gamma n}\ \text{OR}\ \frac{t^{n-1}}{\Gamma(n-1)!}\right.$$

$$\left.\text{if n is an integer and } L^{-1}\left[\frac{1}{s}\right] = 1\right]$$

$$\therefore\ y(t) = e^{-2t}\left\{4\underbrace{L^{-1}\left[\frac{1}{s^2}\right]}_{\frac{t^{2-1}}{(2-1)!}} + 3\underbrace{L^{-1}\left[\frac{1}{s}\right]}_{1}\right\}$$

$$\therefore\ y(t) = e^{-2t}\left\{4\frac{t^{2-1}}{(2-1)!} + 3(1)\right\}$$

$$y(t) = e^{-2t}\left[4\frac{t}{1!} + 3\right]\qquad (\because 1! = 1)$$

$$y(t) = e^{-2t}(4t+3)\ ✓\qquad \ldots\text{Ans.}$$

Ex. 2.10.2 : Using Laplace transform method solve the differential equation : $\dfrac{dy}{dx} - 3y = 9$

When x = 0, y = 0

Soln. :

Given differential equation is,

$$\frac{dy}{dx} - 3y = 9$$

Operate Laplace transform on both sides,

$$L\left[\frac{dy}{dt} - 3y\right] = L[9]$$

➔ **Using linearity property of Laplace transform**

$$\ldots\left[L[a\,f(t) + b\,g(t) - c\,h(t)] = a\,L\,f[f(t)] + b\,L[g(t)] - c\,L[h(t)]\right]$$

$$\therefore\ L\left[\frac{dy}{dx}\right] - 3L[y] = 9\,L[1]$$

➔ **Using Laplace transform of derivative**

$$\ldots\left[L[f'(t)] = s\,f(s) - f(0)\ \text{with}\ L[y] = y(s)\right]$$

$$\therefore\ [sy(s) - y(0)] - 3y(s) = 9L[1]$$

Since when x = 0, y = 0 i.e. y (0) = 0

$$s\,y(s) - 0 - 3\,y(s) = 9\left[\frac{1}{s}\right]\qquad \ldots\left\{\text{since } L[1] = \frac{1}{s}\right\}$$

$$s\,y(s) - 3\,y(s) = \frac{9}{s}$$

$$(s\text{-}3)\,y(s) = \frac{9}{s}$$

$$y(s) = \frac{9}{s(s-3)}\qquad \ldots\text{[Algebraic equation]}$$

$$y(s) = 9\left[\frac{1}{s(s-3)}\right]$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[y(s)] = L^{-1}\left\{\frac{1}{9}\left[\frac{1}{s(s-3)}\right]\right\}\qquad \ldots(1)$$

Now by partial fraction formula,

$$\left\{\frac{1}{(x-a)(x-b)} = \frac{1}{a-b}\left[\frac{1}{x-a} - \frac{1}{x-b}\right]\right\}$$

Compare : s as (s – 0) with (x – a), (s – 3) with (x – b) } a = 0, b = 3

$$\therefore\ \frac{1}{s(s-3)} = \frac{1}{0-3}\left[\frac{1}{s} - \frac{1}{s-3}\right]$$

$$= -\frac{1}{3}\left[\frac{1}{s} - \frac{1}{s-3}\right] = \frac{1}{3}\left[\frac{1}{s-3} - \frac{1}{s}\right]$$

$$\left[\frac{1}{s(s-3)}\right] = \frac{1}{3}\left[\frac{1}{s-3} - \frac{1}{s}\right]$$

Substitute these in Equation (1)

$$L^{-1}[y(s)] = L^{-1}\left\{\frac{1}{9}\times\frac{1}{3}\left[\frac{1}{s-3} - \frac{1}{s}\right]\right\}$$

$$y(t) = L^{-1}\left\{\frac{1}{27}\left[\frac{1}{s-3} - \frac{1}{s}\right]\right\}$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots\left[L^{-1}[a\,f(s) + b\,g(s) - c\,h(s)] = a\,L^{-1}[f(s)] + b\,L^{-1}[g(s)] - cL^{-1}[h(s)]\right]$$

$$y(t) = \frac{1}{27}\left\{L^{-1}\left[\frac{1}{s-3}\right] - L^{-1}\left[\frac{1}{s}\right]\right\}$$

➔ **Using standard results of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s-a}\right] = e^{at}\ \text{and}\ L^{-1}\left[\frac{1}{s}\right] = 1\right]$$

$$y(t) = \frac{1}{27}\left[\underbrace{L^{-1}\left(\frac{1}{s-3}\right)}_{e^{3t}} - \underbrace{L^{-1}\left(\frac{1}{s}\right)}_{1}\right]$$

$$y(t) = \frac{1}{27}[e^{3t} - 1]\ ✓\qquad \ldots\text{Ans.}$$

Ex. 2.10.3 : Solve by Laplace transform $\frac{dy}{dt} + 3y(t) = t$ given $y(0) = 0$

Soln. :

Given differential equation is,

$$\frac{dy}{dt} + 3y(t) = t$$

Operating Laplace transform on both sides,

$$L\left[\frac{dy}{dt} + 3y(t)\right] = L(t)$$

➔ **Using linearity property of Laplace transform**

$$\ldots\left[L[a f(t) \pm b g(t) - c h(t)] = a L f[f(t)] + b L[g(t)] - c L[h(t)]\right]$$

$$\therefore L\left[\frac{dy}{dt}\right] + 3L[y(t)] = L[t]$$

➔ **Using Laplace transform of derivative and standard results**

$$\ldots\left[L[f'(t)] = s f(s) - f(0) \text{ with } L[t] = \frac{1}{s^2}\right]$$

$$\text{Here } L[y(t)] = y(s)$$

$$\therefore [s y(s) - y(0)] + 3y(s) = \frac{1}{s^2}$$

$$\text{Since } y(0) = 0$$

$$s y(s) - 0 + 3 y(s) = \frac{1}{s^2}$$

$$s y(s) + 3 y(s) = \frac{1}{s^2}$$

$$(s+3) y(s) = \frac{1}{s^2}$$

$$y(s) = \left[\frac{1}{s^2(s+3)}\right]$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[y(s)] = L^{-1}\left[\frac{1}{s^2(s+3)}\right] \quad \ldots(1)$$

By partial fraction

$$\frac{1}{s^2(s+3)} = \frac{A}{s+3} + \frac{Bs+C}{s^2} \quad \ldots(2)$$

$$\frac{1}{\cancel{s^2(s+3)}} = \frac{As^2 + (Bs+C)(s+3)}{\cancel{s^2(s+3)}}$$

{by simplification taking L.C.M. of R.H.S.}

$$\therefore I = As^2 + \underbrace{(Bs+C)(s+3)}$$

$$Bs.s + Bs.3 + C.s + C.(3) \quad \ldots(3)$$

...(by distributive property)

$$I = As^2 + Bs^2 + 3Bs + Cs + 3c$$

$$= (As^2 + Bs^2) + (3Bs + Cs) + 3C$$

...(Grouping the terms)

$$I = (A+B)s^2 + (3B+C)s + 3c$$

$$os^2 + os + 1 = (A+B)s^2 + (3B+C)s + 3c$$

Equating coefficients of same power s.

$$\therefore 0 = A + B \quad \Big| \quad 0 = 3B + C \quad \Big| \quad 1 = 3C$$

$$\frac{1}{3} = C \; ; \; C = \frac{1}{3}$$

$$0 = 3B + \frac{1}{3}$$

$$\therefore -\frac{1}{3} = 3B$$

$$\therefore \frac{-1}{9} = B$$

$$\therefore B = \frac{-1}{9}$$

$$\therefore 0 = A + \left(\frac{-1}{9}\right)$$

$$\frac{1}{9} = A$$

$$A = \frac{1}{9}$$

Substitute these $\left(A = \frac{1}{9}, B = \frac{-1}{9}, C = \frac{1}{3}\right)$ in Equation (2),

$$\frac{1}{s^2(s+3)} = \frac{\frac{1}{9}}{s+3} + \frac{\frac{-1}{9}s + \frac{1}{3}}{s^2}$$

$$\frac{-\frac{1}{9}s}{s^2} + \frac{\frac{1}{3}}{s^2}$$

$$\left(\text{By, } \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}\right)$$

$$= \frac{1}{9}\left(\frac{1}{s+3}\right) + \frac{-\frac{1}{9}\cancel{s}}{s^{\cancel{2}}} + \frac{\frac{1}{3}}{s^2}$$

$$\frac{1}{s^2(s+3)} = \frac{1}{9}\left(\frac{1}{s+3}\right) - \frac{1}{9}\frac{1}{s} + \frac{1}{3}\frac{1}{s^2}$$

Taking inverse Laplace transform of both sides,

$$\therefore L^{-1}\left[\frac{1}{s^2(s+3)}\right] = L^{-1}\left[\frac{1}{9}\left(\frac{1}{s+3}\right) - \frac{1}{9}\left(\frac{1}{s}\right) + \frac{1}{3}\left(\frac{1}{s^2}\right)\right]$$

➔ **Using linearity property of inverse Laplace transform**

$$\left[L^{-1}[af(s) + b g(s) - ch(s)] = aL^{-1}[f(s)] + bL^{-1}[g(s)] - cL^{-1}[n(s)]\right]$$

$$\therefore L^{-1}\left[\frac{1}{s^2(s+3)}\right] = \frac{1}{9}L^{-1}\left(\frac{1}{s+3}\right) - \frac{1}{9}L^{-1}\left(\frac{1}{s}\right) + \frac{1}{3}\left(\frac{1}{s^2}\right)$$

➔ **Using standard result of inverse Laplace transform**

$$\left[L^{-1}\left(\frac{1}{s-a}\right) = e^{at}, L^{-1}\left(\frac{1}{s}\right) = 1 \text{ and } L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{\Gamma n} \text{ or } \frac{t^{n-1}}{(n-1)!} \text{ if n is integer}\right]$$

$$L^{-1}\left[\frac{1}{s^2(s+3)}\right] = \frac{1}{9}\underbrace{L^{-1}\left(\frac{1}{s-(-3)}\right)}_{e^{-3t}} - \frac{1}{9}\underbrace{L^{-1}\left(\frac{1}{s}\right)}_{1} + \frac{1}{3}\underbrace{L^{-1}\left(\frac{1}{s^2}\right)}_{\frac{t^{2-1}}{(2-1)!}}$$

$$= \frac{1}{9}e^{-3t} - \frac{1}{9}(1) + \frac{1}{3}\frac{t^{2-1}}{(2-1)!}$$

$$= \frac{1}{9}e^{-3t} - \frac{1}{9} + \frac{1}{3}\frac{t}{1!}$$

$$L^{-1}\left[\frac{1}{s^2(s+3)}\right] = \frac{1}{9}(e^{-3t} - 1) + \frac{1}{3}(t)$$

Substitute these in Equation (1),

$$\therefore L^{-1}[y(s)] = \frac{1}{9}(e^{-3t} - 1) + \frac{t}{3}$$

$$y(t) = \frac{1}{9}(e^{-3t} - 1) + \frac{t}{3} \checkmark \quad \ldots\text{Ans.}$$

Ex. 2.10.4 : Solve the differential equation using Laplace transform

$y''(t) + y(t) = t$; $y(0) = 1$, $y'(0) = -2$.

Soln. :

Given differential Equation is,

$$y''(t) + y(t) = t$$

Operate Laplace transform of both sides,

$$L[y''(t) + y(t)] = L[t]$$

➔ **Using linearity property of Laplace transform**

$$\ldots\left\{L[a f(t) + b g(t) - c h(t)] = aL[f(t)] + bL[g(t)] - cL[h(t)]\right\}$$

$$L[y''(t)] + L[y(t)] = L[t]$$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L\left[\frac{d^2y}{dt^2}\right] = s^2 y(s) - s y(0) - y'(0) \; ; \right.$$

$$L\left[\frac{dy}{dt}\right] = sy(s) - y(0) \text{ and } L[y(t)] = y(s) \; ;$$

$$\left. L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} \text{ OR } \frac{n!}{s^{n+1}} \text{ if n is integer}\right]$$

$$\underbrace{L[y''(t)]}_{s^2y(s) - sy(0) - y'(0)} + \underbrace{L[y(t)]}_{y(s)} = \underbrace{L[t]}_{\frac{1}{s^2}}$$

$$s^2 y(s) - sy(0) - y'(0) + y(s) = \frac{1}{s^2}$$

Since, $y(0) = 1$, $y'(0) = -2$ ...(Given)

$$s^2 y(s) - s(1) - (-2) + y(s) = \frac{1}{s^2}$$

$$s^2 y(s) - s + 2 + y(s) = \frac{1}{s^2}$$

$$[s^2 y(s) + y(s)] + (-s + 2) = \frac{1}{s^2} \quad \ldots\text{(Grouping the terms)}$$

$$(s^2 + 1) y(s) + (-s + 2) = \frac{1}{s^2}$$

$$(s^2 + 1) y(s) = \frac{1}{s^2} + s - 2$$

$$y(s) = \frac{1}{s^2(s^2+1)} + \frac{s-2}{(s^2+1)} \quad \ldots(1)$$

By partial fraction :

$$\frac{1}{s^2(s^2+1)} = \frac{1}{X(X+1)} \qquad \text{Where, } X = s^2$$

$$= \frac{A}{X} + \frac{B}{X+1} \quad \ldots(2)$$

$$A = \left[\frac{1}{X+1}\right]_{X=0} = \frac{1}{1} = 1$$

$$B = \left[\frac{1}{X}\right]_{X=-1} = \frac{1}{-1} = -1$$

Substitute these values in Equation (2),

$$\therefore \frac{1}{s^2(s^2+1)} = \frac{1}{X(X+1)} = \frac{1}{X} + \frac{-1}{X+1}$$

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1} \qquad (\because X = s^2)$$

Substitute these in Equation (1),

$$\therefore y(s) = \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{s-2}{s^2+1}$$

$$= \frac{1}{s^2} \underbrace{- \frac{1}{s^2+1}} + \frac{s}{s^2+1} \underbrace{- \frac{2}{s^2+1}}$$

$$\frac{-3}{s^2+1}$$

$$y(s) = \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}$$

Taking inverse laplace transform of both sides,

$$L^{-1}[y(s)] = L^{-1}\left[\frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}\right]$$

➔ **Using linearity of inverse Laplace transform**

$$\ldots\left\{L^{-1}[af(s) + bg(s) - ch(s)] = a L^{-1}[f(s)] + bL^{-1}[g(s)] - cL^{-1}[h(s)]\right\}$$

$$L^{-1}[y(s)] = L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{s}{s^2+1}\right] - 3L^{-1}\left[\frac{3}{s^2+1}\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{\Gamma(n+1)} \; ; \; L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at \; ;\right.$$

$$\left. \text{and } L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$L^{-1}[y(s)] = \underbrace{L^{-1}\left[\frac{1}{s^2}\right]}_{\frac{t}{12}} + \underbrace{L^{-1}\left[\frac{s}{s^2+1}\right]}_{\text{cost}} - 3\underbrace{L^{-1}\left[\frac{3}{s^2+1}\right]}_{\sin t}$$

$$y(t) = \frac{t}{1!} + \cos t - 3\sin t$$

$$y(t) = t + \cos t = 3\sin t \checkmark \qquad \text{... Ans.}$$

Ex. 2.10.5 : Solve by Laplace transform method

$$\frac{dy}{dt} + 3y(t) + 2\int_0^t y(t)\,dt = t\,;\ \text{given } y(0) = 0$$

Soln. :

Given differential equation is

$$\frac{dy}{dt} + 3y(t) + 2\int_0^t y(t)\,dt = t$$

Operate Laplace transform of both sides,

$$L\left[\frac{dy}{dt} + 3y(t) + 2\int_0^t y(t)\,dt\right] = L[t]$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[af(s) + bg(s) - ch(s)] = aL^{-1}[f(s)] + bL^{-1}[g(s)] - cL^{-1}[h(s)]\right\}$$

$$L\left[\frac{dy}{dt}\right] + 3L[y(t)] + 2L\left[\int_0^t y(t)\,dt\right] = L[t]$$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L\left[\frac{dy}{dt}\right] = sy(s) - y(0) \text{ and } L[y(t)] = y(s)\right];$$

$$\ldots\left[L\left[\int_0^t f(u)\,du\right] = \frac{1}{s}L[f(t)] = \frac{1}{s}f(s)\right];$$

$$\ldots\left[L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} \text{ OR } \frac{n!}{s^{n+1}} \text{ if } n \text{ is integer}\right]$$

$$\underbrace{L\left[\frac{dy}{dt}\right]}_{sy(s)-y(0)} + \underbrace{3L[y(t)]}_{y(s)} + \underbrace{2L\left[\int_0^t y(t)\,dt\right]}_{\frac{1}{s}y(s)} = \underbrace{L[t]}_{\frac{\Gamma(2)}{s^{1+1}} = \frac{1}{s^2}}$$

$$[sy(s) - y(0)] + 3[y(s)] + 2\left[\frac{1}{s}y(s)\right] = \frac{1}{s^2}$$

since $y(0) = 0$ ... (Given)

$$\therefore\ sy(s) - 0 + 3y(s) + 2\frac{1}{s}y(s) = \frac{1}{s^2}$$

Common out y(s) from L.H.S., we get,

$$\left(s + 3 + \frac{2}{s}\right)y(s) = \frac{1}{s^2}$$

$$\left(\frac{s^2 + 3s + 2}{s}\right)y(s) = \frac{1}{s^2}$$

$$\therefore\ y(s) = \frac{1}{s^2} \times \frac{s}{s^2 + 3s + 2}$$

$$y(s) = \frac{1}{s} \cdot \frac{1}{s^2 + 3s + 2}$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[y(s)] = L^{-1}\left[\frac{1}{s} \cdot \frac{1}{s^2 + 3s + 2}\right]$$

➔ **Using property of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s}f(s)\right] = \int_0^t f(t)\,dt \text{ Where } f(t) = L^{-1}[f(s)]\right]$$

$$y(t) = \int_0^t L^{-1}\left[\frac{1}{s^2 + 3s + 2}\right]dt \qquad \ldots (1)$$

Now consider,

$$f(s) = \frac{1}{s^2 + 3s + 2} \qquad \ldots (2)$$

For $s^2 + 3s$ find third term (T.T) as it becomes perfect square

$$\text{T.T.} = \frac{(\text{second term})^2}{4 \times \text{first Term}} \qquad \begin{bmatrix} s^2 & & 3s \\ \uparrow & + & \uparrow \\ \text{F.T} & & \text{S.T} \end{bmatrix}$$

$$= \frac{(3s)^2}{4 \times s^2} = \frac{9s^2}{4s^2} = \frac{9}{4}$$

$$\text{T.T.} = \frac{9}{4}$$

$$\therefore\ s^2 + 3s = \left(s^2 + 3s + \frac{9}{4}\right) - \frac{9}{4}$$

Substitute these in Equation (2),

$$\therefore\ f(s) = \frac{1}{\left(s^2 + 3s + \frac{9}{4}\right) - \frac{9}{4} + 2}$$

$$f(s) = \frac{1}{\left(s + \frac{3}{2}\right)^2 - \frac{1}{4}} \quad \left\{\begin{array}{l} \because\ a^2 + 2ab + b^2 = (a+b)^2 \\ \text{and } \left(2 - \frac{9}{4}\right) = \frac{8-9}{4} = \frac{-1}{4} \end{array}\right.$$

Taking inverse Laplace transform of both sides,

$$\therefore\ L^{-1}[f(s)] = L^{-1}\left[\frac{1}{\left(s + \frac{3}{2}\right)^2 - \frac{1}{4}}\right]$$

Observe that all s are in the form (s – a) i.e. $\left(s + \frac{3}{2}\right)$

➔ **Using first shifting property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)]\right\}$$

$$\therefore\ L^{-1}[f(s)] = e^{\frac{-3}{2}t}L^{-1}\left[\frac{1}{s^2 - \frac{1}{4}}\right]$$

$$L^{-1}[f(s)] = e^{\frac{-3}{2}t}L^{-1}\left[\frac{1}{s^2 - \left(\frac{1}{2}\right)^2}\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^2 - a^2}\right] = \frac{1}{a}\sinh at\right]$$

$$= e^{\frac{-3}{2}t}\underbrace{L^{-1}\left[\frac{1}{s^2 - \left(\frac{1}{2}\right)^2}\right]}_{\downarrow}$$

$$\frac{1}{\left(\frac{1}{2}\right)}\sinh\frac{1}{2}t$$

$$L^{-1}[f(s)] = e^{\frac{-3}{2}t}\frac{1}{\left(\frac{1}{2}\right)}\sinh\frac{1}{2}t$$

$$L^{-1}\left[\frac{1}{s^2 + 3s + 2}\right] = e^{\frac{-3}{2}t}\,2\sinh\left(\frac{t}{2}\right) \quad \text{[from Equation (2)]}$$

Substitute these in Equation (1),

$$\therefore\ y(t) = \int_0^t e^{\frac{-3}{2}t}\,2\sinh\left(\frac{t}{2}\right)dt$$

$$= 2\int_0^t e^{\frac{-3}{2}t}\left[\frac{e^{\frac{1}{2}t} - e^{\frac{-1}{2}t}}{2}\right]dt$$

$$= 2\int_0^t \frac{1}{2}\left[e^{\left(\frac{-3}{2}+\frac{1}{2}\right)t} - e^{\left(\frac{-3}{2}-\frac{1}{2}\right)t}\right]dt$$

$$y(t) = 2 \times \frac{1}{2}\int_0^t (e^{-t} - e^{-2t})\,dt$$

$$= \int_0^t e^{-t}\,dt - \int_0^t e^{-2t}\,dt$$

➔ **Using standard result of integration :** $\ldots\left[\int e^{at}\,dt = \frac{e^{at}}{a}\right]$

$$= \left[\frac{e^{-t}}{-1}\right]_0^t - \left[\frac{e^{-2t}}{-2}\right]_0^t$$

$$= -\left[e^{-t}\right]_0^t + \frac{1}{2}\left[e^{-2t}\right]_0^t$$

Substitute t as : $\begin{pmatrix}\text{upper}\\ \text{limit}\end{pmatrix} - \begin{pmatrix}\text{lower}\\ \text{limit}\end{pmatrix}$

$$= -\left[e^{-t} - e^0\right] + \frac{1}{2}\left[e^{-2t} - e^0\right]$$

$$= -e^{-t} + 1 + \frac{1}{2}e^{-2t} - \frac{1}{2} \qquad (\because\ e^0 - 1)$$

(By simplification)

$$y(t) = \frac{e^{-2t}}{2} - e^{-t} + \frac{1}{2} \checkmark \qquad \text{...Ans.}$$

Ex. 2.10.6 : Solve by Laplace transform method

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = te^{-t}\,;\ y(0) = 1 \text{ and } y'(0) = 2$$

Soln. :

Given differential equation is,

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = te^{-t}\ ;\ y(0) = 1,\ y'(0) = 2$$

Operate Laplace transform of both sides,

$$L\left[\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y\right] = L[te^{-t}]$$

➔ **Using linearity property of Laplace transform**

$$\ldots\left\{L[af(t) + bg(t) - ch(t)] = aL[f(t)] + bL[g(t)] - cL[h(t)]\right\}$$

$$L\left[\frac{d^2y}{dt^2}\right] + 2L\left[\frac{dy}{dt}\right] + L[y(t)] = L[te^{-t}]$$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L\left[\frac{d^2y}{dt^2}\right] = s^2y(s) - sy(0) - y'(0)\ ;\ L\left[\frac{dy}{dt}\right] = sy(s) - y(0) \text{ and } L[y(t)] = y(s)\right]$$

$$[s^2y(s) - sy(0) - y'(0)] + 2[sy(s) - y(0)] + y(s) = L[e^{-t} \cdot t]$$

since $y(0) = 1$, $y'(0) = 2$

$$\therefore [s^2y(s) - s(1) - 2] + 2[sy(s) - 1] + y(s) = L[e^{-t} \cdot t]$$

$$s^2y(s) - s - 2 + 2sy(s) - 2 + y(s) = L[e^{-t} \cdot t]$$

$$[s^2y(s) + 2sy(s) + y(s)] + (-s - 2 - 2) = L[e^{-t} \cdot t]$$

...(Grouping the terms)

$$(s^2 + 2s + 1)y(s) + (-s - 4) = L[e^{-t} \cdot t]$$

➔ **Using first shift theorem of Laplace transform**

$$\ldots L[e^{at}f(t)] = [L[f(t)]]_{s \to s-a} = [f(s)]_{s \to s-a}$$

$$(s^2 + 2s + 1)y(s) - (s + 4) = [1(t)]_{s \to s+1}$$

➔ **Using Laplace transform of standard function**

$$\ldots\left[L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} \text{ OR } \frac{n!}{s^{n+1}} \text{ if } n \text{ is integer}\right]$$

$$(s+1)^2y(s) - (s+4) = \left[\frac{1}{s^2}\right]_{s \to s+1}$$

$$(s+1)^2y(s) - (s+4) = \frac{1}{(s+1)^2}$$

$$(s+1)^2y(s) = \frac{1}{(s+1)^2} + (s+4)$$

$$\therefore\ y(s) = \frac{1}{(s+1)^2(s+1)^2} + \frac{(s+4)}{(s+1)^2}$$

Write every s as the form of (s + 1)

$$y(s) = \frac{1}{(s+1)^4} + \frac{(s+1)+3}{(s+1)^2}$$

...Note the adjustment

Taking inverse Laplace transform of both sides,

$$L^{-1}[y(s)] = L^{-1}\left\{\frac{1}{(s+1)^4} + \frac{(s+1)+1}{(s+1)^2}\right\}$$

Observe that all s are in the form of (s – a) i.e. (s + 1)

➔ **Using first shifting property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)]\right\}$$

$$\therefore L^{-1}[y(s)] = e^{-t}L^{-1}\left[\frac{1}{s^4}+\frac{s+3}{s^2}\right]$$

$$= e^{-t}L^{-1}\left[\frac{1}{s^4}+\frac{s}{s^2}+\frac{3}{s^2}\right]$$

$$= e^{-t}L^{-1}\left[\frac{1}{s^4}+\frac{1}{s}+\frac{3}{s^2}\right]$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[af(s)+bg(s)-ch(s)] = aL^{-1}[f(s)]+bL^{-1}[g(s)]-cL^{-1}[h(s)]\right\}$$

$$L^{-1}[y(s)] = e^{-t}\left\{L^{-1}\left[\frac{1}{s^4}\right]+L^{-1}\left[\frac{1}{s}\right]+3L^{-1}\left[\frac{1}{s^2}\right]\right\}$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^{n+1}}\right]=\frac{t^n}{\lfloor n+1}\right]$$

$$y(t) = e^{-t}\left[\frac{t^3}{\lfloor 4}+1+3\frac{t}{\lfloor 2}\right]$$

➔ **By property of Gamma function**

$$\ldots\left[\lfloor n+1 = n! \text{ if n is integer}\right]$$

$$y(t) = e^{-t}\left[\frac{t^3}{3!}+1+3\frac{t}{1!}\right]$$

since, $3! = 1\times2\times3 = 6$

$$y(t) = e^{-t}\left(\frac{t^3}{6}+3t+1\right) \checkmark \qquad \ldots\text{Ans.}$$

Ex. 2.10.7 : Solve the following differential equation by Laplace transform :

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y = e^{-t}\sin t$$

Given, $y(0) = 0$; $y'(0) = 1$

Soln. :

Given differential equation is,

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y = e^{-t}\sin t$$

Operate Laplace transform of both sides,

$$L\left[\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y\right] = L[e^{-t}\sin t]$$

➔ **Using linearity property of Laplace transform**

$$\ldots\left\{L[af(t)+bg(t)-ch(t)] = aL[f(t)]+bL[g(t)]-cL[h(t)]\right\}$$

$$L\left[\frac{d^2y}{dt^2}\right]+2L\left[\frac{dy}{dt}\right]+5L[y(t)] = L[e^{-t}\sin t]$$

➔ **Using standard result of Laplace transform**

$$\left\{L\left[\frac{d^2y}{dt^2}\right] = s^2y(s)-sy(0)-y'(0)\ ;\ L\left[\frac{dy}{dt}\right] = sy(s)-y(0) \text{ and } L[y(t)] = y(s)\right\}$$

$$[s^2y(s)-sy(0)-y'(0)]+2[sy(s)-y(0)]+5y(s) = L[e^{-t}\sin t]$$

Since $y(0) = 0$, $y'(0) = 1$

$$\therefore [s^2y(s)-s(0)-1]+2[sy(s)-0]+5y(s) = L[e^{-t}\sin t]$$

$$s^2y(s)-1+2sy(s)+5y(s) = L[e^{-t}\sin t]$$

$$(s^2+2s+5)y(s)-1 = L[e^{-t}\sin t]$$

➔ **Using first shifting theorem of Laplace transform**

$$L[e^{at}f(t)] = [L[f(t)]]_{s\to s-a} = [f(s)]_{s\to s-a}$$

$$\therefore (s^2+2s+5)y(s)-1 = \{L[\sin t]\}_{s\to s-(-1)}$$

$$(s^2+2s+5)y(s)-1 = \left[\frac{1}{s^2+1}\right]_{s\to s+1}$$

$$(s^2+2s+5)y(s) = \frac{1}{(s+1)^2+1}+1$$

$$= \frac{1+(s+1)^2+1}{(s+1)^2+1}$$

...(by simplification)

$$(s^2+2s+5)y(s) = \frac{1+s^2+2s+1+1}{(s^2+2s+1)+1}$$

$$(s^2+2s+5)y(s) = \frac{s^2+2s+3}{s^2+2s+2}$$

$$y(s) = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

Taking inverse Laplace transform of both sides,

$$\therefore L^{-1}[y(s)] = L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right] \qquad \ldots(1)$$

By partial fraction

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{X+3}{(X+2)(X+5)} \qquad \text{Put, } s^2+2s = X$$

$$\frac{X+3}{(X+2)(X+5)} = \frac{A}{(X+2)}+\frac{B}{(X+5)} \qquad \ldots(2)$$

$$A = \left[\frac{X+3}{X+5}\right]_{X=-2} = \frac{-2+3}{-2+5} = \frac{1}{3}$$

$$B = \left[\frac{X+3}{X+2}\right]_{X=-5} = \frac{-5+3}{-5+2} = \frac{-2}{-3} = \frac{2}{3}$$

Substitute these values in Equation (2),

$$\frac{X+3}{(X+2)(X+5)} = \frac{\frac{1}{3}}{X+2}+\frac{\frac{2}{3}}{X+5}$$

Since, $X = s^2+2s$

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{\frac{1}{3}}{s^2+2s+2}+\frac{\frac{2}{3}}{s^2+2s+5}$$

Substitute these in Equation (1),

$$\therefore L^{-1}[y(s)] = L^{-1}\left[\frac{\frac{1}{3}}{s^2+2s+2}+\frac{\frac{2}{3}}{s^2+2s+5}\right] \qquad \ldots(3)$$

Now find Third term (T.T.) of $s^2 + 2s$ such that it become perfect square

$$\text{T.T.} = \frac{(\text{Second Term})^2}{4\times(\text{First Term})} \qquad \begin{bmatrix} s^2 & + & 4s \\ \uparrow & & \uparrow \\ \text{F.T.} & & \text{S.T.} \end{bmatrix}$$

$$= \frac{(2s)^2}{4\times s^2} = \frac{4s^2}{4s^2} = 1$$

$$\text{T.T.} = 1$$

$$\therefore s^2+2s = (s^2+2s+1)-1$$

Substitute these in Equation (3)

$$\therefore L^{-1}[y(s)] = L^{-1}\left[\frac{\frac{1}{3}}{(s^2+2s+1)-1+2}+\frac{\frac{2}{3}}{(s^2+2s+1)-1+5}\right]$$

$$= L^{-1}\left[\frac{\frac{1}{3}}{(s+1)^2+1}+\frac{\frac{2}{3}}{(s+1)^2+4}\right]$$

➔ **Using first shifting property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)]\right\}$$

$$L^{-1}[y(s)] = e^{-t}L^{-1}\left[\frac{\frac{1}{3}}{s^2+1}+\frac{\frac{2}{3}}{s^2+4}\right]$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[af(s)+bg(s)-ch(s)] = aL^{-1}[f(s)]+bL^{-1}[g(s)]-cL^{-1}[h(s)]\right\}$$

$$L^{-1}[y(s)] = e^{-t}\left\{\frac{1}{3}L^{-1}\left[\frac{1}{s^2+1}\right]+\frac{2}{3}L^{-1}\left[\frac{1}{s^2+4}\right]\right\}$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^2+a^2}\right]=\frac{1}{a}\sin at\right]$$

$$y(t) = e^{-t}\left\{\underbrace{\frac{1}{3}L^{-1}\left[\frac{1}{s^2+1}\right]}_{\sin t}+\underbrace{\frac{2}{3}L^{-1}\left[\frac{1}{s^2+(2)^2}\right]}_{\frac{1}{2}\sin 2t}\right\}$$

$$y(t) = e^{-t}\left[\frac{1}{3}\sin t+\frac{2}{3}\cdot\frac{1}{2}\sin 2t\right]$$

$$y(t) = \frac{1}{3}e^{-t}[\sin t+\sin 2t] \checkmark \qquad \ldots\text{Ans.}$$

Exercise

Solve by Laplace transform method

Ex. 1 : $y'+2y = 4te^{-2t}$, $y(0) = -3$

Ans. : $y(t) = 2t^2e^{-2t}-3e^{-2t}$

Ex. 2 : $y'+10y = t^2$, $y(0) = 0$

Ans. : $y = \frac{t^2}{10}-\frac{t}{50}+\frac{1}{500}-\frac{1}{500}e^{-10t}$

Ex. 3 : $y'+2y = te^{-t}$, $y(0) = 2$

Ans. : $y = te^{-t}-e^{-t}+3e^{-2t}$

Ex. 4 : $y-6y = 2\sin 3t$, $y(0) = -1$

Ans. : $y = \frac{-13}{15}e^{6t}-\frac{2}{15}\cos 3t-\frac{4}{15}\sin 3t$

Syllabus Topic : Applications to Find the Solutions of Simultaneous Linear Differential Equations with Constant Coefficients

2.11 Simultaneous Differential Equations by Laplace Transform

Simultaneous differential equations can also solved by Laplace transform method.

Use same steps as differential equation for different equations and solve simultaneous equations.

Ex. 2.11.1 : Solve the simultaneous equations $\frac{dx}{dt}+y = 0$ and $\frac{dy}{dt}-x = 0$

Under the condition $x(0) = 1$, $y(0) = 0$

Soln. :

Given simultaneous equations,

$$\frac{dx}{dt}+y = 0 \text{ and } \frac{dy}{dt}-x = 0$$

Step I : First equation as : $\frac{dx}{dt}+y = 0$

Taking Laplace transform of both sides,

$$L\left[\frac{dx}{dt}+y\right] = 0$$

➔ **Using linearity property of Laplace Transform**

$$\ldots\left[L[af(t)+bg(t)-ch(t)] = aL[f(t)]+bL[g(t)]-cL[h(t)]\right]$$

$$L\left[\frac{dx}{dt}\right]+L[y] = 0$$

Taking inverse Laplace transform of both sides,

$$L\left[\frac{dy}{dt}\right] = sy(s)-y(0) \text{ and } L[y(t)] = y(s)$$

$$[sx(s)-x(0)]+y(s) = 0$$

since $x(0) = 1$

$$[sx(s)-1]+y(s) = 0$$

$$sx(s)+y(s) = 1 \qquad \ldots(1)$$

Step II : Now second equation as : $\frac{dy}{dt}-x = 0$

Taking Laplace transform of both sides,

$$L\left[\frac{dy}{dt} - x\right] = 0$$

➔ **Using linearity property of Laplace Transform**

$$\ldots\left[L[a f(t) + bg(t) - c h(t)] = a L[f(t)] + b L[g(t)] - cL[h(t)]\right]$$

$$L\left[\frac{dy}{dt}\right] - L[x] = 0$$

$$[sy(s) - y(0)] - x(3) = 0$$

$$\text{since } y(0) = 0$$

$$[sy(s) - 0] - x(s) = 0$$

$$sy(s) - x(s) = 0 \qquad \ldots(2)$$

Step III : Now, solve Equations (1) and (2)

$$sx(s) + y(s) = 1 \text{ and } sy(s) - x(s) = 0$$

$$\left.\begin{aligned} sx(s) + y(s) &= 1 \\ -x(s) + sy(s) &= 0 \end{aligned}\right\} \begin{aligned} &\ldots\ldots \times s \\ &\ldots\ldots \times 1 \end{aligned}$$

$$s^2(x)(s) + \cancel{sy(s)} = s$$

$$-x(s) + \cancel{sy(s)} = 0$$

$$+ \quad - \quad -$$

$$(s^2+1)\,x(s) = s$$

$$(s^2+1)\cdot x(s) = s$$

$$x(s) = \frac{2}{s^2+1}$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[x(s)] = L^{-1}\left[\frac{s}{s^2+1}\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at\right]$$

$$x(t) = \cos t \qquad \ldots(3)$$

Step IV : Again solve Equations (1) and (2),

$$\left.\begin{aligned} sx(s) + y(s) &= 1 \\ -x(s) + sy(s) &= 0 \end{aligned}\right\} \begin{aligned} &\ldots X 1 \\ &\ldots X s \end{aligned}$$

$$sx(s) + y(s) = 1$$

$$-sx(s) + s^2 y(s) = 0$$

$$(1+s^2)\,y(s) = 1$$

$$(s^2+1)\,y(s) = 1$$

$$y(s) = \frac{1}{s^2+1}$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[y(s)] = L^{-1}\left[\frac{1}{s^2+1}\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$y(t) = \sin t \checkmark \qquad \ldots(4)$$

Hence from Equations (3) and (4)

$$x(t) = \cos t; \quad y(t) = \sin t \checkmark \qquad \ldots\text{Ans.}$$

This is required solution.

Ex. 2.11.2 : Solve the simultaneous equations

$$\frac{dx}{dt} - y = e^t; \quad \frac{dy}{dt} + x = \sin t \quad \text{given } x(0) = 1,$$

$$y(0) = 0$$

Soln. :

Given simultaneous equations :

$$\frac{dx}{dt} - y = e^t; \quad \frac{dy}{dt} + x = \sin t$$

Step I : $\frac{dx}{dt} - y = e^t$

Operating Laplace transform of both sides,

$$L\left[\frac{dx}{dt} - y\right] = L[e^t]$$

➔ **Using linearity property of Laplace Transform**

$$\ldots\left[L[a f(t) + bg(t) - c h(t)] = a L[f(t)] + b L[g(t)] - cL[h(t)]\right]$$

$$L\left[\frac{dx}{dt}\right] - L[y] = L[e^t]$$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L\left[\frac{dy}{dt}\right] = sy(s) - y(0) \text{ and } L[y(t)] = y(s)\right]$$

➔ **Using Laplace transform of standard functions**

$$\ldots\left[L[e^{at}] = \frac{1}{s-a}\right]$$

$$[sx(s) - x(0)] - y(s) = \frac{1}{s-1}$$

$$\text{since } x(0) = 1$$

$$[sx(s) - 1] - y(s) = \frac{1}{s-1}$$

$$sx(s) - y(s) = \frac{1}{s-1} + 1 = \frac{1+s-1}{s-1}$$

$$sx(s) - y(s) = \frac{s}{s-1} \qquad \ldots(1)$$

Step II : Now, second equation : $\frac{dy}{dt} + x = \sin t$

Operating Laplace transform of both sides,

$$L\left[\frac{dy}{dt} + x\right] = L[\sin t]$$

➔ **Using linearity property of Laplace Transform**

$$\ldots\left[L[a f(t) + bg(t) - c h(t)] = a L[f(t)] + b L[g(t)] - cL[h(t)]\right]$$

$$L\left[\frac{dy}{dt}\right] + L[x] = L[\sin t]$$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L\left[\frac{dy}{dt}\right] = sy(s) - y(0) \text{ and } L[y(t)] = y(s)\right]$$

We know convolution theorem,

$$\ldots\left[L^{-1}[f(s) * g(s)] = \int_0^t f(t-u)\cdot g(u)\,du\right]$$

$$[sy(s) - y(0)] + [x(s)] = \frac{1}{s^2+1}$$

$$\text{since } y(0) = 0$$

$$[sy(s) - 0] + x(s) = \frac{1}{s^2+1}$$

$$\therefore sy(s) + x(s) = \frac{1}{s^2+1} \qquad \ldots(2)$$

Step III : Now solve Equations (1) and (2)

$$sx(s) - y(s) = \frac{s}{s-1} \text{ and } sy(s) + x(s) = \frac{1}{s^2+1}$$

$$\left.\begin{aligned} sx(s) - y(s) &= \frac{s}{s-1} \\ x(s) + sy(s) &= \frac{1}{s^2+1} \end{aligned}\right\} \begin{aligned} &\ldots x s \\ &\ldots x 1 \end{aligned}$$

$$s^2 x(s) - sy(s) = \frac{s^2}{s-1}$$

$$x(s) + sy(s) = \frac{1}{s^2+1}$$

$$(s^2+1)\,x(s) = \frac{s^2}{s-1} + \frac{1}{s^2+1}$$

$$x(s) = \frac{s^2}{(s-1)(s^2+1)} + \frac{1}{(s^2+1)^2} \qquad \ldots(3)$$

Now, By Partial fraction :

$$\frac{s^2}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+c}{s^2+1} \qquad \ldots(4)$$

$$s^2 = A(s^2+1) + (Bs+c)(s-1) \qquad \ldots(5)$$

Put $s - 1 = 0 \Rightarrow s = 1$ in Equation (5)

$$\therefore (1)^2 = A(1+1) + (B(1) + C)(0)$$

$$1 = A(2) \quad \therefore A = \frac{1}{2}$$

Put $s = 0$ in Equation (5)

$$0 = A(0+1) + (B(0) + C)(0-1)$$

$$0 = A + C(-1)$$

$$\text{Since } A = \frac{1}{2}$$

$$0 = \frac{1}{2} - C \quad \therefore C = \frac{1}{2}$$

Put $s = -1$ in Equation (5)

$$(-1)^2 = A((-1)^2+1) + (B(-1) + C)(-1-1)$$

$$1 = A(1+1) + (-B + C)(-2)$$

$$\text{Since } A = \frac{1}{2},\ C = \frac{1}{2}$$

$$1 = \left(\frac{1}{2}\right)(2) + \left(-B + \frac{1}{2}\right)(-2)$$

$$1 = 1 + 2B - \left(\frac{1}{2}\right)(2)$$

$$= \cancel{1} + 2B - \cancel{1}$$

$$2B = 1 \quad \therefore B = \frac{1}{2}$$

Substitute these values in Equation (4)

$$\therefore \frac{s^2}{(s-1)(s^2+1)} = \frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}s + \frac{1}{2}}{s^2+1} \qquad \ldots(6)$$

Substitute these in Equation (3)

$$\therefore x(s) = \frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}s + \frac{1}{2}}{s^2+1} + \frac{1}{(s^2+1)^2}$$

$$= \frac{1}{2}\left[\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}\right] + \frac{1}{(s^2+1)^2}$$

Taking inverse Laplace transform of both sides, we get

$$L^{-1}[x(s)] = L^{-1}\left\{\frac{1}{2}\left[\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}\right] + \frac{1}{(s^2+1)^2}\right\}$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[af(s) + bg(s) - ch(s)] = aL^{-1}[f(s)] + bL^{-1}[g(s)] - cL^{-1}[h(s)]\right\}$$

$$x(t) = \frac{1}{2}\left\{L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]\right\} + L^{-1}\left[\frac{1}{(s^2+1)^2}\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s-a}\right] = e^{at}\right]$$

$$\ldots\left[L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at\right]$$

$$\ldots\left[L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^2} = (\sin at - at\cos at)$$

$$x(t) = \frac{1}{2}[e^t + \cos t + \sin t] + \frac{1}{2}(\sin t - t\cos t)$$

$$= \frac{1}{2}[e^t + \cos t + \sin t + \sin t - t\cos t]$$

$$x(t) = \frac{1}{2}[e^t + \cos t + \sin 2t\, t - t\cos t] \checkmark \quad \ldots\text{Ans.}$$

Step IV : Again from Equation (1) and (2)

$$\left.\begin{aligned} sx(s) - y(s) &= \frac{s}{s-1} \\ x(s) + sy(s) &= \frac{1}{s^2+1} \end{aligned}\right\} \quad \begin{aligned} &\ldots \times 1 \\ &\ldots \times s \end{aligned}$$

$$s\,x(s) - y(s) = \frac{s}{s-1}$$

$$s\,x(s) + s^2 y(s) = \frac{s}{s+1}$$

$$-(s^2+1)\,y(s) = \frac{s}{s-1} - \frac{s}{s^2+1}$$

$$(s^2+1)\,y(s) = -\frac{s}{s-1} + \frac{s}{s^2+1}$$

$$y(s) = \frac{s}{(s^2+1)^2} - \frac{s}{(s-1)(s^2+1)}$$

From Equation (6)

$$y(s) = \frac{s}{(s^2+1)^2} - \frac{1}{2}\left[\frac{1}{s-1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right]$$

Taking inverse Laplace transform of both sides we get

$$L^{-1}[y(s)] = L^{-1}\left\{\frac{s}{(s^2+1)^2} - \frac{1}{2}\left[\frac{1}{s-1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right]\right\}$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[af(s) + bg(s) - ch(s)] = a\,L^{-1}[f(s)] + bL^{-1}[g(s)] - cL^{-1}[h(s)]\right\}$$

$$= L^{-1}\left[\frac{s}{(s^2+1)^2}\right] - \frac{1}{2}\left\{L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left(\frac{s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s^2+1}\right)\right\}$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s-a}\right] = e^{at}\right]$$

$$\ldots\left[L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at\right]$$

$$\ldots\left[L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$\ldots\left[L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a}t\sin at\right]$$

$$\therefore\ y(t) = \frac{1}{2}t\sin t - \frac{1}{2}[e^t - \cos t + \sin t]$$

$$y(t) = \frac{1}{2}[t\sin t - e^t + \cos t - \sin t] \checkmark \qquad \ldots\text{Ans.}$$

Ex. 2.11.3 : Solve the simultaneous equations

$$(D^2-3)\,x - 4y = 0;\ x + (D^2+1)y = 0$$

For $t > 0$, given that $x = y = \frac{dy}{dt} = 0$ and $\frac{dx}{dt} = 2$ at $t = 0$

Soln. :

Given simultaneous equations are

$$(D^2-3)\,x - 4y = 0\,;\ x + (D^2+1)y = 0$$

where $D^2 = \frac{d^2}{dt^2}$

Step I : First equation as : $(D^2-3)x - 4y = 0$

operating Laplace transform of both sides,

$$L[(D^2-3)\,x - 4y] = 0;$$

Substitute t as : $\left(\begin{matrix}\text{upper}\\\text{limit}\end{matrix}\right) - \left(\begin{matrix}\text{lower}\\\text{limit}\end{matrix}\right)$

$$L[D^2x] - 3L[x] - 4L[y] = 0$$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L\left[\frac{dy}{dt}\right] = sy(s) - y(0) \text{ and } L[y(t)] = y(s)\right]$$

$$[s^2x(s) - sx(0) - x'(0)] - 3x(s) - 4y(s) = 0$$

$$\text{Since } x(0) = 0,\ x'(0) = \left(\frac{dx}{dt}\right)_{x=0} = 2$$

$$(s^2x(s) - 0 - 2) - 3x(s) - 4y(s) = 0$$

$$(s^2-3)\,x(s) - 4y(s) = 2 \qquad \ldots(1)$$

Step II : Now, second equation : $x + (D^2+1)\,y = 0$

Operating laplace transform of both sides,

$$L[x + (D^2+1)\,y] = 0$$

Substitute t as : $\left(\begin{matrix}\text{upper}\\\text{limit}\end{matrix}\right) - \left(\begin{matrix}\text{lower}\\\text{limit}\end{matrix}\right)$

$$L[x] + L[D^2y] + L[y] = 0$$

➔ **Using standard result of Laplace transform**

$$\ldots\left[L\left[\frac{dy}{dt}\right] = sy(s) - y(0) \text{ and } L[y(t)] = y(s)\right]$$

$$x(s) + [s^2y(s) - sy(0) - y'(0)]\,y(s) = 0$$

since $y(0) = 0, \frac{dy}{dt} = 0$

$$x(s) + s^2y(s) - 0 - 0 + y(s) = 0$$

$$x(s) + (s^2+1)\,y(s) = 0 \qquad \ldots(2)$$

Step III : Now solve Equations (1) and (2)

$$(s^2-3)\,x(s) - 4y(s) = 2 \text{ and}$$

$$x(s) + (s^2+1)\,y(s) = 0$$

$$\left.\begin{aligned}(s^2-3)\,x(s) - 4y(s) &= 2 \\ x(s) + (s^2+1)\,y(s) &= 0\end{aligned}\right\} \quad \begin{aligned}&\ldots\times(s^2+1)\\&\ldots\times 4\end{aligned}$$

$$(s^2+1)(s^2-3)\,x(s) - 4\cancel{(s^2+1)\,y(s)} = 2(s^2+1)$$

$$4x(s) + 4\cancel{(s^2+1)\,y(s)} = 0$$

$$[(s^2+1)(s^2-3) + 4]\,x(s) = 2(s^2+1)$$

$$x(s) = \frac{2(s^2+1)}{(s^2+1)(s^2-3)+4}$$

$$= \frac{2(s^2+1)}{s^4 - 3s^2 + s^2 - 3 + 4}$$

$$= \frac{2(s^2+1)}{s^4 - 2s^2 + 1}$$

$$x(s) = \frac{2(s^2+1)}{(s^2-1)^2}$$

$$= \frac{2(s^2+1)}{[(s-1)(s+1)]^2}$$

$$x(s) = \frac{2(s^2+1)}{(s-1)^2(s+1)^2}$$

By Partial fraction

$$x(s) = \frac{1}{(s-1)^2} + \frac{1}{(s+1)^2}$$

Operating inverse Laplace transform of both sides.

$$L^{-1}[x(s)] = L^{-1}\left[\frac{1}{(s-1)^2} + \frac{1}{(s+1)^2}\right]$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[af(s) + bg(s) - ch(s)] = a\,L^{-1}[f(s)] + bL^{-1}[g(s)] - cL^{-1}[h(s)]\right\}$$

$$x(t) = L^{-1}\left[\frac{1}{(s-1)^2}\right] + L^{-1}\left[\frac{1}{(s+1)^2}\right]$$

➔ **Using first shifting property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)]\right\}$$

$$x(t) = e^tL^{-1}\left[\frac{1}{s^2}\right] + e^{-t}L^{-1}\left[\frac{1}{s^2}\right]$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{\lfloor n+1}\right]$$

$$x(t) = e^t\left(\frac{t}{\lfloor 2}\right) + e^{-t}\left(\frac{t}{\lfloor 2}\right)$$

$$= e^t t + e^{-t}\cdot t = t(e^t + e^{-t})$$

$$x(t) = t(e^t + e^{-t}) \qquad \ldots(3)$$

Step IV : Again solve Equations (1) and (2)

$$\left.\begin{aligned}(s^2-3)\,x(s) - 4y(s) &= 2 \\ x(s) + (s^2+1)\,y(s) &= 0\end{aligned}\right\} \quad \begin{aligned}&\ldots\times 1\\&\ldots\times(s^2-3)\end{aligned}$$

$$\cancel{(s^2-3)\,x(s)} - 4y(s) = 2$$

$$\cancel{(s^2-3)\,x(s)} + (s^2+1)(s^2-3)\,y(s) = 0$$

$$-[4 + (s^2+1)(s^2-3)]\,y(s) = 2$$

$$[(s^2+1)(s^2-3)]\,y(s) = -2$$

$$y(s) = \frac{-2}{(s^2+1)(s^2-3)} \qquad \ldots(4)$$

➔ **Using formula of partial fraction**

$$\ldots\left\{\frac{1}{(X-a)(X-b)} = \frac{1}{a-b}\left[\frac{1}{X-a} - \frac{1}{X-b}\right]\right\}$$

$$\frac{1}{(s^2+1)(s^2-3)} = \frac{1}{(X+1)(X-3)} \qquad \ldots\text{with } X = s^2$$

$$= \frac{1}{2}\left[\frac{1}{X-3} - \frac{1}{X+1}\right]$$

$$= \frac{1}{2}\left(\frac{1}{s^2-3} - \frac{1}{s^2+1}\right)$$

Substitute these in Equation (4)

$$y(s) = -2\left[\frac{1}{2}\left(\frac{1}{s^2-3} - \frac{1}{s^2+1}\right)\right]$$

$$= -\frac{1}{s^2-3} + \frac{1}{s^2+1}$$

$$y(s) = \frac{1}{s^2+1} - \frac{1}{s^2-3}$$

Taking inverse Laplace transform of both sides,

$$L^{-1}[y(s)] = L^{-1}\left\{\frac{1}{s^2+1} - \frac{1}{s^2-3}\right\}$$

➔ **Using linearity property of inverse Laplace transform**

$$\ldots\left\{L^{-1}[af(s) + bg(s) - ch(s)] = a\,L^{-1}[f(s)] + bL^{-1}[g(s)] - cL^{-1}[h(s)]\right\}$$

$$y(t) = L^{-1}\left(\frac{1}{s^2+1}\right) - L^{-1}\left(\frac{1}{s^2-3}\right)$$

➔ **Using standard result of inverse Laplace transform**

$$\ldots\left[L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

$$\ldots\left[L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at\right]$$

$$y(t) = \underbrace{L^{-1}\left(\frac{1}{s^2+(1)^2}\right)}_{\frac{1}{1}\sin t} - \underbrace{L^{-1}\left(\frac{1}{s^2-(\sqrt{3})^2}\right)}_{\frac{1}{\sqrt{3}}\sinh\sqrt{3}\,t}$$

$$y(t) = \sin t - \frac{1}{\sqrt{3}}\sinh\sqrt{3}\,t \checkmark \qquad \ldots\text{Ans.}$$

Exercise

Solve the following simultaneous equations.

Ex. 1 : $\frac{dx}{dt} + y = \sin t,\ \frac{dy}{dt} + x = \cos t$ given that $x = 2$ and $y = 0$ when $t = 0$

Ans. : $x = e^t + e^{-t},\ y = e^{-t} - e^t + \sin t$

Ex. 2 : $\frac{dx}{dt} + 4y = 0,\ \frac{dy}{dt} - 9x = 0$ Given $x = 2$ and $y = 1$ at $t = 0$

Ans. : $x = \frac{-2}{3}\sin 6t + 2\cos 6t\,;\ y = \cos 6t + 3\sin 6t$

Ex. 3 : Solve the equation :

$$y'' - 3y' + 2y = 4t + e^{3t}, \text{ when } y(0) = 1 \text{ and } y'(0) = -1$$

Ans. : $y = 3 + 2t + \frac{1}{2}(e^{3t} - e^t) - 2e^{2t}$

Chapter Ends...

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Note

UNIT 2

Inverse Laplace Transform

Syllabus

Introductory remarks ; Inverse transforms of some elementary functions ; General methods of finding inverse transforms ; Partial fraction method and Convolution Theorem for finding inverse Laplace transforms ; Applications to find the solutions of linear differential equations and simultaneous linear differential equations with constant coefficients.

➢ Chapter 2 : Inverse Laplace Transform

CHAPTER 2

UNIT II

Inverse Laplace Transform

Multiple Choice Questions

Q. 1 The inverse Laplace transform of f(s – a) { i.e. $L^{-1}[f(s-a)]$ } is _____.

(a) $e^{-at} f(t)$ (b) $\frac{1}{a} f(s/a)$

(c) $e^{at} f(t)$ (d) f(s/a)

Ans. (c)

Q. 2 The inverse Laplace transform of f(s + a) { i.e. $L^{-1}[f(s+a)]$ } is _____.

(a) $e^{-at} f(t)$ (b) $\frac{1}{a} f\left(\frac{s}{a}\right)$

(c) $e^{at} f(t)$ (d) $f\left(\frac{s}{a}\right)$

Ans. : (a)

Q. 3 If f(0) = 0 then the inverse Laplace transform of [sf(s)] i.e. { $L^{-1}[sf(s)]$ } is_____.

(a) $f'(t)$ (b) $f''(t)$

(c) t f(t) (d) $\frac{f(t)}{t}$

Ans. : (a)

Q. 4 If $f(0) = f'(0) = 0$ then $L^{-1}\{s^2 f(s)\} =$ _____.

(a) $f'(t)$ (b) t f(t)

(c) f(t) / t (d) $f''(t)$

Ans. : (d)

Q. 5 If $f(o) = f'(0) = f''(o) =$ _____ $= f^{(n-1)}(0) = 0$. then $L^{-1}\{s^n f(s)\} =$ _____.

(a) $\int_0^\infty \int_0^\infty$ ....n times $\int_0^\infty f(t)\,dt$ (b) $\frac{d^n}{dt^n}[f(t)]$

(c) $t^n f(t)$ (d) $\frac{f(t)}{t^n}$

Ans. : (b)

Q. 6 The value of $L^{-1}\{f^{(n)}(s)\} =$ _____ (where (n) is n^{th} derivative w.r.t. s

(a) $(-1)^n t^n f(t)$ (b) $\frac{d^n}{dt^n}[f(t)]$

(c) $\int_0^\infty \int_0^\infty$ ....n times $\int_0^\infty f(t)\,dt$ (d) $\frac{f(t)}{t^n}$

Ans. : (a)

Q. 7 The inverse Laplace transform of $\left[\frac{d}{ds} f(s)\right]$ is _____.

(a) f(t) (b) t f (t)

(c) – t f(t) (d) $\frac{f(t)}{t}$

Ans. : (c)

Q. 8 The inverse transform of $\frac{d^2}{ds^2}[f(s)]$ is _____.

(a) t f(t) (b) – t f(t)

(c) $t^2 f(t)$ (d) $\frac{f(t)}{t}$

Ans. : (c)

Q. 9 The inverse Laplace transform of $f'''(s)$ is _____.

(a) $-t^3 f(t)$ (b) $t^3 f(t)$

(c) $\frac{f(t)}{t^3}$ (d) $-\frac{f(t)}{t^3}$

Ans. : (a)

Q. 10 The inverse Laplace transform of $\left\{\frac{f(s)}{s}\right\}$ is _____.

(a) $f'(t)$ (b) $\frac{f(t)}{t}$

(c) t f(t) (d) $\int_0^t f(u)\,du$

Ans. : (d)

Q. 11 The inverse Laplace transform of $\left(\frac{1}{s}\right)$ is _____.

(a) 1 (b) t

(c) e^t (d) 0

Ans. : (a)

Q. 12 The inverse Laplace transform of the function $\frac{1}{s-a}$ $(s > a)$ is _____.

(a) e^{-at} (b) e^{at}

(c) t (d) te^{at}

Ans. : (b)

Q. 13 The inverse Laplace transform of the function

$\frac{1}{s+a}$ $(s < -a)$ is _____.

(a) e^{-at} (b) e^{at} (c) t (d) te^{at}

Ans. : (a)

Q. 14 The inverse Laplace transform of $\left(\frac{1}{s^2+a^2}\right)$ is _____.

(a) $\cos at$ (b) $\frac{1}{a}\sin at$

(c) e^{at} (d) e^{-at}

Ans. : (b)

Q. 15 The inverse Laplace transform of the function $f(s) = \frac{s}{s^2+a^2}$ is _____.

(a) $\cos at$ (b) $\sin at$

(c) $a \sin at$ (d) $t \cos at$

Ans. : (a)

Q. 16 The inverse Laplace transform of the function $f(s) = \frac{1}{s^2-a^2}$ $(s > |a|)$ is _____.

(a) $\sin at$ (b) $\cos at$

(c) $\frac{1}{a}\sinh at$ (d) $\cosh at$

Ans. : (c)

Q. 17 The inverse Laplace transform of the function $f(s) = \frac{s}{s^2-a^2}$ $(s > |a|)$ is _____.

(a) $\sin at$ (b) $\cos at$

(c) $\sinh at$ (d) $\cosh at$

Ans. : (d)

Q. 18 The inverse Laplace transform of the function $f(s/a)$ is _____ [if $L^{-1} f(s) = f(t)$]

(a) $a f(at)$ (b) $t f(t)$ (c) $f(t)/(t)$ (d) $a f(t)$

Ans. : (a)

Q. 19 The inverse Laplace transform of $\left\{\frac{1}{s^n}\right\}$ is _____.

(a) t^n (b) $\frac{t^{n-1}}{\Gamma n}$

(c) $e^{at} t$ (d) $\frac{d}{dt}[t^n]$

Ans. : (b)

Q. 20 The inverse Laplace transform of $\left\{\frac{1}{s^2}\right\}$ is _____.

(a) t^2 (b) $2t$ (c) $\frac{2!}{t^2}$ (d) t

Ans. : (d)

Q. 21 The inverse Laplace transform of $\left(\frac{1}{s^3}\right)$ is _____.

(a) $\frac{1}{3}t^3$ (b) $\frac{t^2}{2}$ (c) $\frac{t^3}{\sqrt{3}}$ (d) $\frac{t^3}{3}$

Ans. : (b)

Q. 22 The inverse Laplace transform of $\left(\frac{1}{s^{3/2}}\right)$ is _____.

(a) $2\sqrt{\frac{t}{\pi}}$ (b) $\frac{\sqrt{t}}{\pi}$

(c) $\frac{t}{\sqrt{\pi}}$ (d) $\frac{2}{3}t^{3/2}$

Ans. : (a)

Q. 23 The inverse Laplace transform of $\left(\frac{1}{s^3}\right)$ is _____.

(a) $\frac{1}{3}t^2$ (b) $\frac{t^2}{2}$ (c) $\frac{t^3}{13}$ (d) $\frac{t^3}{3}$

Ans. : (b)

Q. 24 The inverse Laplace transform of $\left(\frac{1}{s^{1/2}}\right)$ is _____.

(a) $\sqrt{\frac{t}{\pi t}}$ (b) $\frac{\sqrt{t}}{\pi}$

(c) $\frac{t}{\sqrt{\pi}}$ (d) $\frac{2}{3}t^{3/2}$

Ans. : (a)

Q. 25 By the convolution theorem, If $L^{-1}[f(s)] = F(t)$, $L^{-1}[G(s)] = G(t)$ then $L^{-1}[f(s) * G(s)] =$ _____.

(a) $\int_0^\infty F(t-u)\,G(u)\,du$ (b) $\int_0^t F(u)\,G(u)\,du$

(c) $\int_0^t F(t-u)\,G(u)\,du$ (d) $\int_0^\infty F(u)\,G(u)\,du$

Ans. : (c)

Q. 26 The inverse Laplace transform of $\frac{1}{s-4}$ is _____.

(a) e^{4t} (b) $t-4$

(c) e^{-4t} (d) $4t$

Ans. : (a)

Q. 27 Inverse Laplace transform of $\frac{1}{s^2+2}$ is _____.

(a) $\frac{1}{2}\sin 2t$ (b) $\sin t$

(c) $\frac{1}{\sqrt{2}}\sin\sqrt{2}\,t$ (d) $\sin 2t$

Ans. : (c)

Q. 28 Inverse Laplace transform of $\frac{s}{s^2+3}$ is _____.

(a) $\frac{1}{\sqrt{3}}\sin\sqrt{3}\,t$ (b) $\cos\sqrt{3}\,t$

(c) $\frac{1}{\sqrt{3}}\cos\sqrt{3}\,t$ (d) $\sin\sqrt{3}\,t$

Ans. : (b)

Q. 29 The inverse Laplace transform of $\frac{1}{s^2-4}$ is _____.

(a) $\frac{1}{2}\cosh 2t$ (b) $\frac{1}{2}\sin 2t$

(c) $\frac{1}{2}\sinh 2t$ (d) $\cosh 2t$

Ans. : (c)

Q. 30 The inverse Laplace transform of $\frac{s}{s^2-2}$ is _____.

(a) $\cosh\sqrt{2}\,t$ (b) $\sin\sqrt{2}\,t$

(c) $\frac{1}{\sqrt{2}}\sin\sqrt{2}\,t$ (d) $\frac{1}{\sqrt{2}}\cos\sqrt{2}\,t$

Ans. : (a)

Q. 31 The inverse Laplace transform of $\frac{1}{(s^2+9)(s^2+16)}$ is _____.

(a) $\cos 3t + \cos 4t$ (b) $\frac{1}{7}\left\{\frac{1}{3}\sin 3t - \frac{1}{4}\sin 4t\right\}$

(c) $\sin 3t - \sin 4t$ (d) $\frac{1}{12}[\sin 3t \sin 4t]$

Ans. : (b)

Q. 32 The inverse Laplace transform of $\left(\frac{1}{\sqrt{s+4}}\right)$ is _____.

(a) $\frac{e^{-4t}}{\sqrt{t}}$ (b) $\frac{e^{4t}}{\sqrt{t}}$

(c) $\frac{e^{-4t}}{\sqrt{\pi t}}$ (d) $\frac{e^{4t}}{\sqrt{\pi t}}$

Ans. : (c)

Q. 33 The inverse Laplace transform of $\frac{1}{s(s+1)}$ is _____.

(a) $e^{-t}+1$ (b) e^{-t}

(c) $-e^{-t}$ (d) $(1-e^{-t})$

Ans. : (d)

Q. 34 The inverse Laplace transform of $\frac{1}{(s^2+1)(s^2-1)}$

(a) $\sin t \cdot \sinh t$ (b) $\frac{1}{2}(\sinh t - \sin t)$

(c) $\cos t \cdot \cosh t$ (d) $\cos t - \cosh t$

Ans. : (b)

Q. 35 The inverse Laplace transform of $\frac{1}{s^2+6s+5}$ is _____.

(a) $e^{-3t}\cosh 2t$ (b) $e^{-3t}\sin 2t$

(c) $e^{-3t}\cos 2t$ (d) $\frac{1}{2}e^{-3t}\sinh 2t$

Ans. : (d)

Q. 36 The inverse Laplace transform of $\frac{1}{(s^2+9)(s^2+7)}$ is _____.

(a) $\frac{1}{2}\{\sin\sqrt{7}\,t - \cos 3t\}$

(b) $\frac{1}{2}\left\{\frac{1}{\sqrt{7}}\sin\sqrt{7}\,t - \frac{1}{3}\sin 3t\right\}$

(c) $\sin 2t - \cos 3t$

(d) $\cos 3t - \sin 2t$

Ans. : (b)

Q. 37 The inverse Laplace transform of $\frac{s^2}{(s+a)^3}$ is _____.

(a) $e^{at}(1+t+t^2)$ (b) $e^{-at}(1+t+t^2)$

(c) $e^{-at}\left[1-2at+\frac{a^2}{2}t^2\right]$ (d) $e^{at}[1+2at+a^2t^2]$

Ans. : (c)

Q. 38 The inverse Laplace transform of $\frac{s+1}{(s-1)(s-2)}$ is _____.

(a) $3e^{2t}-2e^{t}$ (b) $e^{t}+e^{2t}$

(c) $e^{-t}+e^{-2t}$ (d) $3e^{-2t}+2e^{-t}$

Ans. : (a)

Q. 39 The inverse Laplace transform of $\frac{s+4}{s^2+1}$ is _____.

(a) $\cos t + 4\sin t$ (b) $\cos t - 4\sin t$

(c) $e^{-4t}\sin t$ (d) $e^{4t}\sin t$

Ans. : (a)

Q. 40 By Laplace transform the value of $\int_0^\infty e^{-2t}\sin 3t\, dt$ is _____.

(a) $\frac{-2}{3}$ (b) $\frac{4}{9}$ (c) -6 (d) $\frac{3}{13}$

Ans. : (d)

Q. 41 The inverse Laplace transform of $\tan^{-1}(s)$ is _____.

(a) $\frac{\cos t}{t}$ (b) $-t\cos t$

(c) $\frac{-\sin t}{t}$ (d) $t\sin t$

Ans. : (c)

Q. 42 The inverse Laplace transform of $\frac{2s+5}{(s+1)(s-2)}$ is _____.

(a) $e^{-t}+e^{2t}$ (b) $3e^{2t}-e^{-t}$

(c) $e^{-t}-e^{2t}$ (d) $2e^{-t}+5e^{2t}$

Ans. : (b)

Q. 43 The inverse Laplace transform of $\frac{2s^3+13s}{(s^2+9)(s^2+4)}$ is _____.

(a) $\cos 3t + \cos 2t$ (b) $\sin 3t - \sin 2t$

(c) $\cos 3t - \cos 2t$ (d) $\sin 3t + \sin 2t$

Ans. : (a)

Q. 44 The inverse Laplace transform of $\frac{1}{s^2+4s+12}$ is _____.

(a) $\frac{1}{4}e^{2t}\sin 4t$ (b) $e^{-2t}\cos 2t$

(c) $e^{-2t}[\sin 2t + \cos 2t]$ (d) $\frac{1}{2\sqrt{2}}e^{-2t}\sin(2\sqrt{2}\,t)$

Ans. : (d)

Q. 45 The inverse Laplace transform of $\log\left(\frac{s^2+1}{s^2+4}\right)$ is _____.

(a) $\sin t - \sin 2t$ (b) $\cos t + \cos 2t$

(c) $2\left(\frac{\cos 2t - \cos t}{t}\right)$ (d) $2t(\cos 2t + \cos t)$

Ans. : (c)

Q. 46 The inverse Laplace transform of $\log\left(\frac{s+1}{s+2}\right)$ is _____.

(a) $\frac{e^{-2t}-e^{-t}}{t}$ (b) $e^{-2t}-e^{-t}$

(c) $t(e^{-2t}+e^{-t})$ (d) $e^{-2t}+e^{-t}$

Ans. : (a)

Q. 47 The inverse Laplace transform of $\frac{s}{(s-2)^4}$ is _____.

(a) $e^{2t}\left[\frac{t^2}{2}+\frac{t^3}{3}\right]$ (b) $e^{2t}t^3$

(c) $e^{-2t}t^4$ (d) $e^{-2t}[t^2+t^3]$

Ans. : (a)

Q. 48 The inverse Laplace transform of $\frac{1}{s^2-4s+3}$ is _____

(a) $t\cos 4t$ (b) $e^{2t}\sinh t$

(c) $e^{2t}\cosh t$ (d) $t\sin 4t$

Ans. : (b)

Q. 49 The inverse Laplace transform of $\frac{s+1}{s^2+2s-3}$ is _____.

(a) $e^{2t}\sinh t$ (b) $e^{-2t}\sin t$

(c) $e^{t}\cos t$ (d) $e^{-t}\cosh 2t$

Ans. : (d)

Q. 50 The inverse Laplace transform of $\frac{s+2}{s^2+4s+13}$ is _____.

(a) $e^{-2t}\cos 3t$ (b) $e^{-2t}\sin 3t$

(c) $e^{2t}\cos 3t$ (d) $e^{2t}\sin 3t$

Ans. : (a)

Q. 51 The inverse Laplace transform of $\frac{s+2}{s(s^2+4s+8)}$ is _____.

(a) $e^{-2t}\cos 2t$ (b) $\int_0^t e^{-2t}\cos t\, dt$

(c) $e^{2t}\cos t$ (d) $\int_0^t e^{2t}\cos t\, dt$

Ans. : (b)

Q. 52 The inverse Laplace transform of $\frac{2}{s(s^2-2s+5)}$ is _____.

(a) $e^{-t}\int_0^t \sin 2t\, dt$ (b) $\int_0^t e^{-t}\sin 2t\, dt$

(c) $\int_0^t e^{t}\sin 2t\, dt$ (d) $e^{t}\int_0^t \sin 2t\, dt$

Ans. : (c)

Q. 53 The inverse Laplace transform of $\frac{s}{s^2+6s+25}$ is _____.

(a) $e^{-3t}\left[\cos 4t - \frac{3}{4}\sin 4t\right]$ (b) $e^{3t}\cos 4t$

(c) $e^{-3t}[\sin 4t + \cos 3t]$ (d) $e^{3t}\sin 4t$

Ans. : (a)

Q. 54 The inverse Laplace transform of $\frac{3s+7}{s^2-2s-3}$ is _____.

(a) $e^{3t}+e^{-t}$ (b) $e^{3t}-e^{-t}$

(c) $4e^{3t}-e^{-t}$ (d) $3e^{-3t}-e^{t}$

Ans. : (c)

Q. 55 The inverse Laplace transform of $\left(\frac{e^{-2s}}{s^2+4}\right)$ is $F(t) =$ _____.

(a) $\begin{cases}\frac{1}{2}\sin 2t & t>2\\ 0 & t<2\end{cases}$

(b) $\begin{cases}\frac{1}{2}\sin 2(t-2) & t>2\\ 0 & 0<t<2\end{cases}$

(c) $\frac{1}{2}\sin 2t - \frac{1}{s+2}$

(d) $\frac{1}{2}\sin 2(t-2)$

Ans. : (b)

Explanation :

$L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\sin 2t$

$\therefore$ By second shifting property,

$L^{-1}\left[\frac{e^{-2s}}{s^2+4}\right] = F(t)$

Where $F(t) = \begin{cases}\frac{1}{2}\sin 2(t-2) & t>2\\ 0 & 0<t<2\end{cases}$

Q. 56 The inverse Laplace transform of $\left(\frac{e^{-s}s}{s^2+1}\right)$ is $F(t) =$ _____.

(a) $\frac{\cos t}{t+1}$ (b) $\cos(t-1)$

(c) $\begin{cases}\cos t & t>1\\ 0 & t<1\end{cases}$ (d) $\begin{cases}\cos(t-1) & t>1\\ 0 & 0<t<1\end{cases}$

Ans. : (d)

Explanation :

Since, $L^{-1}\left[\frac{s}{s^2+1}\right] = \cos t$

$\therefore$ By second shifting property,

$L^{-1}\left[e^{-s}\frac{s}{s^2+1}\right] = F(t)$

Where $F(t) = \begin{cases}\cos(t-1) & t>1\\ 0 & 0<t<1\end{cases}$

Q. 57 By using Laplace transform for the differential equation $y'' + y = 0$, $y(0) = 1$, $y'(0) = 2$ reduces to $Y(s) =$ _____.

(a) $\frac{s+2}{s^2+1}$ (b) $\frac{s}{s^2+1}$

(c) $\frac{1}{s^2}+\frac{1}{s}$ (d) $\frac{1}{s^2+1}$

Ans. : (a)

Explanation : $y''+y = 0$; $y(0)=1, y'(0)=2$

Operating Laplace transform on both sides

$L[y''+y] = 0$

$\therefore s^2Y(s) - sY(0) - Y'(0) + Y(s) = 0$

$(s^2+1)Y(s) = sY(0) + Y'(0) = s+2$

$$Y(s) = \frac{s+2}{s^2+1}$$

Q. 58 By Laplace transform the differential equation $\frac{dy}{dt} + 3y(t) + 2\int_0^t y(t)\,dt = t$; $y(0) = 0$ reduces to $Y(s) =$ _____.

(a) $\frac{s^2}{s^2+3s+2}$ (b) $\frac{1}{s^2} - \frac{1}{s}$

(c) $\frac{s+3s+2}{s}$ (d) $\frac{1}{s^2+3s+2}$

Ans. : (d)

Explanation :

$$\frac{dy}{dt} + 3y(t) + 2\int_0^t y(t)\,dt = t\ ;\ y(0) = 0$$

Operate Laplace transform

$$s\,Y(s) - Y(0) + 3\,Y(s) + 2\frac{1}{s}\,Y(s) = \frac{1}{s^2}$$

$$\left(s + 3 + \frac{2}{s}\right) Y(s) = \frac{1}{s^2}$$

$$\left(\frac{s^2+3s+2}{s}\right) Y(s) = \frac{1}{s^2}$$

$$Y(s) = \frac{1}{s(s^2+3s+2)}$$

Q. 59 By Laplace transform the differential equation $y'' + 4y = \delta(t-2)$; $y(0) = 0$; $y'(0) = 1$ reduces to $Y(s) =$ _____.

(a) $\frac{e^{-2s}}{s(s^2+4)}$ (b) $\frac{1+s}{s(s^2+4)}$

(c) $\frac{e^{-2s}+1}{s^2+4}$ (d) $\frac{2s}{s^2+4}$

Ans. : (c)

Explanation :

$$y'' + 4y = \delta(t-2)\ ;\ y(0) = 0\ ;\ y'(0) = 1$$

Operate Laplace transform on both sides,

$$s^2 y(s) - s\,y(0) - y'(0) + 4\,y(s) = e^{-2s}$$

$$(s^2+4)\,y(s) = e^{-2s} + s\,y(0) + y'(0)$$

$$= e^{-2s} + 1$$

$$\therefore\ Y(s) = \frac{e^{-2s}+1}{s^2+4}$$

Q. 60 By Laplace transform the differential equation $y'' + 2y' + y = t\,e^{-t}$; $y(0) = 1$, $y'(0) = 2$ reduces to $Y(s) = 0$ _____.

(a) $\frac{s(s^2+1)+1}{(s^2+1)}$ (b) $\frac{1+(s+2)(s+1)^2}{(s+1)^4}$

(c) $\frac{s+2}{(s+1)^4}$ (d) $\frac{s+2}{(s+1)^2}$

Ans. : (b)

Explanation :

$$y'' + 2y' + y = t\,e^{-t}\ ;\ y(0) = 1,\ y'(0) = 2$$

Operate Laplace transform,

$$s^2 y(s) - s\,y(0) - y'(0) + 2\,[s\,y(s) - y(0)] + y(s) = \left[\frac{1}{s^2}\right]_{s \to s+1}$$

$$[s^2 + 2s + 1]\,Y(s) = \frac{1}{(s+1)^2} + s\,y(0) + y'(0)$$

$$(s+1)^2 \cdot Y(s) = \frac{1}{(s+1)^2} + s + 2 = \frac{1+(s+2)(s+1)^2}{(s+1)^2}$$

$$\therefore\ Y(s) = \frac{1+(s+2)+(s+1)^2}{(s+1)^4}$$

Q. 61 By Laplace transform the differential equation $\frac{d^2y}{dt^2} + 9y(t) = 18t$, $y(0) = y'(0) = 0$ reduces to $Y(s) =$ _____.

(a) $\frac{18}{s^2(s^2+9)}$ (b) $\frac{2s}{(s^2+9)}$

(c) $\frac{s}{s^2+9}$ (d) $\frac{18}{(s^2+9)}$

Ans. : (a)

Explanation :

$$\frac{d^2y}{dt^2} + 9\,y(t) = 18t\ ;\ y(0) = y'(0) = 0$$

Operate Laplace transform,

$$s^2 y(s) - s\,y(0) - y'(0) + 9\,y(s) = 18\frac{1}{s^2}$$

$$y(s) = \frac{18}{s^2(s^2+9)}$$

Q. 62 By using Laplace transform the differential equation $y'' + 4y = U(t-4)$; $y(0) = 0$, $y'(0) = 1$ reduces to $Y(s) =$ _____.

(a) $\frac{s}{(s^2+4)}$ (b) $\frac{e^{-4s}+s}{s(s^2+4)}$

(c) $\frac{e^{-4s}+1}{s^2+4}$ (d) $\frac{e^{-4s}+s}{s^2+4}$

Ans. : (b)

Explanation :

$$y'' + 4y = U(t-4)\ ;\ y(0) = 0,\ y'(0) = 1$$

Operate Laplace transform,

$$s^2 Y(s) - s\,y(0) - y'(0) + 4y(s) = \frac{1}{s}\,e^{-4s}$$

$$(s^2+4)\,y(s) = \frac{1}{s}\,e^{-4s} + y'(0) + s\,y(0)$$

$$(s^2+4)\,y(s) = \frac{e^{-4s}}{s} + 1 = \frac{e^{-4s}+s}{s}$$

$$Y(s) = \frac{e^{-4s}+s}{s(s^2+4)}$$

Q. 63 By Laplace transform the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t;$$

$y(0) = 0$, $y'(0) = 0$ reduces to $Y(s) =$ _____.

(a) $\frac{1}{(s+1)^2+1}$ (b) $\frac{s^2+2s+5}{(s+1)^2+1}$

(c) $\frac{1}{s^2+2s+5}$ (d) $\frac{1}{[(s+1)^2+1]\,[s^2+2s+5]}$

Ans. : (d)

Explanation : $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t$

Operate Laplace transform on both sides,

$$s^2 y(s) - s\,y(0) - y'(0) + 2\,[s\,y(s) - y(0)] + 5\,y(s)$$

$$= \left[\frac{1}{s^2+1}\right]_{s \to s+1}$$

$$(s^2+2s+5)\,y(s) = \frac{1}{(s+1)^2+1}$$

$$Y(s) = \frac{1}{[(s+1)^2+1]\,[s^2+2s+5]}$$

Q. 64 By Laplace transform method the differential equation $X'' + 3X' + 2X = t\,\delta(t-1)$; $X(0) = X'(0)$ reduces to $X(s) =$ _____.

(a) $\frac{e^{-s}}{s^2+3s+2}$ (b) $\frac{1}{s(s^2+3s+2)}$

(c) $\frac{1}{s^2+3s+2}$ (d) $\frac{e^{-s}}{s(s^2+3s+2)}$

Ans. : (a)

Explanation : $X'' + 3X' + 2X = t\,\delta(t-1)$;

$X(0) = X'(0) = 0$

Operate Laplace transform,

$$s^2 X(s) - s\,X(0) - X'(0) + 3\,[s\,X(s) - X(0)] + 2X(s) = e^{-s}$$

$$\left[\because L[f(t)\,\delta(t-a)] = e^{-s} f(a)\right]$$

$$(s^2+3s+2)\,X(s) = e^{-s}$$

$$X(s) = \frac{e^{-s}}{s^2+3s+2}.$$

Q. 65 By Laplace transform the equation $y' = -2$; $y(0) = 0$ reduces to $Y(s) =$ _____.

(a) $\frac{-2}{s}$ (b) $\frac{1}{s-2}$

(c) $\frac{-2}{s^2}$ (d) $\frac{1}{s}$

Ans. : (c)

Explanation :

$$y' = -2\ ;\ y(0) = 0$$

Operate Laplace transform,

$$s\,Y(s) - Y(0) = \frac{-2}{s}\ ;$$

$$Y(s) = \frac{-2}{s^2}$$

Q. 66 By Laplace transform the differential equation $y' = -t$; $y(0) = 0$ reduces to $Y(s) =$ _____.

(a) $\frac{-1}{s}$ (b) $\frac{1}{s-1}$

(c) $\frac{-1}{s^2}$ (d) $\frac{-1}{s^3}$

Ans. : (d)

Explanation :

$$y' = -t\ ;\ y(0) = 0$$

Operate Laplace transform on both sides,

$$s\,y(s) - y(0) = \frac{-1}{s^2}\,y(s) = \frac{-1}{s^3}$$

Q. 67 By Laplace transform the differential equation $y' = 1 + t$; $y(0) = 0$ reduces to $Y(s) =$ _____.

(a) $\frac{s+1}{s^2}$ (b) $\frac{s}{s^2+1}$

(c) $\frac{s+1}{s^3}$ (d) $\frac{1}{s(s+1)}$

Ans. : (c)

Explanation :

$$y' = 1 + t\ ;\ y(0) = 0$$

operate Laplace transform on both sides,

$$s\,y(s) - y(0) = \frac{1}{s} + \frac{1}{s^2} = \frac{s+1}{s^2};\ y(s) = \frac{s+1}{s^3}$$

Q. 68 By Laplace transform the differential equation $y' = 3 + 2t$; $y(0) = 0$ reduces to $Y(s) =$ _____.

(a) $\frac{3s+2}{s^3}$ (b) $\frac{5}{(s+2)}$

(c) $\frac{3s+2}{s^2}$ (d) $\frac{1}{s^3}$

Ans. : (a)

Explanation :

$$y' = 3 + 2t\ ;\ y(0) = 0$$

Operate Laplace transform,

$$s\,y(s) - y(0) = \frac{3}{s} + \frac{2}{s^2} = \frac{3s+2}{s^2};$$

$$y(s) = \frac{3s+2}{s^3}$$

Q. 69 By Laplace transform the differential equation $y'' = 1 - t$, $y(0) = y'(0) = 0$ reduces to $y(s) =$ _____.

(a) $\frac{s-1}{s^3}$ (b) $\frac{s+1}{s^4}$

(c) $\frac{s-1}{s^4}$ (d) $\frac{s+1}{s^3}$

Ans. : (c)

Explanation :

$$y'' = 1 - t \;;\; y(0) = y'(0) = 0$$

$$s^2 y(s) - s\,y(0) - y'(0) = \frac{1}{s} - \frac{1}{s^2} = \frac{s-1}{s^2}$$

$$y(s) = \frac{s-1}{s^4}$$

Q. 70 By Laplace transform the differential equation

$y'' = 3 + 2t$; $y(0) = 0$, $y'(0) = 1$ reduces to $y(s) =$ _____.

(a) $\frac{s^2+3s+2}{s^4}$ (b) $\frac{s^2+2s+1}{s^3}$

(c) $\frac{s^2-3s+1}{s^3}$ (d) $\frac{s^2-2}{s^4}$

Ans. : (a)

Explanation :

$$y'' = 3 + 2t \;;\; y(0) = 0,\; y'(0) = 1$$

$$s^2 y(s) - s\,y(0) - y'(0) = \frac{3}{s} + \frac{2}{s^2}$$

$$s^2 y(s) = \frac{3s+2}{s^2} + s\,y(0) + y'(0) = \frac{3s+2}{s^2} + 1$$

$$s^2 y(s) = \frac{s^2+3s+2}{s^2} \Rightarrow Y(s) = \frac{s^2+3s+2}{s^4}$$

Q. 71 By Laplace transform the differential equation

$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 12y = 12e^{-2t}$; $y(0) = 2$, $y'(0) = 6$ reduces to $y(s) =$ _____.

(a) $\frac{2s^2-6}{s^2-3s+12}$ (b) $\frac{2s^2+16s+36}{(s+2)(s^2-3s+12)}$

(c) $\frac{s^2+8s+18}{(s+2)(s^2-3s+12)}$ (d) $\frac{2s^2+6}{s^2-3s+12}$

Ans. : (b)

Explanation :

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 12y = 12e^{-2t} \;;\; y(0) = 2,\; y'(0) = 6$$

Operate Laplace transform,

$$s^2 y(s) - s\,y(0) - y'(0) - 3\,[s\,y(s) - y(0)] + 12\,y(s) = 12\frac{1}{s+2}$$

$$(s^2 - 3s + 12)\,y(s) = \frac{12}{s+2} + s\,y(0) + y'(0) + 3\,y(0)$$

$$= \frac{12}{s+2} + 2s + 6 + 6$$

$$= \frac{12}{s+2} + 2s + 12$$

$$= \frac{12 + 2s(s+2) + 12(s+2)}{s+2}$$

$$(s^2 - 3s + 12)\,y(s) = \frac{2s^2+16s+36}{(s+2)}$$

$$y(s) = \frac{2s^2+16s+36}{(s+2)(s^2-3s+12)}$$

□□□

CHAPTER 3 — UNIT III

Fourier Transform

Syllabus

Syllabus

Definitions – integral transforms ; Fourier integral theorem (without proof) ; Fourier sine and cosine integrals ; Complex form of Fourier integrals ; Fourier sine and cosine transforms ; Properties of Fourier transforms ; Parseval's identity for Fourier Transforms.

3.1 Introduction

In the engineering field some system gives ordinary differential equation and some gives partial differential equation. A Fourier transform can be used to solve partial differential equation, so it is necessary to study Fourier transform.

Prerequisite

(i) Gamma function : The gamma function is defined as

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx \quad (n > 0)$$

[We can use the notation Γn or $\Gamma(n)$]

(ii) $\Gamma(n+1) = n\Gamma n$, If n is fraction

$= n!$ If, n is positive integer

(iii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(iv) $\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx+c) - b\cos(bx+c)]$

(v) $\int e^{ax}\cos(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos(bx+c) + b\sin(bx+c)]$

(vi) $\int_0^\infty e^{-ax}\sin(bx+c)\,dx = \frac{b}{a^2+b^2}$

(vii) $\int_0^\infty e^{-ax}\cos(bx+c)\,dx = \frac{a}{a^2+b^2}$

Syllabus Topic : Integral transforms

3.1.1 Integral Transforms

Definition

An integral transform is any transform T of the form

$$T[f(u)] = \int_{t_1}^{t_2} f(t)\cdot k(t-u)\,dt$$

The input of this transform is a function f(t) and output is another function T[f(u)];

An integral transform is perticular kind of mathematical operator.

There are various useful integral transforms. Each is specified by a choice of the function k of two variables which is called as kernel of the transform.

Some kernels have an associated inverse kernel $k^{-1}(u, t)$ which yields an inverse transform given by

$$f(t) = \int_{u_1}^{u_2} T[f(u)]\cdot k^{-1}(u, t)\,du$$

Syllabus Topic : Fourier's Integral Theorem

3.1.2 Fourier's Integral Theorem

Given :

(i) f is function piecewise continuous on every bounded closed interval of the x -axis.

(ii) At each point the function f has both a left derivative and a right derivative ,

(iii) f is absolutely integrable along the x-axis:

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty \text{ then } \frac{1}{2}[f(x-)+f(x+)]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\left[\int_{-\infty}^{\infty} f(t)\, e^{-i\lambda t}\, dt\right] d\lambda$$

Remark : The above conditions are sufficient but not necessary.

Syllabus Topic : Complex form of Fourier Integrals

3.2 Complex Exponential form of Fourier Series

We have, for a periodic function defined in the interval $-l < x < l$ and satisfiesDirichlet's conditions, the Fourier series as,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\cdot dx,$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\, dx;$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\cdot\sin\frac{n\pi x}{l}\, dx$$

We have,

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}\ ;\ \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\frac{1}{2}\left(e^{i\frac{n\pi x}{l}}+e^{-i\frac{n\pi x}{l}}\right) + b_n\left(\frac{-i}{2}\right)\left(e^{i\frac{n\pi x}{l}}-e^{-i\frac{n\pi x}{l}}\right)\right]$$

$$\left(\because \frac{1}{i} = -i\right)$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{\left(\frac{a_n - ib_n}{2}\right)e^{in\pi x/l} + \left(\frac{a_n+ib_n}{2}\right)e^{-in\pi x/l}\right\}$$

Put, $C_0 = \frac{a_0}{2}$, $C_n = \frac{a_n - ib_n}{2}$, $C_{-n} = \frac{a_n + ib_n}{2}$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l}$$

Where, $C_0 = \frac{a_0}{2}$

$$C_n = \frac{1}{2}(a_n - ib_n)$$

$$= \frac{1}{2}\left\{\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx - i\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx\right\}$$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\left[\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right]dx$$

$$\therefore C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\cdot e^{-i\frac{n\pi x}{l}}dx;$$

Also $$C_{-n} = \frac{1}{2l}\int_{-l}^{l} f(x)\cdot e^{i\frac{n\pi x}{l}}dx$$

$$C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\cdot e^{-i\frac{n\pi x}{l}}dx$$

Hence, the complex exponential form of a fourier series is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi x}{l}}$$

Where, $$C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\cdot e^{-i\frac{n\pi x}{l}}dx$$

3.3 Fourier Integral

The complex exponential form of a fourier series is,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi x}{l}}$$

$$C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\cdot e^{-i\frac{n\pi x}{l}}du$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty}\left\{\frac{1}{2l}\int_{-l}^{l} f(u)\cdot e^{-i\frac{n\pi u}{l}}du\right\}e^{i\frac{n\pi x}{l}}$$

$$= \sum_{n=-\infty}^{\infty}\frac{1}{2l}\int_{-l}^{l} f(u)\, e^{-i\frac{n\pi(u-x)}{l}}du$$

$$= \sum_{n=-\infty}^{\infty}\left\{\frac{1}{2\pi}\left(\frac{\pi}{l}\right)\int_{-l}^{l} f(u)\cdot e^{\frac{-in\pi(u-x)}{l}}du\right\}$$

Put, $\lambda = \frac{n\pi}{l}$

$$\therefore \Delta l = \frac{(n+1)\pi}{l} - \frac{n\pi}{l} = \frac{\pi}{l}$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty}\left\{\frac{1}{2\pi}\int_{-l}^{l} f(u)\cdot e^{\frac{-in\pi(u-x)}{l}}du\right\}\Delta\lambda$$

If $l \to \infty$, then $\lambda \to 0$

$$\therefore f(x) = \lim_{l\to\infty}\sum_{n=-\infty}^{\infty}\left\{\frac{1}{2\pi}\int_{-l}^{l} f(u)\cdot e^{\frac{-in\pi(u-x)}{l}}du\right\}\Delta\lambda$$

$$= \lim_{\Delta\lambda\to 0}\sum_{n=-\infty}^{\infty}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\cdot e^{-i\lambda(u-x)}du\right\}\Delta\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\cdot e^{-i\lambda(u-x)}du\right]d\lambda$$

This integral is called as Fourier integral representation of f(x), in the interval $-\infty < x < \infty$.

Thus, Fourier integral representation of f(x), where $-\infty < x < \infty$ is given by,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\cdot e^{-i\lambda(u-x)}du\, d\lambda$$

3.4 Fourier Transform

We have Fourier integral

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\cdot e^{i\text{ and }\lambda(u-x)}du\, d\lambda \quad ...(1)$$

The double integral on RHS of Equation (1) is in the two variables u and λ with both limits as constant.

By the process of evaluation of double integral,is that when we integrate with respect to one variable,terms of other variable are to be kept constant.

In the integral separating terms of u and λ

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\cdot e^{-i\lambda u + i\lambda x}du\, d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\cdot e^{-i\lambda u}e^{i\lambda x}du\, d\lambda$$

$$\therefore f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\cdot e^{-i\lambda u}du\right]e^{i\lambda x}d\lambda \quad ...(2)$$

In above Equation (2) the integral $\int_{-\infty}^{\infty} f(u)\cdot e^{-i\lambda u}du$, variable is u and after integral we get function of λ. Let this function be denoted by F(λ), say

$$\therefore F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}du \quad ...(3)$$

The function F(λ) is called the Fourier Transform of f(x).

Using value of F(λ) in Equation (2) we get,

$$\therefore f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}d\lambda \quad ...(4)$$

This integeral is called Fourier integral representation of f(x).

The function f(x)is called as the inverse Fourier transform of F(λ).

3.4.1 Definition

Fourier Transform of the function f(x) defined in the interval $(-\infty, \infty)$ is defined as,

$$F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}du \quad ...(3a)$$

and **the inverse Fourier transform of F(λ) or Fourier integral representation of f(x) in the interval** $(-\infty, \infty)$is given by ,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}d\lambda \quad ...(4a)$$

Note : To find Fourier transform of given function we find the value of integral given in Equation (3).

To find inverse Fourier transform of F(λ) or Fourier integral representation of any function f(x),first find the value of F(λ) by using Equation (3a) and then put the value of F(λ) in Equation (4a)

3.4.2 Even Function

If a function f(x) is defined in the interval of the form (– a, a), then we check whether f(x) is even or odd function.

The function is f(x) said to be even if

$$f(-x) = f(x), \qquad \forall x \in (-a, a)$$

For even function graph is symmetric about y-axis

e.g.

(i) cos x [$\because \cos(-x) = \cos x$]

(ii) x^2

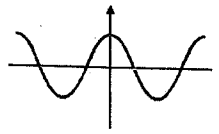


Fig. 3.4.1 : cos x

3.4.3 Odd Function

The Function is said to be odd if

$f(-x) = -f(x), \forall x \in (-a, a)$

For odd function graph is symmetric in opposite quadrant.

e.g.

(i) sinx $[\because \sin(-x) = -\sin x]$

(ii) x

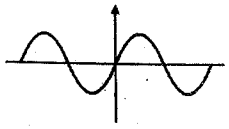


Fig. 3.4.2 : sin x

Note :

$f(x) = e^x$ is neither even nor odd function.

$\therefore e^{ax} \neq e^{ax}$ and $e^{-ax} \neq -e^{ax}$ [i.e. $f(x) \neq f(-x)$ and $f(-x) \neq -f(x)$]

3.4.4 Fourier transform for Even and Odd Function

Even Function : A function f(x)defined in the interval $-\infty < x < \infty$ is said to even function if

$f(-x) = f(x)$. for all $a \in (-\infty, \infty)$

Consider Fourier integral representation of f(x),

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-x)}\, du\, d\lambda \qquad ...(5)$$

➔ **Using standard formula**

$...[e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$

Here $\theta = \lambda(u-x)]$

$$e^{-i\lambda(u-x)} = \cos\lambda(u-x) - i\sin\lambda(u-x)$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\Big[\cos\lambda(u-x) - i\sin\lambda(u-x)\Big]du\, d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\int_{-\infty}^{\infty}\Big[\cos\lambda(u-x)\, d\lambda - i\sin\lambda(u-x)\, d\lambda\Big]du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\Big[\int_{-\infty}^{\infty}\cos\lambda(u-x)\, d\lambda - i\int_{-\infty}^{\infty}\sin\lambda(u-x)\, d\lambda\Big]du$$

...(6)

Now for the variable λ, the term $\cos\lambda(u-x)$ is an even function and $\sin\lambda(u-x)$ is odd function

➔ **Using Standard result of definite integral**

$$...\left[\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{for even function} \\ 0, & \text{for odd function}\end{cases}\right]$$

Therefore in Equation (6),

$$\int_{-\infty}^{\infty}\sin\lambda(u-x)\, d\lambda = 0 \text{ and}$$

$$\int_{-\infty}^{\infty}\cos\lambda(u-x)\, d\lambda = 2\int_0^{\infty}\cos\lambda(u-x)\, d\lambda$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\Big[2\int_0^{\infty}\cos\lambda(u-x)\, d\lambda - i(0)\Big]du$$

$$= \frac{1}{2\pi}(2)\int_{-\infty}^{\infty} f(u)\Big[\int_0^{\infty}\cos\lambda(u-x)\, d\lambda\Big]du$$

$$= \frac{1}{\pi}\int_0^{\infty}\Big[\int_{-\infty}^{\infty} f(u)\cos\lambda(u-x)\, du\Big]d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\Big[\int_{-\infty}^{\infty} f(u)\cos(\lambda u - \lambda x)\, du\Big]d\lambda \qquad ...(7)$$

➔ **Using standard formula**

$...[\cos(A-B) = \cos A\cos B + \sin A\sin B]$

Here $A = \lambda u$, $B = \lambda x$

$$\cos(\lambda u - \lambda x) = \cos\lambda u\cos\lambda x + \sin\lambda u\sin\lambda x$$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\Big[\int_{-\infty}^{\infty} f(u)\,(\cos\lambda u\cos\lambda x + \sin\lambda u\sin\lambda x)\, du\Big]d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\Big[\int_{-\infty}^{\infty} f(u)\,(\cos\lambda u\cos\lambda x)\, du + \int_{-\infty}^{\infty} f(u)\,\sin\lambda u\sin\lambda x)\, du\Big]d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\Big[\int_{-\infty}^{\infty} f(u)\cos\lambda u\, du\Big]\cos\lambda x\, d\lambda$$

$$+ \Big[\int_{-\infty}^{\infty} f(u)\sin\lambda u\, du\Big]\sin\lambda x\, d\lambda \qquad ...(8)$$

The variable of inner integral of Equation (8) is u.

Using standard result in Equation(8)

(1) Product of two even functions is even function.

(2) Product of two odd functions is even function.

(3) Product of one even and one odd function is odd function.

Here in first integral f(u) is even function and $\cos\lambda u$ is **also an even function.**

Therefore f(u) $\cos\lambda u$ is an even function.

Also in second integral f(u) is even function and $\sin\lambda u$ **is an odd function.Therefore**

f(u) $\sin\lambda u$ is an odd function

➔ **Using Standard result of definite integral**

$$...\left[\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{for even function} \\ 0 & \text{for odd function}\end{cases}\right]$$

Therefore in Equation (8),

$$\int_{-\infty}^{\infty} f(u)\sin\lambda u\, du = 0 \text{ and}$$

$$\int_{-\infty}^{\infty} f(u)\cos\lambda u\, du = 2\int_0^{\infty} f(u)\cos\lambda u\, du$$

$$f(x) = \frac{1}{\pi}\int_0^{\infty} 2\Big[\int_0^{\infty} f(u)\cos\lambda u\, du\Big]\cos\lambda x\, d\lambda + \Big[0\Big]\sin\lambda x\, d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(u)\cos\lambda u\cos\lambda x\, du\, d\lambda \qquad ...(9)$$

This integral gives Fourier integral representation of an even function.

Rewriting the integral as

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\Big[\int_0^{\infty} f(u)\cos\lambda u\, du\Big]\cos\lambda x\, d\lambda \qquad ...(10)$$

In this double integral, the integral $\int_0^{\infty} f(u)\cos\lambda u\, du$ is function of λ. We denote this function by $F(\lambda)$ which is called Fourier transform of f(x)

Thus the Fourier transform of an even function f(x) is given by ,

$$F(\lambda) = \int_0^{\infty} f(u)\cos\lambda u\, du$$

Using value of $F(\lambda)$ in Euation (10) we get,

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F(\lambda)\cos\lambda x\, d\lambda$$

This integral is called as inverse Fourier transform of $F(\lambda)$ or Fourier integral representation of an even function.

Odd Function : A function f(x) defined in the interval $-\infty < x < \infty$ is said to odd function if

$f(-x) = -f(x)$.

Consider Fourier integral representation of f(x),

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-x)}\, du\, d\lambda \qquad ...(11)$$

➔ **Using standard formula**

$...[e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta]$

Here $\theta = \lambda(u-x)$

$$e^{-i\lambda(u-x)} = \cos\lambda(u-x) - i\sin\lambda(u-x)$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\Big[\cos\lambda(u-x) - i\sin\lambda(u-x)\Big]du\, d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\int_{-\infty}^{\infty}\Big[\cos\lambda(u-x)\, d\lambda - i\sin\lambda(u-x)\, d\lambda\Big]du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\Big[\int_{-\infty}^{\infty}\cos\lambda(u-x)\, d\lambda - i\int_{-\infty}^{\infty}\sin\lambda(u-x)\, d\lambda\Big]du$$

...(12)

Now for the variable λ, the term $\cos\lambda(u-x)$ is an even function and $\sin\lambda(u-x)$ is odd function

➔ **Using Standard result of definite integral**

$$...\left[\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{for even function} \\ 0 & \text{for odd function}\end{cases}\right]$$

Therefore in Equation (12),

$$\int_{-\infty}^{\infty} \sin\lambda(u-x)\,d\lambda = 0 \text{ and}$$

$$\int_{-\infty}^{\infty} \cos\lambda(u-x)\,d\lambda = 2\int_{0}^{\infty} \cos\lambda(u-x)\,d\lambda$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[2\int_{0}^{\infty}\cos\lambda(u-x)\,d\lambda - i(0)\right]du$$

$$= \frac{1}{2\pi}(2)\int_{-\infty}^{\infty} f(u)\left[\int_{0}^{\infty}\cos\lambda(u-x)\,d\lambda\right]du$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\cos\lambda(u-x)\,du\right]d\lambda$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\cos(\lambda u-\lambda x)\,du\right]d\lambda \quad ...(13)$$

➔ **Using standard formula**

$$...[\cos(A-B) = \cos A\cos B + \sin A\sin B]$$

Here $A = \lambda u$, $B = \lambda x$

$$\cos(\lambda u-\lambda x) = \cos\lambda u\cos\lambda x + \sin\lambda u\sin\lambda x$$

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\,(\cos\lambda u\cos\lambda x + \sin\lambda u\sin\lambda x)\,du\right]d\lambda$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\,(\cos\lambda u\cos\lambda x)\,du\right.$$

$$\left. + \int_{-\infty}^{\infty} f(u)\sin\lambda u\sin\lambda x)\,du\right]d\lambda$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\cos\lambda u\,du\right]\cos\lambda x\,d\lambda$$

$$+ \left[\int_{-\infty}^{\infty} f(u)\sin\lambda u\,du\right]\sin\lambda x\,d\lambda \quad ...(14)$$

The variable of inner integral of Equation (14) is u.

Using standard result in Equation (14)

(1) Product of two even functions is even function.

(2) Product of two odd functions is even function.

(3) Product of one even and one odd function is odd function.

Here in first integral f(u) is odd function and cos λu is an even function.

Therefore f(u) cos λu is an odd function.

Also in second integral f(u) is odd function and sinλu is also an odd function.

Therefore f(u) sinλu is an even function

➔ **Using Standard result of definite integral**

$$...\left[\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx & \text{for even function} \\ 0 & \text{for odd function}\end{cases}\right]$$

Therefore in Equation (14),

$$\int_{-\infty}^{\infty} f(u)\sin\lambda u\,du = 2\int_{0}^{\infty} f(u)\sin\lambda u\,du \text{ and}$$

$$\int_{-\infty}^{\infty} f(u)\cos\lambda u\,du = 0$$

$$f(x) = \frac{1}{\pi}\left[\int_{0}^{\infty}(0)\,d\lambda + \int_{0}^{\infty}2\left[\int_{0}^{\infty} f(u)\sin\lambda u\,du\right]\sin\lambda x\,d\lambda\right]$$

$$= \frac{1}{\pi}\left[(0) + 2\int_{0}^{\infty}\int_{0}^{\infty} f(u)\sin\lambda u\sin\lambda x\,du\,d\lambda\right]$$

$$= \frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty} f(u)\sin\lambda u\sin\lambda x\,du\,d\lambda \quad ...(15)$$

This integral gives Fourier integral representation of an odd function.

Rewriting the integral as

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\left[\int_{0}^{\infty} f(u)\sin\lambda u\,du\right]\sin\lambda x\,d\lambda \quad ...(16)$$

In this double integral, the integral $\int_{0}^{\infty} f(u)\sin\lambda u\,du$ is function of λ.

We denote this function by F(λ) which is called Fourier transform of f(x)

Thus the Fourier transform of an odd function f(x) is given by,

$$F(\lambda) = \int_{0}^{\infty} f(u)\sin\lambda u\,du$$

Using value of F(λ) in Equation (16) we get,

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} F(\lambda)\sin\lambda x\,d\lambda$$

This integral is called as inverse Fourier transform of F(λ) or Fourier integral representation of an odd function.

Fourier transform of an odd function f(x) is given by,

$$F(\lambda) = \int_{0}^{\infty} f(u)\sin\lambda u\,du$$ and the inverse Fourier transforms of F(λ) is given by,

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} F(\lambda)\sin\lambda x\,d\lambda$$

Syllabus Topic : Fourier Sine and Cosine Transforms

3.5 Fourier Sine and Cosine Transforms

3.5.1 Fourier Cosine Transform (FCT)

If f(x) is defined in the interval $0 < x < \infty$, then **Fourier Cosine transform** of f(x) is denoted by $F_C(\lambda)$ and is given by,

$$F_C(\lambda) = \int_{0}^{\infty} f(u)\cos\lambda u\,du$$

and the **inverse Fourier Cosine transform** of $F_C(\lambda)$ is defined as,

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} F_C(\lambda)\cos\lambda x\,d\lambda$$

3.5.2 Fourier Sine Transform (FST)

If f(x) is defined in the interval $0 < x < \infty$, then **Fourier Sine transform** of f(x) is denoted by $F_S(\lambda)$ and is given by,

$$F_S(\lambda) = \int_{0}^{\infty} f(u)\sin\lambda u\,du$$

and the **inverse Fourier sine transform** of $F_S(\lambda)$ is given by,

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} F_S(\lambda)\sin\lambda x\,d\lambda$$

List of Fourier transform to Remember

Sr. No.	Interval	Fourier Transform	Inverse Fourier transform
1.	$-\infty < x < \infty$	$F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\,du$	$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\,d\lambda$
2.	$-\infty < x < \infty$ For Even Function	$F(\lambda) = \int_{0}^{\infty} f(u)\cos\lambda u\,du$	$f(x) = \frac{2}{\pi}\int_{0}^{\infty} F(\lambda)\cos\lambda x\,d\lambda$
3.	$-\infty < x < \infty$ For odd Function	$F(\lambda) = \int_{0}^{\infty} f(u)\sin\lambda u\,du$	$f(x) = \frac{2}{\pi}\int_{0}^{\infty} F(\lambda)\sin\lambda x\,d\lambda$
4.	$0 < x < \infty$ Fourier Cosine Transform	$F_C(\lambda) = \int_{0}^{\infty} f(u)\cos\lambda u\,du$	$f(x) = \frac{2}{\pi}\int_{0}^{\infty} F_C(\lambda)\cos\lambda x\,d\lambda$
5.	$0 < x < \infty$ Fourier Sine Transform	$F_S(\lambda) = \int_{0}^{\infty} f(u)\sin\lambda u\,du$	$f(x) = \frac{2}{\pi}\int_{0}^{\infty} F_S(\lambda)\sin\lambda x\,d\lambda$

Note : $f(x) \longrightarrow F(\lambda) \longrightarrow f(x)$

To find Fourier Transform :

(1) In general fourier transform, the term $\frac{1}{2\pi}$ can be adjusted according to convinient i.e. it can be written in the formula of fourier transform or in the formula of inverse Fourier transform. Also for symmetry it can be written as $\frac{1}{\sqrt{2\pi}}$ in both formulae.

Therefore formula of Fourier transform and inverse Fourier transform can also be written as

$$F(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$$

$$f(x) = \int_{-\infty}^{\infty} f(u)\, e^{i\lambda x}\, d\lambda$$

OR $$F(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{i\lambda x}\, d\lambda$$

similarly, Fourier transform of even function ϕ or Fourier cosine transform can also be written as,

$$F(\lambda) = \frac{2}{\pi}\int_{0}^{\infty} f(u)\cos\lambda u\, du$$

$$f(x) = \int_{0}^{\infty} f(u)\cos\lambda x\, d\lambda$$

OR $$F(\lambda) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(u)\cos\lambda u\, du$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(\lambda)\cos\lambda x\, d\lambda$$

Given f(x)in the interval $(-\infty, \infty)$

Even	odd	neither even nor odd
F.T. $F(\lambda) = \int_0^\infty f(u)\cos\lambda u\, du$	$F(\lambda) = \int_0^\infty f(u)\sin\lambda u\, du$	$F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$
I.F.T. $f(x) = \frac{2}{\pi}\int_0^\infty F(\lambda)\cos\lambda x\, d\lambda$	$f(x) = \frac{2}{\pi}\int_0^\infty F(\lambda)\sin\lambda x\, d\lambda$	$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda$

(ii) $|x| < a$ means $x < a$ and $-x < a$ i.e. $x > -a$

In this case $-a < x < a$

Therefore whenever function f(x) is defined for $|x| < a$, we take value of function f(x) in $-a < x < a$

(iii) $|x| > a$ means $x > a$ and $-x > a$ i.e. $x < -a$

In this case $x < -a$ and $x > a$

Therefore whenever function f(x) is defined for $|x| > a$, we take value of function f(x) in

$x < -a$ and $x > a$

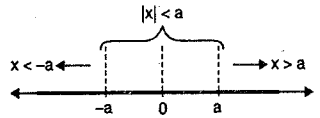


Fig. 3.5.1

(iv) In examples if we want to find Fourier integral representation of f(x), first find Fourier transform F(λ) of f(x) by using appropriate formula and then put value of F(λ) in the formula of corresponding inverse Fourier transform.

(v) When in example it is asked to find the value of given integral by using Fourier integral representation of f(x), first we find the value of same integral in terms of λ by chosing suitable value of x in Fourier integral representation of f(x). Then the variable λ is to be change to x as value of definite integral is independent of variable.

(vi) Value of function at the point of discontinuity is average of left hand limit and right hand limit.

(vii) In practice we have to find out Fourier transform of only three types of functions viz. Exponential, Algebraic polynomial,Trigonometric(sine or cosine). For this we use following methods/Formulae

For exponential function, use any one of following appropriate formula

$$\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}[\,a\sin(bx+c) - b\cos(bx+c)\,]$$

OR

$$\int e^{ax}\cos(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}[\,a\cos(bx+c) + b\sin(bx+c)\,]$$

(I) For algebraic polynomial use generalized rule of integration

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots.$$

Where u is algebraic polynomial

v is exponential / sine / cosine term

Here dash denotes order of derivative and suffix denotes integration.

(II) For trigonometric functions,simplify given integral by using following formulae

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$\Rightarrow \sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

Syllabus Topic : Properties of Fourier Transforms

Properties of Fourier transform

1. **Linearity property**

   If f(x) and g(x) are any functions and a,b are any constants then,

   $F[af(x) + bg(x)] = a\,F[f(x)] + b\,F[g(x)]$

2. **Delay Property**

   $F\{f(x-x_0)\} = e^{-i\lambda x_0}\,F[\lambda]$

3. **Frequency Shifting**

   $F\{f(x)\,e^{-i\lambda x_0}\} = [F(\lambda)]_{\lambda \to \lambda - \lambda_0}$

4. **Scaling Property**

   $F[f(ax)] = \frac{1}{|a|}[F(\lambda)]_{\lambda \to \frac{\lambda}{a}}$

3.5.3 Type - I Examples on Fourier integral representation

Ex.3.5.1 : Find the Fourier integral representation of the Function. $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

and hence (a) Evaluate $\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$

(b) deduce the value of $\int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda$

(c) Find the value of above integrals at $|x| = 1$, which are points of discontinuity of f(x).

Soln. :

Step I : Given function can be written as

Given, $f(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & x > 1 \text{ and } x < -1 \end{cases}$

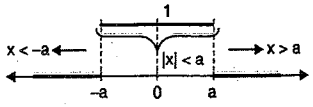


Fig. P. 3.5.1

From figure we observe that graph of f(x) is symmetric about Y-axis therefore f(x) is even function in the interval $-\infty < x < \infty$.

Step II : ∴ The Fourier transform of f(x) is given by

$$F(\lambda) = \int_0^\infty f(u) \cdot \cos \lambda u \, du \qquad ...(1)$$

The limits of integral are 0 to ∞

From figure we observe that in the interval $0 < x < \infty$, value of f(u) is given in two different parts as f(u) = 1 for 0 < x < 1 and f(u) = 0 for x > 1. Therefore integral in Equation (1) can be split as

$$F(\lambda) = \int_0^1 f(u) \cdot \cos \lambda u \, du + \int_1^\infty f(u) \cdot \cos \lambda u \, du$$

$$= \int_0^1 (1) \cos \lambda u \, du + \int_1^\infty (0) \cos \lambda u \, du$$

$$= \int_0^1 \cos \lambda u \, du + 0 = \int_0^1 \cos \lambda u \, du \qquad ...(2)$$

Step III :

➔ **Using standard formula of integration**

$$...\left[\int \cos Ax \, dx = \frac{\sin Ax}{A} \text{ for any constant } A\right]$$

Here $A = \lambda$ and $x = u$

$$\therefore F(\lambda) = \left[\frac{\sin \lambda u}{\lambda}\right]_0^1 = \left[\frac{\sin \lambda u}{\lambda}\right]_{\text{at } u=1} - \left[\frac{\sin \lambda u}{\lambda}\right]_{\text{at } u=0}$$

$$= \left[\frac{\sin \lambda(1)}{\lambda}\right] - \left[\frac{\sin \lambda(0)}{\lambda}\right] = \left[\frac{\sin \lambda}{\lambda}\right] - \left[\frac{\overbrace{\sin(0)}^{0}}{\lambda}\right] \qquad ...(3)$$

Using standard value sin0 = 0 in Equation (3)

$$\therefore F(\lambda) = \frac{\sin \lambda}{\lambda} - 0 = \frac{\sin \lambda}{\lambda}$$

$$F(\lambda) = \frac{\sin \lambda}{\lambda} \qquad ...(4)$$

This is Fourier transform of f(x).

Step IV : The inverse Fourier transform of F(λ) is given by,

$$f(x) = \frac{2}{\pi} \int_0^\infty F(\lambda) \cdot \cos \lambda x \, d\lambda$$

Using value of F(λ) from Equation (4),

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin \lambda}{\lambda} \cos \lambda x \, d\lambda$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda} \, d\lambda \qquad ...(5)$$

Which is required Fourier integral representation.

Step V :

(a) We are required to find the value of integral $\int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda} \, d\lambda$ which we observe in Equation (5),

$$\therefore \text{From Equation (5), } f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda} \, d\lambda$$

$$\therefore \int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda} \, d\lambda = \frac{\pi}{2} f(x),$$

Where $f(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & x > 1 \text{ and } x < -1 \end{cases}$

$$\therefore \int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda} \, d\lambda = \begin{cases} \frac{\pi}{2}, & -1 < x < 1 \\ 0, & x > 1 \text{ and } x < -1 \end{cases} \qquad ...(6)$$

(b) Now we want to find the value of $\int_0^\infty \frac{\sin \lambda}{\lambda} \, d\lambda$

Put, x = 0 on both sides of Equation (6) we get,

$$\int_0^\infty \frac{\sin \lambda \overbrace{\cos \lambda(0)}^{1}}{\lambda} \, d\lambda = \begin{cases} \frac{\pi}{2}, & -1 < x < 1 \\ 0, & x > 1 \text{ and } x < -1 \end{cases} \text{ at } x = 0 \qquad ...(7)$$

On RHS of Equation (7), the value of function is $\frac{\pi}{2}$ when $-1 < x < 1$ and we want value of f(x) at x = 0.

x = 0 lies in the range (− 1, 1) .Therefore $f(x) = \frac{\pi}{2}$ on RHS

$$\therefore \int_0^\infty \frac{\sin \lambda (1)}{\lambda} \, d\lambda = \frac{\pi}{2}$$

$$\therefore \int_0^\infty \frac{\sin \lambda}{\lambda} \, d\lambda = \frac{\pi}{2}$$

(c) at $|x| = 1$ which is point of discontinuity.

$$\left[\int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda\right]_{|x|=1} = \frac{\frac{\pi}{2} + 0}{2} = \frac{\pi}{4} \text{ i.e. } \frac{1}{2}[f(1^-) + f(1^+)] \checkmark \quad ...\text{Ans.}$$

Ex. 3.5.2 : Find the Fourier transform of

$$f(x) = \begin{cases} 1 - x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

Hence evaluate : $\int_0^\infty \left(\frac{x \cos x - \sin x}{x^3}\right) \cos\left(\frac{x}{2}\right) dx.$

Soln. :

Step I : Given function can be written as,

$$f(x) = \begin{cases} 1 - x^2, & -1 < x < 1 \\ 0, & x > 1 \text{ and } x < -1 \end{cases}$$

From definition of function, $f(0) = 1 - (0)^2 = 1 - 0 = 1$, $f(1) = 1 - (1)^2 = 1 - 1 = 0$. Thus f(x) decrease from 1 to 0 as x increases from 0 to 1.

Similarly $f(0) = 1 - (0)^2 = 1 - 0 = 1$, $f(-1) = 1 - (-1)^2 = 1 - 1 = 0$

From this graph of function can be drawn as follows.

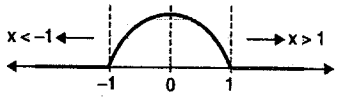


Fig. P. 3.5.2

From figure we observe that graph of f(x) is symmetric about Y-axis therefore f(x) is even function in the interval $-\infty < x < \infty$.

Step II : ∴ The Fourier transform of f(x) is given by

$$F(\lambda) = \int_0^\infty f(u) \cdot \cos \lambda u \, du \qquad ...(1)$$

The limits of integral are 0 to ∞

From figure we observe that in the interval $0 < x < \infty$, value of f(u) is given in two different parts as $f(u) = 1 - x^2$ for 0 < x < 1 and f(u) = 0 for x > 1. Therefore integral in Equation (1) can be split as

$$F(\lambda) = \int_0^1 f(u) \cdot \cos \lambda u \, du + \int_1^\infty f(u) \cdot \cos \lambda u \, du$$

$$= \int_0^1 (1 - u^2) \cos \lambda u \, du + \int_1^\infty (0) \cos \lambda u \, du$$

$$= \int_0^1 (1 - u^2) \cos \lambda u \, du + 0$$

$$= \int_0^1 (1 - u^2) \cos \lambda u \, du \qquad ...(2)$$

➔ **Using formula of generalized rule of integration**

$$...\left[\int uv \, dx = uv_1 - u' v_2 + u'' v_3 - u''' v_4 + ...\right]$$

Where u is algebraic polynomial

v is exponential / sine / cosine term

Here dash denotes order of derivative and suffix denotes integration.

Here $u = 1 - u^2$, $v = \cos \lambda u$

$u' = -2u, u'' = -2, u''' = 0$

$v_1 = \frac{\sin \lambda u}{\lambda}, v_2 = -\frac{\cos \lambda u}{\lambda^2}, v_3 = -\frac{\sin \lambda u}{\lambda^3}$ And so on

$$\therefore F(\lambda) = \left[(1 - u^2)\left(\frac{\sin \lambda u}{\lambda}\right) - (-2u)\left(\frac{-\cos \lambda u}{\lambda^2}\right) + (-2)\left(\frac{-\sin \lambda u}{\lambda^3}\right) + (0)\left(\frac{\cos \lambda u}{\lambda^4}\right)\right]_0^1$$

$$= \left[\frac{1}{\lambda}(1 - u^2) \sin \lambda u - \frac{2}{\lambda^2} u \cos \lambda u + \frac{2}{\lambda^3} \sin \lambda u + 0\right]_0^1$$

$$= \left[\frac{1}{\lambda}(1 - u^2) \sin \lambda u - \frac{2u}{\lambda^2} \cos \lambda u + \frac{2}{\lambda^3} \sin \lambda u\right]_{u=1}$$

$$- \left[\frac{1}{\lambda}(1 - u^2) \sin \lambda u - \frac{2}{\lambda^2} u \cos \lambda u + \frac{2}{\lambda^3} \sin \lambda u\right]_{u=0}$$

$$= \left[\frac{1}{\lambda}(1 - (1)^2) \sin \lambda (1) - \frac{2(1)}{\lambda^2} \cos \lambda(1) + \frac{2}{\lambda^3} \sin \lambda (1)\right]$$

$$- \left[\frac{1}{\lambda}(1 - (0)^2) \sin \lambda (0) - \frac{2(0)}{\lambda^2} \cos \lambda (0) + \frac{2}{\lambda^3} \sin \lambda (0)\right]$$

$$= \left[\frac{1}{\lambda}\underbrace{(1 - 1)}_{0} \sin \lambda - \frac{2}{\lambda^2} \cos \lambda + \frac{2}{\lambda^3} \sin \lambda\right]$$

$$- \left[\frac{1}{\lambda}\underbrace{(1 - 0)}_{0} \underbrace{\sin 0}_{0} - \frac{2(0)}{\lambda^2} \underbrace{\cos 0}_{1} + \frac{2}{\lambda^3} \underbrace{\sin 0}_{0}\right]$$

$$= \left[\frac{1}{\lambda}(0) \sin \lambda - \frac{2}{\lambda^2} \cos \lambda + \frac{2}{\lambda^3} \sin \lambda\right]$$

$$- \left[\frac{1}{\lambda}(1)(0) - \frac{2}{\lambda^2}(1) + \frac{2}{\lambda^3}(0)\right]$$

$$= \left[0 - \frac{2}{\lambda^2} \cos \lambda + \frac{2}{\lambda^3} \sin \lambda\right] - [0 - (0)(1) + (0)]$$

$$= \left[-\frac{2}{\lambda^2}\cos\lambda + \frac{2}{\lambda^3}\sin\lambda\right] - [0]$$

$$= -\frac{2}{\lambda^2}\cos\lambda + \frac{2}{\lambda^3}\sin\lambda$$

$$F(\lambda) = \left[-\frac{2\cos\lambda}{\lambda^2} + \frac{2\sin\lambda}{\lambda^3}\right] \quad ...(4)$$

This is Fourier transform of f(x)

Step III : The inverse Fourier transform of $F(\lambda)$ is given by,

$$f(x) = \frac{2}{\pi}\int_0^\infty F(\lambda)\cos\lambda x \cdot d\lambda \quad ...(5)$$

Using value of $F(\lambda)$ from Equation (4) in Equation (5),

$$f(x) = \frac{2}{\pi}\int_0^\infty\left(\frac{-2\cos\lambda}{\lambda^2} + \frac{2\sin\lambda}{\lambda^3}\right)\cos\lambda x\, d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty\left(\frac{-2\lambda\cos\lambda + 2\sin\lambda}{\lambda^3}\right)\cos\lambda x\, d\lambda \quad ...(6)$$

This is Fourier integral representation of f(x).

Step IV : Giving site on required integral we simplify this integral as,

$$F(x) = \frac{2}{\pi}\int_0^\infty(-2)\left(\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right)\cos\lambda x\, d\lambda$$

$$= \frac{2}{\pi}(-2)\int_0^\infty\left(\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right)\cos\lambda x\, d\lambda$$

$$= -\frac{4}{\pi}\int_0^\infty\left(\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right)\cos\lambda x\, d\lambda$$

$$\therefore \int_0^\infty\left(\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right)\cos\lambda x\, d\lambda = -\frac{\pi}{4}f(x) \quad ...(7)$$

To find the value of integral $\int_0^\infty\left(\frac{x\cos x - \sin x}{x^3}\right)\cos\left(\frac{x}{2}\right)dx$,

first we find the value of same integral in terms of λ, by using proper value of x, i.e. we find the value of integral

$$\int_0^\infty\left(\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right)\cos\left(\frac{\lambda}{2}\right)d\lambda \quad ...(8)$$

Comparing Equations (7) and (8), we observe that integral in Equation (8) is obtained by putting $x = \frac{1}{2}$ in Equation (7)

$$\therefore \int_0^\infty\left(\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right)\cos\lambda\left(\frac{1}{2}\right)d\lambda = -\frac{\pi}{4}f\left(\frac{1}{2}\right) \quad ...(9)$$

$f\left(\frac{1}{2}\right)$ means value of f(x) at $x = \frac{1}{2}$

From given function, $f(x) = \begin{cases} 1 - x^2, & -1 < x < 1 \\ 0, & x > 1 \text{ and } x < -1 \end{cases}$

$x = \frac{1}{2}$ lies in $(-1, 1)$, therefore to find value of f(x) at $x = \frac{1}{2}$ we take $f(x) = 1 - x^2$

$$\therefore f\left(\frac{1}{2}\right) = 1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = -\frac{4-1}{4} = -\frac{3}{4}$$

From Equation (9),

$$\therefore \int_0^\infty\left(\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right)\cos\left(\frac{\lambda}{2}\right)d\lambda = -\frac{\pi}{4}\left(-\frac{3}{4}\right) = \frac{3\pi}{16} \quad ...(10)$$

As value of definite integral is independent on variable, changing the variable λ to x

$$\int_0^\infty\left(\frac{x\cos x - \sin x}{x^3}\right)\cos\left(\frac{x}{2}\right)dx = \frac{3\pi}{16} \checkmark \quad ...\text{Ans.}$$

Ex. 3.5.3 : Show that Fourier transform of $e^{-\frac{x^2}{2}}$ is itself or $e^{-\frac{\lambda^2}{2}}$

Soln. : Step I : Consider function

$$f(x) = e^{-x^2/2}, \quad -\infty < x < \infty$$

We have

$$f(-x) = e^{-(-x)^2/2} = e^{-x^2/2} = f(x)$$

i.e $f(-x) = f(x)$

Therefore given function is an even function.

Step II : The Fourier transform of f(x) is given by

$$F(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cdot\cos\lambda u\, du \quad ...(1)$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cdot\cos\lambda u\, du$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-u^2/2}\cos\lambda u\, du$$

Step III : Consider $I(\lambda) = \int_0^\infty e^{-u^2/2}\cos\lambda u\, du \quad ...(2)$

Using standard rule of DUIS in Equation(2),

If $I(\alpha) = \int_a^b f(x, \alpha)\, dx$ then

$$\frac{dI(\alpha)}{d\alpha} = \int_a^b\left[\frac{\partial}{\partial\alpha}f(x, \alpha)\right]dx \text{ where a and b are constants}$$

Here $x = u$, $\alpha = \lambda$

$$\therefore \frac{dI}{d\lambda} = \int_0^\infty\frac{\partial}{\partial\lambda}\left(e^{-u^2/2}\cos\lambda u\right)du$$

By the process of partial differentiation, when we differentiate given function with respect to one variable, terms of other variable are kept constant.

Here we are differentiating $e^{-u^2/2}\cos\lambda u$, partially with respect to λ. So terms other than λ are to be kept constant.

$$\therefore \frac{dI}{d\lambda} = \int_0^\infty\left(e^{-u^2/2}\right)\frac{\partial}{\partial\lambda}(\cos\lambda u)\, du \quad ...(3)$$

➔ **Using standard formula of derivative**

$$...\left[\frac{d}{dx}(\cos Ax) = -A\sin Ax \text{ for any constant } A\right]$$

Here constant is $A = u$

$$\therefore \frac{dI}{d\lambda} = \int_0^\infty e^{-u^2/2}(-u)\sin\lambda u\, du$$

$$= \int_0^\infty\left(-u\, e^{-u^2/2}\right)\sin\lambda u\, du \quad ...(4)$$

➔ **Using integration by parts**

$$...\left[\int(u.v)\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx\right]$$

Here $u = \sin\lambda u$, $v = (-u\, e^{-u^2/2})$

$$\therefore \frac{dI}{d\lambda} = (\sin\lambda u)\int_0^\infty\left(-u\, e^{-u^2/2}\right)du$$

$$-\int_0^\infty\left[\frac{d}{dx}(\sin\lambda u)\int_0^\infty\left(-u\, e^{-u^2/2}\right)du\right]du \quad ...(5)$$

Step IV : Adjusting the term in the integral of Equation (5) as follows

$$\frac{d}{du}(e^{-u^2/2}) = e^{-u^2/2}\frac{d}{du}\left(-\frac{u^2}{2}\right) = e^{-u^2/2}\left(-\frac{1}{2}(2u)\right) = e^{-u^2/2}(-u)$$

$$\therefore \int e^{-u^2/2}(-u)\, du = e^{-u^2/2}$$

Also $\frac{d}{dx}(\sin Ax) = A\cos Ax$ for any constant A

$$\therefore \frac{d}{dx}(\sin\lambda u) = \lambda\cos\lambda u$$

$$\therefore \frac{dI}{d\lambda} = \left[\sin\lambda u\left(e^{-u^2/2}\right)\right]_0^\infty - \int_0^\infty(\cos\lambda u)(\lambda)\left(e^{-u^2/2}\right)du$$

$$= \left[\sin\lambda(\infty)\underbrace{\left(e^{-\infty^2/2}\right)}_{0} - \underbrace{\sin\lambda(0)}_{0}\underbrace{\left(e^{-0^2/2}\right)}_{1}\right]$$

$$-\int_0^\infty(\cos\lambda u)(\lambda)\left(e^{-u^2/2}\right)du \quad ...(6)$$

Using standard values in Equation (6)

$$e^{-\infty} = 0, \quad \sin 0 = 0$$

$$\therefore \frac{dI}{d\lambda} = (0 - 0) - \lambda\int_0^\infty e^{-u^2/2}\cdot\cos\lambda u\, du$$

$$\therefore \frac{dI}{d\lambda} = -\lambda\int_0^\infty e^{-u^2/2}\cdot\cos\lambda u\, du$$

But from Equation (2), $\int_0^\infty e^{-u^2/2}\cdot\cos\lambda u\, du = I(\lambda)$

$$\frac{dI}{d\lambda} = -\lambda I \quad \Rightarrow \quad \frac{dI}{I} = -\lambda d\lambda \quad ...(7)$$

➔ **Using standard formula**

$$...\left[\int\frac{dx}{x}dx = \log x \text{ and } \int x\,dx = \frac{x^2}{2}\right]$$

$$\therefore \log I = -\frac{\lambda^2}{2} + C \quad \Rightarrow I(\lambda) = e^{-\lambda^2/2 + C} = e^{-\frac{\lambda^2}{2}}\cdot e^C$$

Let $e^C = A$ say

$$\therefore I(\lambda) = A\, e^{-\lambda^2/2} \quad ...(8)$$

Step V : To find value of A

Put $\lambda = 0$ in Equation (8) we get,

$$I(0) = A\, e^0 \Rightarrow I(0) = A \quad ...(9)$$

To get value of A, we find value of integral $I(\lambda)$ at $\lambda = 0$

$$I(\lambda) = \int_0^\infty e^{-u^2/2}\cos\lambda u\, du \quad ...(10)$$

$$\text{Put } u^2/2 = t \Rightarrow u^2 = 2t \Rightarrow 2u\,du = 2\,dt \Rightarrow u\,du = dt \Rightarrow du = \frac{dt}{u}$$

$$\Rightarrow du = \frac{dt}{\sqrt{2t}}$$

$\therefore$ From Equation (10),

$$\therefore \quad I(0) = \int_0^\infty e^{-t}\frac{dt}{\sqrt{2}\sqrt{t}} = \frac{1}{\sqrt{2}}\int_0^\infty e^{-t}(t)^{-1/2}\,dt = \frac{1}{\sqrt{2}}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{\sqrt{2}}\sqrt{\pi} = \sqrt{\frac{\pi}{2}}\left[\text{Gamma function}\,\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$\therefore \quad I(0) = \sqrt{\frac{\pi}{2}}$$

From Equation(9), $\sqrt{\frac{\pi}{2}} = A$

Step VI :From Equation (8), $I(\lambda) = e^{-\frac{\lambda^2}{2}}\sqrt{\frac{\pi}{2}} = \sqrt{\frac{\pi}{2}}\,e^{-\frac{\lambda^2}{2}}$

$$\therefore \quad I(\lambda) = \sqrt{\frac{\pi}{2}}\,e^{-\frac{\lambda^2}{2}}$$

$$\text{i.e.} \quad \int_0^\infty e^{-\frac{u^2}{2}}\cos\lambda u\,du = \sqrt{\frac{\pi}{2}}\,e^{-\frac{\lambda^2}{2}}$$

Put this value in Equation and then by (1),

$$\therefore \quad F(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cos\lambda u\,du$$

$$= \sqrt{\frac{2}{\pi}}\sqrt{\frac{\pi}{2}}\,e^{-\frac{\lambda^2}{2}}$$

$$\therefore \quad F(\lambda) = e^{-\lambda^2/2} \checkmark \qquad \text{...Ans.}$$

Which is required.

Ex. 3.5.4 : Find the Fourier integral for the function.

$$f(x) = \begin{cases} 0 & x<0 \\ e^{-x} & x>0 \\ \frac{1}{2} & x=0 \end{cases}$$

Soln. :

Step I : Given function, $f(x) = \begin{cases} 0 & x<0 \\ e^{-x} & x>0 \\ \frac{1}{2} & x=0 \end{cases}$

(I) We observe that given function is neither even nor odd Therefore Fourier transform of f(x) is,

Step II : $$F(\lambda) = \int_{-\infty}^{\infty} f(u)\cdot e^{-i\lambda u}\,du$$

From definition of given function,we split given integral as

$$F(\lambda) = \int_{-\infty}^{0} f(u)\cdot e^{-i\lambda u}\,du + \int_0^\infty f(u)\cdot e^{-i\lambda u}\,du$$

But $f(x) = 0$ for $x<0$ i.e. $f(x) = 0$ for $-\infty < x < 0$ and

$f(x) = e^{-x}$ for $0 < x < \infty$

$$\therefore \quad F(\lambda) = 0 + \int_0^\infty e^{-u}e^{-i\lambda u}\,du = \int_0^\infty e^{-u-i\lambda u}\,du$$

$$= \int_0^\infty e^{-u(1+i\lambda)}\,du = \int_0^\infty e^{-(1+i\lambda)u}\,du \qquad \text{...(1)}$$

➔ **Using standard formula of integration**

$$\text{...}\left[\int e^{Ax}\,dx = \frac{e^{Ax}}{A} \text{ for any constant A}\right]$$

Here $A = -(1+i\lambda)$

$$\therefore \quad F(\lambda) = \left[\frac{e^{-(1+i\lambda)u}}{-(1+i\lambda)}\right]_0^\infty = \left[\frac{e^{-(1+i\lambda)\infty}}{-(1+i\lambda)} - \frac{e^{-(1+i\lambda)0}}{-(1+i\lambda)}\right]$$

$$= \left[\frac{e^{-\infty}}{-(1+i\lambda)} - \frac{e^0}{-(1+i\lambda)}\right] \qquad \text{...(2)}$$

Using standard values in Equation (2)

$e^{-\infty} = 0$ and $e^0 = 1$

$$F(\lambda) = \left[-\frac{0}{1+i\lambda} - \frac{1}{-(1+i\lambda)}\right]$$

$$\therefore \quad F(\lambda) = \left[0 + \frac{1}{1+i\lambda}\right] = \frac{1}{1+i\lambda} \qquad \text{...(3)}$$

Multiplying and dividing by $1-i\lambda$ on right hand side of Equation (3)

$$\therefore \quad F(\lambda) = \left(\frac{1}{1+i\lambda}\times\frac{1-i\lambda}{1-i\lambda}\right) \qquad \text{...(4)}$$

➔ **Using standard formula**

$$\text{...}[(x+iy)(x-iy) = x^2 + y^2$$

Here $x = 1$, $y = \lambda$]

$$\therefore \quad F(\lambda) = \frac{1-i\lambda}{1+\lambda^2}$$

This is Fourier transform of f(x)

Step III :

(I) The inverse Fourier transform of $F(\lambda)$ is given by ,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\cdot e^{i\lambda x}\,d\lambda$$

Using value of $F(\lambda)$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-i\lambda}{1+\lambda^2}\,e^{i\lambda x}\,d\lambda \qquad \text{...(5)}$$

➔ **Using standard formula**

$$\text{...}[e^{i\theta} = \cos\theta + i\sin\theta$$

Here $\theta = i\lambda x$]

$$E^{i\lambda x} = \cos\lambda x + i\sin\lambda x$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-i\lambda}{1+\lambda^2}\,[\cos\lambda x + i\sin\lambda x]\,d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{1+\lambda^2}(\cos\lambda x + i\sin\lambda x) - \frac{i\lambda}{1+\lambda^2}(\cos\lambda x + i\sin\lambda x)\right]d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{1+\lambda^2}(\cos\lambda x) + \frac{1}{1+\lambda^2}(i\sin\lambda x) - \frac{i\lambda}{1+\lambda^2}(\cos\lambda x) - \frac{i\lambda}{1+\lambda^2}i\sin\lambda x\right]d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{1+\lambda^2}\cos\lambda x + i\frac{1}{1+\lambda^2}\sin\lambda x - i\frac{\lambda}{1+\lambda^2}\cos\lambda x - \underbrace{i^2}_{-(-1)}\frac{\lambda}{1+\lambda^2}\sin\lambda x\right]d\lambda$$

But $i^2 = -1$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{1+\lambda^2}(\cos\lambda x) + i\frac{1}{1+\lambda^2}(\sin\lambda x) - i\frac{\lambda}{1+\lambda^2}(\cos\lambda x) + \frac{\lambda}{1+\lambda^2}\sin\lambda x\right]d\lambda$$

Collecting the real and imaginary parts

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{1+\lambda^2}\cos\lambda x + \frac{1}{1+\lambda^2}(\lambda\sin\lambda x) + i\frac{1}{1+\lambda^2}(\sin\lambda x) - i\frac{1}{1+\lambda^2}\lambda\cos\lambda x\right]d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2} + i\,\frac{\sin\lambda x - \lambda\cos\lambda x}{1+\lambda^2}\right]d\lambda \qquad \text{...(6)}$$

Step IV :

Using standard result in Equation(6)

(1) Product of two even functions is even function.
(2) Product of two odd functions is even function.
(3) Product of one even and one odd function is odd function.
(4) Quotient of two even functions is an even function.
(5) Quotient of two odd functions is an odd function.
(6) Quotient of one even function and one odd function is an odd function.

Here variable of integration is λ.

In first integral,

(i) $\cos\lambda x$ is even function,

(ii) λ is odd function , $\sin\lambda x$ is also odd function therefore $\lambda\sin\lambda x$ is an even function

Therefore $\cos\lambda x + \lambda\sin\lambda x$ is even function

(iii) $1+\lambda^2$ is even function

Therefore $\frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}$ is an even function

In second integral,

(i) $\sin\lambda x$ is odd function

(ii) λ is odd function, $\cos\lambda x$ is even function, therefore $\lambda\cos\lambda x$ is odd function

Therefore $\sin\lambda x - \lambda\cos\lambda x$ is odd function

(iii) $1+\lambda^2$ is even function

Therefore $\frac{\sin\lambda x - \lambda\cos\lambda x}{1+\lambda^2}$ is an odd function

➔ **Using Standard result of definite integral**

$$\text{...}\left[\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{for even function} \\ 0, & \text{for odd function} \end{cases}\right]$$

$$\therefore \int_{-\infty}^{\infty}\frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\,d\lambda = 2\int_0^\infty\frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\,d\lambda$$

$$\text{and} \int_{-\infty}^{\infty}\frac{\sin\lambda x - \lambda\cos\lambda x}{1+\lambda^2}\,d\lambda = 0$$

$$\therefore f(x) = \frac{1}{2\pi}\cdot 2\int_0^\infty\frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\,d\lambda$$

$$\therefore \quad f(x) = \frac{1}{\pi}\int_0^\infty\frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\,d\lambda \checkmark \qquad \text{...Ans.}$$

which is required Fourier integral representation.

Ex. 3.5.5 : If $f(x) = \begin{cases} \sin x & \text{when } 0<x<\pi \\ 0 & \text{when } x<0 \text{ or } x>\pi \end{cases}$

then prove that $f(x) = \frac{1}{\pi}\int_0^\infty\frac{\cos\lambda x + \cos[\lambda(\pi - x)]}{1-\lambda^2}\,d\lambda$

Hence deduce that $\int_0^\infty\frac{\cos\frac{\lambda\pi}{2}}{1-\lambda^2}\,d\lambda = \frac{\pi}{2}$

Soln. :

Step I : Given function,

$$f(x) = \begin{cases} \sin x & \text{when } 0<x<\pi \\ 0 & \text{when } x<0 \text{ or } x>\pi \end{cases}$$

(I) We observe that given function is neither even nor odd

Step II : Therefore Fourier transform of f(x) is,

$$F(\lambda) = \int_{-\infty}^{\infty} f(u)\cdot e^{-i\lambda u}\,du$$

From definition of given function, we split given integral as

$$F(\lambda) = \int_{-\infty}^{0} f(u)\cdot e^{-i\lambda u}\,du + \int_{0}^{\pi} f(u)\cdot e^{-i\lambda u}\,du + \int_{\pi}^{\infty} f(u)\cdot e^{-i\lambda u}\,du$$

But $f(x) = 0$ for $x < 0$ i.e. $f(x) = 0$ for $-\infty < x < 0$ and for $x > \pi$
$f(x) = \sin x$ for $0 < x < \pi$

$$\therefore F(\lambda) = 0 + \int_{0}^{\infty} \sin u\, e^{-i\lambda u}\,du + 0 = \int_{0}^{\pi} \sin u\, e^{-i\lambda u}\,du$$

$$F(\lambda) = \int_{0}^{\pi} e^{-i\lambda u}\sin u\,du \qquad \ldots(1)$$

Step III :

➔ **Using standard formula**

$$\ldots\left[\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx+c) - b\cos(bx+c)]\right]$$

Here $a = -i\lambda$, $b = 1$, $c = 0$

$$F(\lambda) = \left[\frac{e^{-i\lambda u}}{(-i\lambda)^2+(1)^2}(-i\lambda\sin u - \cos u)\right]_0^{\pi}$$

$$= \left[\frac{e^{-i\lambda u}}{(i^2\lambda^2)+(1)^2}(-i\lambda\sin u - \cos u)\right]_0^{\pi}$$

$$= \left[\frac{e^{-i\lambda\pi}}{(i^2\lambda^2)+(1)^2}(-i\lambda\underbrace{\sin\pi}_{0} - \underbrace{\cos\pi}_{(-1)})\right]$$

$$-\left[\frac{e^{-i\lambda 0}}{(i^2\lambda^2)+(1)^2}(-i\lambda\underbrace{\sin 0}_{0} - \underbrace{\cos 0}_{1})\right] \qquad \ldots(2)$$

➔ **Using standard values in Equation (2)**

$\ldots[\cos\pi = -1, \sin\pi = 0, \sin 0 = 0, \cos 0 = 1]$

$$F(\lambda) = \left[\frac{e^{-i\lambda\pi}}{(-\lambda^2)+1}\right](-i\lambda(0) - (-1))$$

$$-\left[\frac{e^{0}}{(-\lambda^2)+1}\right](-i\lambda(0) - (1))$$

$$= \left[\frac{e^{-i\lambda\pi}}{1-\lambda^2}\right](0+1) - \left[\frac{1}{1-\lambda^2}\right](0-1)$$

$$= \left[\frac{e^{-i\lambda\pi}}{1-\lambda^2}\right](1) - \left[\frac{1}{1-\lambda^2}\right](-1)$$

$$= \left[e^{-i\lambda\pi}\frac{1}{1-\lambda^2} + \frac{1}{1-\lambda^2}\right]$$

$$= \frac{1}{1-\lambda^2}[e^{-i\lambda\pi}+1] \qquad \ldots(3)$$

Using standard formula in Equation(3)

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Similarly $e^{-i\theta} = \cos\theta - i\sin\theta$

Here $\theta = i\lambda\pi$

$$e^{-i\lambda\pi} = \cos\lambda\pi - i\sin\lambda\pi$$

$$F(\lambda) = \frac{1}{1-\lambda^2}[(\cos\lambda\pi - i\sin\lambda\pi) + 1]$$

$$= \frac{1}{1-\lambda^2}[(\cos\lambda\pi + 1) - i\sin\lambda\pi]$$

$$F(\lambda) = \frac{(1+\cos\lambda\pi) - i\sin\lambda\pi}{(1-\lambda^2)} \qquad \ldots(4)$$

This is Fourier transform of f(x)

Step IV :

(II) The inverse Fourier transform of $F(\lambda)$ is given by ,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\cdot e^{i\lambda x}\,d\lambda$$

Using value of $F(\lambda)$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{(1+\cos\lambda\pi) - i\sin\lambda\pi}{(1-\lambda^2)}\right](\cos\lambda x + i\sin\lambda x)\,d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{(1+\cos\lambda\pi)(\cos\lambda x) - (i\sin\lambda\pi)(\cos\lambda x) + (1+\cos\lambda\pi)(i\sin\lambda x) + (-i\sin\lambda\pi)(i\sin\lambda x)}{(1-\lambda^2)}\right]d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{(1+\cos\lambda\pi)(\cos\lambda x) - i\sin\lambda\pi\cos\lambda x + i(1+\cos\lambda\pi)(\sin\lambda x) + (-i^2\sin\lambda\pi)(\sin\lambda x)}{(1-\lambda^2)}\right]d\lambda$$

Since $-i^2 = -(-1) = 1$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{(1+\cos\lambda\pi)(\cos\lambda x) + i[(-\sin\lambda\pi\cos\lambda x) + (1+\cos\lambda\pi)(\sin\lambda x)] + (\sin\lambda\pi)(\sin\lambda x)}{(1-\lambda^2)}\right]d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{(1+\cos\lambda\pi)(\cos\lambda x) + (\sin\lambda\pi)(\sin\lambda x)}{(1-\lambda^2)}\right]d\lambda$$

$$+ i\int_{-\infty}^{\infty}\left[\frac{[(-\sin\lambda\pi\cos\lambda x) + (1+\cos\lambda\pi)(\sin\lambda x)]}{(1-\lambda^2)}\right]d\lambda \qquad \ldots(5)$$

Step V : Using standard result in Equation(5)

(1) Product of two even functions is even function.
(2) Product of two odd functions is even function.
(3) Product of one even and one odd function is odd function.
(4) Quotient of two even functions is an even function
(5) Quotient of two odd functions is an odd function
(6) Quotient of one even function and one odd function is an odd function.

Here variable of integration is λ.
In first integral,

(i) $\cos\lambda x$ and $(1+\cos\lambda\pi)$ are even functions, therefore $(1+\cos\lambda\pi)(\cos\lambda x)$ is an even function

(ii) $\sin\lambda\pi$ and $\sin\lambda x$ are odd functions, therefore $(\sin\lambda\pi)(\sin\lambda x)$ is an even function
Therefore $(1+\cos\lambda\pi)(\cos\lambda x) + (\sin\lambda\pi)(\sin\lambda x)$ is an even function

(iii) $1-\lambda^2$ is even function
Therefore $\dfrac{(1+\cos\lambda\pi)(\cos\lambda x) + (\sin\lambda\pi)(\sin\lambda x)}{1-\lambda^2}$ is an even function

In second integral,

(i) $\sin\lambda\pi$ is an odd function, $\cos\lambda x$ is an even functions, therefore $(\sin\lambda\pi)(\cos\lambda x)$ is an odd function

(ii) $(1+\cos\lambda\pi)$ is an even function, $\sin\lambda x$ is odd function, therefore $(1+\cos\lambda\pi)(\sin\lambda x)$ is an odd function
Therefore $(\sin\lambda\pi)(\cos\lambda x) + (1+\cos\lambda\pi)(\sin\lambda x)$ is odd function

(iii) $1-\lambda^2$ is even function
Therefore $\dfrac{(-\sin\lambda\pi)(\cos\lambda x) + (1+\cos\lambda\pi)(\sin\lambda x)}{1-\lambda^2}$ is an odd function

Standard result of definite integral

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx & \text{for even function} \\ 0 & \text{for odd function} \end{cases}$$

$$\therefore \int_{-\infty}^{\infty}\frac{(1+\cos\lambda\pi)(\cos\lambda x) + (\sin\lambda\pi)(\sin\lambda x)}{1-\lambda^2}\,d\lambda$$

$$= 2\int_{0}^{\infty}\frac{(1+\cos\lambda\pi)(\cos\lambda x) + (\sin\lambda\pi)(\sin\lambda x)}{1-\lambda^2}\,d\lambda$$

And $\displaystyle\int_{-\infty}^{\infty}\frac{(-\sin\lambda\pi)(\cos\lambda x) + (1+\cos\lambda\pi)(\sin\lambda x)}{1-\lambda^2}\,d\lambda = 0$

$$\therefore f(x) = \frac{1}{2\pi}\left[2\int_{0}^{\infty}\frac{\cos\lambda x(1+\cos\lambda\pi) + \sin\lambda\pi\sin\lambda x}{(1-\lambda^2)}\,d\lambda + 0\right]$$

$$= \underbrace{\frac{2}{2\pi}}_{\frac{1}{\pi}}\int_{0}^{\infty}\frac{\cos\lambda x(1+\cos\lambda x) + \sin\lambda\pi\sin\lambda x}{1-\lambda^2}\,d\lambda$$

$$\therefore f(x) = \frac{1}{\pi}\int_{0}^{\infty}\frac{\cos\lambda x + \cos\lambda x\cdot\cos\lambda\pi + \sin\lambda\pi\cdot\sin\lambda x}{(1-\lambda^2)}\,d\lambda \qquad \ldots(6)$$

➔ **Using standard formula**

$\ldots[\cos A\cos B + \sin A\sin B = \cos(A-B)]$

Here $A = \lambda x$, $B = \lambda\pi$

$\therefore \cos\lambda x\cos\lambda\pi + \sin\lambda x\sin\lambda\pi = \cos(\lambda x - \lambda\pi)$
$= \cos\lambda(x-\pi) = \cos\lambda(\pi - x)$

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\frac{\cos\lambda x + \cos(\pi - x)\lambda}{(1-\lambda^2)}\,d\lambda \quad \textbf{Hence proved.}$$

Step VI :

(III) Put $x = \dfrac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = \frac{1}{\pi}\int_{0}^{\infty}\frac{\cos\frac{\lambda\pi}{2} + \cos\left(\pi - \frac{\pi}{2}\right)\lambda}{(1-\lambda^2)}\,d\lambda$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\frac{\cos\frac{\lambda\pi}{2} + \cos\frac{\pi}{2}\lambda}{(1-\lambda^2)}\,d\lambda = \frac{1}{\pi}\int_{0}^{\infty}\frac{2\cos\frac{\lambda\pi}{2}}{(1-\lambda^2)}\,d\lambda$$

$$f\left(\frac{\pi}{2}\right) = \frac{2}{\pi}\int_{0}^{\infty}\frac{\cos\frac{\lambda\pi}{2}}{1-\lambda^2}\,d\lambda$$

Step VII :

To find $f\left(\frac{\pi}{2}\right)$, we find value of given function at $x = \frac{\pi}{2}$

Since from definition of given function $f(x) = \sin x$ when $0 < x < \pi$ and $x = \frac{\pi}{2}$ lies in $(0, \pi)$

Therefore $f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

$$\therefore 1 = \frac{2}{\pi}\int_{0}^{\infty}\frac{\cos\frac{\lambda\pi}{2}}{1-\lambda^2}\,d\lambda$$

$$\therefore \int_{0}^{\infty}\frac{\cos\frac{\lambda\pi}{2}}{1-\lambda^2}\,d\lambda = \frac{\pi}{2}\ \checkmark \qquad \textbf{...Hence proved.}$$

Ex. 3.5.6 : Find the Fourier transform of

$$f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ 1, & 0 \le x \le 2 \\ 0, & x < -2 \text{ and } x > 2 \end{cases}$$

Hence show that $\displaystyle\int_{0}^{\infty}\frac{(\cos 2x - 1)\sin 2x}{x}\,dx = \frac{-\pi}{2}$

Soln. :

Step I :

Given function,

$$f(x) = \begin{cases} 1, & -2 \le x \le 0 \\ -1, & 0 \le x \le 2 \\ 0, & x < -2 \text{ and } x > 2 \end{cases}$$

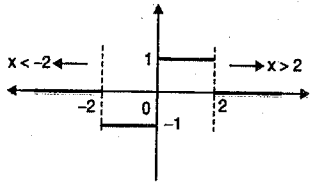


Fig. P. 3.5.6

The graph of function is as shown in the figure.Since graph is symmetric in opposite quadrants,given function is odd function

Step II :

(I) The fourier transform of an odd function is given by ,

$$F(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du$$

From definition of given function,we split given integral as

$$F(\lambda) = \int_0^2 f(u) \sin \lambda u \, du + \int_2^\infty f(u) \sin \lambda u \, du$$

But $f(x) = 0$ for $x > 2$ and $f(x) = 1$ for $0 < x < 2$

$\therefore$ The fourier transform is,

$$F(\lambda) = \int_0^2 (1) \sin \lambda u \, du + 0 = \int_0^2 \sin \lambda u \, du \qquad ...(1)$$

➔ **Using standard formula**

$$...\left[\int \sin Ax \, dx = \frac{-\cos Ax}{x}\right.$$

$$\left.\text{Here } x = u, A = \lambda\right]$$

$$F(\lambda) = \left[\frac{-\cos \lambda u}{\lambda}\right]_0^2$$

$$= \left[\frac{-\cos \lambda(2)}{\lambda}\right] - \left[\frac{-\cos \lambda(0)}{\lambda}\right] = -\frac{\cos 2\lambda}{\lambda} - \left[\frac{-1}{\lambda}\right]$$

$$= \left[\frac{-\cos 2\lambda}{\lambda}\right] - \left[\frac{-\cos 0}{\lambda}\right] \qquad ...(2)$$

$[\because \cos 0 = 1]$

Using standard value in Equation (2)

Cos 0 = 1

$$F(\lambda) = -\frac{\cos 2\lambda}{\lambda} - \left[\frac{-1}{\lambda}\right]$$

$$F(\lambda) = \frac{-\cos 2\lambda}{\lambda} + \frac{1}{\lambda}$$

$$F(\lambda) = \frac{1 - \cos 2\lambda}{\lambda}$$

This is fourier transform of f(x),

Step III :

(II) Now, inverse Fourier transformof F(λ) is,

$$f(x) = \frac{2}{\pi} \int_0^\infty F(\lambda) \sin \lambda x \, d\lambda$$

Using value of F(λ)

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos 2\lambda}{\lambda} \sin \lambda x \, d\lambda$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{-(\cos 2\lambda - 1)}{\lambda} \sin \lambda x \, d\lambda$$

$$= -\frac{2}{\pi} \int_0^\infty \frac{\cos 2\lambda - 1}{\lambda} \sin \lambda x \, d\lambda \qquad ...(3)$$

This is Fourier integral representation of f(x)

Step IV:

(III) To find the value of integral $\int_0^\infty \frac{(\cos 2x - 1)}{x} \sin 2x \, dx$

first we find the value of same integral in terms of λ, by using proper value of, i.e. we find the value of integral $\int_0^\infty \frac{(\cos 2\lambda - 1)}{\lambda} \sin 2\lambda \, d\lambda$, ...(4)

Comparing Equations (3) and (4) , we observe that integral in Equation (4) is obtained by putting $x = 2$ in Equation (3)

Step IV :

Put $x = 2$ in Equation (3),

$$f(2) = -\frac{2}{\pi} \int_0^\infty \frac{\cos 2\lambda - 1}{\lambda} \sin \lambda(2) \, d\lambda \qquad ...(5)$$

To find f(2) , we find value of given function at $x = 2$

Since from definition of given function $f(x) = 1$ when $0 \le x \le 2$ and $x = 2$ lies in $[0,2]$

Therefore $f(2) = 1$

Step V :

$\therefore$ From Equation (5),

$$\therefore 1 = -\frac{2}{\pi} \int_0^\infty \frac{\cos 2\lambda - 1}{\lambda} \sin 2\lambda \, d\lambda$$

$$\therefore \int_0^\infty \frac{(\cos 2\lambda - 1)}{\lambda} \sin 2\lambda \, d\lambda = \frac{-\pi}{2}$$

As value of definite integral is independent on variable, changing the variable λ to x

$$\therefore \int_0^\infty \frac{(\cos 2x - 1)}{x} \sin 2x \, dx = \frac{-\pi}{2} \checkmark \qquad \text{...Hence proved.}$$

Ex. 3.5.7 : Find the Fourier transform of

$$f(x) = \begin{cases} -3 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

Hence evaluate $\int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda$. Deduce the value of $\int_0^\infty \frac{\sin \lambda}{\lambda} d\lambda$

Soln. :

Step I :

Given function can be written as, $f(x) = \begin{cases} -3, & -1 < x < 1 \\ 0, & x > 1 \text{ and } x < -1 \end{cases}$

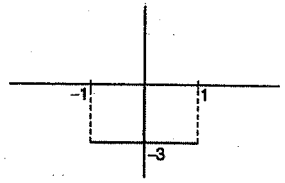


Fig. P. 3.5.7

The graph of function is as shown in the figure.Since graph is symmetric about Y-axis

Therefore given function is an even function

Step II :

(I) The fourier transform of an even function is given by ,

$$F(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du \qquad ...(1)$$

From definition of given function,we split given integral as

$$F(\lambda) = \int_0^1 f(u) \cos \lambda u \, du + \int_1^\infty f(u) \cos \lambda u \, du$$

Since $f(u) = -3$ when $-1 < x < 1$ i.e. when $0 < x < 1$ and $f(u) = 0$ when $x > 1$ i.e. when $1 < x < \infty$

$$F(\lambda) = \int_0^1 (-3) \cos \lambda u \, du + \int_1^\infty (0) \cos \lambda u \, du$$

$$= (-3) \int_0^1 \cos \lambda u \, du + 0 = (-3) \int_0^1 \cos \lambda u \, du \quad ...(2)$$

➔ **Using standard formula of integration**

$$...\left[\int \cos Ax \, dx = \frac{\sin Ax}{A} \text{ for any constant A}\right.$$

$$\left.\text{Here } A = \lambda \text{ and } x = u\right]$$

$$F(\lambda) = (-3)\left[\frac{\sin \lambda u}{\lambda}\right]_0^1 = (-3)\left[\frac{\sin \lambda u}{\lambda}\right]_{at\, u=1} - \left[\frac{\sin \lambda u}{\lambda}\right]_{at\, u=0}$$

$$= (-3)\left[\frac{\sin \lambda(1)}{\lambda}\right] - \left[\frac{\sin \lambda(0)}{\lambda}\right] = (-3)\left[\frac{\sin \lambda}{\lambda}\right] - \left[\frac{\sin(0)}{\lambda}\right] \quad ...(3)$$

Using standard value sin0 = 0 in Equation (3)

$$F(\lambda) = (-3)\frac{\sin \lambda}{\lambda} - 0 = -\frac{3 \sin \lambda}{\lambda}$$

$$F(\lambda) = -\frac{3 \sin \lambda}{\lambda}$$

Step III :

(II) The inverse Fourier transform of F(λ) is given by ,

$$f(x) = \frac{2}{\pi} \int_0^\infty F(\lambda) \cos \lambda x \, d\lambda \qquad ...(4)$$

Using value of F(λ) from Equation (4),

$$f(x) = \frac{2}{\pi} \int_0^\infty -\frac{3 \sin \lambda}{\lambda} \cos \lambda x \, d\lambda$$

$$f(x) = \frac{2}{\pi}(-3) \int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda$$

$$= -\frac{6}{\pi} \int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda \qquad ...(5)$$

Which is required Fourier integral representation.

$$\therefore \int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda = \frac{-\pi}{6} f(x)$$

$$\therefore \int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda = -\left(\frac{\pi}{6}\right)\begin{cases}-3, & -1<x<1\\ 0, & x>1 \text{ and } x<-1\end{cases}$$

$$= -\left(\frac{\pi}{6}\right)(-3)\begin{cases}1, & -1<x<1\\ 0, & x>1 \text{ and } x<-1\end{cases}$$

$$= \left(\frac{\pi}{2}\right)\begin{cases}1, & -1<x<1\\ 0, & x>1 \text{ and } x<-1\end{cases}$$

$$= \begin{cases}\frac{\pi}{2}, & -1<x<1\\ 0, & x>1 \text{ and } x<-1\end{cases} \quad \ldots(6)$$

Step IV :

(III) Now we want to find the value of $\int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda$

Put, $x=0$ on both sides of Equation (6) we get,

$$\int_0^\infty \frac{\sin\lambda\cos\lambda(0)}{\lambda}\,d\lambda = \begin{cases}\frac{\pi}{2} & -1<x<1.\\ 0 & x>1 \text{ and } x<-1\end{cases} \text{ at } x=0 \quad \ldots(7)$$

On RHS of Equation (7), the value of function is $\frac{\pi}{2}$ when $-1<x<1$ and we want value of f(x) at $x=0$.

$x=0$ lies in the range $(-1, 1)$. Therefore $f(x)=\frac{\pi}{2}$ on RHS

$$\therefore \int_0^\infty \frac{\sin\lambda(1)}{\lambda}\,d\lambda = \frac{\pi}{2}$$

$$\therefore \int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda = \frac{\pi}{2} \checkmark \quad \ldots\text{Ans.}$$

3.5.4 Type - II Examples on Fourier Sine/Cosine Transform

Ex. 3.5.8 : Find the Fourier cosine transform of

$$f(x) = \begin{cases}x, & 0<x<1/2\\ 1-x, & 1/2<x<1\\ 0, & x>1\end{cases}$$

Soln. :

Step I :

Given function is, $f(x) = \begin{cases}x, & 0<x<1/2\\ 1-x, & 1/2<x<1\\ 0, & x>1\end{cases}$

We have by formula of Fourier cosine transform (FCT),

Step II :

$$F_C(\lambda) = \int_0^\infty f(u)\cos\lambda u\,du$$

By the definition of given function, we split the integral as,

$$F_C(\lambda) = \int_0^{1/2} f(u)\cos\lambda u\,du + \int_{\frac{1}{2}}^{1} f(u)\cos\lambda u\,du + \int_1^\infty f(u)\cos\lambda u\,du$$

$$F_C(\lambda) = \int_0^{1/2} u\cos\lambda u\,du + \int_{\frac{1}{2}}^{1} (1-u)\cos\lambda u\,du + \int_1^\infty (0)\cos\lambda u\,du \quad \ldots(1)$$

➔ **Using formula of generalized rule of integration in Equation (1)**

$$\ldots\left[\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right]$$

Where u is algebraic polynomial

v is exponential / sine / cosine term

Here dash denotes order of derivative and suffix denotes integration.

Here in first integral $u=u$, $v=\cos\lambda u$

$u'=1, u''=0$

In second integral $u=1-u$, $v=\cos\lambda u$

$u' = -1, u''=0$

$v_1 = \frac{\sin\lambda u}{\lambda}, v_2 = -\frac{\cos\lambda u}{\lambda^2}, v_3 = -\frac{\sin\lambda u}{\lambda^3}$ And so on

Step III :

$$F_C(\lambda) = \left\{(u)\left(\frac{\sin\lambda u}{\lambda}\right) - (1)\left(\frac{-\cos\lambda u}{\lambda^2}\right)\right\}_0^{1/2} + \left\{(1-u)\left(\frac{\sin\lambda u}{\lambda}\right) - (-1)\left(\frac{-\cos\lambda u}{\lambda^2}\right)\right\}_{1/2}^{1}$$

$$= \left\{\frac{1}{\lambda}(u\sin\lambda u) + \left(\frac{1}{\lambda^2}\right)(\cos\lambda u)\right\}_0^{1/2} + \left\{\frac{1}{\lambda}\left((1-u)\sin\lambda u - \frac{1}{\lambda^2}(\cos\lambda u)\right)\right\}_{1/2}^{1}$$

$$= \left\{\frac{1}{\lambda}\left(\frac{1}{2}\sin\lambda\frac{1}{2}\right) + \left(\frac{1}{\lambda^2}\right)\left(\cos\lambda\frac{1}{2}\right)\right\} - \left\{\frac{1}{\lambda}\underbrace{(0)\sin\lambda(0)}_{0} + \left(\frac{1}{\lambda^2}\right)\underbrace{(\cos\lambda(0))}_{1}\right\}$$

$$+\left\{\frac{1}{\lambda}\underbrace{((1-1)\sin\lambda(1))}_{0} - \left(\frac{1}{\lambda^2}\right)(\cos\lambda(1))\right\} - \left\{\frac{1}{\lambda}\left(\underbrace{(1-\tfrac{1}{2})}_{\frac{1}{2}}\sin\lambda\tfrac{1}{2}\right) - \left(\frac{1}{\lambda^2}\right)\left(\cos\lambda\tfrac{1}{2}\right)\right\}$$

$$= \left\{\frac{1}{2\lambda}\sin\frac{\lambda}{2} + \left(\frac{1}{\lambda^2}\left(\cos\frac{\lambda}{2}\right)\right)\right\} - \left\{\underbrace{\frac{1}{\lambda}(0)(0)}_{0} + \left(\frac{1}{\lambda^2}\right)(1)\right\}$$

$$+\left\{\frac{1}{\lambda}\left((0)\sin\lambda - \left(\frac{1}{\lambda^2}\right)(\cos\lambda)\right)\right\} - \left\{\frac{1}{\lambda}\left(\frac{1}{2}\sin\frac{\lambda}{2}\right) - \left(\frac{1}{\lambda^2}\right)\left(\cos\frac{\lambda}{2}\right)\right\}$$

$$= \frac{\sin\frac{\lambda}{2}}{2\lambda} + \frac{\cos\frac{\lambda}{2}}{\lambda^2} - 0 - \frac{1}{\lambda^2} + 0 - \frac{\cos\lambda}{\lambda^2} - \frac{\sin\frac{\lambda}{2}}{2\lambda} + \frac{\cos\frac{\lambda}{2}}{\lambda^2}$$

$$= \underbrace{\left(\frac{\sin\frac{\lambda}{2}}{2\lambda} - \frac{\sin\frac{\lambda}{2}}{2\lambda}\right)}_{0} + \underbrace{\left(\frac{\cos\frac{\lambda}{2}}{\lambda^2} + \frac{\cos\frac{\lambda}{2}}{\lambda^2}\right)}_{\frac{2\cos 2\lambda}{\lambda^2}} - \frac{\cos\lambda}{\lambda^2} - \frac{1}{\lambda^2}$$

$$= 2\,\frac{\cos\lambda/2}{\lambda^2} - \frac{\cos\lambda}{\lambda^2} - \frac{1}{\lambda^2}$$

$$= -\left[\frac{\cos\lambda + (1-2\cos\lambda/2)}{\lambda^2}\right]$$

$$F_C(\lambda) = -\left[\frac{\cos\lambda + (1-2\cos\lambda/2)}{\lambda^2}\right]$$

Ex. 3.5.9 : Find the Fourier sine and cosine transform of $f(x) = \begin{cases}x, & 0\le x\le 1\\ 2-x, & 1\le x\le 2\\ 0, & x>2\end{cases}$

Soln. :

Step I :

Fourier Sine Transform

Given function is, $f(x) = \begin{cases}x, & 0\le x\le 1\\ 2-x, & 1\le x\le 2\\ 0, & x>2\end{cases}$

Step II :

We have by formula of Fourier sine transform (FST),

$$F_S(\lambda) = \int_0^\infty f(u)\sin\lambda u\,du$$

By the definition of given function, we split the integral as,

$$F_S(\lambda) = \int_0^1 f(u)\sin\lambda u\,du + \int_1^2 f(u)\sin\lambda u\,du + \int_2^\infty f(u)\sin\lambda u\,du$$

$$= \int_0^1 u\sin\lambda u\,du + \int_1^2 (2-u)\sin\lambda u\,du + \int_2^\infty (0)\sin\lambda u\,du \quad \ldots(1)$$

Step III :

➔ **Using formula of generalized rule of integration :** $\ldots\left[\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right]$.

Where u is algebraic polynomial

v is exponential / sine / cosine term

Here dash denotes order of derivative and suffix denotes integration.

Here in first integral $u=u$, $v=\sin\lambda u$

$u'=1, u''=0$

In second integral $u=2-u$, $v=\sin\lambda u$

$u'=-1, u''=0$

$v_1 = -\frac{\cos\lambda u}{\lambda}, v_2 = -\frac{\sin\lambda u}{\lambda^2}, v_3 = \frac{\cos\lambda u}{\lambda^3}$ And so on

$$F_S(\lambda) = \left\{(u)\left(\frac{-\cos\lambda u}{\lambda}\right)-(1)\left(\frac{-\sin\lambda u}{\lambda^2}\right)\right\}_0^1+\left\{(2-u)\left(\frac{-\cos\lambda u}{\lambda}\right)-(-1)\left(\frac{-\sin\lambda u}{\lambda^2}\right)\right\}_1^2$$

$$= \left\{\left(-\frac{1}{\lambda}(u\cos\lambda u)\right)+\left(\frac{\sin\lambda u}{\lambda^2}\right)\right\}_0^1+\left\{-\frac{1}{\lambda}(2-u)(\cos\lambda u)-\left(\frac{\sin\lambda u}{\lambda^2}\right)\right\}_1^2$$

$$= \left\{\left(-\frac{1}{\lambda}(1\cos\lambda(1))\right)+\left(\frac{\sin\lambda(1)}{\lambda^2}\right)\right\}-\left\{\left(-\frac{1}{\lambda}(\underbrace{(0)\cos\lambda(0)}_{\cos 0=1})\right)+\left(\frac{\sin\lambda(0)}{\lambda^2}\right)\right\}$$

$$+\left\{-\frac{1}{\lambda}\underbrace{(2-2)}_{0}(\cos\lambda(2))-\left(\frac{\sin\lambda(2)}{\lambda^2}\right)\right\}-\left\{-\frac{1}{\lambda}\underbrace{(2-1)}_{1}(\cos\lambda(1))-\left(\frac{\sin\lambda(1)}{\lambda^2}\right)\right\}$$

$$= \left\{\left(-\frac{1}{\lambda}(\cos\lambda)\right)+\left(\frac{\sin\lambda}{\lambda^2}\right)\right\}-\left\{\left(-\frac{1}{\lambda}(\underbrace{(0)(1)}_{0})\right)+\left(\frac{\underbrace{(0)}_{0}}{\lambda^2}\right)\right\}+\left\{-\frac{1}{\lambda}\underbrace{(0)(\cos 2\lambda)}_{0}-\left(\frac{\sin 2\lambda}{\lambda^2}\right)\right\}-\left\{-\frac{1}{\lambda}(1)(\cos\lambda)-\left(\frac{\sin\lambda}{\lambda^2}\right)\right\}$$

$$= \left\{\left(-\frac{1}{\lambda}\cos\lambda\right)+\left(\frac{\sin\lambda}{\lambda^2}\right)-((0)+0)\right\}+\left\{-(0)-\left(\frac{\sin 2\lambda}{\lambda^2}\right)\right\}-\left\{-\frac{1}{\lambda}(\cos\lambda)-\left(\frac{\sin\lambda}{\lambda^2}\right)\right\}$$

$$= -\frac{\cos\lambda}{\lambda}+\frac{\sin\lambda}{\lambda^2}-\frac{\sin 2\lambda}{\lambda^2}+\frac{\cos\lambda}{\lambda}+\frac{\sin\lambda}{\lambda^2}$$

$$= \underbrace{\left(-\frac{\cos\lambda}{\lambda}+\frac{\cos\lambda}{\lambda}\right)}_{0}+\left(\frac{\sin\lambda}{\lambda^2}+\frac{\sin\lambda}{\lambda^2}\right)-\frac{\sin 2\lambda}{\lambda^2}$$

$$= (0)+\left(2\frac{\sin\lambda}{\lambda^2}\right)-\frac{\sin 2\lambda}{\lambda^2}$$

$$= \frac{2\sin\lambda-\sin 2\lambda}{\lambda^2}$$

$$F_S(\lambda) = \frac{2\sin\lambda-\sin 2\lambda}{\lambda^2}$$

Step IV :

Fourier Cosine Transform

We have by formula of Fourier cosine transform (FCT),

$$F_C(\lambda) = \int_0^\infty f(u)\cos\lambda u\,du$$

By the definition of given function, we split the integral as,

$$F_C(\lambda) = \int_0^1 f(u)\cos\lambda u\,du+\int_1^2 f(u)\cos\lambda u\,du+\int_2^\infty f(u)\cos\lambda u\,du$$

$$= \int_0^1 u\cos\lambda u\,du+\int_1^2 (2-u)\cos\lambda u\,du+\int_2^\infty (0)\cos\lambda u\,du \qquad \text{...(2)}$$

➔ **Using formula of generalized rule of integration :** $\ldots\left[\int uv\,dx = uv_1-u'v_2+u''v_3-u'''v_4+\ldots\right]$

Where u is algebraic polynomial

v is exponential / sine / cosine term

Here dash denotes order of derivative and suffix denotes integration.

Here in first integral $u = u$, $v = \sin\lambda u$

$u' = 1$, $u'' = 0$

In second integral $u = 2-u$, $v = \cos\lambda u$

$u' = -1$, $u'' = 0$

$v_1 = \frac{\sin\lambda u}{\lambda}$, $v_2 = -\frac{\cos\lambda u}{\lambda^2}$, $v_3 = -\frac{\sin\lambda u}{\lambda^3}$ And so on

Step V:

$$F_C(\lambda) = \left\{(u)\left(\frac{\sin\lambda u}{\lambda}\right)-(1)\left(\frac{-\cos\lambda u}{\lambda^2}\right)\right\}_0^1+\left\{(2-u)\left(\frac{\sin\lambda u}{\lambda}\right)-(-1)\left(\frac{-\cos\lambda u}{\lambda^2}\right)\right\}_1^2$$

$$= \left\{\left(\frac{1}{\lambda}(u\sin\lambda u)\right)+\left(\frac{\cos\lambda u}{\lambda^2}\right)\right\}_0^1+\left\{\frac{1}{\lambda}(2-u)(\sin\lambda u)-\left(\frac{\cos\lambda u}{\lambda^2}\right)\right\}_1^2$$

$$= \left\{\left(\frac{1}{\lambda}(1\sin\lambda(1))\right)+\left(\frac{\cos\lambda(1)}{\lambda^2}\right)\right\}-\left\{\left(\frac{1}{\lambda}(\underbrace{(0)\sin\lambda(0)}_{\sin=0})\right)+\left(\frac{\cos\lambda(0)}{\lambda^2}\right)\right\}$$

$$+\left\{\frac{1}{\lambda}\underbrace{(2-2)}_{0}(\sin\lambda(2))-\left(\frac{\cos\lambda(2)}{\lambda 2}\right)\right\}-\left\{\frac{1}{\lambda}\underbrace{(2-1)}_{1}(\sin\lambda(1))-\left(\frac{\cos\lambda(1)}{\lambda 2}\right)\right\}$$

$$= \left\{\left(\frac{1}{\lambda}(\sin\lambda)\right)+\left(\frac{\cos\lambda}{\lambda^2}\right)\right\}-\left\{\left(\frac{1}{\lambda}(\underbrace{(0)(0)}_{0})\right)+\left(\frac{1}{\lambda^2}\right)\right\}+\left\{\frac{1}{\lambda}(0)(\sin 2\lambda)-\left(\frac{\cos 2\lambda}{\lambda^2}\right)\right\}-\left\{\frac{1}{\lambda}(1)(\sin\lambda)-\left(\frac{\cos\lambda}{\lambda^2}\right)\right\}$$

$$= \left\{\left(\frac{1}{\lambda}\sin\lambda\right)+\left(\frac{\cos\lambda}{\lambda^2}\right)-\left((0)+\frac{1}{\lambda^2}\right)\right\}+\left\{(0)-\left(\frac{\cos 2\lambda}{\lambda^2}\right)\right\}-\left\{\frac{1}{\lambda}(\sin\lambda)-\left(\frac{\cos\lambda}{\lambda^2}\right)\right\}$$

$$= \frac{\sin\lambda}{\lambda}+\frac{\cos\lambda}{\lambda^2}-0-\frac{1}{\lambda^2}+0-\frac{\cos 2\lambda}{\lambda^2}-\frac{\sin\lambda}{\lambda}+\frac{\cos\lambda}{\lambda^2}$$

$$= \left(\frac{\sin\lambda}{\lambda}-\frac{\sin\lambda}{\lambda}\right)+\left(\frac{\cos\lambda}{\lambda^2}+\frac{\cos\lambda}{\lambda^2}\right)-\frac{\cos 2\lambda}{\lambda^2}-\frac{1}{\lambda^2}$$

$$= (0)+\left(2\frac{\cos\lambda}{\lambda^2}\right)-\frac{\cos 2\lambda}{\lambda^2}-\frac{1}{\lambda^2}=\frac{2\cos\lambda-\cos 2\lambda-1}{\lambda^2}$$

$$F_C(\lambda) = \frac{2\cos\lambda-(1+\cos 2\lambda)}{\lambda^2} \checkmark \qquad \text{...Ans.}$$

Ex. 3.5.10 : Find the Fourier cosine transform of $f(x) = \begin{cases}1-x^2, & |x|\le 1\\ 0, & |x|>1\end{cases}$; Hence evaluate $\int_0^\infty\left(\frac{x\cos x-\sin x}{x^3}\right)\cos x\,dx$

Soln. :

Step I :

Given function can be written as,

$$f(x) = \begin{cases}1-x^2, & -1\le x\le 1\\ 0, & x>1 \text{ and } x<-1\end{cases}$$

Step II :

(I) The Fourier Cosine transform (FCT) is,

$$F_C(\lambda) = \int_0^\infty f(u)\cos\lambda u\,du \qquad \text{...(1)}$$

By the definition of given function,we split the integral as,

$$F_C(\lambda) = \int_0^1 f(u) \cdot \cos\lambda u\, du + \int_1^\infty f(u) \cdot \cos\lambda u\, du$$

$$= \int_0^1 (1-u^2)\cos\lambda u\, du + \int_1^\infty (0)\cos\lambda u\, du = \int_0^1 (1-u^2)\cos\lambda u\, du + 0$$

$$= \int_0^1 (1-u^2)\cos\lambda u\, du \qquad \ldots(2)$$

Step III :

➔ **Using formula of generalized rule of integration :** $\ldots\left[\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right]$

Where u is algebraic polynomial

v is exponential / sine / cosine term

Here dash denotes order of derivative and suffix denotes integration.

Here $u = 1-u^2$, $v = \cos\lambda u$

$u' = -2u,\ u'' = -2,\ u''' = 0$

$v_1 = \dfrac{\sin\lambda u}{\lambda},\ v_2 = -\dfrac{\cos\lambda u}{\lambda^2},\ v_3 = -\dfrac{\sin\lambda u}{\lambda^3}$ And so on

Step IV :

$$F(\lambda) = \left[(1-u^2)\left(\frac{\sin\lambda u}{\lambda}\right) - (-2u)\left(\frac{-\cos\lambda u}{\lambda^2}\right) + (-2)\left(\frac{-\sin\lambda u}{\lambda^3}\right) + (0)\left(\frac{\cos\lambda u}{\lambda^4}\right)\right]_0^1$$

$$= \left[(1-(1)^2)\left(\frac{\sin\lambda(1)}{\lambda}\right) - (-2(1))\left(\frac{-\cos\lambda(1)}{\lambda^2}\right) + (-2)\left(\frac{-\sin\lambda(1)}{\lambda^3}\right) + (0)\left(\frac{\cos\lambda(1)}{\lambda^4}\right)\right]$$
$$-\left[(1-(0)^2)\left(\frac{\sin\lambda(0)}{\lambda}\right) - (-2(0))\left(\frac{-\cos\lambda(0)}{\lambda^2}\right) + (-2)\left(\frac{-\sin\lambda(0)}{\lambda^3}\right) + (0)\left(\frac{\cos\lambda(0)}{\lambda^4}\right)\right]$$

$$= \left[(1-1)\left(\frac{\sin\lambda}{\lambda}\right) - (-2)\left(\frac{-\cos\lambda}{\lambda^2}\right) + (-2)\left(\frac{-\sin\lambda}{\lambda^3}\right) + (0)\left(\frac{\cos\lambda}{\lambda^4}\right)\right]$$
$$-\left[(1-0)\left(\frac{\sin(0)}{\lambda}\right) - (0)\left(\frac{-\cos(0)}{\lambda^2}\right) + (-2)\left(\frac{-\sin(0)}{\lambda^3}\right) + (0)\left(\frac{\cos(0)}{\lambda^4}\right)\right] \qquad \ldots(3)$$

Using standard values in Equation (3)

$\cos 0 = 1;\ \sin 0 = 0$

$$F(\lambda) = \left[(0)\left(\frac{\sin\lambda}{\lambda}\right) - \left(\frac{2\cos\lambda}{\lambda^2}\right) + \left(\frac{2\sin\lambda}{\lambda^3}\right) + (0)\right] - \left[(1)(0) - (0)(1) + (-2)(0) + (0)(1)\right]$$

$$F(\lambda) = \left[-\frac{2\cos\lambda}{\lambda^2} + \frac{2\sin\lambda}{\lambda^3}\right] \qquad \ldots(4)$$

This is Fourier transform of f(x)

Step V :

The inverse Fourier transform of $F(\lambda)$ is given by ,

$$f(x) = \frac{2}{\pi}\int_0^\infty F(\lambda)\cos\lambda x \cdot d\lambda \qquad \ldots(5)$$

Using value of $F(\lambda)$ from Equation (4) in Equation (5),

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\frac{-2\cos\lambda}{\lambda^2} + \frac{2\sin\lambda}{\lambda^3}\right)\cos\lambda x\, d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\frac{-2\lambda\cos\lambda + 2\sin\lambda}{\lambda^3}\right)\cos\lambda x\, d\lambda \qquad \ldots(6)$$

This is Fourier integral representation of f(x).

Step VI :

Giving site on required integral we simplify this integral as ,

$$F(x) = \frac{2}{\pi}\int_0^\infty (-2)\left(\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right)\cos\lambda x\, d\lambda$$

$$= \frac{2}{\pi}(-2)\int_0^\infty \left(\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right)\cos\lambda x\, d\lambda$$

$$= -\frac{4}{\pi}\int_0^\infty \left(\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right)\cos\lambda x\, d\lambda$$

$$\therefore \int_0^\infty \left(\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right)\cos\lambda x\, d\lambda = -\frac{\pi}{4}f(x) \qquad \ldots(7)$$

Step VII :

To find the value of integral $\displaystyle\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos x\, dx$, first we find the value of same integral in terms of λ, by using proper value of ,i.e. we find the value of integral

$$\int_0^\infty \left(\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right)\cos\lambda\, d\lambda \qquad \ldots(8)$$

Comparing Equations (7) and (8) , we observe that integral in Equation (8) is obtained by putting $x = 1$ in Equation (7)

$$\therefore \int_0^\infty \left(\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right)\cos\lambda(1)\, d\lambda = -\frac{\pi}{4}f(1) \qquad \ldots(9)$$

$f(1)$ means value of $f(x)$ at $x = 1$

From given function , $f(x) = \begin{cases} 1-x^2 & -1 \le x \le 1 \\ 0 & x > 1 \text{ and } x < -1 \end{cases}$,

$x = 1$ lies in $[-1, 1]$, therefore to find value of $f(x)$ at $x = 1$ we take $f(x) = 1 - x^2$

$$\therefore f(1) = 1 - (1)^2 = 1 - 1 = 0$$

From Equation (9),

$$\therefore \int_0^\infty \left(\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right)\cos\lambda\, d\lambda = -\frac{\pi}{4}(0) = 0 \qquad \ldots(10)$$

As value of definite integral is independent on variable, changing the variable λ to x

$$\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos x\,dx = 0 \checkmark \qquad \text{...Ans.}$$

Ex. 3.5.11 : Find the Fourier cosine transform of $f(x) = \begin{cases} \sin x & , |x| < \pi \\ e^{-2x} & , |x| > \pi \end{cases}$

Soln. :

Step I :

Given function can be written as ,

$$f(x) = \begin{cases} \sin x & , -\pi < x < \pi \\ e^{-2x} & , x > \pi \text{ and } x < -\pi \end{cases}$$

Step II :

We have, Fourier cosine transform,

$$F_C(\lambda) = \int_0^\infty f(u)\cos\lambda u\,du$$

By the definition of given function, we split the integral as,

$$F_C(\lambda) = \int_0^\pi f(u)\cos\lambda u\,du + \int_\pi^\infty f(u)\cos\lambda u\,du$$

From definition of given function, $f(x) = \sin x$ for $0 < x < \pi$ and $f(x) = e^{-2x}$ for $x > \pi$

$$F_C(\lambda) = \int_0^\pi \sin u\cos\lambda u\,du + \int_\pi^\infty e^{-2u}\cos\lambda u\,du \qquad \text{...(1)}$$

Step III :

➔ **Using standard formula :** $\left[\sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]\right]$

Here $A = u$, $B = \lambda u$

$$F_C(\lambda) = \int_0^\pi \frac{1}{2}\left[\sin(u+\lambda u) + \sin(u-\lambda u)\right]du + \int_\pi^\infty e^{-2u}\cos\lambda u\,du$$

$$= \int_0^\pi \frac{1}{2}\left[\sin(1+\lambda)u + \sin(1-\lambda)u\right]du + \int_\pi^\infty e^{-2u}\cos\lambda u\,du \qquad \text{...(2)}$$

➔ **Using standard formula In first and second integral :** $\left[\int \sin Ax\,dx = -\frac{\cos Ax}{A} \text{ for any constant } A\right]$

$\left[\int e^{ax}\cos(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx+c) + b\sin(bx+c)\right]\right]$

Here $a = -2$, $b = \lambda$, $c = 0$

$$F_C(\lambda) = \frac{1}{2}\left\{\frac{-\cos(1+\lambda)u}{1+\lambda} - \frac{\cos(1-\lambda)u}{1-\lambda}\right\}_0^\pi + \left\{\frac{e^{-2u}}{4+\lambda^2}\left(-2\cos\lambda u + \lambda\sin\lambda u\right)\right\}_\pi^\infty$$

$$= \frac{1}{2}\left[\left\{\frac{-\cos(1+\lambda)\pi}{1+\lambda} - \frac{\cos(1-\lambda)\pi}{1-\lambda}\right\} - \left\{\frac{-\cos(1+\lambda)0}{1+\lambda} - \frac{\cos(1-\lambda)0}{1-\lambda}\right\}\right]$$

$$+ \left\{\frac{e^{-2(\infty)}}{4+\lambda^2}\left(-2\cos\lambda(\infty) + \lambda\sin\lambda(\infty)\right)\right\} - \left\{\frac{e^{-2(\pi)}}{4+\lambda^2}\left(-2\cos\lambda(\pi) + \lambda\sin\lambda(\pi)\right)\right\}$$

$$= \frac{1}{2}\left[\left\{\frac{-\cos(1+\lambda)\pi}{1+\lambda} - \frac{\cos(1-\lambda)\pi}{1-\lambda}\right\} - \left\{\frac{-\cos(0)}{1+\lambda} - \frac{\cos(0)}{1-\lambda}\right\}\right]$$

$$+ \left\{\frac{e^{-\infty}}{4+\lambda^2}\left(-2\cos\infty + \lambda\sin\infty\right)\right\} - \left\{\frac{e^{-2\pi}}{4+\lambda^2}\left(-2\cos\lambda\pi + \lambda\sin\lambda\pi\right)\right\} \qquad \text{...(3)}$$

➔ **Using standard values**

$e^{-\infty} = 0$, $e^0 = 1$, $\cos 0 = 1$, $\sin 0 = 0$

$$F_C(\lambda) = \frac{1}{2}\left[\left\{\frac{-\cos(1+\lambda)\pi}{1+\lambda} - \frac{\cos(1-\lambda)\pi}{1-\lambda}\right\} - \left\{\frac{-1}{1+\lambda} - \frac{1}{1-\lambda}\right\}\right]$$

$$+ \underbrace{\left\{\frac{0}{4+\lambda^2}\left(-2\cos\infty + \lambda\sin\infty\right)\right\}}_{0} - \left\{\frac{e^{-2\pi}}{4+\lambda^2}\left(-2\cos\lambda\pi + \lambda\sin\lambda\pi\right)\right\}$$

$$= \frac{1}{2}\left[\frac{-\cos(1+\lambda)\pi}{1+\lambda} - \frac{\cos(1-\lambda)\pi}{1-\lambda} + \frac{1}{1+\lambda} + \frac{1}{1-\lambda}\right] + \left[0 - \frac{e^{-2\pi}}{4+\lambda^2}\left(-2\cos\lambda\pi + \lambda\sin\lambda\pi\right)\right] \qquad \text{...(4)}$$

➔ **Using standard formula :** ...$[\cos(A+B) = \cos A\cos B - \sin A\sin B]$

Here $A = \pi$, $B = \lambda\pi$

$$\cos(1+\lambda)\pi = \cos(\pi+\lambda\pi) = \cos\pi\cos\lambda\pi - \sin\pi\sin\lambda\pi$$

Since $\cos\pi = -1$, $\sin\pi = 0$

$$\cos(\pi+\lambda\pi) = (-1)\cos\lambda\pi - (0)\sin\lambda\pi = -\cos\lambda\pi$$

Similarly

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\cos(1-\lambda)\pi = \cos(\pi-\lambda\pi) = (-1)\cos\lambda\pi + (0)\sin\lambda\pi = -\cos\lambda\pi$$

$$F_C(\lambda) = \frac{1}{2}\left[-\left(\frac{-\cos\lambda\pi}{1+\lambda}\right) - \frac{-\cos\lambda\pi}{1-\lambda} + \frac{1}{1+\lambda} + \frac{1}{1-\lambda}\right] - \frac{e^{-2\pi}}{4+\lambda^2}\left(-2\cos\lambda\pi + \lambda\sin\lambda\pi\right)$$

$$= \frac{1}{2}\left[\frac{\cos\lambda\pi}{1+\lambda} + \frac{\cos\lambda\pi}{1-\lambda} + \left(\frac{(1-\lambda)+(1+\lambda)}{(1-\lambda)(1+\lambda)}\right)\right] - \frac{e^{-2\pi}}{4+\lambda^2}\left(-2\cos\lambda\pi + \lambda\sin\lambda\pi\right)$$

$$= \frac{1}{2}\left[\cos\lambda\pi\left(\frac{1}{1+\lambda} + \frac{1}{1-\lambda}\right) + \left(\frac{2}{1-\lambda^2}\right)\right] - \frac{e^{-2\pi}}{4+\lambda^2}\left(-2\cos\lambda\pi + \lambda\sin\lambda\pi\right)$$

$$= \frac{1}{2}\left\{\cos\lambda\pi\left(\frac{2}{1-\lambda^2}\right) + \left(\frac{2}{1-\lambda^2}\right)\right\} - \frac{e^{-2\pi}}{4+\lambda^2}\left(-2\cos\lambda\pi + \lambda\sin\lambda\pi\right)$$

$$= \frac{1}{2}\left(\frac{2}{1-\lambda^2}\right)\left[\cos\lambda\pi + 1\right] - \frac{e^{-2\pi}}{4+\lambda^2}\left(-2\cos\lambda\pi + \lambda\sin\lambda\pi\right)$$

$$F_C(\lambda) = \left(\frac{1}{1-\lambda^2}\right)(1+\cos\lambda\pi) - \frac{e^{-2\pi}}{4+\lambda^2}\left(-2\cos\lambda\pi + \lambda\sin\lambda\pi\right) \checkmark \qquad \text{...Ans.}$$

Ex. 3.5.12 : Find the Fourier sine and cosine transform of $f(x) = \begin{cases} \sin x & , 0 \le x \le a \\ 0 & , x > a \end{cases}$

Soln. :

Step I :

Fourier sine Transform : We have Fourier sine transform,

$$F_s(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du$$

By the definition of given function, we split the integral as,

$$F_s(\lambda) = \int_0^a f(u) \sin \lambda u \, du + \int_a^\infty f(u) \sin \lambda u \, du$$

$$\text{But } f(x) = \sin x, \ 0 \le x \le a \text{ and } f(x) = 0, \ a < x < \infty$$

$$\therefore \ f_s(\lambda) = \int_0^a \sin u \sin \lambda u \, du + \int_a^\infty (0) \sin \lambda u \, du$$

$$= \int_0^a \sin u \sin \lambda u \, du + 0$$

$$= \int_0^a \sin u \sin \lambda u \, du \qquad \ldots(1)$$

Step II :

➔ **Using standard formula :** $\ldots\left[\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] = -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]\right]$

Here $A = u$, $B = \lambda u$

$$F_S(\lambda) = \frac{1}{2}\int_0^a [\cos(u - \lambda u) - \cos(u + \lambda u)] \, du$$

$$= \frac{1}{2}\int_0^a [\cos(1-\lambda)u - \cos(1+\lambda)u] \, du \qquad \ldots(2)$$

Step III:

➔ **Using standard formula In first integral :** $\ldots\left[\int \cos Ax \, dx = \frac{\sin Ax}{A}, \text{ for any constant } A\right]$

$$F_S(\lambda) = \frac{1}{2}\left\{\frac{\sin(1-\lambda)u}{1-\lambda} - \frac{\sin(1+\lambda)u}{1+\lambda}\right\}_0^a$$

$$= \frac{1}{2}\left\{\frac{\sin(1-\lambda)a}{1-\lambda} - \frac{\sin(1+\lambda)a}{1+\lambda}\right\} - \left\{\frac{\sin(1-\lambda)0}{1-\lambda} - \frac{\sin(1+\lambda)0}{1+\lambda}\right\} \qquad \ldots(3)$$

➔ **Using standard formula :** $\ldots[\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B]$

Here $A = a$, $B = \lambda a$

$\sin(a \pm \lambda a) = \sin a \cos \lambda a \pm \cos a \sin \lambda a$

$\sin 0 = 0$

$$F_S(\lambda) = \frac{1}{2}\left\{\frac{1}{1-\lambda}(\sin a \cos \lambda a - \cos a \sin \lambda a) - \frac{1}{1+\lambda}(\sin a \cos \lambda a + \cos a \sin \lambda a)\right\} - \left\{\frac{\sin 0}{1-\lambda} - \frac{\sin 0}{1+\lambda}\right\}$$

$$= \frac{-1}{2(\lambda^2-1)}\{\lambda \sin\lambda a \cos a + \lambda \cos \lambda a \sin a - \sin \lambda a \cos a - \cos \lambda a \sin a$$

$$- \lambda \sin \lambda a \cos a + \lambda \cos \lambda a \sin a - \sin \lambda a \cos a + \cos \lambda a \sin a\}$$

$$= \frac{-1}{2(\lambda^2-1)}[2\lambda \cos \lambda a \sin a - 2 \sin \lambda a \cos a]$$

$$F_S(\lambda) = \frac{\sin \lambda a \cos a - \lambda \cos \lambda a \sin a}{\lambda^2 - 1} \checkmark \qquad \ldots\text{Ans.}$$

Step IV :

Fourier cosine transform

We have, Fourier cosine transform,

$$F_C(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du = \int_0^a \sin u \cos \lambda u \, du = \int_0^a \cos \lambda u \sin u \, du$$

$$= \frac{1}{2}\int_0^a [\sin(\lambda+1)u - \sin(\lambda-1)u) \, du \qquad \{\because \cos x \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]\}$$

$$= \frac{1}{2}\left\{\frac{-\cos(\lambda+1)u}{\lambda+1} + \frac{\cos(\lambda-1)u}{\lambda-1}\right\}_0^a$$

$$= \frac{1}{2}\left\{\frac{-\cos(\lambda+1)a}{\lambda+1} + \frac{\cos(\lambda-1)a}{\lambda-1} + \frac{1}{\lambda+1} - \frac{1}{\lambda-1}\right\}$$

(Use $\cos(A \pm B) = \cos A \cos B \pm \sin A \sin B$ and simplify)

$$= \frac{1}{2(\lambda^2-1)}[2\lambda \sin \lambda a \sin a + 2 \cos \lambda a \cos a - 2]$$

$$F_C(\lambda) = \frac{\lambda \sin \lambda a \sin a + \cos \lambda a \cos a - 1}{\lambda^2 - 1} \checkmark \qquad \ldots\text{Ans.}$$

Ex. 3.5.13 : Find the Fourier cosine Transform of $f(x) = 2e^{-5x} + 5e^{-2x}$

Soln. :

Step I : We have by Fourier cosine transform,

$$F_C(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du$$

$$= \int_0^\infty (2e^{-5u} + 5e^{-2u}) \cos \lambda u \, du$$

$$= 2\int_0^\infty e^{-5u} \cos \lambda u \, du + 5\int_0^\infty e^{-2u} \cos \lambda u \, du \qquad \ldots(1)$$

Step II :

➔ **Using standard formula**

$$\ldots\left[\int_0^\infty e^{-ax} \cos(bx + c) \, dx = \frac{a}{a^2 + b^2}\right]$$

Here In first integral $a = 5$, $b = \lambda$, $c = 0$

In second integral a = 2, b = λ, c = 0

$$F_C(\lambda) = 2\left[\frac{5}{(5)^2+\lambda^2}\right] + 5\left[\frac{2}{(2)^2+\lambda^2}\right]$$

$$= 2\left[\frac{5}{25+\lambda^2}\right] + 5\left[\frac{2}{4+\lambda^2}\right]$$

$$= (2\times 5)\left[\frac{1}{25+\lambda^2}\right] + (5\times 2)\left[\frac{1}{4+\lambda^2}\right]$$

$$= 10\left[\frac{1}{25+\lambda^2}\right] + 10\left[\frac{1}{4+\lambda^2}\right]$$

$$F_C(\lambda) = 10\left[\frac{1}{25+\lambda^2} + \frac{1}{4+\lambda^2}\right]$$

$$= 10\left[\frac{(4+\lambda^2)+(25+\lambda^2)}{(\lambda^2+4)(\lambda^2+25)}\right]$$

$$F_C(\lambda) = 10\left[\frac{2\lambda^2+29}{(\lambda^2+4)(\lambda^2+25)}\right] \checkmark \quad \text{...Ans.}$$

This is required Fourier cosine transform

Ex. 3.5.14 : Find the Fourier sine and cosine transform of the function.

$f(x) = x^{m-1}$

Soln. :

Step I :

We know the formulae of Fourier sine transform $F_S(\lambda)$ and cosine transform $F_C(\lambda)$ of f(x) as,

$$F_S(\lambda) = \int_0^\infty f(u)\sin\lambda u\,du = \int_0^\infty u^{m-1}\sin\lambda u\,du \quad ...(1)$$

$$\text{and } F_C(\lambda) = \int_0^\infty f(u)\cos\lambda u\,du = \int_0^\infty u^{m-1}\cos\lambda u\,du \quad ...(2)$$

Since it is difficult to evaluate both integrals in Equation (1) and Equation (2), for convenience we consider the integral,

$$I = \int_0^\infty f(u)\,e^{-i\lambda u}\,du \quad ...(3)$$

Step II :

➔ **Using standard formula :** $\left[e^{-i\theta} = \cos\theta - i\sin\theta\right]$

Here $\theta = i\lambda\pi$

$$I = \int_0^\infty f(u)\,[\cos\lambda u - i\sin\lambda u]\,du$$

$$= \int_0^\infty f(u)\cos\lambda u\,du - i\int_0^\infty f(u)\sin\lambda u\,du \quad ...(4)$$

Step III :

From Equation (1) and (2) we can write Equation (4) as have,

$$\therefore\ I = \int_0^\infty f(u)\,e^{-i\lambda u}\,du$$

$$= F_C(\lambda) - i\,F_S(\lambda)$$

$$\Rightarrow \int_0^\infty u^{m-1}e^{-i\lambda u}\,du = F_C(\lambda) - i\,F_S(\lambda) \quad ...(5)$$

$$\therefore\ F_C(\lambda) - i\,F_S(\lambda) = \int_0^\infty e^{-i\lambda u}\,u^{m-1}\,du \quad ...(6)$$

Step IV :

If we find the value of integral on RHS of Equation (6) we get values of $F_S(\lambda)$ and $F_C(\lambda)$ both

Put, $i\lambda u = t$ in Equation (6)

$$i\lambda du = dt \Rightarrow du = \frac{dt}{i\lambda}$$

$$\therefore\ F_C(\lambda) - i\,F_S(\lambda) = \int_0^\infty e^{-t}\left(\frac{t}{i\lambda}\right)^{m-1}\frac{dt}{i\lambda}$$

$$= \int_0^\infty e^{-t}t^{m-1}\left(\frac{1}{i\lambda}\right)^{m-1}\left(\frac{1}{i\lambda}\right)dt \quad ...(7)$$

➔ **Using law of index :** $\left[a^m \times a^n = a^{m+n}\right]$

here $a = \frac{1}{i\lambda}$, $m = m-1$, $n = 1$

$$\therefore\ F_C(\lambda) - i\,F_S(\lambda) = \int_0^\infty e^{-t}\,t^{m-1}\left(\frac{1}{i\lambda}\right)^{(m-1)+1}dt$$

$$= \int_0^\infty e^{-t}\,t^{m-1}\left(\frac{1}{i\lambda}\right)^{m}dt$$

$$= \left(\frac{1}{i\lambda}\right)^m\int_0^\infty e^{-t}\,t^{m-1}\,dt \quad ...(8)$$

Step V :

➔ **Using standard formula of Gamma function**

$$\left[\int_0^\infty e^{-t}\,t^{m-1}\,dt = \Gamma m\right]$$

Also $\frac{1}{i} = \frac{-i^2}{i} = -i$

$$\therefore \left(\frac{1}{i\lambda}\right)^m = \left(\frac{1}{i}\right)^m\left(\frac{1}{\lambda}\right)^m = (-i)^m\left(\frac{1}{\lambda}\right)^m$$

$$\therefore\ F_C(\lambda) - i\,F_S(\lambda) = \left(\frac{1}{\lambda}\right)^m(-i)^m\,\Gamma m \quad ...(9)$$

Writing $-i$ as, $-i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

$$F_C(\lambda) - i\,F_S(\lambda) = \frac{1}{\lambda^m}\Gamma m\left[\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right]^m \quad ...(10)$$

➔ **Using standard DeMoivere's theorem**

$$\left[\left[\cos\theta + i\sin\theta\right]^m = \cos m\theta + i\sin m\theta,\ \left[\cos\theta - i\sin\theta\right]^m = \cos m\theta - i\sin m\theta\right]$$

Step VI :

Here $\theta = \frac{\pi}{2}$

$$\therefore\ F_C(\lambda) - i\,F_S(\lambda) = \frac{1}{\lambda^m}\Gamma m\left[\cos\frac{m\pi}{2} - i\sin\frac{m\pi}{2}\right]$$

$$= \frac{1}{\lambda^m}\Gamma m\cos\frac{m\pi}{2} - i\frac{1}{\lambda^m}\Gamma m\sin\frac{m\pi}{2}$$

Equating real and imaginary parts on both sides,

$$\therefore\ F_C(\lambda) = \frac{\cos\frac{m\pi}{2}}{\lambda^m}\Gamma m \checkmark$$

$$\text{and}\quad F_S(\lambda) = \frac{\sin\frac{m\pi}{2}}{\lambda^m}\Gamma m \checkmark \quad \text{...Ans.}$$

3.5.5 Examples on Use of Differentiation Under Integral Sign(DUIS) Rule

Ex. 3.5.15 : Find inverse Fourier sine transform of

$F_S(\lambda) = \frac{1}{\lambda}e^{-a\lambda}, \lambda > 0$

Soln. :

Step I :

We have by the formula of inverse fourier sine transform of $F_S(\lambda)$,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_S(\lambda)\sin\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{1}{\lambda}e^{-a\lambda}\sin\lambda x\,d\lambda \quad ...(1)$$

Consider,

$$I(x) = \int_0^\infty \frac{1}{\lambda}e^{-a\lambda}\sin\lambda x\,d\lambda \quad ...(2)$$

Step II :

➔ **Using standard rule of DUIS**

$$\left[\text{If } I(\alpha) = \int_a^b f(x,\alpha)\,dx \text{ then } \frac{dI(\alpha)}{d\alpha} = \int_a^b\left[\frac{\partial}{\partial\alpha}f(x,\alpha)\right]dx\right]$$

where a and b are constants

Here variable of integral is λ and α = x

$$\text{By DUIS}\ \therefore\ \frac{dI}{dx} = \int_0^\infty\frac{\partial}{\partial x}\left[\frac{1}{\lambda}e^{-a\lambda}\sin\lambda x\right]d\lambda$$

Here variable of derivative is x therefore terms of λ can be treated as constants

$$\therefore\ \frac{dI}{dx} = \int_0^\infty\left(\frac{1}{\lambda}e^{-a\lambda}\right)\underbrace{\frac{\partial}{\partial x}(\sin\lambda x)}_{\lambda\cos\lambda x}d\lambda \quad ...(3)$$

➔ **Using standard formula of derivative**

$$\left[\frac{\partial}{\partial x}(\sin Ax) = A\cos Ax \text{ for any constant } A\right]$$

Here A = λ

$$\therefore\ \frac{dI}{dx} = \int_0^\infty\frac{1}{\lambda}e^{-a\lambda}(\lambda\cos\lambda x)\,d\lambda = \int_0^\infty e^{-a\lambda}\cos\lambda x\,d\lambda \quad ...(4)$$

Step III :

➔ **Using standard formula**

$$\left[\int_0^\infty e^{-ax}\cos(bx+c)\,dx = \frac{a}{a^2+b^2}\right]$$

Here a = a, b = x

$$\therefore\ \frac{dI}{dx} = \frac{a}{a^2+x^2}$$

$$dI = \frac{a}{a^2+x^2}dx$$

This differential equation is in Variable separable Form.

Step IV :

On integrating both sides,

$$\therefore\ \int dI = \int\frac{a}{a^2+x^2}dx + A\ (\text{A integral constant})$$

$$= a\int\frac{1}{a^2+x^2}dx + A \quad ...(5)$$

➔ **Using standard formula of integration**

$$\left[\int\frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$I(x) = a\left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right] + A$$

$$\therefore I(x) = \tan^{-1}\left(\frac{x}{a}\right) + A \quad ...(6)$$

Step V:

We find value of constant of integration A

For this, Put $x = 0$ in Equation (6)

$$I(0) = \tan^{-1}\left(\frac{0}{a}\right) + A \Rightarrow I(0) = \tan^{-1} 0 + A \Rightarrow I(0)$$

$$= 0 + A \Rightarrow I(0) = A \quad ...(7)$$

But from Equation (2)

$$I(0) = \int_0^\infty \frac{1}{\lambda} e^{-a\lambda} \sin\lambda(0)\, d\lambda \Rightarrow I(0) = \int_0^\infty \frac{1}{\lambda} e^{-a\lambda} \underbrace{\sin(0)}_{0}\, d\lambda$$

$$\Rightarrow I(0) = 0 \quad ...(8)$$

From Equation (7) and (8) we get, $A = 0$

Step VI:

$\therefore$ From Equation (6), $I(x) = \tan^{-1}\left(\frac{x}{a}\right)$

Put this value in Equation (1)

$$\therefore f(x) = \frac{2}{\pi}\tan^{-1}\left(\frac{x}{a}\right) \checkmark \quad ...\text{Ans.}$$

Ex. 3.5.16 : Find Fourier sine transform of $\frac{e^{-ax}}{x}$

Soln. :

Step I :

Given,

$$f(x) = \frac{e^{-ax}}{x}$$

We have by Fourier sine transform,

$$F_S(\lambda) = \int_0^\infty f(u)\sin\lambda u\, du$$

$$= \int_0^\infty \frac{e^{-au}}{u}\sin\lambda u\, du \quad ...(1)$$

Consider,

$$I(\lambda) = \int_0^\infty \frac{e^{-au}}{u}\sin\lambda u\, du \quad ...(2)$$

Step III :

➔ **Using standard rule of DUIS**

$$...\left[\text{If } I(\alpha) = \int_a^b f(x,\alpha)\,dx \text{ then } \frac{dI(\alpha)}{d\alpha} = \int_a^b \left[\frac{\partial}{\partial\alpha} f(x,\alpha)\right] dx\right.$$

$$\left.\text{where a and b are constants}\right]$$

Here variable of integral is u

$$\text{By DUIS} \quad \frac{dI}{d\lambda} = \int_0^\infty \frac{\partial}{\partial\lambda}\left[\frac{e^{-au}}{u}\sin\lambda u\right] du$$

Here variable of derivative is λ therefore terms of u can be treated as constants

$$\therefore \frac{dI}{d\lambda} = \int_0^\infty \left(\frac{e^{-au}}{u}\right)\frac{\partial}{\partial\lambda}(\sin\lambda u)\, du \quad ...(3)$$

➔ **Using standard formula of derivative**

$$...\left[\frac{\partial}{\partial x}(\sin Ax) = A\cos Ax \text{ for any constant } A\right]$$

Here A = u

$$\therefore \frac{dI}{d\lambda} = \int_0^\infty \frac{e^{-au}}{u}(u\cos\lambda u)\, du$$

$$\frac{dI}{d\lambda} = \int_0^\infty e^{-au}\cos\lambda u\, du \quad ...(4)$$

➔ **Using standard formula**

$$...\left[\int_0^\infty e^{-ax}\cos(bx+c)\,dx = \frac{a}{a^2+b^2}\right]$$

Here $a = a$, $b = \lambda$

$$\frac{dI}{d\lambda} = \frac{a}{a^2+\lambda^2}$$

$$dI = \frac{a}{a^2+\lambda^2}\, d\lambda$$

This differential equation is in Variable separable Form.

Step IV :

On integrating both sides,

$$\therefore \int dI = \int \frac{a}{a^2+\lambda^2}\, d\lambda + A \quad (\text{A integral constant})$$

$$= a\int \frac{1}{a^2+\lambda^2}\, d\lambda \quad ...(5)$$

➔ **Using standard formula of integration**

$$...\left[\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

Here variable of integration is λ

$$\therefore x = \lambda$$

$$\therefore I(\lambda) = a\left(\frac{1}{a}\tan^{-1}\frac{\lambda}{a}\right) + A$$

$$\therefore I(\lambda) = \tan^{-1}\left(\frac{\lambda}{a}\right) + A \quad ...(6)$$

Step V :

We find value of constant of integration A

For this, Put $\lambda = 0$ in Equation (6)

$$I(0) = \tan^{-1}\left(\frac{0}{a}\right) + A \Rightarrow I(0) = \tan^{-1} 0 + A \Rightarrow I(0)$$

$$= 0 + A \Rightarrow I(0) = A \quad ...(7)$$

But from Equation (2)

$$I(0) = \int_0^\infty \frac{e^{-au}}{u}\sin\lambda(0)\, du \Rightarrow I(0) = \int_0^\infty \frac{e^{-au}}{u}\underbrace{\sin(0)}_{0}\, du$$

$$= \int_0^\infty \frac{e^{-au}}{u}(0)\, du$$

$$\Rightarrow I(0) = 0 \quad ...(8)$$

From Equation (7) and (8) we get, $A = 0$

Step VI :

$\therefore$ From Equation (6), $I(\lambda) = \tan^{-1}\left(\frac{\lambda}{a}\right)$

Put in Equation (1), $F_S(\lambda) = \tan^{-1}\left(\frac{\lambda}{a}\right)$

Ex. 3.5.17 : Find inverse Fourier cosine transform find f(x) if

$$F_C(\lambda) = \frac{\sin a\lambda}{\lambda}$$

Soln. :

Step I :

Given,

$$F_C(\lambda) = \frac{\sin a\lambda}{\lambda}$$

We have by the formula of inverse Fourier cosine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_C(\lambda)\cos\lambda x\, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin a\lambda}{\lambda}\cos\lambda x\, d\lambda \quad ...(1)$$

Step II :

➔ **Using standard formula**

$$...\left[\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]\right]$$

Here $A = a\lambda$, $B = \lambda x$

$$\sin a\lambda\cos\lambda x = \frac{1}{2}[\sin(a\lambda+\lambda x) + \sin(a\lambda-\lambda x)]$$

$$= \frac{1}{2}[\sin(a+x)\lambda + \sin(a-x)\lambda]$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin(a+x)\lambda + \sin(a-x)\lambda}{\lambda}\, d\lambda$$

$$= \frac{2}{\pi}\left[\int_0^\infty \frac{\sin(a+x)\lambda}{\lambda}\, d\lambda + \int_0^\infty \frac{\sin(a-x)\lambda}{\lambda}\, d\lambda\right] \quad ...(2)$$

Step III :

Using standard formula in Equation(2)

$$\int_0^\infty \frac{\sin a\lambda}{\lambda}\, d\lambda = \begin{cases} \frac{\pi}{2}, & x > a \\ -\frac{\pi}{2}, & x < a \end{cases}$$

From Equation (2),

$$f(x) = \frac{2}{\pi}\begin{cases} \frac{\pi}{2}+\frac{\pi}{2}, & a+x>0;\ a-x>0 \\ -\frac{\pi}{2}-\left(-\frac{\pi}{2}\right), & a+x<0;\ a-x<0 \end{cases}$$

$$= \frac{2}{\pi}\begin{cases} 2\left(\frac{\pi}{2}\right), & x>-a;\ a>x \\ -\frac{\pi}{2}+\frac{\pi}{2}, & x<-a;\ a<x \end{cases}$$

$$= \frac{2}{\pi}\begin{cases} \pi, & -a<x<a \\ 0, & x<-a;\ x>a \end{cases}$$

$$= \begin{cases} \pi, & -a<x<a \\ 0, & x<-a;\ x>a \end{cases} \quad ...\text{Ans.}$$

3.5.6 Type - IV Examples on to Establish the given Integral

Note : In these type of examples we observe the term under integral sign.

(1) If integral involves the term $\sin\lambda x$ then we find Fourier sine transform of the function on LHS of given

equation i.e we find $F_S(\lambda)$.This value of $F_S(\lambda)$we put in the formula of inverse Fourier sine transform and then simplify.

(2) If integral involves the term $\cos \lambda x$ then we find Fourier cosine transform of the function on LHS of given equation i.e we find $F_C(\lambda)$.This value of $F_C(\lambda)$we put in the formula of inverse Fourier cosine transform and then simplify.

(3) If integral involves both the terms $\sin \lambda x$ and $\cos \lambda x$ then we find general Fourier transform of the function on LHS of given equation i.e we find $F(\lambda)$.This value of $F(\lambda)$we put in the formula of inverse Fourier transform and then simplify.

Ex. 3.5.18 : Establish the following relation

$$e^{-2x} - e^{-3x} = \frac{10}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{(9+\lambda^2)(4+\lambda^2)}\, d\lambda,\ x > 0$$

Soln. :

Step I :

Since integral contains the term $\sin \lambda x$ we find Fourier sine integral representation of the function on LHS

i.e. $f(x) = e^{-2x} - e^{-3x}$ and $x > 0$

Step II :

(I) By the formula of Fourier sine transform,

$$F_S(\lambda) = \int_0^\infty f(u) \sin \lambda u\, du = \int_0^\infty \left(e^{-2u} - e^{-3u}\right) \sin \lambda u\, du$$

$$= \int_0^\infty e^{-2u} \sin \lambda u\, du - \int_0^\infty e^{-3u} \sin \lambda u\, du \qquad ...(1)$$

➔ **Using standard formula**

$$...\left[\int_0^\infty e^{-ax} \sin (bx + c)\, dx = \frac{b}{a^2 + b^2}\right]$$

Here in first integral, $a = 2, b = \lambda, c = 0$

Here in first integral, $a = 3, b = \lambda, c = 0$

$$F_S(\lambda) = \frac{\lambda}{(2)^2 + \lambda^2} - \frac{\lambda}{(3)^2 + \lambda^2}$$

$$= \frac{\lambda}{4 + \lambda^2} - \frac{\lambda}{9 + \lambda^2}$$

$$= \frac{\lambda(9 + \lambda^2) - \lambda(4 + \lambda^2)}{(4 + \lambda^2)(9 + \lambda^2)}$$

$$= \frac{(9\lambda + \lambda^3) - (4\lambda + \lambda^3)}{(4 + \lambda^2)(9 + \lambda^2)} = \frac{(9\lambda - 4\lambda) + (\lambda^3 - \lambda^3)}{(4 + \lambda^2)(9 + \lambda^2)}$$

$$= \frac{5\lambda + 0}{(4 + \lambda^2)(9 + \lambda^2)} = \frac{5\lambda}{(4 + \lambda^2)(9 + \lambda^2)}$$

$$F_S(\lambda) = \frac{5\lambda}{(4 + \lambda^2)(9 + \lambda^2)} \qquad ...(2)$$

Step III :

(II) Now by the formula of inverse Fourier sine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty F_S(\lambda) \sin \lambda x\, d\lambda$$

using value of $F_S(\lambda)$ from Equation (2)

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{5\lambda}{(4 + \lambda^2)(9 + \lambda^2)} \sin \lambda x\, d\lambda$$

$$= \frac{2}{\pi}(5)\int_0^\infty \frac{\lambda}{(4 + \lambda^2)(9 + \lambda^2)} \sin \lambda x\, d\lambda$$

$$\therefore\ f(x) = \frac{10}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{(4 + \lambda^2)(9 + \lambda^2)}\, d\lambda$$

But since $f(x) = e^{-2x} - e^{-3x}$, we have

$$\therefore\ e^{-2x} - e^{-3x} = \frac{10}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{(4 + \lambda^2)(9 + \lambda^2)}\, d\lambda \checkmark$$ **...Hence Proved.**

Which is required integral

Ex. 3.5.19 : Using Fourier integral representation prove that

$$e^{-x} - e^{-2x} = \frac{6}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{(\lambda^2 + 1)(\lambda^2 + 4)}\, d\lambda,\ x > 0$$

Soln. :

Step I :

Since integral contains the term $\sin \lambda x$ we find Fourier sine integral representation of the function on LHS

i.e. $f(x) = e^{-x} - e^{-2x}$ and $x > 0$

(I) By the formula of Fourier sine transform,

$$F_S(\lambda) = \int_0^\infty f(u) \sin \lambda u\, du = \int_0^\infty \left(e^{-u} - e^{-2u}\right) \sin \lambda u\, du$$

$$= \int_0^\infty e^{-u} \sin \lambda u\, du - \int_0^\infty e^{-2u} \sin \lambda u\, du \qquad ...(1)$$

Step II :

➔ **Using standard formula**

$$...\left[\int_0^\infty e^{-ax} \sin (bx + c)\, dx = \frac{b}{a^2 + b^2}\right]$$

Here in first integral, $a = 1, b = \lambda, c = 0$

Here in first integral, $a = 2, b = \lambda, c = 0$

Step III :

$$F_S(\lambda) = \frac{\lambda}{(1)^2 + \lambda^2} - \frac{\lambda}{(2)^2 + \lambda^2}$$

$$= \frac{\lambda}{1 + \lambda^2} - \frac{\lambda}{4 + \lambda^2}$$

$$= \frac{\lambda(4 + \lambda^2) - \lambda(1 + \lambda^2)}{(1 + \lambda^2)(4 + \lambda^2)}$$

$$= \frac{(4\lambda + \lambda^3) - (\lambda + \lambda^3)}{(1 + \lambda^2)(4 + \lambda^2)}$$

$$= \frac{(4\lambda - \lambda) + (\lambda^3 - \lambda^3)}{(1 + \lambda^2)(4 + \lambda^2)} = \frac{3\lambda + 0}{(1 + \lambda^2)(4 + \lambda^2)}$$

$$= \frac{3\lambda}{(1 + \lambda^2)(4 + \lambda^2)}$$

$$F_S(\lambda) = \frac{3\lambda}{(1 + \lambda^2)(4 + \lambda^2)}$$

Step IV :

(II) Now by the formula of inverse Fourier sine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty F_S(\lambda) \sin \lambda x\, d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{3\lambda}{(1 + \lambda^2)(4 + \lambda^2)} \sin \lambda x\, d\lambda$$

$$= \frac{2}{\pi}(3)\int_0^\infty \frac{\lambda}{(1 + \lambda^2)(4 + \lambda^2)} \sin \lambda x\, d\lambda$$

$$f(x) = \frac{6}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{(1 + \lambda^2)(4 + \lambda^2)}\, d\lambda$$

But since $f(x) = e^{-x} - e^{-2x}$, we have

$$\therefore\ e^{-x} - e^{-2x} = \frac{6}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{(1 + \lambda^2)(4 + \lambda^2)}\, d\lambda \checkmark$$ **...Hence Proved.**

Ex. 3.5.20 : Using Fourier integral representation show that

$$\int_0^\infty \frac{\cos \frac{\pi\lambda}{2} \cos \lambda x}{1 - \lambda^2}\, d\lambda = \begin{cases} \frac{\pi}{2}\cos x & |x| \le \frac{\pi}{2} \\ 0 & |x| > \frac{\pi}{2} \end{cases}$$

Soln. :

Step I :

Since integral contains the term $\cos \lambda x$ we find Fourier cosine integral representation of the function on RHS

i.e. $f(x) = \begin{cases} \frac{\pi}{2}\cos x & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 0 & x < -\frac{\pi}{2} \text{ and } x > \frac{\pi}{2} \end{cases}$

Step II :

(I) By the formula of Fourier cosine transform,

$$F_C(\lambda) = \int_0^\infty f(u) \cos \lambda u\, du$$

By the definition of given function,we split the integral as

$$= \int_0^{\pi/2} f(u) \cos \lambda u\, du + \int_{\pi/2}^\infty f(u) \cos \lambda u\, du$$

$$= \int_0^{\pi/2} \frac{\pi}{2} \cos u \cos \lambda u\, du + \int_{\pi/2}^\infty (0) \cos \lambda u\, du$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \cos u \cos \lambda u\, du + 0$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \cos u \cos \lambda u\, du \qquad ...(1)$$

Step III :

➔ **Using standard formula**

$$...\left[\cos A \cos B = \frac{1}{2}\left[\cos(A + B) + \cos(A - B)\right]\right]$$

Here $A = u, B = \lambda u$

$$\cos u \cos \lambda u = \frac{1}{2}\left[\cos (u + \lambda u) + \cos (u - \lambda u)\right]$$

$$= \frac{1}{2}\left[\cos (1 + \lambda) u + \cos (1 - \lambda) u\right]$$

$$\therefore\ F_C(\lambda) = \frac{\pi}{2}\frac{1}{2}\int_0^{\pi/2} \left[\cos (1 + \lambda) u + \cos (1 - \lambda) u\right] du \qquad ...(2)$$

Step IV :

➔ **Using standard formula of integration**

$$\ldots\left[\int \cos Ax\,dx = \frac{\sin Ax}{A}\right]$$

Here in first integral $A = 1+\lambda$
In second integral $A = 1-\lambda$

$$F_C(\lambda) = \frac{\pi}{4}\left\{\frac{\sin(1+\lambda)u}{1+\lambda} + \frac{\sin(1-\lambda)u}{1-\lambda}\right\}_0^{\pi/2}$$

$$= \frac{\pi}{4}\left[\frac{\sin(1+\lambda)\frac{\pi}{2}}{1+\lambda} + \frac{\sin(1-\lambda)\frac{\pi}{2}}{1-\lambda}\right] - \left[\frac{\sin(1+\lambda)(0)}{1+\lambda} + \frac{\sin(1-\lambda)(0)}{1-\lambda}\right]$$

$$= \frac{\pi}{4}\left[\frac{\sin\left(\frac{\pi}{2}+\frac{\lambda\pi}{2}\right)}{1+\lambda} + \frac{\sin\left(\frac{\pi}{2}-\frac{\lambda\pi}{2}\right)}{1-\lambda}\right] - \left[\frac{\sin(0)}{1+\lambda} + \frac{\sin(0)}{1-\lambda}\right]$$

Since $\sin 0 = 0$

$$F_C(\lambda) = \frac{\pi}{4}\left[\frac{\sin\left(\frac{\pi}{2}+\frac{\lambda\pi}{2}\right)}{1+\lambda} + \frac{\sin\left(\frac{\pi}{2}-\frac{\lambda\pi}{2}\right)}{1-\lambda}\right] - 0$$

$$= \frac{\pi}{4}\left[\frac{\sin\left(\frac{\pi}{2}+\frac{\lambda\pi}{2}\right)}{1+\lambda} + \frac{\sin\left(\frac{\pi}{2}-\frac{\lambda\pi}{2}\right)}{1-\lambda}\right] \quad \ldots(3)$$

Step V :

Using standard trigonometric formula in Equation (3)

$\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$ and $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$

$$F_C(\lambda) = \frac{\pi}{4}\left[\frac{\cos\frac{\lambda\pi}{2}}{1+\lambda} + \frac{\cos\frac{\lambda\pi}{2}}{1-\lambda}\right]$$

$$F_C(\lambda) = \frac{\pi}{4}\cos\frac{\lambda\pi}{2}\left[\frac{1}{1+\lambda} + \frac{1}{1-\lambda}\right]$$

$$= \frac{\pi}{4}\cos\frac{\lambda\pi}{2}\left[\frac{(1-\lambda)+(1+\lambda)}{(1+\lambda)(1-\lambda)}\right] \quad \ldots(4)$$

➔ **Using standard identity :** $\ldots\left[(a+b)(a-b) = a^2 - b^2\right]$

Here $a = 1, b = \lambda$

$\therefore (1+\lambda)(1-\lambda) = 1-\lambda^2$

$$F_C(\lambda) = \frac{\pi}{4}\cos\frac{\lambda\pi}{2}\left[\frac{2}{1-\lambda^2}\right]$$

$$= \left(\frac{\pi}{4}\right)\left[\frac{2}{1-\lambda^2}\right]\cos\frac{\lambda\pi}{2}$$

$$F_C(\lambda) = \frac{\pi}{2}\left(\frac{\cos\frac{\lambda\pi}{2}}{1-\lambda^2}\right) \quad \ldots(5)$$

Step VI :

(II) The inverse Fourier cosine transform is,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_C(\lambda)\cos\lambda x\,d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\pi}{2}\frac{\cos\frac{\lambda\pi}{2}}{1-\lambda^2}\cos\lambda x\,d\lambda.$$

$$= \underbrace{\left(\frac{2}{\pi}\right)\left(\frac{\pi}{2}\right)}_{1}\int_0^\infty \frac{\cos\frac{\lambda\pi}{2}\cos\lambda x}{1-\lambda^2}\,d\lambda$$

$$= \int_0^\infty \frac{\cos\frac{\lambda\pi}{2}\cos\lambda x}{1-\lambda^2}\,d\lambda$$

$$f(x) = \int_0^\infty \frac{\cos\frac{\lambda\pi}{2}\cos\lambda x}{1-\lambda^2}\,d\lambda$$

Step VII :

But since $f(x) = \begin{cases} \frac{\pi}{2}\cos x, & |x| \le \frac{\pi}{2} \\ 0, & |x| > \frac{\pi}{2} \end{cases}$, we have

$$\therefore \int_0^\infty \frac{\cos\frac{\lambda\pi}{2}\cos\lambda x}{1-\lambda^2}\,d\lambda = \begin{cases} \frac{\pi}{2}\cos x, & |x| \le \frac{\pi}{2} \\ 0, & |x| > \frac{\pi}{2} \end{cases} \checkmark$$

...Hence Proved

Ex. 3.5.21 : Using Fourier integral representation show that

$$\int_0^\infty \frac{2\cos\lambda x}{1+\lambda^2}\,d\lambda = \begin{cases} 0, & x<0 \\ \pi e^{-x}, & x>0 \end{cases}$$

Soln. :

Step I :

Since integral contains the term $\cos\lambda x$ we find Fourier cosine integral representation of the function on RHS

i.e. $f(x) = \begin{cases} 0, & x<0 \\ \pi e^{-x}, & x>0 \end{cases}$

Step II :

(I) By the formula of Fourier cosine transform,

$$F_C(\lambda) = \int_0^\infty f(u)\cos\lambda u\,du$$

$$= \int_0^\infty \pi e^{-u}\cos\lambda u\,du$$

$$= \pi\int_0^\infty e^{-u}\cos\lambda u\,du \quad \ldots(1)$$

➔ **Using standard formula**

$$\ldots\left[\int_0^\infty e^{-ax}\cos(bx+c)\,dx = \frac{a}{a^2+b^2}\right]$$

Here, $a = 1, b = \lambda, c = 0$

$$F_C(\lambda) = \pi\left[\frac{1}{1+\lambda^2}\right]$$

Step III :

(II) The inverse Fourier cosine transform is,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_C(\lambda)\cos\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty\left(\frac{\pi}{1+\lambda^2}\right)\cos\lambda x\,d\lambda$$

$$= \underbrace{\frac{2}{\pi}(\pi)}_{2}\int_0^\infty\left(\frac{1}{1+\lambda^2}\right)\cos\lambda x\,d\lambda$$

$$f(x) = 2\int_0^\infty\left(\frac{1}{1+\lambda^2}\right)\cos\lambda x\,d\lambda$$

But since $f(x) = \begin{cases} 0, & x<0 \\ \pi e^{-x}, & x>0 \end{cases}$

$$\therefore \int_0^\infty \frac{2\cos\lambda x}{1+\lambda^2}\,d\lambda = f(x) = \begin{cases} 0, & x<0 \\ \pi e^{-x}, & x>0 \end{cases} \checkmark$$

...Hence Proved.

Ex. 3.5.22 : Using Fourier integral representation show that

$$\int_0^\infty \frac{2\sin\lambda x - \lambda\cos\lambda x}{\pi\lambda^3}\,d\lambda = \begin{cases} x - x^2, & x>0 \\ 0, & x<0 \end{cases}$$

Soln. :

Step I:

Since integral contains both terms $\sin\lambda x$ and $\cos\lambda x$, we find general Fourier integral representation of the function on RHS i.e. $f(x) = \begin{cases} x - x^2, & x>0 \\ 0, & x<0 \end{cases}$

Step II :

(I) By the formula of general Fourier transform,

$$F(\lambda) = \int_{-\infty}^{\infty} f(u)\,e^{-i\lambda u}\,du$$

By the definition of given function, we split the integral as

$$= \int_{-\infty}^{0} f(u)\,e^{-i\lambda u}\,du + \int_0^\infty f(u)\,e^{-i\lambda u}\,du$$

$$= \int_{-\infty}^{0} (0)\,e^{-i\lambda u}\,du + \int_0^\infty \underset{\substack{\uparrow \\ u}}{(u-u^2)}\,\underset{\substack{\uparrow \\ v}}{e^{-i\lambda u}}\,du$$

$$= \int_0^\infty (u-u^2)\,e^{-i\lambda u}\,du \quad \ldots(1)$$

Step III :

➔ **Using formula of generalized rule of integration**

$$\ldots\left[\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right]$$

Where u is algebraic polynomial

v is exponential / sine / cosine term

Here dash denotes order of derivative and suffix denotes integration.

Here $u = u - u^2$, $v = e^{-i\lambda u}$

$u' = 1 - 2u$, $u'' = -2$, $u''' = 0$

$v_1 = \frac{e^{-i\lambda u}}{-i\lambda}$, $v_2 = \frac{e^{-i\lambda u}}{(-i\lambda)^2} = \frac{e^{-i\lambda u}}{(-i)^2(\lambda)^2} = \frac{e^{-i\lambda u}}{(-1)(\lambda^2)} = -\frac{e^{-i\lambda u}}{(\lambda^2)}$,

$v_3 = \frac{e^{-i\lambda u}}{(-i\lambda)^3} = \frac{e^{-i\lambda u}}{(-i)^3(\lambda)^3} = \frac{e^{-i\lambda u}}{(-i)(-i)^2(\lambda)^3} = \frac{e^{-i\lambda u}}{(-i)(i)^2(\lambda)^3}$

$= \frac{e^{-i\lambda u}}{(-i)(-1)(\lambda)^3} = \frac{e^{-i\lambda u}}{(i)(\lambda)^3}$ and so on

$$F(\lambda) = \left\{\underset{\substack{\uparrow \\ u}}{(u-u^2)}\underset{\substack{\uparrow \\ v_1}}{\left(\frac{e^{-i\lambda u}}{-i\lambda}\right)} - \underset{\substack{\uparrow \\ u'}}{(1-2u)}\underset{\substack{\uparrow \\ v_2}}{\left(\frac{e^{-i\lambda u}}{-\lambda^2}\right)} + \underset{\substack{\uparrow \\ u''}}{(-2)}\underset{\substack{\uparrow \\ v_3}}{\left(\frac{e^{-i\lambda u}}{i\lambda^3}\right)}\right\}_0^\infty$$

$$= \left\{(\infty-\infty^2)\left(\frac{e^{-i\lambda(\infty)}}{-i\lambda}\right) - (1-2(\infty))\left(\frac{e^{-i\lambda(\infty)}}{-\lambda^2}\right) + (-2)\left(\frac{e^{-i\lambda(\infty)}}{i\lambda^3}\right)\right\}$$
$$-\left\{(0-0^2)\left(\frac{e^{-i\lambda(0)}}{-i\lambda}\right) - (1-2(0))\left(\frac{e^{-i\lambda(0)}}{-\lambda^2}\right) + (-2)\left(\frac{e^{-i\lambda(0)}}{i\lambda^3}\right)\right\}$$

$$F(\lambda) = \left\{\infty\left(\frac{e^{-\infty}}{-i\lambda}\right) - (1-\infty)\left(\frac{e^{-\infty}}{-\lambda^2}\right) + (-2)\left(\frac{e^{-\infty}}{i\lambda^3}\right)\right\}$$
$$-\left\{0\left(\frac{e^{-0}}{-i\lambda}\right) - (1-0)\left(\frac{e^{-0}}{-\lambda^2}\right) + (-2)\left(\frac{e^{-0}}{i\lambda^3}\right)\right\} \quad \ldots(2)$$

Using standard values in Equation (2)

$e^{-\infty} = 0,\ e^{0} = 1$

$$\therefore F(\lambda) = \{\underbrace{(\infty)(0)}_{0} - \underbrace{(1-\infty)(0)}_{0} + \underbrace{(-2)(0)}_{0}\}$$

$$-\left\{\underbrace{(0)\left(\frac{1}{-i\lambda}\right)}_{0} - (1-0)\left(\frac{1}{-\lambda^2}\right) + (-2)\left(\frac{1}{i\lambda^3}\right)\right\}$$

$$= (0-0+0) - \left(0 + \frac{1}{\lambda^2} - \frac{2}{i\lambda^3}\right)$$

$$= 0 - \frac{1}{\lambda^2} + \frac{2}{i\lambda^3}$$

$$F(\lambda) = -\frac{1}{\lambda^2} + \frac{2}{i\lambda^3} \qquad \ldots(3)$$

Step IV:

➔ **Using standard value:** $\ldots\left[\frac{1}{i} = -i\right]$

$$\frac{2}{i\lambda^3} = -i\left(\frac{2}{\lambda^3}\right) = \left(-\frac{2i}{\lambda^3}\right)$$

$$F(\lambda) = \frac{-1}{\lambda^2} - \frac{2i}{\lambda^3} = \frac{-\lambda}{\lambda^2\lambda} - \frac{2i}{\lambda^3} = \frac{-\lambda}{\lambda^3} - \frac{2i}{\lambda^3} = \frac{-\lambda-2i}{\lambda^3}$$

$$= \frac{-(\lambda+2i)}{\lambda^3}$$

Step V:

(II) Now, by the formula of inverse Fourier transform,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda$$

Using value of $F(\lambda)$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-(\lambda+2i)}{\lambda^3} e^{i\lambda x}\, d\lambda \qquad \ldots(4)$$

➔ **Using standard formula:** $\ldots\left[e^{i\theta} = \cos\theta + i\sin\theta\right]$

Here $\theta = i\lambda\pi$

$e^{i\lambda\pi} = \cos\lambda\pi + i\sin\lambda\pi$

Step VI:

$$f(x) = \frac{-1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\lambda+2i}{\lambda^3}\right)(\cos\lambda x + i\sin\lambda x)\, d\lambda$$

$$= \frac{-1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\lambda(\cos\lambda x + i\sin\lambda x) + 2i(\cos\lambda x + i\sin\lambda x)}{\lambda^3}\right) d\lambda$$

$$= \frac{-1}{2\pi}\int_{-\infty}^{\infty}\frac{\lambda\cos\lambda x + i\lambda\sin\lambda x + 2i\cos\lambda x + 2i^2\sin\lambda x}{\lambda^3}\, d\lambda$$

$$= \frac{-1}{2\pi}\int_{-\infty}^{\infty}\frac{\lambda\cos\lambda x + i\lambda\sin\lambda x + 2i\cos\lambda x - 2\sin\lambda x}{\lambda^3}\, d\lambda$$

$$= \frac{-1}{2\pi}\int_{-\infty}^{\infty}\frac{(\lambda\cos\lambda x - 2\sin\lambda x) + i(\lambda\sin\lambda x + 2\cos\lambda x)}{\lambda^3}\, d\lambda$$

$$= \underbrace{\frac{-1}{2\pi}\int_{-\infty}^{\infty}\frac{(\lambda\cos\lambda x - 2\sin\lambda x)}{\lambda^3}\, d\lambda}_{\text{Even function}} + i\underbrace{\int_{-\infty}^{\infty}\frac{(\lambda\sin\lambda x + 2\cos\lambda x)}{\lambda^3}\, d\lambda}_{\text{Odd function}} \qquad \ldots(5)$$

StepVII:

Using standard result in Equation(5)

(1) Product of two even functions is even function.
(2) Product of two odd functions is even function.
(3) Product of one even and one odd function is odd function.
(4) Quotient of two even functions is an even function.
(5) Quotient of two odd functions is an odd function.
(6) Quotient of one even function and one odd function is an odd function.

Here variable of integration is λ.

In first integral,

(i) $\cos\lambda x$ is even functions and λ is an odd function, therefore $\lambda\cos\lambda x$ is an odd function

(ii) $\sin\lambda x$ is an odd function. Therefore $\lambda\cos\lambda x - \sin\lambda x$ also odd function.

(iii) $\lambda\cos\lambda x - \sin\lambda x$ is odd function, λ^3 is an odd function therefore $\frac{\lambda\cos\lambda x - 2\sin\lambda x}{\lambda^3}$ is an even function

In second integral,

(i) $\sin\lambda x$ is an odd function and λ is also an odd function therefore $\lambda\sin\lambda x$ is even function

(ii) $\cos\lambda x$ is an odd functions,

Therefore $(\lambda\sin\lambda x + 2\cos\lambda x)$ is an even function

(iii) λ^3 is an odd function

Therefore $\frac{\lambda\sin\lambda x + 2\cos\lambda x}{\lambda^3}$ is an odd function

Standard result of definite integral

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{for even function} \\ 0 & \text{for odd function} \end{cases}$$

$$\int_{-\infty}^{\infty}\frac{(\lambda\cos\lambda x - 2\sin\lambda x)}{\lambda^3}\, d\lambda = 2\int_0^{\infty}\frac{(\lambda\cos\lambda x - 2\sin\lambda x)}{\lambda^3}\, d\lambda$$

$$\int_{-\infty}^{\infty}\frac{(\lambda\sin\lambda x + 2\cos\lambda x)}{\lambda^3}\, d\lambda = 0$$

Step VIII:

$$\therefore f(x) = \frac{-1}{2\pi}(2)\int_0^{\infty}\frac{(\lambda\cos\lambda x - 2\sin\lambda x)}{\lambda^3}\, d\lambda + i(0)$$

$$= \frac{-1}{2\pi}(2)\int_0^{\infty}\frac{-(2\sin\lambda x - \lambda\cos\lambda x)}{\lambda^3}\, d\lambda$$

$$= \left(-\frac{1}{2\pi}\right)(2)(-)\int_0^{\infty}\frac{2\sin\lambda x - \lambda\cos\lambda x}{\lambda^3}\, d\lambda$$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\frac{2\sin\lambda x - \lambda\cos\lambda x}{\lambda^3}\, d\lambda$$

$$\therefore \frac{1}{\pi}\int_0^{\infty}\frac{2\sin\lambda x - \lambda\cos\lambda x}{\pi\lambda^3}\, d\lambda = f(x)$$

But since $f(x) = \begin{cases} x - x^2 & , x > 0 \\ 0 & , x < 0 \end{cases}$, we get

$$\text{i.e.} \int_0^{\infty}\frac{2\sin\lambda x - \lambda\cos\lambda x}{\pi\lambda^3}\, d\lambda = \begin{cases} x - x^2 & x > 0 \\ 0 & x < 0 \end{cases} \checkmark$$

...Hence Proved.

Ex. 3.5.23: Using Fourier integral representation show that

$$\int_0^{\infty}\frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\, d\lambda = \begin{cases} 0 & x < 0 \\ \pi e^{-x} & , x > 0 \\ \pi/2 & x = 0 \end{cases}$$

Soln.:

Step I:

Since integral contains both terms $\sin\lambda x$ and $\cos\lambda x$, we find general Fourier integral representation of the function on RHS i.e. $f(x) = \begin{cases} 0 & x < 0 \\ \pi e^{-x} & , x > 0 \\ \pi/2 & x = 0 \end{cases}$

Step II:

(I) By the formula of general Fourier transform,

$$F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$$

By the definition of given function, we split the integral as

$$= \int_{-\infty}^{0} f(u)\, e^{-i\lambda u}\, du + \int_0^{\infty} f(u)\, e^{-i\lambda u}\, du$$

$$= \int_{-\infty}^{0} (0)\, e^{-i\lambda u}\, du + \int_0^{\infty} \pi e^{-u} e^{-i\lambda u}\, du$$

$$= 0 + \int_0^{\infty} \pi e^{-u} e^{-i\lambda u}\, du$$

$$= \int_0^{\infty} \pi \underset{\substack{\uparrow \\ a^m}}{e^{-u}}\, \underset{\substack{\uparrow \\ a^n}}{e^{-i\lambda u}}\, du \qquad \ldots(1)$$

Step III:

➔ **Using index rule:** $\ldots\left[a^m \times a^n = a^{m+n}\right]$

Here $a = e,\ m = -u,\ n = -i\lambda u$

$$e^{-u} e^{-i\lambda u} = e^{-u + (-i\lambda u)} = e^{-u(1+i\lambda)} = e^{-(1+i\lambda)u}$$

$$F(\lambda) = \pi\int_0^{\infty} e^{-(1+i\lambda)u}\, du \qquad \ldots(2)$$

➔ **Using standard formula of integration**

$$\ldots\left[\int e^{Ax}\, dx = \frac{e^{Ax}}{A} \text{ for any constant } A\right]$$

Here $A = -(1+i\lambda)$

$$F(\lambda) = \pi\left[\frac{e^{-(1+i\lambda)u}}{-(1+i\lambda)}\right]_0^{\infty}$$

$$= \pi\left[\frac{e^{-(1+i\lambda)(\infty)}}{-(1+i\lambda)} - \frac{e^{-(1+i\lambda)(0)}}{-(1+i\lambda)}\right]$$

$$= \pi\left[\frac{e^{-(\infty)}}{-(1+i\lambda)} - \frac{e^{(0)}}{-(1+i\lambda)}\right] \qquad \ldots(3)$$

Using standard values in Equation (3)

$e^{-\infty} = 0,\ e^{0} = 1$

$$F(\lambda) = \pi\left[\frac{0}{-(1+i\lambda)} - \frac{1}{-(1+i\lambda)}\right]$$

$$= \pi\left[0 + \frac{1}{(1+i\lambda)}\right]$$

$$= \pi\left[\frac{1}{1+i\lambda}\right]$$

$$F(\lambda) = \pi\left(\frac{1-i\lambda}{1+\lambda^2}\right) \qquad \left(\because \frac{1}{a+ib}\ \frac{a-ib}{a^2+b^2}\right)$$

Step IV :

(II) Now by using formula of inverse Fourier Transform,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda$$

Using the value of $F(\lambda)$

$$= \frac{1}{2\pi}(\pi)\int_{-\infty}^{\infty}\left(\frac{1-i\lambda}{1+\lambda^2}\right) e^{i\lambda x}\, d\lambda \qquad ...(4)$$

➔ **Using standard formula :** $\left[e^{i\theta} = \cos\theta + i\sin\theta\right]$

Here $\theta = i\lambda x$

$e^{i\lambda x} = \cos\lambda x + i\sin\lambda x$

Step V :

$$f(x) = \frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{1-i\lambda}{1+\lambda^2}\right)(\cos\lambda x + i\sin\lambda x)\, d\lambda$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{1(\cos\lambda x + i\sin\lambda x) - i\lambda(\cos\lambda x + i\sin\lambda x)}{1+\lambda^2}\, d\lambda$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos\lambda x + i\sin\lambda x - i\lambda\cos\lambda x - i^2\lambda\sin\lambda x}{1+\lambda^2}\, d\lambda$$

Since $i^2 = -1$

$$\therefore f(x) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos\lambda x + i\sin\lambda x - i\lambda\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos\lambda x + i\sin\lambda x - i\lambda\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\, d\lambda$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{(\cos\lambda x + \lambda\sin\lambda x) + i(\sin\lambda x - \lambda\cos\lambda x)}{1+\lambda^2}\, d\lambda$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\underbrace{\frac{(\cos\lambda x + \lambda\sin\lambda x)}{1+\lambda^2}}_{\text{Even function}}\, d\lambda$$

$$+\frac{1}{2}(i)\int_{-\infty}^{\infty}\underbrace{\frac{(\sin\lambda x - \lambda\cos\lambda x)}{1+\lambda^2}}_{\text{Odd function}}\, d\lambda \qquad ...(5)$$

Step VI :

Using standard result in Equation(5)

(1) Product of two even functions is even function.
(2) Product of two odd functions is even function.
(3) Product of one even and one odd function is odd function.
(4) Quotient of two even functions is an even function
(5) Quotient of two odd functions is an odd function
(6) Quotient of one even function and one odd function is an odd function

Here variable of integration is λ.

In first integral,

(i) $\cos\lambda x$ is even functions

(ii) λ is an odd function, $\sin\lambda x$ is an odd function. Therefore $\lambda\sin\lambda x$ is an even function.

Therefore $\cos\lambda x + \lambda\sin\lambda x$ is even function

(iii) $1+\lambda^2$ is even function, therefore $\dfrac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}$ is an even function

In second integral,

(i) $\sin\lambda x$ is an odd function

(ii) λ is an odd function, $\cos\lambda x$ is an even function ,Therefore $\lambda\cos\lambda x$ is an odd function,

Therefore $(\sin\lambda x - \lambda\cos\lambda x)$ is an odd function

(iii) $1+\lambda^2$ is even function, therefore $\dfrac{\sin\lambda x - \lambda\cos\lambda x}{1+\lambda^2}$ is an odd function

Standard result of definite integral

$$\int_{-a}^{a} f(x)\, dx = \begin{cases} 2\int_0^a f(x)\, dx & \text{for even function} \\ 0 & \text{for odd function} \end{cases}$$

$$\int_{-\infty}^{\infty}\frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\, d\lambda = 2\int_0^{\infty}\frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\, d\lambda$$

$$\int_{-\infty}^{\infty}\frac{\sin\lambda x - \lambda\cos\lambda x}{1+\lambda^2}\, d\lambda = 0$$

$$f(x) = \frac{1}{2}\left[2\int_0^{\infty}\frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\, d\lambda\right] + i(0)$$

$$= \int_0^{\infty}\frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\, d\lambda$$

$$\therefore \int_0^{\infty}\frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\, d\lambda = f(x)$$

But $f(x) = \begin{cases} 0 & x<0 \\ \pi e^{-x} & x>0 \\ \pi/2 & x=0 \end{cases}$, we get

$$\therefore \int_0^{\infty}\frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\, d\lambda = \begin{cases} 0 & x<0 \\ \pi e^{-x} & x>0 \\ \pi/2 & x=0 \end{cases} \checkmark$$

...Hence Proved.

Ex. 3.5.24 : By considering Fourier sine transform of e^{-mx} $(m>0)$ prove that

$$\int_0^{\infty}\frac{\lambda\sin\lambda x}{\lambda^2+m^2}\, d\lambda = \frac{\pi}{2}e^{-mx},\ m>0,\ x>0$$

Hence deduce that $\displaystyle\int_0^{\infty}\frac{x\sin x}{x^2+m^2}\, dx = \frac{\pi}{2}e^{-m}$

Soln. :

Step I :

Consider,

$$f(x) = \frac{\pi}{2}e^{-mx},\ m>0,\ x>0$$

Since integral contains $\sin\lambda x$ and $x>0$, use Fourier sine transform,

Step II :

(I) We have, Fourier sine transform as,

$$F_S(\lambda) = \int_0^{\infty} f(u)\sin\lambda u\, du = \int_0^{\infty}\frac{\pi}{2}e^{-mu}\sin\lambda u\, du \qquad ...(1)$$

$$F_S(\lambda) = \frac{\pi}{2}\int_0^{\infty} e^{-mu}\sin\lambda u\, du$$

➔ **Using Standard formula**

$$\left[\int_0^{\infty} e^{-ax}\sin(bx+c)\, dx = \frac{b}{a^2+b^2}\right]$$

Here $x = u, a = m, b = \lambda$

$$\therefore F_S(\lambda) = \frac{\pi}{2}\left[\frac{\lambda}{m^2+\lambda^2}\right]$$

$$\therefore F_S(\lambda) = \frac{\pi}{2}\left[\frac{\lambda}{\lambda^2+m^2}\right]$$

Step III :

(II) Now, by using the formula of inverse Fourier sine transform,

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_S(\lambda)\sin\lambda x\, d\lambda$$

Using value of $F_S(\lambda)$

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\underbrace{\frac{\pi}{2}\left[\frac{\lambda}{\lambda^2+m^2}\right]}_{F_S(\lambda)}\sin\lambda x\, d\lambda$$

$$= \underbrace{\left(\frac{2}{\pi}\right)\left(\frac{\pi}{2}\right)}_{1}\int_0^{\infty}\left[\frac{\lambda}{\lambda^2+m^2}\right]\sin\lambda x\, d\lambda$$

$$= \int_0^{\infty}\left[\frac{\lambda}{\lambda^2+m^2}\right]\sin\lambda x\, d\lambda$$

$$\Rightarrow \int_0^{\infty}\frac{\lambda}{\lambda^2+m^2}\sin\lambda x\, d\lambda = f(x)$$

But since $f(x) = \dfrac{\pi}{2}e^{-mx}$

we get,

$$\int_0^{\infty}\frac{\lambda\sin\lambda x}{\lambda^2+m^2}\, d\lambda = \frac{\pi}{2}e^{-mx}\ m>0,\ x>0 \quad ...(2)$$

Step IV:

To find the value of given integral $\displaystyle\int_0^{\infty}\frac{x\sin x}{x^2+m^2}\, dx$, first we find the value of similar integral in terms of λ by using proper value of x.

i.e. we find the value of integral

$$\int_0^{\infty}\frac{\lambda\sin\lambda}{\lambda^2+m^2}\, d\lambda \qquad ...(3)$$

Compairing Equation (2) and (3), we observe that integral (3) is obtained by taking x = 1 in Equation (2)

$$\therefore \int_0^{\infty}\frac{\lambda\sin\lambda}{\lambda^2+m^2}\, d\lambda = \frac{\pi}{2}e^{-m} \qquad ...(4)$$

Since value of definite integral is independent of variable, so changing variable λ to x in Equation (4),

$$\int_0^\infty \frac{x \sin x}{x^2+m^2}\, dx = \frac{\pi}{2} e^{-m} \checkmark \qquad \text{...Hence Proved.}$$

Ex. 3.5.25 : By considering Fourier cosine transform of e^{-mx} $(m > 0)$ prove that

$$\int_0^\infty \frac{\cos \lambda x}{\lambda^2+m^2}\, d\lambda = \frac{\pi}{2m} e^{-mx}, \quad m > 0,\ x > 0$$

Hence deduce that $\displaystyle\int_0^\infty \frac{\cos x}{m^2+x^2}\, dx = \frac{\pi}{2m} e^{-m}$

Soln. :

Step I :

Consider,

$$f(x) = \frac{\pi}{2m} e^{-mx}, \quad m > 0,\ x > 0$$

Since integral contains $\cos \lambda x$ and $x > 0$, use Fourier cosine transform,

Step II :

(I) We have, Fourier cosine transform as,

$$F_C(\lambda) = \int_0^\infty f(u) \cos \lambda u\, du$$

$$= \int_0^\infty \frac{\pi}{2m} e^{-mu} \cos \lambda u\, du \qquad \text{...(1)}$$

$$F_C(\lambda) = \frac{\pi}{2m} \int_0^\infty e^{-mu} \cos \lambda u\, du$$

Step III :

➔ **Using Standard formula**

$$\text{...}\left[\int_0^\infty e^{-ax} \cos(bx+c)\, dx = \frac{a}{a^2+b^2}\right]$$

Here $x = u, a = m, b = \lambda$

$$\therefore\ F_C(\lambda) = \frac{\pi}{2m}\left[\frac{m}{m^2+\lambda^2}\right]$$

$$\therefore\ F_C(\lambda) = \frac{\pi}{2}\left[\frac{1}{m^2+\lambda^2}\right]$$

Step IV :

(II) Now, by using the formula of inverse Fourier cosine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_C(\lambda) \cos \lambda x\, d\lambda$$

Using value of $F_C(\lambda)$

$$f(x) = \frac{2}{\pi}\int_0^\infty \underbrace{\frac{\pi}{2}\left[\frac{1}{m^2+\lambda^2}\right]}_{F_C(\lambda)} \cos \lambda x\, d\lambda$$

$$= \underbrace{\left(\frac{2}{\pi}\right)\left(\frac{\pi}{2}\right)}_{1} \int_0^\infty \left[\frac{1}{m^2+\lambda^2}\right] \cos \lambda x\, d\lambda$$

$$= \int_0^\infty \left[\frac{1}{m^2+\lambda^2}\right] \cos \lambda x\, d\lambda$$

$$\Rightarrow \int_0^\infty \frac{1}{m^2+\lambda^2} \cos \lambda x\, d\lambda = f(x)$$

Step V :

But since $f(x) = \frac{\pi}{2m} e^{-mx}$, we get

$$\int_0^\infty \frac{\cos \lambda x}{m^2+\lambda^2}\, d\lambda = \frac{\pi}{2m} e^{-mx} \qquad m > 0,\ x > 0 \qquad \text{...(2)}$$

Step VI :

To find the value of given integral $\displaystyle\int_0^\infty \frac{\cos x}{m^2+x^2}\, dx$, first we find the value of similar integral in terms of λ by using proper value of x. i.e. we find the value of integral

$$\int_0^\infty \frac{\cos \lambda}{m^2+\lambda^2}\, d\lambda \qquad \text{...(3)}$$

Compairing Equation (2) and (3), we observe that integral (3) is obtained by taking $x = 1$ in Equation (2)

$$\therefore\ \int_0^\infty \frac{\cos \lambda}{m^2+\lambda^2}\, d\lambda = \frac{\pi}{2m} e^{-m} \qquad \text{...(4)}$$

Step VII :

Since value of definite integral is independent of variable, so changing variable λ to x in Equation (4),

$$\int_0^\infty \frac{\cos x}{m^2+x^2}\, dx = \frac{\pi}{2m} e^{-m} \checkmark \qquad \text{...Hence Proved.}$$

3.5.7 Type-V Examples on Solution of Integral Equation

Ex. 3.5.26 : Solve the integral equation

$$\int_0^\infty f(x) \cos \lambda x\, dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda > 1 \end{cases}$$

Hence show that $\displaystyle\int_0^\infty \frac{\sin^2 z}{z^2}\, dz = \frac{\pi}{2}$

Soln. :

Consider given equation,

$$\int_0^\infty f(x) \cos \lambda x\, dx = \begin{cases} 1-\lambda & 0 \le \lambda \le 1 \\ 0, & \lambda > 1 \end{cases} \qquad \text{...(1)}$$

We know Fourier cosine transform of f(x) is given by,

$$F_C(\lambda) = \int_0^\infty f(u) \cos \lambda u\, du \qquad \text{...(2)}$$

Since value of definite integral is independent of variable,so changing variable u to x in Equation (2),

$$F_C(\lambda) = \int_0^\infty f(x) \cos \lambda x\, dx$$

$$\text{OR} \int_0^\infty f(x) \cos \lambda x\, dx = F_C(\lambda) \qquad \text{...(3)}$$

Comparing Equation (1) and (3) we observe that the RHS of Equation (1) is Fourier cosine transform of an unknown function f(x) which we want to find out.

$$\therefore\ F_C(\lambda) = \begin{cases} 1-\lambda & 0 \le \lambda \le 1 \\ 0, & \lambda \ge 1 \end{cases} \qquad \text{...(4)}$$

In this case value of f(x) is given by inverse Fourier cosine transform.

(I) We have, by inverse Fourier cosine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_C(\lambda) \cos \lambda x\, d\lambda$$

by definition of $F_C(\lambda)$, we observe that

$$F_C(\lambda) = 1-\lambda \text{ if } 0 \le \lambda \le 1 \text{ and } F_C(\lambda) = 0 \text{ if } \lambda > 1$$

$$\therefore\ f(x) = \frac{2}{\pi}\left[\int_0^1 F_C(\lambda) \cos \lambda x\, d\lambda + \int_1^\infty F_C(\lambda) \cos \lambda x\, d\lambda\right]$$

Using value of $F_C(\lambda)$

$$f(x) = \frac{2}{\pi}\left[\int_0^1 (1-\lambda) \cos \lambda x\, d\lambda + \int_1^\infty 0 \cos \lambda x\, d\lambda\right]$$

$$\therefore\ f(x) = \frac{2}{\pi}\int_0^1 (1-\lambda) \cos \lambda x\, d\lambda \qquad \text{...(5)}$$

➔ **Using formula of generalized rule of integration**

$$\text{...}\left[\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right]$$

Where u is algebraic polynomial

v is exponential / sine / cosine term

Here dash denotes order of derivative and suffix denotes integration.

Here variable of integration is λ

$$u = 1-\lambda, \qquad v = \cos \lambda x$$

$$u' = -1, \qquad u'' = 0$$

$$v_1 = \frac{\sin \lambda x}{x}, \qquad v_2 = -\frac{\cos \lambda x}{x^2}$$

$$f(x) = \frac{2}{\pi}\left\{\underset{\substack{\uparrow \\ u}}{(1-\lambda)}\underset{\substack{\uparrow \\ v_1}}{\left(\frac{\sin \lambda x}{x}\right)} - \underset{\substack{\uparrow \\ u'}}{(-1)}\underset{\substack{\uparrow \\ v_2}}{\left(\frac{-\cos \lambda x}{x^2}\right)}\right\}_0^1$$

$$= \frac{2}{\pi}\left\{(1-\lambda)\left(\frac{\sin \lambda x}{x}\right) - \left(\frac{\cos \lambda x}{x^2}\right)\right\}_0^1$$

$$= \frac{2}{\pi}\left[\left\{\underbrace{(1-1)}_{0}\left(\frac{\sin (1)x}{x}\right) - \left(\frac{\cos (1)x}{x^2}\right)\right\} - \left\{\underbrace{(1-0)}_{0}\left(\frac{\sin (0)x}{x}\right) - \left(\frac{\cos (0)x}{x^2}\right)\right\}\right]$$

$$= \frac{2}{\pi}\left[\left\{\underbrace{(0)\left(\frac{\sin x}{x}\right)}_{0} - \left(\frac{\cos x}{x^2}\right)\right\} - \left\{\underbrace{(1)\left(\frac{\sin 0}{x}\right)}_{0} - \left(\frac{\cos 0}{x^2}\right)\right\}\right] \quad ...(6)$$

Using standard values in Equation (6)

$\cos 0 = 1$, $\sin 0 = 0$

$$f(x) = \frac{2}{\pi}\left[\left\{0 - \left(\frac{\cos x}{x^2}\right)\right\} - \left\{\underbrace{(1)(0)}_{0} - \left(\frac{1}{x^2}\right)\right\}\right]$$

$$= \frac{2}{\pi}\left\{-\frac{\cos x}{x^2} - 0 + \frac{1}{x^2}\right\}$$

$$f(x) = \frac{2}{\pi}\left[-\frac{\cos x}{x^2} + \frac{1}{x^2}\right]$$

$$f(x) = \frac{2}{\pi}\left(\frac{1-\cos x}{x^2}\right) \quad ...(7)$$

(II) To evaluate given integral

Now again consider Fourier cosine transform of f(x),

$$F_C(\lambda) = \int_0^\infty f(u)\cos\lambda u\, du$$

But from Equation (7), $f(x) = \frac{2}{\pi}\left(\frac{1-\cos x}{x^2}\right)$

$$f_C(\lambda) = \int_0^\infty \frac{2}{\pi}\left(\frac{1-\cos u}{u^2}\right)\cos\lambda u\, du \quad ...(8)$$

➔ **Using Standard formula**

$$...\left[\sin^2 x = \frac{1-\cos 2x}{2} \text{ OR } \frac{1-\cos 2x}{2} = \sin^2 x\right]$$

$\Rightarrow 1 - \cos 2x = 2\sin^2 x \Rightarrow 1 - \cos x = 2\sin^2\frac{x}{2}$

$$\therefore f_C(\lambda) = \frac{2}{\pi}\int_0^\infty \frac{2\sin^2\frac{u}{2}}{u^2}\cos\lambda u\, du$$

$$f_C(\lambda) = \frac{2}{\pi}(2)\int_0^\infty \frac{\sin^2\frac{u}{2}}{u^2}\cos\lambda u\, du$$

$$= \frac{4}{\pi}\int_0^\infty \frac{\sin^2\frac{u}{2}}{u^2}\cos\lambda u\, du \quad ...(9)$$

Put, $\lambda = 0$ in Equation (9)

$$f_C(0) = \frac{4}{\pi}\int_0^\infty \frac{\sin^2\frac{u}{2}}{u^2}\cos(0)\, u\, du$$

$$= \frac{4}{\pi}\int_0^\infty \frac{\sin^2\frac{u}{2}}{u^2}\underbrace{\cos 0}_{1}\, du \quad ...(10)$$

Using standard value in Equation (10)

$\cos 0 = 1$

$$f_C(0) = \frac{4}{\pi}\int_0^\infty \frac{\sin^2\frac{u}{2}}{u^2}\, du \quad ...(11)$$

But from Equation (4),

$$f_C(\lambda) = \begin{cases} 1-\lambda & 0 \le \lambda \le 1 \\ 0 & , \lambda > 0 \end{cases},$$

we have $f_C(0) = 1 - 0 = 1$

From Equation (11),

$$1 = \frac{4}{\pi}\int_0^\infty \frac{\sin^2\frac{u}{2}}{u^2}\, du$$

$$\Rightarrow \int_0^\infty \frac{\sin^2\frac{u}{2}}{u^2}\, du = \frac{\pi}{4} \quad ...(12)$$

Put $\frac{u}{2} = z \Rightarrow u = 2z \Rightarrow du = 2\, dz$

$$\therefore \int_0^\infty \frac{\sin^2 z}{4z^2}\, 2\, dz = \frac{\pi}{4}$$

$$\Rightarrow \frac{2}{4}\int_0^\infty \frac{\sin^2 z}{z^2}\, dz = \frac{\pi}{4}$$

$$\Rightarrow \int_0^\infty \frac{\sin^2 z}{z^2}\, dz = \underbrace{\left(\frac{\pi}{4}\right)\left(\frac{4}{2}\right)}_{\frac{\pi}{2}} \Rightarrow \int_0^\infty \frac{\sin^2 z}{z^2}\, dz = \frac{\pi}{2}$$

Ex. 3.5.27 : Solve the integral equation $\int_0^\infty f(x)\sin\lambda x\, dx = \begin{cases} \lambda - 1 & 0 \le \lambda \le 1 \\ \lambda + 1, & 1 \le \lambda \le 2 \\ 0 & \lambda > 2 \end{cases}$

Soln. : Consider given equation,

$$\int_0^\infty f(x)\sin\lambda x\, dx = \begin{cases} \lambda - 1 & 0 \le \lambda \le 1 \\ \lambda + 1, & 1 \le \lambda \le 2 \\ 0 & \lambda > 2 \end{cases} \quad ...(1)$$

We know Fourier sine transform of f(x) is given by,

$$F_S(\lambda) = \int_0^\infty f(u)\sin\lambda u\, du \quad ...(2)$$

Since value of definite integral is independent of variable, so changing variable u to x in Equation (2),

$$F_S(\lambda) = \int_0^\infty f(x)\sin\lambda x\, dx$$

$$\text{OR } \int_0^\infty f(x)\sin\lambda x\, dx = F_S(\lambda) \quad ...(3)$$

Comparing Equation (1) and (3) we observe that the RHS of Equation (1) is Fourier sine transform of the an unknown function f(x) which we want to find out.

$$\therefore F_S(\lambda) = \begin{cases} \lambda - 1 & 0 \le \lambda \le 1 \\ \lambda + 1, & 1 \le \lambda \le 2 \\ 0 & \lambda > 2 \end{cases} \quad ...(4)$$

In this case value of f(x) is given by inverse Fourier sine transform.

(I) We have, by inverse Fourier sine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_S(\lambda)\sin\lambda x\, d\lambda$$

By the definition of function $F_S(\lambda)$, we split the integral on RHS as,

$$f(x) = \frac{2}{\pi}\left[\int_0^1 F_S(\lambda)\sin\lambda x\, d\lambda + \int_1^2 F_S(\lambda)\sin\lambda x\, d\lambda + \int_2^\infty F_S(\lambda)\sin\lambda x\, d\lambda\right]$$

Using values of function $F_S(\lambda)$ in respective interval,

$$f(x) = \frac{2}{\pi}\left[\int_0^1 (\lambda - 1)\sin\lambda x\, d\lambda + \int_1^2 (\lambda + 1)\sin\lambda x\, d\lambda + + \int_2^\infty (0)\sin\lambda x\, d\lambda\right]$$

$$= \frac{2}{\pi}\left[\int_0^1 (\lambda - 1)\sin\lambda x\, d\lambda + \int_1^2 (\lambda + 1)\sin\lambda x\, d\lambda\right] \quad ...(5)$$

➔ **Using formula of generalized rule of integration :** $...\left[\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...\right]$

Where u is algebraic polynomial

v is exponential / sine / cosine term

Here dash denotes order of derivative and suffix denotes integration.

Here variable of integration is λ

In first integral,

$$u = \lambda - 1, \qquad v = \sin\lambda x$$
$$u' = 1, \qquad u'' = 0$$
$$v_1 = -\frac{\cos\lambda x}{x}, \qquad v_2 = -\frac{\sin\lambda x}{x^2}$$

In second integral,

$$u = \lambda + 1, \qquad v = \sin\lambda x$$
$$u' = 1, \qquad u'' = 0$$
$$v_1 = -\frac{\cos\lambda x}{x}, \qquad v_2 = -\frac{\sin\lambda x}{x^2}$$

$$f(x) = \frac{2}{\pi}\left[\int_0^1 (\lambda-1)\sin\lambda x\, d\lambda + \int_1^2 (\lambda+1)\sin\lambda x\, d\lambda\right]$$

$$= \frac{2}{\pi}\left\{\left[(\lambda-1)\underset{\substack{\uparrow\\ u}}{}\left(\underset{\substack{\uparrow\\ v_1}}{\frac{-\cos\lambda x}{x}}\right) - (1)\left(\underset{\substack{\uparrow\\ v_2}}{\frac{-\sin\lambda x}{x^2}}\right)\right]_0^1 + \left[(\lambda+1)\underset{\substack{\uparrow\\ u}}{}\left(\underset{\substack{\uparrow\\ v_1}}{\frac{-\cos\lambda x}{x}}\right) - (1)\left(\underset{\substack{\uparrow\\ v_2}}{\frac{-\sin\lambda x}{x^2}}\right)\right]_1^2\right\}$$

$$= \frac{2}{\pi}\left\{\left[-\frac{1}{x}(\lambda-1)(\cos\lambda x) + \frac{1}{x^2}(\sin\lambda x)\right]_0^1 + \left[-\frac{1}{x}(\lambda+1)(\cos\lambda x) + \frac{1}{x^2}(\sin\lambda x)\right]_1^2\right\}$$

$$= \frac{2}{\pi}\left\{\left[-\frac{1}{x}\underbrace{(1-1)}_{0}\underbrace{(\cos(1)x)}_{x} + \frac{1}{x^2}\underbrace{(\sin(1)x)}_{x}\right] - \left[-\frac{1}{x}\underbrace{(0-1)}_{-1}\underbrace{(\cos(0)x)}_{0} + \frac{1}{x^2}\underbrace{(\sin(0)x)}_{0}\right]\right.$$
$$\left. + \left[-\frac{1}{x}\underbrace{(2+1)}_{3}\underbrace{(\cos(2)x)}_{2x} + \frac{1}{x^2}\underbrace{(\sin(2)x)}_{2x}\right] - \left[-\frac{1}{x}\underbrace{(1+1)}_{2}\underbrace{(\cos(1)x)}_{x} + \frac{1}{x^2}\underbrace{(\sin(1)x)}_{x}\right]\right\}$$

$$= \frac{2}{\pi}\left\{\left[-\frac{1}{x}\underbrace{(0)(\cos x)}_{0} + \frac{1}{x^2}(\sin x)\right] - \left[-\frac{1}{x}(-1)\underbrace{(\cos 0)}_{1} + \frac{1}{x^2}\underbrace{(\sin 0)}_{0}\right]\right.$$
$$\left. + \left[-\frac{1}{x}(3)(\cos 2x) + \frac{1}{x^2}(\sin 2x)\right] - \left[-\frac{1}{x}(2)(\cos x) + \frac{1}{x^2}(\sin x)\right]\right\}$$

$$= \frac{2}{\pi}\left\{\left[0 + \frac{\sin x}{x^2}\right] - \left[-\frac{1}{x}(-1)(1) + \underbrace{\frac{1}{x^2}(0)}_{0}\right] + \left[-\frac{1}{x}(3)(\cos 2x) + \frac{1}{x^2}(\sin 2x)\right] - \left[-\frac{1}{x}(2)(\cos x) + \frac{1}{x^2}(\sin x)\right]\right\}$$

$$= \frac{2}{\pi}\left\{\left[\frac{\sin x}{x^2}\right] - \left[\frac{1}{x} + 0\right] + \left[-\frac{3}{x}\cos 2x + \frac{\sin 2x}{x^2}\right] - \left[-\frac{2}{x}(\cos x) + \frac{\sin x}{x^2}\right]\right\}$$

$$= \frac{2}{\pi}\left\{\left[\frac{\sin x}{x^2} - \frac{1}{x} - \frac{3}{x}\cos 2x + \frac{\sin 2x}{x^2} + \frac{2}{x}(\cos x) - \frac{\sin x}{x^2}\right]\right.$$

$$f(x) = \frac{2}{\pi}\left\{\frac{\sin x}{x^2} - \frac{1}{x} - \frac{3\cos 2x}{x} + \frac{\sin 2x}{x^2} + \frac{2\cos x}{x} - \frac{\sin x}{x^2}\right\} = \frac{2}{\pi}\left\{\frac{1}{x}(-1 - 3\cos 2x + 2\cos x) + \frac{1}{x^2}(\sin x + \sin 2x - \sin x)\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{x}(2\cos x - 3\cos 2x - 1) + \frac{1}{x^2}(\cancel{\sin x} + \sin 2x - \cancel{\sin x})\right\}$$

$$\therefore f(x) = \frac{2}{\pi}\left[\frac{2\cos x - 3\cos 2x - 1}{x} + \frac{\sin 2x}{x^2}\right] \checkmark \qquad \text{...Ans.}$$

Which is required.

Ex. 3.5.28 : Solve the integral equation

$$\int_0^\infty f(x)\sin\lambda x\, dx = \begin{cases} 1-\lambda & 0 \le \lambda \le 1 \\ 0 & \lambda > 1 \end{cases}$$

Soln. :

Consider given equation,

$$\int_0^\infty f(x)\sin\lambda x\, dx = \begin{cases} 1-\lambda & 0 \le \lambda \le 1 \\ 0 & \lambda > 1 \end{cases} \qquad \text{...(1)}$$

We know Fourier sine transform of f(x) is given by,

$$F_S(\lambda) = \int_0^\infty f(u)\sin\lambda u\, du \qquad \text{...(2)}$$

Since value of definite integral is independent of variable, so changing variable u to x in Equation (2),

$$F_S(\lambda) = \int_0^\infty f(x)\sin\lambda x\, dx$$

$$\text{OR} \int_0^\infty f(x)\sin\lambda x\, dx = F_S(\lambda) \qquad \text{...(3)}$$

Comparing Equation (1) and (3) we observe that the RHS of Equation (1) is Fourier sine transform of an unknown function f(x) which we want to find out.

$$\therefore F_S(\lambda) = \begin{cases} 1-\lambda & 0 \le \lambda \le 1 \\ 0 & \lambda > 1 \end{cases} \qquad \text{...(4)}$$

In this case value of f(x) is given by inverse Fourier sine transform.

(I) We have, by inverse Fourier sine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_S(\lambda)\sin\lambda x\, d\lambda$$

$$= \frac{2}{\pi}\left[\int_0^1 f_S(\lambda)\sin\lambda x\, d\lambda + \int_0^\infty F_S(\lambda)\sin\lambda x\, d\lambda\right]$$

$$= \frac{2}{\pi}\left[\int_0^1 (1-\lambda)\sin\lambda x\, d\lambda + \int_1^\infty 0\sin\lambda x\, d\lambda\right]$$

Using value of $F_S(\lambda)$

$$f(x) = \frac{2}{\pi}\int_0^1 (1-\lambda)\sin\lambda x\, d\lambda \qquad \text{...(5)}$$

➔ **Using formula of generalized rule of integration**

$$\text{...}\left[\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right]$$

Where u is algebraic polynomial

v is exponential / sine / cosine term

Here dash denotes order of derivative and suffix denotes integration.

Here variable of integration is λ

$u = 1-\lambda$, $v = \sin\lambda x$

$u' = -1$, $u'' = 0$

$$v_1 = -\frac{\cos\lambda x}{x}, \quad v_2 = -\frac{\sin\lambda x}{x^2}$$

$$f(x) = \frac{2}{\pi}\left[\underset{\substack{\uparrow\\ u}}{(1-\lambda)}\underset{\substack{\uparrow\\ v_1}}{\left(\frac{-\cos\lambda x}{x}\right)} - \underset{\substack{\uparrow\\ u'}}{(-1)}\underset{\substack{\uparrow\\ v_2}}{\left(\frac{-\sin\lambda x}{x^2}\right)}\right]_0^1$$

$$= \frac{2}{\pi}\left\{-\left(\frac{1}{x}\right)(1-\lambda)(\cos\lambda x) - \left(\frac{\sin\lambda x}{x^2}\right)\right\}_0^1$$

$$= \frac{2}{\pi}\left\{\left[-\frac{1}{x}\underbrace{(1-1)}_{0}\underbrace{(\cos(1)x)}_{x} - \frac{1}{x^2}\underbrace{(\sin(1)x)}_{x}\right]\right.$$
$$\left. - \left[-\frac{1}{x}\underbrace{(1-0)}_{1}\underbrace{(\cos(0)x)}_{0} - \frac{1}{x^2}\underbrace{(\sin(0)x)}_{0}\right]\right\}$$

$$= \frac{2}{\pi}\left\{\left[\underbrace{-\frac{1}{x}(0)(\cos x)}_{0} - \frac{\sin x}{x^2}\right] - \left[-\frac{1}{x}(1)\underbrace{(\cos(0)}_{1} - \underbrace{\frac{1}{x^2}(0)}_{0})\right]\right.$$

$$= \frac{2}{\pi}\left\{\left[0 - \frac{\sin x}{x^2}\right] - \left[-\frac{1}{x}(1)(1) - 0\right]\right.$$

$$= \frac{2}{\pi}\left[-\frac{\sin x}{x^2} + \frac{1}{x}\right] = \frac{2}{\pi}\left[-\frac{\sin x}{x^2} + \frac{x}{x^2}\right] = \frac{2}{\pi}\left[-\frac{\sin x + x}{x^2}\right]$$

$$f(x) = \frac{2}{\pi}\left[\frac{x - \sin x}{x^2}\right] \checkmark \qquad \text{...Ans.}$$

Which is required.

Ex. 3.5.29 : Solve the integral equation

$$\int_0^\infty f(x)\sin\lambda x\, dx = \begin{cases} 1 & 0 \le \lambda \le 1 \\ 2 & 1 \le \lambda \le 2 \\ 0 & \lambda > 2 \end{cases}$$

Soln. :

Consider given equation,

$$\int_0^\infty f(x)\sin\lambda x\, dx = \begin{cases} 1 & 0 \le \lambda \le 1 \\ 2 & 1 \le \lambda \le 2 \\ 0 & \lambda > 2 \end{cases} \qquad \text{...(1)}$$

We know Fourier sine transform of f(x) is given by,

$$F_S(\lambda) = \int_0^\infty f(u)\sin\lambda u\, du \qquad \text{...(2)}$$

Since value of definite integral is independent of variable,so changing variable u to x in Equation (2),

$$F_S(\lambda) = \int_0^\infty f(x)\sin\lambda x\,dx$$

$$\text{OR} \int_0^\infty f(x)\sin\lambda x\,dx = F_S(\lambda) \qquad ...(3)$$

Comparing Equation (1) and (3) we observe that the RHS of Equation (1) is Fourier sine transform of the an unknown function f(x) which we want to find out.

$$\therefore F_S(\lambda) = \begin{cases} 1 & 0\le\lambda\le1 \\ 2, & 1\le\lambda\le2 \\ 0 & \lambda>2 \end{cases} \qquad ...(4)$$

In this case value of f(x) is given by inverse Fourier sine transform.

(I) We have, by inverse Fourier sine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_S(\lambda)\sin\lambda x\,d\lambda$$

By the definition of function $F_S(\lambda)$, we split the integral on RHS as,

$$f(x) = \frac{2}{\pi}\left[\int_0^1 F_S(\lambda)\sin\lambda x\,d\lambda + \int_1^2 F_S(\lambda)\sin\lambda x\,d\lambda + \int_2^\infty F_S(\lambda)\sin\lambda x\,d\lambda\right]$$

Using values of function $F_S(\lambda)$ in respective interval,

$$f(x) = \frac{2}{\pi}\left[\int_0^1 (1)\sin\lambda x\,d\lambda + \int_1^2 (2)\sin\lambda x\,d\lambda + + \int_2^\infty (0)\sin\lambda x\,d\lambda\right]$$

$$= \frac{2}{\pi}\left[\int_0^1 \sin\lambda x\,d\lambda + 2\int_1^2 \sin\lambda x\,d\lambda\right] \qquad ...(5)$$

➔ **Using standard formula of integration**

$$...\left[\int \sin\lambda x\,d\lambda = -\frac{\cos\lambda x}{x}\right]$$

$$f(x) = \frac{2}{\pi}\left\{\left[\frac{-\cos\lambda x}{x}\right]_0^1 + 2\left[\frac{-\cos\lambda x}{x}\right]_1^2\right\}$$

$$= \frac{2}{\pi}\left\{\left[-\frac{1}{x}(\cos\lambda x)\right]_0^1 + \left[-\frac{2}{x}(\cos\lambda x)\right]_1^2\right\}$$

$$= \frac{2}{\pi}\left\{\left[-\frac{1}{x}(\underbrace{\cos(1)x}_{x} - \underbrace{\cos(0)x}_{0})\right] + \left[-\frac{2}{x}(\underbrace{\cos(2)x}_{2x} - \underbrace{\cos(1)x}_{x})\right]\right\}$$

$$= \frac{2}{\pi}\left\{\left[-\frac{1}{x}(\cos x - \underbrace{\cos 0}_{1})\right] + \left[-\frac{2}{x}(\cos 2x - \cos x)\right]\right\}$$

$$= \frac{2}{\pi}\left\{\left[-\frac{1}{x}(\cos x - 1) - \frac{2}{x}(\cos 2x) + \frac{2}{x}\cos x\right]\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{1}{x}\cos x + \frac{1}{x} - \frac{2\cos 2x}{x} + \frac{2}{x}\cos x\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-\cos x + 1 - 2\cos 2x + 2\cos x}{x}\right\}$$

$$= \frac{2}{\pi}\left[\frac{-\cos x + 2\cos x + 1 - 2\cos 2x}{x}\right]$$

$$f(x) = \frac{2}{\pi}\left[\frac{\cos x - 2\cos 2x + 1}{x}\right] ✓ \qquad ...\text{Ans.}$$

Which is required.

Ex. 3.5.30 : Solve the integral equation

$$\int_0^\infty f(u)\sin\lambda u\,du = \begin{cases} 1-\lambda & 0\le\lambda\le1 \\ 2, & 1\le\lambda\le2 \\ 0 & \lambda>2 \end{cases}$$

Soln. :

Consider given equation,

$$\int_0^\infty f(x)\sin\lambda x\,dx = \begin{cases} 1-\lambda & 0\le\lambda\le1 \\ 2, & 1\le\lambda\le2 \\ 0 & \lambda>2 \end{cases} \qquad ...(1)$$

We know Fourier sine transform of f(x) is given by,

$$F_S(\lambda) = \int_0^\infty f(u)\sin\lambda u\,du \qquad ...(2)$$

Since value of definite integral is independent of variable,so changing variable u to x in Equation (2),

$$F_S(\lambda) = \int_0^\infty f(x)\sin\lambda x\,dx$$

$$\text{OR} \int_0^\infty f(x)\sin\lambda x\,dx = F_S(\lambda) \qquad ...(3)$$

Comparing Equation (1) and (3) we observe that the RHS of Equation (1) is Fourier sine transform of the an unknown function f(x) which we want to find out.

$$\therefore F_S(\lambda) = \begin{cases} 1-\lambda & 0\le\lambda\le1 \\ 2, & 1\le\lambda\le2 \\ 0 & \lambda>2 \end{cases} \qquad ...(4)$$

In this case value of f(x) is given by inverse Fourier sine transform.

(I) We have, by inverse Fourier sine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_S(\lambda)\sin\lambda x\,d\lambda$$

By the definition of function $F_S(\lambda)$, we split the integral on RHS as,

$$f(x) = \frac{2}{\pi}\left[\int_0^1 F_S(\lambda)\sin\lambda x\,d\lambda + \int_1^2 F_S(\lambda)\sin\lambda x\,d\lambda + + \int_2^\infty F_S(\lambda)\sin\lambda x\,d\lambda\right]$$

Using values of function $F_S(\lambda)$ in respective interval,

$$f(x) = \frac{2}{\pi}\left[\int_0^1 (1-\lambda)\sin\lambda x\,d\lambda + \int_1^2 (2)\sin\lambda x\,d\lambda + + \int_2^\infty (0)\sin\lambda x\,d\lambda\right]$$

$$= \frac{2}{\pi}\left[\int_0^1 (1-\lambda)\sin\lambda x\,d\lambda + 2\int_1^2 \sin\lambda x\,d\lambda\right] \qquad ...(5)$$

➔ **Using formula of generalized rule of integration**

$$...\left[\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...\right]$$

Where u is algebraic polynomial

v is exponential / sine / cosine term

Here dash denotes order of derivative and suffix denotes integration.

Here variable of integration is λ

$u = 1-\lambda,\ v = \sin\lambda x$

$u' = -1, u'' = 0$

$v_1 = -\frac{\cos\lambda x}{x},\ v_2 = -\frac{\sin\lambda x}{x^2}$

In the second integral,

➔ **Using standard formula of integration**

$$...\left[\int \sin\lambda x\,d\lambda = -\frac{\cos\lambda x}{x}\right]$$

$$f(x) = \frac{2}{\pi}\left\{\left[\underset{\uparrow\ u}{(1-\lambda)}\underset{\uparrow\ v_1}{\left(\frac{-\cos\lambda x}{x}\right)} - \underset{\uparrow\ u'}{(-1)}\underset{\uparrow\ v_2}{\left(\frac{-\sin\lambda x}{x^2}\right)}\right]_0^1 + 2\left[\frac{-\cos\lambda x}{x}\right]_1^2\right\}$$

$$= \frac{2}{\pi}\left\{\left[-\frac{1}{x}(1-\lambda)(\cos\lambda x) - \frac{\sin\lambda x}{x^2}\right]_0^1 + \left[-\frac{2}{x}(\cos\lambda x)\right]_1^2\right\}$$

$$= \frac{2}{\pi}\left\{\left[-\frac{1}{x}\underbrace{(1-1)}_{0}\underbrace{(\cos(1)x)}_{x} - \frac{\sin(1)x}{x^2}\right] - \left[-\frac{1}{x}\underbrace{(1-0)}_{1}\underbrace{(\cos(0)x)}_{0} - \frac{\sin(0)x}{x^2}\right] + \left[-\frac{2}{x}\underbrace{(\cos(2)x)}_{2x} - \left[-\frac{2}{x}\underbrace{(\cos(1)x)}_{x}\right]\right]\right\}$$

$$= \frac{2}{\pi}\left\{\left[\underbrace{-\frac{1}{x}(0)(\cos x)}_{0} - \frac{\sin x}{x^2}\right] - \left[-\frac{1}{x}(1)\underbrace{(\cos 0)}_{0} - \underbrace{\frac{\sin 0}{x^2}}_{0}\right] + \left[-\frac{2}{x}(\cos 2x)\right] - \left[-\frac{2}{x}(\cos x)\right]\right\}$$

$$= \frac{2}{\pi}\left\{\left[0 - \frac{\sin x}{x^2}\right] - \left[-\frac{1}{x}(1)(1) - 0\right] + \left[-\frac{2}{x}(\cos 2x)\right] - \left[-\frac{2}{x}(\cos x)\right]\right\}$$

$$= \frac{2}{\pi}\left\{\left[-\frac{\sin x}{x^2}\right] + \frac{1}{x} + \left[-\frac{2}{x}(\cos 2x)\right] - \left[-\frac{2}{x}(\cos x)\right]\right\}$$

$$= \frac{2}{\pi}\left[-\frac{\sin x}{x^2} + \frac{1}{x} - \frac{2\cos 2x}{x} + \frac{2\cos x}{x}\right]$$

$$= \frac{2}{\pi}\left\{\frac{1}{x}(1 - 2\cos 2x + 2\cos x) - \frac{\sin x}{x^2}\right\}$$

$$f(x) = \frac{2}{\pi}\left\{\frac{1 - 2\cos 2x + \cos x}{x} - \frac{\sin x}{x^2}\right\} ✓ \qquad ...\text{Ans.}$$

Which is required.

Ex.3.5.31 : Solve the integral equation

$$\int_0^\infty f(x)\sin\lambda x\,dx = \begin{cases} 2-\lambda & 0\le\lambda\le2 \\ 0 & \lambda>2 \end{cases}$$

Soln. :

Consider given equation,

$$\int_0^\infty f(x)\sin\lambda x\,d\lambda = \begin{cases} 2-\lambda & 0\le\lambda\le2 \\ 0 & \lambda>2 \end{cases} \qquad ...(1)$$

We know Fourier sine transform of f(x) is given by,

$$F_S(\lambda) = \int_0^\infty f(u)\sin\lambda u\,du \qquad ...(2)$$

Since value of definite integral is independent of variable,so changing variable u to x in Equation (2),

$$F_S(\lambda) = \int_0^\infty f(x)\sin\lambda x\,dx$$

$$\text{OR} \int_0^\infty f(x)\sin\lambda x\,dx = F_S(\lambda) \qquad ...(3)$$

Comparing Equation (1) and (3) we observe that the RHS of Equation (1) is Fourier sine transform of an unknown function f(x) which we want to find out.

$$\therefore F_S(\lambda) = \begin{cases} 2-\lambda & 0\le\lambda\le2 \\ 0, & \lambda>2 \end{cases} \qquad ...(4)$$

In this case value of f(x) is given by inverse Fourier sine transform.

(I) We have, by inverse Fourier sine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_S(\lambda)\sin\lambda x\,d\lambda$$

Using value of $F_S(\lambda)$, we split the integrals as,

$$f(x) = \frac{2}{\pi}\left[\int_0^2 F_S(\lambda)\sin\lambda x\,d\lambda + \int_2^\infty F_S(\lambda)\sin\lambda x\,d\lambda\right]$$

$$= \frac{2}{\pi}\left[\int_0^2 (2-\lambda)\sin\lambda x\, d\lambda + \int_2^\infty (0)\sin\lambda x\, d\lambda\right]$$

$$= \frac{2}{\pi}\left[\int_0^2 (2-\lambda)\sin\lambda x\, d\lambda\right] \quad ...(5)$$

➔ **Using formula of generalized rule of integration**

$$...\left[\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...\right]$$

Where u is algebraic polynomial

v is exponential / sine / cosine term

Here dash denotes order of derivative and suffix denotes integration.

Here variable of integration is λ

$u = 2-\lambda,\ v = \sin\lambda x$

$u' = -1,\ u'' = 0$

$v_1 = -\frac{\cos\lambda x}{x},\ v_2 = -\frac{\sin\lambda x}{x^2}$

$$f(x) = \frac{2}{\pi}\left[\underset{u}{(2-\lambda)}\underset{v_1}{\left(\frac{-\cos\lambda x}{x}\right)} - \underset{u'}{(-1)}\underset{v_2}{\left(\frac{-\sin\lambda x}{x^2}\right)}\right]_0^2$$

$$= \frac{2}{\pi}\left\{-\left(\frac{1}{x}\right)(2-\lambda)(\cos\lambda x) - \left(\frac{\sin\lambda x}{x^2}\right)\right\}_0^2$$

$$= \frac{2}{\pi}\left\{\left[-\frac{1}{x}\underbrace{(2-2)}_{0}\underbrace{(\cos(2)x)}_{2x} - \frac{1}{x^2}\underbrace{(\sin(2)x)}_{2x}\right]\right.$$

$$\left.-\left[-\frac{1}{x}\underbrace{(2-0)}_{2}\underbrace{(\cos(0)x)}_{0} - \frac{1}{x^2}\underbrace{(\sin(0)x)}_{0}\right]\right\}$$

$$= \frac{2}{\pi}\left\{\left[-\frac{1}{x}\underbrace{(0)(\cos 2x)}_{0} - \frac{\sin 2x}{x^2}\right] - \left[-\frac{1}{x}\underbrace{(2)(\cos(0)}_{1} - \frac{1}{x^2}\underbrace{(0))}_{0}\right]\right\}$$

$$= \frac{2}{\pi}\left\{\left[0 - \frac{\sin 2x}{x^2}\right] - \left[-\frac{1}{x}(2)(1) - 0\right]\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{\sin 2x}{x^2} - \left[-\frac{2}{x}\right]\right\}$$

$$= \frac{2}{\pi}\left[-\frac{\sin 2x}{x^2} + \frac{2}{x}\right]$$

Multiplying and dividing by x in second term

$$f(x) = \frac{2}{\pi}\left[-\frac{\sin 2x}{x^2} + \frac{2}{x}\times\frac{x}{x}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\sin 2x}{x^2} + \frac{2x}{x^2}\right] = \frac{2}{\pi}\left[-\frac{\sin 2x + 2x}{x^2}\right]$$

$$f(x) = \frac{2}{\pi}\left[\frac{2x - \sin 2x}{x^2}\right] \checkmark \quad ...\text{Ans.}$$

Which is required.

Ex. 3.5.32 : Solve the following integral equation :

$$\int_0^\infty f(x)\cos\lambda x\, dx = e^{-\lambda},\ \lambda > 0$$

Soln. :

Consider given equation,

$$\int_0^\infty f(x)\cos\lambda x\, dx = e^{-\lambda},\ \lambda > 0 \quad ...(1)$$

We know Fourier cosine transform of f(x) is given by,

$$F_C(\lambda) = \int_0^\infty f(u)\cos\lambda u\, du \quad ...(2)$$

Since value of definite integral is independent of variable,so changing variable u to x in Equation (2),

$$F_C(\lambda) = \int_0^\infty f(x)\cos\lambda x\, dx$$

$$\text{OR} \int_0^\infty f(x)\cos\lambda x\, dx = F_C(\lambda) \quad ...(3)$$

Comparing Equation (1) and (3) we observe that the RHS of Equation (1) is Fourier cosine transform of an unknown function f(x) which we want to find out.

$$\therefore F_C(\lambda) = e^{-\lambda} \quad ...(4)$$

In this case value of f(x) is given by inverse Fourier cosine transform.

(I) We have, by inverse Fourier cosine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_C(\lambda)\cos\lambda x\, d\lambda$$

Using value of $F_C(\lambda)$

$$f(x) = \frac{2}{\pi}\int_0^\infty e^{-\lambda}\cos\lambda x\, d\lambda \quad ...(5)$$

➔ **Using Standard formula**

$$...\left[\int_0^\infty e^{-ax}\cos(bx+c)\, dx = \frac{a}{a^2+b^2}\right]$$

Here $x = \lambda,\ a = 1,\ b = x$

$$f(x) = \frac{2}{\pi}\left[\frac{1}{1+x^2}\right] \checkmark \quad ...\text{Ans.}$$

Which is required.

Ex. 3.5.33 : Solve the integral equation

$$\int_0^\infty f(x)\sin\lambda x\, dx = \frac{e^{-3\lambda}}{\lambda}$$

Soln. :

Consider given equation, $\int_0^\infty f(x)\sin\lambda x\, d\lambda = \frac{e^{-3\lambda}}{\lambda}$ ...(1)

We know Fourier sine transform of f(x) is given by,

$$F_S(\lambda) = \int_0^\infty f(u)\sin\lambda u\, du \quad ...(2)$$

Since value of definite integral is independent of variable,so changing variable u to x in Equation (2),

$$F_S(\lambda) = \int_0^\infty f(x)\sin\lambda x\, dx$$

$$\text{OR} \int_0^\infty f(x)\sin\lambda x\, dx = F_S(\lambda) \quad ...(3)$$

Comparing Equation (1) and (3) we observe that the RHS of Equation (1) is Fourier sine transform of an unknown function f(x) which we want to find out.

$$\therefore F_S(\lambda) = \frac{e^{-3\lambda}}{\lambda} \quad ...(4)$$

In this case value of f(x) is given by inverse Fourier sine transform.

(I) We have, by inverse Fourier sine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_S(\lambda)\sin\lambda x\, d\lambda$$

Using value of $F_S(\lambda)$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{e^{-3\lambda}}{\lambda}\sin\lambda x\, d\lambda \quad ...(5)$$

Consider,

$$I(x) = \int_0^\infty \frac{e^{-3\lambda}}{\lambda}\sin\lambda x\, d\lambda \quad ...(6)$$

➔ **Using standard rule of DUIS**

$$...\left[\text{If } I(\alpha) = \int_a^b f(x,\alpha)\, dx \text{ then } \frac{dI(\alpha)}{d\alpha} = \int_a^b\left[\frac{\partial}{\partial\alpha}f(x,\alpha)\right] dx\right.$$

$$\left.\text{where a and b are constants}\right]$$

Here variable of integral is λ

By DUIS

$$\frac{dI}{dx} = \int_0^\infty \frac{\partial}{\partial x}\left[\frac{e^{-3\lambda}}{\lambda}\sin\lambda x\right] d\lambda$$

Here variable of derivative is x therefore terms of λ can be treated as constants

$$\therefore \frac{dI}{dx} = \int_0^\infty \left(\frac{e^{-3\lambda}}{\lambda}\right)\frac{\partial}{\partial x}(\sin\lambda x)\, d\lambda \quad ...(7)$$

➔ **Using standard formula of derivative**

$$...\left[\frac{\partial}{\partial x}(\sin Ax) = A\cos Ax \text{ for any constant } A\right]$$

Here $A = \lambda$

$$\therefore \frac{dI}{dx} = \int_0^\infty \frac{e^{-3\lambda}}{\lambda}(\lambda\cos\lambda x)\, dx$$

$$= \int_0^\infty \frac{e^{-3\lambda}}{\cancel{\lambda}}(\cancel{\lambda}\cos\lambda x)\, d\lambda$$

$$\frac{dI}{dx} = \int_0^\infty e^{-3\lambda}\cos\lambda x\, dx \quad ...(8)$$

➔ **Using standard formula**

$$...\left[\int_0^\infty e^{-ax}\cos(bx+c)\, dx = \frac{a}{a^2+b^2}\right]$$

Here $a = 3,\ b = x$

$$\frac{dI}{dx} = \frac{3}{(3)^2+x^2} = \frac{3}{9+x^2}$$

$$dI = \frac{3}{9+x^2}\, dx$$

This differential equation is in Variable separable Form.On integrating both sides,

$$\therefore \int dI = \int \frac{3}{9+x^2}\, dx + A \text{ (A integral constant)}$$

$$= 3\int \frac{1}{3^2+x^2}\, dx + A \quad ...(9)$$

➔ **Using standard formula of integration**

$$...\left[\int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

Here variable of integration is λ

$$\therefore a = 3$$

$$I(x) = 3\left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right] + A = \cancel{3}\left(\frac{1}{\cancel{3}}\right)\tan^{-1}\frac{x}{3} + A$$

$$\therefore I(x) = \tan^{-1}\left(\frac{x}{3}\right) + A \quad ...(10)$$

We find value of constant of integration A

For this, Put $x = 0$ in Equation (10)

$$I(0) = \tan^{-1}\left(\frac{0}{3}\right) + A \Rightarrow I(0) = \tan^{-1} 0 + A$$

$$\Rightarrow I(0) = 0 + A \Rightarrow I(0) = A \qquad ...(11)$$

But from Equation (6)

$$I(0) = \int_0^\infty \frac{e^{-3\lambda}}{\lambda} \sin\lambda(0)\, dx \Rightarrow I(0) = \int_0^\infty \frac{e^{-3\lambda}}{\lambda} \underbrace{\sin(0)}_{0}\, dx$$

$$= \int_0^\infty \underbrace{\frac{e^{-3\lambda}}{\lambda}(0)}_{0}\, dx = \int_0^\infty 0\, dx = 0$$

$$\Rightarrow I(0) = 0 \qquad ...(12)$$

From Equation (11) and (12) we get, $A = 0$

$\therefore$ From Equation (10),

$$I(x) = \tan^{-1}\left(\frac{x}{3}\right)$$

Put value of I(x) in Equation (5),

$$f(x) = \frac{2}{\pi}\tan^{-1}\left(\frac{x}{3}\right) \checkmark \qquad ...\text{Ans.}$$

Which is required.

Syllabus Topic : Parseval's identity

3.5.8 Parseval's Identity

We know that, Fourier transform of f(x) is given by

$$F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$$

$\therefore$ Fourier transform of g(x) is given by

$$G(\lambda) = \int_{-\infty}^{\infty} g(u)\, e^{-i\lambda u}\, du$$

The inverse Fourier transform of $F(\lambda)$ and $G(\lambda)$ are respectively given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda \text{ and}$$

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\lambda)\, e^{i\lambda x}\, d\lambda$$

$$\therefore\ g^*(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G^*(\lambda)\, e^{-i\lambda x}\, d\lambda$$

then $\displaystyle\int_{-\infty}^{\infty} f(x)\, g^*(x)\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, G^*(\lambda)\, d\lambda$

Proof :

Consider, $\displaystyle\int_{-\infty}^{\infty} f(x)\, g^*(x)\, dx$

$$= \int_{-\infty}^{\infty} f(x)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} G^*(\lambda)\, e^{-i\lambda x}\, d\lambda\right] dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G^*(\lambda)\left[\int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx\right] d\lambda \qquad ...(1)$$

But $\displaystyle\int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx = F(\lambda)$

$\therefore$ From Equation (1)

$$\int_{-\infty}^{\infty} f(x)\, g^*(x)\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} G^*(\lambda)\, F(\lambda)\, d\lambda$$

Hence prove.

In particular, if

$$f(x) = g(x)$$

$$\int_{-\infty}^{\infty} f(x)\, f^*(x)\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, F^*(\lambda)\, d\lambda$$

$$\underbrace{\int_{-\infty}^{\infty} |f(x)|^2\, dx}_{\text{energy of f(x)}} = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\lambda)|^2\, d\lambda}_{\text{energy spectral density of (x)}}$$

Exercise

Ex. 1 : Using Fourier integral representation show that,

$$\int_0^\infty \left(\frac{1-\cos\lambda\pi}{\lambda}\right)\sin\lambda x\, d\lambda = \begin{cases} \frac{\pi}{2} & 0 < x < \pi \\ 0 & x > \pi \end{cases}$$

Ex. 2 : Using Fourier integral representation show that,

$$\int_0^\infty \frac{\lambda^3 \sin\lambda x}{\lambda^4 + 4}\, d\lambda = \frac{\pi}{2} e^{-x}\cos x \text{ where } x > 0.$$

Ex. 3 : Using the Fourier integral representation show that,

$$\int_0^\infty \frac{\sin\lambda \cos x\lambda}{\lambda}\, d\lambda = \frac{\pi}{2} \text{ when } 0 \le x \le 1.$$

Ex. 4 : Find Fourier transform of $f(x) = \begin{cases} \frac{\pi}{2}\sin x & |x| < \pi \\ 0 & |x| > \pi \end{cases}$.

Ex. 5 : Find the transform of $f(x) = \begin{cases} a - |x| & |x| \le a \\ 0 & |x| > a \end{cases}$.

Ex. 6 : Find Fourier transform of $f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$.

Ans. : $F(\lambda) = \dfrac{2\sin a\lambda}{\lambda}$

Ex. 7 : Find the Fourier transform of

$$f(x) = \begin{cases} a^2 - x^2 & |x| \le a \\ 0 & |x| > a \end{cases}.$$

Ans. : $F(\lambda) = 2\left\{\dfrac{(a^2\lambda^2 - 2)\sin a\lambda + 2a\lambda\cos a\lambda}{\lambda^3}\right\}$

Ex. 8 : Find the Fourier transform of $f(x) = \begin{cases} \frac{1}{2a} & |x| \le a \\ 0 & |x| > a \end{cases}$.

Ans. : $\dfrac{\sin\lambda a}{\lambda a}$

Ex. 9 : Find the Fourier cosine transform of $f(x) = e^{-ax^2}$; $a > 0 \cdot x > 0$.

Ex. 10 : Prove that the Sine Fourier transform of $f(x) = \dfrac{1}{x}$ is $\sqrt{\dfrac{\pi}{2}}$.

Ex. 11 : Show that the Fourier transform of $f(x) = e^{-|x|}$ is $\dfrac{2}{1+\lambda^2}$.

Ex. 12 : Find the Fourier cosine transform of

$$f(x) = \begin{cases} \cos x & 0 < x < a \\ 0 & x \ge a \end{cases}.$$

Ans. : $F(\lambda) = \dfrac{1}{2}\left\{\dfrac{\sin(1+\lambda)a}{1-\lambda} + \dfrac{\sin(1-\lambda)a}{1-\lambda}\right\}$

Ex. 13 : Find the Fourier sine transform of $f(x) = \dfrac{x}{a^2 + x^2}$.

Ans. : $\sqrt{\dfrac{\pi}{2}}\, e^{-a\lambda}$ OR $e^{-a\lambda}$

Ex. 14 : Find the Fourier sine transform of $f(x) = \dfrac{1}{x(x^2 + a^2)}$.

Ans. : $\dfrac{\pi}{2a^2}(1 - e^{-a\lambda})$

Ex. 15 : Using Fourier cosine transform find f(x) is

$$F_C(\lambda) = \begin{cases} \sqrt{\frac{2}{\pi}}\left(a - \frac{\lambda}{2}\right) & \\ 0 & \lambda \le 2a \end{cases}.$$

Ex. 16 : Solve the integral equation $\displaystyle\int_0^\infty f(x)\sin\lambda x\, dx = \frac{\lambda}{\lambda^2 + k^2}$.

Ans. : $f(x) = e^{-kx}, x > 0$

Chapter Ends...

□□□

Note

UNIT 3

Fourier Transform

Syllabus

Definitions – integral transforms ; Fourier integral theorem (without proof) ; Fourier sine and cosine integrals ; Complex form of Fourier integrals ; Fourier sine and cosine transforms ; Properties of Fourier transforms ; Parseval's identity for Fourier Transforms.

➤ Chapter 3 : Fourier Transform

CHAPTER 3

UNIT II

Fourier Transform

Multiple Choice Questions

Q. 1 The Fourier integral representation of f(x) in the interval $(-\infty, \infty)$ is

(a) $\frac{2}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(x-u)}\, du\, d\lambda$

(b) $\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-x)}\, du\, d\lambda$

(c) $\int_{0}^{\infty} f(u)\, e^{-i\lambda u}\, du \cdot \int_{0}^{\infty} f(\lambda)\, e^{i\lambda x}\, d\lambda$

(d) $\int_{0}^{\infty} f(u)\, F(\lambda)\, e^{i\lambda u}\, du\, d\lambda$

Ans. : (b)

Q. 2 The Fourier transform of the function f(x) defined in the interval $(-\infty, \infty)$ is ____.

(a) $2\int_{0}^{\infty} f(u)\, e^{-i\lambda u}\, du$

(b) $\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$

(c) $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda$

(d) $2\int_{0}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda$

Ans. : (b)

Q. 3 The inverse Fourier transform of $F(\lambda)$ in the interval $(-\infty, \infty)$ is defined as _____.

(a) $2\int_{0}^{\infty} f(u)\, e^{-i\lambda u}\, du$ (b) $\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$

(c) $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda$ (d) $2\int_{0}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda$

Ans. : (c)

Q. 4 The Fourier transform of the even function f(x) defined in $(-\infty, \infty)$ is $F(\lambda) =$ ____

(a) $\int_{0}^{\infty} f(u) \cos \lambda u\, du$ (b) $\int_{0}^{\infty} f(u) \sin \lambda u\, du$

(c) $\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$ (d) $\int_{-\infty}^{\infty} f(u)\, e^{i\lambda u}\, d\lambda$

Ans. : (a)

Q. 5 The Fourier transform of odd function f(x) defined in $(-\infty, \infty)$ is $F(\lambda) =$ _____

(a) $\int_{0}^{\infty} f(u) \cos \lambda u\, du$ (b) $\int_{0}^{\infty} f(u) \sin \lambda u\, du$

(c) $\int_{-\infty}^{\infty} f(u)\, e^{-ixu}\, du$ (d) $\int_{-\infty}^{\infty} f(u)\, e^{i\lambda u}\, d\lambda$

Ans. : (b)

Q. 6 The Fourier cosine transform of the function f(x) defined in $(0, \infty)$ is $F_C(\lambda) =$ ____.

(a) $\int_{0}^{\infty} f(u) \cos\lambda u\, du$ (b) $\frac{1}{2\pi}\int_{0}^{\infty} f(x) \cos\lambda\, dx$

(c) $2\int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx$ (d) $\frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$

Ans. : (a)

Q. 7 The Fourier sine transform of the function f(x) defined in $(0, \infty)$ is $F_S(\lambda) =$ ____.

(a) $\int_{-\infty}^{\infty} f(u) \sin u\, du$ (b) $\int_{0}^{\infty} f(u) \sin \lambda u\, du$

(c) $\frac{1}{2\pi}\int_0^\infty f(u)\sin\lambda u\,du$ (d) $\frac{1}{2\pi}\int_{-\infty}^\infty f(u)\sin\lambda u\,du$

Ans. : (b)

Q. 8 Fourier transform of the function $f(x)=\begin{cases}1 & , |x|<a\\ 0 & , |x|>a\end{cases}$ is $F(\lambda)=$____.

(a) $2\cos\lambda a$ (b) $\frac{\sin\lambda a}{\lambda}$, $\lambda\neq 0$

(c) $2\lambda\sin\lambda a$ (d) $2\lambda\cos\lambda a$

Ans. : (b)

Explanation : Given function is even function

$$\therefore\ F(\lambda) = \int_0^\infty f(u)\cos\lambda u\,du = \int_0^a \cos\lambda u\,du$$

$$=\left[\frac{\sin\lambda u}{\lambda}\right]_0^a = \frac{\sin\lambda a}{\lambda}$$

Q. 9 The Fourier transform of the function

$f(x)=\begin{cases}1-x^2 & |x|\le 1\\ 0 & |x|>1\end{cases}$ is $F(\lambda)=$____.

(a) $\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^2}$ (b) $\frac{\cos\lambda-\lambda\sin\lambda}{\lambda^2}$

(c) $2\left(\frac{\cos\lambda-\lambda\sin\lambda}{\lambda^3}\right)$ (d) $2\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right)$

Ans. : (d)

Q. 10 The Fourier transform of the function

$f(x)=\begin{cases}-3 & , |x|<1\\ 0 & , |x|>1\end{cases}$ is $F(\lambda)=$____

(a) $\frac{\sin\lambda}{\lambda}$ (b) $\frac{-3\sin\lambda}{\lambda}$

(c) $\frac{3\cos\lambda}{\lambda}$ (d) $\frac{\cos\lambda}{\lambda^2}$

Ans. : (b)

Q. 11 The Fourier transform of the function

$f(x)=\begin{cases}1 & , -2\le x\le 0\\ -1 & , 0\le x\le 2\end{cases}$ is $F(\lambda)=$____

(a) $\frac{\cos 2\lambda-1}{\lambda}$ (b) $\frac{1+\cos 2\lambda}{\lambda}$

(c) $\frac{1+\sin 2\lambda}{\lambda}$ (d) $\frac{\sin 2\lambda-1}{\lambda}$

Ans. : (a)

Q. 12 The Fourier sine transform of the function

$f(x)=e^{-ax}$; $a>0$ is $F_S(\lambda)=$____

(a) $\frac{-\lambda}{a^2+\lambda^2}$ (b) $\frac{-a}{a^2+\lambda^2}$

(c) $\frac{\lambda}{a^2+\lambda^2}$ (d) $\frac{a}{a^2+\lambda^2}$

Ans. : (c)

Explanation : Fourier sine transform (FST) is,

$$F_S(\lambda) = \int_0^\infty e^{-ax}\sin\lambda u\,du = \frac{\lambda}{a^2+\lambda^2} \quad \text{[By formula]}$$

Q. 13 The Fourier cosine transform of the function

$f(x)=e^{-ax}$; $a>0$ is $F_C(\lambda)=$____.

(a) $\frac{-\lambda}{a^2+\lambda^2}$ (b) $\frac{-a}{a^2+\lambda^2}$

(c) $\frac{\lambda}{a^2+\lambda^2}$ (d) $\frac{a}{a^2+\lambda^2}$

Ans. : (d)

Explanation : The Fourier Cosine Transform (FCT)

$$F_C(\lambda) = \int_0^\infty e^{-ax}\cos\lambda u\,du = \frac{a}{a^2+\lambda^2} \quad \text{[By formula]}$$

Q. 14 The Fourier transform of the function

$f(x)=\begin{cases}x & , x>0\\ 0 & , x<0\end{cases}$ is $F(\lambda)=$____.

(a) $\frac{1}{\lambda}$ (b) $\frac{-1}{\lambda^2}$ (c) $\frac{-1}{\lambda}$ (d) $\frac{1}{\lambda^2}$

Ans. : (b)

Explanation : Fourier transform (FT) is,

$$F(\lambda) = \int_0^\infty u\,e^{-i\lambda u}\,du$$

$$=\left\{(u)\left(\frac{e^{-i\lambda u}}{-i\lambda}\right)-(1)\left(\frac{e^{-i\lambda u}}{(-i\lambda^2)}\right)\right\}_0^\infty = 0-\frac{1}{\lambda^2}=-\frac{1}{\lambda^2}$$

Q. 15 The Fourier transform of the function :

$f(x)=\begin{cases}x^2 & x>0\\ 0 & x<0\end{cases}$ is $F(\lambda)=$____.

(a) $\frac{2i}{\lambda^3}$ (b) λ^2 (c) $\frac{1}{\lambda^2}$ (d) $\frac{2}{\lambda^3}$

Ans. : (a)

Explanation : The Fourier transform is,

$$F(\lambda) = \int_0^\infty u^2 e^{-i\lambda u}\,du$$

$$=\left\{(u)^2\left(\frac{e^{-i\lambda u}}{-i\lambda}\right)-(2u)\left(\frac{e^{-i\lambda u}}{-\lambda^2}\right)+(2)\left(\frac{e^{-i\lambda u}}{i\lambda^3}\right)\right\}_0^\infty$$

$$=0-\left[\frac{2}{i\lambda^3}\right]=\frac{2i}{\lambda^3}$$

Q. 16 The Fourier Transform of the even function

$f(x)=\begin{cases}1 & , |x|\le 1\\ 0 & , |x|>1\end{cases}$ is $F(\lambda)=\frac{\sin\lambda}{\lambda}$. using this, value of

$\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$ is ____.

(a) $2\pi f(x)$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{2}f(x)$ (d) $\frac{2}{\pi}f(x)$

Ans. : (c)

Q. 17 If Fourier Transform of the odd function $f(x)=\begin{cases}1 & , -2\le x\le 0\\ -1 & , 0\le x\le 0\end{cases}$ is $\left[\frac{\cos 2\lambda-1}{\lambda}\right]$ then using Fourier integral representation value of

$\int_0^\infty \frac{(\cos 2x-1)\sin 2x}{x}\,dx$ is _____.

(a) $\frac{\pi}{2}$ (b) $\frac{2}{\pi}$ (c) $\frac{-2}{\pi}$ (d) $\frac{-\pi}{2}$

Ans. : (d)

Q. 18 The Fourier Cosine Transform of the function

$f(x)=\begin{cases}0 & , x<0\\ \pi/2 & , x=0\\ \pi e^{-x} & , x>0\end{cases}$ is $F_C(\lambda)=$__.

(a) $\pi(1+\lambda^2)$ (b) $\frac{\pi}{1+\lambda^2}$

(c) $\frac{\pi\lambda}{1+\lambda^2}$ (d) $\pi\lambda(1+\lambda^2)$

Ans. : (b)

Explanation : Fourier transform (FCT) is,

$$F_C(\lambda) = \int_0^\infty \pi e^{-u}\cos\lambda u\,du = \pi\int_0^\infty e^{-u}\cos\lambda u\,du$$

$$=\pi\left[\frac{1}{1+\lambda^2}\right]$$

Q. 19 The Fourier Sine Transform of the function $f(x)=e^{-2x}\sinh x$, $(x>0)$ is $F_s(\lambda)=$__.

(a) $\frac{4\lambda}{(\lambda^2+1)(\lambda^2+9)}$ (b) $\frac{4}{(\lambda^2+1)(\lambda^2+9)}$

(c) $\frac{\lambda(\lambda^2+5)}{(\lambda^2+1)(\lambda^2+9)}$ (d) $\frac{(\lambda^2+5)}{(\lambda^2+1)(\lambda^2+9)}$

Ans. : (a)

Explanation :

$$f(x) = e^{-2x}\sinh x \quad (x>0) = e^{-2x}\left[\frac{e^x-e^{-x}}{2}\right]$$

$$=\frac{1}{2}\left[e^{-x}-e^{-3x}\right]$$

$\therefore$ Fourier Sine Transform [FST] is,

$$F_S(\lambda) = \int_0^\infty \frac{1}{2}(e^{-u}-e^{-3u})\sin\lambda u\,du$$

$$=\frac{1}{2}\left\{\int_0^\infty e^{-u}\sin\lambda u\,du-\int_0^\infty e^{-3u}\sin\lambda u\,du\right\}$$

$$=\frac{1}{2}\left[\frac{\lambda}{1+\lambda^2}-\frac{\lambda}{9+\lambda^2}\right]=\frac{\lambda}{2}\left[\frac{8}{(\lambda^2+1)(\lambda^2+9)}\right]$$

$$=\frac{4\lambda}{(\lambda^2+1)(\lambda^2+9)}$$

Q. 20 The Fourier Cosine Transform of the function $f(x)=e^{-2x}-e^{-3x}$ $(x>0)$ is $F_C(\lambda)=$____.

(a) $\frac{\lambda^2}{(\lambda^2+4)(\lambda^2+9)}$ (b) $\frac{6-\lambda^2}{(\lambda^2+4)(\lambda^2+9)}$

(c) $\frac{\lambda^2-6}{(\lambda^2+4)(\lambda^2+9)}$ (d) $\frac{-\lambda^2}{(\lambda^2+9)(\lambda^2+4)}$

Explanation : $f(x)=e^{-2x}-e^{-3x}$ $x>0$

Ans. : (b)

Fourier Cosine Transform (FCS)) is,

$$F(\lambda) = \int_0^\infty e^{-2u}\cos\lambda u\,du-\int_0^\infty e^{-3u}\cos\lambda u\,du$$

$$=\frac{2}{\lambda^2+4}-\frac{3}{\lambda^2+9}=\frac{-\lambda^2+6}{(\lambda^2+4)(\lambda^2+9)}$$

Q. 21 If Integral Equation is $\int_0^\infty f(x)\sin\lambda x\,dx = e^{-\lambda}$; $\lambda>0$ by using Inverse Fourier Sine transform [IFST] with $F_S(\lambda)=e^{-\lambda}$, $\lambda>0$ the value of $f(x)=$_____.

(a) $\frac{2}{\pi}\left(\frac{x+1}{x-1}\right)$ (b) $\frac{\pi}{2}\left(\frac{x}{1+x^2}\right)$

(c) $\frac{2}{\pi}\left(\frac{x}{1+x^2}\right)$ (d) $\frac{\pi}{2}\left(\frac{x}{1-x^2}\right)$

Ans. : (c)

Explanation : $F_S(\lambda)=e^{-\lambda}$ $\lambda>0$

$\therefore$ Inverse Fourier Sine Transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_S(\lambda)\sin\lambda x\,d\lambda = \frac{2}{\pi}\int_0^\infty e^{-\lambda}\sin\lambda x\,d\lambda$$

$$=\frac{2}{\pi}\left[\frac{x}{1+x^2}\right]$$

Q. 22 The Fourier Cosine Transform of the function $f(x)=e^{-3x}\cosh x$ $(x>0)$ is $F_C(\lambda)=$____.

(a) $\frac{\lambda}{(\lambda^2+4)(\lambda^2+16)}$ (b) $\frac{3(\lambda^2+8)}{(\lambda^2+4)(\lambda^2+16)}$

(c) $\frac{\lambda^2-8}{(\lambda^2+4)(\lambda^2+16)}$ (d) $\frac{\lambda^2+7}{(\lambda^2+4)(\lambda^2-16)}$

Ans. : (b)

Explanation : $f(x)=e^{-3x}\cosh x$ $(x>0)$

$$f(x) = e^{-3x}\left[\frac{e^x+e^{-x}}{2}\right]=\frac{1}{2}\left[e^{-2x}+e^{-4x}\right]$$

$\therefore$ Fourier Cosine Transform (FCT) is,

$$F_C(\lambda) = \int_0^\infty \frac{1}{2}\left[e^{-2u}+e^{-4u}\right]\cos\lambda u\,du$$

$$= \frac{1}{2}\left[\frac{2}{4+\lambda^2}+\frac{4}{16+\lambda^2}\right] = \frac{3(\lambda^2+8)}{(\lambda^2+4)(\lambda^2+16)}$$

Q. 23 The Fourier Sine Transform of the function $f(x) = e^{-3x}\cosh x\ (x>0)$ is $F_S(\lambda) =$ _____.

(a) $\frac{3(\lambda^2+8)}{(\lambda^2+4)(\lambda^2+16)}$ (b) $\frac{8}{(\lambda^2+4)(\lambda^2+16)}$

(c) $\frac{\lambda^2-8}{(\lambda^2+4)(\lambda^2+16)}$ (d) $\frac{6\lambda}{(\lambda^2+4)(\lambda^2+16)}$

Ans. : (d)

Explanation : $f(x) = e^{-3x}\cosh x = \frac{1}{2}\left[e^{-2x}+e^{-4x}\right]$

$\therefore$ Fourier Sine Transform (FST) is,

$$F_S(\lambda) = \frac{1}{2}\int_0^\infty (e^{-2u}+e^{-4u})\sin\lambda u\, du$$

$$= \frac{1}{2}\left[\frac{\lambda}{\lambda^2+4}+\frac{\lambda}{\lambda^2+16}\right] = \frac{(\lambda^2+10)}{(\lambda^2+4)(\lambda^2+16)}$$

Q. 24 If Fourier Cosine Transform of f(x) is $F_C(\lambda) = e^{-\lambda}, \lambda>0$ then Inverse Fourier Cosine Transform of F (λ) is $f(x) =$ _____.

(a) $e^{-x}\cos x$ (b) $\frac{2}{\pi}\left(\frac{1}{1+x^2}\right)$

(c) $\frac{x}{1+x^2}$ (d) $\frac{2}{\pi}\left[\frac{x}{1+x^2}\right]$

Ans. : (b)

Q. 25 If $F_C(\lambda) = \begin{cases} 1-\lambda; & 0\le\lambda\le1 \\ 0 & \lambda\ge1 \end{cases}$ then inverse Fourier Cosine Transform of $F_C(\lambda)$ is $f(x) =$ ___.

(a) $\frac{2}{\pi}\left(\frac{1-\cos x}{x^2}\right)$ (b) $\frac{2}{\pi}\left(\frac{1+\cos x}{x^2}\right)$

(c) $\frac{2}{\pi}\left(\frac{1-\sin x}{x^2}\right)$ (d) $\frac{2}{\pi}\left(\frac{1+\sin x}{x^2}\right)$

Ans. : (a)

Q. 26 If $F_S(\lambda) = \begin{cases} 1-\lambda; & 0\le\lambda\le1 \\ 0\ ; & \lambda\ge1 \end{cases}$ then Inverse Fourier Sine Transform $F_S(\lambda)$ is f(x) _____.

(a) $\frac{2}{\pi}\left(\frac{\sin x}{x^2}\right)$ (b) $\frac{2}{\pi}\left(\frac{x-\cos x}{x^2}\right)$

(c) $\frac{2}{\pi}\left(\frac{1-\cos x}{x^2}\right)$ (d) $\frac{2}{\pi}\left(\frac{x-\sin x}{x^2}\right)$

Ans. : (d)

Q. 27 If $F_S(\lambda) = \begin{cases} 1 & 0\le\lambda\le1 \\ 2 & 0\le\lambda\le2 \\ 0 & \lambda>2 \end{cases}$ then Inverse Fourier Sine Transform (IFST) of $F_S(\lambda)$ is $f(x) =$ _____.

(a) $\frac{2}{\pi}\left[\frac{\cos x+2\cos x}{x^2}\right]$

(b) $\frac{2}{\pi}\left[\frac{1-2\cos 2x+\cos x}{x}\right]$

(c) $\frac{2}{\pi}\left[\frac{\sin x-2\sin 2x}{x}\right]$

(d) $\frac{2}{\pi}\left[\frac{\cos x+2\cos x}{x}\right]$

Ans. : (b)

Q. 28 If $F_S(\lambda) = \begin{cases} 2-\lambda; & 0\le\lambda\le2 \\ 0 & \lambda>2 \end{cases}$ then Inverse Fourier Sine Transform of $F_S(\lambda)$ is $f(x) =$ _____.

(a) $\frac{2}{\pi}\left(\frac{2-\sin x}{x^2}\right)$ (b) $\frac{2}{\pi}\left[\frac{2x-\sin 2x}{x^2}\right]$

(c) $\frac{2}{\pi}\left(\frac{\sin 2x}{x^2}\right)$ (d) $\frac{2}{\pi}\left[\frac{2x-\cos 2x}{x^2}\right]$

Ans. : (b)

Q. 29 In the Fourier sine integral representation

$$e^{-2x}-e^{-3x} = \frac{10}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{(9+\lambda^2)(4+\lambda^2)}d\lambda,\ F_S(\lambda) \text{ is}$$

_____.

(a) $\frac{10}{\pi}\left[\frac{\lambda}{(9+\lambda^2)(4+\lambda^2)}\right]$

(b) $\frac{\lambda}{(9+\lambda^2)(4+\lambda^2)}$

(c) $\frac{\lambda\sin\lambda x}{(9+\lambda^2)(4+\lambda^2)}$

(d) $\frac{5\lambda}{(4+\lambda^2)(9+\lambda^2)}$

Ans. : (d)

Q. 30 In the Fourier integral representation

$$\frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{1-i\lambda}{1+i\lambda}\right)e^{i\lambda x}\,d\lambda = \begin{cases} 0 & x<0 \\ \pi e^{-x} & x>0 \\ \pi/2 & x=0 \end{cases}$$ F(λ) is

_____.

(a) $\frac{1-i\lambda}{1+i\lambda}$ (b) $\pi\left(\frac{1-i\lambda}{1+i\lambda}\right)$

(c) $\left(\frac{1-i\lambda}{1+i\lambda}\right)e^{i\lambda x}$ (d) $\frac{1}{2}\left(\frac{1-i\lambda}{1+i\lambda}\right)$

Ans. : (b)

Q. 31 In the Fourier integral representation

$$\frac{-1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\lambda+2i}{\lambda^3}\right)e^{-i\lambda x}\,d\lambda = \begin{cases} x-x^2 & x>0 \\ 0 & x<0 \end{cases}$$

F(λ) is _____.

(a) $\frac{-1}{2\pi}\left(\frac{\lambda+2i}{\lambda^3}\right)$ (b) $\frac{\lambda+2i}{\lambda^3}e^{-i\lambda x}$

(c) $-\left(\frac{\lambda+2i}{\lambda^3}\right)$ (d) $\frac{-1}{2\pi}\left(\frac{\lambda+2i}{\lambda^3}\right)\cos\lambda x$

Ans. : (c)

Q. 32 In the Fourier cosine integral representation

$$\int_0^\infty \frac{2\cos\lambda x\,d\lambda}{1+\lambda^2} = \begin{cases} 0 & x<0 \\ \pi e^{-x} & x>0 \end{cases};\ F_C(\lambda) \text{ is } _____.$$

(a) $\frac{1}{1+\lambda^2}$ (b) $\frac{\pi}{1+\lambda^2}$

(c) $\frac{2\cos\lambda x}{1-\lambda^2}$ (d) $\frac{2}{1+\lambda^2}$

Ans. : (b)

Q. 33 In the Fourier integral representation

$$\int_0^\infty \frac{\cos\frac{\pi\lambda}{2}\cos\lambda x}{1-\lambda^2}d\lambda = \begin{cases} \frac{\pi}{2}\cos x & |x|<\frac{\pi}{2} \\ 0 & |x|>\frac{\pi}{2} \end{cases}$$

The value of F(λ) is _____.

(a) $\frac{\cos\lambda x}{1-\lambda^2}$ (b) $\frac{\cos\frac{\pi\lambda}{2}}{1-\lambda^2}$

(c) $\frac{\cos\frac{\pi\lambda}{2}\cos\lambda x}{1-\lambda^2}$ (d) $\frac{\pi}{2}\left(\frac{\cos\frac{\lambda\pi}{2}}{1-\lambda^2}\right)$

Ans. : (d)

Q. 34 In the Fourier sine integral representation

$$e^{-x}-e^{-2x} = \frac{6}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{(\lambda^2+1)(\lambda^2+4)}d\lambda,$$ the value of $F_S(\lambda)$ is ____.

(a) $\frac{6\lambda}{\pi(\lambda^2+1)(\lambda^2+4)}$ (b) $\frac{3\lambda}{(\lambda^2+1)(\lambda^2+4)}$

(c) $\frac{\lambda\sin\lambda x}{(\lambda^2+1)(\lambda^2+4)}$ (d) $\frac{6}{\pi}\left[\frac{\lambda\sin\lambda x}{(\lambda^2+1)(\lambda^2+4)}\right]$

Ans. : (b)

Q. 35 The Fourier transform of the function

$$f(x) = \begin{cases} 0 & x<0 \\ e^{-x} & x>0 \\ \frac{1}{2} & x=0 \end{cases}$$ is $F(\lambda) =$ _____.

(a) $\frac{1+i\lambda}{1+\lambda^2}$ (b) $\frac{1-i\lambda}{1+\lambda^2}$ (c) $\frac{1+i\lambda}{1-i\lambda}$ (d) $\frac{1-\lambda}{1+\lambda}$

Ans. : (b)

Q. 36 The Fourier transform of

$$f(x) = \begin{cases} \sin x & 0<x<\pi \\ 0 & x<0 \text{ or } x>\pi \end{cases}$$ is $F(\lambda) =$ _____.

(a) $\int_0^\pi e^{-i\lambda u}\sin u\,du$ (b) $\int_{-\infty}^{\infty}\sin x\sin\lambda x\,dx$

(c) $\int_0^\infty \sin u\sin\lambda u\,du$ (d) $\int_0^\pi e^{-i\lambda u}\sin x\,d\lambda$

Ans. : (a)

Q. 37 The Fourier Cosine Transform of the function

$f(x) = e^{-2x}\sinh x$, $(x>0)$ is $F_C(\lambda) =$ _____.

(a) $\frac{1}{2}\left[\frac{1}{1+\lambda^2}+\frac{3}{9+\lambda^2}\right]$ (b) $\frac{1}{2}\left[\frac{\lambda}{1+\lambda^2}-\frac{\lambda}{9+\lambda^2}\right]$

(c) $\frac{1}{2}\left[\frac{\lambda}{1+\lambda^2}+\frac{\lambda}{9+\lambda^2}\right]$ (d) $\frac{1}{2}\left[\frac{1}{1+\lambda^2}-\frac{3}{9+\lambda^2}\right]$

Ans. : (d)

Explanation :

Given, $f(x) = e^{-2x}\sinh x = e^{-2x}\left[\frac{e^x-e^{-x}}{2}\right] = \frac{e^{-x}-e^{-3x}}{2}$

$\therefore$ Fourier cosine transform is, $F_C(\lambda) = \int_0^\infty f(u)\cos\lambda u\,du$

$$= \int_0^\infty \frac{e^{-u}-e^{-3u}}{2}\cos\lambda u\,du$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-u}\cos\lambda u\,du - \int_0^\infty e^{-3u}\cos\lambda u\,du\right]$$

$$= \frac{1}{2}\left[\frac{1}{1+\lambda^2}-\frac{3}{9+\lambda^2}\right]$$

Q. 38 The Fourier Sine Transform of the function $f(x) = e^{-2x}-e^{-3x}\ (x>0)$ is $F_S(\lambda) =$ ____.

(a) $\frac{\lambda}{\lambda^2+4}-\frac{\lambda}{\lambda^2+9}$ (b) $\frac{2}{\lambda^2+4}-\frac{3}{\lambda^2+9}$

(c) $\frac{\lambda}{\lambda^2+4}+\frac{\lambda}{\lambda^2+9}$ (d) $\frac{2}{\lambda^2+4}+\frac{3}{\lambda^2+9}$

Ans. : (a)

Q. 39 The Fourier Cosine Transform of $f(x) = 2e^{-5x}+5e^{-2x}$ is $F_S(\lambda) =$ _____.

(a) $10\left[\frac{\lambda}{\lambda^2+25}-\frac{1}{\lambda^2+4}\right]$

(b) $\frac{1}{\lambda^2+25}-\frac{1}{\lambda^2+4}$

(c) $10\left[\frac{\lambda}{\lambda^2+25}+\frac{1}{\lambda^2+4}\right]$

(d) $\frac{\lambda}{\lambda^2+25}-\frac{\lambda}{\lambda^2+4}$

Ans. : (c)

Q. 40 Fourier Sine Transform of $f(x) = e^{-x}-e^{-2x}\ (x>0)$ is $F_S(\lambda) =$ _____.

(a) $\frac{1}{\lambda^2+1}-\frac{1}{\lambda^2+4}$ (b) $\frac{\lambda}{1+\lambda^2}-\frac{\lambda}{4+\lambda^2}$

(c) $\frac{1}{\lambda^2+1}+\frac{2}{\lambda^2+4}$ (d) $\frac{\lambda}{1+\lambda^2}+\frac{2\lambda}{4+\lambda^2}$

Ans. : (b)

Q. 41 The Fourier Cosine Transform of

$$f(x)=\begin{cases}0 & , x<0\\ \pi e^{-x} & , x>0\end{cases} \text{ is } F_C(\lambda)=_____.$$

(a) $\frac{\lambda}{1-\lambda^2}$ (b) $\frac{\pi}{1+\lambda^2}$ (c) $\frac{\lambda}{1+\lambda^2}$ (d) $\frac{\pi}{1-\lambda^2}$

Ans. : (b)

Q. 42 The Fourier transform of $f(x)=\begin{cases}1 & , x>0\\ 0 & , x<0\end{cases}$ is $F(\lambda)=_____$.

(a) λ (b) $\frac{1}{i\lambda}$ (c) $\frac{i}{\lambda}$ (d) $i\lambda$

Ans. : (b)

Explanation : Given, $f(x)=\begin{cases}1 & x>0\\ 0 & x<0\end{cases}$

We have Fourier transform is,

$$F(\lambda)=\int_0^\infty (1)\,e^{-i\lambda u}\,du=\left[\frac{e^{-i\lambda u}}{-i\lambda}\right]_0^\infty=\left[0+\frac{1}{i\lambda}\right]=\frac{1}{i\lambda}$$

Q. 43 If Fourier transform of $f(x)=\begin{cases}1-x^2 & , |x|\le 1\\ 0 & , |x|>1\end{cases}$ is $F(\lambda)=\left[\frac{2\sin\lambda-2\lambda\cos\lambda}{\lambda^3}\right]$ then the value of

$\int_0^\infty \frac{\lambda\cos\lambda-\sin\lambda}{\lambda^3}\cos\lambda x\,d\lambda$ is $_____$.

(a) $\pi f(x)$ (b) $\frac{\pi}{2}f(x)$

(c) $-\frac{\pi}{4}f(x)$ (d) $-\pi f(x)$

Ans. : (c)

Q. 44 The Fourier integral representation of

$$f(x)=\begin{cases}\sin x & , 0<x<\pi\\ 0 & , x<0 \text{ or } x>\pi\end{cases} \text{ is,}$$

$f(x)=\frac{1}{\pi}\int_0^\infty \frac{\cos\lambda x+\cos(\pi-x)\lambda}{1-\lambda^2}d\lambda$ then the value of

$\int_0^\infty \frac{\cos\frac{\lambda\pi}{2}}{1-\lambda^2}d\lambda$ is $_____$.

(a) $\frac{2}{\pi}$ (b) $\frac{\pi}{2}$ (c) π (d) $\frac{1}{2\pi}$

Ans. : (b)

Q. 45 The Fourier integral representation of

$$f(x)=\begin{cases}1-x^2 & , |x|\le 1\\ 0 & , |x|>1\end{cases} \text{ is,}$$

$f(x)=\frac{4}{\pi}\int_0^\infty \frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\cos\lambda x\,d\lambda$ then the value of

$\int_0^\infty \frac{x\cos x-\sin x}{x^3}\cos\frac{x}{2}\,dx$ is $_____$.

(a) $\frac{3\pi}{16}$ (b) $\frac{\pi}{2}$ (c) $\frac{-3\pi}{16}$ (d) $\frac{-\pi}{2}$

Ans. : (c)

Q. 46 The Fourier integral representation of

$$f(x)=\begin{cases}1 & , -2\le x\le 0\\ -1 & , 0\le x\le 2\end{cases} \text{ is,}$$

$f(x)=\frac{2}{\pi}\int_0^\infty\left(\frac{\cos 2\lambda-1}{\lambda}\right)\sin\lambda x\,d\lambda$ then the value of

$\int_0^\infty\left(\frac{\cos 2\lambda-1}{x}\right)\sin 2x\,dx$ is $_____$.

(a) π (b) $\frac{\pi}{2}$ (c) $-\pi$ (d) $\frac{-\pi}{2}$

Ans. : (d)

Q. 47 The Fourier integral representation of

$$f(x)=\begin{cases}-3 & , |x|<1\\ 0 & , |x|>1\end{cases} \text{ is,}$$

$f(x)=\frac{-6}{\pi}\int_0^\infty \frac{\sin\lambda\cos\lambda x\,dx}{\lambda}d\lambda$ then the value of

$\int_0^\infty \frac{\sin\lambda}{\lambda}d\lambda$ is $_____$.

(a) $\frac{6}{\pi}$ (b) $\frac{\pi}{2}$ (c) $-\frac{6}{\pi}$ (d) $-\frac{\pi}{2}$

Ans. : (b)

Q. 48 The solution of the integral equation

$$\int_0^\infty f(x)\cos\lambda x\,dx=\begin{cases}1-\lambda & , 0\le\lambda\le 1\\ 0 & , \lambda\ge 1\end{cases}$$

is $f(x)=_____$.

(a) $\frac{\pi}{2}\left(\frac{1+\sin x}{x^2}\right)$ (b) $\frac{2}{\pi}\left(\frac{1-\sin x}{x^2}\right)$

(c) $\frac{2}{\pi}\left(\frac{1-\cos x}{x^2}\right)$ (d) $\frac{\pi}{2}\left(\frac{1+\cos x}{x^2}\right)$

Ans. : (c)

Q. 49 For the integral equation

$$\int_0^\infty f(x)\cos\lambda x\,dx=\begin{cases}1-\lambda & , 0\le\lambda\le 1\\ 0 & , \lambda\ge 1\end{cases}$$

The Fourier cosine transform is,

$F(\lambda)=\frac{4}{\pi}\int_0^\infty \frac{\sin^2 u/2}{u^2}\cos\lambda x\,du$ then value of $\int_0^\infty \frac{\sin^2 z}{z^2}dz$ is $_____$.

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) $\frac{\pi}{4}$

Ans. : (b)

Q. 50 The solution of the integral equation

$$\int_0^\infty f(x)\sin\lambda x\,dx=\begin{cases}\lambda-1 & , 0\le\lambda\le 1\\ \lambda+1 & , 1\le\lambda\le 2\\ 0 & , \lambda>2\end{cases}$$

is $f(x)=_____$.

(a) $\frac{2}{\pi}\left[\frac{2\cos x-3\cos 2x-1}{x}+\frac{\sin 2x}{x^2}\right]$

(b) $\frac{2}{\pi}\left[\frac{2\sin x-3\sin 2x-1}{x}+\frac{\cos 2x}{x^2}\right]$

(c) $\frac{2}{\pi}\left[\frac{2\sin 2x-\cos 2x}{x^2}\right]$

(d) $\frac{2}{\pi}\left[\frac{2\cos x-\sin x}{x}+\frac{\sin 2x}{x^2}\right]$

Ans. : (a)

Q. 51 The solution of integral equation

$$\int_0^\infty f(x)\sin\lambda x\,d\lambda=\begin{cases}1-\lambda & 0\le\lambda\le 1\\ 0 & \lambda\ge 1\end{cases}$$

is $f(x)=_____$.

(a) $\frac{\pi}{2}\left(\frac{x-\cos x}{x^2}\right)$ (b) $\frac{2}{\pi}\left(\frac{\sin x}{x^2}\right)$

(c) $\frac{2}{\pi}\left[\frac{x-\sin x}{x^2}\right]$ (d) $\frac{\pi}{2}\left(\frac{\cos x}{x^2}\right)$

Ans. : (c)

Q. 52 The solution of integral equation

$$\int_0^\infty f(x)\sin\lambda x\,dx=\begin{cases}1 & , 0\le\lambda\le 1\\ 2 & , 1\le\lambda\le 2\\ 0 & , \lambda>2\end{cases}$$

is $f(x)=_____$.

(a) $\frac{\pi}{2}\left[\frac{\cos x-2\cos 2x+1}{x}\right]$

(b) $\frac{2}{\pi}\left[\frac{\sin x-2\sin 2x+1}{x}\right]$

(c) $\frac{2}{\pi}\left[\frac{\cos x-2\sin 2x+1}{x}\right]$

(d) $\frac{2}{\pi}\left[\frac{\sin x-2\cos 2x+1}{x}\right]$

Ans. : (a)

Q. 53 The solution of integral equation

$$\int_0^\infty f(x)\sin\lambda x\,dx=\begin{cases}2-\lambda & , 0\le\lambda\le 2\\ 0 & , \lambda\ge 2\end{cases} \text{ is } F(x)=___.$$

(a) $\frac{2}{\pi}\left[\frac{\sin 2x}{x^2}\right]$ (b) $\frac{2}{\pi}\left[\frac{2x-\cos 2x}{x^2}\right]$

(c) $\frac{2}{\pi}\left[\frac{\cos 2x}{x^2}\right]$ (d) $\frac{2}{\pi}\left[\frac{2x-\sin 2x}{x^2}\right]$

Ans. : (d)

Q. 54 The Fourier transform of the function

$$f(x)=\begin{cases}\sin x & , x>0\\ 0 & , x<0\end{cases} \text{ is } F(\lambda)=_____.$$

(a) $\frac{i\lambda}{1-\lambda^2}$ (b) $\frac{1}{1-\lambda^2}$

(c) $\frac{-i\lambda}{1-\lambda^2}$ (d) $\frac{\lambda}{1-\lambda^2}$

Ans. : (b)

Explanation : Fourier transform of f(x) is,

$$F(\lambda)=\int_{-\infty}^{\infty} f(u)\,e^{-i\lambda u}\,du=\int_0^\infty \sin u\,e^{-i\lambda u}\,du$$

$$=\int_0^\infty e^{-i\lambda u}\sin u\,du=\frac{1}{(i\lambda)^2+1}=\frac{1}{1-\lambda^2}$$

Q. 55 The Fourier transform of the function

$$f(x)=\begin{cases}\cos x & , x>0\\ 0 & , x<0\end{cases} \text{ is } F(\lambda)=_____.$$

(a) $\frac{\lambda}{1-\lambda^2}$ (b) $\frac{1}{1-\lambda^2}$

(c) $\frac{i\lambda}{1-\lambda^2}$ (d) $\frac{-i\lambda}{1-\lambda^2}$

Ans. : (c)

Explanation : Fourier transform of f(x) is,

$$F(\lambda)=\int_{-\infty}^{\infty} f(u)\,e^{-i\lambda u}\,du=\int_0^\infty \cos u\,e^{-i\lambda u}\,du$$

$$=\int_0^\infty e^{-i\lambda u}\cos u\,du=\frac{i\lambda}{(i\lambda)^2+1}=\frac{i\lambda}{1-\lambda^2}$$

Q. 56 The Fourier integral representation of

$$f(x)=\begin{cases}0 & , x<0\\ e^{-x} & , x>0\\ \frac{1}{2} & , x=0\end{cases}$$

with $F(\lambda)=\frac{1-i\lambda}{1+\lambda^2}$ is $f(x)=$ _____.

(a) $\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-i\lambda}{1+\lambda^2}[\cos\lambda x+i\sin\lambda x]\,d\lambda$

(b) $\frac{2}{\pi}\int_{0}^{\infty}\frac{1-i\lambda}{1+\lambda^2}\cos\lambda x\,d\lambda$

(c) $\frac{2}{\pi}\int_{0}^{\infty}\frac{1-i\lambda}{1+\lambda^2}\sin\lambda x\,d\lambda$

(d) $\frac{2}{\pi}\int_{0}^{\infty}\frac{1-i\lambda}{1+\lambda^2}e^{i\lambda x}\,d\lambda$

Ans. : (a)

Q. 57 For the function $f(x)=\begin{cases}1 & , |x|<1\\ 0 & , |x|>1\end{cases}$ the Fourier transform $F(\lambda)=\frac{\sin\lambda}{\lambda}$ then the Fourier integral representation is $f(x)$ _____.

(a) $\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin\lambda\sin\lambda x}{\lambda}\,d\lambda$ (b) $\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$

(c) $\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin\lambda}{\lambda}e^{-i\lambda u}\,d\lambda$ (d) $\frac{1}{2\pi}\int_{0}^{\infty}\frac{\sin\lambda}{\lambda}e^{-i\lambda u}\,d\lambda$

Ans. : (b)

Q. 58 The Fourier transform of $f(x)=\begin{cases}1-x^2 & |x|\le 1\\ 0 & |x|>1\end{cases}$

$F(\lambda)=2\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right)$ then the Fourier integral representation is $f(x)=$ ____.

(a) $\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\sin\lambda x\,d\lambda$

(b) $\frac{4}{\pi}\int_{0}^{\infty}\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\sin\lambda x\,d\lambda$

(c) $\frac{4}{\pi}\int_{0}^{\infty}\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\cos\lambda x\,d\lambda$

(d) $\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\cos\lambda x\,d\lambda$

Ans. : (c)

Q. 59 For the function $f(x)=\begin{cases}1 & -2\le x<0\\ -1 & 0<x\le 2\end{cases}$ the Fourier transform $F(\lambda)=\frac{\cos 2\lambda-1}{\lambda}$ then the Fourier integral representation is $f(x)$ _____.

(a) $\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\cos 2\lambda-1}{\lambda}e^{-i\lambda u}\,d\lambda$

(b) $\frac{2}{\pi}\int_{0}^{\infty}\frac{\cos 2\lambda-1}{\lambda}\sin\lambda x\,d\lambda$

(c) $\frac{2}{\pi}\int_{0}^{\infty}\frac{\cos 2\lambda-1}{\lambda}\cos\lambda x\,d\lambda$

(d) $\frac{1}{2\pi}\int_{0}^{\infty}\frac{\cos 2\lambda-1}{\lambda}\,d\lambda$

Ans. : (b)

Q. 60 For the function $f(x)=\begin{cases}-3 & , |x|<1\\ 0 & , |x|>1\end{cases}$ the Fourier transform $F(\lambda)=\frac{-3\sin\lambda}{\lambda}$ then the Fourier integral representation is $f(x)=$ _____.

(a) $\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin\lambda}{\lambda}\cos\lambda x\,d\lambda$ (b) $\frac{-6}{\pi}\int_{0}^{\infty}\frac{\sin\lambda}{\lambda}\sin\lambda x\,d\lambda$

(c) $\frac{-6}{\pi}\int_{0}^{\infty}\frac{\sin\lambda}{\lambda}\cos\lambda x\,d\lambda$ (d) $\frac{-3}{2\pi}\int_{-\infty}^{\infty}\frac{\sin\lambda}{\lambda}e^{-i\lambda x}\,d\lambda$

Ans. : (c)

Q. 61 For an even function $f(x)$ in the interval $(-\infty,\infty)$, the Fourier integral representation is $f(x)=$ _____.

(a) $\frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty}f(u)\sin\lambda u\cos\lambda x\,du\,d\lambda$

(b) $\frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty}f(u)\sin\lambda u\sin\lambda x\,du\,d\lambda$

(c) $\frac{1}{2\pi}\int_{0}^{\infty}\int_{0}^{\infty}f(u)\cos\lambda u\cos\lambda x\,du\,d\lambda$

(d) $\frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty}f(u)\cos\lambda u\cos\lambda x\,du\,d\lambda$

Ans. : (d)

Q. 62 For an odd function $f(x)$ in the interval $(-\infty,\infty)$, the Fourier integral representation is $f(x)=$ _____.

(a) $\frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty}f(u)\sin\lambda u\cos\lambda x\,du\,d\lambda$

(b) $\frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty}f(u)\sin\lambda u\sin\lambda x\,du\,d\lambda$

(c) $\frac{1}{2\pi}\int_{0}^{\infty}\int_{0}^{\infty}f(u)\cos\lambda u\cos\lambda x\,du\,d\lambda$

(d) $\frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty}f(u)\cos\lambda u\cos\lambda x\,du\,d\lambda$

Ans. : (b)

Q. 63 In the Fourier sine integral representation.

$$\int_{0}^{\infty}\frac{1-\cos\lambda\pi}{\lambda}\sin\lambda x\,d\lambda=\begin{cases}\frac{\pi}{2} & 0<x<\pi\\ 0 & x>\pi\end{cases}$$

The Fourier sine transform $F_S(\lambda)=$ _____.

(a) $\frac{1-\cos\lambda\pi}{\lambda}$ (b) $\frac{2}{\pi}\left[\frac{1-\cos\lambda\pi}{\lambda}\right]$

(c) $\frac{1}{2\pi}\left[\frac{1-\cos\lambda\pi}{\lambda}\right]$ (d) $\frac{\pi}{2}\left[\frac{1-\cos\lambda\pi}{\lambda}\right]$

Ans. : (d)

Explanation : Since Fourier sine integral representation is,

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}F_S(\lambda)\sin\lambda x\,d\lambda$$

$$=\int_{0}^{\infty}\frac{1-\cos\lambda x}{\lambda}\sin\lambda x\,d\lambda$$

$$\therefore\ F_S(\lambda)=\frac{\pi}{2}\left[\frac{1-\cos\lambda\pi}{\lambda}\right]$$

Q. 64 In the Fourier sine integral representation

$\int_{0}^{\infty}\frac{\lambda^3\sin\lambda x}{\lambda^4+4}\,d\lambda=\frac{\pi}{2}e^{-x}\cos x\ (x>0)$ the Fourier sine transform $F_S(\lambda)$ is ——.

(a) $\frac{\pi}{2}\left[\frac{\lambda^3}{\lambda^4+4}\right]$ (b) $\frac{2}{\pi}\left[\frac{\lambda^3\sin\lambda x}{\lambda^4+4}\right]$

(c) $\frac{2}{\pi}\left[\frac{\lambda^3}{\lambda^4+4}\right]$ (d) $\frac{\lambda^3}{\lambda^4+4}$

Ans. : (a)

Explanation : The Fourier sine integral representation is,

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}F_S(\lambda)\sin\lambda x\,d\lambda=\int_{0}^{\infty}\frac{\lambda^3\sin\lambda x}{\lambda^4+4}\,d\lambda$$

$$\therefore\ F_S(\lambda)=\frac{\pi}{2}\left[\frac{\lambda^3}{\lambda^4+4}\right]$$

Q. 65 In the Fourier cosine integral representation

$f(x)=\int_{0}^{\infty}\frac{\sin\lambda\cos x\lambda}{\lambda}\,d\lambda$, the Fourier cosine transform $F_C(\lambda)$ is _____.

(a) $\frac{\sin\lambda}{\lambda}$ (b) $\frac{\pi}{2}\left(\frac{\sin\lambda}{\lambda}\right)$

(c) $\frac{1}{2\pi}\left(\frac{\sin\lambda}{\lambda}\right)$ (d) $\frac{2}{\pi}\left(\frac{\sin\lambda}{\lambda}\right)$

Ans. : (b)

Explanation : Fourier cosine integral representation is,

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}F_C(\lambda)\cos\lambda x\,d\lambda=\int_{0}^{\infty}\frac{\sin\lambda}{\lambda}\cos x\lambda\,d\lambda$$

$$\therefore\ F_C(\lambda)=\frac{\pi}{2}\left(\frac{\sin\lambda}{\lambda}\right)$$

□□□

Note

CHAPTER 4

UNIT 4

Partial Differential Equations and Their Applications

Syllabus

Formation of Partial differential equations by eliminating arbitrary constants and functions; Equations solvable by direct integration; Linear equations of first order (Lagrange's linear equations); Method of separation of variables – applications to find solutions of one dimensional heat flow equation $\left(\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}\right)$, and two dimensional heat flow equation (i.e. Laplace equation : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$).

4.1 Introduction

Mathematical model of any physical phenomena containing more than one independent variables related to partial differential equations.

Partial differential equation occur in many practical situation. Such as epidemiology (prey models of AIDS). Traffic flow studies, the analysis of economics, theory of stochastic processes, radioactive disintegration, noise in communication system, gas dynamics, wave motion; radiation, conduction of energy, potential theory and many more. So the study of PDE is very much essential.

Let $z = f(x, y)$

$$\therefore \quad p = \frac{\partial z}{\partial x} \quad \text{and} \quad q = \frac{\partial z}{\partial y}$$

Here, z is dependent variable and x, y are independent variables.

The general solution of partial differential equation in a region R involving arbitrary constants or arbitrary functions.

Syllabus Topic : Formation of Partial Differential Equations by Eliminating Arbitrary Constants and Functions

4.2 Formation of Partial Differential Equations

Partial differential equation can be formed by elimination of arbitrary constants or by the elimination of an arbitrary functions.

4.2.1 Formation of Partial differential Equation by Eliminating an Arbitrary Constants

Partial differential equation can be obtained by eliminating an arbitrary constants. In the solution, if the number of an arbitrary constants is equal to number of independent variables, we get first order partial differential equation.

If the number of arbitrary constant in the solution is more than the number of independent variables. Then we get second or higher order partial differential equations.

Ex. 4.2.1 : Form a Partial differential equation from

$$z = ax + by + ab$$

Soln. :

Step I

Given function $z = ax + by + ab$ ...(1)

Here, z – is dependent variable ;

x, y – are independent variables ;

a, b – are arbitrary constants.

Also, number of independent variables is equal to number of arbitrary constants (i.e. n = 2)

We get first order Partial differential equation

Step II : Differentiate Equation (1) Partially w.r.t. x, keeping y as constant

$$p = \frac{\partial z}{\partial x} = a + 0 + 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = a \quad \text{...(i)}$$

Differentiate Equation (1) Partially w.r.t. y, keeping x as constants

$$\therefore \quad q = \frac{\partial z}{\partial y} = 0 + b + 0$$

$$\Rightarrow \quad \frac{\partial z}{\partial y} = b$$

Step III : Substitute the values of a = p and b = q in Equation (1), we get

$$z = px + qy + pq$$

$$\text{i.e.} \quad z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) \checkmark \quad \text{...Ans.}$$

is the required first order partial differential equation.

Ex. 4.2.2 : Form a Partial differential equation from

$$z = ax + by + a^2 + b^2$$

Soln. :

Step I

Given, $z = ax + by + a^2 + b^2$ ...(1)

Here, z – is dependent variable ;

x, y – are independent variables ;

a, b – are arbitrary constants.

Also, number of independent variables equal to number of arbitrary constants (i.e. = 2)

We get first order P.D.E.

Step II : Differentiate Equation (1) Partially w.r.t. x, keeping y as constant.

$$\therefore \quad p = \frac{\partial z}{\partial x} = a + 0 + 0 + 0$$

$$\Rightarrow \quad p = a$$

And differentiate Equation (1) Partially w.r.t. y, keeping x as constants

$$q = \frac{\partial z}{\partial y} = 0 + b + 0 + 0$$

$$q = b$$

Step III : By putting the values of a = p and b = q in Equation (1), we get

$$z = px + qy + p^2 + q^2$$

$$\Rightarrow \quad z = x\cdot\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \checkmark \text{...Ans.}$$

is the required first order partial differential equation

Ex. 4.2.3 : Find the Partial differential equation of the family of planes, whose sum of x, y, z intercepts is equal to unity.

Soln. :

Step I : If a, b, c be the x, y, z intercepts then equation of family of planes is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Given, sum of x, y, z intercepts is unity

i.e. $a + b + c = 1$

$\Rightarrow c = 1 - a - b$

...(we may write $a = 1 - b - c$ or $b = 1 - a - c$)

Above Equation becomes

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{1-a-b} = 1 \quad \text{...(1)}$$

Here, consider z - as dependent variable and x, y - are independent variables. And a, b - are arbitrary constants

Arbitrary function = Arbitrary constants

We get first order P.D.E.

Step II :

∴ Differentiate Equation (1) Partially w.r.t x, keeping y as constant

$$\frac{1}{a} + 0 + \frac{\frac{\partial z}{\partial x}}{(1-a-b)} = 0$$

$$\Rightarrow \quad \frac{p}{(1-a-b)} = -\frac{1}{a} \quad \text{...(2)}$$

$$\left(\because p = \frac{\partial z}{\partial x}\right)$$

Differentiate Equation (1) partially w.r.t. y, keeping x – as constant

$$0 + \frac{1}{b} + \frac{\frac{\partial z}{\partial y}}{(1-a-b)} = 0$$

$$\Rightarrow \quad \frac{\frac{\partial z}{\partial y}}{1-a-b} = -\frac{1}{b}$$

$$\frac{q}{1-a-b} = -\frac{1}{b} \quad \text{...(3)}$$

$$\left(\because q = \frac{\partial z}{\partial y}\right)$$

Step III :

Equation (2) ÷ Equation (3)

$$\frac{\left(\frac{p}{1-a-b}\right)}{\left(\frac{q}{1-a-b}\right)} = \frac{\left(-\frac{1}{a}\right)}{\left(-\frac{1}{b}\right)}$$

$$\Rightarrow \quad \frac{p}{q} \times \left(\frac{1-a-b}{1-a-b}\right) = \left(\frac{1}{a}\right) \times \left(\frac{b}{1}\right)$$

$$\Rightarrow \quad \frac{p}{q} = \frac{b}{a} \quad \text{...(4)}$$

Also from (2), $\frac{a \cdot p}{(1-a-b)} = -1$

$\Rightarrow ap = -(1-a-b)$

$\Rightarrow ap = a + b - 1$

$$ap = a + \left(\frac{ap}{q}\right) - 1 \quad \text{...from (4)}$$

$$\Rightarrow \quad ap - a - \frac{ap}{q} = -1$$

$$a\left(p - 1 - \frac{p}{q}\right) = -1$$

$$\Rightarrow a\left(1 - p + \frac{p}{q}\right) = 1$$

$$\frac{a(q - pq + p)}{q} = 1$$

$$a = \frac{q}{(p + q - pq)} \quad \text{...(5)}$$

Similarly, from Equations (3) and (4), we get

$$b = \left(\frac{p}{p + q - pq}\right) \quad \text{...(6)}$$

Step IV :

Now, substitute the values of a and b in Equation (1)

$$\frac{x}{\left[\frac{q}{p+q-pq}\right]} + \frac{y}{\left[\frac{p}{p+q-pq}\right]} + \frac{z}{\left[1 - \frac{q}{(p+q-pq)} - \frac{p}{(p+q-pq)}\right]} = 1$$

$$\Rightarrow \left(\frac{p+q-pq}{q}\right)x + \left(\frac{p+q-pq}{p}\right)y + \frac{z}{\left[\frac{\cancel{p} + \cancel{q} - pq - \cancel{q} - \cancel{p}}{p+q-pq}\right]} = 1$$

$$\Rightarrow \left(\frac{p+q-pq}{q}\right)x + \left(\frac{p+q-pq}{p}\right)y + \left(\frac{p+q-pq}{-pq}\right)z = 1$$

$$\Rightarrow (p+q-pq)\left[\frac{1}{q}\cdot x + \frac{1}{p}\cdot y - \frac{z}{pq}\right] = 1$$

$$\text{i.e.} \quad \frac{x}{q} + \frac{y}{p} - \frac{z}{pq} = \frac{1}{p+q-pq}$$

or multiplying both sides by pq

$$p\cancel{q}\cdot\frac{x}{\cancel{q}} + \cancel{p}q\cdot\frac{y}{\cancel{p}} - \cancel{p}\cancel{q}\cdot\frac{z}{\cancel{p}\cancel{q}} = \frac{pq}{p+q-pq}$$

$$\Rightarrow \quad px + qy - z = \frac{pq}{p+q-pq} \checkmark$$

...Ans.

is the required first order partial differential equation

$$\left(\because \text{Here, } p = \frac{dz}{dx}\ ;\ q = \frac{\partial z}{\partial y}\right)$$

Ex. 4.2.4 : Form a Partial differential equation from

$$(x-a)^2 + (y-b)^2 + z^2 = 1$$

Soln. :

Step I :

Given function

$$(x-a)^2 + (y-b)^2 + z^2 = 1 \quad \text{...(1)}$$

Here, consider z – as dependent variable ;

x, y – are independent variables ;

a, b – are arbitrary constants.

Also, note that, number of independent variables is equal to number of constants.

We get first order P.D.E.

Step II : Differentiate Equation (1) partially w.r.t. x, keeping y as constant

$$\cancel{2}(x-a) + 0 + \cancel{2}z\cdot\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow (x-a) = -z\frac{\partial z}{\partial x} \quad \text{...(2)}$$

Differentiate Equation (1) partially w.r.t. y, keeping x as constant.

$$0 + \cancel{2}(y-b) + \cancel{2}z\cdot\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow (y-b) = -z\frac{\partial z}{\partial y} \quad \text{...(3)}$$

Step III : Substitute the values of (x – a) and (y – b) in Equation (1), we get

$$\left(-z\frac{\partial z}{\partial x}\right)^2 + \left(-z\frac{\partial z}{\partial y}\right)^2 + z^2 = 1$$

$$\Rightarrow z^2p^2 + z^2q^2 + z^2 = 1$$

$$\Rightarrow z^2(p^2 + q^2 + 1) = 1 \checkmark \quad \text{...Ans.}$$

be the required first order partial differential equation

Ex. 4.2.5 : Form a Partial differential equation from

$$z = xy + y\sqrt{x^2 - a^2} + b$$

Soln. :

Step I : The given function is

$$z = xy + y\sqrt{x^2 - a^2} + b \quad \text{...(1)}$$

Here, z - is dependent variable ;

x, y – are the independent variables ;

a, b – are the arbitrary constant.

Also note that, number of independent variables is equal number of arbitrary constants (i.e. = 2)

We get first order P.D.E.

Step II : Differentiate Equation (1) partially w.r.t. x, keeping y as constant.

➔ **Using standard result**

$$\left[\frac{d}{dx}\sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}}\cdot\frac{d}{dx}f(x)\right] \text{...}$$

$$\frac{\partial z}{\partial x} = y + y\cdot\frac{1}{2\sqrt{x^2 - a^2}}\cdot\frac{\partial}{\partial x}(x^2 - a^2) + 0$$

$$\frac{\partial z}{\partial x} = y + y\cdot\frac{1}{\cancel{2}\sqrt{x^2 - a^2}}\cancel{2}x + 0$$

$$\therefore \quad p = y + \frac{xy}{\sqrt{x^2 - a^2}}$$

$$\Rightarrow \quad p - y = \frac{xy}{\sqrt{x^2 - a^2}}$$

$$\Rightarrow \sqrt{x^2 - a^2} = \frac{xy}{p - y} \quad \text{...(2)}$$

Also, differentiate Equation (1) Partially w.r.t. y, keeping 'x' as constant

$$\frac{\partial z}{\partial y} = x + \sqrt{x^2 - a^2} + 0$$

$$\Rightarrow \quad q - x = \sqrt{x^2 - a^2}$$

$$\Rightarrow \quad q - x = \frac{xy}{p - y} \quad \text{...from (2)}$$

$\therefore (q-x)(p-y) = xy$

$\Rightarrow pq - qy - px + \cancel{xy} = \cancel{xy}$

$\therefore \quad px + qy = pq$ ✓ ...Ans.

is the required first order partial differential equation

Ex. 4.2.6 : Form a Partial differential equation from

$$z = ax^2 + bxy + cy^2$$

Soln. :

Step I : The given function is,

$$z = ax^2 + bxy + cy^2 \quad ...(1)$$

Here, z – is dependent variables

x, y – are independent variables

a, b, c – are arbitrary constants.

Also note that, the number of arbitrary constants are greater than number of independent variables.

So, we get second or higher order partial differential equation

Step II : Differentiate Equation (1) partially w.r.t. x, keeping other variables are constants.

$$\frac{\partial z}{\partial x} = 2ax + by + 0$$

Again differentiate $\frac{\partial z}{\partial x}$ w.r.t. x, keeping y as constant

$$\frac{\partial^2 z}{\partial x^2} = 2a \Rightarrow r = 2a \Rightarrow a = \frac{r}{2}$$

Differentiate Equation (1) w.r.t. y, keeping other variables as constant

$$\frac{\partial z}{\partial y} = 0 + bx + 2cy$$

Again differentiate $\frac{\partial z}{\partial y}$ w.r.t. y, keeping x as constant

$$\frac{\partial^2 z}{\partial y^2} = 0 + 2c \Rightarrow t = 2c \Rightarrow c = \frac{t}{2}$$

Also, differentiate $\frac{\partial z}{\partial y}$ w.r.t. x, keeping y as constant

$$\frac{\partial^2 z}{\partial x \partial y} = b + 0 \Rightarrow b = s$$

Step III : Substitute the values of a, b, c in Equation (1), we get

$$z = \left(\frac{r}{2}\right) x^2 + s\,xy + \frac{t}{2}(y^2)$$

$$2z = rx^2 + 2xy\,s + t\,y^2 \checkmark \quad ...Ans.$$

Where, $r = \frac{\partial^2 z}{\partial x^2}$; $s = \frac{\partial^2 z}{\partial x \partial y}$; $t = \frac{\partial^2 z}{\partial y^2}$

is the required second order partial differential equation.

Ex. 4.2.7 : Form a Partial differential equation from

$$z = ax\,e^y + \frac{1}{2}a^2 e^{2y} + b$$

Soln. :

Step I : Given function

$$z = ax\,e^y + \frac{1}{2}a^2 e^{2y} + b \quad ...(1)$$

Here, z – is dependent variable ;

x, y – are independent variables ;

a, b – are arbitrary constants

Note that, number of independent variables is equal to number of arbitrary constant (i.e. = 2)

We get first order partial differential equation.

Step II : Differentiate Equation (1) partially w.r.t. x, keeping y as constant

$$\frac{\partial z}{\partial x} = ae^y + 0 + 0$$

$$\Rightarrow p = ae^y \quad ...(2)$$

And differentiate Equation (1) w.r.t y, keeping x as constant

$$\frac{\partial z}{\partial y} = ax\,e^y + \frac{1}{\cancel{2}}a^2(\cancel{2}\,e^{2y}) + 0$$

$$\therefore \quad q = x(p) + (ae^y)^2$$

$$q = xp + (p)^2 \quad ...\text{from (2)}$$

$$\Rightarrow q = px + p^2 \checkmark \quad ...Ans.$$

be the required first order partial differential equation

Ex. 4.2.8 : Find the partial differential equation of all spheres whose center's lies on the z-axis

Soln. :

Step I : Equation of sphere whose centre at any point (a, b, c) is

$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$ where, R is radius

Now, equation of sphere whose centre lies on z-axis i.e. at (0, 0, c) and radius R is

$$x^2 + y^2 + (z-c)^2 = R^2 \quad ...(1)$$

Let, z - be the dependent variable and x, y are independent variables. R, C - be the arbitrary constant

Also the number of independent variables is equal to number of constants, we get first order partial differential equation.

Therefore, eliminate C and R from (1), we get first order P.D.E.

Step II : Differentiate Equation (1) Partially w.r.t. x, keeping y-as constant

$$\cancel{2}x + 0 + \cancel{2}(z-c)\cdot\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow (z-c)\frac{\partial z}{\partial x} = -x$$

$$\Rightarrow (z-c) = \frac{-x}{\left(\frac{\partial z}{\partial x}\right)} = -\frac{x}{p} \quad ...(2)$$

Also, differentiate (1) partially, w.r.t. y keeping x as constant

$$0 + \cancel{2}y + \cancel{2}(z-c)\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow y + (z-c)\cdot q = 0$$

$$\Rightarrow y + \left(\frac{-x}{p}\right) q = 0 \quad ...\text{from(2)}$$

$$\Rightarrow py - xq = 0 \checkmark \quad ...Ans.$$

is the required first order Partial differential equation

Ex. 4.2.9 : Find the Partial differential equation of all spheres of radius '3' units having their centres in the xy – plane

Soln. :

Step I : The equation of sphere whose centre at any point p(a, b, c) and radius 'R' is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$$

Here, centre lies in xy – plane, take c = 0 and radius R = 3

∴ Equation of sphere is :

$$(x-a)^2 + (y-b)^2 + z^2 = 9 \quad ...(1)$$

Let, z – be dependent variable ;

x, y – are independent variables,

a, b – are arbitrary constants.

Also note that number of independent variables is equal to number of arbitrary constants. (i.e. = 2)

Eliminate constants a, b, we get first order partial differential equation.

Step II : Differentiate Equation (1) partially w.r.t. x, keeping y as constant

$$\cancel{2}(x-a) + 0 + \cancel{2}z\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow (x-a) = -zp \quad ...(2)$$

Differentiate Equation (1) Partially w.r.t. y, keeping x as constant

$$0 + \cancel{2}(y-b) + \cancel{2}z\frac{\partial z}{\partial y} = 0$$

$$(y-b) = -zq \quad ...(3)$$

Step III : By using Equations (2) and (3), Equation (1) becomes

$$(-zp)^2 + (-zq)^2 + z^2 = 9$$

$$\Rightarrow z^2[p^2 + q^2 + 1] = 9 \checkmark \quad ...Ans.$$

is the required first order partial differential equation.

Ex. 4.2.10 : Find the Partial differential equation of all planes which are at 'a' distance from origin.

Soln. :

Step I : Equation of plane in normal form is

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p$$

Where, cosα, cosβ, cosγ are direction consines of any normal line from origin to the plane.

p – is the perpendicular distance from origin to the plane

Let, $\cos\alpha = l$; $\cos\beta = m$; $\cos\gamma = n$ and given p = a

Also we know $l^2 + m^2 + n^2 = 1$

$\therefore n^2 = 1 - l^2 - m^2$

or $n = \sqrt{1 - l^2 - m^2}$

∴ Equation of plane is,

$lx + my + nz = a$

or

$$lx + my + \left(\sqrt{1 - l^2 - m^2}\right) z = a \quad ...(1)$$

Let,

z – is dependent variable ;

x, y – are independent variables ;

l, m – are arbitrary constants, and

a is any fixed constant.

Note that, number of independent variables is equal to number of arbitrary constants (i.e. = 2). Eliminate l and m from (1), we get first order PDE

Step II : Differentiate Equation (1) partially, w.r.t x, keeping y as constant

$$\therefore \quad l + 0 + \left(\sqrt{1 - l^2 - m^2}\right)\cdot\frac{\partial z}{\partial x} = 0 \quad ...(2)$$

Also, differentiate Equation (1) Partially w.r.t. y, keeping x as constant

$$\therefore 0 + m + \left(\sqrt{1 - l^2 - m^2}\right)\cdot\frac{\partial z}{\partial y} = 0 \quad ...(3)$$

Step III : From Equations (2) and (3)

$$l = -\left(\sqrt{1 - l^2 - m^2}\right) p \quad ...(4)$$

$$\text{and} \quad m = -\left(\sqrt{1 - l^2 - m^2}\right)\cdot q \quad ...(5)$$

Squaring and adding,

$$l^2 + m^2 = (1 - l^2 - m^2)p^2 + (1 - l^2 - m^2)q^2$$

$$(l^2 + m^2) = (1 - l^2 - m^2)(p^2 + q^2)$$

$$= [1 - (l^2 + m^2)](p^2 + q^2)$$

$$= (p^2 + q^2) - (l^2 + m^2)(p^2 + q^2)$$

$$\Rightarrow (l^2 + m^2) + (l^2 + m^2)(p^2 + q^2) = p^2 + q^2$$

$$\Rightarrow (l^2 + m^2)[1 + p^2 + q^2] = p^2 + q^2$$

$$\Rightarrow (l^2 + m^2) = \frac{p^2 + q^2}{1 + p^2 + q^2}$$

Also $1 - l^2 - m^2 = 1 - (l^2 + m^2)$

$$= 1 - \frac{(p^2 + q^2)}{(1 + p^2 + q^2)}$$

$$= \frac{1 + \cancel{p^2} + \cancel{q^2} - \cancel{p^2} - \cancel{q^2}}{(1 + p^2 + q^2)}$$

$$1 - l^2 - m^2 = \frac{1}{(1+p^2+q^2)} \quad ...(6)$$

Step IV : Therefore Equation (4) becomes

$$l = -\left(\frac{1}{\sqrt{1+p^2+q^2}}\right) p$$

and Equation (5) becomes

$$m = -\left(\frac{1}{\sqrt{1+p^2+q^2}}\right) \cdot q$$

Substitute the values of l, m and $1 - l^2 - m^2$ in Equation (1), we get

$$-\left(\frac{1}{\sqrt{1+p^2+q^2}}\right) p x - \left(\frac{1}{\sqrt{1+p^2+q^2}}\right) qy + \left(\frac{1}{\sqrt{1+p^2+q^2}}\right) z = a$$

$$\Rightarrow -px - qy + z = a\sqrt{1+p^2+q^2}$$

$$\Rightarrow z = px + qy + a\sqrt{1+p^2+q^2} \checkmark \quad ...\text{Ans.}$$

is the required first order partial differential equation.

Ex. 4.2.11 : Form a partial differential equation of set of all right circular cones whose axes coinside with z-axis.

Soln. :

Step I : The set of all right circular cones whose axes coinsides with z-axis is represent by the equation

$$x^2 + y^2 = (z-c)^2 \tan^2 \alpha \quad ...(1)$$

Let

z – is dependent variable,

x, y – are independent variables,

c, α – are any arbitrary constants.

Eliminate these two arbitrary constants from (1), we get first order partial differential equation.

Step II : Differentiate Equation (1) partially, w.r.t x, keeping y as constant

$$\cancel{2} x + 0 = \cancel{2} (z-c) \tan^2 \alpha \frac{\partial z}{\partial x}$$

$$\Rightarrow x = p(z-c)\tan^2 \alpha \quad ...(2)$$

Also, differentiate Equation (1) partially w.r.t y, keeping y as constant

$$\therefore 0 + \cancel{2} y = \cancel{2} (z-c)\tan^2 \alpha \cdot \frac{\partial z}{\partial y}$$

$$\Rightarrow y = q(z-c)\tan^2 \alpha \quad ...(3)$$

Step III : To eliminate c and α, Multiply Equation (2) by q and Equation (3) by 'p' and subtraction, we get

$$qx = pq(z-c)\tan^2 \alpha$$
$$py = pq(z-c)\tan^2 \alpha$$
$$- \quad -$$
$$\Rightarrow xq - py = 0$$

or $py - qx = 0 \checkmark$ ...Ans.

is the required first order partial differential equation

Ex. 4.2.12 : Eliminate arbitrary constants a, b and form a partial differential equation from

(i) $z = (x+a)(y+b)$

(ii) $2z = (ax+y)^2 + b$

(iii) $ax^2 + by^2 + z^2 = 1$

Soln. :

(i)

Step I : Given,

$$z = (x+a)(y+b) \quad ...(1)$$

Here, z – is dependent variable

x, y – are independent variables.

a, b – are arbitrary constants.

By eliminating arbitrary constants a, b from (1), we get first order partial differential equation.

Step II : Differentiate Equation (1) partially w.r.t. x, keeping y as constant,

$$P = \frac{\partial z}{\partial x} = (y+b) \quad ...(2)$$

Differentiate Equation (1) partially w.r.t. x, keeping x as constant.

$$q = \frac{\partial z}{\partial y} = (x+a) \quad ...(3)$$

From Equations (2) and (3), Equation (1) becomes

$$z = qp$$

be the required first order partial differential equation.

(ii)

Step I :

Given, $2z = (ax+y)^2 + b^2$ ...(1)

Here, z – is dependent variable ;

x, y – are independent variables ;

a, b – are arbitrary constants

By eliminating arbitrary constants a, b from Equation (1), we get first order partial differential equation.

Step II : Differentiate Equation (1) partially, w.r.t. x, keeping y as constant.

$$\cancel{2} \frac{\partial z}{\partial x} = \cancel{2} (ax+y) \cdot (a)$$

$$\Rightarrow p = (ax+y)a \quad ...(2)$$

Differentiate Equation (1) partially, w.r.t. y, keeping x as constant.

$$\cancel{2} \frac{\partial z}{\partial y} = \cancel{2} (ax+y)$$

$$\Rightarrow q = (ax+y) \quad ...(3)$$

Step III : x × Equation (2) + y × Equation (3), we get

$$xp = (ax+y)(ax)$$
$$+ yq = (ax+y)(y)$$
$$(xp+yq) = (ax+y)(ax) + (ax+y)(y)$$
$$= (ax+y)(ax+y)$$

$$\Rightarrow xp + yq = (ax+y)^2$$
$$= q^2 \quad ...\text{From (3)}$$

$$\Rightarrow xp + yq = q^2$$ be the required first order partial differential equation

(iii)

Step I :

Given, $ax^2 + by^2 + z^2 = 1$ ...(1)

Let z – be dependent variable,

x, y – are independent variables,

a, b – are arbitrary constants.

By eliminating arbitrary constants a, b from Equation (1), we get, first order partial differential equation.

Step II : Differentiating Equation (1) partially w.r.t. x, keeping y as constant.

$$\cancel{2} ax + 0 + \cancel{2} z \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow ax + pz = 0$$

$$\Rightarrow a = -\frac{pz}{x} \quad ...(2)$$

Also, differentiate Equation (1) partially w.r.t. y, keeping x as constant.

$$0 + \cancel{2} by + \cancel{2} z \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow by + zq = 0$$

$$\Rightarrow b = \left(-\frac{zq}{y}\right) \quad ...(3)$$

Step III : Substitute the values of a and b in Equation (1), we get,

$$\left(-\frac{zp}{\cancel{x}}\right)x^2 + \left(-\frac{zp}{\cancel{y}}\right)y^2 + z^2 = 1$$

$$\Rightarrow -xzp - yzq + z^2 = 1$$

$$\Rightarrow xzp + yzq - z^2 = -1$$

$$\Rightarrow xzp + yzq = z^2 - 1 \checkmark \quad ...\text{Ans.}$$

is the required first order partial differential equation.

Ex. 4.2.13 : From the PDE by eliminating the arbitrary constants from

(i) $z = (x^2+a)(y^2+b)$ **Ans. : 4xyz = pq**

(ii) $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ **Ans. : 2z = px + qy**

(iii) Ellipsoide : $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1$

Ans. : $xz(r) + x(p^2) - zp = 0$ or $yz(t^2) + y(t)^2 - zt = 0$

Soln. :

(i)

Given, $z = (x^2+a)(y^2+b)$ ...(1)

Here, z – is a dependent variable ;

x, y – are independent variables ;

a, b – are arbitrary constants.

By eliminating arbitrary constants a, b, from (1), we get first order P.D.E.

Step I : Differentiate z partially w.r.t. x, keeping y as constant.

$$p = \frac{\partial z}{\partial x} = (2x) \cdot (y^2+b)$$

$$\Rightarrow (y^2+b) = \frac{p}{2x} \quad ...(2)$$

And differentiate z partially w.r.t. y, keeping x as constant.

$$q = \frac{\partial z}{\partial y} = (2y)(x^2+a)$$

$$\Rightarrow (x^2+a) = \frac{q}{2y} \quad ...(3)$$

From Equations (2) and (3), Equation (1) becomes.

$$z = \left(\frac{p}{2x}\right)\left(\frac{q}{2y}\right)$$

$$\Rightarrow pq = 4xyz$$ be the first order P.D.E.

(ii)

Step I :

Given : $2z = \frac{x^2}{a^2} + \frac{y^2}{a^2}$ ...(1)

Here, z – is dependent variable,

x, y – are independent variables ;

a, b – are arbitrary constants.

By eliminating arbitrary constants a, b from Equation (1), we get first order P.D.E.

Step II : Differentiate Equation (1) partially w.r.t. x, keeping y as constant.

$$\cancel{2} \frac{\partial z}{\partial x} = \cancel{2} \frac{x}{a^2} + 0 \Rightarrow \frac{1}{a^2} = \frac{p}{x} \quad ...(2)$$

and differentiate Equation (1) partially w.r.t. y, keeping x as constant.

$$\cancel{2} \frac{\partial z}{\partial y} = 0 + \frac{\cancel{2} y}{b^2} \Rightarrow \frac{1}{b^2} = \frac{q}{y} \quad ...(3)$$

Step III : By using Equations (2) and (3), Equation (1) becomes.

$$2z = \left(\frac{p}{x}\right)x^2 + \left(\frac{q}{y}\right)y^2$$

$$\Rightarrow 2z = px + qy$$

Be the required first order PDE.

(iii)

Step I :

Given : $\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{c^2}=1$

Let z – be dependent variable

x, y – are independent variables ;

a, b, c – are arbitrary constants.

Here, number of arbitrary constants are greater than number of independent variables, we get, second or higher order PDE.

Step II : Differentiate Equation (1) partially w.r.t. x, keeping y as constant.

$$\frac{1}{a^2}\cdot \cancel{2}x+0+\frac{\cancel{2}z}{c^2}\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow c^2x+a^2z\frac{\partial z}{\partial x} = 0 \qquad ...(2)$$

and differentiate Equation (1) partially w.r.t. y, keeping y as constant.

$$\therefore 0+\frac{1}{b^2}\cancel{2}y+\frac{\cancel{2}z}{c^2}\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow c^2y+b^2z\frac{\partial z}{\partial y} = 0 \qquad ...(3)$$

Step III :

Again differentiate Equation (2) partially, w.r.t. x, we get

$$c^2+a^2\left(z\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow\left(\frac{c^2}{a^2}\right)+z\frac{\partial^2 z}{\partial x^2}+\left(\frac{\partial z}{\partial x}\right)^2 = 0 \qquad ...(4)$$

But from Equation (2) $\frac{c^2}{a^2}=-\left(\frac{z}{x}\cdot\frac{\partial z}{\partial x}\right)$

Substitute this value of $\left(\frac{c^2}{a^2}\right)$ in Equation (4), we get

$$-\left(\frac{z}{x}\cdot\frac{\partial z}{\partial x}\right)+z\frac{\partial^2 z}{\partial x^2}+\left(\frac{\partial z}{\partial x}\right)^2 = 0$$

$$\Rightarrow xz\frac{\partial^2 z}{\partial x^2}+x\left(\frac{\partial z}{\partial x}\right)^2-z\cdot\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow xz\,r+xp^2-z\cdot p = 0 \checkmark \qquad ...\text{Ans.}$$

is the required second order P.D.E.

Ex. 4.2.14 : Form the partial differential equation from

(i) $(x-a)^2+(y-b)^2=z^2\cot^2\alpha$

(ii) $z=a\cdot\log\left\{\frac{b(y-1)}{1-x}\right\}$

Soln. : (i)

Step I :

Given : $(x-a)^2+(y-b)^2 = z^2\cot^2\alpha$ ...(1)

Let z – be dependent variable

x, y are independent variables,

a, b are arbitrary constants

and α – is any fixed constant.

By eliminating arbitrary constants a, b from Equation (1), we get first order P.D.E.

Step II : Differentiate Equation (1) partially, w.r.t. x, keeping y as constant.

$$\cancel{2}(x-a)+0 = \cancel{2}z\frac{\partial z}{\partial x}\cdot\cot^2\alpha$$

$$\Rightarrow (x-a) = z\,p\cot^2\alpha \qquad ...(2)$$

And differentiate Equation (1) partially w.r.t. y, keeping y as constant.

$$0+\cancel{2}(y-b) = \cancel{2}z\frac{\partial z}{\partial y}\cot^2\alpha$$

$$\Rightarrow (y-b) = z\,q\cot^2\alpha \qquad ...(3)$$

Step III : Substitute the values of (x – a) and (y – b) in Equation (1), we get

$$\cancel{z^2}\,p^2\cot^4\alpha+\cancel{z^2}\,q^2\cot^4\alpha = \cancel{z^2}\cot^2\alpha$$

$$\cot^4\alpha\,[p^2+q^2] = \cot^2\alpha$$

$$\Rightarrow \quad p^2+q^2 = \frac{\cot^2\alpha}{\cot^4\alpha}=\frac{1}{\cot^2\alpha}$$

$$\therefore\ p^2+q^2 = \tan^2\alpha \qquad ...\text{Ans.}$$

is the required first order P.D.E.

(ii)

Step I :

Given $z = a\log\left\{\frac{b(y-1)}{1-x}\right\}$

$$z = a\,[\log b(y-1)-\log(1-x)]$$

$$z = a\log b+a\log(y-1)-a\cdot\log(1-x) \quad ...(1)$$

Here, z - is dependent variable,

x, y - are independent variables;

a, b - are arbitrary constant.

By eliminating arbitrary constants a, b from (1), we get first order P.D.E.

Step II : Differentiate Equation (1) partially, w.r.t. x, keeping y as constant.

$$p = \frac{\partial z}{\partial x}=0+0-\frac{a\cdot 1}{(1-x)}(-1)$$

$$\Rightarrow \quad p = \frac{a}{(1-x)}$$

$$\Rightarrow \quad a = p(1-x)$$

And differentiate Equation (1) partially w.r.t. y, keeping x as constant.

$$q = \frac{\partial z}{\partial y}=0+\frac{a\cdot 1}{(y-1)}-0$$

$$q = \frac{a}{(y-1)}=\frac{1}{(y-1)}(p(1-x))$$

$$(y-1)\,q = p\,(1-x) \checkmark \qquad ...\text{Ans.}$$

is the required first order P.D.E.

4.2.2 Formation of Partial Differential Equation by Eliminating an Arbitrary Function

Partial differential equations can be obtained by eliminating an arbitrary functions.

The order of partial differential is equal to the number of arbitrary functions to be eliminated.

i.e. if the number of arbitrary functions are n to be eliminate, then we get n^{th} order partial differential equations.

Let, u and v are any two given functions of x, y and z.

We have to form a partial differential equation by eliminating an arbitrary function f, we get first order PDE.

Formation of Partial Differential Equations from F(u, v) = 0

Let, F – be any arbitrary function of u and v such that

$$F(u, v) = 0 \qquad ...(1)$$

where, u, v are functions of x, y, z. Equation (1) is the implicit form ,so here, z - is a dependent variable on x, y.

$$F\to u, v\to x, y, \underset{x\ y}{\underset{\wedge}{z}}$$

We have to form a partial differential equation by eliminating arbitrary function F.

We differentiate F partially w.r.t. y, keeping x as constant, we get

$$\left[\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right]+\left[\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right]=0$$

$$\Rightarrow\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot p\right]+\frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot q\right]=0$$

And differentiate F partially w.r.t. y, keeping x as constant.

$$\left[\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right]+\left[\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right]=0$$

$$\Rightarrow\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot q\right]+\frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot q\right]=0 \qquad ...(3)$$

To eliminate $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$ from Equations (2) and (3), by Cramer's rule,

$$\begin{vmatrix} \frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot p & \frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot p \\ \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q & \frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q \end{vmatrix}=0$$

$$\Rightarrow\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot p\right)\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot q\right)-\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)\cdot\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0$$

$$\Rightarrow\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}+q\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}+p\frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial z}+pq\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial z}$$

$$-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}-p\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-q\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-pq\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial z}=0$$

$$\Rightarrow -p\left[\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial z}\right]-q\left[\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}\right]+\left[\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\right]=0$$

$$\Rightarrow \quad -pP-qQ+R = 0$$

$$\text{or } pP+qQ = R$$

where, $p = \frac{\partial(u,v)}{x(y,z)}$; $\quad Q = \frac{\partial(u,v)}{x(z,x)}$

$$R = \frac{\partial(u,v)}{x(x,y)}$$

Hence, for the function F(u,v) = 0, the partial differential equation is

$$pP+qQ = R \checkmark \qquad ...\text{Ans.}$$

Ex. 4.2.15 : Form a partial differential equation (by eliminating arbitrary function) from the relation.

$$z=y^2+2f\left(\frac{1}{x}+\log y\right)$$

Soln. :

Step I : The given function is

$$z = y^2+2f\left(\frac{1}{x}+\log y\right) \qquad ...(1)$$

Here, z – is dependent variable,

x, y – are independent variable and,

f – is an arbitrary function.

Step II : To eliminate arbitrary function f, differentiate z partially w.r.t. x, keeping y as constant

$$p = \frac{\partial z}{\partial x}=0+2f'\left(\frac{1}{x}+\log y\right)\cdot\frac{\partial}{\partial x}\left(\frac{1}{x}+\log y\right)$$

$$p = 2f'\left(\frac{1}{x}+\log y\right)\cdot\left(-\frac{1}{x^2}+0\right)$$

$$\Rightarrow -p\,x^2 = 2f'\left(\frac{1}{x}+\log y\right) \qquad ...(2)$$

Again differentiate Equation (1) partially w.r.t. y, keeping x as constant.

$$q = \frac{\partial z}{\partial y}=2y+2f'\left(\frac{1}{x}+\log y\right)\cdot\frac{\partial}{\partial y}\left(\frac{1}{x}+\log y\right)$$

$$\therefore\ q = 2y+2f'\left(\frac{1}{x}+\log y\right)\left(0+\frac{1}{y}\right)$$

$$y\cdot q = 2y^2+2f'\left(\frac{1}{x}+\log y\right)$$

$$\Rightarrow qy-2y^2 = 2f'\left(\frac{1}{x}+\log y\right) \qquad ...(3)$$

Step III : From Equations (2) and (3), we get,

$$-p\,x^2 = qy-2y^2$$

$$\Rightarrow \quad px^2+qy = 2y^2 \checkmark \qquad ...\text{Ans.}$$

is the required first order partial differential equation.

Ex. 4.2.16 : Form a partial differential equation by eliminating an arbitrary function from

$$z=(x+y)\,\phi\,(x^2-y^2)$$

Soln. :

Step I : Given : $z=(x+y)\,\phi\,(x^2-y^2)$ ...(1)

Here, z – is the dependent variable ;

x, y – are independent variables and

ϕ – is an arbitrary function.

We have to eliminate one arbitrary function, we get first order partial differential equations.

Step II : To differentiate Equation (1) partially w.r.t. x, keeping y as constant

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[(x+y)\,\phi(x^2-y^2)]$$

➔ **Using standard result :** ...[d (uv) = u dv + v du]

➔ **Using chain rule :** ...$\left[d[f(g(x))] = f'(g(x)) \cdot \frac{d}{dx}g(x)\right]$

$$p = (x+y)\cdot\phi'(x^2-y^2)(2x) + \phi(x^2-y^2)\cdot(1) \quad ...(2)$$

Also differentiate Equation (1) partially, w.r.t. y keeping x as constant.

$$q = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[(x+y)\,\phi(x^2-y^2)]$$

➔ **Using standard formula :** ...[d(uv) = u dv + vdu]

$$= (x+y)\cdot\underbrace{\frac{\partial}{\partial y}(\phi(x^2-y^2))}_{\phi'(x^2-y^2)\cdot(-2y)} + \phi(x^2-y^2)\cdot\underbrace{\frac{\partial}{\partial y}(x+y)}_{(1)}$$

$$q = (x+y)\cdot\phi'(x^2-y^2)(-2y) + \phi(x^2-y^2)(1)$$

$$\therefore q = -2y(x+y)\,\phi'(x^2-y^2) + \phi(x^2-y^2) \quad ...(3)$$

Step III : From (1) : $\phi(x^2-y^2) = \frac{z}{x+y}$

Equation (2) becomes,

$$p = (x+y)\,\phi'(x^2-y^2)(2x) + \frac{z}{(x+y)}$$

$$\Rightarrow p - \frac{z}{x+y} = (x+y)\,\phi'(x^2-y^2)(2x) \quad ...(4)$$

And Equation (3) becomes

$$q = -2y(x+y)\,\phi'(x^2-y^2) + \frac{z}{(x+y)}$$

$$\Rightarrow q - \frac{z}{(x+y)} = (-2y)(x+y)\,\phi'(x^2-y^2) \quad ...(5)$$

Equation (4) ÷ Equation (5)

$$\frac{p-\left(\frac{z}{x+y}\right)}{q-\left(\frac{z}{x+y}\right)} = \frac{(x+y)\,\phi'(x^2-y^2)(2x)}{(x+y)\,\phi'(x^2-y^2)(-2y)}$$

$$\Rightarrow \frac{\left(p-\frac{z}{x+y}\right)}{\left(q-\frac{z}{x+y}\right)} = -\frac{x}{y}$$

$$\Rightarrow \frac{p(x+y)-z}{q(x+y)-z} = -\frac{x}{y}$$

$$[p(x+y)-z]\,y = -x[q(x+y)-z]$$

$$\Rightarrow py(x+y) - zy = -xq(x+y) + zx$$

$$\Rightarrow (py+xq)(x+y) = z(x+y)$$

$$\Rightarrow \mathbf{py + xq = z} \checkmark \quad ...\text{Ans.}$$

be the required first order partial differential equation.

Ex. 4.2.17 : Form a partial differential equation from

$$z = \phi(x+at) + \psi(x-at)$$

Soln. :

Step I : Given, $z = \phi(x+at) + \psi(x-at)$ ...(1)

Here, z – is dependent variable and ϕ, ψ - are the arbitrary function. Also, x, y are independent variables. So, the number of arbitrary functions are two, we get second order P.D.E.

Step II : Differentiate Equation (1) partially w.r.t. x two times, keeping t as constant.

$$p = \frac{\partial z}{\partial x} = \phi'(x+at) + \psi'(x-at)$$

and

$$r = \frac{\partial^2 z}{\partial x^2} = \phi''(x+at) + \psi''(x-at) \quad ...(2)$$

Also, Differentiate Equation (1) partially w.r.t. t, two times, keeping x as constant.

$$\frac{\partial z}{\partial t} = \phi'(x+at)\cdot\underbrace{\frac{\partial}{\partial t}(x+at)}_{(a)} + \psi'(x-at)\underbrace{\frac{\partial}{\partial t}(x-at)}_{(-a)}$$

$$\frac{\partial z}{\partial t} = \phi'(x+at)(a) + \psi'(x-at)(-a)$$

And

$$\frac{\partial^2 z}{\partial t^2} = \phi''(x+at)(a^2) + \psi'(x-at)(-a)^2$$

$$\frac{\partial^2 z}{\partial t^2} = a^2[\phi''(x+at) + \psi''(x-at)] \quad ...(3)$$

Step III : From Equations (2), Equation (3) becomes

$$\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2}$$

$$\text{or} \quad \frac{\partial^2 z}{\partial t^2} = \mathbf{a^2 r} \checkmark \quad ...\text{Ans.}$$

is the required second order partial differential equation.

Ex. 4.2.18 : Form a partial differential equation from

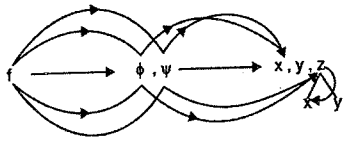
$$f(x^2+y^2, z-xy) = 0$$

Soln. :

Step I : Given : $f(x^2+y^2,\ z-xy) = 0$

Let, $\phi = x^2+y^2$, $\psi = z-xy$ ...(1)

$\therefore$ $f(\phi, \psi) = 0$ is an implicit function



Here, z - is a function of x and y

Here, z – is dependent variable,

x, y – are independent variables,

f – be arbitrary function

Differentiate f w.r.t. x, keeping y as constant and note that z - is function of x, y.

$$\therefore \left[\frac{\partial f}{\partial \phi}\cdot\frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial \phi}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{\partial z}{\partial x}\right] + \left[\frac{\partial f}{\partial \psi}\cdot\frac{\partial \psi}{\partial x} + \frac{\partial f}{\partial \psi}\frac{\partial \psi}{\partial z}\cdot\frac{\partial z}{\partial x}\right] = 0$$

$$\Rightarrow \left[\frac{\partial f}{\partial \phi}\cdot(2x) + \frac{\partial f}{\partial \phi}\cdot(0)\cdot p\right] + \left[\frac{\partial f}{\partial \psi}\cdot(-y) + \frac{\partial f}{\partial \psi}\cdot(1)\cdot p\right] = 0$$

$$\Rightarrow \frac{\partial f}{\partial \phi}(2x) + \frac{\partial f}{\partial \psi}(-y+p) = 0 \quad ...(2)$$

Also differentiate f partially w.r.t. y, keeping x as constant, remember that z is a function of x and y.

$$\therefore \left[\frac{\partial f}{\partial \phi}\cdot\frac{\partial \phi}{\partial y} + \frac{\partial f}{\partial \phi}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{\partial z}{\partial y}\right] + \left[\frac{\partial f}{\partial \psi}\cdot\frac{\partial \psi}{\partial y} + \frac{\partial f}{\partial \psi}\cdot\frac{\partial \psi}{\partial z}\cdot\frac{\partial z}{\partial y}\right] = 0$$

$$\Rightarrow \left[\frac{\partial f}{\partial \phi}(2y) + \frac{\partial f}{\partial \phi}(0)\cdot q\right] + \left[\frac{\partial f}{\partial \psi}\cdot(-x) + \frac{\partial f}{\partial \psi}\cdot(1)\cdot q\right] = 0$$

$$\Rightarrow \frac{\partial f}{\partial \phi}(2y) + \frac{\partial f}{\partial \psi}(-x+q) = 0 \quad ...(3)$$

Step III : Eliminate $\frac{\partial f}{\partial \phi}$ and $\frac{\partial f}{\partial \psi}$ From Equations (2) and (3), by using Cramer's rule,

"If $a x_1 + b y_1 = 0$ and $a x_2 + b y_2 = 0$ then

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = 0\text{"}$$

Here, $a = \frac{\partial f}{\partial \phi}$; $b = \frac{\partial f}{\partial \psi}$

$$\therefore \begin{vmatrix} 2x & -y+p \\ 2y & -x+q \end{vmatrix} = 0$$

$$\Rightarrow (2x)(-x+q) - (2y)(-y+p) = 0$$

$$\Rightarrow -\cancel{2}x^2 + \cancel{2}xq + \cancel{2}y^2 - \cancel{2}yp = 0$$

$$\Rightarrow \mathbf{xq - yp = x^2 - y^2} \checkmark \quad ...\text{Ans.}$$

be the required first order partial differential equation.

Ex. 4.2.19 : Form a partial differential equation from.

$$u = f(x+it) + f_2(x-it)$$

Soln. :

Step I : Given

$$u = f_1(x+it) + f_2(x-it) \quad ...(1)$$

Here, u – is dependent variable ;

x, t – are independent variables.

f_1, f_2 – are arbitrary functions

We get second order partial differential equation.

Step II : Differentiate Equation (1) partially w.r.t. x two times, keeping t as constant

$$\frac{\partial u}{\partial x} = f_1'(x+it) + f_2'(x-it)$$

$$\text{and} \quad \frac{\partial^2 u}{\partial x^2} = f_1''(x+it) + f_2''(x-it) \quad ...(2)$$

Differentiate Equation (1) partially w.r.t. t, two times, keeping x as constant.

$$\frac{\partial u}{\partial t} = f_1'(x+it)\cdot\underbrace{\frac{\partial}{\partial y}(it)}_{(i)} + f_2'(x-it)\cdot\underbrace{\frac{\partial}{\partial y}(-it)}_{(-i)}$$

$$= (i)[f_1'(x+it) - f_2'(x-it)]$$

$$\text{and} \quad \frac{\partial^2 u}{\partial t^2} = (i)\left[f_1''(x+it)\underbrace{\frac{\partial}{\partial t}(it)}_{(i)} - f_2''(x-it)\underbrace{\frac{\partial}{\partial t}(-it)}_{(-i)}\right]$$

$$= (i)^2[f_1''(x+it) + f_2''(x-it)] \quad ...(i^2 = -1)$$

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2 u}{\partial x^2} \quad ...\text{From Equation (2)}$$

$$\Rightarrow \mathbf{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0} \checkmark \quad ...\text{Ans.}$$

is the required second order partial differential equation.

Ex. 4.2.20 : Form a partial differential equation from :

$$u = x f(x+t) + g(x+t)$$

Soln. :

Step I : Given : $u = x f(x+t) + g(x+t)$ ...(1)

Here u – is dependent variables

f, g – are arbitrary functions.

x, t – are independent variables

We get second order partial differential equation.

Step II : Differentiate Equation (1) partially w.r.t. x two times, keeping t as constant.

$$\frac{\partial u}{\partial x} = x\cdot\frac{\partial}{\partial x}[f(x+t)] + f(x+t)\cdot\frac{\partial}{\partial x}(x) + g'(x+t)$$

$$\frac{\partial u}{\partial x} = x f'(x+t) + f(x,t) + g'(x+t)$$

$$\text{And} \quad \frac{\partial^2 u}{\partial x^2} = x f''(x+t)\cdot + f'(x+t) + f'(x+t) + g''(x+t) \quad ...(2)$$

$$= x f''(x+t) + 2f'(x+t) + g''(x+t)$$

And differentiate Equation (1) partially w.r.t. t, two times, keeping x as constant

$$\frac{\partial u}{\partial t} = x f'(x+t) + g'(x+t)$$

And

$$\frac{\partial^2 u}{\partial t^2} = x f''(x+t) + g''(x+t) \quad ...(3)$$

Step III : By using Equation (3), Equation (2) becomes,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + 2f'(x+t)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 2f'(x+t) \quad ...(4)$$

Step IV : Now eliminate f' function. Differentiate $\frac{\partial u}{\partial t}$ w.r.t. x

$$\Rightarrow \frac{\partial^2 u}{\partial x\,\partial t} = f'(x+t) + x f''(x+t) + g''(x+t)$$

$$= \frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2}\right] + \frac{\partial^2 u}{\partial t^2} \quad \text{...from (4) and (3)}$$

Hence

$$\frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2}\right] + \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x \partial t} = 0 \checkmark \quad \text{...Ans.}$$

is required second order partial differential equation.

Ex. 4.2.21 : From a partial differential equation from

$$f(x+y+z,\ x^2+y^2+z^2) = 0$$

Soln. :

Step I : Given

$$f(x+y+z,\ x^2+y^2+z^2) = 0$$

Let $u = x+y+z;\ v = x^2+y^2+z^2$

$\therefore\ f(u, v) = 0$ It is an implicit function.

Step II :

$$f \longrightarrow u,\ v \longrightarrow x, y, z \quad (x,\ y) \quad \text{...(1)}$$

Differentiate f partially w.r.t. x, keeping y as constant and note that z-is a function of x, y

$$\therefore \left[\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right] + \left[\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right] = 0$$

$$\Rightarrow \left[\frac{\partial f}{\partial u}(1) + \frac{\partial f}{\partial u}(1)\cdot p\right] + \left[\frac{\partial f}{\partial v}\cdot(2x) + \frac{\partial f}{\partial v}\cdot(2z)\cdot p\right] = 0$$

$$\Rightarrow \frac{\partial f}{\partial u}(1+p) + \frac{\partial f}{\partial v}(2x+2zp) = 0 \quad \text{...(2)}$$

Differentiate f partially w.r.t. y, keeping x as constant and remember that z is a function of x and y.

$$\therefore \left[\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right] + \left[\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right] = 0$$

$$\Rightarrow \left[\frac{\partial f}{\partial u}(1) + \frac{\partial f}{\partial u}(1)\cdot q\right] + \left[\frac{\partial f}{\partial v}\cdot(2y) + \frac{\partial f}{\partial v}\cdot(2z)\cdot q\right] = 0$$

$$\Rightarrow \frac{\partial f}{\partial u}(1+q) + \frac{\partial f}{\partial v}(2y+2zq) = 0 \quad \text{...(3)}$$

Step III : By eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (2), and (3),

"By Cramer's rule,

if $a x_1 + b y_1 = 0$ and $a x_2 + b y_2 = 0$

then, $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = 0$"

Here, $a = \frac{\partial f}{\partial u}$ and $b = \frac{\partial f}{\partial v}$

$$\therefore \begin{vmatrix} 1+p & 2x+2zp \\ 1+q & 2y+2zq \end{vmatrix} = 0$$

$$\Rightarrow (1+p)(2y+2zq) - (1+q)(2x+2zp) = 0$$

$$\Rightarrow 2y + 2zq + 2py + \cancel{2zpq} - 2x - 2zp - 2qx - \cancel{2zpq} = 0$$

$$\Rightarrow y + zq + py - x - pz - qx = 0$$

$$\Rightarrow (y-z)p + (z-x)q = x - y \checkmark \quad \text{...Ans.}$$

Which is required first order partial differential equation.

Alternate Method

Soln. :

Step I : Given, $F(x+y+z,\ x^2+y^2+z^2) = 0$

Let, $u = x+y+z;\ v = x^2+y^2+z^2$

$\therefore\ F(u, v) = 0$

Since the P.D.E. of $F(u, v) = 0$ is of the

From, $pP + qQ = R$ ...(1)

Where $p = \frac{\partial(u, v)}{\partial(y, z)}$; $Q = \frac{\partial(u, v)}{\partial(z, x)}$;

$R = \frac{\partial(u, v)}{\partial(x, y)}$

$$\therefore P = \frac{\partial(u, v)}{\partial(y, z)} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2y & 2z \end{vmatrix} = 2(z-y)$$

$$Q = \frac{\partial(u, v)}{\partial(z, x)} = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2z & 2x \end{vmatrix} = 2(x-z)$$

$$R = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2x & 2y \end{vmatrix} = 2(y-x)$$

Equation (1) becomes

$$\cancel{2}(z-y)p + \cancel{2}(x-z)q = \cancel{2}(y-x)$$

$$\Rightarrow (z-y)p + (x-z)q = (y-x)$$

Or

$$(y-z)p + (z-x)q = (x-y) \checkmark \quad \text{...Ans.}$$

be the required first order P.D.E.

Ex. 4.2.22 : Eliminate the arbitrary function from $z = xy + f(x^2+y^2)$ and form a partial differential equation.

Soln. :

Step I : Given, $z = xy + f(x^2+y^2)$ ...(1)

Here z – is dependent variable;

x, y – are independent variables;

f – is an arbitrary function

By eliminating arbitrary function f, from Equation (1), we get first order P.D.E.

Step II : Differentiate Equation (1) partially w.r.t. x, keeping y as constant

$$p = \frac{\partial z}{\partial x} = y + f'(x^2+y^2)\cdot\underbrace{\frac{\partial}{\partial y}(x^2+y^2)}_{(2x)}$$

$$\Rightarrow p = y + (2x)\cdot f'(x^2+y^2) \quad \text{...(2)}$$

Also, differentiate Equation (1) partially w.r.t. y, keeping x as constant.

$$q = \frac{\partial z}{\partial y} = x + f'(x^2+y^2)\cdot\frac{\partial}{\partial y}\underbrace{(x^2+y^2)}_{(2y)}$$

$$\Rightarrow q = x + (2y)f'(x^2+y^2) \quad \text{...(3)}$$

Step III : y × Equation (2) – x × Equation (3), we get

$$yp - xq = [y^2 + 2xy f'(x^2+y^2)] - [x^2 + 2xy f'(x^2+y^2)]$$

$$\Rightarrow yp - xq = (y^2 - x^2) + [2xy f'(x^2+y^2) - 2xy f'(x^2+y^2)]$$

$$yp - xq = (y^2 - x^2) \checkmark \quad \text{...Ans.}$$

Is the required first order P.D.E.

Ex. 4.2.23 : Eliminate an arbitrary function from

$z = f\left(\frac{xy}{z}\right)$ and form a partial differential equation

Soln. :

Step I : Given, $z = f\left(\frac{xy}{z}\right)$ ...(1)

Here, z – is dependent variable;

x, y – are independent variable;

and f – is any arbitrary function

By eliminating arbitrary function f from Equation (1), we get first order P.D.E.

Step II : Differentiate Equation (1) partially w.r.t. x, keeping y as constant

$$p = \frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right)\cdot\frac{\partial}{\partial x}\left(\frac{xy}{z}\right)$$

(∵ On R.H.S. z – is independent variable)

$$p = f'\left(\frac{xy}{z}\right)\cdot\left(\frac{y}{z}\right) \quad \text{...(2)}$$

Also, differentiate Equation (1) partially w.r.t. y, keeping x as constant.

$$q = \frac{\partial z}{\partial y} = f'\left(\frac{xy}{z}\right)\cdot\underbrace{\left(\frac{\partial}{\partial y}\left(\frac{xy}{z}\right)\right)}_{\left(\frac{x}{z}\right)}$$

$$q = f'\left(\frac{xy}{z}\right)\cdot\left(\frac{y}{z}\right) \quad \text{...(3)}$$

Step III : To eliminate $f'\left(\frac{xy}{z}\right)$, From Equations (2) and (3), we get

$$x\cdot p - yq = \frac{xy}{z}\cdot f'\left(\frac{xy}{z}\right) - \frac{xy}{z}\cdot f'\left(\frac{xy}{z}\right)$$

$$xp - yq = 0 \checkmark \quad \text{...Ans.}$$

is the required first order P.D.E.

Ex. 4.2.24 : Form a partial differential equation by eliminating arbitrary function from

$z = x + y + f(xy)$

Soln. :

Step I : Given : $z = x + y + f(xy)$ ...(1)

Here, z – is dependent variable;

x, y – are independent variables;

f – is any arbitrary function.

By eliminating f from Equation (1) we get, first order P.D.E.

Step II : Differentiate Equation (1) partially w.r.t. x, keeping y as constant

$$p = \frac{\partial z}{\partial y} = 1 + 0 + f'(xy)\cdot y$$

$$\Rightarrow p = 1 + y f'(xy) \quad \text{...(2)}$$

Also, differentiate Equation (1) partially w.r.t. y, keeping x as constant

$$q = \frac{\partial z}{\partial y} = 0 + 1 + f'(xy)\cdot x$$

$$\Rightarrow q = 1 + x f'(xy) \quad \text{...(3)}$$

Step III : To eliminate f' from Equations (2) and (3)

$$px - qy = (x + \cancel{xy f'(xy)}) - (y + \cancel{xy f'(xy)})$$

$$px - qy = (x - y) \checkmark \quad \text{...Ans.}$$

is the required first order P.D.E.

Ex. 4.2.25 : Form the partial differential equation by eliminating an arbitrary function From

(i) $z = f(x^2 - y^2)$

(ii) $z = f(y/x)$

Soln. : (i)

Step I : Given : $z = f(x^2 - y^2)$ ...(1)

Here, z – is dependent variable;

x, y – are independent variables and

f – is any arbitrary function

By eliminating function f from Equation (1), we get first order P.D.E.

Step II : Differentiate Equation (1) partially w.r.t. x, keeping y as constant

$$p = \frac{\partial z}{\partial x} = f'(x^2 - y^2)\cdot(2x) \quad \text{...(2)}$$

Also, differentiate Equation (1) partially w.r.t. y, keeping x as constant

$$q = \frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y) \quad \text{...(3)}$$

Step III : To eliminate f', from Equation (2) and (3)

$$yp + xq = 2xy f'(x^2 - y^2) - 2xy f'(x^2 - y^2)$$

$$\Rightarrow yp + xq = 0 \checkmark \quad \text{...Ans.}$$

is the required first order P.D.E.

(ii)

Step I : Given, $z = f\left(\frac{y}{x}\right)$ ...(1)

Here, z – is dependent variable ;

x, y – are independent variables ;

f – is arbitrary function

By eliminating function f, from Equation (1), we get first order P.D.E.

Step I : Differentiate Equation (1) partially w.r.t. x, keeping y as constant

$$p = \frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right) \cdot \underbrace{\frac{\partial}{\partial x}\left(\frac{y}{x}\right)}_{(-y/x^2)}$$

$$= f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$x^2 p = -y \cdot f'\left(\frac{y}{x}\right) \quad ...(2)$$

Also, differentiate Equation (1) partially w.r.t. y, keeping x as constant

$$q = \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right) \cdot \underbrace{\frac{\partial}{\partial x}\left(\frac{y}{x}\right)}_{(1/x)}$$

$$= f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$\Rightarrow xq = f'\left(\frac{y}{x}\right) \quad ...(3)$$

Step III : To eliminate f', from Equation (2) and (3) we have

$$x^2 p + yxq = -y f'\left(\cancel{\frac{y}{x}}\right) + y f'\left(\cancel{\frac{y}{x}}\right)$$

$$= 0$$

$$\Rightarrow x^2 p + xyq = 0 \quad \Rightarrow \quad px + qy = 0 \checkmark \quad ...\text{Ans.}$$

be the required first order P.D.E.

Ex. 4.2.26 : Form a partial differential equation from

(i) $f(x^2 + y^2 + z^2,\ z^2 - 2xy) = 0$

(ii) $f(x^2 + y^2,\ z - xy) = 0$

(iii) $f(xy + z^2,\ x + y + z) = 0$

Soln. : (i)

Step I : Given,

$$f(x^2 + y^2 + z^2,\ z^2 - 2xy) = 0$$

Let $u = x^2 + y^2 + z^2$; $v = z^2 - 2xy$

$\therefore\ f(u, v) = 0$ ...(1)

Step II :

Since, the solution of f(u, v) = 0, is a first order PDE of the form

$pp + qQ = R$ ...(2)

where, $P = \frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} 2y & 2z \\ -2x & 2z \end{vmatrix}$

$= 4yz + 4xz = 4z(x + y)$

$Q = \frac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} 2z & 2x \\ 2z & -2y \end{vmatrix}$

$= -4yz - 4zx = -4z(x + y)$

and $R = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ -2y & -2x \end{vmatrix}$

$= -4x^2 + 4y^2 = 4(y^2 - x^2)$

Step III : Substitute the values of P, Q, R in Equation (2), we get

$$\cancel{4}z(x + y)p - \cancel{4}z(x + y)q = \cancel{4}(y^2 - x^2)$$

$$z(x \cancel{+} y)[p - q] = (y - x)(y \cancel{+} x)$$

$$\Rightarrow z(p - q) = (y - x) \checkmark \quad ...\text{Ans.}$$

be the required first order P.D.E.

(ii)

Step I : Given, $f(x^2 + y^2,\ z - xy) = 0$

Let $u = x^2 + y^2$; $v = z - xy$

$\therefore\ f(u, v) = 0$ ...(1)

Step II : Since, the solution of f (u, v) = 0, is the first order P.D.E. is of the form,

$pP + qQ = R$ ...(2)

where, $P = \frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} 2y & 0 \\ -x & 1 \end{vmatrix}$

$= (2y) - 0 = (2y)$

$Q = \frac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} 0 & 2x \\ 1 & -y \end{vmatrix}$

$= 0 - (2x) = -2x$

$R = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ -y & -x \end{vmatrix}$

$= -2x^2 + 2y^2 = 2(y^2 - x^2)$

Step III : Substituting the values of P, Q R in Equation (2), we get

$$\cancel{2}yp - \cancel{2}xq = \cancel{2}(y^2 - x^2)$$

$$\Rightarrow yp - xq = (y^2 - x^2) \checkmark \quad ...\text{Ans.}$$

be the required first order P.D.E.

(iii)

Step I : Given, $f(xy + z^2,\ x + y + z) = 0$

Let, $u = xy + z^2$; $v = x + y + z$

$\therefore\ f(u, v) = 0$ ...(1)

Step II : Since, the solution of f (u, v) = 0 is a first order, PDE is of the form

$pP + qQ = R$ ...(2)

where, $P = \frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} x & 2z \\ 1 & 1 \end{vmatrix}$

$\therefore\ P = (x - 2z)$

$Q = \frac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} 2z & y \\ 1 & 1 \end{vmatrix}$

$= (2z - y)$

and $R = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & 1 \end{vmatrix}$

$= (y - x)$

Step III : Substituting the value of P, Q, R in Equation (1), we get,

$$(x - 2z)p - (2z - y)q = (y - x) \checkmark \quad ...\text{Ans.}$$

be the required first order P.D.E.

Ex. 4.2.27 : Form a partial differential equation by eliminating arbitrary functions from.

(i) $xyz = \phi(x + y + z)$

(ii) $z = f_1(x) \cdot f_2(y)$

(iii) $z = y f(x) + x g(y)$

(iv) $z = f_1(y + 2x) + f_2(y - 3x)$

(v) $z = f(x - y)$

Soln. :

(i)

Step I : Given, $xyz = \phi(x + y + z)$ ...(1)

Let z - is a dependent variable ;

x, y - are independent variables ;

ϕ - is any arbitrary function.

By eliminating function ϕ from Equation (1), we get first order P.D.E.

Step II : Differentiate Equation (1) partially w.r.t. x, keeping y as constant

➔ **Using standard result :** ...[d (u, v) = u dv + v du]

$$y\left[\frac{\partial z}{\partial x}x + z(1)\right] = \phi'(x + y + z) \cdot \left[1 + \frac{\partial z}{\partial x}\right]$$

$$\Rightarrow y(xp + z) = \phi'(x + y + z)(1 + P)$$

$$\Rightarrow (xpy + yz) = (1 + p)\phi'(x + y + z) \quad ...(2)$$

Also, differentiate Equation (1) partially w.r.t. y, keeping x as constant

➔ **Using standard result :** ...[d (uv) = u dv + vdu]

$$x\left[y\frac{\partial z}{\partial y} + z(1)\right] = \phi'(x + y + z)\left(1 + \frac{\partial z}{\partial y}\right)$$

$$\Rightarrow x[yq + z] = (1 + q)\phi'(x + y + z) \quad ...(3)$$

Step III : To eliminate ϕ' from Equation (2) and (3)

(1 + q) × Equation (2) – (1 + p) × Equation (3), we get,

$$(1 + q)(xpy + yz) - (1 + p)(xyq + xz)$$

$$= \cancel{(1 + p)(1 + q)\phi'(x + y + z)} - \cancel{(1 + p)(1 + q)\phi'(x + y + z)} = 0$$

$$\Rightarrow [xpy + yz + \cancel{xyq} + yzq] - [xyq + xz + \cancel{xyp} + xzp] = 0$$

$$\Rightarrow x(y - z)p + y(z - x)q + z(y - x) = 0$$

$$\Rightarrow x(y - z)p + y(z - x)q = -z(y - x) = z(x - y)$$

is the required first order P.D.E.

(ii)

Step I : Given, $z = f_1(x) \cdot f_2(y)$ ...(1)

Here, z – is dependent variable

x, y – are independent variables

f_1, f_2 – are arbitrary function

by eliminating function f_1 and f_2 from Equation (1), We get second order PDE.

Step II : Differentiate Equation (1) partially w.r.t. x, keeping y as constant

$$p = \frac{\partial z}{\partial x} = f_1'(x) \cdot f_2(y) \quad ...(2)$$

Also, differentiate Equation (1) partially w.r.t. y, keeping x as constant

$$q = \frac{\partial z}{\partial y} = f_1(x) \cdot f_2'(y) \quad ...(3)$$

Again differentiate $\frac{\partial z}{\partial y}$ w.r.t. x, keeping y as constant.

$$\frac{\partial^2 z}{\partial x \partial y} = f_1'(x) \cdot f_2'(y)$$

and $$z\frac{\partial^2 z}{\partial x \partial y} = f_1(x) f_2(y) \cdot f_1'(x) f_2'(y) \quad ...(4)$$

Step III : From Equations (2), (3) and (4), we get

$$z\frac{\partial^2 z}{\partial x \partial y} = [f_1(x) \cdot f_2'(y)] \cdot [f_2(y) f_1'(x)]$$

$$= \left(\frac{\partial z}{\partial y}\right) \cdot \left(\frac{\partial z}{\partial x}\right) \text{ from Equations (2) and (3)}$$

$$\Rightarrow z\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} \checkmark \qquad \text{...Ans.}$$

be the required second order P.D.E.

(iii)

Step I: Given

$$z = y\,f(x) + x\,g(y) \qquad ...(1)$$

Here, z – is dependent variable;

x, y – are independent variables;

f, g – are arbitrary functions.

To eliminate functions f and g from Equation (1), we get second order P.D.E.

Step II: Differentiating Equation (1) partially w.r.t. x, keeping y as constant

$$p = \frac{\partial z}{\partial x} = y\,f'(x) + g(y) \qquad ...(2)$$

Also, differentiate Equation (1) partially w.r.t. y, keeping x as constant

$$q = \frac{\partial z}{\partial y} = f(x) + x\,g'(y) \qquad ...(3)$$

Step III: Differentiate equation $\left(\frac{\partial z}{\partial x}\right)$ w.r.t. y, keeping x as constant

$$\frac{\partial^2 z}{\partial y\,\partial x} = f'(x) + g'(y)$$

$$xy\frac{\partial^2 z}{\partial x\,\partial y} = xy\,[f'(x) + g'(y)] \qquad ...(4)$$

and from Equations (2) and (3)

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy\,f'(x) + x\,g(y) + y\,f(x) + xy\,g'(y)$$

$$= xy\,[f'(x) + g'(y)] + y\,f(x) + x\,g(y)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy\frac{\partial^2 z}{\partial x\,\partial y} + z$$

...from Equations (4) and (1)

$$\text{or}\quad xy\frac{\partial^2 z}{\partial x\,\partial y} - x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} + z = 0 \checkmark \qquad \text{...Ans.}$$

is the required second order P.D.E.

(iv)

Step I: Given, $z = f_1(y + 2x) + f_2(y - 3x)$ ...(1)

Here, z - is dependent variable;

x, y - are independent variables;

f_1, f_2 - are arbitrary functions

By eliminating functions f_1, f_2 from Equation (1), we get second order PDE

Step II: Differentiate Equation (1) partially w.r.t. x, two times, keeping y as constant

$$\frac{\partial z}{\partial x} = f_1'(y + 2x)(2) + f_2'(y - 3x)(-3)$$

$$\text{And}\quad \frac{\partial^2 z}{\partial x^2} = f_1''(y + 2x)(4) + f_2''(y - 3x)(9) \qquad ...(2)$$

Also, differentiate Equation (1) partially w.r.t. y, two times, keeping x as constant

$$\frac{\partial z}{\partial y} = f_1'(y + 2x) + f_2'(y - 3x)$$

$$\text{and}\quad \frac{\partial^2 z}{\partial y^2} = f_1''(y + 2x) + f_2''(y - 3x) \qquad ...(3)$$

Differentiate $\frac{\partial z}{\partial y}$ partially w.r.t. x, keeping y as constant

$$\frac{\partial^2 z}{\partial x\,\partial y} = f_1''(y + 2x)\cdot 2 + f_2''(y - 3x)(-3) \qquad ...(4)$$

Step III: To eliminate f_1'' and f_2'', from Equations (2), (3) and (4), we have

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\,\partial y} = 4\cdot f_1''(y + 2x) + 9\,f_2''(y - 3x)$$

$$+ 2\,f_1''(y + 2x) - 3\,f_2''(y - 3x)$$

$$= 6\,[f_1''(y + 2x) + f_2''(y - 3x)]$$

$$= 6\frac{\partial^2 z}{\partial y^2} \qquad \text{...from (3)}$$

$$\text{Hence,}\quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\,\partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0 \checkmark \qquad \text{...Ans.}$$

be the required second order P.D.E.

(v)

Step I: Given, $z = f(x - y)$ ...(1)

Here, z - is dependent variable;

x, y - are independent variables;

f - is arbitrary function.

By eliminating function f from Equation (1), we get first order P.D.E.

Step II: Differentiate Equation (1) partially w.r.t. x and y

$$p = \frac{\partial z}{\partial x} = f'(x - y) \text{ and } q = \frac{\partial z}{\partial y} = -f'(x - y)$$

Adding, we get

$$p + q = f'(x - y) - f'(x - y) = 0$$

$$\Rightarrow p + q = 0 \checkmark \qquad \text{...Ans.}$$

be the required first order P.D.E.

Exercise

Ex. 1: Form a partial differential equations from the following

(i) $z = f(x + 4t) + g(x - 4t)$

(ii) $z = x\,f_1(x + t) + f_2(x + t)$

(iii) $F(xy + z^2, x + y + z) = 0$

(iv) $z = e^{my}\,\phi(x - y)$

(v) $z = f(x) + e^y\,g(x)$

(vi) $z = x^2 f(y) + y^2 g(x)$

(vii) $z = f(x^2 - y^2)$

(viii) $v = \frac{1}{r}[f(r - at) + g(r + at)]$

Ans.:

(i) $16\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial t^2} = 0$

(ii) $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x\partial t} + \frac{\partial^2 z}{\partial t^2} = 0$

(iii) $p(x - 2z) + q(2z - y) = y - x$

(iv) $p + q = mz$

(v) $\frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial y}$

(vi) $xy\frac{\partial^2 z}{\partial x^2} = 2(px + qy - 2z)$

(vii) $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$

(viii) $\frac{\partial^2 v}{\partial t^2} = \frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)$

4.3 Solutions of Partial Differential Equations

Any relation of the form $f(x, y, z, a, b) = 0$ containing two arbitrary constants a, b is a solution of first order partial differential equations of the form.

$$f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0$$

- Solution is a relation between dependent and independent variables which satisfies the given partial differential equations.
- Solution does not contain any differentials

Complete Integral:

The solution is of the form $f(x, y, z, a, b) = 0$ is called complete solution or complete integral.

General integral/solution:

If the solution contains an arbitrary functions, then such solution is called general solution or general integral.

Particular Integral / particular solution:

Any solution which is obtained from the complete integrals by assigning a particular values to the arbitrary constants is called particular solution / Integral

The unique solution of a PDE corresponding to physical problem, must satisfy certain other boundary and initial conditions.

Ex. 4.3.1: Show that $e^{-n^2t}\sin nx$ is a solution of the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

Soln.:

Step I:

$$\text{Let,}\quad u = e^{-n^2t}\sin nx \qquad ...(1)$$

Differentiate u partially, w.r.t. t, keeping x as constant.

$$\frac{\partial u}{\partial t} = e^{-n^2t}(-n^2)\cdot\sin nx$$

$$\frac{\partial u}{\partial t} = -n^2 e^{-n^2t}\cdot\sin nx \qquad ...(2)$$

Again differentiate u partially, w.r.t. x, keeping t as constant

$$\frac{\partial u}{\partial x} = e^{-n^2t}\cdot\cos nx\,(n)$$

$$\text{and}\quad \frac{\partial^2 u}{\partial x^2} = -n^2 e^{-n^2t}\cdot\sin nx \qquad ...(3)$$

From Equations (2) and (3), we get,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

This shows that $u = e^{-n^2t}\sin nx$ is a solution of the given P.D.E. ✓ Hence Proved.

Ex. 4.3.2: Show that $z = f(x^2 + y^2)$ is a solution of

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$$

Soln.:

Step I: Given, $z = f(x^2 + y^2)$ ...(1)

Differentiate Equation (1) partially, w.r.t. x, keeping y as constant.

$$\frac{\partial z}{\partial x} = f'(x^2 + y^2)(2x)$$

Multiply both sides by y,

$$y\frac{\partial z}{\partial x} = 2xy\,f'(x^2 + y^2) \qquad ...(2)$$

Differentiate Equation (1) partially, w.r.t. y, keeping x as constant

$$\frac{\partial z}{\partial y} = f'(x^2 + y^2)(2y)$$

Multiplying by x,

$$x\frac{\partial z}{\partial y} = 2xy\,f'(x^2 + y^2) \qquad ...(3)$$

Step II: Equation (2) – Equation (3), we get

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 2xy\,f'(x^2 + y^2) - 2xy\,f'(x^2 + y^2)$$

$$= 0$$

This shows that $z = f(x^2 + y^2)$ is the solution of given P.D.E. ✓ Hence Proved.

Ex. 4.3.3: Show that $u = \cos kx\cdot\sinh ky$ is a solution of Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Soln.:

Step I: Given $u = \cos kx\cdot\sinh ky$ ...(1)

Differentiate 'u' partially w.r.t. x two times, keeping y as constant.

$$\frac{\partial u}{\partial x} = \underbrace{\frac{\partial}{\partial x}(\cos kx)}_{(-\sin kx \cdot k)} \cdot \sinh ky$$

$$\frac{\partial u}{\partial x} = -k \sin kx \cdot \sinh ky$$

and $$\frac{\partial^2 u}{\partial x^2} = -k \underbrace{\frac{\partial}{\partial x}(\sin kx)}_{(k \cos kx)} \cdot \sinh ky$$

$$\frac{\partial^2 u}{\partial x^2} = -k^2 \cos kx \cdot \sinh ky \qquad ...(2)$$

Step II : Differentiate Equations (1) partially two times w.r.t. y, keeping x as constant

$$\frac{\partial z}{\partial y} = \cos kx \cdot \underbrace{\frac{\partial}{\partial y}\sinh ky}_{(\cosh ky) \cdot k}$$

$$\frac{\partial z}{\partial y} = k \cos kx \cdot \cosh ky$$

and $$\frac{\partial^2 z}{\partial y^2} = k \cdot \cos kx \cdot \underbrace{\frac{\partial}{\partial y}(\cosh ky)}_{(k \sinh ky)}$$

$$\frac{\partial^2 z}{\partial y^2} = k^2 \cos kx \cdot \sinh ky \qquad ...(3)$$

Step III : Adding Equations (2) and (3), we get,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = [-k^2 \cancel{\cos kx} \cdot \sinh ky] + [k^2 \cancel{\cos kx} \cdot \sinh ky]$$

$$= 0$$

This shows that given u is a solution of given P.D.F. ✓ Hence Proved.

Ex. 4.3.4 : Shows that if u_1 and u_2 are two solutions of the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = A\frac{\partial^2 u}{\partial t^2} + B\frac{\partial u}{\partial t}$ then $c_1 u_1 + c_2 u_2$ is also solution.

Soln. :

Step I : Given u_1 and u_2 are solution of the given PDE.

Therefore, $$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} = A\frac{\partial^2 u_1}{\partial t^2} + B\frac{\partial u_1}{\partial t} \qquad ...(1)$$

and $$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_2}{\partial z^2} = A\frac{\partial^2 u_2}{\partial t^2} + B\frac{\partial u_2}{\partial t} \qquad ...(2)$$

Step II : Now, we have to check $c_1 u_1 + c_2 u_2$ is also solution of given equation.

$$\therefore \frac{\partial^2}{\partial x^2}(c_1 u_1 + c_2 u_2) + \frac{\partial^2}{\partial y^2}(c_1 u_1 + c_2 u_2) + \frac{\partial^2}{\partial z^2}(c_1 u_1 + c_2 u_2)$$

$$= c_1\frac{\partial^2 u_1}{\partial x^2} + c_2\frac{\partial^2 u_2}{\partial x^2} + c_1\frac{\partial^2 u_1}{\partial y^2} + c_2\frac{\partial^2 u_2}{\partial y^2} + c_1\frac{\partial^2 u_1}{\partial z^2} + c_2\frac{\partial^2 u_2}{\partial z^2}$$

$$= c_1\left[\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2}\right] + c_2\left[\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_2}{\partial z^2}\right]$$

$$= c_1\left[A\frac{\partial^2 u_1}{\partial t^2} + B\frac{\partial u_1}{\partial t}\right] + c_2\left[A\frac{\partial^2 u_2}{\partial t^2} + B\frac{\partial u_2}{\partial t}\right]$$

$$= \left[c_1 A\frac{\partial^2 u_1}{\partial t^2} + c_2 A\frac{\partial^2 u_2}{\partial t^2}\right] + \left[c_1 B\frac{\partial u_1}{dt} + c_2 B\frac{\partial u_2}{\partial t}\right]$$

$$= A\frac{\partial^2}{\partial t^2}[c_1 u_1 + c_2 u_2] + B\frac{\partial}{\partial t}[c_1 u_1 + c_2 u_2]$$

This shows that $c_1 u_1 + c_2 u_2$ is also solution of given P.D.E.

Also note that, if $u_1, u_2, u_3, \ldots, u_n$ are independent solutions then $c_1 u_1 + c_2 u_2 + \ldots + c_n u_n$ is also the solution. ✓

Hence Proved.

Syllabus Topic : Equations Solvable by Direct Integration

4.3.1 Solution of Partial Differential Equation - By Direct Integration

Ex. 4.3.5 : Solve : $\frac{\partial^2 z}{\partial x^2} = xy$

Soln. :

Step I : Given partial differential equation :

$$\frac{\partial^2 z}{\partial x^2} = xy \qquad ...(1)$$

Here, z - is dependent variable ;

x, y - are independent variables.

Step II : Integrating Equation (1) w.r.t. x, keeping y as constant

$$\frac{\partial z}{\partial x} = \int xy\, dx + f_1(y)$$

Where f(y) be any arbitrary function (Integrating constant)

$$\therefore \frac{\partial z}{\partial x} = y\left(\frac{x^2}{2}\right) + f_1(y)$$

Step III : Again integrate $\frac{\partial z}{\partial x}$ w.r.t. x, keeping y as constant.

$$\therefore z = \frac{y}{2}\int x^2 dx + \int f_1(y)\, dx + f_2(y)$$

Where, $f_2(y)$ is any arbitrary function (as integrating constant)

$$z = \frac{y}{2}\left(\frac{x^3}{3}\right) + f_1(y)\, x + f_2(y)$$

$$\Rightarrow \quad z = \frac{x^3 y}{6} + x f_1(y) + f_2(y) \checkmark \qquad ...\text{Ans.}$$

be the required solution of the given partial differential equation.

Ex. 4.3.6 : Solve : $\frac{\partial^2 z}{\partial x\, \partial y} = \frac{1}{xy}$

Soln. :

Step I : Given partial differential equation is

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{1}{xy} \qquad ...(1)$$

Here, z - is dependent variable ;

x, y - are independent variables.

Step II : Integrating Equation (1) w.r.t. x, keeping y as constant.

$$\therefore \frac{\partial z}{\partial y} = \int \frac{1}{xy} dx + f(y)$$

Where, f(y) be any arbitrary function (as integrating constants.)

$$\therefore \frac{\partial z}{\partial y} = \frac{1}{y}\int \frac{1}{x} dx + f(y)$$

$$\frac{\partial z}{\partial y} = \frac{1}{y}\log(x) + f(y)$$

Step III : Again integrating $\frac{\partial z}{\partial y}$ w.r.t. y, keeping x as constant

$$z = \int \frac{1}{y}\log x\, dy + \underbrace{\int f(y)\, dy}_{g(y)} + \phi(x)$$

where, $\phi(x)$ is any arbitrary function (as integrating constant.)

$$z = \log x \int \frac{1}{y} dy + g(y) + \phi(x)$$

$$z = \log x \cdot \log(y) + g(y) + \phi(x) \checkmark \qquad ...\text{Ans.}$$

($\because$ Here, $g(y) = \int f(y)\, dy$ is any general function of y)

be the required solution of given partial differential equation.

Ex. 4.3.7 : Solve : $\frac{\partial^2 z}{\partial x\, \partial y} = 2x + 2y$

Soln. :

Step I : Given partial differential equation is

$$\frac{\partial^2 z}{\partial x\, \partial y} = 2x + 2y \qquad ...(1)$$

Here, z - is dependent variable ;

x, y - are independent variables.

Step II : Integrating Equation (1) w.r.t. x, keeping y as constant.

$$\frac{\partial z}{\partial y} = \int (2x + 2y)\, dx + f(y)$$

Where, f(y) is any arbitrary function of y (as integrating constant.)

$$\frac{\partial z}{\partial y} = \cancel{2}\frac{x^2}{\cancel{2}} + 2y \cdot x + f(y)$$

$$= x^2 + 2xy + f(y)$$

Step III : Integrating $\frac{\partial z}{\partial y}$ w.r.t. y, keeping x as constant

$$z = x^2 \int dy + 2x \int y\, dy + \underbrace{\int f(y)\, dy}_{g(y)} + \phi(x)$$

where, $\phi(x)$ is any arbitrary function (like any arbitrary constant)

$$z = x^2 y + 2x\left(\frac{y^2}{2}\right) + g(y) + \phi(x)$$

$$\Rightarrow \quad z = x^2 y + xy^2 + g(y) + \phi(x) \checkmark \qquad ...\text{Ans.}$$

be the required solution of given partial differential equation

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Ex. 4.3.8 : Find the general integral or solve : $\frac{\partial^2 z}{\partial y^2} = \sin(xy)$

Soln. :

Step I : Given partial differential equation is

$$\frac{\partial^2 z}{\partial y^2} = \sin(xy) \qquad ...(1)$$

Here, z - is dependent variable ;

x, y - are independent variables.

Step II : Integrating Equation (1) w.r.t. y, keeping x as constant

$$\frac{\partial z}{\partial y} = \int \sin(xy)\, dy + f(x)$$

where, f(x) is arbitrary function (like integrating constant.)

$$\frac{\partial z}{\partial y} = \left(\frac{-\cos xy}{x}\right) + f(x)$$

Step III : Again integrating $\frac{\partial z}{\partial y}$ w.r.t.y, keeping x as constant.

$$\therefore z = -\frac{1}{x}\int \cos xy\, dy + \int f(x)\, dy + \phi(x)$$

Where, $\phi(x)$ is any arbitrary function (like integrating constant)

$$z = -\frac{1}{x}\left(\frac{\sin xy}{x}\right) + f(x)\, y + \phi(x)$$

$$\Rightarrow \quad z = -\frac{1}{x^2}\sin xy + y\, f(x) + \phi(x) \checkmark \qquad ...\text{Ans.}$$

be the required solution of given partial differential equation.

Ex. 4.3.9 : Find the general integral of $\frac{\partial^2 u}{\partial y\, \partial x} = 4x \sin(3xy)$.

Soln. :

Step I : Given partial differential equation is

$$\frac{\partial^2 u}{\partial y\, \partial x} = 4x \sin(3xy) \qquad ...(1)$$

Here, z - is dependent variable,

x, y - are independent variables.

Step II : Integrating Equation (1) w.r.t. y, keeping x as constant

$$\frac{\partial u}{\partial x} = \int 4x \sin(3xy)\, dy + f(x)$$

Where, f(x) is any arbitrary function (like integrating constant)

$$\frac{\partial u}{\partial x} = \frac{4x(-\cos(3xy))}{3x} + f(x)$$

$$\frac{\partial u}{\partial x} = -\frac{4}{3}\cos(3xy) + f(x)$$

Step III : Integrating $\frac{\partial u}{\partial x}$ w.r.t. x, keeping y as constant

$$\therefore\ u = -\frac{4}{3}\int \cos(3xy)\, dx + \int f(x)\, dx + \phi(y)$$

Where, $\phi(y)$ is any arbitrary function (like integral constant)

$$\therefore\ u = -\frac{4}{3}\left(\frac{\sin(3xy)}{3y}\right) + x f(x) + \phi(y)$$

$$\Rightarrow\ u = -\frac{4}{9}\cdot\frac{\sin(3xy)}{y} + x f(x) + \phi(y) \checkmark \quad \text{...Ans.}$$

be the required solution of given partial differential equation.

Ex. 4.3.10 : Solve : $\log\left[\frac{\partial^2 z}{\partial x\, \partial y}\right] = x + y.$

Soln. :

Step I : Given partial differential equation is

$$\log\left[\frac{\partial^2 z}{\partial x\, \partial y}\right] = x + y$$

$$\Rightarrow \frac{\partial^2 z}{\partial x\, \partial y} = e^{x+y} \quad \text{...(1)}$$

$(\because\ \log z = b \Rightarrow z = e^b)$

Here, z – is dependent variable ;

x, y - are independent variables.

Step II : Integrating Equation (1) w.r.t. x, keeping y as constant

$$\frac{\partial z}{\partial y} = \int e^{x+y}\, dx + f(y)$$

Where, f(y) is any arbitrary function (like integrating constant)

➔ **Using law of indices :** ...[$e^{x+y} = e^x \cdot e^y$]

$$\frac{\partial z}{\partial y} = \int e^x \cdot e^y\, dx + f(y)$$

$$= e^y \int (e^x)\, dx + f(y)$$

$$\frac{\partial z}{\partial y} = e^y \cdot e^x + f(y)$$

Step III : Integrating $\frac{\partial z}{\partial y}$ w.r.t. y, keeping x as constant

$$z = \int e^x \cdot e^y\, dy + \underbrace{\int f(y)\, dy}_{g(y)} + \phi(x)$$

where, $\phi(x)$ is any arbitrary function (like integrating constant)

$$z = e^x \int e^y\, dy + g(y) + \phi(x)$$

$$z = e^x \cdot e^y + g(y) + \phi(x)$$

$$\Rightarrow\ z = e^{x+y} + g(y) + \phi(x) \quad (\because e^x \cdot e^y = e^{x+y}) \checkmark$$

...Ans.

be the required solution of given partial differential equation.

Ex. 4.3.11 : Find the general integral of $\frac{\partial^2 u}{\partial x\, \partial t} = e^{-t} \cdot \cos x.$

Soln. :

Step I : Given partial differential equation is

$$\frac{\partial^2 u}{\partial x\, \partial t} = e^{-t} \cdot \cos x \quad \text{...(1)}$$

Here, u - is a dependent variable

x, t - are independent variables

Step II : Integrating Equation (1) w.r.t. to x, keeping t as constant

$$\frac{\partial u}{\partial t} = \int e^{-t} \cos x\, dx + f(t)$$

Where, f(t) is any arbitrary function of t (like integrating constant)

$$\therefore\ \frac{\partial u}{\partial t} = e^{-t} \int \cos x\, dx + f(t)$$

$$\Rightarrow\ \frac{\partial u}{\partial t} = e^{-t}(\sin x) + f(t)$$

Step III : Integrating $\frac{\partial u}{\partial t}$ w.r.t. t, keeping x as constant

$$\therefore\ u = \int e^{-t} \cdot \sin x\, dt + \underbrace{\int f(t)\, dt}_{g(t)} + \phi(x)$$

Where, $\phi(x)$ is any arbitrary function of x (like integrating constant)

$$\therefore\ u = \sin x \int e^{-t}\, dt + g(t) + \phi(x)$$

$$u = \sin x(-e^{-t}) + g(t) + \phi(x)$$

$$\Rightarrow\ u = -e^{-t}\sin x + g(t) + \phi(x) \checkmark \quad \text{...Ans.}$$

be the required solution of given partial differential equation.

Ex. 4.3.12 : Solve : $\frac{\partial^3 z}{\partial x^2 \cdot \partial y} = \cos(2x + 3y).$

Soln. :

Step I : Given partial differential equation is

$$\frac{\partial^3 z}{\partial x^2 \cdot \partial y} = \cos(2x + 3y) \quad \text{...(1)}$$

Here, z – is dependent variable

x, y - are independent variables

Step II : Integrating Equation (1) w.r.t. x, keeping y as constant

$$\therefore\ \frac{\partial^2 z}{\partial x \cdot \partial y} = \int \cos(2x + 3y)\, dx + f(y)$$

Where, f(y) be any arbitrary function of y (like integrating constant)

$$\therefore\ \frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{\sin(2x + 3y)}{2} + f(y) \quad \text{...(2)}$$

Step III : Integrating $\frac{\partial^2 z}{\partial x \cdot \partial y}$ w.r.t. x, keeping y as constant

$$\frac{\partial z}{\partial y} = \frac{1}{2}\int \sin(2x + 3y)\, dx + \int f(y)\, dx + h(y)$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}\left(\frac{-\cos(2x + 3y)}{2}\right) + x f(y) + h(y)$$

Again, integrate $\frac{\partial z}{\partial y}$ w.r.t. y, keeping x as constant.

$$z = -\frac{1}{4}\int \cos(2x + 3y)\, dy + x \underbrace{\int f(y)\, dy}_{g(y)} + \underbrace{\int h(y)\, dy}_{\phi(y)} + \psi(x)$$

Where, $\psi(x)$ is any arbitrary function of x (like integrating constant)

$$\therefore\ z = -\frac{1}{4}\frac{(\sin(2x + 3y))}{3} + x\, g(y) + \phi(y) + \psi(x)$$

$$\Rightarrow\ z = -\frac{1}{12}\sin(2x + 3y) + x\, g(y) + \phi(y) + \psi(x) \checkmark$$

...Ans.

be the required solution of given partial differential equation.

Ex. 4.3.13 : Find the general integral of $\frac{\partial^2 z}{\partial x^2} = a^2 z$, given that when $x = 0, \frac{\partial z}{\partial x} = a \sin y$ and $\frac{\partial z}{\partial y} = 0.$

Soln. :

Step I : Given partial differential equation is

$$\frac{\partial^2 z}{\partial x^2} = a^2 z \quad \text{...(1)}$$

Given conditions :

(i) at $x = 0$, $\frac{\partial z}{\partial x} = a \sin y$

(ii) at $x = 0$, $\frac{\partial z}{\partial y} = 0$

Here, z – is dependent variable ;

x, y - are independent variables.

($\because$ partial derivative indicates that z is a function of two variables x and y.)

Step II : If z is a function of single variable x only, then the differential equation is linear ordinary differential equation $\frac{\partial^2 z}{\partial x^2} = a^2 z \Rightarrow (D^2 - a^2) z = 0$,

where, $D = \frac{\partial}{\partial z}$ and roots are $D = \pm a$, solution is

$$z = c_1 e^{ax} + c_2 e^{-ax}$$

Here, c_1 and c_2 are arbitrary constants.

But, here z is a function of two independent variables x and y, so c_1 and c_2 becomes arbitrary functions of y. (the independent variable kept constant)

Hence, the solution becomes,

$$z = f_1(y)\, e^{ax} + f_2(y)\, e^{-ax} \quad \text{...(2)}$$

$$\frac{\partial z}{\partial x} = a f_1(y)\, e^{ax} - a f_2(y)\, e^{-ax} \quad \text{...(3)}$$

$$\text{and } \frac{\partial z}{\partial y} = f_1'(y)\, e^{ax} + f_2'(y)\, e^{-ax} \quad \text{...(4)}$$

Step III : By using given condition (i)

as $x = 0$; $\frac{\partial z}{\partial y} = a \sin y$

$\therefore$ from Equation (3),

$$a \sin y = a f_1(y) - a f_2(y)$$

$$\Rightarrow\ f_1(y) - f_2(y) = \sin y \quad \text{...(5)}$$

Also by condition (ii),

As $x = 0$, $\frac{\partial z}{\partial y} = 0$

Therefore, from Equation (4), we get

$$0 = f_1'(y) + f_2'(y) \quad \text{.... } (\because e^0 = 1)$$

$$\Rightarrow f_1'(y) + f_2'(y) = 0 \quad \text{... (6)}$$

Now Differentiating Equation (5) partially, w.r.t. y and add with Equation (6)

$$\therefore f_1'(y) - f_2'(y) = \cos y$$

$$+ f_1'(y) + f_2'(y) = 0$$

$$2.\, f_1'(y) = \cos y \Rightarrow f_1'(y) = \frac{\cos y}{2}$$

Integrating w.r.t.y, keeping x as constant

$$f_1(y) = \frac{\sin y}{2} + \phi(x)$$

substitute the value of $f_1(y)$ in Equation (5) we get

$$\frac{\sin y}{2} + \phi(x) - f_2(y) = \sin y$$

$$\Rightarrow \frac{\sin y}{2} + \phi(x) - \sin y = f_2(y)$$

$$f_2(y) = \phi(x) - \frac{\sin y}{2}$$

By substituting the values of f_1 and f_2 in Equation (2), we have

$$z = \left[\frac{\sin y}{2} + \phi(x)\right] e^{ax} + \left[\phi(x) - \frac{\sin y}{2}\right] e^{-ax}$$

$$= \frac{\sin y}{2} e^{ax} + \phi(x) e^{ax} + \phi(x) e^{-ax} - \frac{\sin y}{2} e^{-ax}$$

$$= \sin y \left[\frac{e^{ax} - e^{-ax}}{2}\right] + 2\phi(x)\left[\frac{e^{ax} + e^{-ax}}{2}\right]$$

$$z = \sin y \cdot \sinh ax + 2\phi(x) \cos ax \checkmark \quad \text{...Ans.}$$

be the required solution of given partial differential equation

Ex. 4.3.14 : Solve : $\frac{\partial^2 z}{\partial x \partial y} = \sin x, \sin y$, given that

$$\frac{\partial z}{\partial y} = -z \sin y \text{ when } x = 0 \text{ and } z = 0$$

when y is an odd multiples of $\frac{\pi}{2}$

Soln. :

Step I :

Given partial differential equation is,

$$\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y \quad \text{...(1)}$$

Here, z - is dependent variable,

x, y - are independent variables.

Given, conditions are

(i) At $x = 0$; $\frac{\partial z}{\partial y} = -2 \sin y$

(ii) At $y = (2n+1)\frac{\pi}{2}$; $z = 0$

Step II : Integrating Equation (1) w.r.t. x, keeping y as constant,

$$\frac{\partial z}{\partial y} = \int \sin x \cdot \sin y \, dx + f(y)$$

where, f(y) is any arbitrary function of (y) (like integral constant)

$$\therefore \frac{\partial z}{\partial y} = (-\cos x) \sin y + f(y) \quad \text{...(2)}$$

By condition (i), put $x = 0$; $\frac{\partial z}{\partial y} = -2 \sin y$

$\therefore -2 \sin y = -\sin y + f(y)$ .... ($\because \cos \theta = 1$)

$\Rightarrow f(y) = -\sin y$

Equation (2) becomes

$$\frac{\partial z}{\partial y} = -\cos x \sin y - \sin y$$

Step III : Integrating $\frac{\partial z}{\partial y}$ w.r.t. y, keeping x-as constant.

$$z = -\int (\cos x \cdot \sin y + \sin y) \, dy + f(x)$$

Where, f(x) – be any arbitrary function of x (like integrating constant)

$$z = -[-\cos x \cdot \cos y - \cos y] + f(x)$$

$$z = \cos x \cos y + \cos y + f(x) \quad \text{...(3)}$$

By condition (ii) at $y = (2n+1)\frac{\pi}{2}$; $z = 0$

$$0 = \cos x \cdot \left(\cos (2n+1)\frac{\pi}{2}\right) + \cos (2n+1)\frac{\pi}{2} + f(x)$$

$$\Rightarrow \quad 0 = 0 + 0 + f(x)$$

$$\Rightarrow \quad f(x) = 0 \quad \left(\because \cos (2n+1)\frac{\pi}{2} = 0\right)$$

Equation (3) becomes

$$z = \cos x \cdot \cos y + \cos y \checkmark \quad \text{...Ans.}$$

be the required solution of given partial differential equation

Ex. 4.3.15 : Solve : $\frac{\partial^2 z}{\partial x^2} + z = 0$, given that when $x = 0$, $z = e^y$ and $\frac{\partial z}{\partial x} = 1$

Soln. :

Step I : Given partial differential equation is

$$\frac{\partial^2 z}{\partial x^2} + z = 0 \quad \text{...(1)}$$

Here, z - is dependent variable,

x, y - are independent variables.

($\because$ partial derivative indicates that z is a function of two variable x, y)

Step II : If z is a function of x alone, then Equation (1) is a linear ordinary differential equation and its solution is

$$\frac{d^2 z}{dx^2} + z = 0 \Rightarrow (D^2 + 1) z = 0$$

where, $D = \frac{d}{dz}$

Auxiliary equation is $D^2 + 1 = 0 \therefore$ Roots are $D = \pm i$

All roots are complex.

Therefore, complementary function is

$$= C_1 \cos x + C_2 \sin x$$

and

Particular Integral (P.I.) $= \frac{0}{D^2 + 1} = 0$

The general solution is $z = C_1 \cos x + C_2 \sin x$

where, C_1 and C_2 are arbitrary constants.

But, here z is a function of two variables x and y, therefore, C_1, C_2 are arbitrary functions of y (the independent variable kept as constant)

Step III : The solution of the given P. D.E. is

$$z = f(y) \cos x + \phi(y) \sin x \quad \text{...(2)}$$

differentiate partially w.r.t. x, keeping y as constant.

$$\frac{\partial z}{\partial x} = -f(y) \sin x + \cos x \, \phi(y) \quad \text{...(3)}$$

By given conditions

when $x = 0$, $z = e^y$ and $\frac{\partial z}{\partial x} = 1$

$\therefore$ put $x = 0$ in (2)

$e^y = f(y) + 0$ ($\because \sin 0 = 0, \cos 0 = 1$)

$\therefore f(y) = e^y$

And from Equation (3)

$1 = 0 + \phi(y) \Rightarrow \phi(y) = 1$

Equation (2) becomes

$$z = e^y \cos x + \sin x \checkmark \quad \text{...Ans.}$$

be the required solution of given P.D.E.

Ex. 4.3.16 : Solve $\frac{\partial^3 z}{\partial x^2 \partial y} + 18 xy^2 + \sin (2x - y) = 0$

Soln. :

Step I : Given partial differential equation is

$$\frac{\partial^3 z}{\partial x^2 \partial y} + 18 xy^2 + \sin (2x - y) = 0 \quad \text{...(1)}$$

Here, z – is dependent variable ;

x, y – are independent variables.

Step II : Integrating Equation (1) w.r.t. x, keeping y as constant.

$$\frac{\partial^2 z}{\partial x \partial y} + 18y^2 \int x \, dx + \int \sin (2x - y) \, dx = f(y)$$

where, f(y) – is any arbitrary function (like integrating constant)

$$\frac{\partial^2 z}{\partial x \partial y} + \cancel{18}^{9} y^2 \left(\frac{x^2}{\cancel{2}}\right) - \frac{\cos(2x - y)}{2} = f(y)$$

Again integrating $\frac{\partial^2 z}{\partial x \partial y}$ w.r.t. x, keeping y as constant

$$\frac{\partial z}{\partial y} + 9y^2 \int x^2 \, dx - \frac{1}{2} \int \cos (2x - y) \, dx = \int f(y) \, dx + \phi(y)$$

where, $\phi(y)$ is any arbitrary function (like integrating constant)

$$\frac{\partial z}{\partial y} + \cancel{9}^{3} y^2 \left(\frac{x^3}{3}\right) - \frac{1}{2}\left(\frac{\sin (2x - y)}{2}\right) = f(y) x + \phi(y)$$

Step III :

Integrate $\frac{\partial z}{\partial y}$ w.r.t. y, keeping x as constant.

$$z + 3x^3 \int y^2 \, dy - \frac{1}{4} \int \sin (2x - y) \, dy$$

$$= x \underbrace{\int f(y) \, dy}_{h(y)} + \underbrace{\int \phi(y) \, dy}_{g(y)} + \psi(x)$$

where, $\psi(x)$ is any arbitrary function (like integrating constants)

$$z + \cancel{3}x^3 \left(\frac{y^3}{\cancel{3}}\right) - \frac{1}{4}\frac{(-\cos (2x - y))}{(-1)}$$

$$= x \cdot h(y) + g(y) + \psi(x)$$

$$\Rightarrow z + x^3 y^3 - \frac{1}{4} \cos (2x - y) = xh(y) + g(y) + \psi(x) \checkmark \quad \text{...Ans.}$$

be the required solution of given partials differential equation.

Syllabus Topic : Linear Equations of First Order (Lagrange's Linear Equations)

4.4 Linear Partial Differential Equations of First Order (Lagrange's Linear Equations)

If the first order partial derivatives $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$ are in first degree and they are not multiple together, then such PDE is called a linear partial differential equation of the first order.

The standard form of a linear partial differential equation of first order is

$Pp + Qq = R$,

where P, Q, R are functions of x, y, z

It is also called Lagrange's linear equations.

Working rule to solve Lagrange's Linear equations :

Step I : Write the given linear PDE and compare with standard form $Pp + Qq = R$

Find the values of P, Q, R

Step II : Write the Auxiliary equations or subsidiary equations as $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Step III : These type of symmetrical simultaneous differential equations have two solutions

$u(x, y, z) = c_1$ and

$v(x, y, z) = c_2$

For solutions, use Grouping method or Set of multiplier methods and find these two solutions.

Step IV : The required general integral / solution is

$F(c_1, c_2) = 0$

or $F(u, v) = 0$

or $u = f(v)$

Ex. 4.4.1 : Find the general integral of the linear partial differential equation

$y^2 p - xy q = x (z - 2y)$

Soln. :

Step I : The given linear partial differential equation

$$y^2 p - xy\,q = x(z-2y) \qquad ...(1)$$

The given linear PDE is of the form

$$pP + qQ = R \qquad ...\text{form}$$

Where,

$$P = y^2 ; Q = -xy ; R = x(z-2y)$$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{i.e.} \quad \frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)} \qquad ...(2)$$

For first solution :

By the method of grouping, From first two members

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$\Rightarrow \quad x\,dx = -y\,dy$$

$$\Rightarrow \quad x\,dx + y\,dy = 0$$

Variables are separated, so, integrating,

$$\int x\,dx + \int y\,dy = c$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} = c \Rightarrow x^2 + y^2 = c_1 \qquad ...(3)$$

For second solution :

Step III : From last two members of Equation (2)

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\Rightarrow (z-2y)\,dy = -y\,dz$$

$$\Rightarrow z\,dy - 2y\,dy + y\,dz = 0$$

$$\Rightarrow d(yz) - 2y\,dy = 0$$

Integrating, we get,

$$\int d(yz) - 2\int y\,dy = c_2$$

$$\Rightarrow \quad yz - 2\frac{y^2}{2} = c_2$$

$$\text{i.e.} \quad yz - y^2 = c_2 \qquad ...(4)$$

Step IV : By using Equations (3) and (4),

Hence the required general integral is

$$F(C_1, C_2) = 0$$

$$F(x^2 + y^2, yz - y^2) = 0$$

$$\text{OR} \quad x^2 + y^2 = f(yz - y^2) \checkmark \qquad ...\text{Ans.}$$

Ex. 4.4.2 : Find the general integral of the linear partial differential equation :

$$(y + zx)\,p - (x + yz)\,q = x^2 - y^2.$$

Soln. :

Step I : The given linear partial differential equation

$$(y + zx)\,p - (x + yz)\,q = x^2 - y^2 \qquad ...(1)$$

The given linear PDE is of the form

$$pP + qQ = R \qquad ...\text{form}$$

where, $P = (y + zx) ; Q = -(x + yz) ; R = x^2 - y^2$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{i.e.} \quad \frac{dx}{(y+zx)} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2 - y^2} \qquad ...(2)$$

For the first solution :

Let x, y, 0 be the set of multipliers

$$\text{Each ratio} = \frac{x\,dx + y\,dy + 0\,dz}{x(y+zx) - y(x+yz) + 0(x^2 - y^2)}$$

$$= \frac{x\,dx + y\,dy}{\cancel{xy} + x^2 z - \cancel{xy} - y^2 z}$$

$$= \frac{x\,dx + y\,dy}{z(x^2 - y^2)} \qquad ...(3)$$

Compare Equation (3) with third ratio of Equation (2)

$$\therefore \quad \frac{x\,dx + y\,dy}{z(x^2 - y^2)} = \frac{dz}{(x^2 - y^2)}$$

$$\Rightarrow \quad x\,dx + y\,dy = z\,dz$$

Integrating, we get,

$$\frac{x^2}{2} + \frac{y^2}{2} = \frac{z^2}{2} + C$$

$$\Rightarrow \quad x^2 + y^2 - z^2 = C_1 \qquad ...(4)$$

For second solution :

Let y, x and 0 be the set of multipliers

$$\therefore \text{Each ratio} = \frac{y\,dx + x\,dy + 0\,dz}{y(y+zx) - x(x+yz) + 0(x^2 - y^2)}$$

$$= \frac{y\,dx + x\,dy}{y^2 + z\cancel{xy} - x^2 - \cancel{xyz} + 0}$$

$$= \frac{y\,dx + x\,dy}{-(x^2 - y^2)} \qquad ...(5)$$

Compare Equation (5) with third ratio of Equation (2),

$$\therefore \quad \frac{y\,dx + x\,dy}{-(\cancel{x^2 - y^2})} = \frac{dz}{(\cancel{x^2 - y^2})}$$

$$\Rightarrow \quad y\,dx + x\,dy = -dz \Rightarrow d(xy) = -dz$$

Integrating, we get,

$$xy = -z + C$$

$$\Rightarrow \quad xy + z = C_2 \qquad ...(6)$$

Step III : Hence the required general integral is

$$F(C_1, C_2) = 0$$

$$F(x^2 + y^2 - z^2,\ xy + z) = 0$$

$$\text{OR} \quad (x^2 + y^2 - z^2) = f(xy + z) \checkmark \qquad ...\text{Ans.}$$

Ex. 4.4.3 : Use Lagrange's method to solve partial differential equation is given in the form of

$$\begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & -1 \end{vmatrix} = 0$$

Where, $z = f(x, y)$

Soln. :

Step I : The given linear partial differential equation

$$\begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & -1 \end{vmatrix} = 0$$

$$\Rightarrow x\left[-\beta - \gamma\frac{\partial z}{\partial y}\right] - y\left[-\alpha - \gamma\frac{\partial z}{\partial x}\right] + z\left[\alpha\frac{\partial z}{\partial y} - \beta\frac{\partial z}{\partial x}\right] = 0$$

$$\Rightarrow -x\beta - x\gamma\frac{\partial z}{\partial y} + \alpha y + y\gamma\frac{\partial z}{\partial x} + z\alpha\frac{\partial z}{\partial y} - z\beta\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow (y\gamma - z\beta)\frac{\partial z}{\partial x} + (z\alpha - x\gamma)\frac{\partial z}{\partial y} = x\beta - y\alpha \qquad ...(1)$$

The given linear PDE is of the form

$$pP + qQ = R \qquad ...\text{form}$$

where, $P = (y\gamma - z\beta) ; Q = (z\alpha - x\gamma) ;$ and $R = x\beta - y\alpha$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\therefore \quad \frac{dx}{y\gamma - z\beta} = \frac{dy}{z\alpha - x\gamma} = \frac{dz}{x\beta - y\alpha} \qquad ...(2)$$

By the method of set of multipliers

For first solution :

Consider x, y, z be the first set of multipliers such that

$$xP + yQ + zR = 0$$

$$\therefore \text{Each ratio} = \frac{x\,dx + y\,dy + z\,dz}{x(y\gamma - z\beta) + y(z\alpha - x\gamma) + z(x\beta - y\gamma)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{\cancel{xy\gamma} - \cancel{xz\beta} + \cancel{yz\alpha} - \cancel{xy\gamma} + \cancel{xz\beta} - \cancel{yz\alpha}}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\Rightarrow x\,dx + y\,dy + z\,dz = 0$$

Integrating, $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C$

$$\Rightarrow \quad x^2 + y^2 + z^2 = C_1 \qquad ...(3)$$

Step III : For second solution :

Let us consider α, β, γ be the second set of multipliers from Equation (2),

$$\therefore \text{Each ratio} = \frac{\alpha\,dx + \beta\,dy + \gamma\,dz}{\alpha(y\gamma - z\beta) + \beta(z\alpha - x\gamma) + \gamma(x\beta - y\alpha)}$$

$$= \frac{\alpha\,dx + \beta\,dy + \gamma\,dz}{\cancel{\alpha\gamma y} - \cancel{\alpha\beta z} + \cancel{\alpha\beta z} - \cancel{\beta\gamma x} + \cancel{\gamma\beta x} - \cancel{\alpha\gamma y}}$$

$$= \frac{\alpha\,dx + \beta\,dy + \gamma\,dz}{0}$$

$$\Rightarrow \alpha\,dx + \beta\,dy + \gamma\,dz = 0$$

Integrating,

$$\alpha x + \beta y + \gamma z = C_2 \qquad ...(4)$$

Step IV : Hence the required general integral is

$$F(C_1, C_2) = 0$$

$$F(x^2 + y^2 + z^2, \alpha x + \beta y + \gamma z) = 0$$

$$\text{OR} \quad x^2 + y^2 + z^2 = F(\alpha x + \beta y + \gamma z) \checkmark \qquad ...\text{Ans.}$$

Ex. 4.4.4 : Find the general integrals of the linear PDE :

$$pz - qz = z^2 + (x + y)^2$$

Soln. :

Step I : The given linear partial differential equation

$$pz - qz = z^2 + (x + y)^2 \qquad ...(1)$$

The given linear PDE is of the ...form

$$pP + qQ = R \text{ form}$$

Where, $P = z ; Q = -z ; R = z^2 + (x + y)^2$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\therefore \quad \frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x + y)^2} \qquad ...(2)$$

For first solution :

consider first two ratios, from Equation (2)

$$\frac{dx}{\cancel{z}} = \frac{dy}{-\cancel{z}} \Rightarrow dx + dy = 0$$

Integrating, we get,

$$x + y = C_1 \qquad ...(3)$$

For second solution :

Consider second and third ratio from Equation (2), we get,

$$\frac{dy}{-z} = \frac{dz}{z^2 + (x + y)^2}$$

$$\Rightarrow \quad -dy = \frac{z\,dz}{z^2 + C_1^2} \qquad ...\text{from Equation (3)}$$

➔ **Using standard result :** $\left[\int \frac{f'(x)}{f(x)}dx = \log f(x)\right]$

Integrating we get,

$$-y = \frac{1}{2}\int \frac{2z\,dz}{z^2+C_1^2} + C_2$$

$$-y = \frac{1}{2}\log[z^2+C_1^2] + C_2$$

$$-2y = \log\left(z^2+C_1^2\right) + C_2$$

$$\therefore\ 2y + \log\left(z^2+C_1^2\right) = C_2 \qquad ...(4)$$

Step III : Hence the required general integral is

$$F(C_1, C_2) = 0$$

$$F\left(x+y,\ 2y+\log\left(z^2+C_1^2\right)\right) = 0$$

i.e. $F = [x+y,\ 2y+\log(z^2+(x+y)^2)] = 0$

OR $x+y = F(2y+\log(z^2+(x+y)^2)) = 0$ ✓ **...Ans.**

Ex. 4.4.5 : Find the general integral of the linear partial of the linear partial differential equation.

$$(x^2-yz)p + (y^2-zx)q = (z^2-xy)$$

Soln. :

Step I : The given linear partial differential equation

$$(x^2-yz)p + (y^2-zx)q = (z^2-xy) \qquad ...(1)$$

The given linear PDE is of the form

$$pP + qQ = R \text{ form}$$

Where, $P = (x^2-yz)$; $Q = (y^2-zx)$; $R = (z^2-xy)$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation

i.e. Auxiliary equations are :

$$\therefore\ \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\therefore\ \frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy} \qquad ...(2)$$

For first solution :

Subtracting first two and last two ratios and then equating them, we get,

$$\frac{dx-dy}{(x^2-yz)-(y^2-zx)} = \frac{dy-dz}{(y^2-zx)-(z^2-xy)}$$

$$\Rightarrow \frac{dx-dy}{(x^2-y^2)+z(x-y)} = \frac{dy-dz}{(y^2-z^2)+x(y-z)}$$

$$\Rightarrow \frac{dx-dy}{(x-y)(x+y)+z(x-y)} = \frac{dy-dz}{(y-z)(y+z)+x(y-z)}$$

$$\therefore\ \frac{dx-dy}{(x-y)\cancel{(x+y+z)}} = \frac{dy-dz}{(y-z)\cancel{(x+y+z)}}$$

$$\Rightarrow \frac{dx-dy}{(x-y)} = \frac{dy-dz}{(y-z)}$$

Integrating, we get,

$$\log(x-y) = \log(y-z) + \log C_1$$

$$\log(x-y) - \log(y-z) = \log C_1$$

➔ **Using standard results :** $\left[\log(A) - \log(B) = \log\left(\frac{A}{B}\right)\right]$

$$\log\left(\frac{x-y}{y-z}\right) = \log C_1$$

Remove log on both sides,

$$\left(\frac{x-y}{y-z}\right) = C_1 \qquad ...(3)$$

For second solution :

Subtract third and first ratio of Equation (2) and equating with $\frac{dx-dy}{(x-y)(x+y+z)}$

$$\frac{dz-dx}{(z^2-xy)-(x^2-yz)} = \frac{dx-dy}{(x-y)(x+y+z)}$$

$$\Rightarrow \frac{dz-dx}{(z^2-x^2)+y(z-x)} = \frac{dx-dy}{(x-y)(x+y+z)}$$

$$\Rightarrow \frac{dz-dx}{(z-x)(z+x)+y(z-x)} = \frac{dx-dy}{(x-y)(x+y+z)}$$

$$\Rightarrow \frac{dz-dx}{(z-x)(x+y+z)} = \frac{dx-dy}{(x-y)(x+y+z)}$$

Integrating, we get,

$$\log(z-x) = \log(x-y) + \log C_2$$

$$\log(z-x) - \log(x-y) = \log C_2$$

➔ **Using Standard results :** $\left[\log(A) - \log(B) = \log\left(\frac{A}{B}\right)\right]$

$$\Rightarrow \log\left(\frac{z-x}{x-y}\right) = \log C_2$$

Remove log on both sides,

$$\frac{z-x}{x-y} = C_2 \qquad ...(4)$$

Step III : From Equation (3) and (4),

The required general integral is

$$F(C_1, C_2) = 0$$

$$F\left(\frac{x-y}{y-z}, \frac{z-x}{x-y}\right) = 0$$

OR $\frac{x-y}{y-z} = F\left(\frac{z-x}{x-y}\right)$ ✓ **...Ans.**

Ex. 4.4.6 : Find the integral surface (general solution) of the partial differential equation :

$$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = (x+y)z$$

Soln. :

Step I : The given linear partial differential equation

$$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = (x+y)z$$

$$\Rightarrow x^2p + y^2q = (x+y)z \qquad ...(1)$$

The given linear PDE is of the form

$$pP + qQ = R \qquad ...\text{form}$$

where, $P = x^2$; $Q = y^2$; $R = (x+y)z$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\therefore\ \frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} \qquad ...(2)$$

For first solution :

From equation (2),

consider the first two ratios,

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Variables are separated, integrating, we get,

$$-\frac{1}{x} = -\frac{1}{y} + C_1$$

$$\Rightarrow -\frac{1}{x} + \frac{1}{y} = C_1$$

$$\Rightarrow \frac{-y+x}{xy} = C$$

$$\therefore\ \frac{y-x}{xy} = -C_1,\ \text{Hence, } \frac{y-x}{xy} = C_1 \qquad ...(3)$$

Step III : For second solution :

Let the set of multiplier is 1, – 1, 0, from Equation (2)

$$\text{Each ratio} = \frac{dx-dy}{x^2-y^2} = \frac{dx-dy}{(x+y)(x-y)}$$

Now equating this ratio with third ratio of Equation (2)

$$\therefore\ \frac{dx-dy}{(x-y)\cancel{(x+y)}} = \frac{dz}{\cancel{(x+y)}z}$$

V. S. form integrating,

$$\therefore\ \int\frac{dx-dy}{(x-y)} = \int\frac{dz}{z} + \log C_2$$

$$\log(x-y) = \log z + \log C_2$$

$$\Rightarrow \log(x-y) - \log z = \log C_2$$

➔ **Using standard result :** $\left[\log A - \log B = \log\left(\frac{A}{B}\right)\right]$

$$\therefore\ \log\left(\frac{x-y}{z}\right) = \log C_2$$

$$\frac{x-y}{z} = C_2 \qquad ...(4)$$

Step IV : By using (3) and (4),

The required general integral is

$$F(C_1, C_2) = 0$$

$$\therefore\ F\left(\frac{y-x}{xy}, \frac{x-y}{z}\right) = 0$$

OR $\frac{y-x}{xy} = F\left(\frac{x-y}{z}\right)$ ✓ **...Ans.**

Ex. 4.4.7 : Find the general integrals of $\frac{y^2z}{x}\cdot P + xzq = y^2$

Soln. :

Step I : The given linear partial differential equation

$$\therefore\ \frac{y^2z}{x}\cdot P + xzq = y^2 \qquad ...(1)$$

The given linear PDE is of the form

$$pP + qQ = R \qquad ...\text{form}$$

where, $P = \frac{y^2z}{x}$, $Q = xz$; $R = y^2$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\therefore\ \frac{dx}{\left(\frac{y^2z}{x}\right)} = \frac{dy}{xz} = \frac{dz}{y^2} \qquad ...(2)$$

For first solution :

Consider first and third ratio of Equation (2)

$$\frac{dx}{\left(\frac{y^2z}{x}\right)} = \frac{dz}{y^2}$$

$$\Rightarrow x\,dx = z\,dz$$

$$\Rightarrow x\,dx - z\,dz = 0$$

V.S. form, integrating

$$\int x\,dx - \int z\,dz = C$$

$$\Rightarrow \frac{x^2}{2} - \frac{z^2}{2} = C^2$$

$$\Rightarrow x^2 - z^2 = C_1 \qquad ...(3)$$

Step II : For second solution :

Consider first and second ratio of Equation (2)

$$\frac{dx}{\left(\frac{y^2\cancel{z}}{x}\right)} = \frac{dy}{x\cancel{z}}$$

$$\Rightarrow x^2\,dx = y^2\,dy$$

$$\Rightarrow x^2\,dx - y^2\,dy = 0$$

Integrating

$$\int x^2\,dx - \int y^2\,dy = C$$

$$\Rightarrow \frac{x^3}{3} - \frac{y^3}{3} = C$$

$$\Rightarrow \quad x^3 - y^3 = C_2$$

$$\Rightarrow \quad x^3 - y^3 = C_2 \qquad ...(4)$$

Step IV : From Equation (3) and (4),

The required general integral is

$$F(C_1, C_2) = 0$$

$$F(x^3 - y^3, x^2 - y^2) = 0$$

$$\text{OR} \quad x^3 - y^3 = F(x^2 - y^2) \checkmark \qquad ...\text{Ans.}$$

Ex. 4.4.8 : Find the general solution of the partial differential equations :

$$(y+1)p + (x+1)q = z$$

Soln. :

Step I : The given linear partial differential equation

$$\therefore \quad (y+1)P + (x+1)q = z \qquad ...(1)$$

The given linear PDE is of the form

$$pP + qQ = R \qquad ...\text{form}$$

where, $P = (y+1)$; $Q = (x+1)$; $R = z$

Step II :

The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y+1} = \frac{dy}{x+1} = \frac{dz}{z} \qquad ...(2)$$

For first solution :

Consider the first two ratio, of Equation (2)

$$\frac{dx}{y+1} = \frac{dy}{x+1}$$

$$\Rightarrow \quad (x+1)\,dx - (y+1)\,dy = 0$$

v.s. form, Integrating

$$\int (x+1)\,dx - \int (y+1)\,dy = C$$

$$\Rightarrow \quad \frac{x^2}{2} + x - \left(\frac{y^2}{2} + y\right) = C$$

$$\Rightarrow \quad \frac{x^2}{2} - \frac{y^2}{2} + x - y = C$$

$$x^2 - y^2 + 2x - 2y = C_1 \qquad ...(3)$$

Step III : for second solution : Add first two ration of Equation (2) and by equating with third ratio of Equation (2).

$$\frac{dx - dy}{y + 1 - (x+1)} = \frac{dz}{z}$$

$$\Rightarrow \quad \frac{dx - dy}{y - x} = \frac{dz}{z}$$

$$\Rightarrow \quad -\left[\frac{dx - dy}{(x-y)}\right] = \frac{dz}{z}$$

v.s. form integration

$$-\log(x-y) = \log z + \log C$$

$$-\log(x-y) - \log z = \log C$$

$$-[\log(x-y) + \log z] = \log C$$

$$\Rightarrow -\log[(x-y)\cdot z] = \log C$$

$$...(\because \log(A) + \log(B) = \log(AB))$$

$$\Rightarrow \log[(x-y)\cdot z] = -\log C = \log C^{-1}$$

$$= \log C_1$$

$$\therefore (x-y)\cdot z = C_1 \qquad ...(4)$$

Step IV : From Equation (3) and (4),

The required general integral is

$$F(C_1, C_2) = 0$$

$$F(x^2 - y^2 + 2x - 2y ; z(x-y)) = 0$$

$$\text{Or} \quad (x^2 - y^2 + 2x - 2y) = f(zx - zy) \checkmark ...\text{Ans.}$$

Ex. 4.4.9 : Find the general integral of the linear PDE : $xp - yq = z$

Soln. :

Step I : The given linear partial differential equation

$$xp - yq = z \qquad ...(1)$$

The given linear PDE is or the form

$$pP + qQ = R \qquad ...\text{form}$$

where, $P = x$; $Q = -y$; $R = z$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\therefore \frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{z} \qquad ...(2)$$

For first solution :

Consider first two ratio :

$$\frac{dx}{x} = \frac{dy}{-y}$$

$$\Rightarrow \quad \log x = -\log y + \log C_1$$

$$\Rightarrow \quad \log x + \log y = \log C_1$$

➔ **Using standard result :** ...[$\log A + \log B = \log(AB)$]

$$\therefore \log(xy) = \log C_1$$

$$\Rightarrow \quad xy = C_1 \qquad ...(3)$$

and for second solution consider first and third ratio, of Equation (2)

$$\frac{dx}{x} = \frac{dz}{z} \Rightarrow \text{Integrating, we get}$$

$$\log x = \log z + \log C$$

$$\log(x) - \log z = \log C$$

➔ **Using standard result :** ...$\left[\log A - \log B = \log\left(\frac{A}{B}\right)\right]$

$$\Rightarrow \quad \log\left(\frac{x}{z}\right) = \log C_2$$

$$\Rightarrow \quad \frac{x}{z} = C_2 \qquad ...(4)$$

Step III : From Equation (3) and (4),

The required general integral is

$$F(C_1, C_2) = 0$$

$$\therefore f\left(xy, \frac{x}{z}\right) = 0$$

$$\text{OR} \quad xy = f(x/z) \checkmark \qquad ...\text{Ans.}$$

Ex. 4.4.10 : Solve the following linear PDE :

$$(mz - ny)p + (nx - lz)q = ly - mx$$

Soln. :

Step I : The given linear partial differential equation

$$\therefore (mz - ny)p + (nx - lz)q = ly - mx \qquad ...(1)$$

The given linear PDE is of the form

$$pP + qQ = R \qquad ...\text{form}$$

where, $P = mz - ny$; $Q = nx - lz$; $R = ly - mx$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\therefore \frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} \qquad ...(2)$$

For first solution : from Equation (2)

By the set of multiplier method,

Let, x, y, z be the set of multiplier such that,

$$\text{Each ratio} = \frac{x\,dx + y\,dy + z\,dz}{x(mz - ny) + y(nx - lz) + z(ly - mx)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{\cancel{mxz} - \cancel{nxy} + \cancel{nxy} - \cancel{lyz} + \cancel{lyz} + \cancel{mxz}}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\Rightarrow x\,dx + y\,dy + z\,dz = 0$$

v.s. from, integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_1 \Rightarrow x^2 + y^2 + z^2 = C_1 \qquad ...(3)$$

For second solution : from Equation (2)

Let l, m, n be the set of multipliers such that,

$$\text{each ratio} = \frac{l\,dx + m\,dy + n\,dz}{l(mz - ny) + m(nx - lz) + z(ly - mx)}$$

$$= \frac{l\,dx + m\,dy + n\,dz}{\cancel{lmz} - \cancel{lny} + \cancel{mnx} - \cancel{mlz} + \cancel{nly} - \cancel{mnx}}$$

$$= \frac{l\,dx + m\,dy + n\,dz}{0}$$

$$\Rightarrow \quad l\,dx + m\,dy + n\,dz = 0$$

Integrating, we get

$$\int l\,dx + \int m\,dy + \int n\,dz = C_2$$

$$\Rightarrow \quad lx + my + nz = C_2 \qquad ...(4)$$

Step III : From Equation (3) and (4), the required general integral is

$$F(C_1, C_2) = 0$$

$$F(x^2 + y^2 + z^2 ; lx + my + nz) = 0$$

OR

$$x^2 + y^2 + z^2 = f(lx + my + nz) \checkmark \qquad ...\text{Ans.}$$

Ex. 4.4.11 : Solve the first order linear PDE :

$$x^2(y - z)p + y^2(z - x)q = z^2(x - y)$$

Soln. :

Step I : The given linear partial differential equation

$$x^2(y - z)p + y^2(z - x)q = z^2(x - y) \qquad ...(1)$$

The given linear PDE is of form

$$pP + qQ = R \qquad ...\text{form}$$

where, $P = x^2(y - z)$; $Q = y^2(z - x)$; $R = z^2(x - y)$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{i.e.} \quad \frac{dx}{x^2(y - z)} = \frac{dy}{y^2(z - x)} = \frac{dz}{z^2(x - y)} \qquad ...(2)$$

For the first solution :

Let $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ be the set of multipliers : such that

$$\text{Each ratio} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{\cancel{(y-z)} + \cancel{(z-x)} + \cancel{(x-y)}}$$

$$= \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$$\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

v.s. form, integrating

$$\int \frac{1}{x^2}dx + \int \frac{1}{y^2}dy + \int \frac{1}{z^2}dz = C$$

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = C$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = C_1 \qquad ...(3)$$

And for second solution :

Let, $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ be the set of multiplier such that

$$\text{Each ratio} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{x(y - z) + y(z - x) + z(x - y)}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\cancel{xy} - \cancel{xz} + \cancel{yz} - \cancel{yx} + \cancel{zx} - \cancel{zy}}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating, $\int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = \log C_2$

$\log x + \log y + \log z = \log C_2$

➔ **Using standard result:** ...[$\log A + \log B + \log C = \log(ABC)$]

$\therefore \log(xyz) = \log C_2$

$xyz = C_2$ ...(4)

Step III: From Equation (3) and (4), the required general integral is

$F(C_1, C_2) = 0$

$\therefore F\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right) = 0$

OR $\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = f(xyz)$ ✓ ...Ans.

Ex. 4.4.12: Find the general integral of the linear PDE $(x^2 - y^2 - z^2)p + 2xyq = 2xz$

Soln.:

Step I: The given linear partial differential equation

$(x^2 - y^2 - z^2)p + 2xy\,q = 2xz$ ...(1)

The given linear PDE is or the form

$pP + qQ = R$ ...form

where, $P = x^2 - y^2 - z^2$; $Q = 2xy$; $R = 2xz$

Step II: The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation

i.e. Auxiliary equations are:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e. $\frac{dx}{(x^2 - y^2 - z^2)} = \frac{dy}{2xy} = \frac{dz}{2xz}$ ...(2)

For first solution:

Let us considers, second and third ratio

$$\frac{dy}{2\cancel{x}y} = \frac{dz}{2\cancel{x}z}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

v.s. form integrating

$$\int \frac{dy}{y} = \int \frac{dz}{z} + \log C$$

$\log y = \log z + \log C_1$

$\log y - \log z = \log C_1$

➔ **Using standard formula:** ...$\left[\log(A) - \log(B) = \log\left(\frac{A}{B}\right)\right]$

$\therefore \log\left(\frac{y}{z}\right) = \log C_1$

$\frac{y}{z} = C_1$ ...(3)

And for second solution: From Equation (2)

Let x, y, z be the set of multipliers

$$\text{Each ratio} = \frac{xdx + ydy + zdz}{x(x^2 - y^2 - z^2) + 2xy^2 + 2xz^2}$$

$$= \frac{xdx + ydy + zdz}{x(x^2 - y^2 - z^2 + 2y^2 + 2z^2)}$$

$$= \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)} \quad ...(4)$$

By equating above ratio (4) with second ratio of Equation (2)

$$\therefore \frac{xdx + ydy + zdz}{\cancel{x}(x^2 + y^2 + z^2)} = \frac{dy}{2\cancel{x}y}$$

Integrating, we get $\frac{1}{2}\int \frac{2xdx + 2ydy + 2zdz}{(x^2 + y^2 + z^2)} = \int \frac{dy}{2y} + \log C_2$

➔ **Using standard result:** ...$\left[\int \frac{f'(x)}{f(x)}dx = \log f(x)\right]$

$\therefore \cancel{\frac{1}{2}} \log(x^2 + y^2 + z^2) = \cancel{\frac{1}{2}} \log y + \log C_2$

$\log(x^2 + y^2 + z^2) - \log y = \log C_2$

➔ **Using standard result:** ...$\left[\log(A) - \log(B) = \log\left(\frac{A}{B}\right)\right]$

$\therefore \log\left(\frac{x^2 + y^2 + z^2}{y}\right) = \log C_2$

$\left(\frac{x^2 + y^2 + z^2}{y}\right) = C_2$ ...(5)

Step III: From Equations (3) and (5), the required general integral is

$F(C_1, C_2) = 0$

$F\left(\frac{y}{z}, \frac{x^2 + y^2 + z^2}{y}\right) = 0$

OR $\frac{x^2 + y^2 + z^2}{y} = F\left(\frac{y}{z}\right)$ ✓ ...Ans.

Ex. 4.4.13: Find the general integral of the first order linear PDE. $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$

Soln.:

Step I: The given linear partial differential equation

$(z^2 - 2yz - y^2)p + (xy + zx)q = (xy - zx)$ ...(1)

The given linear PDE is of the form

$pP + qQ = R$ ...form

where, $P = z^2 - 2yz - y^2$; $Q = (xy + zx)$; $R = (xy - zx)$

Step II: The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation

i.e. Auxiliary equations are:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e. $\frac{dx}{(z^2 - 2yz - y^2)} = \frac{dy}{(xy + zx)} = \frac{dz}{(xy - zx)}$ ...(2)

For first solution:

Let x, y, z be the set of multipliers such that

$$\text{Each ratio} = \frac{xdx + ydy + zdz}{x(z^2 - 2yz - y^2) + y(xy + zx) + z(xy - zx)}$$

$$= \frac{xdx + ydy + zdz}{\cancel{xz^2} - 2\cancel{xyz} - \cancel{y^2x} + \cancel{xy^2} + \cancel{zyx} + \cancel{zxy} - \cancel{z^2x}}$$

$$= \frac{xdx + ydy + zdz}{0}$$

$x\,dx + y\,dy + z\,dz = 0$

Integrating, we get,

$\int xdx + \int ydy + \int zdz = C$

$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C$

$x^2 + y^2 + z^2 = C_1$ ...(3)

and for second solution:

Consider second and third ratio of Equation (2)

$$\therefore \frac{dy}{xy + zx} = \frac{dz}{xy - zx}$$

$$\frac{dy}{\cancel{x}(y + z)} = \frac{dz}{\cancel{x}(y - z)}$$

Cross multiplication:

$(y - z)dy = (y + z)dz$

$ydy - zdy - ydz - zdz = 0$

$ydy - d(yz) - zdz = 0$

Integrating, we get

$\int ydy - \int d(yz) - \int zdz = C_2$

$\frac{y^2}{2} - yz - \frac{z^2}{2} = C_2$

$y^2 - 2yz - z^2 = C_2$ ...(4)

Step III: From Equation (3) and (4), the required general integral is

$F(C_1, C_2) = 0$

$F(x^2 + y^2 + z^2, y^2 - 2yz - z^2) = 0$

OR $(x^2 + y^2 + z^2) = f(y^2 - 2yz - z^2)$ ✓ ...Ans.

Ex. 4.4.14: Solve the following

(i) $2p + 3q = 1$

(ii) $xp + yq = 3z$

(iii) $(y - z)p + (x - y)q = z - x$

(iv) $p\tan x + q\tan y = \tan z$

(v) $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

(vi) $(z - y)p + (x - z)q = y - x$

(vii) $yzp + zxq = xy$

(viii) $(y + z)p + (z + x)q = x + y$

(ix) $y^2p - xyq = x(z - 2y)$

(x) $x(y - z)p + y(z - x)q = z(x - y)$

(xi) $\left(\frac{y - z}{yz}\right)p + \left(\frac{z - x}{zx}\right)q = \frac{x - y}{xy}$

(xii) $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$

(xiii) $x^2p + y^2q = (x + y)z$

(xiv) $z(xp - yq) = y^2 - x^2$

(xv) $(y^3x - 2x^4)p + (2y^4 - x^3y)q = 9z(x^3 - y^3)$

(xvi) $x_1p_1 + 2x_2p_2 + 2x_3p_3 + 4x_4p_4 = 0$

(xvii) $(z - y)p + (x - z)q = y - x$

Soln.:

(i)

Step I: The given linear partial differential equation

$2p + 3q = 1$ ...(1)

The given linear PDE is of the form

$pP + qQ = R$

where, $P = 2; Q = 3;\ R = 1$

Step II: The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation

i.e. Auxiliary equations are:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e. $\frac{dx}{2} = \frac{dy}{3} = \frac{dz}{1}$ ...(2)

For First solution:

let $\frac{dx}{2} = \frac{dy}{3} \Rightarrow 3dx - 2dy = 0$

integrating,

$3x - 2y = C_1$ ...(3)

For second solution:

Consider, $\frac{dy}{3} = \frac{dz}{1}$

$dy = 3dz$

Integrating we get

$$y = 3z + C_2$$

$$y - 3z = C_2 \quad ...(4)$$

Step III : From Equations (3) and (4)

The required general integral is

$$F(C_1, C_2) = 0$$

$$F(3x - 2y,\ y - 3z) = 0$$

OR

$$(3x - 2y) = F(y - 3z) \checkmark \quad ...\text{Ans.}$$

(ii)

Step I : The given linear partial differential equation

$$xp + yq = 3z \quad ...(1)$$

The given linear PDE is of the form

$$pP + qQ = R$$

where, $P = x;\ Q = y;\ R = 3z$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{i.e. } \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3z} \quad ...(2)$$

For First solution :

let us consider, $\frac{dx}{x} = \frac{dy}{y}$

integrating, we get,

$$\int \frac{dx}{x} = \int \frac{dy}{y} + \log C_1$$

$$\therefore \log x = \log y + \log C_1$$

$$\log x - \log y = \log C_1$$

➔ **Using standard result :** $...\left[\log(A) - \log(B) = \log\left(\frac{A}{B}\right)\right]$

$$\therefore \log\left(\frac{x}{y}\right) = \log C_1$$

$$\frac{x}{y} = C_1$$

And for second solution

Let us consider, $\frac{dy}{y} = \frac{dz}{3z}$ from Equation (2)

$$3\frac{dy}{y} - \frac{dz}{z} = 0$$

Integrating $3\int \frac{dy}{y} - \int \frac{dz}{z} = \log C_2$

$$3\log y - \log z = \log C_2$$

$$\log y^3 - \log z = \log C_2$$

➔ **Using standard result :** $...\left[\log(A) - \log(B) = \log\left(\frac{A}{B}\right)\right]$

$$\therefore \log\left(\frac{y^3}{z}\right) = \log C_2$$

$$\frac{y^3}{z} = C_2 \quad ...(A)$$

Step III : From Equation (3) and (4), the general integral is

$$F\left(\frac{x}{y}, \frac{y^3}{z}\right) = 0$$

$$\frac{x}{y} = F\left(\frac{y^3}{z}\right) \checkmark \quad ...\text{Ans.}$$

(iii)

Step I : The given linear partial differential equation

$$(y - z)p + (x - y)q = (z - x) \quad ...(1)$$

The given linear PDE is of the form

$$pP + qQ = R$$

where, $p = y - z;\ Q = x - y;\ R = z - x$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{i.e. } \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3z}$$

For first solution :

Let us consider the set of multipliers are 1, 1, 1 such that

$$\text{Each ratio} = \frac{dx + dy + dz}{(y - z) + (x - y) + (z - x)}$$

$$= \frac{dx + dy + dz}{0}$$

$$dx + dy + dz = 0$$

integrating, $x + y + z = C_1 \quad ...(3)$

And for second solution : from Equation (2)

Let, x, z, y be the set of multipliers such that

$$\text{Each ratio} = \frac{xdx + zdy + ydz}{x(y - z) + z(x - y) + y(z - x)}$$

$$= \frac{xdx + d(yz)}{xy - xz + xz - yz + yz - xy}$$

$$= \frac{xdx + d(yz)}{0}$$

$$xdx + d(yz) = 0$$

Integrating, we get

$$\int xdx + \int d(yz) = C$$

$$\frac{x^2}{2} + yz = C \quad ...(4)$$

Step III : From Equations (3), (4), the required general integral is

$$F(C_1, C_2) = 0$$

$$F\left(x + y + z;\ \frac{x^2}{2} + yz\right) = 0$$

$$\text{Or } \left(\frac{x^2}{2} + yz\right) = F(x + y + z) \checkmark \quad ...\text{Ans.}$$

(iv)

Step I : The given linear partial differential equation

$$p\tan x + q\tan y = \tan z \quad ...(1)$$

The given linear PDE is of the form

$$pP + qQ = R$$

where, $P = \tan x\ ;\ Q = \tan y\ ;\ R = \tan z$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{i.e. } \frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z} \quad ...(2)$$

For first solution : from Equation (2)

consider,

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

$$\cot x\, dx = \cot y\, dy \quad ...\left(\because \frac{1}{\tan\theta} = \cot\theta\right)$$

and $\cot\theta = \frac{\cos\theta}{\sin\theta}$

$$\left(\frac{\cos x}{\sin x}\right) dx = \frac{\cos y}{\sin y} dy$$

Integrating, we get

$$\int \frac{\cos x}{\sin x} dx - \int \frac{\cos y}{\sin y} dy = \log C_1$$

$$...\left[\int \frac{f'(x)}{f(x)} dx = \log f(x)\right]$$

$$\therefore \log\sin x - \log\sin y = \log C_1$$

➔ **Using standard results :** $...\left[\log(A) - \log(B) = \log\left(\frac{A}{B}\right)\right]$

$$\therefore \log\left(\frac{\sin x}{\sin y}\right) = \log C_1$$

$$\frac{\sin x}{\sin y} = C_1 \quad ...(3)$$

And for second solution :

from Equations (2) consider,

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$\cot y\, dy = \cot z\, dz$$

integrating, $\int \cot y\, dy = \int \cot z\, dz + \log C_2$

$$\int \frac{\cos y}{\sin y} dy - \int \frac{\cos z}{\sin z} dz = \log C_2$$

$$\log\sin y - \log\sin z = \log C_2$$

$$\log\left(\frac{\sin y}{\sin z}\right) = \log C_2$$

$$\text{hence, } \frac{\sin y}{\sin z} = C_2 \quad ...(4)$$

Step III : From Equation (3) and (4), the required general integral is

$$F(C_1, C_2) = 0$$

$$F\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$$

$$\text{OR } \frac{\sin x}{\sin y} = F\left(\frac{\sin y}{\sin z}\right) \checkmark \quad ...\text{Ans.}$$

(v)

Step I : The given linear partial differential equation

$$p\sqrt{x} + q\sqrt{y} = \sqrt{z} \quad ...(1)$$

The given linear PDE is of the form

$$pP + qQ = R$$

where, $p = \sqrt{x};\ Q = \sqrt{y}\ ;\ R = \sqrt{z}$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the Auxiliary equation.

i.e. Auxiliary equations are : $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\text{i.e. } \frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}} \quad ...(2)$$

For first solution :

from Equation (2), consider,

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}}$$

$$\text{Integrating, } \int \frac{dx}{\sqrt{x}} - \int \frac{dy}{\sqrt{y}} = C_1$$

$$\int x^{\frac{-1}{2}} dx - \int y^{\frac{-1}{2}} dy = C_1$$

$$\frac{x^{\frac{1}{2}}}{\frac{1}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = C_1$$

$$2\sqrt{x} - 2\sqrt{y} = C_1$$

$$\sqrt{x} - \sqrt{y} = C_1 \quad ...(3)$$

And for second solution :

from Equation (2)

$$\text{Consider, } \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

Integrating,

$$\int \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}} + C_2$$

$$\left(\because \text{ standard result : } \int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\right)$$

$$\therefore\ 2\sqrt{y} - 2\sqrt{z} = C_2$$

$$\therefore\ \sqrt{y} - \sqrt{z} = C_2 \qquad \ldots(4)$$

Step III : From Equation (3) and (4), the required general integral is

$$F(C_1, C_2) = 0$$

$$F\left(\sqrt{x}-\sqrt{y}, \sqrt{y}-\sqrt{z}\right) = 0$$

$$\text{Or} \quad \left(\sqrt{x}-\sqrt{y}\right) = F\left(\sqrt{y}-\sqrt{z}\right)$$

(vi)

Step I : The given linear partial differential equation

$$(z-y)\,p + (x-z)\,q = (y-x) \qquad \ldots(1)$$

The given linear PDE is of the form

$$pP + qQ = R$$

where, $P = z-y$; $Q = x-z$; $R = y-x$

Step II : **The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation**

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{i.e.} \quad \frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x} \qquad \ldots(2)$$

For first solution :

from Equation (2)

let 1, 1, 1 be the set of multipliers such that

$$\text{Each ratio} = \frac{dx+dy+dz}{(\cancel{z}-\cancel{y})+(\cancel{x}-\cancel{z})+(\cancel{y}-\cancel{x})} = \frac{dx+dy+dz}{0}$$

$$\Rightarrow\ dx + dy + dz = 0$$

Integrating, we get

$$x + y + z = 4 \qquad \ldots(3)$$

and for second solution :

from Equation (2),

let, x, y, z be the set of multiplier such that

$$\text{Each ratio} = \frac{xdx+ydy+zdz}{x(z-y)+y(x-z)+z(y-x)} = \frac{xdx+ydy+zdz}{\cancel{xz}-\cancel{xy}+\cancel{xy}-\cancel{yz}+\cancel{yz}-\cancel{xz}} = \frac{xdx+ydy+zdz}{0}$$

$$\Rightarrow\ xdx + ydy + zdz = 0$$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_2$$

$$x^2 + y^2 + z^2 = C_2 \qquad \ldots(4)$$

Step III : From Equations (3) and (4), the required general integral is

$$F(C_1, C_2) = 0$$

$$F\left(x+y+z\ ;\ x^2+y^2+z^2\right) = 0$$

$$\text{or} \quad (x^2+y^2+z^2) = F(x+y+z) \checkmark \qquad \ldots\textbf{Ans.}$$

(VII)

Step I : The given linear partial differential equation.

$$yzp + zxq = xy \qquad \ldots(1)$$

The given linear PDE is of the

$$pP + qQ = R \text{ form}$$

Where, $P = yz$; $Q = zx$; $R = xy$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation.

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{i.e.} \quad \frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy} \qquad \ldots(2)$$

For First solution : From Equation (2), Consider,

$$\frac{dx}{y\cancel{z}} = \frac{dy}{\cancel{z}x}$$

$$\Rightarrow \quad \frac{dx}{y} = \frac{dy}{x} \Rightarrow xdx = y\,dy$$

$$\therefore\ x\,dx - y\,dy = 0$$

Integrating,

$$\int x\,dx - \int y\,dy = C$$

$$\Rightarrow \quad \frac{x^2}{2} - \frac{y^2}{2} = C$$

$$\Rightarrow \quad x^2 - y^2 = C_1 \qquad \ldots(3)$$

And for second solution : from Equation (2), Consider,

$$\frac{dy}{z\cancel{x}} = \frac{dz}{\cancel{x}y}$$

$$\Rightarrow \quad ydy = zdz$$

$$\Rightarrow \quad y\,dy - z\,dz = 0$$

Integrating we get

$$\int y\,dy - \int z\,dz = C$$

$$\Rightarrow \quad \frac{y^2}{2} - \frac{z^2}{2} = C$$

$$\Rightarrow \quad y^2 - z^2 = C_2 \qquad \ldots(4)$$

Step III : From Equation (3) and (4), the required general integral is

$$F(C_1, C_2) = 0$$

$$F\left(x^2-y^2\ ;\ y^2-z^2\right) = 0$$

$$\text{or} \quad (x^2-y^2) = f\left(y^2-z^2\right) \checkmark \qquad \ldots\textbf{Ans.}$$

(VIII)

Step I : The given linear partial differential equation.

$$(y+z)\,p + (z+x)\,q = x+y \qquad \ldots(1)$$

The given linear PDE is of the form

$$pP + qQ = R$$

Where, $P = (y+z)$; $Q = (z+x)$; $R = (x+y)$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation.

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{i.e.} \quad \frac{dx}{(y+z)} = \frac{dy}{(z+x)} = \frac{dz}{(x+y)} \qquad \ldots(2)$$

For first solution :

From Equation (2) substract first two ratio and last two ratio and equating them,

$$\therefore\ \frac{dx-dy}{(y+z)-(z+x)} = \frac{dy-dz}{(z+x)-(x+y)}$$

$$\therefore\ \frac{dx-dy}{y+\cancel{z}-\cancel{z}-x} = \frac{dy-dz}{z+\cancel{x}-\cancel{x}-y}$$

$$\Rightarrow \quad +\left(\frac{dx-dy}{(x-y)}\right) = +\left(\frac{dy-dz}{(y-z)}\right)$$

Integrating, we get,

$$\int \frac{dx-dy}{x-y} = \int \frac{dy-dz}{y-z} + \log C_1$$

$$\Rightarrow \log(x-y) - \log(y-z) = \log C_1$$

➔ **Using standard result :** $\ldots\left[\log A - \log B = \log\left(\frac{A}{B}\right)\right]$

$$\therefore\ \log\left(\frac{x-y}{y-z}\right) = \log C_1$$

$$\Rightarrow \quad \frac{x-y}{y-z} = C_1 \qquad \ldots(3)$$

And for second solution, from Equation (2),

By using first two ratio,

$$\text{Each ratio} = \frac{dx-dy}{(y+\cancel{z})-(\cancel{z}+x)} = \frac{dx-dy}{-(x-y)} \qquad \ldots(4)$$

and by adding all ratio,

$$\therefore\ \text{Each ratio} = \frac{dx+dy+dz}{y+z+z+x+x+y} = \frac{dx+dy+dz}{2(x+y+z)} \qquad \ldots(5)$$

By Equating Equation (4) and (5),

$$\frac{dx-dy}{-(x-y)} = \frac{dx+dy+dz}{2(x+y+z)}$$

Integrating,

$$-\int \frac{dx-dy}{(x-y)} = \frac{1}{2}\int \frac{dx+dy+dz}{x+y+z} + \log C_2$$

$$\Rightarrow \quad -\log(x-y) = \frac{1}{2}\log(x+y+z) + \log C_2$$

$$\Rightarrow \quad -2\log(x-y) = \log(x+y+z) + \log C_2$$

$$\Rightarrow \quad +\log(x-y)^2 + \log(x+y+z) = \log C_2$$

➔ **Using standard result :** $\ldots[\log A + \log B = \log(A \cdot B)]$

$$\therefore\ \log\left[(x-y)^2 \cdot (x+y+z)\right] = \log C_2$$

$$\Rightarrow \quad (x-y)^2(x+y+z) = C_2 \qquad \ldots(6)$$

Step III : From Equation (3) and (6), the required general integral is,

$$F(C_1, C_2) = 0$$

$$F\left(\frac{x-y}{y-z}, (x-y)^2(x+y+z)\right) = 0$$

$$\text{or} \quad \frac{x-y}{y-z} = f\left[(x-y^2)^2(x+y+z)\right] \checkmark \qquad \ldots\textbf{Ans.}$$

(IX)

Step I : The given linear partial differential equation.

$$y^2p - xyq = x(z-2y) \qquad \ldots(1)$$

The given linear PDE is of the form

$$pP + qQ = R$$

Where, $P = y^2$; $Q = -xy$; $R = x(z-2y)$

Step II The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation.

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{i.e.} \quad \frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)} \qquad \ldots(2)$$

For first solution : from Equation (2),

$$\text{Consider,} \quad \frac{dx}{y^{\cancel{2}}} = \frac{dy}{-x\cancel{y}} \Rightarrow +\,x\,dx + y\,dy = 0$$

$$\text{Integrating,} \quad \int x\,dx + \int y\,dy = C$$

$$\Rightarrow \quad \frac{x^2}{2} + \frac{y^2}{2} = C$$

$$\Rightarrow \quad x^2 + y^2 = C_1 \qquad \ldots(3)$$

And for second solution : from Equation (2),

$$\text{Consider,} \quad \frac{dy}{-\cancel{x}y} = \frac{dz}{\cancel{x}(z-2y)}$$

$$\Rightarrow \quad (z-2y)\,dy = -y\,dz$$

$$\Rightarrow \quad z\,dy - 2y\,dy + y\,dz = 0$$

$$\Rightarrow \quad d(yz) - 2y\,dy = 0$$

Integrating, we get

$$\int d(yz) + 2\int y\,(dy) = C_2$$

$$yz - \cancel{2}\frac{y^2}{\cancel{2}} = C_2$$

$$\Rightarrow \quad yz - y^2 = C_2 \qquad ...(4)$$

Step III : From Equation (3) and (4), the required general integral is,

$$F(C_1, C_2) = 0$$

$$F(x^2+y^2,\ yz-y^2) = 0 \quad \text{or}$$

$$x^2+y^2 = f(yz-y^2) \checkmark \qquad ...\textbf{Ans.}$$

(X)

Step I : The given linear partial differential equation.

$$x(y-z)p + y(z-x)q = z(x-y) \qquad ...(1)$$

The given linear PDE is of the form

$$pP + qQ = R$$

Where, $P = x(y-z)$; $Q = y(z-x)$; $R = z(x-y)$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation.

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{i.e.} \quad \frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \qquad ...(2)$$

For first solution : From Equation (2),

Let 1, 1, 1 be the set of multipliers such that,

$$\text{Each ratio} = \frac{1\,dx + 1\,dy + 1\,dz}{x(y-z)+y(z-x)+z(x-y)}$$

$$= \frac{dx+dy+dz}{\cancel{xy}-\cancel{xz}+\cancel{yz}-\cancel{yx}+\cancel{zx}-\cancel{zy}}$$

$$= \frac{dx+dy+dz}{0}$$

$$\Rightarrow \quad dx+dy+dz = 0$$

Integrating, we get,

$$\Rightarrow \quad x+y+z = C_1 \qquad ...(3)$$

And for second solution : from Equation (2),

Let $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ be the set of multipliers such that,

$$\text{Each ratio} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\frac{1}{\cancel{x}}(y-z)+\frac{1}{\cancel{y}}(z-x)+\frac{1}{\cancel{z}}(x-y)}$$

$$= \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\cancel{y}-\cancel{z}+\cancel{z}-\cancel{x}+\cancel{x}-\cancel{y}}$$

$$= \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$$\Rightarrow \quad \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz = 0$$

Integrating,

$$\int\frac{1}{x}dx + \int\frac{1}{y}dy + \int\frac{1}{z}dz = \log C_2$$

$$\Rightarrow \quad \log x + \log y + \log z = \log C_2$$

➔ **Using standard result**

$$...[\log(A)+\log(B)+\log(C) = \log(ABC)]$$

$$\therefore \quad \log(xyz) = \log C_2$$

$$\Rightarrow \quad xyz = C_2 \qquad ...(4)$$

Step III : From Equation (3) and (4), the required general integral is,

$$F(C_1, C_2) = 0$$

$$F(x^2+y^2+z^2,\ xyz) = 0 \quad \text{or}$$

$$(x^2+y^2+z^2) = f(xyz) \checkmark \qquad ...\textbf{Ans.}$$

(XI)

Step I : The given linear partial differential equation.

$$\left(\frac{y-z}{yz}\right)p + \left(\frac{z-x}{zx}\right)q = \left(\frac{x-y}{xy}\right) \qquad ...(1)$$

The given linear PDE is of the form

$$pP + qQ = R \quad \text{form}$$

Where, $P = \left(\frac{y-z}{yz}\right)$; $Q = \left(\frac{z-x}{zx}\right)$; $R = \left(\frac{x-y}{xy}\right)$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation.

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{i.e.} \quad \frac{dx}{\left(\frac{y-z}{yz}\right)} = \frac{dy}{\left(\frac{z-x}{zx}\right)} = \frac{dz}{\left(\frac{x-y}{xy}\right)} \qquad ...(2)$$

For first solution : from Equation (2),

Let $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ be the set of multiplier

$$\therefore \quad \text{Each ratio} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\frac{1}{x}\left(\frac{y-z}{yz}\right)+\frac{1}{y}\left(\frac{z-x}{zx}\right)+\frac{1}{z}\left(\frac{x-y}{xy}\right)}$$

$$= \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\left(\frac{y-z}{xyz}\right)+\left(\frac{z-x}{xyz}\right)+\left(\frac{x-y}{xyz}\right)}$$

$$= \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\frac{\cancel{y}-\cancel{z}+\cancel{z}-\cancel{x}+\cancel{x}-\cancel{y}}{xyz}}$$

$$= \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\frac{0}{(xyz)}}$$

$$= \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$$\Rightarrow \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz = 0$$

Integrating, we get,

$$\int\frac{1}{x}dx + \int\frac{1}{y}dy + \int\frac{1}{z}dz = \log C_1$$

$$\Rightarrow \quad \log x + \log y + \log z = \log C_1$$

$$\log(xyz) = \log C_1$$

$$\Rightarrow \quad xyz = C_1 \qquad ...(3)$$

And for second solution : from Equation (2),

Let 1, 1, 1 be the set of multiplier.

$$\therefore \quad \text{Each ratio} = \frac{dx+dy+dz}{\left(\frac{y-z}{yz}\right)+\left(\frac{z-x}{zx}\right)+\left(\frac{x-y}{xy}\right)}$$

$$= \frac{dx+dy+dz}{\frac{\cancel{y}}{\cancel{y}z}-\frac{\cancel{z}}{\cancel{z}y}+\frac{\cancel{z}}{\cancel{z}x}-\frac{\cancel{x}}{zx}+\frac{\cancel{x}}{\cancel{x}y}-\frac{\cancel{y}}{x\cancel{y}}}$$

$$= \frac{dx+dy+dz}{\left(\frac{1}{\cancel{z}}-\frac{1}{\cancel{y}}+\frac{1}{\cancel{x}}-\frac{1}{\cancel{z}}+\frac{1}{\cancel{y}}-\frac{1}{\cancel{x}}\right)}$$

$$= \frac{dx+dy+dz}{0}$$

$$\Rightarrow dx+dy+dz = 0$$

Integrating, we get

$$x+y+z = C_2 \qquad ...(4)$$

Step III : From Equations (3) and (4), the required general integral is

$$F(C_1, C_2) = 0$$

$$F(xyz,\ x+y+z) = 0 \quad \text{or} \quad x+y+z = f(xyz) \checkmark \qquad ...\textbf{Ans.}$$

(XII)

Step I : The given linear partial differential equation.

$$x(z^2-y^2)p + y(x^2-z^2)q = z(y^2-x^2) \qquad ...(1)$$

The given linear PDE is of the form

$$pP + qQ = R$$

Where, $P = x(z^2-y^2)$; $Q = y(x^2-z^2)$

$R = z(y^2-x^2)$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation.

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{i.e.} \quad \frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2-z^2)} = \frac{dz}{z(y^2-x^2)} \qquad ...(2)$$

For first solution : From Equation (2),

Let $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ be the set of multipliers

$$\therefore \quad \text{Each ratio} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{(z^2-y^2+x^2-y^2+y^2-x^2) = 0}$$

$$\Rightarrow \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz = 0$$

Variables are separated, Integrating

$$\log x + \log y + \log z = \log C_1$$

➔ **Using standard result**

$$...[\log(a)+\log(b)+\log(c) = \log(abc)]$$

$$\therefore \quad \log(xyz) = \log C_1$$

$$\Rightarrow \quad xyz = C_1 \qquad ...(3)$$

For second solution : From Equation (2)

Let x, y, z be the set of multipliers

$$\therefore \text{Each ratio} = \frac{x\,dx+y\,dy+z\,dz}{x^2(z^2-y^2)+y^2(x^2-z^2)+z^2(y^2-z^2)}$$

$$= \frac{x\,dx+y\,dy+z\,dz}{0}$$

$$\Rightarrow x\,dx+y\,dy+z\,dz = 0$$

Integrating,

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = C_2$$

$$\Rightarrow x^2+y^2+z^2 = C_2 \qquad ...(4)$$

Step III : By using Equations (3) and (4) we get the solution

$$u(f_1, f_2) = 0$$

$$\text{i.e.} \quad u(xyz,\ x^2+y^2+z^2) = 0$$

(XIII)

Step I : The given linear partial differential equation.

$$x^2p + y^2q = (x+y)z \qquad ...(1)$$

The given linear PDE is of the form

$$pP + qQ = R$$

Where, $P = x^2$; $Q = y^2$; $R = (x+y)z$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation.

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e. $\dfrac{dx}{x^2} = \dfrac{dy}{y^2} = \dfrac{dz}{(x+y)z}$ ...(2)

For first solution : from Equation (2),

Consider, $\dfrac{dx}{x^2} = \dfrac{dy}{y^2}$

Integrating,

$$\int \frac{1}{x^2}dx - \int \frac{1}{y^2}dy = C_1$$

$$\Rightarrow \int x^{-2} - \int y^{-2}dy = C_1$$

$$\therefore \frac{x^{-1}}{-1} - \left(\frac{y^{-1}}{-1}\right) = C_1$$

$$\Rightarrow -\frac{1}{x} + \frac{1}{y} = C_1$$

$$\Rightarrow \frac{(-y+x)}{xy} = C_1 \Rightarrow \frac{x-y}{xy} = C_1 \quad ...(3)$$

And for second solution : from Equation (2),

$$\frac{dx-dy}{x^2-y^2} = \frac{dz}{z(x+y)}$$

$$\Rightarrow \frac{dx-dy}{(x-y)(x+y)} = \frac{dz}{z(x+y)}$$

$$\Rightarrow \frac{dx-dy}{(x-y)} = \frac{dz}{z}$$

Integrating,

$$\Rightarrow \int \frac{dx-dy}{(x-y)} = \int \frac{dz}{z} + \log C_2$$

$$\therefore \log(x-y) - \log z = \log C_2$$

➔ **Using standard result :** $\ldots\left[\log A - \log B = \left(\dfrac{A}{B}\right)\right]$

$$\therefore \log\left(\frac{x-y}{z}\right) = \log C_2$$

$$\Rightarrow \frac{x-y}{z} = C_2 \quad ...(4)$$

Step III : From Equations (3) and (4), the required general integral is

$$F(C_1, C_2) = 0$$

$$F\left(\frac{x-y}{xy}, \frac{x-y}{z}\right) = 0$$

or $\dfrac{x-y}{xy} = f\left(\dfrac{x-y}{z}\right)$ ✓ ...Ans.

(XIV)

Step I : The given linear partial differential equation.

$$z(xp - yq) = y^2 - x^2$$

$$\Rightarrow xp - yq = \left(\frac{y^2-x^2}{z}\right) \quad ...(1)$$

The given linear PDE is of the form

$$pP + qQ = R$$

Where, $P = x;\ Q = -y;\ R = \left(\dfrac{y^2-x^2}{z}\right)$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation.

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e. $\dfrac{dx}{x} = \dfrac{dy}{-y} = \dfrac{dz}{\left(\dfrac{y^2-x^2}{z}\right)}$ ...(2)

For first solution : from Equation (2),

Consider, $\dfrac{dx}{x} = \dfrac{dy}{-y}$

Integrating,

$$\log x = -\log y + \log C_1$$

$$\Rightarrow \log x + \log y = \log C_1$$

➔ **Using standard result :** $\ldots[\log A + \log B = \log(AB)]$

$$\therefore \log(xy) = \log C_1$$

$$\Rightarrow xy = C_1 \quad ...(3)$$

And for second solution : from Equation (2),

Let x, y, z be the set of multiplier such that,

$$\text{Each ratio} = \frac{xdx + ydy + zdz}{x^2 - y^2 + \cancel{z}\left(\dfrac{y^2-x^2}{\cancel{z}}\right)}$$

$$= \frac{x\,dx + ydy + z\,dz}{0}$$

$$\Rightarrow x\,dx + ydy + z\,dz = 0$$

Integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C$$

$$\Rightarrow x^2 + y^2 + z^2 = C_2 \quad ...(4)$$

Step III :

From Equations (3) and (4), the required general integral is,

$$F(c_1, c_2) = 0$$

$$F(xy, x^2 + y^2 + z^2) = 0$$

or $x^2 + y^2 + z^2 = F(xy)$ ✓ ...Ans.

XV)

Step I : The given linear partial differential equation

$$(y^3x - 2x^4)p + (2y^4 - x^3y)q = 9z(x^3 - y^3) \quad ...(1)$$

The given linear PDE is of the form

$$pP + qQ = R$$

where, $P = y^3x - 2x^4;\quad Q = (2y^4 - x^3y);$

$R = 9z(x^3 - y^3)$

Step II : **The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation**

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e. $\dfrac{dx}{y^3x - 2x^4} = \dfrac{dy}{2y^4 - x^3y} = \dfrac{dz}{9z(x^3 - y^3)}$ ...(2)

For first solution : from Equation (2),

Let $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{3z}$ be the set of multiplier such that

$$\text{Each ratio} = \frac{\dfrac{1}{x}dx + \dfrac{1}{y}dy + \dfrac{1}{3z}dz}{\dfrac{1}{x}(y^3x - 2x^4) + \dfrac{1}{y}(2y^4 - x^3y) + \dfrac{1}{3\cancel{z}}9\cancel{z}(x^3 - y^3)}$$

$$= \frac{\dfrac{1}{x}dx - \dfrac{1}{y}dy + \dfrac{1}{3z}dz}{\cancel{y^3} - \cancel{2x^3} + \cancel{2y^3} - \cancel{x^3} + \cancel{3x^3} - \cancel{3y^3}}$$

$$= \frac{\dfrac{1}{x}dx + \dfrac{1}{y}dy + \dfrac{1}{3z}dz}{0}$$

$$\Rightarrow \frac{1}{x}dx - \frac{1}{y}dy + \frac{1}{3z}dz = 0$$

Integrating,

$$\int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{3z}dz = \log C_1$$

$$\Rightarrow \log x + \log y + \frac{1}{3}\log z = \log C_1$$

$$\log x + \log y + \log z^{1/3} = \log C_1$$

➔ **Using standard result**

$$\ldots\left[\log A + \log B + \log C = \log(ABC)\right]$$

$$\therefore \log\left(xyz^{1/3}\right) = \log C_1$$

$$\Rightarrow xyz^{1/3} = C_1 \quad ...(3)$$

And for second solution : from Equation (2),

$$\text{Each ratio} = \frac{x^2dx + y^2dy}{x^2(y^3x - 2x^4) + y^2(2y^4 - x^3y)}$$

$$= \frac{x^2dx + y^2dy}{\cancel{y^3x^3} - 2x^6 + 2y^6 - \cancel{x^3y^3}}$$

$$= \frac{x^2dx + y^2dx}{2(y^6 - x^6)}$$

➔ **Using standard result :** $\ldots\left[x^2 - y^2 = (x-y)(x+y)\right]$

$$= \frac{x^2dx + y^2dy}{2(y^3 - x^3)(y^3 + x^3)} \quad ...(4)$$

Equating this ratio (4) with third ratio of Equation (2),

$$\therefore \frac{x^2dx + y^2dy}{2(y^3 - x^3)(y^3 + x^3)} = \frac{dz}{9z(x^3 - y^3)}$$

$$\Rightarrow \frac{x^2dx + y^2dy}{-2(x^3 - y^3)(x^3 + y^3)} = \frac{dz}{9z(x^3 - y^3)}$$

$$-3\left[\frac{3x^2dx + 3y^2dy}{(x^3 + y^3)}\right] = 2\frac{dz}{z}$$

Integrating, we get

$$-3\int \frac{3x^2dx + 3y^2dy}{(x^3 + y^3)} = 2\int \frac{dz}{z} + \log C_2$$

$$\Rightarrow -\log(x^3 + y^3)^3 - \log z^2 = \log C_2$$

$$\Rightarrow \log(x^3 + y^3)^3 + \log z^2 = \log C_2$$

$$\log\left[(x^3 + y^3)^3 \cdot z^2\right] = \log C_2$$

$$\Rightarrow (x^3 + y^3)^3 z^2 = C_2 \quad ...(5)$$

Step III : From Equation (3) and (5) **the required general integral is,**

$$F(C_1, C_2) = 0$$

$$F(xyz^{1/3}, (x^3 + y^3)z^2) = 0$$

or $xyz^{1/3} = f(x^3 + y^3)z^3$ ✓ ...Ans.

(XVI)

Step I : The given linear partial differential equation

$$x_1P_1 + 2x_2P_2 + 3x_3P_3 + 4x_4P_4 = 0 \quad ...(1)$$

The given linear PDE is of the form

$$pP + qQ = R$$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation.

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e. $\dfrac{dx_1}{x_1} = \dfrac{dx_2}{2x_2} = \dfrac{dx_3}{3x_3} = \dfrac{dx_4}{4x_4}$ ...(2)

For first solution : from Equation (2),

Consider, $\dfrac{dx_1}{x_1} = \dfrac{dx_2}{2x_2} \Rightarrow \log x_1 = \dfrac{1}{2}\log x_2 + C$

$$2\log x_1 - \log x_2 = \log C_2$$

$$\Rightarrow \log\left(x_1^2\right) - \log x_2 = \log C_2$$

$$\Rightarrow \left(\frac{x_1^2}{x_2}\right) = C_2 \quad ...(3)$$

For second solution : from Equations (2),

Consider, $\dfrac{dx_1}{x_1} = \dfrac{dx_3}{3x_3}$

$$\Rightarrow 3\frac{dx_1}{x_1} = \frac{dx_3}{x_3} \Rightarrow 3\log x_1 - \log x_3 = \log C_2$$

$$\Rightarrow \log x_1^3 - \log x_3 = \log C_2$$

$$\log\left(\frac{x_1^3}{x_3}\right) = \log C_2 \Rightarrow \frac{x_1^3}{x_3} = C_2 \qquad ...(4)$$

and for third solution, from Equation (2)

Consider,

$$\frac{dx_1}{x_1} = \frac{dx_4}{4x_4} \Rightarrow 4\frac{dx_1}{x_1} = \frac{dx_4}{x_4}$$

Integrating, $4\log x_1 - \log x_4 = \log C_3$

$$\Rightarrow \log x_1^4 - \log x_4 = \log C_3$$

$$\text{i.e.} \quad \frac{x_1^4}{x_4} = C_3 \qquad ...(5)$$

Step III : From Equations (3), (4) and (5), the required general integral is

$$f(C_1, C_2, C_3) = 0$$

$$\therefore \quad F\left(\frac{x_1^2}{x_2}, \frac{x_1^3}{x_3}, \frac{x_1^4}{x_4}\right) = 0 \checkmark \qquad \textbf{...Ans.}$$

Note

In the differential equation,

$Pp + Qq = R$, there are two independent variables x and y, so after solving this, we get two independent solutions.

Satisfying given D.E : $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

This idea is extended for linear differential equations of first order having more than two independent variables.

If $u_1 = c_1$, $u_2 = c_2$, $u_3 = c_3$, ... $u_n = c_n$ are n independent solutions of

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \frac{dx_3}{P_3} = \ldots = \frac{dx_n}{P_n} = \frac{dz}{R}$$

Then the general solution of the PDE of

$$P_1\frac{\partial z}{\partial x_1} + P_2\frac{\partial z}{\partial x_2} + \ldots + P_n\frac{\partial z}{\partial x_n} = R \text{ is}$$

$$\phi(u_1, u_2, \ldots, u_n) = 0$$

(XVII)

Step I : The given linear partial differential equation

$$(z-y)p + (x-z)q = y - x \qquad ...(1)$$

The given linear PDE is of the form

$$pP + qQ = R$$

where, $P = z - y$; $Q = (x - z)$; $R = (y - x)$

Step II : The integral surface of the given P.D.E. is generated by the integral curves of the auxiliary equation

i.e. Auxiliary equations are :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{i.e.} \quad \frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x} \qquad ...(2)$$

For first solution : from Equation (2),

let 1, 1, 1 be the set of multiplier such that,

$$\text{each ratio} = \frac{1dx + 1dy + 1dz}{z - y + x - z + y - x} = \frac{dx + dy + dz}{0}$$

$$\Rightarrow dx + dy + dz = 0$$

Integrating,

$$x + y + z = C_1 \qquad ...(3)$$

And for second solution : from Equation (2),

Let x, y, z be the set of multiplier, such that

$$\text{Each ratio} = \frac{x\,dx + y\,dy + z\,dz}{x(z-y) + y(x-z) + z(y-x)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{\cancel{xz} - \cancel{xy} + \cancel{xy} - \cancel{yz} + \cancel{yz} - \cancel{xz}}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\Rightarrow \quad x\,dx + y\,dy + z\,dz = 0$$

Integrating,

$$\int x\,dx + y\,dy + z\,dz = C_2$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_2$$

$$\Rightarrow \quad x^2 + y^2 + z^2 = C_2 \qquad ...(4)$$

Step III : From Equation (3) and (4), the required general integral is

$$F(C_1, C_2) = 0$$

$$F(x + y + z, x^2 + y^2 + z^2) = 0 \quad \text{Or}$$

$$x + y + z = f(x^2 + y^2 + z^2) = 0 \checkmark \qquad \textbf{...Ans.}$$

Exercise

Solve the following

Ex. 1 : $p - q = \log(x + y)$

Ans. : $x\log(x + y) - z = f(x + y)$

Ex. 2 : $xp - yq = y^2 - x^2$

Ans. : $(x^2 + y^2) + 2z = \log(x + y)$

Ex. 3 : $(y^2 - z^2)p - xyq + zx = 0$

Ans. : $f\left(\frac{y}{z}, x^2 + y^2 + z^2\right) = 0$

Ex. 4 : $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$

Ans. : $f\left(\frac{y}{z}, \frac{z}{x} - \frac{y}{x} + x^2\right) = 0$

Ex. 5 : $P\cos(x + y) + q\sin(x + 1) = z$

$[\cos(x + y) + \sin(x + y)]\,c^{y-x}$

Ans. : $\phi\left[z\sqrt{2}\cdot\tan\left(\frac{x+y}{2} + \frac{\pi}{8}\right)\right]$

4.5 Solution of Partial Differential Equations

- The general solution of the Partial differential equation in a region R involves arbitrary constants or arbitrary functions. Since, the physical problems will satisfy condition at the boundary of the region R known as boundary conditions and if these conditions are given at time $t = 0$ then these conditions are known as initial conditions.
- The differential equation together with boundary conditions known as boundary value problems (BVP) and Initial value problems (IVB).
- The method of solution of Partial differential equations is different from the method of solution of the ordinary differential equations.
- Method of separation of variables is a powerful tool to solve linear Partial differential equations. In this method, the particular solution of the Partial differential equation which satisfy all or some boundary conditions and then adjust them till the remaining conditions are also satisfied.
- A combination of these particular solutions gives the solution of given problem (superposition principle).
- The general solution of the partial differential equations must contains either arbitrary constants equal to twice the order of equation or arbitrary function is equal to the order of partial differential equation or if both (arbitrary constants and functions) are involved then the sum of arbitrary constants divided by two plus number of arbitrary functions is equal to the order of partial differential equation.
- Particularly, we study the following types of partial differential equations :

(i) One Dimensional Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

(ii) One Dimensional Heat Flow Equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$

(iii) Two Dimensional Heat Flow Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Syllabus Topic : Method of Separation of Variables

4.5.1 Method of Separation of Variables :

- Generally, in ordinary linear differential equations, we first find the general solutions (as C. F. + P. I.) and then determine the values of arbitrary constants by using initial conditions.
- But the method is different for the problems involving partial differential equations. Since, boundary value (initial or final) problems generally are linear partial differential equations. These type of P.D.E. can be solved by using special method called "Method of Separation of Variables."
- Since, the general solutions of partial differential equation contains arbitrary function which are difficult to adjust. So, in this method, right from beginning, we try to find the particular solution for given P.D.E. which satisfy all or some conditions (initial and boundary) and adjust these solutions so as they (solutions) satisfy remaining conditions. The combinations of these particular solutions be the solution of the given P.D.E.
- In this method, we consider the dependent variable as the product of several functions each of which involves only one of the independent variables. We get several ordinary differential equations. Find its solutions and then product of all solutions is the solution of given P.D.E.

4.5.2 Working Rule :

Step I : Write the trial solution of the given P.D.E.

Step II : Differentiate the solution from Step – I, as per the requirement.

Step III : Substitute the values of all differentials in the given P.D.E. and find the suitable ordinary differential equations.

Step IV : Solve the linear ordinary differential equation.

Use :

(i) If all roots $m_1, m_2, m_3, \ldots, m_n$ of Auxiliary Equation (A.E.) are real and non repeated roots then solution,

$$= C_1 e^{m_1 x} + C_2 e^{m_2 x} + \ldots + C_n e^{m_n x}$$

(ii) If real roots are repeated i.e. if $m_1 = m_2$ and other roots are of type (i) then solution,

$$= \left[(C_1 x + C_2)e^{m_1 x} + C_3 e^{m_3 x} + C_4 e^{m_4 x} + \ldots + C_n e^{m_n x}\right]$$

If $m_1 = m_2 = m_3$ then solution

$$= (C_1 x^2 + C_2 x + C_3)e^{m_1 x} + C_4 e^{m_4 x} + \ldots + C_n e^{m_n x}$$

(iii) **Complex Roots :**

If the roots of A. E. are complex i.e. if $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ and other roots are of type (i) then

$$\text{Solution} = e^{\alpha x}[C_1\cos\beta x + C_2\sin\beta x] + C_3 e^{m_3 x} + \ldots + C_n e^{m_n x}$$

(iv) **Complex roots are repeated :**

If $m_1 = m_2 = \alpha + i\beta$ and $m_3 = m_4 = \alpha - i\beta$ and other roots are of type (i) then

Solution is

$$= e^{\alpha x}\left[(C_1 x + C_2)\cos\beta x + (C_3 x + C_4)\sin\beta x\right] + C_5 e^{m_5 x} + \ldots + C_n e^{m_n x}.$$

Step V : Substitute the solutions in trial solution, we get the required solution of given partial differential equation.

Ex. 4.5.1 : Solve by the method of separation of variables :

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

Soln. :

Step I : The given P.D.E. is,

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0 \quad \ldots(1)$$

Equation (1) shows that, z - is a function of two variables x and y.

Let, the solution be

$$Z = X(x)\cdot Y(y)$$

$$\text{i.e. } Z = XY \quad \ldots(2)$$

Here, X – is a function of x only and Y – is a function of y only.

Step II : Differentiate Equation (1)

Partially w.r.t. x, keeping y as constant

$$\frac{\partial z}{\partial x} = \frac{\partial X}{\partial x} Y = X'Y; \quad \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 X}{\partial x^2} Y = X''Y \quad \ldots(3)$$

Also, differentiate Equation (1) w.r.t. y, keeping x as constant

$$\frac{\partial z}{\partial y} = X\frac{\partial Y}{\partial y} = XY' \quad \ldots(4)$$

Step III : By using Equations (3) and (4), Equation (1) becomes

$$X''Y - 2X'Y + XY' = 0$$

$$\Rightarrow \quad (X'' - 2X')Y = -XY'$$

$$\Rightarrow \quad \frac{X'' - 2X'}{X} = -\frac{Y'}{Y} \quad \ldots(5)$$

Now, variables are separated.

Since, x and y are independent variables. When x changes, y remains constant and vice versa. So that Equation (5) is true only when L.H.S. and R.H.S, must equal to the same constant k (Say)

$$\therefore \frac{X'' - 2X'}{X} = k \quad \text{and} \quad \frac{-Y'}{Y} = k$$

$$\Rightarrow \quad X'' - 2X' - kX = 0 \quad \ldots(6)$$

$$\text{and} \quad Y' + kY = 0 \quad \ldots(7)$$

Equations (6) and (7) are ordinary differential equations.

Step IV : From Equation (6),

$$X'' - 2X' - kX = 0$$

It is like, $\frac{d^2X}{dx^2} - 2\frac{dX}{dx} - kX = 0$

Auxiliary Equation (A.E.) is

$$m^2 - 2m - k = 0$$

Where, $m = \frac{d}{dx}$

➔ **Using standard result**

$$\ldots\left[\text{If } ax^2 + bx + C = 0 \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right]$$

$$\therefore \quad m = \frac{2 \pm \sqrt{4 - 4\times 1\times(-k)}}{2\times 1}$$

$$m = \frac{2 \pm 2\sqrt{1+k}}{2}$$

$$= 1 \pm \sqrt{1+k}$$

Solution is

$$X = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$X(x) = C_1 e^{\left(1+\sqrt{1+k}\right)x} + C_2 e^{\left(1-\sqrt{1+k}\right)x} \quad \ldots(8)$$

Also, from Equation (7),

$$Y' + kY = 0$$

It is like,

$$\frac{dY}{dy} = -kY$$

$$\Rightarrow \quad \frac{dY}{Y} = -kdy$$

Variables are separated, Integrating,

$$\log Y = -ky + C$$

$$Y = e^{-ky + C} = e^{-ky}\cdot e^{C}$$

$$Y = A\, e^{-ky} \quad \ldots(9)$$

Where, $A = e^{C}$.

Step V : From Equations (8) and (9), Equation (2) becomes,

$$z = \left[C_1 e^{\left(1+\sqrt{1+k}\right)x} + C_2 e^{\left(1-\sqrt{1+k}\right)x}\right]\cdot Ae^{-ky} \checkmark \quad \ldots\text{Ans.}$$

Which is the complete solution of Equation (1)

Ex. 4.5.2 : Solve using the method of separation of variables,

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u, \quad \text{Given, } u(x, 0) = 6e^{-3x}$$

Soln. :

Step I : The given differential equation is,

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \quad \ldots(1)$$

This shows that, u - is a function of two independent variables x and t.

Let the solution be,

$$u = X(x)\cdot T(t)$$

$$\text{i.e.} \quad u = XT \quad \ldots(2)$$

Here, X – is the function of x only ;

T – is the function of t only.

Step II : Differentiate u, partially w.r.t. x and t.

$$\frac{\partial u}{\partial x} = X\cdot\frac{\partial X}{\partial x}\cdot T = X'T$$

$$\text{and} \quad \frac{\partial u}{\partial t} = X\cdot\frac{\partial T}{\partial t} = XT'$$

Step III : Substitute the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ in Equation (1),

$$X'T = 2XT' + XT$$

$$\Rightarrow \quad (X' - X)T = 2XT'$$

$$\Rightarrow \quad \frac{(X' - X)}{X} = 2\frac{T'}{T} \quad \ldots(3)$$

Since, x and t are independent variables, Equation (3) is true if L. H. S. and R. H. S. is equal to the same constant k (say).

$$\therefore \quad \frac{X' - X}{X} = \frac{2T'}{T} = k \text{ (Say)}$$

$$\Rightarrow \quad \frac{X' - X}{X} = k \Rightarrow \frac{X'}{X} - 1 - k = 0$$

$$\text{i.e.} \quad \frac{dX}{X} - (1+k)\,dx = 0 \quad \ldots(4)$$

$$\text{and} \quad \frac{2T'}{T} = k \Rightarrow \frac{T'}{T} = \frac{k}{2}$$

$$\Rightarrow \frac{dT}{T} - \frac{k}{2}\,dt = 0 \quad \ldots(5)$$

Equations (4) and (5) are ordinary differential equations.

Step IV : From Equation (4),

$$\frac{dX}{X} - (1+k)\,dx = 0$$

Integrating, $\log X - (1+k)x = C_1$

$$\Rightarrow \quad \log X = (1+k)x + C_2$$

$$X = e^{(1+k)x + C_2}$$

$$X = e^{(1+k)x}\cdot e^{C_2} = e^{(1+k)x}\cdot A \quad \ldots(6)$$

And from Equation (5),

$$\frac{dT}{T} - \frac{k}{2}\,dt = 0$$

Integrating, $\log T - \frac{k}{2}t = C_2$

$$\log T = \frac{k}{2}t + C_2$$

$$T = e^{\frac{k}{2}t + C_2} = e^{\frac{k}{2}t}\cdot e^{C_2} = Be^{\frac{k}{2}t} \quad \ldots(7)$$

Step V : Substitute the values of X and T from *Equations (6) and (7) in Equation (2), we get*

$$u = Ae^{(1+k)x}\cdot Be^{\frac{k}{2}t}$$

$$\text{i.e.} \quad u(x, t) = Ce^{(1+k)x}\cdot e^{\frac{k}{2}t} \quad \ldots(8)$$

where, $C = AB$

Given $u(x, 0) = 6e^{-3x}$

i.e. put $t = 0$ in Equation (8)

$$6e^{-3x} = u(x, 0) = Ce^{(1+k)x}\cdot 1$$

$$\Rightarrow \quad C = 6 \quad \text{and} \quad (1+k) = -3 \Rightarrow 1 + k = -3$$

$$\Rightarrow \quad k = -3 - 1 = -4$$

Equation (8) becomes,

$$u(x, t) = 6e^{-3x}e^{-2t} = 6e^{-(3x+2t)} \checkmark \quad \ldots\text{Ans.}$$

be the required solution of given P.D.E.

Ex. 4.5.3 : Solve by using the method of separation of variables.

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}, \quad \text{Given} \quad u(0, y) = 8e^{-3y}$$

Soln. :

Step I : The given partial differential equation is,

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y} \quad \ldots(1)$$

This shows that u is a function of two independent variables x and y

Let, the solution be,

$$u = X(x)\,Y(y)$$

$$\therefore u = XY \quad \ldots(2)$$

Here, X – be the function of x only and

Y – be the function of y only

Step II : Differentiate u partially w.r.t. x and y.

$$\frac{\partial u}{\partial x} = \frac{\partial X}{\partial x}\cdot Y = X'Y$$

$$\text{and} \quad \frac{\partial u}{\partial y} = X\frac{\partial Y}{\partial y} = XY'$$

Step III : Substitute the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in Equation (1)

$$X'Y = 4XY'$$

$$\Rightarrow \quad \frac{X'}{X} = \frac{4Y'}{Y} \quad \ldots(3)$$

Since, x and y are independent variables. Equation (3) is true if L.H.S and R.H.S. is equal to the same constant k (say).

$$\therefore \frac{X'}{X} = \frac{4Y'}{Y} = k \text{ (say)}$$

$$\Rightarrow \quad \frac{X'}{X} = k \quad \ldots(4)$$

$$\text{and} \quad \frac{4Y'}{Y} = k \quad \ldots(5)$$

Equations (4) and (5) are ordinary differential equations.

Step IV : From Equation (4),

$$\frac{X'}{X} = K$$

Integrating, $\int \frac{X'}{X} dx = \int k\, dx + C_1$

$\Rightarrow \quad \log X = kx + C_1$

$\Rightarrow \quad X = e^{kx + C_1} = e^{kx} \cdot e^{C_1} = A e^{kx}$ ...(6)

where $A = e^{C_1}$

And from Equation (5),

$$\frac{4Y'}{Y} = K$$

Integrating, $4\int \frac{Y'}{Y} dy = \int ky + C_2$

$\Rightarrow \quad 4 \log Y = ky + C_2$

$\Rightarrow \quad \log Y = \frac{1}{4}(ky + C_2)$

$$Y = e^{\frac{1}{4}(ky + C_2)} = e^{\frac{k}{4}y} \cdot e^{\frac{C_2}{4}}$$

$$Y = Be^{\frac{k}{4}y} \quad ...(7)$$

where, $B = e^{\frac{C_2}{4}}$

Step V : From Equations (6) and (7), Equation (2) becomes

$$u(x, y) = A e^{kx} \cdot Be^{\frac{k}{4}y}$$

$\Rightarrow \quad u(x, y) = C \cdot e^{kx + \frac{k}{4}y}$ ...(8)

where, $C = AB$

Given $u(0, y) = 8e^{-3y}$

i.e. Put $x = 0$ in Equation (8),

$8e^{-3y} = u(0, y) = Ce^{\frac{k}{4}y}$

$\Rightarrow \quad C = 8$ and $-3 = \frac{k}{4} \Rightarrow k = -12$

Substitute the values of C and k, in Equation (8), we get

$u(x, y) = 8e^{-12x - 3y}$, is the required solution of given P.D.E. ✓ **...Ans.**

Ex. 4.5.4 : Solve by the method of variable separable.

$$4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u, \text{ given, } u = 3e^{-y} - e^{-5y}, \text{ when } x = 0$$

Soln. :

Step I : The given partial differential equation is,

$$4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u \quad ...(1)$$

This shows that u is a function of two independent variables x and y.

Let the solution be,

$u = X(x) \cdot Y(y)$

i.e. $u = XY$ ...(2)

Here, X – be the function of x alone.

Y – be the function of y alone.

Step II : Differentiate u, partially w.r.t. x and y,

$$\frac{\partial u}{\partial x} = \frac{\partial X}{\partial x} \cdot Y = X'Y$$

and $\frac{\partial u}{\partial y} = X \cdot \frac{\partial Y}{\partial y} = XY'$

Step III : Substitute the values of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ and u in the Equation (1)

$\therefore \quad 4X'Y + XY' = 3XY$

$\Rightarrow \quad \frac{4X'}{X} + \frac{Y'}{Y} = 3$

$\Rightarrow \quad 4\frac{X'}{X} = 3 - \frac{Y'}{Y}$ ...(3)

Since, x and y are independent variables Equation (3) is true if both L.H.S. and R.H.S. is equal to the same constant (say) k

$\therefore \quad 4\frac{X'}{X} = \left(3 - \frac{Y'}{Y}\right) = k$

$4\frac{X'}{X} = k \Rightarrow \frac{X'}{X} = \frac{k}{4}$ ...(4)

and $3 - \frac{Y'}{y} = k \Rightarrow \frac{Y'}{Y} = 3 - k$ ...(5)

Equations (4) and (5) are ordinary differential equations

Step IV :

From Equation (4), $\frac{X'}{X} = \frac{k}{4}$

Integrating, $\int \frac{X'}{X} dx = \int \frac{k}{4} dx + C_1$

$\text{Log } X = \frac{k}{4}x + C_1$

$\Rightarrow \quad X = e^{\frac{k}{4}x + C_1} = e^{\frac{k}{4}x} \cdot e^{C_1}$

$\therefore \quad X = Ae^{\frac{k}{4}x}$ ...(6)

where, $A = e^{C_1}$

And from Equation (6),

$$\frac{Y'}{Y} = (3 - k)$$

Integrating, $\int \frac{Y'}{Y} dy = \int (3 - k)\, dy + C_2$

$\text{Log } Y = (3 - k)y + C_2$

$\therefore \quad Y = e^{(3-k)y + C_2}$

$= e^{(3-k)y} \cdot e^{C_2}$

$Y = Be^{(3-k)y}$ ...(7)

where, $B = e^{C_2}$

Step V : By using Equations (6) and (7), Equation (2) can be written as

$$u = Ae^{\frac{k}{4}x} Be^{(3-k)y}$$

$u(x, y) = Ce^{\frac{k}{4}x + (3-k)y}$ ...(8)

where, $C = AB$

$u(0, y) = Ce^{(3-k)y}$

Given, $u(0, y) = 3e^{-y} - e^{-5y}$

i.e. put $x = 0$ in Equation (8)

$u(0, y) = Ce^{(3-k)y}$

$3e^{-y} - e^{-5y} = Ce^{(3-K)y}$

Comparing both sides

$C = 3$ and $3 - k = -1$

or $C = -1$ and $3 - k = -5$

$\Rightarrow \quad C = 3$ and $k = 4$

or $C = -1$ and $k = 8$

$\therefore$ Equation (8) becomes,

$u(x, y) = 3e^{x-y} - e^{2x-5y}$ ✓ **...Ans.**

which is the required solution of given P.D.E.

Ex. 4.5.5 : Solve by the method of separation of variables

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0, \text{ where, } u(x, 0) = 4e^{-x}$$

Soln. :

Step I : The given partial differential equation is

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0 \quad ...(1)$$

This shows that u is a function of two independent variables x and y.

Let, the solution be

$u = X(x)\, Y(y)$

i.e. $u = XY$ ...(2)

Here, X – is a function of x only and

Y – is a function of y only

Step II : Differentiate u, partially w.r.t. x and y

$$\frac{\partial u}{\partial x} = \frac{\partial X}{\partial x} \cdot Y = X'Y$$

and $\frac{\partial u}{\partial y} = X \cdot \frac{\partial Y}{\partial y} = XY'$

Step III : Substitute the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in the Equation (1)

$\therefore \quad 3X'Y + 2XY' = 0$

$\Rightarrow \quad 3X'Y = -2XY'$

$\Rightarrow \quad 3\frac{X'}{X} = \frac{-2Y'}{Y}$ ...(3)

Since, x and y are independent variables. Equation (3) is true if L.H.S. and R.H.S. is equal to the same constant k (say)

$\therefore \quad \frac{3X'}{X} = \frac{-2Y'}{Y} = k$ (say)

$\frac{3X'}{X} = k \Rightarrow \frac{X'}{X} = \frac{k}{3}$ ...(4)

and $\frac{-2Y'}{Y} = k \Rightarrow \frac{Y'}{Y} = \frac{-k}{2}$ ...(5)

Equation (4) and (5) are ordinary differential equations

Step IV : From Equation (4),

$$\frac{X'}{X} = \frac{k}{3}$$

Integrating, $\log X = \frac{k}{3}x + C_1$

$\therefore \quad X = e^{\frac{k}{3}x + C_1} = e^{\frac{k}{3}x} \cdot e^{C_1}$

$X = Ae^{\frac{k}{3}x}$ ...(6)

where, $A = e^{C_1}$

And from Equation (5)

$$\frac{Y'}{Y} = \frac{-k}{2}$$

Integrating, $\log Y = \frac{-k}{2}y + C_2$

$\therefore \quad Y = e^{\frac{-k}{2}y + C_2} = e^{\frac{-k}{2}y} \cdot e^{C_2}$

$Y = Be^{\frac{-K}{2}y}$ ...(7)

where, $B = e^{C_2}$

Step V : Substitute the values of X and Y from Equations (6) and (7), in Equation (2), we get

$u = Ae^{\frac{k}{3}x} \cdot Be^{\frac{-k}{2}y}$

$u = Ce^{\frac{k}{3}x - \frac{k}{2}y}$ ...(8)

where, $C = AB$

Given : $y = 0$; $u = 4e^{-x}$

put, $y = 0$ in Equation (8)

$\therefore \quad 4e^{-x} = C \cdot e^{\frac{k}{3}x}$

$\Rightarrow C = 4; \quad \frac{k}{3} = -1$

$\Rightarrow C = 4; \quad k = -3$

$\therefore$ Equation (8) becomes

$u = 4e^{-x + \frac{3}{2}y}$ ✓ **...Ans.**

Is the required solution of given P.D.E.

Ex. 4.5.6 : Find a solution of the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u$ in the form $u = f(x)\,g(y)$

Soln. :

Step I : The given partial differential equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u \qquad ...(1)$$

Equation (1) shows that u is a function of two independent variables x and y

Let the solution be,

$$u(x, y) = X(x)\,Y(y)$$

$$\Rightarrow \quad u = XY \qquad ...(2)$$

Here, X – be the function of x only

Y – be the function of y only

Step II : Differentiate Equation (2) partially w.r.t. x and y

$$\frac{\partial u}{\partial x} = \frac{\partial X}{\partial x} \cdot Y = X'Y;$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 X}{\partial x^2} \cdot Y = X''Y; \text{ and}$$

$$\frac{\partial u}{\partial y} = X \cdot \frac{\partial Y}{\partial y} = XY'$$

Step III : Substitute the values of $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial u}{\partial y}$ in Equation (1)

$$X''Y = XY' + 2XY = X(Y' + 2Y)$$

$$\frac{X''}{X} = \frac{Y' + 2Y}{Y} \qquad ...(3)$$

Since, x and y are independent variables. Equation (3) is true if L.H.S. and R.H.S. is equal to the same constant k(say)

$$\therefore \frac{X''}{X} = \frac{Y' + 2Y}{Y} = k \text{ (say)}$$

$$\frac{X''}{X} = k \Rightarrow X'' - kX = 0 \qquad ...(4)$$

and $$\frac{Y' + 2Y}{Y} = k \Rightarrow \frac{Y'}{Y} + 2 - k = 0 \qquad ...(5)$$

Equations (4) and (5) are ordinary differential equations

Step IV : From Equation (4),

$$X'' - kX = 0$$

It is like, $$(D^2 - k)X = 0$$

where, $D^2 \equiv X'' \equiv \frac{d^2X}{dx^2}$

$\therefore$ Auxiliary equation is, $D^2 - k = 0$

$$D = \pm\sqrt{k}$$

$\therefore$ Solution is,

$$X = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$X = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x} \qquad ...(6)$$

And from Equation (5)

$$\frac{Y'}{Y} + (2 - k) = 0$$

Integrating w.r.t. y

$$\int \frac{Y'}{Y} dy + \int (2 - k)\, dy = C_3$$

$$\log Y + (2 - k)y = C_3$$

$$\log Y = (k - 2)y + C_3$$

$$Y = e^{(k-2)y + C_3} = e^{(k-2)y} \cdot e^{C_3}$$

$$Y = Ae^{(k-2)y} \qquad ...(7)$$

where, $A = e^{C_3}$

Step V : By using Equations (6) and (7), Equation (2) becomes,

$$u(x, y) = \left[C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}\right] \cdot Ae^{(k-2)y}$$

$$u(x, y) = \left(C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}\right) \cdot e^{(k-2)y} \qquad ...(8)$$

4.6 Applications - Partial Differential Equations

4.6.1 Introduction

- Many physical processes fundamental to science and engineering are governed by partial differential equations. The application of partial differential equations is much wider than other differential equations.
- Maxwell's equations forms the basis of electromagnetic theory, aeronautical engineering governs the equations of fluid flow, Acoustics (the study of ground water flows in civil engineering), fluid handeling devices in mechanical engineering, combustion processes in chemical engineering, quantum mechanics, stress analysis in civil and mechanical engineering. They also arise in epidemiology (for example multivariable predator / prey modes of AIDS), traffic flow along on highway, telephone traffic, the analysis of economics and so on.
- In fact their study is important and essential to understand the nature of solutions. We will primarily concerned in this part with three partial differential equations called wave equations. $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$; Heat equations $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, and Laplace (or potential) equation : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

- Each of these we find the solutions under different boundary and initial conditions.
- Sometimes a certain conditions are not applicable, in such cases, the most general solution is written as the sum of particular solutions and the constants can be determined by using Fourier series, so as to satisfy the remaining conditions.

4.6.2 Prerequisite

(i) Half Range sine series in the interval $(0, l)$ is

$$f(x) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where, $$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

(ii) Half Range sine series in the interval $(0, \pi)$ is

$$f(x) = \sum_{n-1}^{\infty} b_n \sin nx$$

where, $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx dx$$

(iii) Half Range cosine series in the interval $(0, l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where, $$a_0 = \frac{2}{l}\int_0^l f(x)\, dx$$

and $$a_n = \frac{2}{l}\int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

(iv) Half Range cosine series in the interval $(0, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where, $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

and $$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx$$

(v) Integration by parts : (By LIATE Rule)

1. $$\int uv\, dx = u\int v\, dx - \int\left[\frac{du}{dx} \cdot \int v\, dx\right] dx$$
2. If first function is a polynomial in x then

$$\int uv\, dx = u v_1 - u' v_2 + u'' v_3 - u''' v_4 + ...$$

dashes denotes derivatives and suffixes denotes integration.

(vi) $\int e^{ax} \cdot \sin bx\, dx = \frac{e^{ax}}{(a^2 + b^2)}[a \sin bx - b \cos bx]$

(vii) $\int e^{ax} \cdot \cos bx\, dx = \frac{e^{ax}}{(a^2 + b^2)}[a \cos bx + b \sin bx]$

(viii) $\cos n\pi = (-1)^n$; $\sin n\pi = 0$

$\cos\frac{n\pi}{2}$; $\sin\left(\frac{n\pi}{2}\right)$

(ix) $\sin 3\theta = 3 \sin\theta - 4 \sin^3\theta$

$\cos 3\theta = 4 \cos^3\theta - 3 \cos\theta$

4.7 Modelling of Vibrating String / Wave Equation

(One – dimensional wave equation) or (Vibrations of stretched string)

- Let us consider a uniform elastic string of length L, such as guitar string stretched tightly between two points. O and A on X-axis. We want to describe the motion of the string if it is given a small displacement and released in a plane. When string starts to vibrate, assume that the motion takes place in the XY-plane in such a manner that each point on the string moves in a direction perpendicular (transverse vibrations). Here O as origin, OA – as X-axis and perpendiculars line through O as Y-axis.
- Let, y(x, t) – be the vertical displacement of any point as a function of distance x and time t. the graph of function y(x, t) of x, is the shape of the string of that time.

We will find the equation of motion for the stretched string under the following assumptions :

- The string stretched under tension T(x, t) in the string always acts tangentially to the string.
- The tension T(x, t) is so large that the damping forces (air resistance and weight of the string) can be neglected. (i.e. the effect of gravity neglected.)
- The magnitude of tension ||T|| is the same at all paints.
- No other twice act on the string
- The displacement y and the slope $\frac{dy}{dx}$ are small so that their higher powers can be neglected.
- The string is perfectly flexible and bends easily.
- The string is homogeneous, that is its mass per unit length 'm' is constant.
- The motion takes place entirely in the xy-plane and each individual particles of the string moves only vertically. (Perpendicular to the equilibrium position OA of the string.)

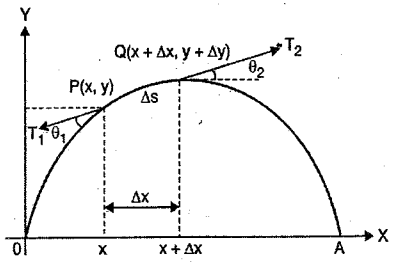


Fig. 4.7.1

Now, consider a typical segment PQ of the string between p(x, y) to Q (x + Δx, y + Δy) having length Δs.

Let m-be the mass per unit length of the string.

mΔs – be the total mass of segment PQ.

Let, the motion of the element PQ of the string between the points P and Q, where, the tangents makes angle θ_1 and θ_2 with X-axis.

$\frac{\partial^2 y}{\partial t^2}$ - be the acceleration of the element in upward (vertical) direction.

By Newtron's second law of motion.

mass x acceleration = net force

$$(m\Delta s)\frac{\partial^2 y}{\partial t^2} = T_2 \sin(\theta_2) - T_1 \sin(\theta_1)$$

Now, the tension T_1 and T_2 are tensions at P and Q respectively in tangential direction

$\therefore\ |T_1| = |T_2| = T$ and

$m\Delta s = m\Delta x$ on $[x + \Delta x, x]$

since, there is no motion in horizontal direction

$$\therefore\ T_1 \cos\theta_1 = T_2 \cos\theta_2 = T \text{ (constant)}$$

$$\therefore\ \frac{m\,\Delta x}{T}\frac{\partial^2 y}{\partial t^2} = \frac{T_2 \sin\theta_2}{T_2 \cos\theta_2} - \frac{T_1 \sin\theta_1}{T_1 \cos\theta_1}$$

$$\Rightarrow\ \Delta x \frac{\partial^2 y}{\partial t^2} = \frac{T}{m}(\tan\theta_2 - \tan\theta_1)$$

$$\therefore\ \frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left[\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\Delta x}\right]$$

($\because$ $\tan\theta_1$ and $\tan\theta_2$ are the slopes of string curve at p and Q respectively)

As $\Delta x \to 0$; $Q \to P$, we get

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}\ ;\ \text{where, } c^2 = \frac{T}{m}$$

This is the required partial differential equation of vibration of stretched string or one dimensional wave equation.

OR

By the Newton's second law of motion.

Mass element × acceleration in y direction = net force in y – direction

$$m\Delta s \frac{\partial^2 y}{\partial t^2}(\bar{x}, t) = T(x + \Delta x, t)\sin(\theta_2) - T(x, t)\sin(\theta_1)$$

where, $\bar{x}$ - be the centre of mass of the segment and

$T(x, t) = ||T(x, t)|| =$ magnitude of T

As $\rho\Delta s = \rho\Delta x$ for the first approximation.

$$\therefore\ \frac{T(x + \Delta x, t)\sin\theta_2 - T(x, t)\sin\theta_1}{\Delta x} = m\frac{\partial^2 y}{\partial t^2}(\bar{x}, t)$$

Now, $V(x, t) = T(x, t)\sin\theta$ – be the vertical component of the tension

$$\therefore\ \frac{V(x + \Delta x, t) - V(x, t)}{\Delta x} = m\frac{\partial^2 y}{\partial t^2}(\bar{x}, t)$$

In the limiting form,

As $\Delta x \to 0$, ; $\bar{x} \to x$ i.e. $Q \to P$

$$\frac{\partial v}{\partial x} = m\frac{\partial^2 y}{\partial t^2} \qquad \ldots(1)$$

Since, the horizontal component of the tension is

$$T(x, t)\cos\theta = h(x, t)$$

$$\therefore\ T(x, t) = \frac{h(x, t)}{\cos\theta}$$

$$\text{Hence, } V(x, t) = h(x, t)\frac{\sin\theta}{\cos\theta}$$

$$= h(x, t)\tan\theta$$

$$V(x, t) = h\frac{\partial y}{\partial x}$$

Substitute in Equation (1),

$$\frac{\partial}{\partial x}\left(h\frac{\partial y}{\partial x}\right) = m\frac{\partial^2 y}{\partial t^2}$$

Since, the horizontal components of the tension of the segment is zero so, $h(x + \Delta x, t) - h(x, t) = 0$

i.e. h is independent of x.

$$\therefore\ h\frac{\partial^2 y}{\partial x^2} = m\frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow\ \frac{\partial^2 y}{\partial t^2} = \frac{h}{m}\frac{\partial^2 y}{\partial x^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad \ldots(2)$$

$$\text{Where, } c^2 = \frac{h}{m}$$

Be the required one-dimensional wave equation.

The boundary and initial conditions, which the equation $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial t^2}$ has to satisfy are :

(I) Boundary conditions : If the ends of the string are fixed

(i) $y(0, t) = 0\ ;\ \forall\ t$

(ii) $y(L, t) = 0\ ;\ \forall\ t$

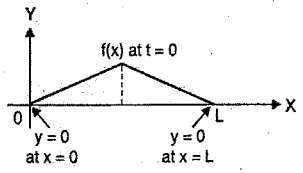


Fig. 4.7.2

(II) Initial conditions : (Initial means at t = 0)

(i) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x)$ for $0 \le x \le L$

(ii) $y(x, 0) = f(x)$ for $0 \le x \le L$

Here f(x) and g(x) are given functions satisfying certain conditions.

Note :

- If the string is fixed at its ends, then the initial position function satisfying

  $f(0) = f(L) = 0$

- If the string is released from rest, then initial velocity is zero.

  i.e. $g(x) = 0$ at $t = 0$

  i.e. $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

- If any external force of magnitude F units of force per unit length is acting on the string in the vertical direction, then Equation (2) becomes,

  $$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} + \frac{1}{m}F$$

- Two dimensional make equation is

  $$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

- Here, z (x, y, t) be the vertical displacement of a membrane (for example, vibrations of drum surface)

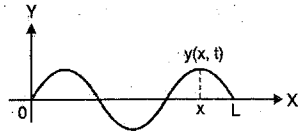


Fig. 4.7.3

4.7.1 Solution of One-dimensional Wave Equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}\ :$$

We can find the solution of one-dimensional wave equation by the method of separation of variables.

Given, one dimensional wave equation is,

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad \ldots(1)$$

With boundary conditions,

(i) $y(0, t) = 0$, for all t

(ii) $y(L, t) = 0$, for all t

And initial conditions

(iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0,\ 0 \le x \le L$

(Here, the string is released from rest)

(iv) $y(x, 0) = f(x)\ ;\ 0 \le x \le L$

Let the solution of (1) is,

$$y = X(x)\,T(t)$$

$$\therefore\ y = X\,T \qquad \ldots(2)$$

Here, X – is a function of x only and

T – is a function of t only

Differentiate Equation (2) partially, w.r.t. x and t

$$\frac{\partial y}{\partial x} = \frac{\partial X}{\partial x}T = X'T$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 X}{\partial x^2}\cdot T = X''T \quad \text{and}$$

$$\frac{\partial y}{\partial t} = X\frac{\partial T}{\partial t} = XT'$$

$$\frac{\partial^2 y}{\partial t^2} = X\frac{\partial^2 T}{\partial t^2} = XT''$$

Equation (1) becomes,

$$XT'' = c^2 X''T$$

$$\frac{T''}{c^2 T} = \frac{X''}{X} \qquad \ldots(3)$$

Since, x and t are independent variables. Equation (3) is true only if L.H.S. = R.H.S. = constant

$$\therefore\ \frac{T''}{c^2 T} = \frac{X''}{X} = K \text{ (say)}$$

(Here, K is any constant, – K is convenient but customary)

$$\Rightarrow \frac{X''}{X} = K \Rightarrow X'' - kX = 0 \qquad \ldots(4)$$

and $$\frac{T''}{c^2 T} = K \Rightarrow T'' - Kc^2 T = 0 \qquad \ldots(5)$$

Equation (4) and (5) are two ordinary differential equations for solving Equations (4) and (5),

(i) If K is positive (say) $K = m^2$

Then from (4), we get, $X = c_1 e^{mx} + c_2 e^{-mx}$ and from (5), we get, $T = c_3 e^{cmt} + c_4 e^{-cmt}$

(ii) If K is negative (say) $K = -m^2$

Then from (4), we get, $X = c_1 \cos mx + c_2 \sin mx$

and from (5), we get, $T = c_3 \cos mct + c_4 \sin mct$

(iii) If K is zero. K = 0

$\therefore$ From (4), $X = c_1 x + c_2$

and from (5), $T = c_3 t + c_4$

Hence, by using superposition principle, we get various possible solutions of wave Equation (1) are from Equation (2)

$$y = (c_1 e^{mx} + c_2 e^{-mx})(c_3 e^{cmt} + c_4 e^{-cmt}) \quad ...(6)$$

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct) \quad ...(7)$$

and $$y = (c_1 x + c_2)(c_3 t + c_4) \quad ...(8)$$

Eigen value

The numbers m for which the differential equation $y'' + my = 0$ has a non Trivial (non-zero) solution is called an **Eigen value.**

Eigen function

A corresponding non-Trivial solution is called **Eigen function.**

Case (i) : K is positive (say) $K = m^2$

The solution, from (4), $y = c_1 \cos mx + c_2 \sin mx$

By boundary condition $y(0) = 0 ; \forall t$

$\therefore\ 0 = c_1 + 0 \Rightarrow c_1 = 0$

and $y(x) = c_2 \sin mx$

by $y(L) = 0 ; \forall t$

$0 = c_2 \sin(mL) \Rightarrow$ To avoid Trivial solution

We need $c_2 \neq 0$. We must chose m so that

$\sin(mL) = 0 \Rightarrow mL = n\pi$

$\therefore\ m = \frac{n\pi}{L}$

$\therefore\ K = m^2 = \frac{n^2\pi^2}{L^2}$ For n = 1, 2, 3,....

Each of these numbers are **Eigen value** and corresponding each functions are **Eigen functions.**

$$y_n(x) = c_n \sin\left(\frac{n\pi}{L}x\right)$$

where $c_n \neq 0$

Case (ii) : K is negative (say) $K = -m^2, \forall t, m > 0$

from (4), solution is,

$0 = c_1 e^{mx} + c_2 e^{-mx}$

since, $y(0) = 0 \Rightarrow 0 = c_1 + c_2$

and $c_2 = -c_1$

$y = c_1 e^{mx} - c_1 e^{-mx} = 2c_1 [\sin h mx]$

but for other end $y(L) = 0$

$y(L) = 2c_1 \sin h(mL) = 0$

here, $\sin h\, mL = 0$ only if $mL = 0$ but here $m > 0$

therefore, $c_1 = 0$

and if $c_1 = 0$ then we get, **Trivial solution**

Case (iii) : K is zero $\therefore K = 0$

from (4), $y = c_1 x + c_2$

since $y(0) = 0 \Rightarrow 0 = c_2$ and

$y(L) = 0 \Rightarrow 0 = c_1(L) \Rightarrow c_1 = 0$

we get **Trivial solution**

$\therefore$ From case (ii) and (iii), we get Trivial solution i.e. values of m are not **Eigen values.** From case (i), values of m are **Eigen values**

Out of these three solutions (6), (7), (8) we have to select non-Trivial solution i.e. for case (i).

(Also, here we are discussing wave equation, its solution must contain trigonometric terms which are periodic)

Hence the most suitable general solution of one-dimensional wave Equation (1), which satisfies all boundary and initial conditions is.

$$y(x, t) = (c_1 \cos mx + c_2 \sin mx) \cdot (c_3 \cos mct + c_4 \sin mct) \quad ...(9)$$

Now, use boundary condition,

(i) $y(0, t) = 0$ for all t

$0 = (c_1 + 0)(c_3 \cos mct + c_4 \sin mct)$

$\Rightarrow c_1 = 0$

Equation (9) becomes

$$y(x, t) = (c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct) \quad ...(10)$$

(ii) By condition $y(L, t) = 0, \forall t$

$0 = c_2 \sin mL\ [c_3 \cos mct + c_4 \sin mct]$

$\Rightarrow$ For non Trivial solution, $c_2 \neq 0$

Only, $\sin mL = 0 \Rightarrow mL = n\pi$

$m = \frac{n\pi}{L}$; n = 1, 2, 3

Equation (10) become

$$y(x, t) = c_2 \sin\left(\frac{n\pi}{L}x\right) \cdot \left[c_3 \cos\frac{n\pi}{L}ct + c_4 \sin\frac{n\pi}{L}ct\right]$$

where, n = 1, 2, 3....

$$y(x, t) = \sum_{n=0}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left[b_n \cos\frac{n\pi}{L}ct + a_n \sin\frac{n\pi}{L}ct\right] \quad ...(11)$$

where $c_2 c_3 = b_n$ and $c_2 c_4 = a_n$

Now, by initial conditions :

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$$

$\therefore$ Differentiate (11) w.r.t. t

$$\frac{\partial y}{\partial t} = \sum_{n=0}^{\infty} \sin\left(\frac{n\pi}{L}\right)x \cdot \left[-b_n \cdot \left(\frac{L}{n\pi c}\right) \cdot \sin\left(\frac{n\pi}{L}ct\right) + a_n\left(\frac{L}{n\pi c}\right)\cos\left(\frac{n\pi}{L}ct\right)\right]$$

$$\therefore 0 = \sum_{n=0}^{\infty} \sin\left(\frac{n\pi}{L}\right)x\left[0 + a_n\left(\frac{L}{n\pi c}\right)\right]$$

$\sin\frac{n\pi}{L}x \neq 0$ and $a_n = 0$

Equation (11) becomes

$$y(x, t) = \sum_{n=0}^{\infty} bn \sin\left(\frac{n\pi}{L}x\right) \cdot \cos\left(\frac{n\pi}{L}ct\right) \quad ...(12)$$

Now by conditions,

$y(x, 0) = f(x)$ for $0 \le x \le L$

$$\therefore f(x) = \sum_{n=0}^{\infty} bn \sin\left(\frac{n\pi}{L}x\right) \quad \text{for } 0 \le x \le L$$

Which is Half range Fourier Sine series in $0 \le x \le L$

$$\therefore b_n = \frac{2}{L}\int_0^L f(x) \cdot \sin\left(\frac{n\pi}{L}\right)x\, dx$$

Hence, if we know the initial displacement f(x) then we get the complete solution (12)

Therefore, the most suitable general solution of one dimensional wave equation or vibration of stretched string is

$$y(x,t) = (C_1 \cos mx + C_2 \sin mx) \cdot (C_3 \cos cmt + C_4 \sin cmt)$$

To solve one dimensional wave equation

$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$, we use

(i) $y(0, t) = 0 \ ; \forall t$

(ii) $y(L, t) = 0 \ ; \forall t$

(iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

(iv) $y(x,0) = f(x)$ in $0 \le x \le L$

Note : For vibration of stretched string, the solution of one dimensional wave equation or stretched string, we should directly assume the most suitable solution Equation

4.7.1.1 Procedure to Solve One Dimensional Wave Equation

Step I : Write the given conditions

Step II : Write the most suitable solution.

$y(x,t) = [C_1 \cos mx + C_2 \sin mx]$
$[C_3 \cos mct + C_4 \sin mct]$

Step III : Apply condition (i), we get $C_1 = 0$

Step IV : Put $C_1 = 0$ in step - II solution and then differentiate y w.r.t t $\left(\text{i.e. find } \frac{\partial y}{\partial t}\right)$

Step V : Apply the condition, at $t = 0, \frac{\partial y}{\partial t} = 0$, we get $C_4 = 0$

Step VI : Then apply condition (ii), $x = l, y = 0$ (probabaly) we get, $\sin ml = 0$

$m = \frac{n\pi}{l}$; n = 0, 1, 2, 3, ...

put this value of m in the equation, which we get from step VI.

Step VII : Apply last condition, and then use half range sine series, we get the value of b_n.

Step VIII : Put the value of b_n in the equation which we get from step VI.

We get required displacement y at any time t.

Ex. 4.7.1 : If $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ represents the vibrations of a stretched string of length l fixed at both ends, find the solution with boundary conditions,

(i) $y(0, t) = 0 ; \forall t$

(ii) $y(l, t) = 0 \ ; \forall t$

and initial conditions

(iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 ; \forall x$

(iv) $y(x, 0) = k(lx - x^2) ; 0 \le x \le l$

Soln. : Let y(x, t) be the displacement at any time t.

Step I :

The differential equation of vibration of stretched string is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ and the given conditions are

(i) $y(0, t) = 0 ; \forall t$; (ii) $y(l, t) = 0 ; \forall t$

(iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 ; \forall x$ (iv) $y(x, 0) = k(lx - x^2) ; 0 \le x \le l$

Step II :

The most suitable solution is,

$$y(x, t) = (C_1 \cos mx + C_2 \sin mx)(C_3 \cos mct + C_4 \sin mct) \quad ...(1)$$

Step III :

By condition (i), $y(0, t) = 0$

$\therefore$ put $x = 0$, $y = 0$ in Equation (1)

$\because\ 0 = (C_1 + 0)(C_3 \cos mct + C_4 \sin mct)$

$\Rightarrow C_1 = 0$

Only. ($\because$ If we take bracketed term is equal to zero, then displacement vanishes)

Equation (1) becomes,

$$y(x, t) = C_2 \sin mx (C_3 \cos mct + C_4 \sin mct) \quad ...(2)$$

For condition (iii), differentiate Equation (2) partially w.r.t t, keeping x as constant

Step IV :

$\frac{\partial y}{\partial t} = C_2 \sin mx [-mc C_3 \sin mct + mc C_4 \cos mct]$

Step V :

By condition (iii), $\frac{\partial y}{\partial t} = 0$ at $t = 0$

put, $t = 0$ in $\frac{\partial y}{\partial t}$ equation

$\therefore\ 0 = C_2 \sin mx (0 + mc C_4)$

... ($\because \sin 0 = 0 ; \cos 0 = 1$)

$\Rightarrow C_2 C_4 = 0$ or $mc \sin mx = 0$

$\Rightarrow C_4 = 0$ only

($\because$ If we take $C_2 = 0$ then Equation (2) becomes zero)

Equation (2) becomes,

$$y(x,t) = C_2 C_3 \sin mx \cdot \cos mct \qquad ...(3)$$

Step VI :

By condition (ii), $x = l, y = 0$

Put, $x = l$ in Equation (3)

$\therefore\ 0 = C_2\ C_3 \sin ml \cos mct$

$\Rightarrow C_2 C_3 = 0$ OR $\sin ml = 0$ OR $\cos mct = 0$

Only $\sin ml = 0$

($\because$ If we take, $C_2 C_3 = 0$ and $\cos mct = 0$ then, displacement vanishes)

$\Rightarrow ml = n\pi \ ;\ n = 0, 1, 2, 3, ...$

$$m = \frac{n\pi}{l} \ ;\ n = 0, 1, 2, 3, ...$$

Equation (3) becomes,

$$y(x,t) = C_2 C_3 \sin\left(\frac{n\pi}{l}\right)x \cdot \cos\left(\frac{n\pi}{l}c\right)t;$$

$$n = 0, 1, 2, 3, ...$$

($\because$ These are the solutions of one dimensional wave equation. These functions are called **Eigen Functions** corresponding to the **Eigen values** $\left(\lambda_n = \frac{n\pi c}{L}\right)$ of the vibrating string.)

Let, $C_2 C_3 = b_n$ and take sum of all terms

$$y(x,t) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right)x \cdot \cos\left(\frac{n\pi}{l}c\right)t \qquad ...(4)$$

Step VII :

Now, by using condition (iv) ;

$$y(x,0) = k(lx - x^2)\ ;\ 0 \le x \le l$$

$\therefore$ Put $t = 0$ in Equation (4)

$$k(lx - x^2) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right)x \ ;\ \text{in} \quad 0 \le x \le l$$

Which is half range sine series in $0 \le x \le l$.

$$\left(\because f(x) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right)x\right)$$

Where,

$$\therefore\ b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi}{l}\right)x\,dx$$

$$= \frac{2}{l}\int_0^l \underbrace{k(lx - x^2)}_{\text{I}}\ \underbrace{\sin\left(\frac{n\pi}{l}\right)x}_{\text{II}}\,dx$$

➔ **Using standard result**

$$...\left[\int uv\,dx = u''v_1 - u'v_2 + u''v_3 - u'''v_4 + ...\right]$$

$$= \frac{2}{l}\left[k(lx - x^2)\left(\frac{-\cos\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)}\right) - (k(l - 2x))\left(\frac{-\sin\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)^2}\right) + (-2k)\left(\frac{\cos\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\left\{0 - 0 - \frac{2k(-1)^n}{\left(\frac{n\pi}{l}\right)^3}\right\} - \left\{0 - 0 - \frac{2k}{\left(\frac{n\pi}{l}\right)^3}\right\}\right];$$

$$(\because \sin 0 = 0\ ;\ \cos 0 = 1\ ;\ \sin n\pi = 0,\ \cos n\pi = (-1)^n)$$

$$= \frac{2}{l}\left[-2k\left(\frac{l}{n\pi}\right)^3(-1)^n + 2k\left(\frac{l}{n\pi}\right)^3\right]$$

$$= \frac{2}{l} \times 2k\frac{l^3}{n^3\pi^3}\left[1 - (-1)^n\right]$$

$$b_n = \begin{cases} \frac{8kl^2}{n^3\pi^3}\ ; & \text{if n is an odd number} \\ 0\ ; & \text{if n is an even number} \end{cases}$$

$$\left(\because \text{Odd number} = 2n + 1\ ;\ n = 0, 1, 2, 3....\ \ = 2n - 1\ ;\ n = 1, 2, 3....\right)$$

Step VIII :

Put the value of b_n in Equation (4), we get

$$y(x,t) = \sum_{n=0}^{\infty} \frac{8kl^2}{\pi^3(2n+1)^3} \cdot \sin\left(\frac{(2n+1)\pi}{l}\right)x \cdot \cos\left(\frac{(2n+1)\pi c}{l}\right)t$$

$$\therefore\ y(x,t) = \frac{8kl^2}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3} \cdot \sin\left(\frac{(2n+1)\pi}{l}\right)x \cdot \cos\left(\frac{(2n+1)\pi c}{l}\right)t \checkmark \ \textbf{...Ans.}$$

Which is the required deflection of the string, at any time t and at any distance x.

Ex. 4.7.2 : A string is stretched and fastened to two points L aparts. Motion is started by displacing the string in the form $u = a \sin\frac{\pi x}{L}$ from which it is released at time $t = 0$. Find the displacement $u(x, t)$ from one end.

Soln. :

Let $u(x, t)$ - be the displacement of the string at any time t.

Step I :

The differential equation of vibration of stretched string is $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ and the conditions are

(i) $u(0, t) = 0\ ;\ \forall t$

(ii) $u(l, t) = 0\ ;\ \forall t$

(iii) $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0\ ;\ \forall x$

(iv) $u(x, 0) = a \sin\frac{\pi x}{L}\ ;\ 0 \le x \le L$

Step II : The most suitable solution is

$$u(x,t) = [C_1 \cos mx + C_2 \sin mx] \cdot [C_3 \cos mct + C_4 \sin mct] \qquad ...(1)$$

Step III : By condition (i) ; $u = 0$ when $x = 0$

put $x = 0$ in Equation (1)

Equation (1) becomes,

$0 = (C_1 + 0)(C_3 \cos mct + C_4 \sin mct)$

$\Rightarrow C_1 = 0$ only

($\because$ If we take bracketed term is equal to zero, then displacement vanishes i.e. we get Trivial solution)

Equation (1) becomes,

$$u(x,t) = C_2 \sin mx\,[C_3 \cos mct + C_4 \sin mct] \qquad ...(2)$$

Step IV : For condition (iii), differentiate Equation (2) partially w. r. t. t, keeping x as constant

$$\frac{\partial u}{\partial t} = C_2 \sin mx\,[-mc\,C_3 \sin mct + mc\,C_4 \cos mct]$$

Now, by condition (iii), $\frac{\partial u}{\partial t} = 0$ when $t = 0$

$\therefore$ put, $t = 0$ in equation $\frac{\partial u}{\partial t}$

$\therefore\ 0 = C_2 \sin mx\,[0 + mc\,C_4]$

$\Rightarrow C_4 = 0$

($\because$ if $C_2 = 0$, the Equation (2) completely becomes zero)

$\therefore$ Equation (2) becomes

$$u(x,t) = C_2 C_3 \sin mx \cos mct \qquad ...(3)$$

Step V :

By condition (ii), $u = 0$ when $x = L$

put $x = L$ in Equation (3)

$0 = C_2 C_3 \sin mL \cos mct$

$\therefore\ C_2 C_3 = 0$ OR $\sin mL = 0$ OR $\cos mct = 0$

$\Rightarrow$ only $\sin mL = 0$

($\because$ If we take $C_2 C_3 = 0$ or $\cos mct = 0$, then we get Trivial solution)

$mL = n\pi\ ;\ n = 0, 1, 2, 3, ...$

$$\Rightarrow m = \frac{n\pi}{L}\ ;\ n = 0, 1, 2, 3, ...$$

Put in Equation (3),

$$\therefore\ u(x,t) = C_2 C_3 \sin\left(\frac{n\pi}{L}\right)x \cdot \cos\left(\frac{n\pi}{L}c\right)t\ ;$$

where, $n = 0, 1, 2, 3, ...$

put $C_2 C_3 = b_n$ and sum of all terms

$$\Rightarrow u(x,t) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x \cdot \cos\left(\frac{n\pi}{L}c\right)t \qquad ...(4)$$

Step VI :

Now, by condition (iv),

$$u(x,t) = a \sin\frac{\pi}{L}x\ ;\ \text{For } t = 0 \text{ in } 0 \le x \le L$$

put $t = 0$ in Equation (4),

$$a \sin\frac{\pi}{L}x = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x\ ;\ \text{in } 0 \le x \le L \qquad ...(5)$$

Which is half range sine series in $0 \le x \le L$

(Here on both sides, sine terms are present. So not require to solve by the half range formula. We can solve by comparing both sides).

From Equation (5),

$$+ .. + 0 \sin\frac{3\pi}{L} + 0 \sin\frac{2\pi}{L} + a \sin\frac{\pi}{L}x = b_1 \sin\frac{\pi}{L}x + b_2 \sin\frac{2\pi}{L}x + b_3 \sin\frac{3\pi}{L}x + ...$$

Comparing both sides, we get

$\Rightarrow\ b_1 = a\ ;\quad b_2 = 0, b_3 = 0, b_4 = 0, ;.....$

Step VII :

Equation (4) becomes

$$u(x,t) = b_1 \sin\left(\frac{\pi}{L}\right)x \cdot \cos\left(\frac{\pi}{L}c\right)t + b_2 \sin\left(\frac{2\pi}{L}\right)x \cdot \cos\left(\frac{2\pi}{L}c\right)t + b_3 \sin\left(\frac{3\pi}{L}\right)x \cdot \cos\left(\frac{3\pi}{L}c\right)t + ...$$

$$u(x,t) = a \sin\left(\frac{\pi}{L}\right)x \cdot \cos\left(\frac{\pi}{L}c\right)t + 0 + 0 + ...$$

$$\Rightarrow u(x,t) = a \sin\left(\frac{\pi}{L}\right)x \cdot \cos\left(\frac{\pi}{L}c\right)t \checkmark \qquad \textbf{...Ans.}$$

Which is the required displacement of the string at any time t and at any distance x.

Syllabus Topic : One Dimensional Heat Flow Equations

4.8 One Dimensional Heat Flow Equation

Let us consider a thin rod or wire of length L, with constant density ρ, and constant cross-sectional area A, is coincide with X-axis from 0 to L

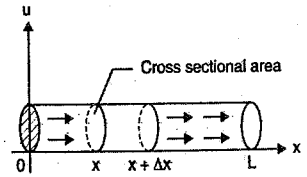


Fig. 4.8.1

Let us suppose that

- The flow of heat within the rod in the x-direction only
- The latered, or curved, surface of the rod is insulated. So heat cannot radiate from these surfaces.

- No heat is being generated within the rod.
- The rod is homogeneous, that is, its mass per unit volume (density) ρ – is the constant.
- μ - specific heat and k-thermal conductivity of the material of the rod are constants.
- The temperature u (x, t) on the cross-section of the rod is perpendicular to the x-axis.

Since,

(i) The quantity of heat a in a element of mass 'm' is

$$Q = \mu m u \quad ...(1)$$

and

(ii) The rate of heat flow $Q_1 \left(=\frac{\partial Q}{\partial t}\right)$ through the cross section show in figure is proportional to the cross-sectional areal A and the temperature gradient $\left(\frac{\partial u}{\partial x}\right)$

$$\frac{\partial Q}{\partial t} \propto A\frac{\partial u}{\partial x}$$

i.e. $$\frac{\partial Q}{\partial t} = -kA\frac{\partial u}{\partial x} \quad ...(2)$$

minus sign indicates, heat flows in the direction of decreasing temperature. If the circular region of the rod between x and x + Δx is very small, then u(x, t) be the approximate temperature at each point in the interval.

Since, mass of the circular element is $m = \rho A\,\Delta x$

$\therefore$ from Equation (1)

$$Q = \rho u A \Delta x\, u \quad ...(3)$$

When the heat flows in the positive x-direction, then the net rate of heat flows in the cross sectional area is

$$\frac{\partial Q}{\partial t} = \left[-kA\frac{\partial u(x,t)}{\partial x}\right] - \left[-kA\frac{\partial(x+\Delta x,t)}{\partial x}\right]$$

i.e.

$$\frac{\partial Q}{\partial t} = kA\left[\frac{\partial[x+\Delta x,t]}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right] \quad ...(4)$$

Differentiate Equation (3) w.r.t. t, then the net rate

$$\frac{\partial Q}{\partial t} = (\mu\rho\, A\,\Delta x)\cdot\frac{\partial u}{\partial t} \quad ...(5)$$

Now, from Equation (4) and (5) we get

$$(\mu\rho\, A\,\Delta x)\cdot\frac{\partial u}{\partial t} = kA\left[\frac{\partial u(x+\Delta x,t)}{\partial x} - \frac{\partial u(x\,t)}{\partial x}\right]$$

i.e. $$\frac{\partial u}{\partial t} = \frac{k}{\mu\rho}\left[\frac{\frac{\partial u(x+\Delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}}{\Delta x}\right]$$

At the limiting econdition $\Delta_x \to 0$

$$\frac{\partial u}{\partial t} = \frac{k}{\mu\rho}\frac{\partial^2 u}{\partial x^2}$$

Let, $C^2 = \frac{k}{\mu\rho}$ is positive constant called thermal diffusivity.

$\therefore$ The differential equation of one dimensional heat equation is

$$\frac{\partial u}{\partial t} = C^2\frac{\partial^2 u}{\partial x^2} \quad ...(6)$$

Where, u (x, t) – be the temperature in the rod or wire at any distance x and at any time t.

For unique solution from Equation (6) we need some initial and boundary condition. Specifying the information at the ends of the rod at all time and temperature through out the rod at time t = 0

$\therefore$ For one-dimentional heat flow partial differential Equation (6), boundary and initial conditions are

(I) (i) $u(0, t) = T_1$ (ii) $u(L, t) = T_2$; For all t

(iii) $u(x, 0) = f(x)$, for $0 \le x \le L$

Where, T_1, T_2 are constants. (zero or non zero)

(II) Some time, if there is no heat at the ends of rod / bar then we use

$$\frac{\partial u}{\partial x}(0, t) = 0\ ;\forall t$$

and

$$\frac{\partial u}{\partial x}(L, t) = 0\ ;\forall t$$

(Here, ends are insulated)

(III) If the left end of rod/bar is at constant temperature and right end is insulated then

$$u(0, t) = T\ ;\forall t \text{ and}$$

$$\frac{\partial u}{\partial x}(L, t) = 0\ ;\forall t$$

4.8.1 Solution of One Dimensional Heat Equation

Since, the one dimentional heat flow equation is

$$\frac{\partial u}{\partial t} = C^2\frac{\partial^2 u}{\partial x^2} \quad ...(1)$$

Let $u(x, t) = X(x)\,T(t)$ ...(2)

be the solution,

Here, X(x) – be a function of 'x' only.

T(t) – is a function of t only.

i.e. X and T are independent to each other.

Differentiate Equation (2) w.r. t. x two times and w.r.t one time 't' and put these values in Equation (1)

We get,

$$X(x)\,T'(t) = C^2 X''(x)\,T(t)$$

$$\Rightarrow \frac{X''}{X} = \frac{T'}{C^2 T} = \lambda \text{ (say)}$$

Where, λ - is any proportionality constant.

$$\Rightarrow \frac{X''}{X} = \lambda \text{ and } \frac{T'}{C^2 T} = \lambda$$

i.e. $X'' - \lambda X = 0$

and $T' - \lambda C^2 T = 0$

We get two ordinary differential equations

$$\Rightarrow \frac{d^2X}{dx^2} - \lambda X = 0 \quad ...(3)$$

and

$$\frac{dT}{dt} - \lambda C^2 T = 0 \quad ...(4)$$

(i) If λ is positive and $\lambda = p^2$ (say). Differential Equations (3), and (4) becomes

$$\frac{d^2X}{dx^2} + p^2 X = 0 \text{ and } \frac{dT}{dt} + p^2C^2T = 0$$

A.E. are : $D^2 - P^2 = 0$ and $D' + P^2C^2 = 0$

$\Rightarrow$ D = IP and $D' = P^2C^2$

$\therefore$ Solution (C.F.) is

$$X = C_1e^{px} + C_2e^{-px} \text{ and } T = C_3e^{p^2c^2t}$$

(ii) It λ is negative and $\lambda = -P^2$ (say). Differential Equations (3) and (4) becomes $\frac{d^2X}{dx^2} + P^2X = 0$ and $\frac{dT}{dt} + P^2C^2T = 0$

A.E. are

$D^2 + P^2 = 0$ and $D' + P^2C^2 = 0$

$\Rightarrow$ D = $\pm$P and $D' = -P^2C^2$

Solution (C.F.) is

$$X = C_1\cos Px + C_2\sin Px \text{ and } T = e^{-P^2C^2t}$$

(iii) If λ is zero i.e. $\lambda = 0$

Equation (3) and (4) becomes

$$\frac{d^2X}{dx^2} = 0 \quad \text{and} \quad \frac{dT}{dt} = 0$$

A.E are $D^2 = 0$ and $D' = 0$

$\Rightarrow$ D = 0, 0 and $D' = 0$

Solution (C.F.) is

$$X = (C_1x + C_2) \text{ and } T = C_3$$

The various possible solutions of the heat Equation (1) are,

Equation (2) becomes.

$$u(x, t) = (C_1e^{px} + C_2e^{-px})\cdot C_3e^{c^2p^2t} \quad ...(5)$$

$$u(x, t) = (C_1\cos px + C_2\sin px)\cdot C_3e^{-c^2p^2t} \quad ...(6)$$

$$u(x, t) = (C_1x + C_2)\,C_3 \quad ...(7)$$

Now by boundary and initial conditions

Let the ends of the bar/rod are kept at temperature zero

i.e. $u(0, t) = 0;\ \forall t$ and

$u(L, t) = 0;\ \forall t$

$u(x, 0) = f(x)$ in $0 \le x \le L$

By using boundary conditions, from (5) and (7), we get Trivial solutions (zero solutions) for any value of t > 0. It is not possible. Hence, the most suitable solution which satisfies the boundary and initial conditions, is Equation (6)

i.e. $$u(x, t) = (C_1\cos px + C_2\sin px)\,C_3e^{-c^2p^2t}$$

(Also, by the fact that, as time increases, temperature decreases to zero it is possible only with the solution (6))

[Now apply the physical conditions to solution (6)

As u(0, t) = 0, Equation (6) becomes

$$0 = (C_1 + 0)\,C_3\ e^{-c^2p^2t}$$

$C_1 = 0$ only

(if $C_3 = 0$ or $e^{-c^2p^2t} = 0$, we get trival solution, it is not true)

Equation (6) becomes,

$$u(x, t) = C_2(\sin p\,x)\,C_3e^{-c^2p^2t} \quad ...(8)$$

By condition $u(L, t) = 0\ ;\ \forall t$

$$0 = (C_2\sin P\,L)\,C_3e^{-c^2p^2t}$$

$\Rightarrow C_2 = 0$ or $\sin pL = 0$ or $C_3 = 0$

Or $e^{-c^2p^2t} = 0$

It is true only sin pL = 0

(for others, we get zero solution)

$\therefore \sin pL = 0 \Rightarrow pL = n\pi\ ;\ n = 1, 2, 3 ...$

$$\Rightarrow p = \frac{n\pi}{L}\ ;\ n = 1, 2, 3, ...$$

Equation (8) becomes

$$u(x, t) = C_2C_3\sin\left(\frac{n\pi}{L}\right)x\cdot e^{-c^2\left(\frac{n\pi}{L}\right)^2t}$$

where, n = 1, 2, 3, ...

Let $C_3, C_3 = b_n$ and

Using superposition principle (Adding all solutions), we get

$$u(x, t) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi}{L}\right)x\cdot e^{-c^2\left(\frac{n\pi}{L}\right)^2t} \quad ...(9)$$

Now, by using initial condition

$u(x, 0) = f(x)$ in $0 \le x \le L$, we get

$$f(x) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi}{L}\right)x \text{ in } 0 \le x \le L$$

Which is half range sine series in (0, L)

$$\text{Where, } b_n = \frac{2}{L}\int_0^L f(x)\cdot\sin\left(\frac{n\pi}{L}\right)x\,dx$$

Substitute the values of b_n in Equation (9), we get the required temperatures at any distance x and at any time t.]

By considering the physical nature of the problem of one dimensional heat equation the temperature u(x,t) must be decreases as increase of time t. Therefore the most suitable solution of one dimensional heat flow is

$$u(x,t) = (C_1\cos mx + C_2\sin mx)\,e^{-c^2m^2t}$$

To solve one dimensional heat flow equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \text{ we use}$$

(i) u(x,t) is finite i.e. $u \to 0$ as $t \to \infty$

(ii) Insulated boundary or end means $\frac{\partial u}{\partial x} = 0$ at that end

(iii) $u(0, t) = 0$
(iv) $u(L, t) = 0$ } if conditions are not mentioned in the problems.

4.8.1.1 Procedure to Solve One Dimensional Heat Flow Equation

Step I : Write the given initial conditions.

Step II : Write the most suitable solution of one dimensional heat flow.

Step III : Apply condition (ii) as u = 0, when x = 0, we get $C_1 = 0$

Step IV : Apply condition (iii) as x = l, u = 0, we get the value of m.

Step V : Put the value of m in equation which we get from step – III

Step VI : Apply condition (iv) as t = 0, u = f(x) and then use half range Fourier sine series (if required, to find b_n.)

Step VII: Find b_n and put this value of b_n in equation which we get from step – V, we get required temperature.

Note : Steady state condition - It means the temperature u depends only upon x not on time t.

Ex. 4.8.1 : Find the solution of the equation $\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2}$

Such that (i) $V \neq \infty$ when $t \to \infty$.

(ii) $\frac{\partial V}{\partial x} = 0$ when $x = 0$; $\forall t$

(iii) $V = 0$ when $x = l$; $\forall t$

(iv) $V = V_0$

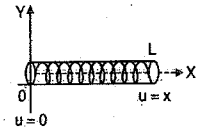
when t = 0 for all values of x between 0 to l.

Soln. :

Step I :

Let V(x, t) - be the temperature at any time t.

The most suitable solution of the given heat equation is



$$V(x, t) = [C_1 \cos mx + C_2 \sin mx]\, e^{-km^2 t} \quad \ldots(1)$$

Step II : Given conditions are,

(i) V is finite; $\forall t$

(ii) $\frac{\partial V}{\partial x} = 0$; when x = 0 ; $\forall t$

(iii) $V(l, t) = 0$; $\forall t$

(iv) $V = V_0$ when t = 0, $\forall x ; 0 \le x \le l$

By condition (i) : as $t \to \infty$, $V \to 0$; i.e. V is finite, $\forall$ t.

Step III :

For condition (ii)

Differentiate Equation (1) partially w.r.t x, keeping t as constant

$$\frac{\partial V}{\partial x} = [-m C_1 \sin mx + m C_2 \cos mx]\, e^{-km^2 t}$$

But by condition (ii), $\frac{\partial V}{\partial x} = 0$, when x = 0

put x = 0 in $\frac{\partial V}{\partial x}$ equation,

$$\therefore\ 0 = (0 + m C_2)\, e^{-km^2 t} = 0$$

$\Rightarrow m C_2 = 0$ or $e^{-km^2 t} = 0$

($\because$ If we equate $e^{-Km^2 t}$ with zero, then temperature become zero)

So $e^{-km^2 t} \neq 0$. Hence $C_2 = 0$

Equation (1) becomes

$$V(x, t) = C_1 \cos mx\, e^{-m^2 kt} \quad \ldots(2)$$

Step IV :

By condition (iii) $V(l, t) = 0$. Put x = l in Equation (2),

$$\Rightarrow 0 = C_1 \cos ml\, e^{-m^2 kt}$$

$\Rightarrow C_1 = 0$ **OR** $\cos ml = 0$ **OR** $e^{-km^2 t} = 0$

But if $C_1 = 0$ and $e^{-km^2 t} = 0$,

then Equation (2) becomes zero.

Therefore, only $\cos ml = 0$

$\Rightarrow ml = \frac{n\pi}{2}$; where n is an odd integer

$\left(\because \cos\frac{n\pi}{2} = 0 \text{ if n is an odd integer}\right)$

$$\Rightarrow m = \frac{n\pi}{2l} \quad \Rightarrow m = \frac{(2n+1)}{l}\frac{\pi}{2} ; n = 0, 1, 2, 3, \ldots.$$

Equation (2) becomes

$$V(x, t) = C_1 \cos\left(\frac{2n+1}{l}\cdot\frac{\pi}{2}\right) x \cdot e^{-\left[\frac{2n+1}{l}\right]^2 \cdot \frac{\pi^2}{4} kt}$$

where, n = 0, 1, 2, 3 ....

put, $C_1 = a_n$ and sum of all terms

$$V(x, t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{2n+1}{l}\cdot\frac{\pi}{2}\right) x \cdot e^{-\frac{(2n+1)^2}{l^2}\cdot\frac{\pi^2}{4} kt} \quad \ldots(3)$$

Step V : By condition (iv), as t = 0, $V = V_0$, $0 < x < l$.

$$\therefore\ V_0 = \sum_{n=0}^{\infty} a_n \cos\left(\frac{2n+1}{l}\cdot\frac{\pi}{2}\right) x \quad 0 < x < l$$

Which is half range cosine series, like

$$f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right) x.$$

Step VI :

$$\text{Where, } a_n = \frac{2}{l}\int_0^l f(x)\cdot \cos\frac{n\pi}{L} x\, dx$$

$$a_n = \frac{2}{l}\int_0^l V_0 \cos\left(\frac{2n+1}{l}\cdot\frac{\pi}{2} x\right) dx$$

$$= \frac{2V_0}{l}\left[\frac{\sin\left(\frac{2n+1}{l}\right)\cdot\frac{\pi}{2}x}{\left(\frac{2n+1}{l}\right)\cdot\frac{\pi}{2}}\right]_0^l$$

$$= \left[\frac{4V_0}{\pi}\frac{1}{(2n+1)}\cdot\sin(2n+1)\cdot\frac{\pi}{2} - 0\right]$$

$$= \frac{4V_0}{\pi}\frac{1\cdot(-1)^n}{(2n+1)} \quad \text{put in Equation (3)}$$

$$\left(\begin{aligned} &\because \sin(2n+1)\frac{\pi}{2} = \sin\left(n\pi + \frac{\pi}{2}\right) \\ &= \sin n\pi \cdot \cos\frac{\pi}{2} + \cos n\pi \cdot \sin\frac{\pi}{2} \\ &= 0 + (-1)^n \cdot 1 = (-1)^n \end{aligned}\right)$$

$$V(x, t) = \frac{4V_0}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\cos\frac{(2n+1)\pi x}{2l}\, e^{\frac{-(2n+1)^2 n^2 kt}{4l^2}}$$

✓ ...Ans.

Which is the required temperature of the rod at any distance x and at any time t.

Ex. 4.8.2 : Solve : $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$

If (i) u is finite; $\forall t$

(ii) $u(0, t) = 0$; $\forall t$

(iii) $u(L, t) = 0$; $\forall t$

(iv) $u(x, 0) = u_0$, $0 \le x \le L$.

Where, L being a length of a bar.

Soln. :

Step I : Let u(x, t) – be the temperature at any time t.

The most suitable solution of one dimensional heat flow equation is

$$u(x, t) = [C_1 \cos mx + C_2 \sin mx]\, e^{-c^2 m^2 t} \quad \ldots(1)$$

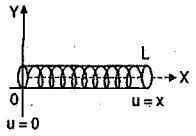


Fig. P. 4.8.2

By condition (i) as $t \to \infty$; $u \to 0$,

hence, u is finite.

Step II :

By condition (ii), u = 0 as x = 0

$$0 = (C_1 + 0)\, e^{-c^2 m^2 t} \Rightarrow C_1 = 0$$

($\because$ If we equate the exponential term to zero them from Equation (1) ,u – becomes zero)

Equation (1) becomes

$$u(x, t) = C_2 \sin mx\, e^{-c^2 m^2 t} \quad \ldots(2)$$

Step III :

By condition (iii), u = 0 as x = L

Put x = L in Equation (2),

$$\therefore\ 0 = C_2 \sin mL\, e^{-m^2 c^2 t}$$

$\Rightarrow C_2 \neq 0$, $e^{-m^2 c^2 t} \neq 0$; Only $\sin mL = 0$

$\Rightarrow mL = n\pi$; n = 0, 1, 2, 3, ...

$$\Rightarrow m = \frac{n\pi}{L} \quad ; n = 1, 2, 3, \ldots$$

Equation (2) becomes,

$$u(x, t) = C_2 \sin\left(\frac{n\pi}{L}\right) x \cdot e^{-\left(\frac{n\pi}{L}\right)^2 \cdot c^2 t} ; n = 1, 2, 3, \ldots.$$

put $C_2 = b_n$ and sum of all terms

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right) x \cdot e^{-\left(\frac{n\pi}{L}\right)^2 \cdot c^2 t} \quad \ldots(3)$$

Step IV :

By condition (iv), $u = u_0$; for t = 0 in $0 \le x \le L$

$$\therefore\ u_0 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right) x \quad \text{in} \quad 0 \le x \le L$$

Which is half range sine series like,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L} x$$

Step V :

Where,

$$\therefore\ b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L} x\, dx = \frac{2}{L}\int_0^L u_0 \sin\frac{n\pi}{L} x\, dx$$

$$= \frac{2}{L} u_0 \left(\frac{-\cos\frac{n\pi}{L} x}{\frac{n\pi}{L}}\right)_0^L$$

$$= \frac{-2}{L} u_0 \left(\frac{L}{n\pi}\right)[\cos n\pi - 1]$$

($\because \cos n\pi = (-1)^n$; $\cos 0 = 1$)

$$b_n = \frac{2u_0}{n\pi}[1 - (-1)^n]$$

Equation (3) becomes,

$$u(x,t) = \frac{2u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} [1-(-1)^n] \sin\left(\frac{n\pi}{L}\right) x \cdot e^{-\left(\frac{n\pi}{L}\right)^2 \cdot c^2 t}$$

✓ ...Ans.

is the required temperature in the rod at any distance x and at any time t.

Ex. 4.8.3 : The temperature at any point of a insulated metal rod of one meter length is governed by the differential equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$.

Find u(x, t) subject to the following conditions:

(i) $u(0,t) = 0°C;\ \forall t$

(ii) $u(1,t) = 0°C;\ \forall t$

(iii) $u(x, 0) = 50°$ C and hence find the temperature in the middle of the rod at any subsequent time.

Soln. :

Step I : Let, u(x, t) - be the temperature at any time t.

The most suitable solution of one dimension heat flow equation is

$$u(x,t) = [C_1 \cos mx + C_2 \sin mx]\, e^{-m^2c^2t} \quad ...(1)$$

Step II : Given conditions are

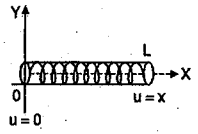


Fig. P. 4.8.3

(i) $u(0,t) = 0;\ \forall t$

(ii) $u(L,t) = 0;\ \forall t$

(iii) $u(x,0) = 50;$ in $0 \le x \le 1$

Step III :

By condition (i), u = 0, when x = 0

Put, x = 0 in Equation (1)

$$0 = (C_1 + 0)\, e^{-m^2c^2t} \Rightarrow C_1 = 0$$

(∵ If we equate the exponential term to zero, then temperature is zero/Trivial solution)

Equation (1) becomes

$$u(x,t) = C_2 \sin mx\, e^{-m^2c^2t} \quad ...(2)$$

Step IV :

By condition (ii) u = 0 when x = 1

put x = 1 in Equation (2)

$$\therefore\ 0 = C_2 \sin m1\, e^{-m^2c^2t}$$

$C_2 \ne 0;\ e^{-m^2c^2t} \ne 0$; **Only sin m1 = 0**

$\Rightarrow m1 = n\pi$; $n = 1, 2, 3, \ldots$

$$\Rightarrow m = \frac{n\pi}{1} \quad ;\ n = 1, 2, 3, \ldots$$

Equation (2) becomes

$$u(x,t) = C_2 \sin n\pi\, x \cdot e^{(n\pi)^2 \cdot c^2 t} \quad ;\ n = 1, 2, 3, \ldots$$

put $c_2 = b_n$ and sum, all the terms

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi)\, x \cdot e^{(n\pi)^2 \cdot c^2 t} \quad ...(3)$$

Step V :

By condition (iii),

$u = 50;$ for $0 \le x \le 1$ at t = 0

put, t = 0 in Equation (3)

$$\therefore\ 50 = \sum_{n=1}^{\infty} b_n \sin(nt)\, x; \quad \text{in } 0 \le x \le 1$$

Which is half range sine series in $0 \le x \le 1$

$$\text{It is like, } f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right) x$$

Step VI :

Where,

$$\therefore\ b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}\right) x\, dx$$

$$= \frac{2}{1}\int_0^1 50 \sin(n\pi)x\, dx \quad (\because L = 1)$$

$$= 100\left[\frac{-\cos(n\pi)\, x}{(n\pi)}\right]_0^1 = -100\frac{1}{n\pi}[\cos n\pi - 1]$$

$$b_n = 100\frac{1}{n\pi}[1-(-1)^n]$$

$(\because \cos(n\pi) = (-1)^n;\ \cos 0 = 1)$

Step VII :

Equation (3) becomes

$$u(x,t) = \frac{100}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}[1-(-1)^n] \cdot \sin(n\pi)\, x \cdot e^{-(n\pi)^2 \cdot c^2 t} \quad ...(4)$$

which is the temperature at any time t and at any distance x.

Now, when $x = \frac{1}{2}$; (at the middle)

$$\therefore\ u(x,t) = \frac{100}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}[1-(-1)^n] \cdot \sin\left(\frac{n\pi}{2}\right) e^{-(n\pi)^2 \cdot c^2 t}$$

✓ ...Ans.

is the required temperature in the middle of the rod at any subsequent time.

Ex. 4.8.4 : Solve $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$; where u (x, t) satisfies the following conditions.

(i) $u(0,t) = 0;\ \forall t$

(ii) $u(L,t) = 0;\ \forall t$

(iii) $u(x,0) = x;$ for $0 < x < L$

(iv) $u(x, \infty)$ is finite, $\forall x$

Soln. :

Step I : Let, u(x, t) - be the temperature at any time t.

Here, given $c^2 = a^2 \Rightarrow c = a$

The most suitable solution of one dimensional heat flow equation is,

$$u(x,t) = [C_1 \cos mx + C_2 \sin mx]\, e^{-m^2a^2t} \quad ...(1)$$

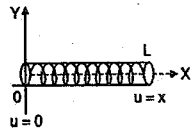


Fig. P. 4.8.4

Given conditions are,

(i) $u(0,t) = 0;\ \forall t$

(ii) $u(L,t) = 0;\ \forall t$

(iii) $u(x,0) = 50;$ in $0 \le x \le L$

Step II :

By condition (i), u = 0, when x = 0

Put, x = 0 in Equation (1)

$$0 = (C_1 + 0)\, e^{-m^2a^2t} \Rightarrow C_1 = 0$$

(∴ By putting $e^{m^2a^2t} = 0$, Equation (1) becomes zero, it is not possible everytime)

Equation (1) becomes

$$u(x,t) = C_2 \sin mx\, e^{-m^2a^2t} \quad ...(2)$$

Step III :

By condition (ii) as u = 0 when x = L

Put, x = L in Equation (2),

$$\therefore\ 0 = C_2 \sin mL\, e^{-m^2a^2t}$$

$\Rightarrow C_2 \ne 0$ and $e^{-m^2a^2t} \ne 0$;

Only sin mL = 0

$\Rightarrow mL = n\pi$; $n = 1, 2, 3, \ldots$

$\Rightarrow m = \frac{n\pi}{L}$; $n = 1, 2, 3, \ldots$

Equation (2) becomes

$$u(x,t) = C_2 \sin\left(\frac{n\pi}{L}\right) x \cdot e^{\left(\frac{n\pi}{L}\right)^2 \cdot a^2 t} \quad ;\ n = 1, 2, 3, \ldots$$

Put $C_2 = b_n$ and sum of all the terms

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right) x \cdot e^{\left(\frac{n\pi}{L}\right)^2 \cdot a^2 t} \quad ...(3)$$

Step IV :

By condition (iii) u = x at t = 0 in $0 \le x \le L$

Put, t = 0 in Equation (3)

∴ From Equation (3),

$$x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right) x \ ;\ 0 \le x \le L$$

which is half range sine series in $0 \le x \le L$,

$$\text{It is like } f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right) x \quad \text{in } 0 \le x \le L$$

Step V :

Where,

$$\therefore\ b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L} x\, dx = \frac{2}{L}\int_0^L x \sin\frac{n\pi}{L} x\, dx$$

➔ **Using standard result**

$$...\left[\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right]$$

$$= \frac{2}{L}\left\{(x)\left(\frac{-\cos\left(\frac{n\pi}{L}\right)x}{\left(\frac{n\pi}{L}\right)}\right) - (1)\left(\frac{-\sin\left(\frac{n\pi}{L}\right)x}{\left(\frac{n\pi}{L}\right)^2}\right)\right\}_0^L$$

$$= \frac{2}{L}\left[-\frac{L}{n\pi}(L) \cdot \cos n\pi + 0 + 0 - 0\right] \quad (\because \sin n\pi = 0)$$

$$b_n = \frac{-2L}{n\pi}(-1)^n \quad (\because \cos n\pi = (-1)^n)$$

Equation (3) becomes

$$u(x,t) = \frac{-2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\frac{n\pi}{L} x\, e^{-\left(\frac{n\pi}{L}\right)^2 \cdot a^2 t}$$

✓ ...Ans.

is the required temperature at any time t and at distance x.

Ex. 4.8.5 : A homogeneous rod of conducting material of length 100 cm has its ends are kept at zero temperature and the temperature initially is,

$$u(x,0) = \begin{cases} x & ;\ 0 \le x \le 50 \\ 100 - x & ;\ 50 \le x \le 100 \end{cases}$$

Find the temperature u (x, t) at any time.

Soln. :

Step I : Let, u(x, t) - be the temperature at any time t.

The partial differential equation of one dimensional heat flow is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Given conditions are,

(i) $u(0, t) = 0 ; \forall t$

(ii) $u(100, t) = 0 ; \forall t$

(iii) $u(x, 0) = \begin{cases} x & ; 0 \le x \le 50 \\ 100 - x & ; 50 \le x \le 100 \end{cases}$

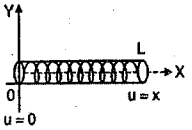


Fig. P. 4.8.5

The most suitable solution is

$$u(x, t) = [C_1 \cos mx + C_2 \sin mx]\, e^{-m^2c^2t} \quad \ldots(1)$$

Step II :

By condition (i), $u = 0$, $x = 0$

Put, $x = 0$ in Equation (1)

$$\therefore 0 = (C_1 + 0)\, e^{-m^2c^2t} \Rightarrow C_1 = 0$$

($\because$ If we equate the exponential term to zero, then from Equation (1), u vanishes)

Equation (1) becomes

$$u(x, t) = C_2 \sin mx\, e^{-m^2c^2t} \quad \ldots(2)$$

Step III :

By condition (ii); $u = 0$ as $x = 100$

Put, $x = 100$ in Equation (2),

$$\therefore 0 = C_2 \sin m\,100\, e^{-m^2c^2t}$$

But $C_2 \ne 0$ OR $e^{-m^2c^2t} \ne 0$;

Only $\sin m100 = 0$

$$\Rightarrow m100 = n\pi$$

$$m = \frac{n\pi}{100} \; ; \; n = 1, 2, 3, \ldots.$$

Equation (2) becomes,

$$u(x, t) = C_2 \sin \frac{n\pi}{100} x \cdot e^{-\left(\frac{n\pi}{L}\right)^2 \cdot c^2 t} \; ; \; n = 1, 2, 3, \ldots$$

Put $c_2 = b_n$ and take sum of all terms

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{100}\right) \cdot e^{-\left(\frac{n\pi}{100}\right)^2 \cdot c^2 t} \quad \ldots(3)$$

Step IV :

By condition (iii), at $t = 0$,

$$u = \begin{cases} x ; & 0 \le x \le 50 \\ 100 - x ; & 50 \le x \le 100 \end{cases}$$

Put, $t = 0$ in Equation (3)

$$\therefore \left.\begin{matrix} x ; & 0 \le x \le 50 \\ 100 - x ; & 50 \le x \le 100 \end{matrix}\right\} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{100}\right) x$$

Which is half range sine series in $0 \le x \le 100$ like

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right) x \text{ in } 0 \le x \le L$$

Step V :

Where,

$$\therefore b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{100} x\, dx = \frac{2}{100}\int_0^{100} f(x) \sin\left(\frac{n\pi}{100}\right) x\, dx$$

$$\ldots(\because \text{ Here, } L = 100)$$

$$= \frac{1}{50}\left[\int_0^{50} f(x) \sin\left(\frac{n\pi}{100}\right) x\, dx + \int_{50}^{100} f(x) \sin\left(\frac{n\pi}{100}\right) x\, dx\right]$$

$$= \frac{1}{50}\left[\int_0^{50} x \sin\left(\frac{n\pi}{100}\right) x\, dx + \int_{50}^{100} (100 - x) \cdot \sin\left(\frac{n\pi}{100}\right) x\, dx\right]$$

➔ **Using standard results**

$$\ldots\left[\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots\right]$$

$$b_n = \frac{1}{50}\left\{\left[(x)\left(\frac{-\cos\left(\frac{n\pi}{100}\right)x}{\left(\frac{n\pi}{100}\right)}\right) - (1)\left(\frac{-\sin\left(\frac{n\pi}{100}\right)x}{\left(\frac{n\pi}{100}\right)^2}\right)\right]_0^{50}\right.$$

$$\left. + \left[(100 - x)\left(\frac{-\cos\left(\frac{n\pi}{100}\right)x}{\left(\frac{n\pi}{100}\right)}\right) - (-1)\left(\frac{-\sin\left(\frac{n\pi}{100}\right)x}{\left(\frac{n\pi}{100}\right)^2}\right)\right]_{50}^{100}\right\}$$

$$b_n = \frac{1}{50}\left\{\left[-(50)\frac{n\pi}{2} \cdot \cancel{\cos\frac{n\pi}{2}} + \left(\frac{100}{n\pi}\right)^2 \sin\frac{n\pi}{2} - 0\right]\right.$$

$$\left. + [0 - 0] - \left[-(50)\cancel{\left(\frac{100}{n\pi}\right)} \cos\frac{n\pi}{2} - \left(\frac{100}{n\pi}\right)^2 \sin\frac{n\pi}{2}\right]\right\}$$

$$= \frac{1}{50}\left[2 \cdot \left(\frac{100}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right)\right] = \frac{10000}{25(n\pi)^2} \sin\frac{n\pi}{2}$$

$$b_n = \frac{400}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right)$$

Equation (3) becomes

$$u(x, t) = \sum_{n=1}^{\infty} \frac{400}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{100}\right) x \cdot e^{-\left(\frac{n\pi}{100}\right)^2 \cdot c^2 t}$$

✓ ...Ans.

is the required temperature at any time t and at any distance x.

Ex. 4.8.6 : Solve : $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$

if (i) $u(0, t) = 0$

(ii) $u(l, t) = 0$

(iii) $u(x, t)$ is bounded and

(iv) $u(x, 0) = \frac{u_0}{L} x$, for $0 \le x \le L$.

OR

A bar of length l with insulated sides is initially at a uniform temperature $\frac{u_0 x}{l}$. Both the ends of the bar are kept at zero temperature. Find the temperature of the bar at any time t. Use $\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$

Soln. :

Step I : Let, $u(x, t)$ - be the temperature at any time t,

The given condition are,

(i) $u(x, t)$ is bounded as $t \to \infty$

(ii) $u(0, t) = 0 ; \forall t$;

(iii) $u(L, t) = 0 ; \forall t$

(iv) $u(x, 0) = \frac{u_0}{L} x$ for $0 \le x \le L$

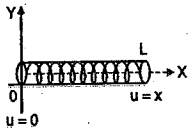


Fig. P. 4.8.6

The most suitable solution is

$$u(x, t) = [C_1 \cos mx + C_2 \sin mx]\, e^{-m^2kt} \quad \ldots(1)$$

$$\ldots(\because \text{ Here } c^2 = K)$$

Step II :

By condition (i) as $t \to \infty$; $u \to 0$

$\therefore$ u is bounded.

Step III :

By condition (ii), as $x = 0$, $u = 0$

Put, $x = 0$ in Equation (1), we get,

$$0 = [C_1 + 0]\, e^{-m^2kt} \Rightarrow C_1 = 0$$

($\because$ If we equate the exponential term to zero, from (1), we get zero solution)

$\therefore$ Equation (1) becomes

$$u(x, t) = C_2 \sin mx\, e^{-km^2t} \quad \ldots(2)$$

Step IV :

by condition (iii) as $x = L$; $u = 0$

put $x = L$ in Equation (2)

$$\Rightarrow 0 = C_2 \sin(ml)\, e^{-m^2kt}$$

$$\Rightarrow C_2 \ne 0 ;$$

only, $\sin ml = 0$

$$\Rightarrow ml = n\pi$$

$$\Rightarrow m = \frac{n\pi}{l} \; ; \; n = 0, 1, 2, 3, \ldots$$

Equation (2) becomes

$$u(x, t) = C_2 \sin\left(\frac{n\pi}{l}\right) x \cdot e^{-k\left(\frac{n\pi}{l}\right)^2 t} ; \; n = 1, 2, 3 \ldots\ldots$$

Put $c_2 = b_n$ and sum all the terms

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right) x \cdot e^{-k\left(\frac{n\pi}{l}\right)^2 t} \quad \ldots(3)$$

Step V :

By condition (iv) as $t = 0$; $u = \frac{u_0}{L} x$ in $0 \le x \le L$

Put, $t = 0$ in Equation (3)

$$\frac{u_0}{l} x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l} x ; \quad \text{for } 0 \le x \le L$$

Which is half range sine series,

$$\text{It is like, } f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L} x \text{ in } 0 \le x \le L$$

Step VI :

Where,

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L} x \cdot dx$$

$$= \frac{2}{l}\int_0^l \frac{u_0}{l} x \cdot \sin\left(\frac{n\pi}{l}\right) x\, dx$$

➔ **Using standard result**

$$\ldots\left[\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots\right]$$

$$= \frac{2u_0}{l^2}\left[(x)\left(\frac{-\cos\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)}\right) - (1)\left(\frac{-\sin\frac{n\pi}{l}x}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$b_n = \frac{2u_0}{l^2}\left[-(l)\left(\frac{l}{n\pi}\right)\cos(n\pi) - 0 - 0\right]$$

$$\ldots(\because \sin n\pi = 0; \sin 0 = 0; \cos n\pi = (-1)^n)$$

$$b_n = -\frac{2u_0}{n\pi}(-1)^n$$

Step VII :

Equation (3) becomes,

$$u(x, t) = -\frac{2u_0}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\left(\frac{n\pi}{l}\right) x \cdot e^{-k\left(\frac{n\pi}{l}\right)^2 t}$$

✓ ...Ans.

is required temperature at any time t and at any distance x.

Ex. 4.8.7 : A rod of length l with insulated sides is initially at a uniform temperature x. Both the ends of rod are kept at zero temperature. Find the temperature at any point and at any time t.

Soln. :

Step I : Let u(x, t) - be the temperature at any time t.

The differential equation of the conduction of heat of the rod is,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

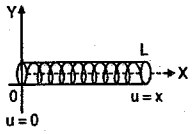


Fig. P. 4.8.7

The boundary conditions are

(i) $u(0, t) = 0 ; \forall t$

(ii) $u(l, t) = 0 ; \forall t$

(iii) $u(x, 0) = x ; 0 \le x \le l$

Step II :

The most suitable solution is,

$$u(x, t) = [C_1 \cos mx + C_2 \sin mx]\, e^{-m^2 c^2 t} \quad ...(1)$$

Step III :

By condition (i) as $x = 0, u = 0$

Put, $x = 0$ in Equation (1)

$$\therefore \quad 0 = [C_1 + 0]\, e^{-m^2 c^2 t}$$

$$\Rightarrow \quad C_1 = 0 \quad \text{only}$$

($\because$ If we equate the exponential term to zero, we get trivial solution. But is not true at any time and at any distance)

$\therefore$ Equation (1) becomes,

$$u(x, t) = C_2 \sin mx\; e^{-m^2 c^2 t} \quad ...(2)$$

Step IV :

By condition (ii) as $x = l,\ u = 0$

Put $x = l$ in Equation (2)

$$\therefore \quad 0 = C_2 \sin ml\; e^{-m^2 c^2 t}$$

$C_2 \ne 0, \quad e^{-m^2 c^2 t} \ne 0$

It is possible only if $\sin ml = 0$

$$\Rightarrow ml = n\pi \ ; \ n = 1, 2, 3, ...$$

$$\Rightarrow m = \frac{n\pi}{l} \ ; \ n = 1, 2, 3, ...$$

Equation (2) becomes,

$$u(x, t) = C_2 \sin\left(\frac{n\pi}{l}\right) x \cdot e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t} \ ; \ n = 1, 2, 3, ...$$

put, $C_2 = b_n$ and sum, all the terms

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right) x \cdot e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t} \quad ...(3)$$

Step V :

By condition (iii), as $t = 0, u = x$; in $0 \le x \le l$

Put, $t = 0$ in Equation (3)

$$\therefore \quad x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right) x \quad \text{in } 0 \le x \le l$$

which is half range sine series in $0 \le x \le l$

Step VI :

Where,

$$\therefore \quad b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}\right) x\, dx = \frac{2}{l}\int_0^l x \sin\left(\frac{n\pi}{l}\right) x\, dx$$

➔ **Using standard result**

$$...\left[\int uv\,dx = uv_1 - u' v_2 + u'' v_3 - u''' v_4 + ...\right]$$

$$= \frac{2}{l}\left[(x)\left(\frac{-\cos\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)}\right) - (1)\left(\frac{-\sin\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$= \frac{2}{l}\left[-(l)\left(\frac{l}{n\pi}\right)\cos n\pi + 0 - 0 - 0\right] = -\frac{2l}{n\pi}(-1)^n$$

$$...(\because \cos n\pi = 0; \cos n\pi = (-1)^n, \sin 0 = 0)$$

$$\therefore \quad b_n = \frac{2l}{\pi}\frac{(-1)^{n+1}}{n}$$

Put this value of b_n in Equation (3) we get,

$$u(x, t) = \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} \cdot \sin\left(\frac{n\pi}{l}\right) x\; e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t}$$

which is required temperature at any time t.

Ex. 4.8.8 : An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state condition prevail. If B is suddenly reduced to 0°C and maintained at 0°C. Find the temperature at a distance x from A at any time t.

Soln. :

Step I : Let u(x, t) - be the temperature at any distance x and at any time t. The differential equation of conduction of heat flow in one-dimension is,

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2} \quad ...(1)$$

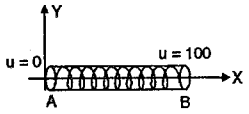


Fig. P. 4.8.8

Given, until steady state condition prevail

i.e. when $t = 0$, the heat function u is independent of time i.e. $\frac{\partial u}{\partial t} = 0$

$\therefore$ Differential Equation (1) becomes,

$$\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow (D^2)u = 0 \Rightarrow D = 0, 0$$

Here, $D \equiv \frac{\partial}{\partial x}$

Step II :

It's general solution is $u = C_1 x + C_2$ ...(2)

By given condition, when $\begin{cases} u = 0 \ ; & x = 0 \\ u = 100 \ ; & x = l \end{cases}$

Put $x = 0$ in Equation (2)

$$\therefore \quad 0 = 0 + C_2 \Rightarrow C_2 = 0$$

$\therefore \quad u = C_1 x$; Now $x = l, u = 100$

$$\Rightarrow 100 = C_1 \Rightarrow C_1 = \frac{100}{l}$$

Equation (2) becomes,

$$u = \frac{100}{l} x$$

Hence, the initial condition, $u(x, 0) = \frac{100}{l} x$

Now, Suddenly the end B reduces the temperature to 0.

Step III :

The boundary conditions are,

(i) $u(0, t) = 0 ; \forall t$

(ii) $u(l, t) = 0 ; \forall t$

(iii) $u(x, 0) = \frac{100x}{l}$ in $0 \le x \le l$

The most suitable solution of Equation (1) is,

$$u(x, t) = [C_1 \cos mx + C_2 \sin mx]\,[e^{-C^2 m^2 t}] \quad ...(3)$$

Step IV :

By condition (i), as $x = 0, u = 0$

Put, $x = 0$ in Equation (3)

$$\therefore \quad 0 = (C_1 + 0) \cdot e^{-C^2 m^2 t}$$

$$\Rightarrow \quad C_1 = 0$$

($\because$ If we equate exponential term to zero, we get zero solution. It is not possible always)

Equation (3) becomes,

$$u(x, t) = C_2 \sin mx\; e^{-C^2 m^2 t} \quad ...(4)$$

Step V :

By condition (ii) as $x = l,\ u = 0$

Put, $x = l$ in Equation (4)

$$\therefore \quad 0 = C_2 \sin ml \cdot e^{-C^2 m^2 t}$$

Here $C_2 \ne 0$; $e^{-C^2 m^2 t} \ne 0$

It is possible only if $\sin ml = 0$

$$\Rightarrow ml = n\pi \ ; \ n = 1, 2, 3, ...$$

$$\Rightarrow m = \frac{n\pi}{l} \ ; \ n = 1, 2, 3, ...$$

Equation (4) becomes,

$$u(x, t) = C_2 \sin\left(\frac{n\pi}{l}\right) x \cdot e^{-C^2\left(\frac{n\pi}{l}\right)^2 t} \ ; \ n = 1, 2, 3, ...$$

put, $C_2 = b_n$ and takes the summation of all terms

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right) x \cdot e^{-c^2\left(\frac{n\pi}{l}\right)^2 t} \quad ...(5)$$

Step VI :

By condition (iii), as $t = 0$, $u = \frac{100x}{l}$ in $0 \le x \le l$

Put, $t = 0$ in Equation (5)

$$\therefore \frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right) x \quad \text{in } 0 \le x \le l$$

Which is half range sine series in $0 \le x \le l$

It is like, $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right) x$; in $0 \le x \le L$

Step VII :

Here

$$\therefore \quad b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}\right) x\, dx$$

$$= \frac{2}{l}\int_0^l \frac{100x}{l} \cdot \sin\left(\frac{n\pi}{l}\right) x\, dx$$

➔ **Using standard results**

$$...\left[\int uv\,dx = uv_1 - u' v_2 + u'' v_3 - u''' v_4 + ...\right]$$

$$= \frac{200}{l^2}\left[(x)\left(\frac{-\cos\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)}\right) - (1)\left(\frac{-\sin\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$= \frac{200}{l^2}\left[-(l)\left(\frac{l}{n\pi}\right) \cdot \cos n\pi + 0 - 0 - 0\right]$$

$$= -\frac{200}{\pi}\frac{(-1)^n}{n} \quad ...(\because \sin n\pi = 0; \sin 0 = 0)$$

$$\therefore \quad b_n = \frac{200}{\pi}\frac{(-1)^{n+1}}{n}$$

Equation (5) becomes,

$$u(x, t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{l}\right) x \cdot e^{-c^2\left(\frac{n\pi}{l}\right)^2 t}$$

✓ ...Ans.

Which is the required temperature at a distance x form end A at any time t.

Ex. 4.8.9 : Solve : $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ if :

(i) $u(x, \infty)$ is finite

(ii) $u(0, t) = 0 \quad , \forall t$

(iii) $u(l, t) = 0 \quad , \forall t$

(iv) $u(x, 0) = 20x \ , 0 < x < L$

Soln. :

Step I :

Let, u (x, t) - be the temperature at any distance x and at any time t.

Given conditions are

(i) $u(x, \infty)$ is finite

(ii) $u(0, t) = 0, \quad \forall t$

(iii) $u(l, t) = 0, \quad \forall t$

(iv) $u(x, 0) = 20x \quad , 0 < x < l$

The most suitable solution is,

$$u(x, t) = [C_1 \cos mx + C_2 \sin mx] e^{-m^2 t} \qquad ...(1)$$

$$...(\because \text{ Here, } C^2 = 1)$$

Step II : By condition (i), u is finite for $t \to \infty$

$\therefore$ as $t \to \infty$, $u \to 0$, Hence, u is finite

Step III : By condition (ii), $u = 0$ as $x = 0$

Put, $x = 0$ in Equation (1)

$$0 = C_1 e^{-m^2 t} \Rightarrow \mathbf{C_1 = 0} \text{ only}$$

($\because$ If we equate exponential term to zero, we get trivial solution)

Equation (1) becomes

$$u(x, t) = C_2 \sin mx \cdot e^{-m^2 t} \qquad ...(2)$$

Step IV :

By condition (iii) $u = 0$ as $x = l$

Put, $x = l$ in Equation (2)

$$0 = C_2 \sin ml \cdot e^{-m^2 t}$$

$\Rightarrow C_2 \neq 0$ or $\sin ml = 0$ or $e^{-m^2 t} \neq 0$

But only $\sin ml = 0$

$\Rightarrow ml = n\pi \ ; n = 0, 1, 2, 3, ...$

$\Rightarrow m = \frac{n\pi}{l} \ ; n = 0, 1, 2, 3, ...$

Equation (2) becomes,

$$u(x, t) = C_2 \sin\left(\frac{n\pi}{l}\right) x e^{-\left(\frac{n\pi}{l}\right)^2 t} \ ; n = 0, 1, 2, 3 ....$$

Put, $C_2 = b_n$ and take sum of all the terms

$$u(x, t) = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi}{l} x \cdot e^{-\left(\frac{n\pi}{l}\right)^2 t} \qquad ...(3)$$

Step V :

Now, by condition (iv), as $t = 0$; $u = 20x$ in $0 < x < l$

$$\therefore 20x = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi}{l} x \ ; \quad \text{for } 0 < x < l$$

which is half range sine series.

$$\text{It is like } f(x) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right) x \text{ in } 0 \le x \le L$$

Step VI :

$$\therefore b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi}{l} x \, dx$$

$$= \frac{2}{l}\int_0^l 20x \cdot \sin \frac{n\pi}{l} x \, dx$$

➔ **Using standard result**

$$...\left[\int uv dx = uv_1 - u' v_1 + u'' v_2 - u''' v_3 + ,....\right]$$

$$= \frac{40}{l}\left[(x)\frac{\left(-\cos\left(\frac{n\pi}{l}\right)x\right)}{\left(\frac{n\pi}{l}\right)} - (1)\frac{\left(-\sin\left(\frac{n\pi}{l}\right)x\right)}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l$$

$$= \frac{40}{l}\left[-\frac{1}{n\pi}\cos(n\pi) + 0 + 0 + 0\right]$$

$$...(\because \sin n\pi = 0; \sin 0 = 0)$$

$$b_n = \frac{-40}{l\, n\pi}(-1)^n \ ; \qquad n = 0, 1, 2, 3, ...$$

$$...(\because \cos n\pi = (-1)^n)$$

Equation (3) becomes

$$u(x, t) = \frac{-40}{l\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cdot \sin\left(\frac{n\pi}{l}\right) x e^{-\left(\frac{n\pi}{l}\right)^2 t} \ \checkmark ...\text{Ans.}$$

which is required solution.

Ex. 4.8.10 : Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ if (i) U is finite $\forall t$;

(ii) $U = 0$ when $x = 0, \pi, \forall t$

(iii) $U = \pi x - x^2$ when $t = 0$ and $0 \le x \le \pi$.

Soln. :

Step I : Let u(x, t) – be the temperature at any distance x and at any time t.

Given conditions are,

(i) U is finite ; $\forall t$

(ii) $U = 0$ when $x = 0$; $\forall t$

(iii) $U = 0$ when $x = \pi$; $\forall t$

(iv) $U = \pi x - x^2$; when $t = 0$ and $0 \le x \le \pi$.

The most suitable solution is,

$$U(x, t) = [C_1 \cos mx + C_2 \sin mx] e^{-m^2 t} \qquad ...(1)$$

$$...(\text{Here, } c^2 = 1)$$

Step II :

By condition (i), U is finite $\forall t$ i.e. as $t \to \infty$; $U \to 0$

By condition (ii) as $u = 0$ for $x = 0$

Put $x = 0$ in Equation (1)

$$\therefore 0 = (C_1 + 0) e^{-m^2 t} \Rightarrow C_1 = 0$$

Put in Equation (1)

$$U(x, t) = C_2 \sin mx \, e^{-m^2 t} \qquad ...(2)$$

Step III : By condition (iii) as $x = \pi$, $U = 0$

Put $x = \pi$ in Equation (2)

$$\therefore 0 = C_2 \sin m\pi \, e^{-m^2 t}$$

$\Rightarrow C_2 = 0$ or $\sin m\pi = 0$ or $e^{-m^2 t} = 0$.

Only $\sin m\pi = 0 \Rightarrow m\pi = 0, n\pi,$

$\Rightarrow m = n \ ; \ n = 0, 1, 2, 3....$

$\therefore$ Equation (2) becomes

$$U(x, t) = C_2 \sin nx \, e^{-n^2 t} ; \quad n = 0, 1, 2, 3....$$

Put $c_2 = b_n$ and take sum of all the terms

$$U(x, t) = \sum_{n=0}^{\infty} b_n \sin nx \, e^{-n^2 t} \qquad ...(3)$$

Step IV : Now by condition (iv) as $t = 0$, $U = \pi x - x^2$;

for $0 \le x \le \pi$

$$\therefore \pi x - x^2 = \Sigma b_n \sin nx ; \quad 0 \le x \le \pi$$

Which is half range sine series, where

$$\therefore b_n = \frac{2}{l}\int_0^l f(x) \sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \sin nx \, dx$$

➔ **Using standard result**

$$...\left[\int uv' dx = uv_1 - u' v_2 + u'' v_3 - u''' v_4 + ...\right]$$

$$= \frac{2}{\pi}\left\{(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[0 + 0 - \frac{2}{n^3}(-1)^n\right] - \left[0 + 0 - 2\frac{1}{n^3}\right]\right\}$$

$$...(\sin 0 = 0; \sin n\pi = 0)$$

$$b_n = \frac{2}{\pi} \times \frac{2}{n^3}[1 - (-1)^n] \qquad ....(\because \cos n\pi = (-1)^n)$$

$$= \frac{4}{\pi n^3} \times 0 \qquad ...\text{if n is an even integer}$$

$$= \frac{4}{\pi n^3} \times (2) \qquad ...\text{if n is an odd integer}$$

$$b_n = \begin{cases} 0 .... \text{if n is an even integer} \\ \frac{8}{\pi n^3} ... \text{if n is an odd integer} \end{cases}$$

Step V : Equation (3) becomes,

$$u(x, t) = \sum_{n=1,3,5}^{\infty} \left(\frac{8}{\pi n^3}\right) \sin nx \, e^{-n^2 t}$$

OR

$$u(x, t) = \sum_{n=0}^{\infty} \frac{8}{\pi} \frac{1}{(2n+1)^3} \sin(2n+1)x \cdot e^{-(2n+1)^2 t} \ \checkmark \ ...\text{Ans.}$$

$$(\because n = 2n + 1)$$

Which is required temperature at any time.

Ex. 4.8.11 : Solve $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ for conduction of heat along a rod without radiation, subject to the following conditions.

(i) U is not infinite as $t \to \infty$.

(ii) $\frac{\partial u}{\partial x} = 0$ for $x = 0$, $x = L$ i.e. ends are insulated i.e. no heat flows through the ends and

(iii) $u = Lx - x^2$ when $t = 0$ in $0 \le x \le L$.

Soln. : Step I : Let, u(x, t) – be the temperature at any time,

The given conditions are,

(i) u is finite as $t \to \infty$

(ii) $\frac{\partial u}{\partial x} = 0$ as $x = 0$; $\forall t$

(iii) $\frac{\partial u}{\partial x} = 0$ as $x = L$

(iv) $u = Lx - x^2$ when $t = 0$ in $0 \le x \le L$

The most suitable solution is

$$u(x, t) = (C_1 \cos mx + C_2 \sin mx) e^{-m^2 kt} \qquad ...(1)$$

Step II : By condition (i), as $t \to \infty$, $u = 0$

$\therefore$ u is finite.

Step III : For condition (ii), $\frac{\partial u}{\partial x} = 0$, as $x = 0$

Differentiate Equation (1) w.r.t x

$$\frac{\partial u}{\partial x} = [-C_1 m \sin mx + C_2 m \cos mx] e^{-m^2 kt}$$

Put $x = 0$ in above equation,

$$0 = (0 + C_2 m) e^{-m^2 kt} \Rightarrow \mathbf{C_2 = 0}$$

$\therefore$ From Equation (1)

$$u(x, t) = C_1 \cos mx \, e^{-m^2 kt} \qquad ...(2)$$

Step IV : For condition (iii), $\frac{\partial u}{\partial x} = 0$ as $x = L$

Differentiate Equation (2) w.r.t x

$$\frac{\partial u}{\partial x} = -m \cdot C_1 \sin(mx) e^{-m^2 kt}$$

$$\therefore 0 = -m \cdot C_1 \sin ml \, e^{-m^2 kt}$$

Put $x = L$ in equation $\frac{\partial u}{\partial x}$

$\Rightarrow$ only $\sin ml = 0$

$\Rightarrow ml = n\pi$

$\Rightarrow m = \frac{n\pi}{l} \qquad \therefore n = 0, 1, 2, 3, ...$

Put, $C_1 = a_n$ and take sum of all terms

Equation (2) becomes

$$u(x,t) = C_1 \cos\frac{n\pi}{l}x \cdot e^{-\left(\frac{n\pi}{l}\right)^2 kt} \; ; \; n = 0, 1, 2, 3, \ldots$$

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x \; e^{-\left(\frac{n\pi}{l}\right)^2 kt} \qquad \ldots(3)$$

Step V : By condition (iv) as $t = 0$; $u = lx - x^2$; for $0 \le x \le l$

$$(lx - x^2) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x \cdot \text{in} \quad 0 \le x \le l$$

Which is half range cosine series, where,

$$a_n = \frac{2}{l}\int_0^L f(x)\cos\frac{n\pi}{l}x$$

$$a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi}{l}x\,dx$$

➔ **Using standard result**

$$\ldots\left[\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right]$$

$$= \frac{2}{l}\left[(lx - x^2)\left(\frac{\sin\frac{n\pi}{l}x}{\left(\frac{n\pi}{l}\right)}\right) - (l - 2x)\left(\frac{-\cos\frac{n\pi}{l}x}{\left(\frac{n\pi}{l}\right)^2}\right) + (-2)\left(\frac{-\sin\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^l$$

$$\ldots(\sin 0 = 0;\ \sin n\pi = 0)$$

$$= \frac{2}{l}\left[\left\{0 + (-l)\left(\frac{l}{n\pi}\right)^2\cos n\pi - 0\right\} - \left\{0 + l\cdot\left(\frac{l}{n\pi}\right)^2(1) + 0 \ldots\right\}\right]$$

$$= \frac{2}{l}\left[\frac{-l^3}{(n\pi)^2}(-1)^n + \frac{l^3}{(n\pi)^2}\right]$$

$$a_n = \frac{2l^2}{(n\pi)^2}\left[1 - (-1)^n\right]$$

Equation (3) becomes,

$$u(x,t) = \frac{2l^2}{\pi^2}\sum_{n=0}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cdot\cos\left(\frac{n\pi}{l}\right)x \cdot e^{-\left(\frac{n\pi}{l}\right)^2 kt} \checkmark$$

Ex. 4.8.12 : The equation for the conduction of heat along a bar of length l is $\frac{\partial\theta}{\partial t} = k\frac{\partial^2\theta}{\partial x^2}$ neglecting radiation. Find an expression for θ if the ends of the bar are maintained at zero temperature and if initially the temperature is T at the centre of the bars and falls uniformally to zero at its ends.

Soln. : Step I : Let, $u(x, t)$ – be the temperature of any time t,

The most suitable solution is,

$$u(x,t) = [C_1\cos mx + C_2\sin mx]\,e^{-km^2t} \qquad \ldots(1)$$

The given conditions are

(i) $\theta(0, t) = 0 ; \forall t$

(ii) $\theta(l, t) = 0 ; \forall t$

(iii) $\theta(x, 0) = ?$

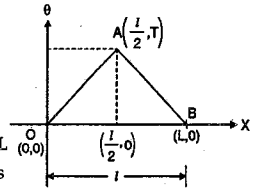


Fig. P. 4.8.12

For condition (iii) :

At $t = 0$ in $0 \le x \le L$

Equation of line OA is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\Rightarrow \frac{y - 0}{T - 0} = \frac{x - 0}{\left(\frac{l}{2}\right) - 0} \Rightarrow y = \frac{2}{l}\,T\,x \text{ in } 0 \le x \le l/2$$

Equation of line AB

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \Rightarrow \frac{y - T}{0 - T} = \frac{x - \frac{l}{2}}{l - \frac{l}{2}}$$

$$\Rightarrow y - T = -\frac{2T}{l}\left[x - \frac{l}{2}\right]$$

$$y = T - \frac{2T}{l}x + T = 2T - \frac{2T}{l}x = \frac{2T}{l}[l - x]$$

$$\therefore\ y = \frac{2T}{l}(l - x) ; \frac{l}{2} \le x \le l.$$

Hence, Condition (iii) is,

$$\theta(x, 0) = \begin{cases} \frac{2T}{l}x & ; 0 \le x \le \frac{l}{2} \\ \frac{2T}{l}(l - x) & ; \frac{l}{2} \le x \le l \end{cases}$$

Step II : Now, by condition (i) $\theta = 0$, as $x = 0$

$\therefore$ From Equation (1),

$$0 = (C_1 + 0)\,e^{-m^2kt} \Rightarrow C_1 = 0$$

Equation (1) becomes

$$\theta(x,t) = C_2\sin mx\,e^{-m^2kt} \qquad \ldots(2)$$

Step III : By condition (ii), $\theta = 0$ as $x = l$

Put $x = l$ in Equation (2)

$$\therefore\ 0 = C_2\sin ml \cdot e^{-m^2kt}$$

$$\Rightarrow \text{only } \sin ml = 0$$

$$\Rightarrow ml = n\pi \; ; \; n = 1, 2, 3, \ldots.$$

$$\Rightarrow m = \frac{n\pi}{l} \; ; \; n = 1, 2, 3, \ldots.$$

$\therefore$ Equation (2) becomes,

$$\theta(x,t) = C_2\sin\frac{n\pi}{l}x \cdot e^{-\left(\frac{n\pi}{l}\right)^2 kt} \; ; \; n = 0, 1, 2, 3, \ldots.$$

Put $C_2 = b_n$ and take sum of all terms.

$$\theta(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right)x \cdot e^{-\left(\frac{n\pi}{l}\right)^2 kt} \qquad \ldots(3)$$

Step IV : Now by condition (iii),

$$\text{As } t = 0 ;\ \theta = \begin{cases} \frac{2T}{l}x & ; 0 \le x \le \frac{l}{2} \\ \frac{2T}{l}(l - x) & ; \frac{l}{2} \le x \le l \end{cases}$$

Put $t = 0$ in Equation (3)

$$\therefore \left.\begin{matrix} \frac{2T}{l}x & ; 0 \le x \le \frac{l}{2} \\ \frac{2T}{l}(l - x) & ; \frac{l}{2} \le x \le l \end{matrix}\right\} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x$$

Which is half range sine series in

where, $0 \le x \le L$.

$$\therefore\ b_n = \frac{2}{L}\int_0^L f(x), \sin\frac{n\pi}{l}x\,dx$$

$$= \frac{2}{L}\left[\int_0^{\frac{l}{2}} f(x)\sin\frac{n\pi}{l}x\,dx + \int_{\frac{l}{2}}^{l} f(x)\sin\frac{n\pi}{l}x\,dx\right]$$

$$= \frac{2}{l}\left\{\int_0^{\frac{l}{2}}\frac{2T}{l}x\sin\frac{n\pi}{l}x\,dx + \int_{\frac{l}{2}}^{l}\frac{2T}{l}(l - x)\sin\frac{n\pi}{l}x\,dx\right\}$$

$$= \frac{4T}{l^2}\left\{\int_0^{\frac{l}{2}} x\sin\frac{n\pi}{l}x\,dx + \int_{\frac{l}{2}}^{l}(l - x)\sin\left(\frac{n\pi}{l}\right)x\,dx\right\}$$

➔ **Using standard result**

$$\ldots\left[\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right]$$

$$= \frac{4T}{l^2}\left\{\left[(x)\left(\frac{-\cos\frac{n\pi}{l}x}{\left(\frac{n\pi}{l}\right)}\right) - (1)\left(\frac{-\sin\frac{n\pi}{l}x}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^{\frac{l}{2}} + \left[(l - x)\left(\frac{-\cos\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)}\right) - (-1)\left(\frac{-\sin\frac{n\pi}{l}x}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_{\frac{l}{2}}^{l}\right\}$$

$$= \frac{4T}{l^2}\left\{\left[-\left(\frac{l}{2}\right)\left(\frac{l}{n\pi}\right)\cos\frac{n\pi}{2} + \left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi}{2}\right] - 0 + 0 - \left[\left(\frac{l}{2}\right)\left(\frac{-l}{n\pi}\right)\cos\left(\frac{n\pi}{2}\right) - \left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi}{2}\right]\right\}$$

$$\ldots(\because\ \sin 0 = 0)$$

$$= \frac{4T}{l^2}\left[2\frac{l^2}{(n\pi)^2}\cdot\sin\frac{n\pi}{2}\right]$$

$$b_n = \frac{8T}{(n\pi)^2}\sin\frac{n\pi}{2}$$

Put this value in Equation (3) we get,

$$\theta(x,t) = \frac{8T}{\pi^2}\sum_{n=1}^{\infty}\sin\frac{n\pi}{n^2}\sin\left(\frac{n\pi}{l}\right)x \cdot e^{-\left(\frac{n\pi}{l}\right)^2 kt} \checkmark$$

Ex. 4.8.13 : Solve $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ For the conduction of heat along a homogeneous rod of length l with insulated sides is initially at a uniform temperature u_0. Its ends are cooled to 0°C and are kept at that temperature. Find the temperature $u(x, t)$ at any time t.

Soln. :

Step I : Let $u(x, t)$ – be the temperature,

The given conditions are,

(i) $u(0, t) = 0 ; \forall t$

(ii) $u(l, t) = 0 ; \forall t$

(iii) $u(x, 0) = u_0 ; \forall x$

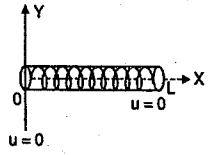


Fig. P. 4.8.13

The most suitable solution is,

$$u(x,t) = [C_1\cos mx + C_2\sin mx]\,e^{-m^2kt} \qquad \ldots(1)$$

Step II : By condition (i) as $x = 0$, $u = 0$

Put $x = 0$ in Equation (1)

$$\therefore\ 0 = (C_1 + 0)\,e^{-m^2kt}$$

$$\Rightarrow C_1 = 0$$

Equation (1) becomes,

$$u(x,t) = C_2\sin mx\,e^{-m^2kt} \qquad \ldots(2)$$

Step III : By condition (ii) as $x = l$, $u = 0$

Put $x = l$ in Equation (2)

$$\therefore\ 0 = C_2\sin ml\,e^{-m^2kt}$$

It is possible only if $\sin ml = 0$

$$\Rightarrow ml = n\pi \; ; \; n = 1, 2, 3, \ldots\ldots$$

$$\Rightarrow m = \frac{n\pi}{l} \; ; \; n = 1, 2, 3, \ldots\ldots$$

Equation (2) becomes,

$$u(x,t) = C_2\sin\left(\frac{n\pi}{l}\right)x\,e^{-\left(\frac{n\pi}{l}\right)^2 kt} \; ; \; n = 1, 2, 3, \ldots..$$

Put $C_2 = b_n$ and take sum of all terms

$$u(x,t) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi}{l}\right)x\,e^{-\left(\frac{n\pi}{l}\right)^2 kt} \qquad \ldots(3)$$

Step IV : By condition (iii), as $t = 0$, $u = u_0$ in $0 \le x \le l$

$$u_0 = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi}{l}\right)x \quad ; \quad \text{in } 0 \le x \le l$$

which is half range sine series, where,

$$\therefore\ b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx$$

$$= \frac{2}{l}\int_0^l u_0 \sin\left(\frac{n\pi}{l}\right)x\,dx$$

$$= \frac{2u_0}{l}\left(\frac{-\cos\left(\frac{n\pi}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right)_0^l$$

$$= -\frac{2u_0}{l}\left(\frac{l}{n\pi}\right)[\cos n\pi - 1]$$

$$\therefore\ b_n = \frac{2u_0}{\pi}\left[\frac{1-(-1)^n}{n}\right]$$

Put this value of b_n in Equation (3) we get,

$$u(x,t) = \frac{2u_0}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]\sin\left(\frac{n\pi}{l}\right)$$

$$x\cdot e^{-\left(\frac{n\pi}{l}\right)^2 kt} \checkmark \qquad \text{...Ans.}$$

which is required temperature at any time t.

Ex. 4.8.14 : Solve : $\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$ subject to the following boundary conditions :

(i) $u(0,t) = 0\ ;\ \forall t$

(ii) $u(l,t) = 0\ ;\ \forall t$

(iii) $u(x,0) = \begin{cases} x & ;\ 0 \le x \le l/2 \\ l-x & ;\ l/2 \le x \le l \end{cases}$

(iv) $u(x,\infty)$ is not infinite.

Soln. : Step I : Let $u(x,t)$ – be the temperature.

The given conditions are,

(i) $u(0,t) = 0\ ;\ \forall t$

(ii) $u(l,t) = 0\ ;\ \forall t$

(iii) $u(x,0) = \begin{cases} x & ;\ 0 \le x \le l/2 \\ l-x & ;\ l/2 \le x \le l \end{cases}$

(iv) $u(x,\infty)$ is not infinite.

The most suitable solution is,

$$u(x,t) = [C_1\cos mx + C_2\sin mx]\,e^{-m^2a^2t} \qquad ...(1)$$

Step II : By condition (iv) as $t \to \infty$, $u \to 0$

Hence, u is not infinite.

Step III : *By condition (i), as* $x = 0, u = 0$

Put $x = 0$ in Equation (1)

$$\therefore\ 0 = [C_1 + 0]\,e^{-m^2a^2t}$$

$$\Rightarrow\ C_1 = 0$$

Equation (1) becomes,

$$u(x,t) = C_2\sin mx\ e^{-m^2a^2t} \qquad ...(2)$$

Step IV : By condition (ii), as $x = l$, $u = 0$

Put $x = l$ in Equation (2)

$$\therefore\ 0 = C_2\sin l\cdot e^{-m^2a^2t}$$

It is possible only if $\sin ml = n\pi$; $n = 1, 2, 3, ...$

$$\Rightarrow\ ml = n\pi\ ;\ n = 1, 2, 3, ...$$

$$\Rightarrow\ m = \frac{n\pi}{l}\ ;\ n = 1, 2, 3, ...$$

Equation (2) becomes,

$$u(x,t) = C_2\sin\left(\frac{n\pi}{l}\right)x\cdot e^{-\left(\frac{n\pi}{l}\right)^2a^2t}\ ;\ n = 1, 2, 3, ...$$

Put $C_2 = b_n$ and take sum of all terms.

$$u(x,t) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi}{l}\right)x\cdot e^{-\left(\frac{n\pi}{l}\right)^2a^2t} \qquad ...(3)$$

Step V : Now, by condition (iii), as $t = 0$,

$$u = \begin{cases} x & ;\ 0 \le x \le l/2 \\ l-x & ;\ l/2 \le x \le l \end{cases}$$

Put $t = 0$ in Equation (3)

$$\therefore\ \left.\begin{matrix} x & ;\ 0 \le x \le l/2 \\ l-x & ;\ l/2 \le x \le l \end{matrix}\right\} = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi}{l}\right)x$$

Which is half range sine series in $0 \le x \le l$, where

$$\therefore\ b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx$$

$$= \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx + \int_{l/2}^{l} f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx\right]$$

$$= \frac{2}{l}\left[\int_0^{l/2} x\sin\left(\frac{n\pi}{l}\right)x\,dx + \int_{l/2}^{l}(l-x)\sin\left(\frac{n\pi}{l}\right)x\,dx\right]$$

➔ **Using standard result**

$$...\left[\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...\right]$$

$$= \frac{2}{l}\left\{\left[(x)\left(\frac{-\cos\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)}\right) - (1)\left(\frac{-\sin\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^{l/2}\right.$$

$$\left. + \left[(l-x)\left(\frac{-\cos\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)}\right) - (-1)\left(\frac{-\sin\left(\frac{n\pi}{l}x\right)}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_{l/2}^{l}\right\}$$

$$= \frac{2}{l}\left\{-\left(\frac{l}{2}\right)\left(\frac{l}{n\pi}\right)\cos\left(\frac{n\pi}{2}\right) + \left(\frac{l}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right) - 0 + 0 + \left(\frac{l}{2}\right)\left(\frac{l}{n\pi}\right)\cos\left(\frac{n\pi}{2}\right) + \left(\frac{l}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right)\right\}$$

$$\therefore\ b_n = \frac{2}{l}\left[2\times\frac{l^2}{n^2\pi^2}\cdot\sin\left(\frac{n\pi}{2}\right)\right]$$

$$\therefore\ b_n = 4\,\frac{l}{\pi^2}\,\frac{\sin\frac{n\pi}{2}}{n^2}$$

Equation (3) becomes,

$$u(x,t) = \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}\sin\left(\frac{n\pi}{l}\right)x\cdot e^{-a^2\left(\frac{n\pi}{l}\right)^2t} \checkmark$$

Ex. 4.8.15 : An insulated rod of leight l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C and maintained these after. Find the temperature at distance x and at time t.

Soln. :

Step I : Let $u(x,t)$ be the temperature at any distance x and at any time t.

Here, the solution of one dimensional heat flow differential equation

$$\frac{\partial u}{\partial t} = C^2\frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

Involves two solutions

$$u(x,t) = u_s(x) + u_t(x,t) \qquad ...(2)$$

Where, $u_s(x) =$ is the steady state condition part, which is independent of time

and $u_t(x,t) =$ is the transient solution which is depends on time (as time increases, temperature decreases)

Step II :

Part I :

First we have to find the initial condition by using steady state conditions,

Where, $\frac{\partial u}{\partial t} = 0$ ($\because$ u is not depend on time)

$\therefore$ Equation (1) becomes

$$\frac{\partial^2 u}{\partial x^2} = 0$$

$$\Rightarrow\ D = 0, 0 \qquad \left(\because D = \frac{\partial^2}{\partial x^2}\right)$$

$$C.F. = Ax + B$$

and $P.I. = 0$

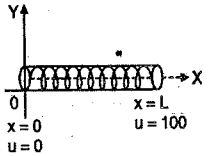


Fig. P. 4.8.15

General solution is,

$$u(x) = C.F. + F.I.$$

$$u(x) = Ax + B \qquad ...(3)$$

In steady state condition,

$$u(0) = 0 \text{ and } u(l) = 100$$

Put $x = 0$ in (3) ; $u(0) = B$

$$\Rightarrow 0 = B \Rightarrow B = 0$$

and put $x = l$ in Equation (3),

$$\Rightarrow\ u(l) = Al + B$$

$$\Rightarrow\ 100 = Al + 0 \Rightarrow A = \frac{100}{l}$$

Equation (3) becomes,

$$u(x) = \frac{100}{l}x$$

i.e. Initially at $t = 0$, $u = \frac{100}{l}x \Rightarrow u(x,0) = \frac{100x}{l}$ ...(4)

Part II :

Step IV

Here, we have find the boundary conditions,

Given, $u(0,t) = 20$; $u(l,t) = 80$ ...(5)

Like part I, put $x = 0$ in Equation (3)

$$u(0) = 0 + B$$

$$\Rightarrow\ 20 = B \Rightarrow B = 20$$

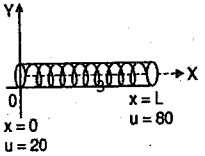


Fig. P. 4.8.15(a)

and put $x = l$ in Equation (3),

$$\Rightarrow\ u(l) = Al + B$$

$$\Rightarrow\ 80 = Al + 20 \Rightarrow A = \frac{60}{l}$$

Equation (3) becomes, solution for steady state condition is,

$$u_s(x) = \frac{60}{l}x + 20 \qquad ...(6)$$

Step V : Now, By using Equation (2),

$$u_t(x,t) = u(x,t) - u_s(x) \qquad ...(7)$$

Put, $x = 0$ in Equation (7) and use Equation (5) and (6)

$$u_t(0,t) = u(0,t) - u_s(0)$$
$$= 20 - 20$$
$$...(\because \text{ by putting } x = 0 \text{ in (5), we get } u_s(0) = 0)$$
$$u_t(0,t) = 0 \qquad ...(8)$$

and put $n = l$ is in Equation (7) and using (5) and Equation (6)

$$u_t(l,t) = u(l,t) - u_s(l)$$
$$= 80 - (60 + 20) = 0$$
$$\therefore u_t(l,t) = 0 \qquad ...(9)$$

Now, put $t = 0$ in Equation (7) and using Equations (4) and (6).

$$u_t(x,0) = u(x,0) - u_s(x)$$
$$= \left(\frac{100x}{l}\right) - \left(\frac{60x}{l} + 20\right)$$
$$u_t(x,0) = \frac{40x}{l} - 20 \qquad ...(10)$$

Part III :

Step VI : To find solution of transient part.

Now the most suitable solution for transient part is,

$$u_t(x,t) = [C_1 \cos mx + C_2 \sin mx]\,[e^{-c^2m^2t}] \qquad ...(11)$$

Step VII : By using Equation (8), as $x = 0$; $u_t = 0$

Put $x = 0$ in Equation (11)

$$\therefore \quad 0 = (C_1 + 0)\, e^{-c^2m^2t}$$
$$\Rightarrow \quad C_1 = 0$$

Equation (11) becomes,

$$u_t(x,t) = C_2 \sin mx\; e^{-c^2m^2t} \qquad ...(12)$$

Step VIII : By using Equation (9) as $x = l$; $u_t = 0$

Put $x = l$ in Equation (12)

$$\therefore \quad 0 = C_2 \sin ml\; e^{-c^2m^2t}$$

It is possible **only if** $\sin ml = 0$

$$\Rightarrow \quad ml = n\pi \;;\; n = 1, 2, 3, ...$$
$$m = \frac{n\pi}{l} \;;\; n = 1, 2, 3, ...$$

Equation (12) becomes,

$$u_t(x,t) = C_2 \sin\left(\frac{n\pi}{l}\right)x \cdot e^{-C^2\left(\frac{n\pi}{l}\right)^2 t} \;;\; n = 1, 2, 3, ...$$

Put $C_2 = b_n$ and take sum of all terms

$$u_t(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right)x \cdot e^{-C^2\left(\frac{n\pi}{l}\right)^2 t} \qquad ...(13)$$

Step IX :

Now, by Equation (10), as $t = 0$, $u_t = \frac{40x}{l} - 20$;

in $0 \le x \le l$

Put $t = 0$ in (13)

$$\therefore \left(\frac{40x}{l} - 20\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right)x \quad ; \text{ in } 0 \le x \le l$$

Which is half range sine series ; in $0 \le x \le l$, where

$$\therefore \quad b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}\right)x\,dx$$
$$= \frac{2}{l}\int_0^l \left(\frac{40x}{l} - 20\right) \cdot \sin\left(\frac{n\pi}{l}\right)x\,dx$$

➔ **Using standard result**

$$...\left[\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...\right]$$

$$= \frac{2}{l}\left[\left(\frac{40x}{l} - 20\right)\left(\frac{-\cos\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)}\right) - \left(\frac{40}{l}\right)\left(\frac{-\sin\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$\therefore b_n = \frac{2}{l}\left[-(20)\left(\frac{l}{n\pi}\right)\cos n\pi + 0 + (-20)\left(\frac{l}{n\pi}\right)\cdot 1 - 0\right]$$
$$= \frac{2}{l}\cdot(-20)\left(\frac{l}{n\pi}\right)[1 + \cos n\pi]$$
$$b_n = \frac{-40}{n\pi}[1 + (-1)^n]$$

Step X :

Put this value of b_n in Equation (13),

$$u_t(x,t) = \frac{-40}{n}\sum_{n=1}^{\infty}\left[\frac{1 + (-1)^n}{n}\right] \cdot \sin\frac{n\pi}{l}x\, e^{-C^2\left(\frac{n\pi}{l}\right)^2 t} \qquad ...(14)$$

Therefore, by using Equations (6) and (14) Equation (2) becomes,

$$u(x,t) = \left(\frac{60x}{l} + 20\right) - \frac{40}{\pi}\sum_{n=1}^{\infty}\left[\frac{1 + (-1)^n}{n}\right] \cdot \sin\left(\frac{n\pi}{l}\right)x\, e^{-C^2\left(\frac{n\pi}{l}\right)^2 t}$$

Which is the required temperature at any distance x and at any time t.

Exercise

Ex. 1 : Solve the equation : $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, with boundary conditions : $u(x,t) = 3 \sin n\pi x$, $u(0,t) = 0$; $u(1,t) = 0$, where $0 < x < 1, t > 0$

Ans. : $u(x,t) = 3\sum_{n=1}^{\infty} \sin n\pi x \cdot e^{-n^2\pi^2 t}$

Ex. 2 : Find the solution of $\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$, given that $V = V_0 \sin nt$, when $x = 0$ for all values of t and $V = 0$ when x is very large.

Ans. : $V(x,t) = V_0\, e^{\sqrt{\left(\frac{n}{2k}\right)}x} \cdot \sin\left[nt - \sqrt{\left(\frac{n}{2k}\right)}x\right]$

Ex. 3 : Solve : $\frac{\partial \theta}{\partial t} = k\frac{\partial^2 \theta}{\partial x^2}$, such that

(i) θ is finite, when $t \to \infty$

(ii) $\frac{\partial \theta}{\partial t} = 0$ when $x = 0$ and $\theta = 0$ when $x = l$ for all t

(iii) $\theta = \theta_0$ when $t = 0$ for all x ; $0 \le x \le l$

Ans. :

$$\theta = \frac{4\theta_0}{\pi}\left[e^{-\left(\frac{\pi}{2l}\right)^2 kt} \cdot \cos\left(\frac{\pi x}{2l}\right) - \frac{1}{3}e^{-\left(\frac{3\pi}{2l}\right)^2 kt} \cdot \cos\left(\frac{3\pi x}{2l}\right) + \frac{1}{5}\cdot e^{-\left(\frac{5x}{2l}\right)^2 kt} \cos\left(\frac{5\pi}{2l}x\right) .....\right]$$

Syllabus Topic: Two Dimensional Heat Flow Equations

4.9 Two Dimensional Heat Flow Equations (Laplace Equations in two Dimensions)

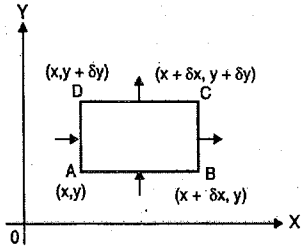


Fig. 4.9.1

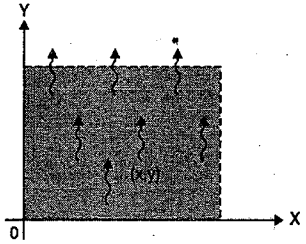


Fig. 4.9.2

Let us consider a uniform thick α (cm) metal plate with density ρ (gr/cm^3), specific heat s (cal/gram deg) and thermal conductivity k (cal / cm . sec. deg).

Let the flow of heat in XOY plane. Consider, the temperature u(x, y) at any point is independent of the z-cordinate and depends on x, y and time t. Then the flow is called two dimensional heat flow.

Let the heat flow is in XY plane only and zero along the normal to XOY – plane (i.e. zero along z-direction)

Consider a rectangular plate ABCD with sides δx and δy with thickness α (cm).

As we discussed a case in one-diamensional heat equation, the amount of heat that enters into the plate from sides AB and AD per second is,

$$Q = -k\alpha\,\delta x\left(\frac{\partial u}{\partial y}\right)_y \quad \text{and} \quad Q = -k\alpha\,\delta y\left(\frac{\partial u}{\partial x}\right)_x$$

respectively,

Also, the heat flow out through the sides CD and BC per second is

$$Q = -k\alpha\,\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} \quad \text{and} \quad Q = -k\alpha\,\delta y\left(\frac{\partial u}{\partial y}\right)_{y+\delta x}$$

Respectively

Hence, the total gain of heat by the rectangular plate ABCD per second is

$$Q = (\text{Along AB, AD}) - (\text{Along CD, BC})$$
$$Q = -k\alpha\,\delta x\left(\frac{\partial u}{\partial y}\right)_{yx} - k\alpha\,\delta y\left(\frac{\partial u}{\partial x}\right)_x + k\alpha\,\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} + k\alpha\,\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\partial x}$$
$$= k\alpha\,\delta x\,\delta y\left[\frac{\left(\frac{\partial u}{\partial y}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] \qquad ...(1)$$

Also, the quantity of heat gain by the plate element is

$$Q = s\rho\,\delta x\,\delta y\,u$$

Differentiate w.r.t time t, we get the total rate of gain of heat by the plate element is

$$= S\rho\,\delta x\,\delta y\frac{\partial u}{\partial t} \qquad ...(2)$$

From Equation (1) and (2), we get

$$k\,\alpha\ \delta x\,\delta y\left[\frac{\left(\frac{\partial u}{\partial y}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_{x}}{\delta x}\right]+\left[\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}-\left(\frac{\partial u}{\partial y}\right)_{y}}{\delta y}\right]$$

$$= S\rho\,\delta x\,\delta y\frac{\partial u}{\partial t}$$

Dividing both sides by $\alpha\ \delta x\ \delta y$ and by considering limiting conditions as $\delta x \to 0$, $\delta y \to 0$, we get

$$\frac{k}{s\rho}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right] = \frac{\partial u}{\partial t}$$

$$\Rightarrow C^2\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right] = \frac{\partial u}{\partial t} \qquad ...(3)$$

Where, $C^2 = \frac{k}{s\rho}$

Hence, $C^2\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right] = \frac{\partial u}{\partial t}$ is the differential equation of heat flow in two dimension.

If the temperature u(x, y) is independent of time t, i.e. the steady state condition, u is independent of time,

$$\therefore \frac{\partial u}{\partial t} = 0$$

Hence, we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(4)$$

is the Laplace Equation in two dimensions of heat flow, where u(x, y) – be temperature distribution in x and y direction.

4.9.1 Solution of Laplace Equation in Two Dimensions

Since, the partial differential equation of two dimensional heat flow is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(1)$$

Let the solution is

$$u(x, y) = X(x)\,Y(y) \qquad ...(2)$$

Here, X – is a function of x only.

Y – is a function of y only.

Differentiate Equation (2), partially w.r.t x and y (two times)

$$\frac{\partial u}{\partial x} = X'Y \quad \text{and} \quad \frac{\partial u}{\partial y} = XY'$$

$$\text{and } \frac{\partial^2 u}{\partial x^2} = X''Y \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = XY''$$

Equation (1) becomes

$$X''Y + XY'' = 0 \Rightarrow \frac{X''}{x} = -\frac{Y''}{y}$$

Here, in Equation (3)

L.H.S. is a function of x only and R.H.S is a function of y only

So, X and Y are independent to each other. Each equal to K

$$\therefore \frac{X''}{X} - \frac{Y''}{Y} = K \text{ (say)}$$

$$\Rightarrow \frac{X''}{X} = k \quad \text{and} \quad \frac{-Y''}{Y}\ k \text{ (say)}$$

We get, the ordinary differential equations.

$$\frac{\partial^2 X}{\partial x^2} - kX = 0 \quad \text{and} \quad \frac{\partial^2 Y}{\partial y^2} + kY = 0 \qquad ...(4)$$

Now solution depends upon the nature of k.

(i) If k is positive, i.e. let $k = p^2$ solution is

$$X = C_1 e^{px} + C_2 e^{-px}\ ;\ Y = C_3 \cos py + C_4 \sin py$$

(ii) If k is negative ; i.e. let $k = -P^2$, solution is

$$X = C_1 \cos px + C_2 \sin px\ ;\ Y = C_3 e^{py} + C_4 e^{-py}$$

(iii) If k is zero i.e. k = 0, solution is

$$X = C_1 X + C_2\ ;\ Y = C_3 y + C_4$$

Hence the various solutions are substitute in Equation (2), the temperature u(x, y) is

$$u(x, y) = [C_1 e^{px} + C_2 e^{-px}]\,[C_3 \cos py + C_4 \sin py] \qquad ...(5)$$

$$u(x, y) = [C_1 \cos px + C_2 \sin px]\,[C_3 e^{py} + C_4 e^{-py}] \qquad ...(6)$$

and

$$u(x, y) = (C_1 x + C_2)\,(C_3 y + C_4) \qquad ...(7)$$

Note : (i) Use the exponential form $y = C_1 e^{px} + C_2 e^{-px}$ or $C_3 e^{py} + C_4 e^{-py}$ when the domain of x or y is an infinite or semi-infinite interval.

(ii) Use the hyperbolic form

$x = C_1 \cosh px + C_2 \sinh px$ or

$y = C_3 \cosh py + C_4 \sinh py$

when domain of x or y is a finite interval.

(iii) If the ends are insulated then use $\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0$

and $\left(\frac{\partial u}{\partial x}\right)_{x=a} = 0$ for $0 < y < b$

Or

$\left(\frac{\partial u}{\partial y}\right)_{y=0} =$ and $\left(\frac{\partial u}{\partial y}\right)_{y=b} = 0$ for $0 < y < a$

The most suitable general solution will depend on boundary conditions of the problem.

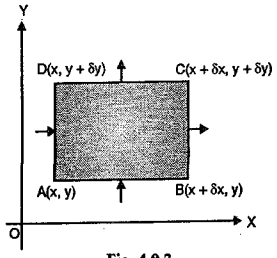


Fig. 4.9.3

Use the following conditions :

(i) If $u(x,y) = 0$ as $y \to \infty$, $\forall x$ in (0,L) then the most suitable solution is,

$$u(x,y) = (C_1 \cos mx + C_2 \sin mx)\ (C_3 e^{my} + C_4 e^{-my})$$

$$= (C_1 \cos mx + C_2 \sin mx) \cdot [(C_3 (\cosh my + \sinh my) + C_4 (\cosh my - \sinh my)]$$

$$= (C_1 \cos mx + C_2 \sin mx)\,(C_3 + C_4) \cosh my + (C_3 - C_4) \sinh my) \cdot$$

In general,

$$u(x,y) = (C_1 \cos mx + C_2 \sin mx) \cdot (C_3 \cosh my + C_4 \sinh my)$$

(ii) If $u(x,y) = 0$ as $x \to \infty$, $\forall y$ in (0,L) then the most suitable solution is,

$$u(x,y) = (C_1 e^{mx} + C_2 e^{-mx})\,(C_3 \cos my + c_4 \sin my)$$

OR

$$u(x,y) = (C_1 \cosh mx + C_2 \sinh mx) \cdot (C_3 \cos my + C_4 \sin my)$$

4.9.1.1 Procedure to Solve Two Dimensional Heat Flow Equation

Step I : Write the given initial conditions of two dimensional heat flow equation.

Step II : Write the most suitable general solution.

Step III : Use condition either $y \to \infty$ or $x \to \infty$

Step IV : Use condition (ii) u = 0 as x = 0 or y = 0

Step V : Use condition (iii) u = 0 as x = *l* or y = *l*.

Step VI : Put the value of m in the equation which we get from step IV.

Step VII : Use condition (iv) as y = 0 or x = 0 for u.

Step VIII : Use half range sine series for the interval $0 \le x \le l$ or $0 \le y \le l$ and find b_n.

Step IX : Put b_n in equation which is obtained from the step VI.

We get required temperature of plate.

Ex. 4.9.1 : A thin sheet of metal, bounded by the X – axis and the lines x = 0 and x = 1 and stretching to infinity in the Y direction has its upper and lower faces perfectly insulated and its vertical edges and the edge at infinity are maintained at the constant temperature 0°C, while over the base temperature of 100°C is maintained. Find steady state temperature u(x, y).

Soln. :

Step I :

Let u (x, y) - be the temperature,

The partial differential equation of two dimensional heat flow is,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Subject to the conditions

(i) $u(x, \infty) = 0;\ \forall x$

(ii) $u(0, y) = 0;\ \forall y$

(iii) $u(1, y) = 0;\ \forall y$

(iv) $u(x, 0) = 100;\quad 0 \le x \le 1$

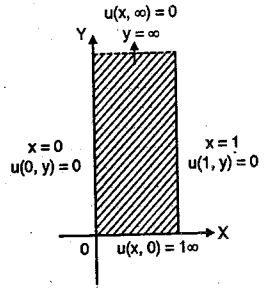


Fig. P. 4.9.1

Step II :

As $y \to \infty$, the most suitable solution is

$$u(x, y) = (C_1 \cos mx + C_2 \sin mx)\,(C_3 e^{my} + C_4 e^{-my}) \qquad ...(1)$$

Step III :

By condition (i), as $y \to \infty$, u = 0

Put, $y \to \infty$ in Equation (1)

$$\therefore\ 0 = (C_1 \cos mx + C_2 \sin mx)\,(C_3 e^{my} + 0)$$

Since, u is finite only if $C_3 = 0$

...($\because C_3 e^{my} = 0 \Rightarrow C_3 = 0$ and $e^{my} \ne 0$ as $y \to +\infty$. Also the bracketed term never zero)

Equation (1) becomes

$$u(x, y) = [C_1 \cos mx + C_2 \sin mx]\,C_4 e^{-my} \qquad ...(2)$$

Step IV :

By condition (ii), u = 0 as x = 0

Put, x = 0 in Equation (2), we get

$$0 = (C_1 + 0)\,C_4 e^{-my} \Rightarrow C_1 = 0 \text{ only}$$

($\because$ Again if we equate C_4 with zero, then the temperature u(x, y) becomes zero)

Equation (2) becomes

$$u(x, y) = C_2 C_4 \sin mx\, e^{-my} \qquad ...(3)$$

Step V :

By condition (iii), u = 0 as x = 1

Put, x = 1 in Equation (3), we get

$$0 = C_2 C_4 \sin m1\, e^{-my}$$

$\Rightarrow C_2C_4 \neq 0$

$\left(\because \text{ if } C_2C_4 = 0 \text{ and } e^{-my} = 0 \text{ then completely, } u = 0\right)$

$\therefore$ only $\sin m = 0$

$\Rightarrow m = n\pi \quad ; n = 0, 1, 2, 3, \ldots.$

Equation (3) becomes

Step VI :

$$u(x, y) = C_2 C_4 \sin(n\pi) x e^{-n\pi y} \quad ; n = 1, 2, 3, \ldots.$$

Put, $C_2 C_4 = b_n$ and take sum of all terms

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin(n\pi) x e^{-n\pi y} \qquad \ldots(4)$$

Step VII :

Now by condition (iv), as $y = 0$; $u = 100$ in $0 \le x \le 1$

Put $y = 0$ in Equation (4),

$\therefore$ From Equation (4) we get,

$$100 = \sum_{n=1}^{\infty} b_n \sin n\pi x, \qquad \text{in } 0 \le x \le 1$$

Which is half range sine series in $0 \le x \le 1$.

$$\text{It is like } f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right) x \qquad \text{in } 0 \le x \le L$$

Where,

$$\therefore b_n = \frac{2}{L}\int_0^L f(x) \sin n\pi x \, dx$$

$$= \frac{2}{1}\int_0^1 100 \sin n\pi x \, dx$$

$$= 200\left(-\frac{\cos n\pi x}{n\pi}\right)_0^1$$

$$= \frac{200}{n\pi}\left[-(-1)^n + 1\right] \quad \ldots(\because \cos n\pi = (-1)^n, \cos 0 = 1)$$

$$b_n = \frac{200}{n\pi}\left(1 - (-1)^n\right)$$

Step VIII :

Equation (4) becomes,

$$u(x, y) = \frac{200}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right] \cdot \sin n\pi x e^{-n\pi y} \checkmark \quad \ldots\text{Ans.}$$

which is required temperature, in the plate at distance x and y

Ex. 4.9.2 : A rectangular plate with insulated surface is 10 cm wide and so long to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $y = 0$ is given by,

$$y = \begin{cases} 20x & \text{for } 0 \le x \le 5 \\ 20(10 - x) & \text{for } 5 \le x \le 10 \end{cases}$$

and the two long edges $x = 0$, $x = 10$ as well as other short edge are at 0°C, then prove that the temperature $u(x, y)$ at any point (x, y) is given by

$$u(x, y) = \frac{800}{\pi^2}\sum_{n^2}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}\sin\frac{n\pi}{10}x \cdot e^{-\frac{n\pi}{10}y}$$

Soln. :

Step I : Let $u(x, y)$ - be the temperature at any point (x, y)

The given conditions are

(i) $u(x, \infty) = 0$; $\forall x$

(ii) $u(0, y) = 0$; $\forall y$

(iii) $u(10, y) = 0$; $\forall y$

(iv) $u(x, 0) = \begin{cases} 20x & ; 0 \le x \le 5 \\ 20(10 - x) & ; 5 \le x \le 10 \end{cases}$

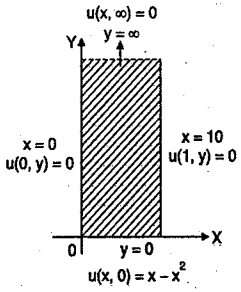


Fig. P. 4.9.2

Step II :

As $y \to \infty$

The most suitable solution is,

$$u(x, y) = [C_1 \cos mx + C_2 \sin mx][C_3 e^{my} + C_4 e^{-my}] \qquad \ldots(1)$$

Step III :

By condition (i) as $y \to \infty$, u must finite. i.e. $u = 0$

Put, $y \to \infty$ in Equation (1)

$$0 = [C_1 \cos mx + C_2 \sin mx][C_3 e^{my} + 0]$$

U becomes finite only if $\Rightarrow C_3 = 0$

$\ldots(\because$ If $[C_1 \cos mx + C_2 \sin mx] = 0$ then from Equation (1) $u = 0$. It is not possible always)

Equation (1) becomes,

$$\therefore u(x, y) = [C_1 \cos mx + C_2 \sin mx] C_4 e^{-my} \qquad \ldots(2)$$

Step IV :

By condition (ii), as $x = 0$; $u = 0$

Put, $x = 0$ in Equation (2),

$$0 = (C_1 + 0) C_4 e^{-my} \Rightarrow C_1 = 0$$

$\ldots(\because$ If $C_4 = 0$ then $u(x, y) = 0$. It is not possible at every point)

From Equation (2),

$$u(x, y) = C_2C_4 \sin mx \, e^{-my} \qquad \ldots(3)$$

Step V :

By condition (iii), as $x = 10$, $u = 0$

Put, $x = 10$ in Equation (3)

$$0 = C_2 C_4 \sin m10 \cdot e^{-my}$$

$$\Rightarrow C_2C_4 \neq 0 \; ; e^{-my} \neq 0$$

only $\sin m10 = 0$

$$\Rightarrow m10 = n\pi \quad ; n = 0, 1, 2, 3, \ldots\ldots$$

$$\Rightarrow m = \frac{n\pi}{10} \quad ; n = 0, 1, 2, 3. \ldots.$$

Equation (3) becomes,

$$u(x, y) = C_2 C_4 \sin\frac{n\pi}{10}x \cdot e^{-\frac{n\pi}{10}y} \; ; \qquad n = 0, 1, 2, 3 \ldots$$

Put $C_2 C_4 = b_n$ and sum all terms

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{10}x \cdot e^{-\frac{n\pi}{10}y} \qquad \ldots(4)$$

Step VI :

By condition (iv) as : $y = 0$, $u = \begin{cases} 20x, & 0 \le x \le 5 \\ 20(10 - x); & 5 \le x \le 10 \end{cases}$

$$\therefore \left.\begin{matrix} 20x \; ; 0 \le x \le 5 \\ 20(10 - x) \; ; 5 \le x \le 10 \end{matrix}\right\} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{10}x$$

Which is half range sine series.

$$\text{It is like : } f(x) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x \quad ; \qquad 0 \le x \le L$$

Step VII :

Where,

$$\therefore b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{10}x \, dx = \frac{2}{10}\int_0^{10} f(x)\sin\frac{n\pi}{10}x \, dx$$

$$= \frac{1}{5}\left[\int_0^5 f(x)\sin\frac{n\pi}{10}x \, dx + \int_5^{10} f(x)\sin\frac{n\pi}{10}x \, dx\right]$$

$$= \frac{1}{5}\left[\int_0^5 20x\sin\frac{n\pi}{10}x \, dx + \int_5^{10} 20(10 - x)\sin\frac{n\pi}{10}x \, dx\right]$$

➔ **Using standard result**

$$\ldots\left[\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots\right]$$

$$= \frac{1}{5}\left\{\left[(20x)\left(\frac{-\cos\frac{n\pi}{10}x}{\left(\frac{n\pi}{10}\right)}\right) - (20)\left(\frac{-\sin\frac{n\pi}{10}x}{\left(\frac{n\pi}{10}\right)^2}\right)\right]_0^5 + \left[20(10 - x)\left(\frac{-\cos\frac{n\pi}{10}x}{\frac{n\pi}{10}}\right) - 20(-1)\left(\frac{-\sin\frac{n\pi}{10}x}{\left(\frac{n\pi}{10}\right)^2}\right)\right]_5^{10}\right\}$$

$$= \frac{1}{5}\left\{\left[-100\left(\frac{10}{n\pi}\right)\cos\frac{n\pi}{2} + 20\left(\frac{10}{n\pi}\right)^2\sin\frac{n\pi}{2}\right] - 0 - \left[-100\left(\frac{10}{n\pi}\right)\cos\left(\frac{n\pi}{10}\right) - 20\left(\frac{10}{n\pi}\right)^2\sin\frac{n\pi}{2}\right]\right\} \ldots(\because \sin 0 = 0)$$

$$= \frac{1}{5}\left[2 \times 20\left(\frac{10}{n\pi}\right)^2\sin\frac{n\pi}{2}\right]$$

$$b_n = \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Step VIII :

Put the value of b_n in Equation (4),

$$u(x, y) = \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2} \cdot \sin\frac{n\pi}{2}x \cdot e^{-\frac{n\pi}{10}y} \checkmark \ldots\text{Ans.}$$

is the required temperature at any point (x, y)

Ex. 4.9.3 : Solve the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ with condition

(i) $u = 0$ when $y \to +\infty$ for all x.

(ii) $u = 0$ when $x = 0$, for all values of y

(iii) $u = 0$ when $x = 1$, for all values of y

(iv) $u = x(1 - x)$ when $y = 0$ for $0 \le x \le 1$.

Soln. :

Step I : Let $u(x, y)$ - be the temperature at any point (x, y).

As $y \to \infty$ for all x

The most suitable solution of two dimensional heat flow equation is,

$$u(x, y) = [C_1 \cos mx + C_2 \sin mx][C_3 e^{my} + C_4 e^{-my}] \qquad \ldots(1)$$

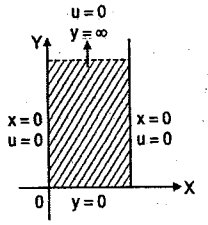


Fig. P. 4.9.3

Step II :

By condition (i) as $y \to \infty$; $u \to 0$ (finite)

Put $t \to \infty$ in Equation (1)

$$0 = [C_1 \cos mx + C_2 \sin mx][C_3 e^{my} + 0]$$

only $C_3 = 0$

($\because$ First bracketed term never zero)

Equation (1) becomes,

$$u(x, y) = [C_1 \cos mx + C_2 \sin mx] C_4 e^{-my} \quad ...(2)$$

Step III :

By condition (ii), as $x = 0, u = 0$

Put, $x = 0$ in Equation (2)

$$\therefore 0 = [C_1 + 0] \cdot C_4 e^{-my}$$

$$\Rightarrow C_1 = 0 \text{ only}$$

($\because$ If we equate $C_4 e^{-my}$ with zero, then from Equation (2), temperature becomes zero)

Equation (2) becomes,

$$u(x, y) = C_2 C_4 \cdot \sin mx \cdot e^{-my} \quad ...(3)$$

Step IV :

By condition (iii) ; as $x = 1, u = 0$

Put, $x = 1$ in Equation (3),

$$\therefore 0 = C_2 C_4 \cdot \sin m \cdot 1 \cdot e^{-my}$$

$$\Rightarrow C_2 C_4 \neq 0; \; e^{-my} \neq 0 \quad \text{only } \sin m = 0$$

$$\Rightarrow m = n\pi \quad ; n = 0, 1, 2, 3. ...$$

Equation (3) becomes,

$$u(x, y) = C_2 C_4 \cdot \sin n\pi x \cdot e^{-n\pi y} \quad ; n = 0, 1, 2, 3, ...$$

Put, $C_2 C_4 = b_n$ and take sum of all terms

Hence,

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin n\pi x \cdot e^{-n\pi y} \quad ...(4).$$

Step V :

Now, by condition (iv) as, $y = 0$, $u = x(1 - x)$ in $0 \le x \le 1$

Put, $y = 0$ in Equation (4)

$$\therefore x(1 - x) = \sum_{n=1}^{\infty} b_n \sin n\pi x \cdot 1 \quad \text{in } 0 \le x \le 1$$

Which is half range sine series ; in $0 \le x \le 1$

$$\text{It is like : } f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \quad \text{in } 0 \le x \le L$$

Step VI

Where,

$$\therefore b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

$$= \frac{2}{1} \int_0^1 (x - x^2) \sin n\pi x \, dx$$

➔ **Using standard result**

$$...\left[\int uv dx = uv_1 - u' v_2 + u'' v_3 - u''' v_4 + ...\right]$$

$$= 2\left\{(x - x^2)\left(\frac{-\cos n\pi x}{n\pi}\right) - (1 - 2x)\left(\frac{-\sin n\pi x}{(n\pi)^2}\right) + (-2)\left(\frac{\cos n\pi x}{(n\pi)^3}\right)\right\}_0^1$$

$$= 2\left\{\left[0 - 0 - 2\frac{\cos n\pi}{(n\pi)^3}\right] - \left[0 - 0 - 2 \cdot \frac{1}{(n\pi)^3}\right]\right\}$$

$$...(\because \sin n\pi = 0 ; \sin 0 = 0)$$

$$\therefore b_n = 2\left[\frac{-2(-1)^n}{(n\pi)^3} + \frac{2}{(n\pi)^3}\right] = \frac{4}{(n\pi)^3}[1 - (-1)^n]$$

Equation (4) becomes,

$$u(x, y) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{[1 - (-1)^n]}{n^3} \cdot \sin n\pi x \cdot e^{-n\pi y} \checkmark \; ...\text{Ans.}$$

Which is the required temperature at any point (x, y).

Ex. 4.9.4 : A rectangular plate with insulated surface is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along short edge $y = 0$ is given by $u(x, 0) = 100 \sin\left(\frac{\pi x}{10}\right)$, while the two long edges $x = 0$ and $x = 10$ as well as the other short edge are kept at 0° C. Find steady state temperature u (x, y).

Soln. :

Step I : Let u(x, y) - be the temperature at any point (x, y).

The given conditions are,

(i) $u(x, \infty) = 0 \; ; \forall x$

(ii) $u(0, y) = 0 \; ; \forall y$

(iii) $u(10, y) = 0 \; ; \forall y$

(iv) $u(x, 0) = 100 \sin\left(\frac{\pi x}{10}\right) \; ; \; 0 \le x \le 10$

Step II :

As $y \to \infty$ for all x,

The most suitable solution is,

$$u(x, y) = [C_1 \cos mx + C_2 \sin mx][C_3 e^{my} + C_4 e^{-my}] \quad ...(1)$$

By condition (i) as $y \to \infty, u \to 0$

As $y \to \infty$ from Equation (1) becomes

$$\therefore 0 = [C_1 \cos mx + C_2 \sin mx][C_3 e^{my} + 0]$$

$$\Rightarrow \text{only } C_3 = 0$$

$\therefore$ Equation (1) becomes,

$$u(x, y) = [C_1 \cos mx + C_2 \sin mx] C_4 e^{-my} \quad ...(2)$$

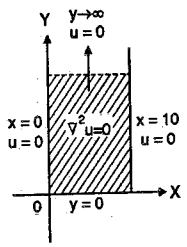


Fig. P. 4.9.4

Step III :

By condition (ii) as $x = 0, u = 0$

Put, $x = 0$ in Equation (2)

$$\therefore 0 = (C_1 + 0) C_4 e^{-my}$$

$$\Rightarrow C_1 C_4 = 0 \quad \text{and} \quad C_4 \neq 0$$

$$\Rightarrow C_1 = 0 \quad \text{only}$$

Equation (2) becomes,

$$u(x, y) = C_2 C_4 \sin mx \cdot e^{-my} \quad ...(3)$$

Step IV :

By condition (iii), as $x = 10, u \to 0$

Put, $x = 10$ in Equation (3)

$$\therefore 0 = C_2 C_4 \sin m10 \; e^{-my}$$

$$\Rightarrow C_2 C_4 \neq 0 \text{ OR } e^{-my} \neq 0$$

$$\text{Only } \sin 10m = 0 \Rightarrow 10m = n\pi$$

$$\Rightarrow m = \frac{n\pi}{10} \quad ; n = 1, 2, 3, ...$$

Equation (3) becomes,

$$u(x, y) = C_2 C_4 \sin \frac{n\pi}{10} x \cdot e^{-\left(\frac{n\pi}{10}\right) y} \quad ; n = 1, 2, 3, ...$$

Put, $C_2 C_4 = b_n$ and take sum of all the terms

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{10} x \cdot e^{-\left(\frac{n\pi}{10}\right) y} \quad ...(4)$$

Step V :

Now, by condition (iv) as $y = 0$; $u = 100 \sin \frac{\pi}{10} x$;

in $0 \le x \le 10$

Put, $y = 0$ in Equation (4)

$$\therefore 100 \sin\left(\frac{\pi}{10} x\right) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{10} x; \quad \text{in } 0 \le x \le 10$$

$$= b_1 \sin \frac{\pi}{10} x + b_2 \sin \frac{2\pi}{10} x + ...$$

Comparing both sides

$$\Rightarrow b_1 = 100 \; ; \; b_2, b_3, ..., b_n = 0$$

Step VI :

Equation (4) becomes,

$$u(x, y) = b_1 \sin \frac{\pi}{10} x \cdot e^{-\frac{\pi}{10} y} + b_2 \sin \frac{2\pi}{10} x \, e^{-\frac{2\pi}{10} y} + ...$$

$$u(x, y) = 100 \sin \frac{\pi}{10} x \cdot e^{-\frac{\pi}{10} y} \checkmark \quad ...\text{Ans.}$$

is the required temperature at any point (x, y)

Ex. 4.9.5 : An infinitely long uniform metal plate is enclosed between lines $y = 0$ and $y = L$ for $x > 0$. The temperature is zero along the edges : $y = 0$, $y = L$ and at infinity. If the edge $x = 0$ is kept at a constant temperature u_0. Find the temperature distribution u (x, y).

Soln. :

Step I : Let u (x, y) - be the temperature at any point x, y.

The given conditions are,

(i) $(\infty, y) = 0 \; ; \forall y$ (ii) $(x, 0) = 0 \; ; \forall x$

(iii) $(x, L) = 0 \; ; \forall x$ (iv) $(0, y) = u_0 \; ; \; 0 \le y \le L$

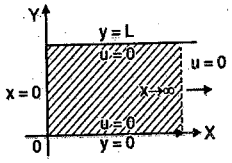


Fig. P. 4.9.5

As $x \to \infty$ for all y

The most suitable solution is,

$$u(x, y) = [C_1 e^{mx} + C_2 e^{-mx}][C_3 \cos my + C_4 \sin my] \quad ...(1)$$

Step II :

By condition (i) as $x \to \infty, u \to 0$

As $x \to \infty$, Equation (1) becomes

$$\therefore 0 = [C_1 e^{mx} + 0][C_3 \cos my + C_4 \sin my]$$

$u \to 0$ if $C_1 = 0$

($\because$ If $[C_3 \cos my + C_4 \sin my] = 0$, then from Equation (1), $u = 0$, but it is not possible)

$\therefore$ Equation (1) becomes,

$$u(x, y) = C_2 e^{-mx}[C_3 \cos my + C_4 \sin my] \quad ...(2)$$

Step III :

By condition (ii) as $y = 0$; $u = 0$

Put $y = 0$ in Equation (2)

$$\therefore 0 = C_2 e^{-mx}[C_3 + 0]$$

$$\Rightarrow C_3 = 0$$

($\because$ $C_2 \neq 0, e^{-mx} \neq 0$. If we equate these terms to zero, we get zero temperature always)

Equation (2) becomes,

$$u(x, y) = C_2 e^{-mx}[C_4 \sin my] \quad ...(3)$$

Step IV :

By condition (iii), as $y = L, u \to 0$

Put y = L in Equation (3)

$\therefore\ 0 = C_2 C_4\, e^{-mx} \sin mL$

$\Rightarrow C_2 C_4 \neq 0,\ e^{-mx} \neq 0$ only, $\sin mL = 0$

$\Rightarrow mL = n\pi;\ n = 1, 2, 3, \ldots$

$\Rightarrow m = \dfrac{n\pi}{L}\ ;\ n = 1, 2, 3, \ldots\ldots$

Put $m = \dfrac{n\pi}{L}$ in Equation (3),

$$u(x, y) = C_2 C_4\, e^{-\left(\frac{n\pi}{L}\right)x} \cdot \sin\left(\frac{n\pi}{L}\right) y\ ;\ n = 1, 2, 3, \ldots$$

Put, $C_2C_4 = b_n$ and take sum of all the terms

$$u(x, y) = \sum_{n=1}^{\infty} b_n \cdot e^{-\left(\frac{n\pi}{L}\right)x} \cdot \sin\left(\frac{n\pi}{L}\right) y \qquad \ldots(4)$$

Step V :

Now, by condition (iv) as $x = 0$; $u = u_0$ in $0 \le y \le L$

Put, $x = 0$ in Equation (4)

$$\therefore\ u_0 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right) y\ ;\ \text{in } 0 \le y \le L$$

which is half range sine series. It is like

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}\, x\ ;\ \text{in } 0 \le x \le L$$

Step VI :

Where, $$b_n = \frac{2}{L} \int_0^L f(y) \sin\left(\frac{n\pi}{L}\right) y\, dy$$

$$= \frac{2}{L} \int_0^L u_0 \sin\left(\frac{n\pi}{L}\right) y\, dy$$

$$b_n = \frac{2}{L} \cdot u_0 \left[\frac{-\cos\left(\frac{n\pi}{L}\right) y}{\left(\frac{n\pi}{L}\right)}\right]_0^L$$

$$= \frac{-2}{L} u_0 \left(\frac{L}{n\pi}\right) [\cos n\pi - 1]$$

$$= -2\, \frac{u_0}{n\pi} [(-1)^n - 1] \qquad \ldots(\because \cos n\pi = (-1)^n)$$

$$b_n = \frac{2 u_0}{n\pi} [\, 1 - (-1)^n \,]$$

Equation (4) becomes,

$$u(x, y) = \frac{2 u_0}{\pi} \sum_{n=1}^{\infty} \left[\frac{1 - (-1)^n}{n}\right] \sin\left(\frac{n\pi}{L}\right) y \cdot e^{-\left(\frac{n\pi}{L}\right)x}$$

✓ ...Ans.

Which is required temperature at any point (x, y).

Ex. 4.9.6 : A rectangular plate is bounded by $x = 0$, $x = a$, $y = 0$, $y = b$. Its surfaces are insulated and temperature along three edges $x = 0$, $x = a$, $y = 0$ is maintained at 0° C. While the fourth edge $y = b$ is maintained at constant temperature u_0, until steady state is reached. Find u (x, y).

Soln. :

Step I : Let, u(x, y) – be the temperature at any point.

The given conditions are,

(i) $u(0, y) = 0°\ C;\ \forall y$

(ii) $u(a, y) = 0°\ C;\ \forall y$

(iii) $u(x, 0) = 0;\ \forall x$

(iv) $u(x, b) = u_0$ in $0 \le x \le a$

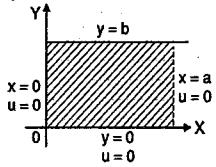


Fig. P. 4.9.6

Step II :

Here, all the sides are finite i.e. plate is bounded. Also, only at y = b ; temperature changes

The most suitable solution is,

$$u(x, y) = [C_1 \cos mx + C_2 \sin mx]\,[C_3 e^{my} + C_4 e^{-my}] \qquad \ldots(1)$$

...(Here, we may use solution in hyperbolic function form)

Step III :

By condition (i) as $x = 0$, $u = 0$

Put, $x = 0$ in Equation (1)

$\therefore\ 0 = [C_1 + 0]\,[C_3 e^{my} + C_4 e^{-my}]$

$\Rightarrow C_1 = 0 \qquad \ldots(\because (C_3 e^{my} + C_4 e^{-my}) \neq 0)$

Equation (1) becomes,

$$u(x, y) = C_2 \sin mx\,[C_3 e^{my} + C_4 e^{-my}] \qquad \ldots(2)$$

Step IV :

By condition (ii) as $x = a$, $u = 0$

Put, $x = a$ in Equation (2)

$\therefore\ 0 = C_2 \sin ma\,[C_3 e^{my} + C_4 e^{-my}]$

$\Rightarrow C_2 \neq 0\ ;\ (C_3 e^{my} + C_4 e^{-my}) \neq 0;$

only $\sin ma = 0$

$\Rightarrow ma = n\pi\ ;$

$\Rightarrow m = \dfrac{n\pi}{a}\ ;\ n = 1, 2, 3, \ldots\ldots$

Equation (2) becomes,

$$u(x,y) = C_2 \sin\left(\frac{n\pi}{a}\right) x \left[C_3\, e^{\frac{n\pi}{a}y} + C_4\, e^{-\frac{n\pi}{a}y}\right];$$

$$n = 1, 2, 3, \ldots\ldots \qquad \ldots(3)$$

Step V :

By condition (iii), as $y = 0$, $u = 0$

Put, $y = 0$ in Equation (3),

$\therefore\ 0 = C_2 \sin\left(\dfrac{n\pi}{a}\right) x\,[C_3 + C_4]$

$C_2 \neq 0\ ;\ \sin\left(\dfrac{n\pi}{a}\right) \neq 0$

This is possible only $C_3 + C_4 = 0 \Rightarrow C_4 = -C_3$

Equation (3) becomes,

$$u(x, y) = C_2 \sin\left(\frac{n\pi}{a}\right) x \left[C_3 e^{\frac{n\pi}{a}y} - C_3 e^{-\frac{n\pi}{a}y}\right];$$

$n = 1, 2, 3, \ldots\ldots$

$$= C_2 C_3 \sin\left(\frac{n\pi}{a}\right) x \cdot \left[e^{\frac{n\pi}{a}y} - e^{-\frac{n\pi}{a}y}\right] \text{ for } n = 1, 2, 3, \ldots$$

$$= C_2 C_3 \sin\left(\frac{n\pi}{a}\right) x \cdot 2\left[\frac{e^{\frac{n\pi}{a}y} - e^{-\frac{n\pi}{a}y}}{2}\right];\quad n = 1, 2, 3, \ldots\ldots$$

$$= 2 C_2 C_3 \sin\left(\frac{n\pi}{a}\right) x \cdot \sinh\left(\frac{n\pi}{a}\right) y \qquad n = 1, 2, 3, \ldots\ldots$$

$$\ldots\left(\because \text{Use}: \frac{e^{\theta} - e^{-\theta}}{2} = \sinh \theta\right)$$

Put, $C_2 C_3 = b_n$ and take sum of all the terms

$$u(x, y) = 2 \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{a}\right) x \cdot \sinh\left(\frac{n\pi}{a}\right) y \qquad \ldots(4)$$

Step VI :

By condition (iv), as $y = b$, $u = u_0$ in $0 \le x \le a$

Put, $y = b$ in Equation (4)

$$\Rightarrow u_0 = 2 \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{a}\right) x \cdot \sinh\left(\frac{n\pi b}{a}\right)$$

$$\frac{u_0}{2 \sinh\left(\frac{n\pi b}{a}\right)} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{a}\right) x$$

Which is half range sine series in $0 \le x \le a$,

It is like, $$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{a}\right) x \quad \text{in } 0 \le x \le L$$

$$\therefore\ b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{a}\, x\, dx$$

$$= \frac{2}{a} \int_0^a \frac{u_0}{2 \sinh\left(\frac{n\pi}{a} b\right)} \cdot \sin \frac{n\pi}{a}\, x\, dx$$

$$= \frac{u_0}{a \sinh\left(n \frac{\pi b}{a}\right)} \int_0^a \sin\left(\frac{n\pi}{a}\right) x\, dx$$

$$= \frac{u_0}{a \sinh\left(\frac{n\pi b}{a}\right)} \cdot \left(\frac{-\cos\left(\frac{n\pi}{a}\right) x}{\frac{n\pi}{a}}\right)_0^a$$

$$\therefore\ b_n = -\frac{u_0}{\sinh\left(\frac{n\pi b}{a}\right)} \cdot \frac{[\cos n\pi - 1]}{n\pi}$$

$$\therefore\ b_n = \frac{u_0}{\pi} \left[\frac{1 - (-1)^n}{n \sinh\left(\frac{n\pi}{a} b\right)}\right]$$

Put this value of b_n in Equation (4), we get

$$u(x, y) = \frac{2 u_0}{\pi} \sum_{n=1}^{\infty} \left[\frac{1 - (-1)^n}{n \sinh\left(\frac{n\pi b}{a}\right)}\right] \sin\left(\frac{n\pi}{a} x\right) \sinh\left(\frac{n\pi}{a}\right) y$$

✓ ...Ans.

Which is the required temperature at (x, y)

Ex. 4.9.7 : An infinitely long plane uniform plate is bounded by two parallel edges in the y-direction and an end at right angles to them. The breadth of the plate is π. This end is maintained at temperature u_0 at all points and other edges at zero temperature. Find the steady-state temperature function u(x, y).

Soln. :

Step I : Let, u (x, y) – be the temperature any point (x, y)

The given conditions are,

(i) $u(x, \infty) = 0,\ \forall x$

(ii) $u(0, y) = 0,\ \forall y$

(iii) $u(\pi, y) = 0,\ \forall y$

(iv) $u(x, 0) = u_0$ in $0 \le x \le \pi$

Step II :

As $y \to \infty$; The most suitable solution is,

$$u(x, y) = [C_1 \cos mx + C_2 \sin mx]\,[C_3 e^{my} + C_4 e^{-my}] \qquad \ldots(1)$$

Step III :

By condition (i)

As $y \to \infty$, $u \to 0$ from Equation (1)

$\therefore\ 0 = [C_1 \cos mx + C_2 \sin mx]\,[C_3 e^{my} + 0]$

$u \to$ finite only if $C_3 = 0$

$\ldots(\because (C_1 \cos mx + C_2 \sin mx) \neq 0)$

Equation (1) becomes,

$$u(x, y) = [C_1 \cos mx + C_2 \sin mx]\,[C_4 e^{-my}] \qquad \ldots(2)$$

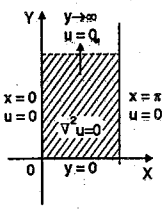


Fig. P. 4.9.7

Step IV :

By condition (ii) as $x = 0, u = 0$

Put $x = 0$ in Equation (2)

$\therefore \quad 0 = [C_1 + 0]\, C_4 e^{-my}$

$C_4 \neq 0 ; e^{-my} \neq 0$

$\Rightarrow$ **only** $C_1 = 0$

Equation (2) becomes,

$u(x, y) = C_2 C_4 \sin mx\, e^{-my}$ ...(3)

Step V :

By condition (iii), as $x = \pi$, $u = 0$

$\therefore \quad 0 = C_2 C_4 \sin m\pi \cdot e^{-my}$

$\Rightarrow \quad C_2 C_4 \neq 0 ; \; e^{-my} \neq 0$

This is possible **only if** $\sin m\pi = 0$

$\Rightarrow \quad m\pi = n\pi$

$\Rightarrow \quad m = n \; ; \; n = 1, 2, 3, \ldots\ldots$

Equation (3) becomes,

$u(x, y) = C_2 C_4 \sin nx \cdot e^{-ny} \quad ; \; n = 1, 2, 3, \ldots\ldots$

put $C_2 C_4 = b_n$ and take sum of all the terms

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin nx \cdot e^{-ny} \quad \ldots(4)$$

Step VI :

Now by condition (iv), as $y = 0$, $u = u_0$ in $0 \le x \le \pi$

$$\therefore \quad u_0 = \sum_{n=1}^{\infty} b_n \sin nx \quad ; \; 0 \le x \le \pi$$

Which is half range sine series in $0 \le x \le \pi$.

$$\text{It is like, } f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right) x \; ; \quad \text{in } 0 \le x \le L$$

Step VII :

Where,

$$\therefore \quad b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}\right) x\, dx = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx\, dx$$

$$= \frac{2u_0}{\pi}\left(\frac{-\cos nx}{n}\right)_0^{\pi} = \frac{-2u_0}{n\pi}\,[\cos n\pi - 1]$$

$$\therefore \quad b_n = \frac{2u_0}{n\pi}\,[1 - (-1)^n] \quad \ldots(\because \cos o = 1; \cos n\pi = (-1)^n)$$

Step VIII :

Put this value of b_n in Equation (4),

$$u(x, y) = \frac{2u_0}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right] \cdot \sin nx \cdot e^{-ny} \checkmark \quad \ldots\textbf{Ans.}$$

which is the required temperature at any point P(x, y)

Ex. 4.9.8 : A square plate has its faces are insulated. The edges $x = 0$ and $x = \pi$ are also insulated while the edge $y = \pi$ is kept at 0°C. The edge $y = 0$ has temperature distribution $u(x, 0) = x^2$. Find the steady state temperature distribution.

Soln. :

Step I : Let, $u(x, y)$ - be the temperature at any point (x, y)

The given conditions are,

(i) $\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0 \quad (\because$ Insulated edge means $\frac{\partial u}{\partial x} = 0)$

(ii) $\left(\frac{\partial u}{\partial x}\right)_{x=\pi} = 0$

(iii) $u(x, \pi) = 0$

(iv) $u(x, 0) = x^2 \; ; \; 0 \le x \le \pi$

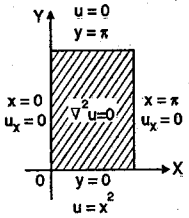


Fig. P. 4.9.8

Step II :

The temperature is different at $y = 0$ edge

The most suitable solution is,

$u(x, y) = [C_1 \cos mx + C_2 \sin mx]\,[C_3 \cosh my + C_4 \sinh my]$ ...(1)

Step III : For condition (i) and (ii),

Differentiate Equation (1) partially w.r.t x, keeping y as constant

$\therefore \quad \frac{\partial u}{\partial x} = [-m C_1 \sin mx + m C_2 \cos mx]\,[C_3 \cosh my + C_4 \sinh my]$ ...(2)

By condition (i), as $x = 0$, $\frac{\partial u}{\partial x} = 0$

Put $x = 0$ in $\frac{\partial u}{\partial x}$ equation

$\therefore \quad 0 = [-m\,0 + m C_2]\,[C_3 \cosh my + C_4 \sinh my]$

$\Rightarrow$ only if $C_2 = 0 \quad \ldots(\because (C_3 \cos h\, my + C_4 \sin h\, my) \neq 0)$

Step V :

And by condition (ii), as $x = \pi$, $\frac{\partial u}{\partial x} = 0$

Put, $x = \pi$ in $\frac{\partial u}{\partial x}$ equation, where $C_2 = 0$

$\therefore \quad 0 = (-m C_1 \sin m\pi)\,(C_3 \cosh my + C_4 \sinh my)$

$C_1 \neq 0, \; (C_3 \cosh my + C_4 \sinh my) \neq 0$

This is possible only if $\sin m\pi = 0$

$\Rightarrow \quad m\pi = n\pi$

$\Rightarrow \quad m = n \quad ; \; n = 1, 2, 3, \ldots\ldots$

$\therefore$ Equation (1) becomes

$u(x, y) = (C_1 \cos nx)\,[C_3 \cosh ny + C_4 \sinh ny]\,;$

$n = 1, 2, 3, \ldots\ldots$ ...(3)

Step VI :

By condition (iii) as $y = \pi$, $u = 0$.

Put $y = \pi$ in Equation (3)

$\therefore \quad 0 = C_1 \cos nx\,[C_3 \cosh n\pi + C_4 \sinh n\pi]$

$\Rightarrow C_1 \cos nx \neq 0$

$\Rightarrow$ **Only if** $[C_3 \cosh n\pi + C_4 \sinh n\pi] = 0$

$$\Rightarrow \quad C_4 = \frac{-C_3 \cosh n\pi}{\sinh n\pi} = -C_3 \coth n\pi$$

Equation (3) becomes,

$u(x, y) = C_1 \cos nx\,[C_3 \cosh ny - C_3 \coth n\pi \cdot \sinh ny]$

$; \; n = 1, 2, 3, \ldots\ldots$

$= C_1 C_3 \cos nx\,[\cos h\, ny - \cot h\, n\pi \cdot \sin h\, y]$

$\ldots n = 1, 2, 3, \ldots.$

Put, $C_1 C_3 = a_n$ and take sum of all the terms

$$u(x, y) = \sum_{n=1}^{\infty} a_n \cos nx\,[\cosh ny - \coth n\pi \sinh ny] \quad \ldots(4)$$

Step VII :

By condition (iv) as, $y = 0$, $u = x^2$ in $0 \le x \le \pi$

Put, $y = 0$ in Equation (4)

$$x^2 = \sum_{n=1}^{\infty} a_n \cos nx\,[1 - 0] \quad \ldots(\because \text{as } \sinh 0 = 0, \cosh 0 = 1)$$

$$x^2 = \sum_{n=1}^{\infty} a_n \cos nx$$

Which is half range cosine series in $0 \le x \le \pi$.

$$\text{It is like } f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right) x \ldots \text{in} \quad 0 \le x \le L$$

Step VIII :

Where,

$$\therefore \quad a_n = \frac{2}{L}\int_0^L f(x) \cos\frac{n\pi}{L}\, x\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$$

➔ **Using standard result**

$$\ldots\left[\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots\right]$$

$$= \frac{2}{\pi}\left\{(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) - (2)\left(\frac{-\sin nx}{n^3}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[0 + \frac{2\pi \cos n\pi}{n^2} + 0\right] - [0 - 0 - 0]\right\}$$

$\ldots(\because \sin n\pi = 0; \sin 0 = 0)$

$$\therefore \quad a_n = \frac{4(-1)^n}{n^2} \quad \ldots(\because \cos n\pi = (-1)^n)$$

Equation (4) becomes,

$$u(x, y) = 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cdot \cos nx$$

$[\cosh ny - \coth n\pi \sinh ny] \checkmark$ ...**Ans.**

is the required temperature distribution at any point P (x, y)

Ex. 4.9.9 : A rectangular metal plate is bounded by $x = 0$, $x = a$, $y = 0$, $y = b$. The edges $x = 0$, $x = a$, $y = b$ are insulated and the edge, $y = 0$ is kept at temperature "$u_0 \cos\frac{\pi x}{a}$".

Find the temperature distribution $u(x, y)$.

Soln. :

Step I : Let, $u(x, y)$ - be the temperature at any point (x, y)

The given boundary conditions are,

(i) $\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0 \quad (\because$ These ends are insulated)

(ii) $\left(\frac{\partial u}{\partial x}\right)_{x=a} = 0$

(iii) $\left(\frac{\partial u}{\partial y}\right)_{y=b} = 0$

(iv) $u(x, 0) = u_0 \cos\frac{\pi x}{a} \; ; \; 0 \le x \le a$

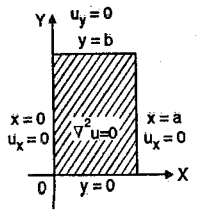


Fig. P. 4.9.9

Step II :

The temperature changes at $y = 0$ edge

The most suitable solution is,

$$u(x, y) = [C_1 \cos mx + C_2 \sin mx] [C_3 \cosh my + C_4 \sinh my] \quad ...(1)$$

For condition (i) and (ii), differentiate Equation (1) partially w.r.t. x, keeping y as constant

$$\therefore \frac{\partial u}{\partial x} = [-m C_1 \sin mx + m C_2 \cos mx] [C_3 \cosh my + C_4 \sinh my]$$

Step III :

By condition (i), as $x = 0$, $\frac{\partial u}{\partial x} = 0$

Put, $x = 0$ in $\frac{\partial u}{\partial x}$ equation

$$\therefore 0 = [0 + m C_2] [C_3 \cosh my + C_4 \sinh my]$$

$\Rightarrow$ only if $C_2 = 0$ ...($\because (C_3 \cos h\, my + C_4 \sin h\, my) \neq 0$)

Step IV :

and by condition (ii), as $x = a$, $\frac{\partial u}{\partial x} = 0$

put, $x = a$ in $\frac{\partial u}{\partial x}$ equation where, $C_2 = 0$

$$\therefore 0 = [-m C_1 \sin ma] [C_3 \cosh my + C_4 \sinh my]$$

$\Rightarrow mC_1 \neq 0$; $(C_3 \cos h\, my + C_4 \sin h\, my) \neq 0$

It is possible **only if** $\sin ma = 0$

$\Rightarrow ma = n\pi$; $n = 1, 2, 3, ...$

$\Rightarrow m = \frac{n\pi}{a}$; $n = 1, 2, 3, ...$

Equation (1) becomes,

$$u(x, y) = C_1 \cos \frac{n\pi}{a} x \left[C_3 \cosh \frac{n\pi}{a} y + C_4 \sinh \frac{n\pi}{a} y\right]$$

as $n = 1, 2, 3, ...$ ...(2)

Step V :

For condition (iii) differentiate Equation (2), partially w.r.t. y, keeping x as constant

$$\frac{\partial u}{\partial y} = C_1 \cos \frac{n\pi}{a} x \cdot \left[\frac{n\pi}{a} C_3 \sinh \frac{n\pi}{a} y + \frac{n\pi}{a} C_4 \cosh \frac{n\pi}{a} y\right]$$

$$...\left(\because \frac{d}{dx}(\cos h\, \theta) = \sin h\, \theta ; \frac{d}{dx}(\sin h\, \theta) = \cos h\, \theta\right)$$

Step VI :

By condition (iii), as $y = b$; $\frac{\partial u}{\partial y} = 0$

Put, $y = b$ in $\frac{\partial u}{\partial y}$ equation

$$\therefore 0 = C_1 \cos \frac{n\pi}{a} x \left[\frac{n\pi}{a} \cdot C_3 \sinh \frac{n\pi}{a} b + \frac{n\pi}{a} C_4 \cosh \frac{n\pi}{a} b\right]$$

It is possible only if,

$$\frac{n\pi}{a} C_3 \sinh \frac{n\pi}{a} b + \frac{n\pi}{a} C_4 \cosh \frac{n\pi}{a} b = 0$$

i.e. if $C_4 = \dfrac{-C_3 \sinh \frac{n\pi}{a} b}{\cosh \frac{n\pi}{a} b} = -C_3 \tan h \frac{n\pi}{a} b$

Equation (2) becomes,

$$u(x, y) = C_1 \cos \frac{n\pi}{a} x \cdot \left[-C_3 \cosh \frac{n\pi}{a} y - C_3 \tan h \frac{n\pi}{a} b \cdot \sinh \frac{n\pi}{a} y\right]$$

$n = 1, 2, 3, ...$

$$= C_1 C_3 \cos \frac{n\pi}{a} x \left[\cosh \frac{n\pi}{a} y - \tan h \left(\frac{n\pi}{a} b\right) \cdot \sinh \frac{n\pi}{a} y\right]$$

$n = 1, 2, 3, ...$

Put, $C_1 C_3 = b_n$ and take sum of all the terms

$$u(x, y) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{a} x \cdot \left[\cosh \frac{n\pi}{a} y - \tan h \left(\frac{n\pi}{a} b\right) \cdot \sinh \frac{n\pi}{a} y\right] \quad ...(3)$$

Step VII :

By condition (iv), as $y = 0$; $u = u_0 \cos \frac{\pi}{a} x$; for $0 \le x \le a$

Put, $y = 0$ in Equation (3),

$$\therefore u_0 \cos \frac{\pi}{a} x = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{a} x [1 - 0] \quad ; 0 \le x \le a$$

Which is half range cosine series. But both sides involves cosine term. So, comparing both sides

$$\therefore u_0 \cos \frac{\pi}{a} x = a_1 \cos \frac{\pi}{a} x + a_2 \cos \frac{2\pi}{a} x + a_3 \cos \frac{3\pi}{a} x + ...$$

By comparing on both sides, we get

$a_1 = u_0$; $a_2 = a_3 = a_4 = ... = a_n = 0$

Equation (3) becomes,

$$u(x, y) = u_0 \cos \frac{\pi}{a} x \cdot \left[\cosh \frac{\pi}{a} y - \tanh\left(\frac{\pi}{a} b\right) \cdot \sinh \frac{\pi}{a} y\right]$$

✓ ...Ans.

which is required temperature at any point (x, y)

Ex. 4.9.10 : A rectangular plate with insulated surface is 1 cm wide and so long to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $y = 0$ is given by $(x - x^2)$ degrees and two long edges $x = 0$, $x = 1$ cm and the other short edge are kept at 0°C, then find the temperature u (x, y).

Soln. : Let u(x, y) be the temperature.

The partial differential equation of two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The given conditions are

(i) $u(x, \infty) = 0$; $\forall x$

(ii) $u(0, y) = 0$; $\forall y$

(iii) $u(1, y) = 0$; $\forall y$

(iv) $u(x, 0) = x - x^2$; $0 \le x \le 1$

As $y \to \infty$, the most suitable solution is

$$u(x, y) = (C_1 \cos mx + C_2 \sin mx)(C_3 e^{my} + C_4 e^{-my}) \quad ...(1)$$

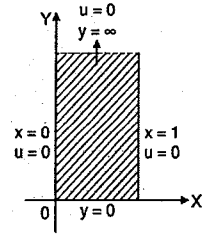


Fig. P. 4.9.10

By condition (i), as $y \to \infty$, $u = 0$

$$\therefore 0 = (C_1 \cos mx + C_2 \sin mx)(C_3 e^{my} + 0)$$

u is finite only if $C_3 = 0$

Equation (1) becomes

$$u(x, y) = [C_1 \cos mx + C_2 \sin mx] C_4 e^{-my} \quad ...(2)$$

By condition (ii), $u = 0$ as $x = 0$

$0 = (C_1 + 0) C_4 e^{-my}$ $\Rightarrow$ $C_1 = 0$ only

Equation (2) becomes

$$u(x, y) = C_2 C_4 \sin mx\, e^{-my} \quad ...(3)$$

By condition (iii), $u = 0$ as $x = 1$

$0 = C_2 C_4 \sin m1\, e^{-my}$

$\Rightarrow C_2 C_4 \neq 0$ ($\because$ if $C_2 C_4 = 0$ then completely $u = 0$)

$\therefore$ only $\sin m = 0$

$\Rightarrow m = n\pi$; $n = 0, 1, 2, 3, ...$

Equation (3) becomes

$$u(x, y) = C_2 C_4 \sin (n\pi) x\, e^{-n\pi y} ; \quad n = 1, 2, 3, ...$$

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin (n\pi) x\, e^{-n\pi y} \quad ....(4)$$

Now by condition (iv) as $y = 0$; $u = x - x^2$

$\therefore$ From Equation (4),

$$(x - x^2) = \sum b_n \sin n\pi x \quad \text{in } 0 \le x \le 1$$

But it is a half range sine series

$$\therefore b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x\, dx$$

$$= \frac{2}{1} \int_0^1 (x - x^2) \sin n\pi x\, dx$$

$$b_n = 2\left[(x - x^2)\left(\frac{-\cos n\pi x}{n\pi}\right) - (1 - 2x)\left(\frac{-\sin n\pi x}{}\right) + (-2)\left(\frac{+\cos n\pi x}{(n\pi)^3}\right)\right]_0^1$$

$$= 2\left[0 + 0 - \frac{\cos n\pi}{(n\pi)^3} + 0 - 0 + \frac{2(1)}{(n\pi)^3}\right]$$

$$b_n = \frac{4}{(n\pi)^3} [1 - (-1)^n]$$

Equation (4) becomes

$$u(x, y) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \left(\frac{1 - (-1)^n}{n}\right) \sin n\pi x \cdot e^{-n\pi y} ✓ \quad ...Ans.$$

is the required temperature.

Ex. 4.9.11 : A square metal plate of side a has edges represented by lines $x = 0$, $x = a$, $y = 0$, $y = a$. The edges $x = a$ and $y = a$ are insulated. The edge $x = 0$ is kept at 0°C and $y = 0$ at u_0 ; where u_0 is constant. Obtain the temperature distribution u (x,y) under steady state conditions.

Soln. :

Step I : Let, u(x, y) – be the temperature at any point (x, y)

The given conditions are,

(i) $u(0, y) = 0$; $\forall y$

(ii) $\left(\frac{\partial u}{\partial x}\right)_{x=a} = 0$; $\forall y$

(iii) $\left(\frac{\partial u}{\partial y}\right)_{y=a} = 0$; $\forall y$

$\therefore$ The ends at $x = a$ and $y = a$ are insulated

(iv) $u(x, 0) = u_0$; $0 \le x \le a$

The most suitable solution is,

$$u(x, y) = [C_1 \cos mx + C_2 \sin mx] [C_3 \cosh my + C_4 \sinh my] \quad ...(1)$$

Step II : By condition (i) as $x = 0$, $u = 0$

Put $x = 0$ in (1)

$$\therefore 0 = [C_1 + 0] [C_3 \cosh my + C_4 \sinh my]$$

$\Rightarrow C_1 = 0$

Equation (1) becomes,

$$u(x, y) = C_2 \sin mx [C_3 \cosh my + C_4 \sinh my] \quad ...(2)$$

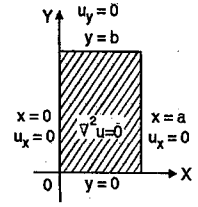


Fig. P. 4.9.11

Step III : For condition (ii), Differentiate Equation (2) w.r.t x

$$\therefore \frac{\partial u}{\partial x} = C_2 m \cos mx [C_3 \cosh my + C_4 \sinh my]$$

By condition (ii), as $x = a$, $\frac{\partial u}{\partial x} = 0$

Put $x = a$ in equation $\frac{\partial u}{\partial x}$

$$\therefore 0 = C_2 m \cos ma [C_3 \cosh my + C_4 \sinh my]$$

This is possible **only if** $\cos ma = 0$

$\Rightarrow ma = \frac{n\pi}{2}$; $n = 1, 3, 5.....$

$$\Rightarrow \quad m = \frac{n\pi}{2a} \quad ; n = 1, 3, 5, \ldots\ldots$$

Equation (2) becomes,

$$u(x, y) = C_2 \sin\left(\frac{n\pi}{2a}\right) x \cdot \left[C_3 \cosh \frac{n\pi}{2a} y + C_4 \sinh \frac{n\pi}{2a} y \right] \quad \ldots(3)$$

$$n = 1, 3, 5, \ldots\ldots$$

Step IV : For condition (iii), differentiate Equation (3) w.r.t y

$$\frac{\partial u}{\partial y} = C_2 \sin\left(\frac{n\pi}{2a}\right) x \left[\frac{n\pi}{2a} C_3 \sinh \frac{n\pi}{2a} y + \frac{n\pi}{2a} C_4 \cos h \left(\frac{n\pi}{2a}\right) y \right]$$

By condition (iii), as $y = a$; $\frac{\partial u}{\partial y} = 0$

Put $y = a$ in equation $\frac{\partial u}{\partial y}$

$$\therefore 0 = C_2 \sin\left(\frac{n\pi}{2a}\right) x \cdot \left[\frac{n\pi}{2a} C_3 \sinh \frac{n\pi}{2} + \frac{n\pi}{2a} C_4 \cosh \frac{n\pi}{2} \right]$$

It is possible only if, $\left(\frac{n\pi}{2a} C_3 \sinh \frac{n\pi}{2} + \frac{n\pi}{2a} C_4 \cosh \frac{n\pi}{2} \right) = 0$

$$\Rightarrow C_4 = -C_3 \frac{\sinh \frac{n\pi}{2}}{\cosh \frac{n\pi}{2}} = -C_3 \tanh \frac{n\pi}{2}$$

Equation (3) becomes,

$$u(x, y) = C_2 \sin\left(\frac{n\pi}{2a}\right) x \cdot \left[C_3 \cosh \frac{n\pi}{2a} y - C_3 \tanh \frac{n\pi}{2} \cdot \sinh \frac{n\pi}{2a} y \right]$$

$$n = 1, 3, 5 \ldots.$$

Put $C_2 = b_n$ and take sum of all terms

$$= \sum_{n=1,3,5}^{\infty} b_n \sin\left(\frac{n\pi}{2a}\right) x \left[\cosh \frac{n\pi}{2a} y - \tanh \frac{n\pi}{2} \cdot \sinh \frac{n\pi}{2a} y \right]$$

$$u(x, y) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{(2n+1)\pi}{2a}\right) x \left[\cosh \frac{(2n+1)\pi}{2a} y - \tanh \frac{(2n+1)\pi}{2} \cdot \sinh \frac{(2n+1)\pi}{2a} y \right]$$

Step V : Now by condition (iv), as $y = 0$, $u = u_0$ in $0 \le x \le a$

Put $y = 0$ in Equation (4)

$$\therefore u_0 = \sum_{n=0}^{\infty} b_n \sin \frac{(2n+1)\pi}{2a} x \cdot [1 - 0]$$

$$\ldots(\because \sinh 0 = 0 \ , \cosh 0 = 1)$$

Which is half range sine series in $0 \le x \le a$, where

$$\therefore b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{(2n+1)\pi}{2a} x \, dx$$

$$= \frac{2}{a} \int_0^a u_0 \sin \frac{(2n+1)\pi}{2a} x \, dx$$

$$= \frac{2 u_0}{a} \left[\frac{-\cos\left(\frac{(2n+1)\pi}{2a}\right) x}{\frac{(2n+1)\pi}{2a}} \right]_0^a$$

$$= \frac{4 u_0}{(2n+1)\pi} \left\{ -\cos \frac{(2n+1)\pi}{2} + 1 \right\}$$

$$\therefore b_n = \frac{4 u_0}{(2n+1)\pi} \left[1 - \cos \frac{(2n+1)\pi}{2} \right]$$

Put in Equation (4),

$$u(x, y) = \frac{4 u_0}{\pi} \sum_{n=0}^{\infty} \left\{ \left[\frac{1 - \cos\left(\frac{(2n+1)\pi}{2}\right)}{(2n+1)} \right] \cdot \sin\left(\frac{(2n+1)\pi}{2a}\right) x \times \left[\cosh \frac{(2n+1)}{2a} \pi y - \tanh \frac{(2n+1)\pi}{2} \times \sinh \frac{(2n+1)}{2a} \pi y \right] \right\}$$

✓ ...Ans.

Ex. 4.9.12 : Solve the two dimensional heat flow equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for the following conditions :

(i) $u(0, y) = 0 ; \forall y$

(ii) $u(1, y) = 0 ; \forall y$

(iii) $u(x, \infty) = 0 ; \forall x$

(iv) $u(x, 0) = \begin{cases} x & ; 0 \le x \le 0.5 \\ 1 - x & ; 0.5 \le x \le 1 \end{cases}$

Soln. :

Step I : Let, $u(x, y)$ – be the temperature at any point (x, y),

The given conditions are,

(i) $u(0, y) = 0 \ ; \ \forall y$

(ii) $u(1, y) = 0 \ ; \ \forall y$

(iii) $y(x, \infty) = 0 \ ; \ \forall x$

(iv) $u(x, 0) = \begin{cases} x & ; 0 \le x \le 0.5 \\ 1 - x & ; 0.5 \le x \le 1 \end{cases}$

As $y \to \infty$, the most suitable solution is,

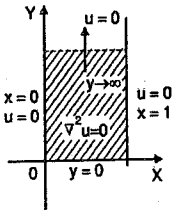


Fig. P. 4.9.12

$$u(x, y) = [C_1 \cos mx + C_2 \sin mx] [C_3 e^{my} + C_4 e^{-my}] \quad \ldots(1)$$

Step II : By condition (i) as $x \to 0$, $u \to 0$

Put $x = 0$ in Equation (1),

$$\therefore 0 = [C_1 + 0] [C_3 e^{my} + C_4 e^{-my}]$$

$$\Rightarrow C_1 = 0$$

Equation (1) becomes,

$$u(x, y) = C_2 \sin mx \ [C_3 e^{my} + C_4 e^{-my}] \quad \ldots(2)$$

Step III : By condition (ii) as $x \to 1$, $u \to 0$

Put $x = 1$ in Equation (2),

$$\therefore 0 = C_2 \sin m \ [C_3 e^{my} + C_4 e^{-my}]$$

This is possible only if $\sin m = 0$

$$\Rightarrow m = n\pi \quad ; n = 1, 2, 3, \ldots\ldots$$

$$u(x, y) = C_2 \sin n\pi x \cdot [C_3 e^{n\pi y} + C_4 e^{-n\pi y}] ; n = 1, 2, 3, \ldots\ldots \quad \ldots(3)$$

Step IV : Now by condition (iii) as $y \to \infty$, $u \to 0$

Put $y = \infty$ in Equation (3),

$$\therefore 0 = C_2 \sin n\pi x \ [C_3 e^{n\pi y} + 0]$$

It is possible only if $\boxed{C_3 = 0}$

$$\therefore u(x, y) = C_2 C_4 \sin n\pi x \ e^{-n\pi y} ; n = 1, 2, 3, \ldots\ldots$$

Put, $C_2 C_3 = b_n$ and take sum of all terms

$$\therefore u(x, y) = \sum_{n=1}^{\infty} b_n \sin n\pi x \cdot e^{-n\pi y} \quad \ldots(4)$$

Step V : By condition (iv), as $y \to 0$,

$$u = \begin{cases} x & ; 0 \le x \le 0.5 \\ 1 - x & ; 0.5 \le x \le 1 \end{cases}$$

Put $y = 0$ in Equation (4),

$$\therefore \left. \begin{matrix} x & ; 0 \le x \le 0.5 \\ 1 - x & ; 1.5 \le x \le 1 \end{matrix} \right\} = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

Which is half range sine series in $0 \le x \le 1$, where

$$\therefore b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx = \frac{2}{1} \int_0^1 f(x) \sin n\pi x \, dx$$

$$= \frac{2}{1} \left[\int_0^{0.5} f(x) \sin n\pi x \, dx + \int_{0.5}^{1} f(x) \sin n\pi x \, dx \right]$$

$$= 2 \left[\int_0^{0.5} x \sin n\pi x \, dx + \int_{0.5}^{1} (1 - x) \sin n\pi x \, dx \right]$$

➜ **Using standard result**

$$\ldots \left[\int uv dx = uv_1 - u' v_2 + u'' v_3 - u''' v_4 + \ldots\ldots \right]$$

$$\therefore b_n = 2 \left\{ \left[(x) \left(\frac{-\cos n\pi x}{n\pi} \right) - (1) \left(\frac{-\sin n\pi x}{(n\pi)^2} \right) \right]_0^{0.5 = 1/2} + \left[(1 - x) \left(\frac{-\cos n\pi x}{n\pi} \right) - (-1) \left(\frac{-\sin n\pi x}{(n\pi)^2} \right) \right]_{0.5 = 1/2}^{1} \right\}$$

$$= 2 \left\{ \left[\frac{-\left(\frac{1}{2}\right)}{n\pi} \left(\frac{\cos n\pi}{2} \right) + \frac{\sin \frac{n\pi}{2}}{(n\pi)^2} - 0 \right] + \left[0 - \left[\frac{-\left(\frac{1}{2}\right)}{n\pi} \cdot \left(\frac{\cos n\pi}{2} \right) - \frac{\sin \frac{n\pi}{2}}{(n\pi)^2} \right] \right] \right\}$$

$$= 2 \left[0 + \frac{2 \sin\left(\frac{n\pi}{2}\right)}{(n\pi)^2} \right]$$

$$\therefore b_n = \frac{4}{\pi^2} \frac{\sin\left(\frac{n\pi}{2}\right)}{(n^2)},$$

Put in Equation (4), we get

$$u(x, y) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{(n^2)} \cdot \sin n\pi x \cdot e^{-n\pi y}$$

✓ ...Ans.

Which is required temperature at any point (x, y).

Ex. 4.9.13 : A rectangular plate bounded by lines $x = 0$, $x = a$, $y = 0$, $y = b$ has its faces insulated while the edges $x = 0$, $x = a$ and $y = b$ are kept at zero degree and lower edge $(y = 0)$ is having temperature distribution. $5 \sin\left(\frac{4\pi}{a}\right) x + 3 \sin\left(\frac{3\pi}{a}\right) x$. Find the steady state temperature distribution.

Soln. :

Step I : Let, $u(x, y)$ – be the temperature at any point (x, y).

The boundary conditions are,

(i) $u(0, y) = 0 \ ; \ \forall y$

(ii) $u(a, y) = 0 \ ; \ \forall y$

(iii) $u(x, b) = 0 \ ; \ \forall x$

(iv) $u(x, 0) = 5 \sin \frac{4\pi}{a} x + 3 \sin\left(\frac{3\pi}{a}\right) x$ in $0 \le x \le a$

as $y \to 0$, $u = f(x)$

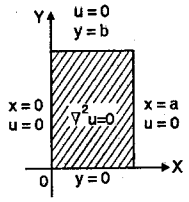


Fig. P. 4.9.13

The most suitable solution is,

$$u(x, y) = [C_1 \cos mx + C_2 \sin mx] [C_3 \cosh my + C_4 \sinh my] \quad \ldots(1)$$

Step II : By condition (i) as $x = 0$, $u = 0$

Put $x = 0$ in Equation (1),

$$\therefore 0 = [C_1 + 0] [C_3 \cosh my + C_4 \sinh my]$$

$$\Rightarrow \text{only } C_1 = 0$$

Equation (1) becomes,

$$u(x, y) = C_2 \sin mx [C_3 \cosh my + C_4 \sinh my] \quad \ldots(2)$$

Step II : By condition (ii) as $x = a, u = 0$

Put $x = a$ in Equation (1),

$\therefore \quad 0 = C_2 \sin ma\,[C_3 \cosh my + C_4 \sinh my]$

$\Rightarrow$ It is possible **only if,** $\sin ma = 0 \Rightarrow ma = n\pi$; $n = 1, 2, 3, \ldots\ldots$

$\Rightarrow \quad m = \dfrac{n\pi}{a}$; $n = 1, 2, 3, \ldots\ldots$

Put in Equation (2),

$$u(x, y) = C_2 \sin\frac{n\pi}{a}x \cdot \left[C_3 \cosh\frac{n\pi}{a}y + C_4 \sinh\frac{n\pi}{a}y\right];$$

$n = 1, 2, 3\ldots\ldots$ ...(3)

Step III : By condition (iii), as $y = b, u = 0$

Put $y = b$ in Equation (3)

$\therefore \quad 0 = C_2 \sin\dfrac{n\pi}{a}x\,[C_3 \cosh n\pi + C_4 \sinh n\pi]$

It is possible only if, $C_3 \cosh n\pi + C_4 \sinh n\pi = 0$

$\Rightarrow \quad C_4 = -C_3 \coth n\pi$; $n = 1, 2, 3$

Put this value of C_4 in Equation (3) we get,

$$u(x, y) = C_2 \sin\left(\frac{n\pi}{a}\right)x \cdot \left[C_3 \cosh\frac{n\pi}{a}y - C_3 \coth n\pi \cdot \sinh\left(\frac{n\pi}{a}\right)y\right]; n = 1, 2, 3, \ldots$$

Put $C_2 C_3 = b_n$ and take sum of all terms

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{a}\right)x \left[\cosh\frac{n\pi}{a}y - \coth n\pi \cdot \sinh\left(\frac{n\pi}{a}\right)y\right]$$

...(4)

Step IV : By condition (iv) as $y = 0$, $u = f(x)$ in $0 \le x \le a$

Put $y = 0$ in Equation (4)

$$\therefore \quad 5 \sin\frac{4\pi}{a}x + 3 \sin\frac{3\pi}{a}x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{a}x\,[1 - 0]$$

...($\because \sinh 0 = 0$; $\cosh 0 = 1$)

$$\therefore 5 \sin\left(\frac{4\pi}{a}x\right) + 3 \sin\left(\frac{3\pi}{a}\right)x = b_1 \sin\frac{\pi}{a}x + b_2 \sin\frac{2\pi}{a}x + b_3 \sin\frac{3\pi}{a}x + b_4 \sin\frac{4\pi}{a}x + b_5 \sin\frac{5\pi}{a}x + \ldots\ldots$$

Comparing both sides, we get

$b_1 = 0,\ b_2 = 0,\ b_3 = 3,\ b_4 = 5;\ b_5 = 0,\ b_6 = 0, \ldots\ldots, b_n = 0$

Equation (4) becomes,

$$u(x, y) = 0 + 0 + 3 \sin\frac{3\pi}{a}x \left[\cosh\frac{3\pi}{a}y - \coth 3\pi \cdot \sinh\left(\frac{3\pi}{a}\right)y\right] + 4 \sin\frac{4\pi}{a}x \left[\cosh\frac{4\pi}{a}y - \coth 4\pi \cdot \sinh\left(\frac{4\pi}{a}\right)y\right] + 0$$

$$u(x, y) = 3 \sin\frac{3\pi}{a}x \left[\cosh\frac{3\pi}{a}y - \coth 3\pi \cdot \sinh\frac{3\pi}{a}y\right] + 5 \sin\frac{4\pi}{a}x \left[\cosh\frac{4\pi}{a}y - \coth 4\pi \cdot \sinh\frac{4\pi}{a}y\right]$$

✓ ...Ans.

Is the required temperature distribution.

Ex. 4.9.14 : The lower side ($y = 0$) of a rectangular metal plate of length "a" and width "b" with insulated upper and lower surface is kept at 100°C while upper side $y = b$ is insulated. If the other two sides are kept at 0°C, find the steady state temperature distribution of the plate.

Soln. :

Step I : Let $u(x, y)$ be the temperature at any point (x, y)

The given boundary conditions are,

(i) $u(x, 0) = 100°\,C$; for $0 \le x \le a$

(ii) $\left(\dfrac{\partial u}{\partial x}\right)_{y=b} = 0$; ($\because y = b$ end insulated)

(iii) $u(0, y) = 0$; $\forall y$

(iv) $u(a, y) = 0$; $\forall y$

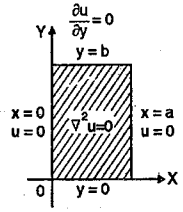


Fig. P. 4.9.14

The most suitable solution is,

$u(x, y) = [C_1 \cos mx + C_2 \sin mx]\,[C_3 \cosh my + C_4 \sinh my]$ ...(1)

Step II : By condition (iii), as $x = 0, u = 0$

Put $x = 0$ in Equation (1),

$\Rightarrow \quad 0 = [C_1 + 0]\,[C_3 \cosh my + C_4 \sinh my]$

$\Rightarrow \quad C_1 = 0$

Equation (i) becomes,

$u(x, y) = C_2 \sin mx\,[C_3 \cosh my + C_4 \sinh my]$ ...(2)

Step III : For condition (ii), differentiate Equation (2), w.r.t. y

$\therefore \quad \dfrac{\partial u}{\partial x} = C_2 \sin mx\,[m C_3 \sinh my + m C_4 \cosh my]$

By condition (ii) as $y = b$, $\dfrac{\partial u}{\partial y} = 0$

Put $y = b$ in Equation (2), we get

$\therefore \quad 0 = C_2 \sin mx\,[m C_3 \sinh mb + m C_4 \cosh mb]$

This is possible only if $m C_3 \sinh mb + m C_4 \cosh mb = 0$

i.e. if $C_4 = -C_3 \tanh mb$

Equation (2) becomes,

$u(x, y) = C_2 \sin mx\,[C_3 \cosh my - C_3 \tanh mb \sinh my]$

$u(x, y) = C_2 C_3 \sin mx\,[\cosh my - \tanh mb \sinh my]$ ...(3)

Step IV : By condition (iv) as, $x = a$, $u = 0$

Put $x = a$ in Equation (3),

$0 = C_2 C_3 \sin ma\,[\cosh my - \tanh mb \sinh my]$

It is possible **only if** $\sin ma = 0$

$\Rightarrow \quad ma = n\pi$; $n = 1, 2, 3, \ldots\ldots$

$\Rightarrow \quad m = \dfrac{n\pi}{a}$; $n = 1, 2, 3, \ldots\ldots$

Equation (3) becomes

$$u(x, y) = C_2 C_3 \sin\frac{n\pi}{a}x \cdot \left[\cosh\frac{n\pi}{a}y - \tanh\frac{n\pi}{a}b \cdot \sinh\frac{n\pi}{a}y\right]; n = 1, 2, 3, \ldots$$

Put $C_2 C_3 = b_n$ and take sum of all terms

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{a}x \left[\cosh\frac{n\pi}{a}y - \tanh\frac{n\pi}{a}b \cdot \sinh\frac{n\pi}{a}y\right]$$

...(4)

Step V : By condition (i) as, $y = 0$, $u = 100$ in $0 \le x \le a$

$$100 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{a}x\,[1 - 0] \quad (\because \sinh 0 = 0, \cosh 0 = 1)$$

Which is half range sine series in $0 \le x \le a$, where

$$\therefore \quad b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L}x\,dx = \frac{2}{a}\int_0^a 100 \sin\frac{n\pi}{a}x\,dx$$

$$= \frac{200}{a}\left(\frac{-\cos\frac{n\pi}{a}x}{\left(\frac{n\pi}{a}\right)}\right)_0^a$$

$$= \frac{200}{n\pi}[-\cos n\pi + 1]$$

$$\therefore \quad b_n = \frac{200}{n\pi}[1 - (-1)^n]$$

Equation (4) becomes,

$$U(x, y) = \frac{200}{\pi}\sum_{n=1}^{\infty}\left[\frac{1 - (-1)^n}{n}\right] \cdot \sin\frac{n\pi}{a}x \cdot \left[\cosh\frac{n\pi}{a}y - \tanh\left(\frac{n\pi}{a}b\right) \cdot \sinh\frac{n\pi}{a}y\right]$$

✓ ...Ans.

is the required temperature distribution.

Ex. 4.9.15 : Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ which satisfies the conditions :

$u(0, y) = u(l, y) = u(x, 0) = 0$

and $u(x, a) = \sin\dfrac{n\pi}{l}x$; $0 \le x \le l$

Soln. :

Step I : Let $u(x, y)$ – be the temperature at any point (x, y)

The given conditions are,

(i) $u(0, y) = 0$; $\forall y$

(ii) $u(l, y) = 0$; $\forall y$

(iii) $u(x, 0) = 0$; $\forall x$

(iv) $u(x, a) = \sin\dfrac{n\pi}{l}x$; $0 \le x \le l$,

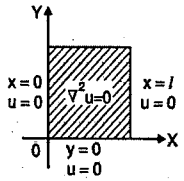


Fig. P. 4.9.15

As $y \to a$, $u = f(x)$

The most suitable solution of $u_{xx} + u_{yy} = 0$ is,

$u(x, y) = [C_1 \cos mx + C_2 \sin mx]\,[C_3 \cosh my + C_4 \sinh my]$ ...(1)

Step II : For condition (i), as $x = 0$, $u = 0$

Put $x = 0$ in Equation (1),

$\therefore \quad 0 = [C_1 + 0]\,[C_3 \cosh my + C_4 \sinh my]$

$\Rightarrow \quad C_1 = 0$

Equation (1) becomes

$\therefore u(x, y) = C_2 \sin mx\,[C_3 \cosh my + C_4 \sinh my]$ ...(2)

Step III : By condition (iii), as $y \to 0$, $u \to 0$

Put $y = 0$ in Equation (2),

$\therefore \quad 0 = C_2 \sin mx\,(C_3 + 0)$

$\Rightarrow \quad C_3 = 0$

Equation (2) becomes,

$u(x, y) = C_2 \sin mx \cdot C_4 \sinh my$ ...(3)

Step IV : By condition (ii) as $x = l$, $u = 0$

Put $x = l$ in Equation (3),

$\therefore \quad 0 = C_2 \sin ml \cdot C_4 \sinh my$

Only $\sin ml = 0$

$\Rightarrow \quad ml = n\pi$; $n = 1, 2, 3, \ldots\ldots$

$\Rightarrow \quad m = \dfrac{n\pi}{l}$; $n = 1, 2, 3, \ldots\ldots$

Equation (3) becomes,

$$u(x, y) = C_2 C_4 \sin \frac{n\pi}{l} x \cdot \sinh \frac{n\pi}{l} y; \qquad n = 1, 2, 3, \ldots$$

Put $C_2 C_4 = b_n$ and take sum of all terms

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cdot \sinh\left(\frac{n\pi}{l}\right) a \qquad \ldots(4)$$

Step V : By condition (iv) as $y \to a$, $u = \sin \frac{n\pi}{l} x$ for $0 \le x \le a$

Put $x = a$ in Equation (4),

$$\therefore \sin \frac{n\pi}{l} x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cdot \sinh \frac{n\pi}{l} a$$

Put $n = 1, 2, 3, \ldots\ldots$ on R.H.S. we get,

$$\therefore \sin \frac{n\pi}{l} x = b_1 \sin \frac{\pi}{l} x \sinh \frac{\pi}{l} a + b_2 \sin \frac{2\pi}{l} x \cdot \sinh \frac{2\pi}{l} a + \ldots\ldots\ldots$$

$$\sin \frac{n\pi}{l} x = b_n \sin \frac{n\pi}{l} x \cdot \sinh \frac{n\pi}{l} a$$

Comparing both sides

$$\Rightarrow \quad b_n = \frac{1}{\sinh \frac{n\pi}{l} a}$$

Equation (4) becomes,

$$\therefore u(x, y) = \frac{1}{\sinh\left(\frac{n\pi a}{l}\right)} \cdot \sin \frac{n\pi}{l} x \cdot \sinh\left(\frac{n\pi}{l}\right) y \checkmark \quad \ldots\text{Ans.}$$

Exercise

Ex. 1 : Solve the equation : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, with conditions :

(i) $u = 0$ when $y \to \infty$ for all x

(ii) $u = 0$ when $x = 0$; for all y

(iii) $u = 0$ when $x = 10$; for all y

(iv) $u = k(1 - x)$ when $y = 0$ for $0 < x < 10$.

Ex. 2 : A rectangular plate with insulated surfaces is 8 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by

$$u(x, 0) = 100 \sin\left(\frac{\pi x}{8}\right); 0 < x < 8$$

while the two edges $x = 0$ and $x = 8$ as well as the other short edge are kept at 0° C. Find the steady – state temperature at any point of the plate.

Ans. : $u(x, y) = 100 \sin e^{\frac{-\pi}{8} y} \cdot \sin\left(\frac{\pi}{8} x\right)$

Ex. 3 : Solve : $\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, for $0 \le x \le \pi$; $0 < y < \pi$, given that $u(0, y) = u(\pi, y) = u(x, \pi) = 0$ and $u(x, 0) = \sin^2 x$.

Ans. : $u(x, y) = \frac{-8}{\pi} \sum_{n=1,3,5}^{\infty} \frac{\sin n x \cdot \sinh n(\pi - y)}{n(n^2 - 4) \sinh n\pi}$

Ex. 4 : Solve : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ within the rectangle $0 \le x \le a$, $0 \le y \le b$. Given : $u(0, y) = u(a, y) = u(x, b) = 0$; and $u(x, 0) = x(a - x)$

Ans. : $u(x, y) = \frac{8a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3}$

$$\cdot \sin\left(\frac{\sin (2n+1) \pi x}{a}\right) \cdot \frac{\sinh\left(\frac{2n+1}{a}\right) \pi \cdot (b - y)}{\sinh\left(\frac{(2n+1) \pi b}{a}\right)}$$

Ex. 5 : A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, 20) = \pi(20 - x)$, when $0 < x < 20$, while all other edges are at 0° C. find the temperature in the given plate.

Ans. : $u(x, y) = \frac{3200}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{2n-1}{20}\right) \pi x \cdot \sinh\left(\frac{(n-1)}{20}\right) \pi k}{(2n-1)^3 \cdot \sinh(2n-1) \pi}$

Ex. 6 : The temperature u is maintained at 0°C along three edges of a square plate of the length 100 cm and the fourth edge is maintained at 100° C until steady state condition prevail. Find the temperature $u(x, y)$ at any point (x, y). Also find the temperature at centre of the plate

Ans. : $u(x, y)$ at centre

$$u(x, y) = \frac{200}{\pi} \cdot \left[\frac{1}{\cosh \frac{\pi}{2}} - \frac{1}{3 \cosh \frac{3\pi}{2}} + \frac{1}{5 \cosh \frac{5\pi}{2}} + \ldots\right]$$

Chapter Ends...

□□□

UNIT 4

Partial Differential Equations and Their Applications

Syllabus

Formation of Partial differential equations by eliminating arbitrary constants and functions; Equations solvable by direct integration; Linear equations of first order (Lagrange's linear equations); Method of separation of variables – applications to find solutions of one dimensional heat flow equation (i.e. $\left(\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}\right)$ Laplace equation : $\frac{\partial^2 u}{\partial x^2} + c^2 \frac{\partial^2 u}{\partial y^2} = 0$).

➢ **Chapter 4 : Partial Differential Equations and Their Applications**

CHAPTER 4

UNIT IV

Partial Differential Equations and Their Applications

Multiple Choice Questions

Q. 1 First order partial differential equation is obtained, by eliminating an arbitrary constants

(a) In the solution, if the number of an arbitrary constants is equal to the number of independent variables

(b) In the solution, if the number of an arbitrary constants is less than the number of independent variables

(c) In the solution, if the number of an arbitrary constants is greater than the number of independent variables

(d) In the solution, if the number of an arbitrary functions is equal to the number of independent variables

Ans. : (a)

Q. 2 Second order partial differential equation is obtained, by eliminating an arbitrary constants

(a) In the solution, if the number of an arbitrary constants is equal to the number of independent variables

(b) In the solution, if the number of arbitrary constant is more than the number of independent variables.

(c) In the solution, if the number of an arbitrary constants is less than the number of independent variables

(d) If the number of an arbitrary functions is equal to the number of independent variables

Ans. : (b)

Q. 3 By eliminating an arbitrary constants from the function $z = ax + by + ab$,

(a) First order Partial differential equation is obtained.

(b) Second order Partial differential equation is obtained.

(c) Third order Partial differential equation is obtained.

(d) Forth order Partial differential equation is obtained.

Ans. : (a)

Q. 4 By eliminating an arbitrary constants from the function $z = ax + by + ab$, with $\frac{\partial z}{\partial x} = a$; $\frac{\partial z}{\partial y} = b$,

Partial differential equation is

(a) $z = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$

(b) $z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial x}\right)$

(c) $z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$

(d) $z = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$

Ans. : (c)

Q. 5 By eliminating an arbitrary constants from the function $(x-a)^2 + (y-b)^2 + z^2 = 1$, with $(x-a) = -z\frac{\partial z}{\partial x}$; $(y-b) = -z\frac{\partial z}{\partial y}$, then the Partial differential equation is

(a) $z^2(p^2 - q^2 + 1) = 1$ (b) $z^2(p^2 + q^2 + 1) = 0$

(c) $z^2(p^2 + q^2) = 1$ (d) $z^2(p^2 + q^2 + 1) = 1$

Ans. : (d)

Q. 6 By eliminating an arbitrary constants from the function $z = ax^2 + bxy + cy^2$, with $a = \frac{r}{2}$; $b = s$; $c = \frac{t}{2}$,

where, $r = \frac{\partial^2 z}{\partial x^2}$; $s = \frac{\partial^2 z}{\partial x\,\partial y}$; $t = \frac{\partial^2 z}{\partial y^2}$, then the Partial differential equation is

(a) $2z = rx^2 + 2xy\ s + t\,y^2$

(b) $z = rx^2 + xy\ s + t\,y^2$

(c) $z = rx^2 - 2xy\ s + t\,y^2$

(d) $z = -rx^2 - 2xy\ s + t\,y^2$

Ans. : (a)

Q. 7 From the given solution, if the number of arbitrary functions are n to be eliminated, then

(a) The $(n + 1)^{th}$ order partial differential equations is obtained.

(b) The $(n - 1)^{th}$ order partial differential equations is obtained.

(c) The $(n - 2)^{th}$ order partial differential equations is obtained.

(d) The nth order partial differential equations is obtained.

Ans. : (d)

Q. 8 By eliminating arbitrary function from the relation $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$, with $-p\,x^2 = 2f'$

$\left(\frac{1}{x}+\log y\right)$ and $qy-2y^2=2f'\left(\frac{1}{x}+\log y\right)$, then the Partial differential equation is

(a) $px^2-qy=0$
(b) $px^2+qy=0$
(c) $px^2+qy=2y^2$
(d) $px^2-qy=2y^2$

Ans. : (c)

Q. 9 By eliminating arbitrary functions from the relation $z=x+y+f(xy)$, with $p=1+yf'(xy)$ and $q=1+xf'(xy)$, then the Partial differential equation is

(a) $px-qy=(x-y)$ (b) $px+qy=(x-y)$
(c) $px-qy=0$ (d) $px+qy=(x+y)$

Ans. : (a)

Q. 10 The solution of the partial differential equation $\frac{\partial^2 z}{\partial x^2}=xy$, with $\frac{\partial z}{\partial x}=y\left(\frac{x^2}{2}\right)+f_1(y)$ is

(a) $z=\frac{x^3y}{6}+xf_1(y)+f_2(y)$
(b) $z=\frac{x^3y}{6}-xf_1(y)+f_2(y)$
(c) $z=\frac{x^3y}{6}+f_1(y)-f_2(y)$
(d) $z=\frac{x^3y}{6}+xf_1(y)$

Ans. : (a)

Q. 11 The solution of the partial differential equation $\log\left[\frac{\partial^2 z}{\partial x\,\partial y}\right]=x+y$, with $\frac{\partial z}{\partial y}=e^y\cdot e^x+f(y)$ is

(a) $z=e^{x-y}+g(y)=\phi(x)$, where, $g(y)=\int f(y)\,dy$
(b) $z=e^{x+y}+g(y).\phi(x)$, where, $g(y)=\int f(y)\,dy$
(c) $z=e^{x+y}+g(y)+\phi(x)$, where, $g(y)=\int f(y)\,dy$
(d) $z=e^{x+y}-g(y)-\phi(x)$, where, $g(y)=\int f(y)\,dy$

Ans. : (c)

Q. 12 The standard form of Lagrange's first order linear partial differential equation is

(a) $Pp-Qq=R$, where P, Q, R are functions of x, y, z
(b) $Pp+Qq=0$, where P, Q are functions of x, y, z
(c) $p-q=0$, where p, q are functions of x, y, z
(d) $Pp+Qq=R$, where P, Q, R are functions of x, y, z

Ans. : (d)

Q. 13 Thefirst integral of the linear partial differential equation $y^2p-xyq=x(z-2y)$, with

$$\frac{dx}{y^2}=\frac{dy}{-xy}=\frac{dz}{x(z-2y)} \text{ is}$$

(a) $x^2-y^2=c_1$
(b) $x^2+y^2=c_1$
(c) $x^3+y^3=c_1$
(d) $x^2=c_1$

Ans. : (b)

Q. 14 The solution of one dimensional heat flow equation $\frac{\partial u}{\partial t}=C^2\frac{\partial^2 u}{\partial x^2}$ is

(a) $u(x,t)=(C_1\cos mx+C_2\sin mx)\,e^{-c^2m^2t}$
(b) $u(x,t)=[C_1\cos mx+C_2\sin mx]\cdot[C_3\cos mct+C_4\sin mct]$
(c) $u(x,t)=(C_1\cosh mx+C_2\sinh mx)\,e^{-c^2m^2t}$
(d) $u(x,t)=[C_1e^{px}+C_2e^{-px}][C_3\cos pt+C_4\sin pt]$

Ans. : (a)

Q. 15 The solution of one dimensional heat flow equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ for $u(x,y)=0$ as $y\to\infty$ is

(a) $u(x,y)=(C_1\cos mx+C_2\sin mx)\,e^{-c^2m^2y}$
(b) $u(x,t)=[C_1e^{py}+C_2e^{-py}]$ $[C_3\cos mcy+C_4\sin mcy]$
(c) $u(x,t)=(C_1\cosh mx+C_2\sinh mx)\,e^{-c^2m^2ys}$
(d) $u(x,t)=[C_1\cos mx+C_2\sin mx][C_3e^{py}+C_4e^{-py}]$

Ans. : (d)

Q. 16 The solution of one dimensional heat flow equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ for $u(x,y)=0$ as $x\to\infty$ is

(a) $u(x,y)=(C_1\cos mx+C_2\sin mx)\,e^{-c^2m^2y}$
(b) $u(x,t)=[C_1\cos mx+C_2\sin mx][C_3e^{py}+C_4e^{-py}]$
(c) $u(x,t)=(C_1\cosh mx+C_2\sinh mx)\,e^{-c^2m^2y}$
(d) $u(x,t)=[C_1e^{px}+C_2e^{-px}][C_3\cos mcy+C_4\sin mcy]$

Ans. : (d)

□□□

CHAPTER 5

UNIT 5

Functions of Complex Variables (Differential Calculus)

Syllabus

Limit and continuity of f(z); Derivative of f(z) ; Analytic functions; Cauchy-Riemann equations in Cartesian and polar forms; Harmonic functions in Cartesian form; Mapping: Translation, magnification and rotation, inversion and reflection , bilinear transformation; Conformal mapping.

5.1 Introduction

Students are familiar about complex numbers. Many engineering problems may be treated and evaluated by methods containing complex numbers and complex functions. Complex functions are functions which produces complex numbers from complex variables. Complex function theory is nothing but the study of analytic functions. It is very powerful technique which is used in conduction of heat flow, fluid dynamics, electro-statics, thermodynamics and electrical fields.

In this topic we will study the analytic functions, harmonic functions.

5.2 Pre-requisite

Let $z=x+iy$ be any complex number, where $x, y\in R$ and $i=\sqrt{-1}$

Polar form :

$$x = r\cos\theta;\ y = r\sin\theta;$$
$$z = r[\cos\theta+i\sin\theta]$$
$$z = re^{i\theta}$$

Where, $r=\sqrt{x^2+y^2}$

- is called modulus or magnitude of complex number $z=x+iy$

$$\therefore\ |z| = r=\sqrt{x^2+y^2} \text{ and}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

- is called argument or amplitude of complex number $z=x+iy$

$$\therefore\ \text{Amp}(z) = \text{Arg}(z)=\theta=\tan^{-1}\left(\frac{y}{x}\right)$$

Since, $z = re^{i\theta}$

General form of logarithm of z is,

$$\text{Log}(z) = \log(re^{i\theta})=\log(re^{i(2n\pi+\theta)})$$
$$\log(z) = \log r+i(2n\pi+\theta)$$
$$= \frac{1}{2}\log(x^2+y^2)+i\left(2n\pi+\tan^{-1}\left(\frac{y}{x}\right)\right)$$

by putting n = 0, we get, the principle value of logarithm of z is

$$\log(z) = \log r+i\theta=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\left(\frac{y}{x}\right)$$

5.2.1 Some Important Curves

$\lvert z\rvert=\lvert z-0\rvert$	:	is a circle having centre at origin
$\lvert z\rvert=1$	:	is a unit circle having centre at origin and radius 1
$\lvert z-z_0\rvert=a$	:	is a circle having centre at z_0 and radius is a
$\lvert z-z_0\rvert<a$	:	means interior part of circle of radius a
$\lvert z-z_0\rvert>a$	:	means exterior part of circle of radius a
$r_1<\lvert z-z_0\rvert<r_2$	:	is a region lies between two concentric circles having centre at z_0 and having radii r_1 and r_2 ($r_1<r_2$), the region is called a open circular ring or open annulus. (See Fig. 5.2.1)

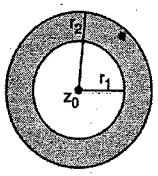


Fig. 5.2.1

- The distance between two points 'z' and 'a' is $|z-a|$. It means a circle 'c' with centre at a and radius r and it can be denoted in the form $|z-a| = r$. Its parametric form is $z = a + re^{i\theta}$; $0 \le \theta \le 2\pi$.
- $|z-a| < r$ - represents the interior port of circle C, such region is called as open circular disk or neighbourhood of point a.
- $|z-a| \le r$ - represents the interior port of circle C and boundary of C itself.

Ellipse : The rectangular form of an ellipse is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- The foci are $(\pm ae, 0)$ and $(0, \pm be)$

$$b^2 = a^2(1-e^2) \quad \text{and} \quad a^2 = b^2(1-e^2).$$

- The sum of distance from any point on the ellipse to the foci must be equal to length of major axis.
- If 'ρ' is the length of major axis and $(\pm a\rho, 0)$ is foci then,

$$|z+a_0| + |z-a_0| = \rho; \quad \text{where } a_0 = ae$$

and if foci are $(0, \pm be)$ then

$$|z+b_0 i| + |z-b_0 i| = \rho \quad \text{where, } b_0 = be$$

5.3 Functions of Complex Variable

5.3.1 Definition

If x and y are real variables, then $z = x + iy$ is called a complex variable. If for each value of the complex variable $z (= x + iy)$ in a given region R, there exists one or more values of another complex variable $w (= u + iv)$, then w is called a function of the complex variable z and is denoted by

$$w = f(z) = u + iv = u(x, y) + iv(x, y)$$

where, u and v are real functions of x and y.

∴ u and v dpends on x and y. It is clear that w depends on $z (= x + iy)$.

Example :

(1) Let $w = \log z = \log r e^{i\theta} = \log r + i\theta$

$$= \log\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right)$$

$$= \log(x^2+y^2)^{\frac{1}{2}} + i\tan^{-1}\left(\frac{y}{x}\right)$$

but $w = f(z) = u + iv$

$$\Rightarrow u + iv = \log(x^2+y^2)^{\frac{1}{2}} + i\tan^{-1}\left(\frac{y}{x}\right)$$

Comparing real and imaginary parts on both sides

$$\Rightarrow u = \log(x^2+y^2)^{\frac{1}{2}}; \quad v = \tan^{-1}\left(\frac{y}{x}\right)$$

(2) Let $w = z^2$

$$\therefore w \ (= u + iv) = (x+iy)^2 = x^2 - y^2 + 2i\,xy$$

Comparing real and imaginary parts on both sides

$$\Rightarrow u = x^2 - y^2; \quad v = 2xy.$$

It is clear that u and v are the functions of real variable x and y.

5.3.2 Single Valued Function

If for each value of z, there exists one and only one value of w, then w is called single - valued function of z, otherwise it is called multi-valued function of z.

e.g. (1) $w = \frac{1}{z}$, z^2 are single valued functions.

(2) $w = \sqrt{z}$ is a multi-valued function.

Let $w = z^2$

Suppose, $z = x + iy$ and

$$w = (x+iy)^2 = x^2 + iy^2 + i2\,xy = (x^2 - y^2) + i\,2xy$$

We get,

$$u = x^2 - y^2 \quad \text{and} \quad v = 2\,xy$$

$x = 2$, $y = 1$, then $u = 3$; $v = 4 \Rightarrow w = 3 + i4$ is a single valued function. In example (1), f(z) defined at all points of the z plane except at $z = 0$ and in example (2), z defined (assumes) at two values of z. In case of a function of a real variable $y = f(x)$, the relation between x and y represents by curve in xoy plane.

To represents $w = f(z)$ graphically, we need two planes : one to represent the point $z = x + iy$ called z-plane or Argand diagram and another to represent $w = f(z)$ called uov plane or the w-plane.

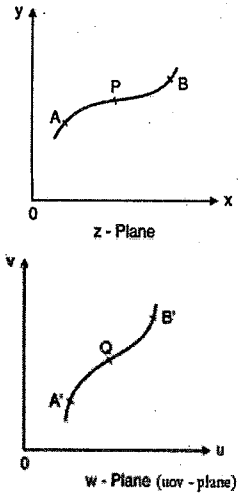


Fig. 5.3.1

The functional relation gives a point Q in the w-plane corresponding to a point P from z-plane and a continuous curve AB in the z-plane is transformed to the curve A′ B′ in the w-plane. The transformation of curves from z-plane to w-plane, we treat it under conformal transformation.

Syllabus Topic : Limit and continuity of f (z)

5.3.3 Limit of f (z)

A function f (z), defined at all points in a neighbourhood of z_0 except at z_0 is said to have the limit w_0 as z tends to z_0 along any path, if to each positive arbitrary number $\epsilon > 0$, there exist a positive number δ such that

$$|f(z) - w_0| < \epsilon \text{ whenever } |z - z_0| < \delta$$

i.e. $(w_0 - \epsilon) < f(z) < (w_0 + \epsilon)$

Whenever, $z_0 - \delta < z < z_0 + \delta$

This means, the value of f (z) can be made as close as to w_0, if we choose the point z close enough to z_0 and we can write

$$\lim_{z \to z_0} f(z) = w_0$$

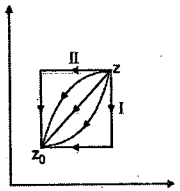


Fig. 5.3.2

Note : In a real variables $x \to x_0$, means x approaches x_0 along the number line, either from left or right of origin. In a function of complex variables $z \to z_0$, means z approaches z_0 along any path, straight or curved. Since, two points z and z_0 in a complex plane can be joined by an infinite number of curves.

When z lies interior of circle of radius a then their images also lies interior of circle with centre at w_0 and radius ∈. This definition says that for each ∈ - neighbourhood of w_0 there exists a δ-neighbourhood of z_0 such that, the images f(z) of all z in δ-neighbourhood with exception at z_0, lie in the ∈-neighbourhood.

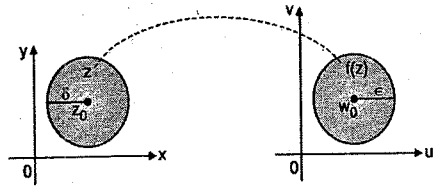


Fig. 5.3.3

Note : Even though all points in the domain $0 < |z - z_0| < \delta$ are to be considered, their images need not constitute the entire neighbourhood $|w - w_0| < \epsilon$.

5.3.4 Continuity of f (z)

A single valued function $w = f(z)$ is said to be continuous at point $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

A function f (z) is said to be continuous in a region R of the z-plane, if it is continuous at every point of the region R of the z-plane. When $w = f(z) = u(x, y) + i\,v(x, y)$ is continuous at $z = z_0$, then u (x, y) and v (x, y) are also continuous at $z = z_0$, i.e. at $x = x_0$ and $y = y_0$; and conversely if u (x, y) and v (x, y) are continuous at (x_0, y_0) then f (z) will also continuous at $z = z_0$. $(= x_0 + iy_0)$

Syllabus Topic : Derivative of f (z)

5.3.5 Derivative of f (z)

If $w = f(z)$ is a single valued function of complex variable $z = x + iy$ then the derivative of $w = f(z)$ is defined as

$$\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z};$$

provided the limit exists and has the same limiting value for all the different paths in which $\delta z \to 0$ i.e. independent of paths.

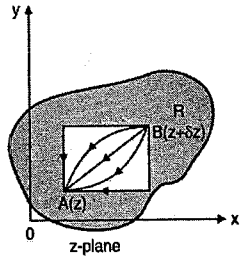


Fig. 5.3.4

Let, A (z) be fixed point and B $(z + \delta z)$ be a neighbouring point. The point B approaches to A along any path, curved or straight, in the given region i.e. δz tends to zero in any manner and suppose if, $\left(\frac{dw}{dz}\right)$ does not exist. Then, it becomes a fundamental problem to determine the necessary and sufficient condition for $\left(\frac{dw}{dz}\right)$ to exist.

Syllabus Topic : Analytic Functions

5.3.6 Analytic Function

A single valued function $W = f(z) = u + iv$ possesses a unique derivative with respect to z at all points in a region R, then it is said to be analytic function or a regular function. It is also called a holomorphic function of z in R, or (Monogenic Function).

5.3.7 Singular Point

A point where the function ceases to be analytic is called a singular point (or the point at which the function does not have a derivative).

Syllabus Topic : Cauchy-Riemann Equations in Cartesian form

5.4 Cauchy-Riemann Equations

5.4.1 Cauchy-Riemann Equation in Cartesian Form

Theorem

The necessary and sufficient conditions for the function ; $w = f(z) = u(x, y) + i\,v(x, y)$ to be analytic function in a region R, are

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

are continuous functions of x and y in the region R.

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

The condition (ii) is known as Cauchy – Riemann Equation or C – R equations.

Proof :

(I) Necessary Condition

Let, $w = f(z) = u(x, y) + iv(x, y)$, be an analytic function in region R, then $\frac{dw}{dz} = f'(z)$ exists uniquely at every point of that region R (i.e. f (z) possesses unique derivative at p (z) in R.).

Let, δx and δy be the small increments in x and y respectively.

$\therefore$ δu and δv be the corresponding increments in u and v, so that

$$\delta z = \delta x + i\delta y; \quad f(z) = u + iv;$$

$$f(z+\delta z) = (u+\delta u) + i(v+\delta v)$$

Then,
$$f'(z) = \frac{dw}{dz} = \lim_{\delta z \to o} \frac{f(z+\delta z) - f(z)}{\delta z}$$

$$= \lim_{\delta z \to o} \frac{(u+\delta u) + i(v+\delta v) - (u+iv)}{\delta z}$$

$$= \lim_{\delta z \to o} \left(\frac{\delta u + i\,\delta v}{\delta z}\right)$$

$$= \lim_{\delta z \to o} \left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right) \quad \text{...(5.4.1)}$$

The limit of Equation (5.4.1) must exists and it is independent of the path in which $\delta z \to 0$ (i.e. along whichever path δx and $\delta y \to 0$). First, consider, $\delta z \to 0$ along a line parallel to x-axis so that $\delta y = 0$ and $\delta z = \delta x$ (or we first assume δz to be wholly real implying that $\delta y = 0$ and $\delta z = \delta x$)

$\therefore$ From Equation (5.4.1),

$$\frac{dw}{dz} = f'(z) = \lim_{\delta x \to o} \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right)$$

$$\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{...(5.4.2)}$$

(Since, f (z) is analytic, $\therefore$ f′(z) exists, and limit on R.H.S. must exists).

Now, consider, $\delta z \to 0$ along a line parallel to y-axis so that $\delta x = 0$ and $\delta z = i\,\delta y$ (or when δz is wholly imaginary ; $\delta x = 0$ and $\delta z = i\,\delta y$)

$\therefore$ From Equation (5.4.1),

$$\frac{dw}{dz} = f'(z) = \lim_{\delta y \to o} \left(\frac{\delta u}{i\delta y} + i\frac{\delta v}{i\delta y}\right) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\frac{dw}{dz} = f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad \left(\because \frac{1}{i} = -i\right) \quad \text{...(5.4.3)}$$

From Equation (5.4.2) and (5.4.3), we have

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Equating real and imaginary part, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

are called Cauchy Riemann equations.

Hence, if f (z) is analytic in region R, partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous and satisfies C – R equations.

Hence, the necessary condition for f(z) to be analytic, C – R equations must be satisfied.

(II) Sufficient Condition

Let, f (z) be a single valued function of complex variable 'Z' possessing partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at each point of the region R and satisfying C – R equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

We shall show that f (z) is analytic i.e. f′(z) exists at every point of the region R i.e. $\delta z \to 0$ in any direction. By Taylor's Theorem for functions of two variables, we have, on neglecting second and higher degree terms of δx and δy.

$$f(z+\delta z) = u(x+\delta x, y+\delta y) + i\,v(x+\delta x, y+\delta y)$$

$$= \left\{u(x,y) + \left[\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\cdot\delta y\right] + \ldots\ldots\right\}$$

$$+ i\left\{v(x,y) + \left[\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right] + \ldots\ldots\right\}$$

$$= [u(x,y) + i\,v(x,y)] + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

$$f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta y$$

$$\left(\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right)$$

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left(i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right) i\,\delta y$$

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right](\delta x + i\delta y) = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta z$$

$$\Rightarrow \lim_{\delta z \to o} \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Thus, f′(z) exists, because, $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ exists.

Hence, f (z) is analytic.

Note :

(1) Real part and imaginary parts of an analytic function $f(z) = u + iv$ are called **conjugate functions**. The relation between two conjugate functions is given by C – R. Equations.

(2) When a function f (z) is known to be analytic, it can be differentiated in the ordinary way.

Thus, $f(z) = z^2 \Rightarrow f'(z) = 2z$

$f(z) = \sin z \Rightarrow f'(z) = \cos z$.

(3) If two functions f (z) and g (z) are analytic in region R, then their sum and product both are analytic in region R. Similarly, $\frac{f(z)}{g(z)}$ is also analytic if $g(z) \neq 0$.

(4) Every polynomial is analytic.

Syllabus Topic : Cauchy-Riemann Equation in Polar Form

5.4.2 Cauchy-Riemann Equation in Polar Form

Let, (r, θ) be the polar co-ordinates of the point whose Cartesian co-ordinates are (x, y)

then, $x = r\cos\theta$; $y = r\sin\theta$

$$\Rightarrow z = x + iy = (r\cos\theta + ir\sin\theta)$$

$$= r[\cos\theta + i\sin\theta] = re^{i\theta}$$

$$f(z) = f(re^{i\theta}) = u + iv \quad \text{...(5.4.4)}$$

Differentiating Equation (5.4.4) partially w.r.t. r, keeping θ as constant, we get,

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})\cdot e^{i\theta} \quad \text{...(5.4.5)}$$

Differentiating Equation (5.4.5) partially w.r.t. θ, keeping r as constant, we get,

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\cdot ire^{i\theta}$$

$$= ir\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \text{...from Equation (5.4.5)}$$

$$= -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}$$

Equating real and imaginary parts on both sides, we get,

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \text{and} \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\cdot\frac{\partial v}{\partial \theta} \quad ; \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad \text{...(5.4.6)}$$

Which are the polar form of Cauchy – Reimann (i.e. C.R.) equations.

Remember : Composition of two analytic function is analytic

$\therefore$ $w = f(g(z))$ is analytic.

Entire Function

An entire function is a function which is analytic at each point in the entire finite plane.

(finite XOY plane). Since, derivative of polynomial exists everywhere

⇒ Every polynomial is analytic.

⇒ Every polynomial is entire function.

Syllabus Topic : Harmonic Function in Cartesian Form

5.5 Harmonic Function in Cartesian Form

The function which satisfies the Laplace equation is known as Harmonic Function.

OR Any solution ϕ of the Laplace equation $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ is called a Harmonic Function.

5.5.1 Milne-Thomson Method

Since, $f(z) = u(x, y) + iv(x, y)$ and $z = x + iy$

Put $x = z$ and $y = 0$

$f(z) = u(z, 0) + i\,v(z, 0)$

Remember

(1) To express any analytic function in terms of z,

Put, $x = z$ and $y = 0$.

(2) If function f (z) is in polar form r and θ

i.e. $f(z) = u(r, \theta) + iv(r, \theta)$,

then to express R.H.S. in terms of z, put $r = z$ and $\theta = 0$.

$\Rightarrow f(z) = u(z, 0) + iv(z, 0)$.

5.5.2 Orthogonal System

Two curves are said to be orthogonal to each other when, they intersect at right angle to each other at their point of intersection.

Equipotential and Stream Function Lines

Note :

1. If $f(z) = \phi(x, y) + i\psi(x, y)$ then the curve $\phi(x, y) = C_1$, is called equipotential lines and the curve $\psi(x, y) = C_2$ is called the stream function line. The function $w = f(z)$ is called complex potential or potential function.
2. Incase of electrostatics, gravitational field the curves $\phi(x, y) = C_1$ and $\psi(x, y) = C_2$ are called equipotential lines and line of force respectively.
3. In heat flow problem, the curve $\phi(x, y) = C_1$ and $\phi(x, y) = C_2$ are known as isothermal and heat flow lines respectively.

Solved Example

Type 1

Ex. 5.5.1 : If $w = \log z$, then find $\left(\frac{dw}{dz}\right)$ and determine, where w is non-analytic.

Soln. : Step I :

Given, $w = \log z$,

$$\therefore \quad u + iv = \log(x + iy) = \log(r e^{i\theta}) = \log(r) + i\theta$$

$$= \log(x^2 + y^2)^{\frac{1}{2}} + i \tan^{-1}\left(\frac{y}{x}\right)$$

Comparing real and imaginary parts on both sides.

$$\Rightarrow u = \log(x^2 + y^2)^{\frac{1}{2}}; \quad v = \tan^{-1}\left(\frac{y}{x}\right)$$

Differentiating 'u' w.r.t. x partially, keeping y as constant.

$$\text{Now,} \quad \frac{\partial u}{\partial x} = \frac{1}{2} \frac{1}{(x^2 + y^2)^{\frac{1}{2}}} \cdot (x^2 + y^2)^{-\frac{1}{2}} \cdot 2x$$

$$= \frac{x}{(x^2 + y^2)} \quad \text{...(i)}$$

Differentiating 'u' w. r. t. y partially, keeping x as constant

$$\frac{\partial u}{\partial y} = \frac{1}{(x^2 + y^2)^{\frac{1}{2}}} \cdot \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}} \cdot 2y$$

$$= \frac{y}{(x^2 + y^2)} \quad \text{...(ii)}$$

Differentiating v w.r.t. x, partially, keeping y as constant

$$\frac{\partial v}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{(x^2 + y^2)}; \quad \text{...(iii)}$$

And differentiating v w.r.t. y partially, keeping x as constant.

$$\frac{\partial v}{\partial y} = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2} \quad \text{...(iv)}$$

From Equation (i) and (iv)

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} = \frac{\partial v}{\partial y}$$

And Form Equation (ii) and (iii)

$$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2} = -\frac{\partial v}{\partial x}$$

Thus, C – R. equation satisfied and it is observed that the partial derivatives are continuous except at (0, 0)

Hence, w is analytic everywhere except at $z = 0$.

$$\text{Now, } f'(z) = \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}$$

$$= \frac{x - iy}{(x^2 + y^2)}$$

...(From Equations (i) and (iii))

$$= \frac{(x - iy)}{(x + iy)(x - iy)}$$

...(since $(a^2 + b^2) = (a + bi)(a - bi)$)

$$= \frac{1}{x + iy} = \frac{1}{z}$$

$\Rightarrow \frac{dw}{dz} = \frac{1}{z}$ $(z \neq 0)$; at $z = 0$, w is non-analytic. ✓ **...Ans.**

Ex. 5.5.2 : Determine which of the following functions are analytic : (i) $\frac{1}{1 - z}$ (ii) $\sin z$

Soln. :

Step I :

(A) Given, $f(z) = \frac{1}{1 - z}$

$$u + iv = \frac{1}{1 - z} = \frac{1}{1 - x - iy} = \frac{1}{(1 - x) - iy}$$

$$= \frac{(1 - x) + iy}{(1 - x)^2 + y^2} \quad \text{...(Rationalization)}$$

Comparing real and imaginary parts on both sides

$$\Rightarrow u = \frac{(1 - x)}{(1 - x)^2 + y^2}; \quad v = \frac{y}{(1 - x)^2 + y^2}$$

Differentiate u partially w.r.t. x, keeping y as constant

➔ **Using standard formula :** $\left[d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}\right]$

$$\frac{\partial u}{\partial x} = \frac{[(1 - x)^2 + y^2] \cdot \frac{d}{dx}(1 - x) - (1 - x) \cdot \frac{d}{dx}[(1 - x)^2 + y^2]}{[(1 - x)^2 + y^2]^2}$$

$$\frac{\partial u}{\partial x} = u_x = \frac{[(1 - x)^2 + y^2](-1) - (1 - x)[2(1 - x)(-1)]}{[(1 - x)^2 + y^2]^2}$$

$$= \frac{-(1 - x)^2 - y^2 + 2(1 - x)^2}{[(1 - x)^2 + y^2]^2} = \frac{(1 - x)^2 - y^2}{[(1 - x)^2 + y^2]^2}$$

Differentiate u partially w.r.t. y, keeping x as constant

$$\frac{\partial u}{\partial y} = u_y = \frac{0 - 2y(1 - x)}{\{(1 - x)^2 + y^2\}^2} = \frac{-2y(1 - x)}{\{(1 - x)^2 + y^2\}^2}$$

Differentiate v partially w.r.t. x, Keeping y as constant

➔ **Using standard result :** $\left[d\left(\frac{u}{v}\right) = \left[\frac{v\,du - u\,dv}{v^2}\right]\right]$

$$\frac{\partial v}{\partial x} = -\frac{y}{[(1 - x)^2 + y^2]} 2(1 - x)(-1) = \frac{2y(1 - x)}{[(1 - x)^2 + y^2]^2}$$

Also, differentiate v partially w.r.t. y, keeping x as constant.

➔ **Using standard result :** $\left[d\left(\frac{u}{v}\right) = \left[\frac{v\,du - u\,dv}{v^2}\right]\right]$

$$\frac{\partial v}{\partial y} = \frac{[(1 - x)^2 + y^2] \cdot 1 - y[0 + 2y]}{[(1 - x)^2 + y^2]^2}$$

$$= \frac{(1 - x)^2 + y^2 - 2y^2}{(1 - x)^2 + y^2} = \frac{(1 - x)^2 - y^2}{[(1 - x)^2 + y^2]^2}$$

By observing $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

$$\text{We get,} \quad \frac{\partial u}{\partial x} = \frac{(1 - x)^2 - y^2}{[(1 - x)^2 + y^2]^2} = \frac{\partial v}{\partial y} \quad \text{and}$$

$$\frac{\partial u}{\partial y} = -2y\frac{(1 - x)}{[(1 - x)^2 + y^2]^2} = -\frac{\partial v}{\partial x}$$

Thus, C – R equations are satisfied and also observed that partial derivatives are continuous everywhere except at $x = 1$ and $y = 0$ i.e. $z = 1$.

...(∵ At $x = 1$ and $y = 0$, denominator becomes ∞.

Means, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ not defined)

Hence f (z) is analytic everywhere except at $z = 1$

Also, by ordinary way, we can check,

Since, $f(z) = \frac{1}{1 - z}$

Differentiate f (z) w.r.t. z

➔ **Using standard result :** $\left[\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}dx\right]$

$$f'(z) = -\frac{1}{(1 - z)^2} \cdot (-1) = \frac{1}{(1 - z)^2} \quad (z \neq 1)$$

(ii) $f(z) = \sin z = \sin(x + iy)$

➔ **Using trigonometric formula**

$$\left[\sin(A + B) = \sin A \cos B + \cos A \sin B\right]$$

Here, $A = x$ and $B = y$

$$u + iv = \sin x \cdot \underset{\cosh y}{\underset{\downarrow}{\cos iy}} + \cos x \cdot \underset{i \sinh y}{\underset{\downarrow}{\sin iy}}$$

$$= (\sin x)(\cosh y) + (\cos x)(i \sinh y)$$

$$= \sin x \cdot \cosh y + i \cos x \cdot \sinh y$$

Comparing real and imaginary parts on both sides

$$\Rightarrow u = \sin x \cosh y; \quad v = \cos x \sinh y$$

Differentiate u and v partially w.r.t. x and y.

➔ **Using standard formula**

$$\left[\frac{d}{dy}(\cosh y) = \sinh y; \ \frac{d}{dy}(\sinh y) = \cosh y\right]$$

$$\therefore \frac{\partial u}{\partial x} = \cos x \cdot \cosh y; \quad \frac{\partial u}{\partial y} = \sin x \cdot \sinh y$$

$$\frac{\partial v}{\partial x} = -\sin x \cdot \sinh y; \quad \frac{\partial v}{\partial y} = \cos x \cdot \cosh y$$

It is observed that

$$\frac{\partial u}{\partial x} = \cos x \cosh y = \frac{\partial v}{\partial y} \quad \text{and}$$

$$\frac{\partial u}{\partial y} = \sin x \sinh y = -\frac{\partial v}{\partial x}$$

$\Rightarrow$ C – R. equations satisfied and also partial derivatives are continuous everywhere.

$\Rightarrow$ f (z) is analytic everywhere.

Also, we differentiate f(z) by ordinary way

$\therefore f'(z) = \cos z$. ✓ **...Ans.**

Ex. 5.5.3 : Prove that the function f (z) defined by $f(z) = \frac{x^3(1 + i) - y^3(1 - i)}{x^2 + y^2}$; $z \neq 0$ and $f(0) = 0$ is continuous and C – R equations are satisfied at the origin, yet $f'(0)$ does not exist.

Soln. :

Step I : Given,

$$f(z) = \frac{x^3(1 + i) - y^3(1 - i)}{x^2 + y^2}$$

for continuous function, $\lim_{z \to o} f(z) = f(z_0)$

$$\lim_{z \to o} f(z) = \lim_{\substack{x \to o \\ y \to o}}\left[\frac{x^3(1 + i) - y^3(1 - i)}{x^2 + y^2}\right]$$

$$= \lim_{y \to o}\left[\frac{-y^3(1 - i)}{y^2}\right]$$

$$= \lim_{y \to o}[-y(1 - i)] = 0$$

Also, $f(0) = 0$ ...(given)

Thus, $f(z) = f(0)$ when $x \to 0$ first and then $y \to 0$ and also vice – versa.

Hence, f (z) is continuous

Now, let both x and y tends to zero simultaneously along the path $y = mx$

$\Rightarrow x \to 0$ and $y \to mx$.

$$\lim_{z\to o} f(z) = \lim_{\substack{y\to mx\\ x\to o}}\left[\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}\right]$$

$$= \lim_{x\to o}\left[\frac{x^3(1+i)-m^3x^3(1-i)}{x^2+m^2x^2}\right]$$

$$= \lim_{x\to o}\left[\frac{x^3(1+i)-m^3(1-i)}{x^2(1+m^2)}\right]$$

$$= \lim_{x\to o}\frac{x[1+i-m^3(1-i)]}{1+m^2} = 0$$

This shows that

$$\lim_{z\to o} f(z) = f(0)\text{; by any path } z\to 0$$

$\therefore$ f (z) is continuous at origin.

Since, $$f(z) = \left[\frac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2}\right]$$

$$= \frac{x^3-y^3}{x^2+y^2} + i\frac{x^3+y^3}{x^2+y^2}$$

$$= u(x,y) + iv(x,y)$$

i.e. $u = \dfrac{x^3-y^3}{x^2+y^2}$; $v = \dfrac{x^3+y^3}{x^2+y^2}$

Since, $f(0) = 0$

$\Rightarrow u(x,y) + iv(x,y) = 0$

$\Rightarrow u(x,y) = 0$ and $v(x,y) = 0$

$\Rightarrow u(0,0) = 0$; $v(0,0) = 0$

$$\frac{\partial u}{\partial x} = \lim_{x\to o}\left[\frac{u(x,0)-u(0,0)}{x}\right]$$

$$= \lim_{x\to o}\left[\frac{\frac{x^3-0}{x^2+0}-\frac{0+0}{0-0}}{x}\right]$$

$$\left(\frac{\partial u}{\partial x}\right)_{(o,o)} = \lim_{x\to o}\frac{(x^3-o)-(o+o)}{x(x^2+o)} = \lim_{x\to o}\left(\frac{x^3}{x^3}\right) = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y\to o}\left[\frac{u(0,y)-u(0,0)}{y}\right]$$

$$= \lim_{y\to o}\left[\frac{\frac{0-y^3}{0+y^2}-\frac{0+0}{0-0}}{y}\right]$$

$$\left(\frac{\partial u}{\partial y}\right)_{(o,o)} = \lim_{y\to o}\frac{(o-y^3)-(o+o)}{y(o+y^2)}$$

$$= \lim_{y\to o}\left(-\frac{y^3}{y^3}\right)$$

$$= -1$$

$$\frac{\partial v}{\partial x} = \lim_{x\to o}\left[\frac{v(x,0)-v(0,0)}{x}\right]$$

$$= \lim_{x\to o}\left[\frac{\frac{x^3+0}{x^2+0}-\left(\frac{0+0}{0+0}\right)}{x}\right]$$

$$\left(\frac{\partial v}{\partial x}\right)_{(o,o)} = \lim_{x\to o}\frac{(x^3+o)-(o+o)}{x(x^2+o)} = \lim_{x\to o}\left(\frac{x^3}{x^3}\right) = 1$$

and $$\frac{\partial v}{\partial y} = \lim_{y\to o}\left[\frac{v(0,y)-v(0,0)}{y}\right]$$

$$= \lim_{y\to o}\left[\frac{\frac{0+y^3}{0+y^2}-\frac{0+0}{0+0}}{y}\right]$$

$$\left(\frac{\partial v}{\partial y}\right)_{(o,o)} = \lim_{y\to o}\frac{(o+y^3)-(o+o)}{y(o+y^2)} = \lim_{y\to o}\left(\frac{y^3}{y^3}\right) = 1$$

$$\Rightarrow \frac{\partial u}{\partial x} = 1 = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -1 = -\frac{\partial v}{\partial x}$$

Hence, C – R equations are satisfied at **origin.**

But $$f'(0) = \lim_{z\to o}\frac{f(z)-f(o)}{z}$$

$$= \lim_{z\to o}\frac{f(z)}{z} = \lim_{z\to o}\frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

If $z\to 0$ along the path $y = mx$ then

$$f'(0) = \lim_{z\to o}\frac{(x^3-m^3x^3)+i(x^3+m^3x^3)}{(x^2+m^2x^2)(x+imx)}$$

$$= \lim_{z\to o}\frac{x^3(1-m^3)+ix^3(1+m^3)}{x^2(1+m^2)\cdot x(1+im)}$$

$$f'(0) = \lim_{z\to o}\frac{(1-m^3)+i(1+m^3)}{(1+m^2)(1+im)}$$

which gives different values as m varies. Therefore $f'(z)$ is not unique at (0, 0).

Thus, f (z) is not analytic at (0, 0) even though it is continuous and satisfies C – R equations at (0, 0). ✓

Hence proved.

Type 2

Ex. 5.5.4 : If $v = 3x^2y - y^3$, then find analytic function f (z) in terms of z.

Soln. :

Step I : Given, f (z) = u + iv is an analytic function

$\therefore$ By C – R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Given, $v = 3x^2y - y^3$

Differentiate v partially w.r.t. x and y

$$\frac{\partial v}{\partial x} = 6xy\ ;\quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

Step II : Since, $f(z) = u + iv$

$$\therefore\ f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

...(by C-R equation)

$$\therefore\ f'(z) = (3x^2-3y^2) + i(6xy)$$

Step III : By Milne-Thomson method, ; Put $x = z$ and $y = 0$

$$\therefore\ f'(z) = 3z^2 + i0 = 3z^2$$

Integrating,

$$\int f'(z)\,dz = \int 3z^2\,dz + C = \cancel{3}\left(\frac{z^3}{\cancel{3}}\right) + C$$

$f(z) = z^3 + C$ is required analytic function

Another method

Step I : Given : $v = 3x^2y - y^3$

Differentiate v partially w. r. t. x, keeping y as constant. Also differentiate w. r. t. y, keeping x as constants

$$\frac{\partial v}{\partial x} = 6xy\ ;\ \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

Step II : Since, u (x, y) is a complex conjugate of v

$$\therefore\ du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad ...(\because u\to x, y)$$

$$= \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy$$

...(By C-R equations $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$)

$$du = \underbrace{(3x^2-3y^2)}_{M}\,dx - \underbrace{6xy}_{N}\,dy$$

$= M\,dx + N\,dy$ form

Step III : $\therefore$ Integrating

$$u = \int M\,dx + \int N\,dy + C$$

y – constant The terms of N free from x

$$u = \int(3x^2-3y^2)\,dx + 0 + C$$

$$u = 3\left(\frac{x^3}{3}\right) - 3y^2(x) + C$$

$u = x^3 - 3y^2x + C$; is a required complex conjugate of v

Step IV : $f(z) = u(x,y) + iv(x,y)$

$$= (x^3 - 3y^2x + C) + i(3x^2y - y^3)$$

Step V : By Milne-Thomson method, put $x = z$ and $y = 0$

$$f(z) = (z^3 - 0 + C) + i(0 - 0)$$

$f(z) = z^3 + C$ ✓ ...Ans.

is a required analytic function.

Ex. 5.5.5 : Show that $u = x\cdot\sin x\cdot\cosh y - y\cos x\sinh y$ is harmonic function and find its harmonic conjugate

Soln. :

Step I : Given : $u = x\sin x\cosh y - y\cos x\cdot\sinh y$ ...(1)

Differentiate u partially w.r.t. x two times, keeping y as constant

$$\therefore\ \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[x\cdot\sin x\cdot\cosh y - y\cos x\cdot\sinh y]$$

$$= \cosh y\left[\frac{\partial}{\partial x}(x\sin x)\right]\sinh y\cdot\frac{\partial}{\partial x}(\cos x)$$

➔ **Using standard result :** ...[d(u v) = u dv + v du]

$$= \cosh y\,[x\cdot(\cos x) + \sin x\,(1)] - y\sinh y\,(-\sin x)$$

$$\therefore\ \frac{\partial u}{\partial x} = x\cdot\cos x\cosh y + \sin x\cosh y + y\sinh y\sin x$$

$$\text{and } \frac{\partial^2 u}{\partial x^2} = \cosh y\frac{\partial}{\partial x}(x\cdot\cos x) + \cosh y\frac{\partial}{\partial x}(\sin x)$$

$$+ y\sinh y\cdot\frac{\partial}{\partial x}(\sin x)$$

$$\frac{\partial^2 u}{\partial x^2} = \cosh y\,[\cos x\cdot 1 - x\sin x] + \cosh y\,(\cos x)$$

$$+ y\sinh y\,(\cos x)$$

$$= \cos x\cdot\cosh y - x\sin x\cdot\cosh y + \cos x\cdot\cosh y$$

$$+ y\cdot\cos x\cdot\sinh y$$

$$\frac{\partial^2 u}{\partial x^2} = 2\cdot\cos x\cdot\cosh y - x\sin x\cosh y + y\cos x\cdot\sinh y \quad ...(2)$$

Now, differentiate u partially w.r.t. y two times, keeping x as constant

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[x\cdot\sin x\cdot\cosh y - y\cos x\cdot\sinh y]$$

$$= x\cdot\sin x\cdot\frac{\partial}{\partial y}(\cosh y) - \cos x\cdot\frac{\partial}{\partial y}(y\sin hy)$$

$$= x\cdot\sin x\cdot\sinh y - \cos x\,[\sinh y\cdot 1 + y\cosh y]$$

$$\frac{\partial u}{\partial y} = x\cdot\sin x\cdot\sinh y - \cos x\cdot\sinh y - \cos x\cdot y\cdot\cosh y$$

and

$$\frac{\partial^2 u}{\partial y^2} = x\cdot\sin x\cdot\frac{\partial}{\partial y}(\sinh y) - \cos x\frac{\partial}{\partial y}(\sinh y) - \cos x\cdot\frac{\partial}{\partial y}(y\cdot\cosh y)$$

$$= x\cdot\sin x\cdot\cosh y - \cos x\cdot\cosh y - \cos x\,(\cosh y + y\sinh y)$$

$$= x\cdot\sin x\cdot\cosh y - 2\cos x\cdot\cosh y - y\cdot\cos x\cdot\sinh y \quad ...(3)$$

Adding equations (2) and (3), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

This shows that u is harmonic function.

Step III : To find harmonic conjugate V :

Since f (z) is harmonic i.e. analytic. So it satisfies C – R equations

By C-R equations ;

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y};\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Step IV : Since, V is harmonic conjugate of u.

and $v = f(x, y)$ ...v → x, y

$$\therefore\ dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}\cdot dy \text{ (by total differentiation)}$$

$$\therefore\ dv = \frac{-\partial u}{dy}dx + \frac{\partial u}{\partial x}dy \qquad \text{(by C.R equations)}$$

$\therefore\ dv = -[x \sin x \sinh y - y \cos x \cosh y - \cos x \sinh y]\ dx$

$\downarrow$

M

$+ [x \cos x \cosh y + \sin x \cosh y + y \sin x \sinh y]\ dy$

$\downarrow$

N

Step V : $dv = M\,dx + N\,dy$ form

$$\therefore\ v = \underbrace{\int M dx}_{\substack{\text{keeping y}\\ \text{as constant}}} + \underbrace{\int N dy}_{\substack{\text{terms of N}\\ \text{free from y}}} + C$$

Step VI :

$v = -\int (x \sin x \sinh y - y \cos x \cosh y - \cos x \sinh y)\, dx + \int o\, dy + C$

$= -[\sinh y \int x \cdot \sin x\, dx - y \cosh y] \int \cos x\, dx - \sinh y \int \cos x\, dx] + C$

➔ **Using standard result**

$$\ldots\left[\int uv\ du = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right]$$

$v = -[\sinh y(-x \cos x + \cos x) - y \sin x \cosh y - \sin x \sinh y] + C$

Hence,

$v = x \cos x \sinh y - \cos x \sinh y + y \sin x \cosh y + \sin x \sinh y$ ✓

Ex. 5.5.6 : Find 'a' such that the function $f(z) = r^2 \cos 2\theta + i\, r^2 \sin(a\theta)$ is analytic function.

Soln. :

Step I : Given,

$f(z) = r^2 \cos 2\theta + i\, r^2 \sin(a\theta)$ is an analytic function

Here, $u = r^2 \cos 2\theta$; $v = r^2 \sin a\theta$

Differentiate u and v partially w.r.t. r keeping θ as constant

$$\frac{\partial u}{\partial r} = 2r \cos 2\theta \ ;\ \frac{\partial v}{\partial r} = 2r \sin(a\theta)$$

Also, differentiate u and v partially w.r.t. θ, keeping r as constant

$$\frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta \ ;\ \frac{\partial v}{\partial \theta} = a r^2 \cos a\theta$$

Step II : Since, f (z) is an analytic function

$\therefore$ By C-R equations in polar form

$$\therefore\ \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Step III : $a r^2 \cos a\theta = 2r^2 \cos 2\theta$

and $-2r^2 \sin 2\theta = -2r^2 \sin a\theta$

Both equations are satisfied for $a = 2$ ✓ **...Ans.**

Ex. 5.5.7 : Find analytic function whose imaginary part $v = \sinh x \cdot \cos y$.

Soln. :

Step I : Given,

$f(z) = u + iv$ is an analytic function

$\therefore$ C – R equations are satisfied

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Given, $v = \sinh x \cos y$

Differentiate v w.r.t. x and y, partially,

$$\frac{\partial v}{\partial x} = \cos y \cdot \underbrace{\frac{\partial}{\partial x}(\sinh x)}_{(\cosh x)} \quad \text{and} \quad \frac{\partial v}{\partial y} = \sinh x \cdot \underbrace{\frac{\partial}{\partial y}(\cos y)}_{(-\sin y)}$$

$$\frac{\partial v}{\partial x} = \cosh x \cos y \quad ; \quad \frac{\partial v}{\partial y} = -\sinh x \sin y$$

Step II : since, $f(z) = u + iv$

Differentiate f(z) w.r.t. x,

$$\therefore\ f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \quad \left(\because \text{by C-R equations } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right)$$

$$= -\sinh x \sin y + i \cosh x \cos y$$

Step III : By Milne-Thomson method; Put $x = z, y = 0$

$$f'(z) = 0 + i \cosh z$$

Integrating, we get

$$\int f'(x)\, dz = i \underbrace{\int \cosh z\, dz}_{\sinh z} + C$$

$f(z) = i \sinh z + C$ is required analytic function ✓ **...Ans.**

Ex. 5.5.8 : If $u = \cosh x \cos y$ then find the harmonic conjugate v such that $f(z) = u + iv$ is analytic function.

Soln. :

Step I : Given : $u = \cosh x \cos y$ ...(1)

Differentiate u partially w.r.t. x, keeping y as constant

$$\frac{\partial u}{\partial x} = \cos y \cdot \underbrace{\frac{\partial}{\partial x}(\cosh x)}_{\sinh x}$$

$$\frac{\partial u}{\partial x} = \sinh x \cos y \qquad \ldots(2)$$

Differentiate u partially w.r.t. y, keeping x as constant

$$\frac{\partial u}{\partial y} = \cosh x \cdot \underbrace{\frac{\partial}{\partial y}(\cos y)}_{(-\sin y)}$$

$$\frac{\partial u}{\partial y} = -\cosh x \sin y \qquad \ldots(3)$$

Step II : Since f(z) is an analytic function.

Therefore, by C – R equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Since, $f(z) = u + iv$

Differentiate w.r.t. x

Step III : $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} + i\left(\frac{-\partial u}{\partial y}\right)$

...(by C – R Equations $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$)

$\therefore\ f'(z) = (\sinh x \cdot \cos y) + i(\cosh x \cdot \sin y)$

(from Equations (2) and (3))

Step IV : By Milne – Thomson method,

Put, $x = z, y = 0$, we get

$$f'(z) = (\sinh z \cdot \underbrace{\cos 0}_{(1)} + i(\cosh z \cdot \underbrace{\sin 0}_{(0)}))$$

$$f'(z) = \sinh z$$

Step V : integrating, w.r.t. z

$$\int f'(z)\, dz = \underbrace{\int \sinh z\, dz}_{(\cosh z)} + C$$

$$f(z) = \cosh z + C \qquad \ldots(4)$$

Which is required analytic function.

Step VI : For harmonic conjugate :

Put, $z = x + iy$ in Equation (4)

$$\therefore\ f(z) = \cosh(x + iy) + c$$

$$= \underbrace{\cosh x \cdot \cosh iy}_{(\cos y)} + \underbrace{\sinh x \cdot \sinh iy}_{(i \sin y)} + C$$

$$f(z) = \cosh x \cdot \cos y + i \sinh x \cdot \sin y + C$$

Step VII : Comparing with given real part $u = \cosh x \cdot \cos y$ we get,

$$V = \sinh x \sin y + C \ ✓ \qquad \textbf{...Ans.}$$

Which is harmonic conjugate of u.

Ex. 5.5.9 : Find the analytic function f (z) whose real part is : $u = r^n \cos n\theta$ and $f(0) = 1$.

Soln. :

Step I : Given : $u = r^n \cos n\theta$

Differentiate u partially w.r.t. r, keeping θ as consent

$$\therefore\ \frac{\partial u}{\partial r} = \cos\theta \cdot \frac{\partial}{\partial r}(r^n)$$

$$\frac{\partial u}{\partial r} = n r^{n-1} \cos n\theta$$

Differentiate u partially w.r.t. θ, keeping r as constant

$$\frac{\partial u}{\partial \theta} = r^n \frac{\partial}{\partial \theta} \cdot (\cos n\theta)$$

$$\frac{\partial u}{\partial \theta} = -n r^n \sin n\theta$$

Step II : $\because$ $f(z) = u + iv$

$$\because\ f'(z) = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)e^{-i\theta} = \left(\frac{\partial u}{\partial r} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right)e^{-i\theta}$$

$$\left(\text{by C-R equations } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= (n r^{n-1} \cos n\theta + i\, n\, r^{n-1} \sin n\theta)\, e^{-i\theta}$$

Step III : By Milne-Thomson method, put $r = z, \theta = 0$

$$f'(z) = n z^{n-1}$$

Integrating $\int f'(z)\, dz = n \int z^{n-1}\, dz + C$

$$= n\left(\frac{z^n}{n}\right) + C$$

$$f(z) = z^n + C$$

For, $z = 0$; $f(0) = 0 + C$; But given, $f(0) = 1$

$\therefore\ C = 1$

$\therefore$ **$f(z) = z^n + 1$ is required analytic function.** ✓ **...Ans.**

Ex. 5.5.10 : If $u = x^3 - 3y^2 x$, find its harmonic conjugate v. Find $f(z) = u + iv$ in terms of z.

Soln. :

Method 1

Step I : Given : $u = x^3 - 3y^2 x$

Differentiate u partially w.r.t. x, keeping y as constant,

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$$

Differentiate u partially w. r. t. y, keeping x as constant.

$$\frac{\partial u}{\partial y} = -6yx$$

Step II : Since, v is a harmonic conjugate of u and $v \to x, y$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$\ldots\left(\because \text{by C – R equations} : \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ and } \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}\right)$$

$$dv = \underbrace{6xy}_{M}\, dx + \underbrace{(3x^2 - 3y^2)}_{N}\, dy = M\,dx + N\,dy \text{ form}$$

Step III : Integrating;

$$v = \underbrace{\int M\,dx}_{\text{y constant}} + \underbrace{\int N\,dy}_{\text{terms free from X}} + C$$

$$v = \underbrace{\int 6xy\,dx}_{\text{y constant}} + \int -3y^2\,dy + C$$

$$v = 6y\left(\frac{x^2}{2}\right) - 3\left(\frac{y^3}{3}\right) + C$$

$$V = 3x^2 y - y^3 + C$$

which is a harmonic conjugate of u

Step IV : Now,

$$f(z) = u + iv = (x^3 - 3y^2 x) + i(3x^2 y - y^3 + C)$$

By Milne-Thomson method; Put $x = z, y = 0$

$$f(z) = (z^3 - 0) + i(0 - 0) + C$$

$f(z) = z^3 + iC$, is required analytic function.

Method 2

Step I : Given,

$$u = x^3 - 3y^2 x$$

Differentiate u partially w. r. t. x, keeping y as constant

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^3) - 3y^2 \frac{\partial}{\partial x}(x)$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$$

Differentiate u partially w. r. t. x, keeping y as constant

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^3) - 3x\frac{\partial}{\partial y}(y^2)$$

$$\frac{\partial u}{\partial y} = -6xy$$

Step II : Since,

$$f(z) = u + iv$$

Differentiate f(z) w. r. t. x

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

(by C-R equations)

$$f'(z) = (3x^2 - 3y^2) + i\,6xy$$

Step III : By Milne-Thomson method, put x = z, y = 0

$$f'(z) = 3z^2$$

Integrating w. r. t. z,

$$\int f'(z)\,dz = \int 3z^2\,dz + C$$

$$f(z) = z^3 + C$$ is an analytic function.

Step IV : For harmonic conjugate,

Put, $z = x + iy$,

we get $f(z) = (x + iy)^3 + C$

➔ **Using standard result**

$$\ldots[(a+b)^3 = a^3 + 3a^2 b + 3a b^2 + b^3]$$

$$f(z) = x^3 - iy^3 + i3x^2y - 3xy^2 + C \qquad \ldots (i^3 = i^2 \cdot i = -i)$$

Hence,

$$f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3 + C)$$

since, given real part $u = x^3 - 3xy^2$

Imaginary part,

$$\therefore\ v = 3x^2y - y^3 + C$$

is required harmonic conjugate function

Method 3

Step I : Given : $u = x^3 - 3y^2 x$

Differentiate, u partially w.r.t. x and y we get,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^3 - 3y^2 x) \text{ and } \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^3 - 3y^2 x)$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2\ ; \qquad \frac{\partial u}{\partial y} = -6xy \qquad \ldots(1)$$

But by C-R equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \ldots(2)$$

Step II : Now, we have to find v

Since, by C – R equations

$$\therefore \frac{\partial v}{\partial y} = \left(\frac{\partial u}{\partial x}\right) = 3x^2 - 3y^2 \qquad \ldots\text{from Equation (i)}$$

Step III : Integrating $\frac{\partial v}{\partial y}$ w. r. t. y, keeping x as constant

$$v = \int (3x^2 - 3y^2)\,dy + f(x)$$

....(Here x is constant)

$$= 3x^2 y - 3\left(\frac{y^3}{3}\right) + f(x)$$

$$\therefore\ v = 3x^2 y - y^3 + f(x) \qquad \ldots(3)$$

Step IV : Differentiating Equation (3) partially w.r.t. x,

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}[3x^2 y - y^3 + f(x)] = 6xy + f'(x) \qquad \ldots(4)$$

But $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ (from C. R. Equation (2))

Step V : ∴ Equation (4) becomes,

$$-\frac{\partial u}{\partial y} = 6xy + f'(x)$$

$$\Rightarrow 6xy = 6xy + f'(x) \qquad \text{(From C – R Equation (1))}$$

Hence, $f'(x) = 0$

Integrating $f(x) = C$

Equation (3) becomes,

$$v = 3x^2 y - y^3 + C$$ is required harmonic function

Step VI : For analytic function,

$$f(z) = u + iv = (x^3 + 3xy^2) + i(3x^2 y - y^3) + C$$

By Milne-Thomson method ; put x = z, y = 0

We get,

$$f(z) = z^3 + C$$ **is required analytic function ✓ ...Ans.**

Ex. 5.5.11 : If $v = \tan^{-1}\left(\frac{y}{x}\right)$, find its harmonic conjugate u. Also find an analytic function f (z) = u + iv in terms of z.

Soln. :

Step I : Given, $v = \tan^{-1}\left(\frac{y}{x}\right)$

Differentiate v partially w. r. t. x, and y

➔ **Using standard formula**

$$\ldots\left[\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}\frac{dx}{dx}\right]$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{1}{\left(\frac{x^2+y^2}{x^2}\right)}\left(\frac{-y}{x^2}\right)$$

$$= \frac{x^2(-y)}{x^2(x^2+y^2)} = \frac{-y}{x^2+y^2};$$

and $$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \left(\frac{1}{\left(\frac{x^2+y^2}{x^2}\right)}\cdot\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

Step II : u (x, y) is a harmonic conjugate of v ;

and $u \rightarrow x, y$

(by the concept of total derivative)

$$du = \underset{\downarrow\ \frac{\partial v}{\partial x}}{\frac{\partial u}{\partial x}}dx + \underset{\downarrow\ \frac{-\partial v}{\partial x}}{\frac{\partial u}{\partial y}}dy = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy$$

(by C-R equations)

Step III : $$du = \underset{\downarrow\ M}{\frac{x}{x^2+y^2}}dx + \underset{\downarrow\ N}{\frac{y}{x^2+y^2}}dy$$

$$du = M\,dx + N\,dy \quad \text{form}$$

Step IV : $$u = \underset{\text{y-constant}}{\int M\,dx} + \underset{\text{Terms of N free from x}}{\int N\,dy} + C$$

$$= \int \frac{x}{x^2+y^2}dx + \int 0\,dy + C$$

➔ **Using standard result :** $$\ldots\left[\int \frac{f'(x)}{f(x)}dx = \log f(x)\right]$$

Here, $f(x) = x^2 + y^2$ and $f'(x) = 2x$

$$\therefore\ u = \frac{1}{2}\int \frac{2x}{x^2+y^2}dx + C$$

(∵ Note Adjustment of 2)

$$u = \frac{1}{2}\log(x^2+y^2) + C$$

Which is a harmonic conjugate of y

Step V : For analytic function; interms of z

$$f(z) = u + iv = \left[\frac{1}{2}\log(x^2+y^2) + C\right] + i\tan^{-1}\left(\frac{y}{x}\right)$$

By Milne-Thomson method, put x = z and y = 0

$$f(z) = \frac{1}{2}\log z^2 + C$$ is required analytic function

Another method

Step I : Given,

$$v = \tan^{-1}\left(\frac{y}{x}\right)$$

Differentiate v partially w.r.t. x, keeping y as constant

$$\frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{d}{dx}\frac{y}{x}\right) = \frac{1}{\left(\frac{x^2+y^2}{x^2}\right)}\cdot\left(-\frac{y}{x^2}\right)$$

$$\frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2}$$

Differentiate v partially w.r.t. y, keeping x as constant

$$\frac{\partial v}{\partial y} = \frac{1}{\left(1+\left(\frac{y}{x}\right)^2\right)}\cdot\frac{d}{dy}\left(\frac{y}{x}\right) = \frac{1}{\left(\frac{x^2+y^2}{x^2}\right)}\cdot\left(\frac{1}{x}\right)$$

$$\frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$$

Step II : Since,

$$f(z) = u + iv$$

$$\therefore\ f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

...(by C-R equations)

$$= \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right)$$

Step III : By Milne-Thomson method, put x = z, y = 0

$$f'(x) = \left(\frac{z}{z^2+0}\right) + i\left(\frac{0}{z^2+0}\right)$$

$$f'(z) = \frac{1}{z} + 0$$

Integrating, $f(z) = \int \frac{1}{z}dz + C$

$$\left(\because \text{ since, } \int \frac{1}{x}dx = \log x + C\right)$$

$$f(z) = \log z + C$$ is an analytic function,

Step IV : For harmonic conjugate,

Put, $z = x + iy$

$$f(z) = \log(x+iy) + C$$

Since, by polar co-ordinate; $z = re^{i\theta}$

$$= \log(re^{i\theta}) + C = \log r + \log e^{i\theta} + C$$

$$\ldots(\because \log(AB) = \log A + \log B)$$

$$= \log r + i\theta + C \qquad \ldots(\log e = 1)$$

$$= \log\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right) + C$$

$$\ldots\left(\because r = \sqrt{x^2+y^2} \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$= \log(x^2 \times y^2)^{1/2} + i\tan^{-1}\left(\frac{y}{x}\right) + C$$

$$f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right) + C$$

since, f(z) = u + iv, and imaginary part given

∴ Take only real part

Step VI : $$\Rightarrow u = \frac{1}{2}\log(x^2+y^2) + C, ✓ \qquad \ldots\text{Ans.}$$

is harmonic conjugate of v

Ex. 5.5.12 : If $u = \frac{1}{2}\log(x^2 + y^2)$, find its harmonic conjugate v such that f(z) = u + iv is an analytic function and hence determine f(z) in terms of z.

Soln. :

Step I : Given : $u = \frac{1}{2}\log(x^2+y^2)$ ...(1)

Differentiate Equation (1) partially w.r.t. x keeping y as constant

$$\frac{\partial y}{\partial x} = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot\underbrace{\frac{d}{dx}(x^2+y^2)}_{(2x)}$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{\cancel{2}}\cdot\frac{1}{x^2+y^2}\cdot\cancel{2}x = \frac{x}{x^2+y^2} \quad ...(2)$$

Differentiate u partially w.r.t. y, keeping x as constant.

$$\frac{\partial u}{\partial y} = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot\underbrace{\frac{d}{dy}(x^2+y^2)}_{(2y)}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\cancel{2}}\cdot\frac{1}{x^2+y^2}\cdot\cancel{2}y = \frac{y}{x^2+y^2} \quad ...(3)$$

Step II : Since, f(z) is an analytic function.

∴ By C-R equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad ...(4)$$

Step III : Since,

$$v \rightarrow x, y$$

by the concept of total derivative,

$$dv = \underbrace{\frac{\partial v}{\partial x}}_{\left(\frac{-\partial u}{\partial y}\right)} dx + \underbrace{\frac{\partial v}{\partial y}}_{\left(\frac{\partial u}{\partial x}\right)} \cdot dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

(From Equation (4))

$$dv = -\underbrace{\frac{y}{x^2+y^2}}_{M}dx + \underbrace{\frac{x}{x^2+y^2}}_{N}dy$$

(From Equations (2), (3))

We get, dv = M dx + N dy form

Step IV : Integrating,

$$v = \underbrace{\int N\,dy}_{\text{x as constant}} + \underbrace{\int M\,dx}_{\text{Terms of M free from y}} + C$$

$$\therefore \quad v = \int \frac{x}{x^2+y^2}dy + \int 0\,dx + C$$

➔ **Using standard result :** $\left[\int \frac{a}{a^2+x^2}dx = \tan^{-1}\left(\frac{x}{a}\right)\right]$

Here, a = x and x = y

$$\therefore \quad v = \tan^{-1}\left(\frac{y}{x}\right) + C$$

Which is the required harmonic conjugate of u

Step V : For analytic function f(z) = u + iv

$$\therefore \quad f(z) = u + iv = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right) + C$$

By Milne – Thomson method, put x = z, y = 0

$$f(z) = \frac{1}{2}\log(z^2+0) + i\tan^{-1}\left(\frac{0}{z}\right) + C$$

$$f(z) = \frac{1}{2}\log(z^2) + C, \checkmark \quad ...\text{Ans.}$$

is the required analytic function f(z).

Ex. 5.5.13 : If $u = 3x^2 - 3y^2 + 2y$, find v such that f(z) = u + iv is analytic. Determine f(z) in terms of z.

Soln. :

Step I : Given : $u = 3x^2 - 3y^2 + 2y$ ...(1)

Differentiate u partially w.r.t. x and y

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(3x^2 - 3y^2 + 2y) \text{ and } \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(3x^2 - 3y^2 + 2y)$$

$$\frac{\partial u}{\partial x} = (6x - 0 + 0) \text{ and } \frac{\partial u}{\partial y} = (0 - 6y + 2)$$

$$\frac{\partial u}{\partial x} = 6x; \quad \frac{\partial u}{\partial y} = -6y + 2 \quad ...(2)$$

Step II : Given, f (z) is an analytic function

Therefore ; by C-R equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad ...(3)$$

Step III : For harmonic conjugate v.

$$\because \quad v \longrightarrow x, \ y$$

Differentiate v, by using the concept of total derivative.

Step IV : $$dv = \underbrace{\frac{\partial v}{\partial x}}_{\left(\frac{-\partial u}{\partial y}\right)}dx + \underbrace{\frac{\partial v}{\partial y}}_{\left(\frac{\partial u}{\partial x}\right)}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

...(from Equation (3))

$$dv = \underbrace{(6y-2)}_{M}\ dx + \underbrace{6x}_{N}\ dy \quad ...\text{(from Equation (2))}$$

Step V : It is of the form, dv = M dx + N dy

$$\therefore v = \underbrace{\int M\,dx}_{\text{y-constant}} + \underbrace{\int N\,dy}_{\text{Terms of N free from x}} + C$$

$$v = \int (6y-2)\,dx + 0 + C$$

$$v = (6yx - 2x) + C$$

$$v = 6xy - 2x + C,$$

is required harmonic conjugate of u

Step VI : For analytic function f (z)

Since, f (z) = u + iv

$$= (3x^2 - 3y^2 + 2y) + i(6xy - 2x) + C$$

Step VII : By Milne-Thomson method put, x = z, y = 0

$$f(z) = (3z^2 - 0 + 0) + i(0 - 2z) + C$$

$$\Rightarrow f(z) = 3z^2 + i(-2z) + C$$

$$f(z) = 3z^2 - 2zi + C \checkmark \quad ...\text{Ans.}$$

is required analytic function

Ex. 5.5.14 : If $u = x^2 - y^2 - y$ then find its harmonic conjugate and corresponding analytic function in terms of z.

Soln. :

Step I : Given, $u = x^2 - y^2 - y$ ...(1)

Differentiate Equation (1) partially w. r. t. x and y

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2 - y^2 - y);$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 - y^2 - y)$$

$$\frac{\partial u}{\partial x} = 2x; \quad \frac{\partial u}{\partial y} = -2y - 1 \quad ...(2)$$

Since, f (z) is an analytic function,

Therefore by C-R equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad ...(3)$$

Step II : For harmonic conjugate v :

$$v \longrightarrow x, y$$

Differentiate v, by the concept of total derivative.

Step III : $$\therefore \quad dv = \underbrace{\frac{\partial v}{\partial x}}_{\frac{-\partial u}{\partial y}}dx + \underbrace{\frac{\partial v}{\partial y}}_{\frac{\partial u}{\partial x}}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

....(from Equation (3))

$$dv = \underbrace{(2y+1)}_{M}\ dx + \underbrace{2x}_{N}\ dy \quad ...\text{(from Equation (2))}$$

Step IV : It is of the form : dv = M dx + N dy

Integrating,

$$\therefore \quad v = \underbrace{\int M\,dx}_{\text{y-constant}} + \underbrace{\int N\,dy}_{\text{Terms of N free from x}} + C$$

$$v = \int (2y+1)\,dx + \int 0\,dy + C$$

$$v = (2y+1)x + 0 + C = (2y+1)x + C$$

$$v = (2xy + x) + C,$$

which is required harmonic conjugate

Step V : For analytic function f (z) :

Since, $f(Z) = u + iv = (x^2 - y^2 - y) + i[2xy + x)] + C$

By Milne-Thomson method, put, x = z, y = 0

$$f(z) = (z^2 - 0 - 0) + i[0 + z] + C$$

$$f(z) = z^2 + zi + C$$

$$f(z) = z^2 + zi + C \checkmark$$

is the required analytic function

Ex. 5.5.15 : Show that the function : $e^x(\cos y + i\sin y)$ is analytic and find f(z) in terms of z.

Soln. :

Step I : Given,

$$f(z) = e^x[\cos y + i\sin y]$$

$$\therefore \quad u + iv = (e^x\cos y + i e^x \sin y)$$

Comparing real and imaginary parts on both sides

$$\Rightarrow \quad u = e^x\cos y; \quad v = e^x\sin y$$

Step II : Differentiate u partially w.r.t. x and y

$$\frac{\partial u}{\partial x} = \cos y\cdot\frac{\partial}{\partial x}(e^x) \Rightarrow \frac{\partial u}{\partial x} = e^x\cos y$$

$$\text{And,} \quad \frac{\partial u}{\partial y} = e^x\underbrace{\frac{\partial}{\partial y}(\cos y)}_{(-\sin y)} \Rightarrow \frac{\partial u}{\partial y} = -e^x\sin y$$

Step III : Differentiate v partially w. r. t. x and y

$$\frac{\partial v}{\partial x} = \sin y\cdot\frac{\partial}{\partial x}(e^x)$$

$$\text{And} \quad \frac{\partial v}{\partial y} = e^x\cdot\underbrace{\frac{\partial}{\partial y}(\sin y)}_{\cos y}$$

$$\Rightarrow \frac{\partial v}{\partial x} = e^x\sin y; \quad \frac{\partial v}{\partial y} = e^x\cos y$$

Step IV : It is observed that,

$$\frac{\partial u}{\partial x} = e^x\cos y = \frac{\partial v}{\partial y}$$

and $\frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$

Thus, C – R equation are satisfied and partial derivatives are continuous. Hence, f (z) is an analytic function.

Step V : For analytic function f (z)

Given, $f(z) = e^x(\cos y + i \sin y)$

By Milne – Thomson's method

Put, $x = z, y = 0$

$\therefore f(z) = e^z(\cos 0 + i \sin 0)$

$f(z) = e^z(1 + 0)$

$f(z) = e^z$

is the required analytic function✓

Hence proved.

Ex. 5.5.16 : Show that $u = y^3 - 3x^2y$ is a harmonic function. Find its harmonic conjugate and the corresponding analytic function.

Soln. :

Step I : For harmonic function, we have to show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Given, $u = y^3 - 3x^2y$

Step II : Differentiate u partially w.r.t. x and y, two times

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(y^3 - 3x^2 y);$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(y^3 - 3x^2 y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = -6xy \; ; \; \frac{\partial u}{\partial y} = 3y^2 - 3x^2 \qquad ...(1)$$

And $\therefore \frac{\partial^2 u}{\partial x^2} = -6y \; ; \; \frac{\partial^2 u}{\partial y^2} = 6y$

Step III :

Adding

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6y + 6y = 0$$

$\therefore u = y^3 - 3x^2y$ is harmonic function.

Step IV : To find the complex conjugate v : since, f(z) is an analytic function, Therefore by C – R equations we have,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \Rightarrow \frac{\partial v}{\partial y} = -6xy$$

...from Equation (1)

Integrating w.r.t. y,

$$v = \int -6xy\, dy + \phi(x),$$

$$v = -\frac{6xy^2}{2} + \phi(x) \quad \because \phi(x) \text{ is a function of } x$$

$$v = -3xy^2 + \phi(x) \qquad ....(2)$$

Step V : Differentiate v w. r. t. x, keeping y as constant

$$\frac{\partial v}{\partial x} = -3y^2 \cdot \frac{\partial}{\partial x}(x) + \phi'(x)$$

$$\frac{\partial v}{\partial x} = -3y^2 + \phi'(x) \; ;$$

Where, $\phi(x)$ is an arbitrary function of x

Step VI : Since, if harmonic functions u and v satisfies C – R. equations then u + iv an analytic function.

$\therefore$ By C – R equations

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$-3y^2 + \phi'(x) = -(3y^2 - 3x^2)$ ...from Equation (1) and (2)

$\Rightarrow -3y^2 + \phi'(x) = -3y^2 + 3x^2 \quad \Rightarrow \phi'(x) = 3x^2,$

Integrating,

$$\phi(x) = \int 3x^2\, dx + C$$

$$\phi(x) = x^3 + C$$

$\therefore$ The complex conjugate,

$$v = -3xy^2 + x^3 + C \qquad \text{from Equation (2)}$$

Step VII : The corresponding analytic function is

$$f(z) = u + iv = (y^3 - 3x^2y) + i(x^3 - 3xy^2) + C$$

By Milne-Thomson method, put $x = z, y = 0$

$$f(z) = (0 - 0) + i(z^3 - 0) + C$$

$\therefore f(z) = i[z^3 + C]$ i.e $f(z) = i z^3 + C$ ✓

is the required analytic function.

Hence proved...

Ex. 5.5.17 : Show that : $v = \sinh x \cos y$ is harmonic and find the corresponding analytic function F(z) and its harmonic conjugate.

Soln. :

Step I : To show v is harmonic, we have to show

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

We have $v = \sinh x \cos y$

Differentiate v partially w.r.t. x and y

$$\frac{\partial v}{\partial x} = \cos y \cdot \underbrace{\frac{\partial}{\partial x}(\sinh x)}_{\cosh x} \text{ and } \frac{\partial v}{\partial y} = \sinh x \cdot \underbrace{\frac{\partial}{\partial y}(\cos y)}_{-\sin y}$$

$$\Rightarrow \frac{\partial v}{\partial x} = \cosh x \cos y \; ; \; \frac{\partial v}{\partial y} = -\sinh x \sin y \qquad ...(1)$$

Step II : Again differentiate $\frac{\partial v}{\partial x}$ w.r.t. x and $\frac{\partial v}{\partial y}$ w.r.t. y

$$\frac{\partial^2 v}{\partial x^2} = \cos y \cdot \underbrace{\frac{\partial}{\partial x}(\cosh x)}_{\sinh x} \text{ and } \frac{\partial^2 v}{\partial x^2} = -\sinh x \cdot \underbrace{\frac{\partial}{\partial y}(\sin y)}_{\cos y}$$

Now, $\frac{\partial^2 v}{\partial x^2} = \sinh x \cos y \; ; \; \frac{\partial^2 v}{\partial y^2} = -\sinh x \cdot \cos y$

Step III : Adding $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \sinh x \cos y - \sinh x \cos y = 0$

This shows that,

$v = \sinh x \cos y$ is harmonic function.

Step IV : For analytic function f(z) = u + iv, we need u

Let, u be its harmonic conjugate function,

Since, $u \longrightarrow x, y$

Differentiate u by the concept of total derivative,

$$du = \underbrace{\frac{\partial u}{\partial x}}_{\left(\frac{\partial v}{\partial y}\right)} \cdot dx + \underbrace{\frac{\partial u}{\partial y}}_{\left(\frac{-\partial v}{\partial x}\right)} \cdot dy = \frac{\partial v}{\partial y} \cdot dx - \frac{\partial v}{\partial x} \cdot dy$$

(by C – R equations)

$$du = \underbrace{(-\sinh x \sin y)}_{M}\, dx - \underbrace{(\cosh x \cos y)}_{N} \cdot dy$$

$du = M\,dx + N\,dy$ form

Step V : Integrating, we get

$$u = \underbrace{\int M\,dx}_{\text{keeping y as constant}} + \underbrace{\int N\,dy}_{\text{Terms of N free from x}} + C$$

$$= \int -\sinh x \cdot \sin y\, dx + \int 0\, dy + C$$

$$= -\sin y \underbrace{\int \sinh x\, dx}_{\cosh x} + C$$

$= (\cosh x \,.\, \sin y) + C$ ... (C is any arbitrary constant)

is the required complex conjugate of v

Step VI : For, the analytic function $f(z) = u + iv$

$\therefore f(z) = -\cosh x \sin y + c + i \sinh x \cos y,$

By Milne – Thomas method,

Put, $x = z$ and $y = 0$

$$\therefore f(z) = -\cosh z \underbrace{(\sin 0)}_{0} + C + i \sinh \underbrace{\cos 0}_{(1)}$$

$$f(z) = 0 + C + i \sinh z$$

$f(z) = i \sinh z + C$ ✓

is an analytic function.

Hence proved.

Ex. 5.5.18 : Determine P such that the function. $f(z) = \frac{1}{2}\log(x^2 + y^2) + i \tan^{-1}\frac{Px}{y}$ be an analytic function.

Soln. :

Step I : If real and imaginary parts of $f(z) = u + iv$ satisfies C – R equation, then f(z) is analytic

Given, $f(z) = u + iv = \frac{1}{2}\log(x^2 + y^2) + i \tan^{-1}\frac{Px}{y}$

Comparing real and imaginary parts on both sides

$$\therefore u = \frac{1}{2}\log(x^2 + y^2) \quad \text{and} \quad v = \tan^{-1}\frac{Px}{y}$$

Step II : Differentiate u and v partially w.r.t. x, keeping y as constant

➔ **Using standard result**

$$...\left[\text{(i) } \frac{d}{dx}\log f(x) = \frac{1}{f(x)} \cdot \frac{d}{dx} f(x)\right.$$

$$\left.\text{(ii) } \frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{1}{1 + \left(\frac{x}{a}\right)^2} \cdot \frac{d}{dx}\left(\frac{x}{a}\right)\right]$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \cdot \left(\frac{1}{2} \cdot \log(x^2 + y^2)\right)$$

$$= \frac{1}{2} \cdot \left(\frac{1}{x^2 + y^2}\right) \cdot \frac{\partial}{\partial x}(x^2 + y^2) = \frac{1}{2} \cdot \left(\frac{1}{x^2 + y^2}\right) \cdot (2x)$$

And $\frac{\partial v}{\partial x} = \frac{1}{1 + \left(\frac{Px}{y}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{Px}{y}\right) = \frac{1 \cdot y^2}{(y^2 + P^2x^2)} \cdot \left(\frac{P}{y}\right)$

$$\therefore \frac{\partial u}{\partial x} = \frac{1 \cdot x}{(x^2 + y^2)}; \quad \frac{\partial v}{\partial x} = \frac{\cdot Py}{y^2 + P^2x^2}$$

Step III : Differentiate u and v w.r.t. y

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{2} \cdot \log(x^2 + y^2)\right) = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot \frac{\partial}{\partial y} \cdot (x^2 + y^2)$$

$$= \frac{1}{\cancel{2}} \frac{1}{(x^2 + y^2)} \cdot \cancel{2}y$$

and $\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{Px}{y}\right)\right)$

$$= \frac{1}{1 + \left(\frac{Px}{y}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{Px}{y}\right)$$

$$= \frac{1}{\frac{(y^2 + P^2x^2)}{y^2}} \cdot \left(\frac{-Px}{y}\right)$$

Hence, $\dfrac{\partial u}{\partial y} = \dfrac{y}{(x^2+y^2)}$; $\dfrac{\partial v}{\partial y} = \dfrac{-Px}{y^2+p^2x^2}$

Step IV : Since, f (z) is an analytic function

Therefore, by C – R equations

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{x}{x^2+y^2} = \frac{-Px}{y^2+P^2x^2} \quad \text{and} \quad \frac{y}{x^2+y^2} = \frac{-Py}{y^2+P^2x^2}$$

Which are satisfied if $P = -1$ ✓ ...Ans.

Ex. 5.5.19 : If $u = \lambda(1+\cos\theta)$. Find v so that $f(z) = u + iv$ is analytical function.

Soln. :

Step I : Given, $u = \lambda(1+\cos\theta)$

Differentiating u partially w.r.t. r and θ

$$\frac{\partial u}{\partial r} = 0; \quad \frac{\partial u}{\partial \theta} = \lambda \cdot \underbrace{\frac{\partial}{\partial\theta}(1+\cos\theta)}_{(-\sin\theta)}$$

$$\frac{\partial u}{\partial r} = 0; \quad \frac{\partial u}{\partial \theta} = -\lambda \sin\theta$$

Step II :

To find complex conjugate v $v \longrightarrow r, \theta$

Since,

$$dv = \frac{\partial v}{\partial r}\cdot dr + \frac{\partial v}{\partial \theta}\cdot d\theta$$

Step III : By C - R. equation in polar form

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\cdot\frac{\partial u}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

$$\therefore \quad dv = -\frac{1}{r}\frac{\partial u}{\partial \theta}\,dr + r\frac{\partial u}{\partial r}\,d\theta$$

$$= -\frac{1}{r}(-\lambda\sin\theta) + r(0) = \frac{\lambda}{r}\sin\theta\,dr + 0$$

Step IV : Integrating w.r.t. r, we get

$$v = \lambda\sin\theta\int\frac{1}{r}\,dr + C$$

$$v = \lambda\sin\theta\log r + C \qquad \left(\because \int\frac{1}{x}dx = \log x\right)$$

$\therefore$ Analytic function is, $f(z) = u + iv$

$$= \lambda(1+\cos\theta) + i\,\lambda\sin\theta\cdot\log r + C$$

By Milne-Thomson method, put $r = z, \theta = 0$

$$\therefore \quad f(z) = \lambda(1+\underbrace{\cos 0}_{(1)}) + i\lambda(\underbrace{\sin(0)}_{(0)})\cdot\log z + C$$

$$f(x) = 2\lambda + C \checkmark \qquad \text{...Ans.}$$

Ex. 5.5.20 : Find the analytic function $w = u + iv$; where, $u = e^{-x}[(x^2-y^2)\cos y + 2xy\sin y]$

Soln. :

Step I : Given : $u = \underbrace{e^{-x}}_{I}\ \underbrace{[(x^2-y^2)\cos y + 2xy\sin y]}_{II}$

Differentiate u partially w. r. t. x, keeping y as constant

$$\frac{\partial u}{\partial x} = \left(\frac{d}{dx}e^{-x}\right)\cdot[(x^2-y^2)\cos y + 2xy\sin y]$$

$$+ e^{-x}\cdot\frac{d}{dx}[(x^2-y^2)\cos y + 2xy\sin y]$$

Step II : $\dfrac{\partial u}{\partial x} = -e^{-x}[(x^2-y^2)\cos y + 2xy\cdot\sin y]$

$$+ e^{-x}[2x\cos y + 2y\sin y] = \phi_1(x, y) \ldots \text{(say)}$$

Differentiate u partially w.r.t. y, keeping x as constant.

$$\frac{\partial u}{\partial y} = e^{-x}\cdot\frac{\partial}{\partial y}\cdot[\underbrace{(x^2-y^2)}_{I}\cdot\underbrace{\cos y}_{II} + \underbrace{2xy}_{I}\,\underbrace{\sin y}_{II}]$$

$$= e^{-x}[(x^2-y^2)\cdot\underbrace{\frac{\partial}{\partial y}(\cos y)}_{(-\sin y)} + \cos y\underbrace{\frac{\partial}{\partial y}\cdot(x^2-y^2)}_{(-2y)}$$

$$+ 2x[\sin y\cdot\underbrace{\frac{\partial}{\partial y}\cdot y}_{(1)} + y\cdot\underbrace{\frac{\partial}{\partial y}\cdot\sin y}_{(\cos y)}]$$

$$\Rightarrow \frac{\partial u}{\partial y} = e^{-x}[-2y\cos y - (x^2-y^2)\cdot\sin y + 2x\sin y$$

$$+ 2xy\cos y]$$

$$= e^{-x}[-(x^2-y^2)\sin y - 2y\cos y + 2x\sin y + 2xy\cos y]$$

$$= \phi_2(x, y) \qquad \text{...(say)}$$

Step III : Since, $w \to F(x, y) = u + iv$

Differentiate w. r. t. x, we get

We know, $\dfrac{dw}{dz} = \dfrac{\partial u}{\partial x} + i\underbrace{\dfrac{\partial v}{\partial x}}_{\left(\frac{-\partial u}{\partial y}\right)} = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$

....By C – R equation $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

$$= \phi_1(x, y) - i\,\phi_2(x, y)$$

Step IV : By Milne – Thomson's method, put $x = z, y = 0$, we get

$$\frac{dw}{dz} = \phi_1(z, 0) - i\,\phi_2(z, 0)$$

$$= -e^{-z}[(z^2-0)\cdot\underbrace{\cos(0)}_{(-1)} + 0] + e^{-z}[2z\cdot\underbrace{\cos(0)}_{(1)} + 0]$$

$$+ e^{-z}[-(z^2+0)\cdot\underbrace{\sin(0)}_{(0)} - 0 + 2Z\underbrace{\sin(0)}_{(0)} + 0]$$

$$= -e^{-z}\cdot z^2 + e^{-z}\cdot 2z + 0$$

$$f'(z) = e^{-z}(2z - z^2)$$

Step V : Integral w.r.t. z

$$f(z) = \int 2z\cdot e^{-z}\,dz - \int z^2\cdot e^{-z}\,dz + C$$

$$\downarrow\ \downarrow \qquad \downarrow\ \downarrow$$

$$u \quad v \qquad u \quad v$$

➔ **Using integrating by parts**

$$\ldots\left[\int uv\,dx = u\,v_1 - u'\,v_2 + u''\,v_3 - \ldots\right]$$

$$= 2\left[z\,\frac{e^{-z}}{(-1)} - (1)\frac{e^{-z}}{1}\right] - \left[z^2\frac{e^{-z}}{(-1)} - 2z\left(\frac{e^{-z}}{1}\right) + 2\cdot\left(\frac{e^{-z}}{-1}\right)\right]$$

$$= e^{-z}\left[-2\cancel{z} - \cancel{z} + z^2 + \cancel{2z} + \cancel{2}\right]$$

$= z^2e^{-z} + C$ **is required analytic function.** ✓ ...Ans.

Ex. 5.5.21 : If $f(z) = u(x, y) + i\,v(x, y)$ is an analytic function, and $u = 3x - 2xy$ then find harmonic conjugate and f (z) interms of z

Soln. :

Step I : Given, $u = 3x - 2xy$

Differentiate u partially w.r.t. x, keeping y as constant

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(3x - 2xy)$$

$$\therefore \frac{\partial u}{\partial x} = 3 - 2y$$

Differentiate u partially w.r.t. y, keeping x as constant.

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(3x - 2xy)$$

$$\therefore \quad \frac{\partial u}{\partial y} = -2x$$

Step II : Since, $v \longrightarrow x, y$

Differentiate v, by using the concept of total derivative

$$dv = \underbrace{\frac{\partial v}{\partial x}}_{\left(-\frac{\partial u}{\partial y}\right)}\cdot dx + \underbrace{\frac{\partial v}{\partial y}}_{\left(\frac{\partial u}{\partial x}\right)}\cdot dy$$

$$= \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)\cdot dy$$

(∵ C-R equations)

$$dv = \underbrace{2x}_{M}\,dx + \underbrace{(3-2y)}_{N}\,dy$$

$dv = M\,dx + N\,dy$ ... from

Step III : Integrating, we get,

$$v = \underbrace{\int M\,dx}_{\text{y-constant}} + \underbrace{\int N\,dy}_{\text{Terms of N free from x}} + c$$

$$v = \int 2x\,dx + \int(3-2y)\,dy + c$$

$$v = 2\left(\frac{x^2}{2}\right) + \left(3y - \frac{2y^2}{2}\right) + c$$

$\therefore v = x^2 + 3y - y^2 + c$; is complex conjugate of u

Step IV : For analytic function

$$f(z) = u + iv = (3x - 2xy) + i(x^2 + 3y - y^2) + c$$

By Milne-Thomson method ; put, $x = z$; $y = 0$, we get

$$f(z) = (3z - 0) + i(z^2 + 0 - 0) + c$$

$$f(z) = 3z - iz^2 + c$$

$f(z) = 3z + iz^2 + c$ **is required analytic function.** ✓ ...Ans.

Ex. 5.5.22 : If $w = \phi + i\psi$ represent the complex potential for an electric field and $\psi = x^2 - y^2 + \dfrac{x}{x^2+y^2}$, determine the function φ. Also find w in terms of z.

Soln. :

Step I : Given, $w = \phi + i\psi$ and $\psi = x^2 - y^2 + \dfrac{x}{x^2+y^2}$,

Step II :

Differentiate ψ partially w.r.t x, keeping y as constant

➔ **Using standard result :** ...[d(uv) = udv + vdu]

$$\therefore \frac{\partial\psi}{\partial x} = 2x + \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} = 2x + \frac{y^2 - x^2}{(x^2+y^2)^2} \quad \ldots(1)$$

Differentiate ψ partially w.r.t y, keeping x as constant

➔ **Using standard result :** $\ldots\left[\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^2}\dfrac{dx}{dx}\right]$

$$\frac{\partial\psi}{\partial y} = -2y + x\left[\frac{-1}{(x^2+y^2)^2\cdot\underbrace{\frac{\partial}{\partial y}(x^2+y^2)}_{(2y)}}\right]$$

$$\frac{\partial\psi}{\partial y} = -2y - \frac{x(2y)}{(x^2+y^2)^2} = -2y - \frac{2xy}{(x^2+y^2)^2} \quad \ldots(2)$$

Step III : Since φ is a complex conjugate

$\therefore \quad \phi \longrightarrow x, y$

Differentiate φ, by using the concept of total derivative.

$$d\phi = \underbrace{\frac{\partial\phi}{\partial x}}_{\left(\frac{\partial\psi}{\partial y}\right)} dx + \underbrace{\frac{\partial\phi}{\partial y}}_{\left(-\frac{\partial\psi}{\partial x}\right)} \cdot dy$$

$$= \frac{\partial\psi}{\partial y} \cdot dx - \frac{\partial\psi}{\partial x} dx \qquad \text{(by C.R. equation)}$$

From Equations (1) and (2)

$$d\phi = \underbrace{\left(-2y - \frac{2xy}{(x^2+y^2)^2}\right)}_{M} dx - \underbrace{\left(2x + \frac{y^2-x^2}{(x^2+y^2)^2}\right)}_{N} dy$$

$$d\phi = M\,dx + N\,dy \qquad \text{... Form}$$

Step IV : This is exact differential equation, integrating

$$\phi = \underbrace{\int M\,dx}_{y \text{ constant}} + \underbrace{\int N\,dy}_{\text{The terms of N free from x}} + C$$

$$\phi = \int\left(-2y - \frac{2xy}{(x^2+y^2)^2}\right) dx + \int 0\,dy + C$$

$$\phi = -2xy - y\int \frac{2x}{(x^2+y^2)^2}\,dx + C$$

Put, $x^2 + y^2 = t$

Derivative w.r.t. t, and y as constant

$$\therefore 2x\,dx = dt$$

$$\phi = -2xy - y\int \frac{dt}{t^2} + C$$

$$= -2xy - y\left(-\frac{1}{t}\right) + C$$

$$= -2xy + \frac{y}{t} + C$$

$$\phi = -2xy + \frac{y}{(x^2+y^2)} + C$$

Step V : Analytic function $f(z) = \phi + i\psi$

$$\therefore w = \phi + i\psi$$

$$= \left(-2xy + \frac{y}{x^2+y^2}\right) + i\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) + C$$

By Milne Thomson's Method, put x = z, y = 0

$$w = (0+0) + i\left(z^2 - 0 + \frac{z}{z^2+0}\right) + C$$

$$\therefore w = 0 + i\left(z^2 + \frac{1}{z}\right) + C$$

i.e. $w = i\left(z^2 + \frac{1}{z}\right) + c$; is the required analytic function. ✓

...Ans.

Ex. 5.5.23 : An electrostatics field in the xy – plane is given by the potential function $\phi = 3x^2y - y^3$, find the stream function and the complex potential function.

Soln. :

Step I : Given $\phi = 3x^2y - y^3$; and $\psi(x,y)$ be an stream function

Step II : By C- R equations,

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

Differentiate ϕ w.r.t. x and y

$$\frac{\partial\phi}{\partial x} = \underbrace{\frac{\partial}{\partial x}(3x^2y - y^3)}_{(6xy)} \text{ and } \frac{\partial\phi}{\partial y} = \underbrace{\frac{\partial}{\partial y}(3x^2y - y^3)}_{(3x^2 - 3y^2)}$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = 6xy \;;\; \frac{\partial\phi}{\partial y} = 3x^2 - 3y^2$$

By C-R equation,

$$\therefore \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} = 3x^2 - 3y^2 \text{ i.e. } \frac{\partial\psi}{\partial x} = -3x^2 + 3y^2 \quad ...(1)$$

Step III : Integrating Equation (1) w.r.t x, keeping y as constant

$$\psi = -3\frac{x^3}{3} + 3y^2x + f(y)$$

$$= -x^3 + 3y^2x + f(y) \qquad ...(3)$$

($\because$ Here, f(y) is a constant function)

Differentiating ψ partially w.r.t. y, keeping x as constant

$$\frac{\partial\psi}{\partial y} = \underbrace{-\frac{\partial}{\partial y}(x^3)}_{(0)} + \underbrace{3x\frac{\partial}{\partial y}(y^2)}_{(2y)} + f'(y)$$

$$\therefore \frac{\partial\psi}{\partial y} = 0 + 6yx + f'(y) \qquad ...(4)$$

Step IV : From Equation (4) we get,

By C-R equations, $\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$

$$6xy = 6xy + f'(y) \Rightarrow f'(y) = 0 \Rightarrow f(y) = c$$

$\therefore$ Equation (3) becomes

$\therefore \psi = -x^3 + 3y^2x + c$, which is required stream function.

Step V : $\therefore$ The complex Potential function : $f(z) = \phi + i\psi$

$$\therefore \quad f(z) = (3x^2y - y^3) + i(-x^3 + 3y^2x + C)$$

Step VI : By Milne Thomson's methods, Put, x = z and y = 0

$$f(z) = (0-0) + i(-z^3 + 0) + c$$

$$f(z) = -iz^3 + C \checkmark \qquad \textbf{...Ans.}$$

Ex. 5.5.24 : If ϕ and ψ are functions of x and y satisfying Laplace equation, show that (s + it) is analytic. where, $s = \frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x}$ and $t = \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y}$

Soln. :

Step I : Since, ϕ and ψ satisfying Laplace equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \quad ...(1) \;;\; \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \quad ...(2)$$

Step II : Given, $s = \frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x}$ and $t = \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y}$

Differentiate s w.r.t x and t w.r.t y

$$\frac{\partial s}{\partial x} = \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial x^2} \;;\; \frac{\partial t}{\partial y} = \frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial y^2}$$

$$\therefore \frac{\partial s}{\partial x} - \frac{\partial t}{\partial y} = \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial y^2}$$

$$= -\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) = 0 \quad \text{(From Equation (2))}$$

This shows that, $\frac{\partial s}{\partial x} = \frac{\partial t}{\partial y}$ ...(3)

Differentiate s w.r.t. y and t w.r.t. x

$$\frac{\partial s}{\partial y} = \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\psi}{\partial x\partial y} \text{ and } \frac{\partial t}{\partial x} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\psi}{\partial y\partial x}$$

$$\therefore \frac{\partial s}{\partial y} + \frac{\partial t}{\partial x} = \frac{\partial^2\phi}{\partial y^2} - \cancel{\frac{\partial^2\psi}{\partial x\partial y}} + \frac{\partial^2\phi}{\partial x^2} + \cancel{\frac{\partial^2\psi}{\partial x\partial y}}$$

$$= \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) = 0 \quad \text{(From Equation (1))}$$

Hence, $\frac{\partial s}{\partial y} = -\frac{\partial t}{\partial x}$ ...(4)

From Equation (3) and Equation (4) we proved that ϕ and ψ satisfies C – R equations. Hence the function (s + it) is analytic. ✓ **Hence Proved**

Ex. 5.5.25 : If u, v are real functions of x, y connected by the relations $u + iv = f(z)$; $z = ix + (1+i)y$. Show that u, v satisfies the equation. $2\frac{\partial^2\phi}{\partial x^2} - 2\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^2\phi}{\partial y^2} = 0$; where, f (z) is an analytic function.

Soln. :

Step I : Since, $z = ix + (1+i)y = y + i(x+y)$

Let, $X = y$ and $Y = x + y$

$$\therefore Z = X + iY \qquad ...(1)$$

Step II : $f(z) = u + iv = u(x,y) + iv(x,y)$

Since, f (z) is an analytical function

$\therefore$ u, v satisfies Laplace equation

$$\left(\begin{array}{l} \because u(X,Y) = \phi(x,y) \\ \quad\;\; v(X,Y) = \psi(x,y) \end{array}\right)$$

$$\frac{\partial^2\phi}{\partial X^2} + \frac{\partial^2\phi}{\partial Y^2} = 0 \qquad ...(2)$$

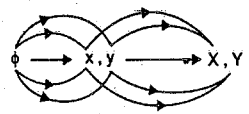
Step III : Now, transform the Equation (2) in terms of x, y by using the relations.

$$y = X;\; x + y = Y.$$

$$\downarrow$$

$$x = y - y = Y - X$$

$$\Rightarrow x = Y - X \quad \text{and} \quad Y = X$$



$$\frac{\partial\phi}{\partial X} = \frac{\partial\phi}{\partial x}\cdot\frac{\partial x}{\partial X} + \frac{\partial\phi}{\partial x}\cdot\frac{\partial y}{\partial X}$$

$$= -1\cdot\frac{\partial\phi}{\partial X} + \frac{\partial\phi}{\partial Y} \ldots\left(\because \frac{\partial x}{\partial X} = -1;\frac{\partial y}{\partial X} = 1\right) \ldots(i)$$

$$\frac{\partial\phi}{\partial Y} = \frac{\partial\phi}{\partial x}\cdot\frac{\partial x}{\partial Y} + \frac{\partial\phi}{\partial y}\cdot\frac{\partial y}{\partial Y}$$

$$= \frac{\partial\phi}{\partial X} + 0 \ldots\left(\because \frac{\partial x}{\partial Y} = 1;\frac{\partial y}{\partial y} = 0\right) \qquad ...(ii)$$

We get, $\frac{\partial}{\partial X} = -\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and $\frac{\partial}{\partial Y} = \frac{\partial}{\partial x}$

$\therefore$ By using principle of equivalent operator

$$\frac{\partial^2\phi}{\partial X^2} = \frac{\partial}{\partial X}\left(\frac{\partial\phi}{\partial X}\right)$$

$$= \left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(-\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\right) \quad \text{...From (i)}$$

$$\Rightarrow \frac{\partial^2\phi}{\partial X^2} = -\frac{\partial}{\partial x}\left[-\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\right] + \frac{\partial}{\partial y}\left[-\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\right]$$

$$= \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^2\phi}{\partial y^2}$$

$$\text{and } \frac{\partial^2\phi}{\partial Y^2} = \frac{\partial}{\partial Y}\left(\frac{\partial\phi}{\partial Y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial^2\phi}{\partial x^2}$$

$$\therefore \frac{\partial^2\phi}{\partial X^2} + \frac{\partial^2\phi}{\partial Y^2} = \left(\frac{\partial^2\phi}{\partial x^2} - 2\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial x^2}\right) = 2\frac{\partial^2\phi}{\partial x^2} - 2\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^2\phi}{\partial y^2}$$

Step VI : Since, $u = \phi$ and $v = \psi$ satisfies Laplace equation, hence

$$\therefore 2\frac{\partial^2\phi}{\partial x^2} - 2\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^2\phi}{\partial y^2} = 0 \checkmark \textbf{ Hence proved}$$

Ex. 5.5.26 : If $w = \phi + i\Psi$ represents the complex potential for an electric field and $\phi = -2xy + \frac{y}{x^2+y^2}$, determine the function Ψ.

Soln. :

Step I : Given; $\phi = -2xy + \frac{y}{x^2+y^2}$

Differentiate ϕ partially w.r.t. x, keeping y as constant

➔ **Using standard result :** $\ldots\left[\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}\frac{dx}{dx}\right]$

$$\frac{\partial\phi}{\partial x} = -2y + y\cdot\frac{\partial}{\partial x}\left(\frac{1}{(x^2+y^2)}\right)$$

$$= -2y + y\left(-\frac{1}{(x^2+y^2)^2}\cdot 2x\right)$$

$$\therefore \quad \frac{\partial \phi}{\partial x} = -2y - \frac{2xy}{(x^2+y^2)^2}$$

Differentiate ϕ partially w.r.t. y, keeping x as constant

➔ **Using standard result:** $\ldots\left[d\left(\frac{u}{v}\right) = \left[\frac{udv - udv}{v^2}\right]\right]$

$$\frac{\partial \phi}{\partial y} = -2x + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

$$= -2x + \left[\frac{(x^2+y^2)\,1 - y\cdot 2y}{(x^2+y^2)^2}\right] = -2x + \frac{x^2-y^2}{(x^2+y^2)^2}$$

Step II: Since, $f(z) = w = \phi + i\psi$

Differentiate w.r.t. x

$$f'(z) = \frac{\partial \phi}{\partial x} + i\underbrace{\frac{\partial \psi}{\partial x}}_{\left(\frac{-\partial \phi}{\partial y}\right)}$$

$$f'(z) = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} \qquad \text{(By C - R equations.)}$$

$$f'(z) = \left[-2y - \frac{2xy}{(x^2+y^2)^2}\right] - i\left[-2x + \frac{x^2-y^2}{(x^2+y^2)^2}\right]$$

Step III: By Milne-Thomson method, put $x = z$, $y = 0$

$$f'(z) = 0 - i\left[-2z + \frac{z^2}{(z^2)^2}\right] = i\left[2z - \frac{1}{z^2}\right]$$

Integrating $f(z) = i\int\left(2z - \frac{1}{z^2}\right)dz + C = i\left(\frac{2z^2}{2} + \frac{1}{z}\right) + C$

$$f(z) = i\left[z^2 + \frac{1}{z}\right] + C$$

Step IV: For complex conjugate, put $z = x + iy$

$$f(z) = i\left[(x+iy)^2 + \frac{1}{(x+iy)}\right]$$

$$= i\left[x^2 + i\,2xy + i^2y^2 + \frac{(x-iy)}{(x+iy)(x-iy)}\right]$$

$$= i\left[x^2 - y^2 + i\,2xy + \frac{x-iy}{(x^2+y^2)}\right]$$

$(\because (a+b)(a-b) = a^2 - b^2$ and $i^2 = -1)$

$$= \left[i\,x^2 - iy^2 - 2xy + \frac{(ix+y)}{(x^2+y^2)}\right]$$

Collect the real and imaginary parts

$$= \left[-2xy + \frac{y}{x^2+y^2}\right] + i\left[x^2 - y^2 + \frac{x}{x^2+y^2}\right] + C$$

Given, $\phi = -2xy + \frac{y}{x^2+y^2}$

$\therefore \Psi = (x^2 - y^2) + \frac{x}{(x^2+y^2)} + C$ is the required function ψ ✓

Hence proved

Ex. 5.5.27: Find the conditions under which $u = ax^3 + bx^2y + cxy^2 + dy^3$ is harmonic function.

Soln.:

Step I: We know, u is harmonic function, i.e. u satisfies Laplace equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Given, $u = ax^3 + bx^2y + cxy^2 + dy^3$

Step II: Differentiating u w.r.t. x and y, two times,

$$\frac{\partial u}{\partial x} = 3ax^2 + 2bxy + cy^2;$$

$$\frac{\partial^2 u}{\partial x^2} = 6ax + 2by \qquad \ldots(1)$$

and

$$\frac{\partial u}{\partial y} = bx^2 + 2cxy + 3dy^2;$$

$$\frac{\partial^2 u}{\partial y^2} = 2cx + 6dy \qquad \ldots(2)$$

Step III: Since, u is harmonic function.

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Step IV: $\Rightarrow (6ax + 2by) + 2cx + 6dy = 0$

$$2x(3a + c) + 2y(b + 3d) = 0$$

$$\Rightarrow 3a + c = 0 \text{ and } b + 3d = 0$$

$$\Rightarrow c = -3a \text{ and } b = -3d \checkmark \qquad \ldots\textbf{Ans.}$$

Ex. 5.5.28: If $v = \frac{-y}{x^2+y^2}$; find u such that $f(z) = u + iv$ is analytic function, also find f(z) in terms of z.

Soln.:

Step I: Given, $v = \frac{-y}{(x^2+y^2)}$

Differentiate v partially w.r.t. x, keeping y as constant

➔ **Using standard result:** $\ldots\left[\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}\frac{dx}{dx}\right]$

$$\frac{\partial v}{\partial x} = -y\cdot\frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2}\right)$$

$$= -y\left(\frac{-1}{(x^2+y^2)^2}\cdot 2x\right)$$

$$\frac{\partial v}{\partial x} = \frac{y\,2x}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

Differentiate v partially w.r.t. y, keeping x as constant

➔ **Using standard result:** $\ldots\left[d\left(\frac{u}{v}\right) = \left[\frac{vdv - udv}{v^2}\right]\right]$

$$\frac{\partial v}{\partial y} = -\left[\frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2}\right] = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Step II: Since $f(z) = u + iv$

Differentiate f(z) w.r.t. x

$$\therefore f'(z) = \underbrace{\frac{\partial u}{\partial x}}_{\frac{\partial v}{\partial y}} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \qquad \text{(by C – R equations)}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2}$$

By Milne-Thomson method put $x = z$, $y = 0$

$$f'(z) = \frac{0 - z^2}{(z^2+0)^2} + i\left(\frac{0}{(z^2+0)^2}\right)$$

$$f'(z) = \frac{-z^2}{(z^2)^2} + 0 = \frac{-1}{z^2}$$

Step IV: Integrating, w.r.t. z, we get

$$f'(z) = \int\frac{-1}{z^2}dz + C$$

$f(z) = \frac{1}{z} + C$; which is required analytic function. ✓ **...Ans.**

Ex. 5.5.29: If $u = \frac{\sin 2x}{\cosh 2y + \cos 2x}$ then find the analytic function f(z), in terms of z.

Soln.:

Step I: Given, $u = \frac{\sin 2x}{\cosh 2y + \cos 2x}$

Differentiate u partially w.r.t. x, keeping y as constant

➔ **Using standard result:** $\ldots\left[d\left(\frac{u}{v}\right) = \left[\frac{vdv - udv}{v^2}\right]\right]$

$$\frac{\partial u}{\partial x} = \frac{(\cosh 2y + \cos 2x)\frac{\partial}{\partial x}(\sin 2x) - \sin 2x\cdot\frac{\partial}{\partial x}(\cosh 2y + \cos 2x)}{(\cosh 2y + \cos 2x)^2}$$

$$\frac{\partial u}{\partial x} = \frac{[\cosh 2y + \cos 2x]\cdot 2\cos 2x - \sin 2x\cdot(-2\sin 2x)}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2\cdot\cosh 2y\cdot\cos 2x + 2(\cos^2 2x + \sin^2 2x)}{(\cosh 2y + \cos 2x)^2}$$

$$\frac{\partial u}{\partial x} = \frac{2\cosh 2y\cdot\cos 2x + 2}{(\cosh 2y + \cos 2x)^2}$$

Differentiate u partially w.r.t. y, keeping x as constant.

➔ **Using standard result:** $\ldots\left[\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}\frac{dx}{dx}\right]$

$$\frac{\partial u}{\partial y} = \sin 2x\cdot\frac{\partial}{\partial y}\left(\frac{1}{\cosh 2y + \cos 2x}\right)$$

$$= \sin 2x\left(-\frac{1}{(\cosh 2y + \cos 2x)^2}\cdot\frac{\partial}{\partial y}(\cosh 2y + \cos 2x)\right)$$

$$= \sin 2x\left(-\frac{1}{(\cosh 2y + \cos 2x)^2}\cdot(2\cdot\sinh 2y + 0)\right)$$

$$\frac{\partial u}{\partial y} = \frac{-2\sin 2x\cdot\sinh 2y}{(\cosh 2y + \cos 2x)^2}$$

Step II: Since, $f(z) = u + i\,v$

$$\because f'(z) = \frac{\partial u}{\partial x} + i\underbrace{\frac{\partial v}{\partial x}}_{\left(\frac{-\partial u}{\partial y}\right)} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{(by C-R equations)}$$

$$= \frac{2\cosh 2y\cos 2x + 2}{(\cosh 2y + \cos 2x)^2} + i\frac{2\sin 2x\sinh 2y}{(\cosh 2y + \cos 2x)^2}$$

Step III: By Milne Thomson method, put $x = z$, $y = 0$

$$f'(z) = \frac{2\cos 2z + 2}{(1 + \cos 2z)^2} + 0$$

$(\because \sin 0 = 0$ and $\cosh 0 = 1$; $\sinh 0 = 0)$

$$= \frac{2}{(1 + \cos 2z)} = \frac{2}{2\cos^2 z} = \sec^2 z$$

Step IV: Integrating, we get,

$$f(z) = \int\sec^2 z\,dz + C$$

$f(z) = \tan z + C$ is the required analytic function. ✓ **...Ans.**

Ex. 5.5.30: Find the analytic function $f(z) = u + iv$ if $v = \left(r - \frac{1}{r}\right)\sin\theta$; $r \neq 0$.

Soln.:

Step I:

Given, $v = \left(r - \frac{1}{r}\right)\sin\theta$

Step II: **Differentiate v partially w.r.t. r and θ**

$$\frac{\partial v}{\partial r} = \sin\theta\cdot\frac{\partial}{\partial r}\left(r - \frac{1}{r}\right) \text{ and } \frac{\partial v}{\partial \theta} = \left(r - \frac{1}{2}\right)\cdot\underbrace{\frac{\partial}{\partial \theta}\sin\theta}_{\cos\theta}$$

$$\frac{\partial v}{\partial r} = \left(1 + \frac{1}{r^2}\right)\sin\theta;\ \frac{\partial v}{\partial \theta} = \cos\theta\left(r - \frac{1}{r}\right) \qquad \ldots(1)$$

$$\left(\because \frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\right)$$

Step III:

Since, $x + i\,y = r\cos\theta + i\,r\sin\theta = r[\cos\theta + i\sin\theta]$

$$z = re^{i\theta} \qquad \ldots\text{(Exponential form)}$$

$$\because f(z) = u + iv$$

Differentiate w.r.t. r

$$f'(z) = \underbrace{\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)}_{\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right)}e^{-i\theta} \qquad \ldots\text{(by 5.4.5)}$$

$$= \left[\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right) + i\frac{\partial v}{\partial r}\right]e^{-i\theta}$$

$$\ldots\text{(by C.R. equation)}$$

$$f'(z) = \left[\frac{1}{r}\cdot\left(r-\frac{1}{r}\right)\cdot\cos\theta + i\left(1+\frac{1}{r}\right)\sin\theta\right]e^{-i\theta}$$

...From Equation (1)

Step V : By Milne Thomson's method, Put, $r = z$, $\theta = 0$

$$= \left[\frac{1}{z}\left(z-\frac{1}{z}\right)\cdot\cos 0 + i\left(1+\frac{1}{z^2}\right)\sin 0\right]e^{-i0}$$

$$\therefore\ f'(z) = \frac{1}{z}\left(z-\frac{1}{z}\right)+0 = \left(1-\frac{1}{z^2}\right)$$

$(\because \cos 0 = 1\ ;\ \sin 0 = 0$ and $e^0 = 1)$

Step VI : Integrating w.r.t. z

$$f(z) = \int\left(1-\frac{1}{z^2}\right)dz + C$$

$$f(z) = \left(z+\frac{1}{z}\right)+C = \frac{z^2+1}{z}+C$$

be the required analytic function f(z). ✓ **...Ans.**

Ex. 5.5.31 : Determine the analytic function f(z) whose real part is $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

Soln. :

Step I : Given, $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

Differentiate u partially w.r.t. x and y

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[x^3 - 3xy^2 + 3x^2 - 3y^2 + 1]\ ;$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[x^3 - 3xy^2 + 3x^2 - 3y^2 + 1]$$

$$\Rightarrow\ \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x\ ;\ \frac{\partial u}{\partial y} = -6xy - 6y$$

Step II : Since, $f(z) = u + iv$

Differentiate f(z) w.r.t. x, we get

$$\therefore\ f'(z) = \frac{\partial u}{\partial x} + \underbrace{i\frac{\partial v}{\partial x}}_{\left(-\frac{\partial u}{\partial y}\right)} = \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right)\ \text{(By C-R Equations)}$$

$$f'(z) = (3x^2 - 3y^2 + 6x) + i(6xy + 6y)$$

Step III : By Milne Thomson's method, Put, $x = z, y = 0$;

$$f'(z) = (3z^2 - 0 + 6z) + i(0)$$

$$f'(z) = (3z^2 + 6z) + i(0)$$

Step IV : Integrating we get,

$$f(z) = \int(3z^2 + 6z)\,dz + C$$

$$= 3\frac{z^3}{3} + 6\frac{z^2}{3} + C$$

$$f(z) = z^3 + 3z^2 + C\ ✓$$

...Ans.

Ex. 5.5.32 : Find the harmonic conjugate of :

$$u = r^3\cos 3\theta + r\sin\theta.$$

Soln. :

Step I : Given, $u = r^3\cos 3\theta + r\sin\theta$ ...(1)

Differentiate u partially w.r.t r and θ

$$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}(r^3\cos 3\theta + r\sin\theta)\ ;$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial}{\partial \theta}[r^3\cos 3\theta + r\sin\theta]$$

$$\Rightarrow\ \frac{\partial u}{\partial r} = 3r^2\cos 3\theta + \sin\theta\ ;$$

$$\text{and}\ \frac{\partial u}{\partial \theta} = -3r^3\sin 3\theta + r\cos\theta \qquad ...(2)$$

$(\because d(r^3) = 3r^2\ ;\ d(\cos 3\theta) = -3\sin 3\theta;\ d(\sin\theta) = \cos\theta)$

Step II : By C-R equation, in polar form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}\ \text{and}\ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Step III : To find complex conjugate v

$$\because\ \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r} = r[3r^2\cos 3\theta + \sin\theta]$$

$$= 3r^2\cos 3\theta + r\sin\theta$$

Step IV : Integrating w.r.t θ, keeping r as constant

$$v = \int(3r^2\cos 3\theta + r\sin\theta)\,d\theta + f(r)$$

$$\therefore\ v = \frac{3r^3\sin 3\theta}{3} - r\cos\theta + f(r) \qquad ...(3)$$

$$\left(\because \int\cos 3\theta\,d\theta = \frac{\sin 3\theta}{3}\right)\ \text{and}\ \int\sin\theta\,d\theta = -\cos\theta$$

Step V : Differentiate Equation (3) w.r.t. r keeping θ as constant

$$\frac{\partial v}{\partial r} = \sin 3\theta\cdot\underbrace{\frac{\partial}{\partial r}(r^3)}_{(3r^2)} - \cos\theta\underbrace{\frac{\partial}{\partial r}(r)}_{(1)} + f'(r)$$

$$\frac{\partial v}{\partial r} = 3r^2\sin 3\theta - \cos\theta + f'(r) \qquad ...(4)$$

Step VI : But by C-R equations, $-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r}$

From Equations (2) and (4)

$$\Rightarrow -\frac{1}{r}[-3r^3\sin 3\theta + r\cos\theta] = 3r^2\sin 3\theta - \cos\theta + f'(r)$$

$$\Rightarrow\ 3r^2\sin 3\theta - \cos\theta = 3r^2\sin 3\theta - \cos\theta + f'(r)$$

$$\Rightarrow\ f'(r) = 0$$

$$\Rightarrow\ f(r) = C\ \text{Put in Equation (3)}$$

$\Rightarrow v = r^3\sin 3\theta - r\cos\theta + C$, is a harmonic conjugate of u. ✓

...Ans.

Type 3

Ex. 5.5.33 : If $f(z) = (x^2 + axy + by^2) + i(px^2 + qxy + y^2)$ is analytic, then find a, b, p and q.

Soln. :

Step I : Given, $f(z) = (x^2 + axy + by^2) + i(px^2 + qxy + y^2)$ is analytic function

Since, $f(z) = u + iv$

$$\Rightarrow\ u = x^2 + axy + by^2;\quad v = px^2 + qxy + y^2$$

Step II : Differentiate u and v w.r.t x, keeping y as constant

$$\frac{\partial u}{\partial x} = 2x + ay\ ;\ \frac{\partial v}{\partial x} = 2px + qy$$

Differentiate u and v w.r.t y, keeping x as constant

$$\frac{\partial u}{\partial y} = ax + 2by\ \text{and}\ \frac{\partial v}{\partial y} = qx + 2y$$

Step III : Since, f (z) is analytic function.

∴ By C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ \text{and}\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\therefore\ 2x + ay = qx + 2y$$

$$\text{and}\quad ax + 2by = -(2px + qy)$$

Step IV : Comparing the coefficients of x and y of above two equations separately, we get

$$a = 2,\quad q = 2,\quad a = -2p\ \text{and}\ 2b = -q$$

$$\therefore\ a = 2,\quad b = -1;\quad p = -1\ \text{and}\ q = 2\ ✓$$ **...Ans.**

Ex. 5.5.34 : If f (z) and $f\left(\bar{z}\right)$ are analytic functions, then prove that f (z) is constant.

Soln. :

Step I : Let, $f(z) = u + iv$

Since, f (z) is analytic function.

∴ By C-R equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad ...(1)$$

$$\text{and}\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad ...(2)$$

Step II : Also, $f\left(\bar{z}\right) = u - iv$

∴ By C-R equations,

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \qquad ...(3)$$

$$\text{and}\ \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \qquad ...(4)$$

Step III : By adding Equations (1) and (3), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}$$

$$2\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial x} = 0 \qquad ...(5)$$

Step IV : By subtracting Equation (2) from Equation (4), we get

$$\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}$$

$$0 = 2\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial x} = 0 \qquad ...(6)$$

Step V : We have, $f(z) = u + iv$

Differentiate w.r.t. x

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \text{...From (5) and (6)}$$

$$f'(z) = 0 \Rightarrow f(z) = \text{Constant}$$

Hence, f (z) is a constant function. ✓

...Hence proved

Ex. 5.5.35 : Prove that, analytic function with constant modulus is constant.

Soln. :

Step I : Given, $f(z) = u + iv$; is an analytic function.

By C-R equation,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ \text{and}\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad ...(1)$$

Step II : Also given that f (z) have constant modulus,

$(\because$ modulus of $f(z) = |f(z)| = \sqrt{u^2 + v^2})$

i.e. $|f(z)| = \text{Constant}$

$$\therefore\ |u + iv| = K\ (\text{say})$$

$$\Rightarrow\ \sqrt{u^2 + v^2} = K$$

Squaring both sides,

$$\Rightarrow\ u^2 + v^2 = K^2 \qquad ...(2)$$

Step III : Differentiating Equation (2) w.r.t. x, keeping y as constant

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$$\Rightarrow\ u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \qquad ...(3)$$

Step IV : Also, differentiating Equation (2) w.r.t. y, we get

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

$$\Rightarrow\ u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

By C-R Equation (1), we get

$$-u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x} = 0$$

$$\ldots\left(\because \text{C-R equation } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x};\ \frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}\right)$$

$$\Rightarrow \quad v\frac{\partial u}{\partial x}-u\frac{\partial v}{\partial x} = 0 \qquad \ldots(4)$$

Step VI : Multiplying Equation (3) by u and Equation (4) by v and add them,

$$\left[u^2\frac{\partial u}{\partial x}+uv\frac{\partial v}{\partial x}\right]+\left[v^2\frac{\partial u}{\partial x}-uv\frac{\partial v}{\partial x}\right] = 0$$

$$\therefore\ (u^2+v^2)\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow\ K^2\frac{\partial u}{\partial x} = 0$$

$$\text{But } K^2 = (u^2+v^2)\neq 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = 0$$

Step VII : Similarly, multiplying Equation (3) by v and Equation (4) by u, and subtract them

$$\left[uv\frac{\partial u}{\partial x}+v^2\frac{\partial v}{\partial x}\right]-\left[uv\frac{\partial u}{\partial x}-u^2\frac{\partial v}{\partial x}\right] = 0$$

$$\Rightarrow \quad \therefore\ (u^2+v^2)\frac{\partial v}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial v}{\partial x} = 0 \quad (\because u^2+v^2\neq 0)$$

Step VIII : Since, $f(z) = u + iv$

Differentiate w.r.t. x

$$f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$f'(z) = 0+i0$$

$$\therefore\ f'(z) = 0$$

Integrating, $f(z) = c$

$\Rightarrow$ **f (z) is a constant function. ✓** **...Hence proved**

Ex. 5.5.36 : Show that analytic function with constant amplitude is constant.

Soln. :

Step I : Given, $f(z) = u + iv$ is an analytic function.

$\therefore$ By C-R equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Step II : Given, f (z) is an analytic function with constant amplitude (argument)

$$\left(\because \text{Amp. } f(z) = \text{Arg } f(z) = \theta = \tan^{-1}\left(\frac{v}{u}\right)\right)$$

$$\therefore \quad \text{amp } f(z) = \tan^{-1}\left(\frac{v}{u}\right) = \text{Constant} \qquad \ldots(\text{Given})$$

$$\text{i.e. } \tan^{-1}\left(\frac{v}{u}\right) = K \text{ (say)}$$

$$\Rightarrow \quad \frac{v}{u} = \tan K$$

$$\Rightarrow \quad v = u\tan K \qquad \ldots(1)$$

Step III : Differentiating Equation (1) partially w.r.t. x, we get

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}\tan K \qquad (\because \tan K \text{ is a constant})$$

$$\Rightarrow \frac{\partial u}{\partial x}\tan K-\frac{\partial v}{\partial x} = 0 \qquad \ldots(2)$$

Step IV : Again, differentiating Equation (1) w.r.t. y, we get

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial y}\tan K$$

$$\Rightarrow \underbrace{\frac{\partial u}{\partial y}}_{-\frac{\partial v}{\partial x}}\tan K-\underbrace{\frac{\partial v}{\partial y}}_{\frac{\partial u}{\partial x}} = 0 \qquad \ldots(\text{by C - R equations})$$

$$\Rightarrow -\frac{\partial v}{\partial x}\tan K-\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}\tan K = 0 \qquad \ldots(3)$$

Step V : Multiplying Equation (3) by tan K, and subtract it from Equation (2)

$$\frac{\partial u}{\partial x}\tan K+\frac{\partial v}{\partial x}\tan^2 K = 0$$

$$\frac{\partial u}{\partial x}\tan K-\frac{\partial v}{\partial x} = 0$$

$$- \qquad +$$

$$0+(1+\tan^2 K)\frac{\partial v}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial v}{\partial x} = 0 \ (\because 1+\tan^2 K\neq 0)$$

Step VI : Similarly, we get $\frac{\partial u}{\partial x} = 0$

Step VII : Since, $f(z) = u + iv$

Differentiate f(z) w.r.t x

$$f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} = 0+i0 = 0$$

$$\therefore f'(z) = 0$$

$$\Rightarrow \quad f(z) = c$$

$\therefore$ f (z) is a constant function. ✓ **Hence proved**

Ex. 5.5.37 : If f(z) is an analytic function and $v^2 = u$ then show that f(z) is constant function.

Soln. :

Step I : Given $f(z) = u + iv$ be an analytic function.

$\therefore$ By C – R equations, $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ...(1)

Step II : Since, $v^2 = u$

i.e. $v^2-u = 0$ ...(2)

Differentiating Equation (2) w.r.t. x and y, we get

$$2v\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x} = 0 \qquad \ldots(3)$$

$$\text{and} \quad 2v\underbrace{\frac{\partial v}{\partial y}}_{\left(\frac{\partial u}{\partial x}\right)}-\underbrace{\frac{\partial u}{\partial y}}_{\left(\frac{-\partial v}{\partial x}\right)} = 0$$

$$\text{We get, } 2v\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial v}{\partial x}+2v\frac{\partial u}{\partial x} = 0 \qquad \ldots(4)$$

Step III : Multiplying Equation (4) by 2v and subtracting with Equation (3)

$$\therefore \left(\cancel{2v\frac{\partial v}{\partial x}}+4v^2\frac{\partial u}{\partial x}\right)-\left(\cancel{2v\frac{\partial v}{\partial x}}-\frac{\partial u}{\partial x}\right) = 0$$

$$\Rightarrow \quad (4v^2+1)\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = 0$$

$$\left(\because 4v^2+1\neq 0 \text{ i.e. } v^2\neq -\frac{1}{4}\right)$$

$$\text{Similarly, we get, } \frac{\partial v}{\partial x} = 0$$

Step IV : Since, $f(z) = u + iv$

Differentiate f(z) w.r.t. x

$$f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$= 0+i0$$

$$f'(z) = 0$$

$$\therefore f(z) = C$$

Hence, f (z) is a constant function. ✓ **Hence proved**

Ex. 5.5.38 : Show that real and imaginary parts of an analytic function are harmonic.

Soln. :

Step I : Given, $f(z) = u + iv$ be an analytic function.

$\therefore$ By C – R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ...(1)

and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ...(2)

Step II : Differentiating Equation (1) partially w.r.t. x and Equation (2) w.r.t. y,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\partial y} \text{ and } \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y\partial x}$$

$$\text{Adding, we get } \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 v}{\partial y\partial x} = 0 \qquad \ldots(3)$$

Step III : Now, differentiating Equation (1) w.r.t. y and Equation (2) w.r.t. x

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 v}{\partial y^2} \text{ and } \frac{\partial^2 u}{\partial x\partial y} = -\frac{\partial^2 v}{\partial x^2}$$

$$\text{Subtracting, we get } \frac{\partial^2 y}{\partial x\partial y}-\frac{\partial^2 y}{\partial x\partial y} = \frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} = 0 \qquad \ldots(4)$$

Step IV : Hence, Equations (3) and (4) shows that the real and imaginary parts of analytic function are harmonic. (i.e. satisfies Laplace equation).

✓ **Hence proved**

Ex. 5.5.39 : Prove that every analytic function f (z) defines two families of curves which forms an orthogonal system.

Soln. :

Step I : Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function.

Step II :

Let us consider, two families of curves ;

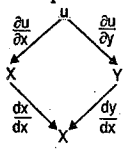
$$u(x, y) = C_1 \qquad \ldots(1)$$

$$v(x, y) = C_2 \qquad \ldots(2)$$

Step III : Differentiating Equation (1) w.r.t. x, partially,

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = 0 \qquad (\because \text{u is implicit function})$$

$$\frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = -\frac{\partial u}{\partial y}$$



$$\therefore \frac{dy}{dx} = -\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} = m_1 \text{ (Say)} \qquad \ldots(3)$$

Step IV : Similarly, from Equation (2), we get

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)} = m_2 \text{ (Say)} \qquad \ldots(4)$$

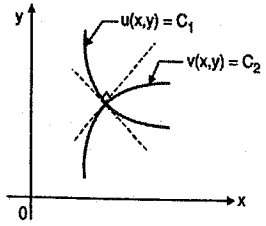


Fig. P. 5.5.39

Step V : From Equations (3) and (4)

$$\therefore m_1 \cdot m_2 = \left(-\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}\right)\cdot\left(-\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}\right) = \left(\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}\right)\left(\frac{-\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}}\right)$$

By C – R equations

$$= -1$$

Step VI : This shows that these two curves intersects at right angles to each other.

Hence, they form an orthogonal system. ✓ **Hence proved**

Type 4

To find the analytic function f(z) = u + iv ; when u + v or u – v given in Cartesian form.

Procedure :

Method 1 :

Step I : Let $f(z) = u + iv$ ...(1)

$i \cdot f(z) = iu - v$ ....($i^2 = -1$) ...(2)

Step II : Adding Equations (1) and (2) ;

$(1+i)\,f(z) = (u - v) + i\,(u + v)$ ...(3)

Step III : Differentiating Equation (3) w.r.t. x

Step IV : $(1+i)\,f'(z) = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right)$ ...(4)

Step V : If u + v given then we get

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}$ ...(5)

and $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$ ...(6)

Step VI : By C-R equation, Equation (6) becomes

$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}$ ...(7)

Step VII : By using Equations (5) and (7), Equation (4) have some value in x and y. Then use Milne-Thomson Method, ; Put x = z, y = 0, we get f′(z) in terms of z.

Step VIII : Integrating, we get ; f(z)

Method 2 :

Step I : To find analytic function f(z) if u + v given. ...(1)

Step II : Differentiate u + v partially w.r.t. x, we get one equation. ...(2)

Step III : Differentiate u + v partially w.r.t. y and use C-R equations, we get another equation. ...(3)

Step IV : Solving Equation (2) and (3), we get the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ or $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$

Step V :

Use $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ or $= \frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y}$

Step VI : Use Milne-Thomson method ; we get f′(z).

Step VII : Then integrating, we get the required analytic function in terms of z.

Ex. 5.5.40 : If $f(z) = u + iv$ is an analytic function, find f(z) if $u - v = x^3 + 3x^2 y - 3xy^2 - y^3$.

Soln. :

Step I : Given, : $u - v = x^3 + 3x^2 y - 3xy^2 - y^3$ ...(1)

Method 1 :

Step II : Differentiating Equation (1) partially w.r.t. x, keeping y as constant

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \underbrace{\frac{\partial}{\partial x}(x^3)}_{(3x^2)} + y\cdot\underbrace{\frac{\partial}{\partial x}(3x^2)}_{(6x)} - y^2\underbrace{\frac{\partial}{\partial x}(3x)}_{(3)} - \underbrace{\frac{\partial}{\partial x}(y^3)}_{(0)}$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = 3x^2 + 6xy - 3y^2 \quad ...(2)$$

Differentiating Equation (1) partially w.r.t y, keeping x as consultant

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = \underbrace{\frac{\partial}{\partial y}(x^3)}_{(0)} + 3x^2\cdot\underbrace{\frac{\partial}{\partial y}(y)}_{(1)} - 3x\cdot\underbrace{\frac{\partial}{\partial y}(y^2)}_{(2y)} - \underbrace{\frac{\partial}{\partial y}(y^3)}_{(3y^2)}$$

$$\Rightarrow \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = 3x^2 - 6xy - 3y^2$$

By C - R equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$,

$$\therefore -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = 3x^2 - 6xy - 3y^2$$

i.e. $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} - 3x^2 + 6xy + 3y^2$ ...(3)

Step III : Since, $f(z) = u + iv$ and $i \cdot f(z) = iu - v$

Adding, $(1+i)\,f(z) = (u - v) + i(u + v)$ ...(4)

Differentiating Equation (4) w.r.t. x

$(1+i)\,f'(z) = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right)$ ...(5)

By using Equations (2) and (3), Equation (5) becomes,

$(1+i)\,f'(z) = (3x^2 + 6xy - 3y^2) + i\,(-3x^2 + 6x + 3y^2)$

Step IV :

By Milne-Thompson method, put x = z, y = 0

$(1+i)\,f'(z) = (3z^2 + 0 - 0) + i\,(-3z^2 + 0 + 0)$

$\therefore (1+i)\,f'(z) = 3z^2 + i\,(-3z^2) = 3z^2(1 - i)$

$$f'(z) = 3z^2\frac{(1-i)}{(1+i)} = \frac{3z^2(1-i)(1-i)}{(1+i)(1-i)} = \frac{3z^2(1-2i-1)}{2}$$

... ($\because (1+i)(1-i) = 1 - i^2 = 1 + 1 = 2$)

$f'(z) = -i\,3z^2$

Step V : Integrating, we get,

$f(z) = -i\,3\int z^2 dz + C$

$= -i\,3\left(\frac{z^3}{3}\right) + C$

$f(z) = -i\,z^3 + C$ is an analytic function.

Method 2 :

Step I : Given, $u - v = x^3 + 3x^2 y - 3xy^2 - y^3$ ...(1)

Step II : Differentiating Equation (1) partially w.r.t. x and y.

$\therefore \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = 3x^2 + 6xy - 3y^2$ ...(2)

and $\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = 3x^2 - 6xy - 3y^2$

Step III : By using C – R equation,

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Above equation becomes

$-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = 3x^2 - 6xy - 3y^2$

i.e. $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) = -3x^2 + 6xy + 3y^2$ ...(3)

Step IV : Adding Equations (2) and (3), we get

$$\left(\frac{\partial u}{\partial x} - \cancel{\frac{\partial v}{\partial x}}\right) + \left(\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial x}}\right) = (3x^2 + 6xy - 3y^2) + (-3x^2 + 6xy + 3y^2)$$

$$\frac{2\partial u}{\partial x} = 12\,xy \qquad \Rightarrow \frac{\partial u}{\partial x} = 6\,xy$$

and subtracting Equations (2) from (3), we get

$$\left(\cancel{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial x}\right) - \left(\cancel{\frac{\partial u}{\partial x}} - \frac{\partial v}{\partial x}\right) = (-3x^2 + \cancel{6xy} + 3y^2) - (3x^2 + \cancel{6xy} - 3y^2)$$

$$\frac{2\partial v}{\partial x} = -6x^2 + 6y^2$$

$$\frac{\partial v}{\partial x} = -3(x^2 - y^2)$$

Step VI : Since, $f(z) = u + iv$

Differentiate f(z) w.r.t. x

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 6xy - i3(x^2 - y^2)$

By Milne-Thomson method, put x = z, y = 0

$f'(z) = 0 - i3\,(z^2 - 0)$

$f'(z) = -i\,3\,z^2$

Step V : Integrating, we get,

$f(z) = -i3\int z^2 dz + C$

$= -i\cancel{3}\left(\frac{z^3}{\cancel{3}}\right) + C$

$f(z) = -i\,z^3 + C$ is an analytic function.

✓...Ans.

Ex. 5.5.41 : If $f(z) = u + iv$ is an analytic function then find f(z) if $u + v = \frac{x}{x^2 + y^2}$

Soln. :

Step I : Given, $f(z) = u + iv$ is an analytic function

$\therefore$ By C-R equation, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ...(1)

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ...(2)

Given, $u + v = \frac{x}{x^2 + y^2}$ ...(3)

Step II : Differentiating Equation (3) w.r.t x and y

➔ **Use standard result :** ...$\left[d\left(\frac{u}{v}\right) = \left[\frac{vdu - udv}{v^2}\right]\right]$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = \frac{(x^2 + y^2)\cdot\frac{\partial}{\partial x}(x) - x\cdot\frac{\partial}{\partial x}(x^2 + y^2)}{(x^2 + y^2)^2}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = \frac{(x^2 + y^2)\,1 - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \quad ...(4)$$

➔ **Using standard result :** ...$\left[\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}\frac{dx}{dx}\right]$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = x\cdot\frac{\partial}{\partial y}\left(\frac{1}{(x^2 + y^2)}\right) = x\left(-\frac{1}{(x^2 + y^2)^2}\cdot 2y\right)$$

$$\Rightarrow \underbrace{\frac{\partial u}{\partial y}}_{\left(\frac{-\partial v}{\partial x}\right)} + \underbrace{\frac{\partial v}{\partial y}}_{\left(\frac{\partial u}{\partial x}\right)} = \frac{-2xy}{(x^2 + y^2)^2}$$

Step III : By C – R Equations (1) and (2)

$-\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} = \frac{-2xy}{(x^2 + y^2)^2}$ ...(5)

Step IV : By adding Equations (4) and (5), we get

$$\left(\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial x}}\right) + \left(\cancel{-\frac{\partial v}{\partial x}} + \frac{\partial u}{\partial x}\right) = \left(\frac{y^2 - x^2}{(x^2 + y^2)^2}\right) + \left(\frac{-2xy}{(x^2 + y^2)^2}\right)$$

$$\Rightarrow 2\,\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{-2xy}{(x^2 + y^2)^2} = \frac{y^2 - x^2 - 2xy}{(x^2 + y^2)^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{2}\left[\frac{y^2 - x^2 - 2xy}{(x^2 + y^2)^2}\right] \quad ...(6)$$

Step V : Subtracting Equations (4) and (5), we get

$$\left(\cancel{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial x}\right) - \left(-\frac{\partial v}{\partial x} + \cancel{\frac{\partial u}{\partial x}}\right) = \left(\frac{y^2 - x^2}{(x^2 + y^2)^2}\right) - \left(-\frac{2xy}{(x^2 + y^2)^2}\right)$$

$$2\,\frac{\partial v}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{2xy}{(x^2 + y^2)^2} = \frac{y^2 - x^2 + 2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{2}\left[\frac{y^2 - x^2 + 2xy}{(x^2 + y^2)^2}\right] \quad ...(7)$$

Since, $f(z) = u + iv$

Differentiating f(z) w.r.t. x

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

From Equations (6) and (7), we get

$$f'(z) = \frac{1}{2}\left[\frac{y^2 - x^2 - 2xy}{(x^2+y^2)^2}\right] + i\frac{1}{2}\left[\frac{y^2 - x^2 + 2xy}{(x^2+y^2)^2}\right]$$

Step VI : By Milne-Theorem method, put $x = z$; $y = 0$

$$\therefore\ f'(z) = \frac{1}{2}\left[\frac{0 - z^2 - 0}{z^4}\right] + \frac{i}{2}\left[\frac{0 - z^2 + 0}{z^4}\right] = -\frac{1}{2}\frac{1}{z^2}[1 + i]$$

Step VII : Integrating both side, we get

$$f(z) = -\frac{1}{2}(1+i)\int\frac{1}{z^2}\,dz + C$$

$$f(z) = -\frac{1}{2}(1+i)\left(-\frac{1}{z}\right) + C$$

$$\mathbf{f(z) = \frac{1}{2}(1+i)\frac{1}{z} + C} \ \checkmark \qquad \text{...Ans.}$$

is a required analytic function

Ex. 5.5.42 : If $f(z) = u + iv$ be an analytic function, find $f(z)$ if $u + v = r^2(\cos 2\theta + \sin 2\theta)$

Soln. :

Step I : Since, $f(z) = u + iv$ is an analytic function then by C – R equations in polar form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \qquad \text{...(1)}$$

Given, $u + v = r^2[\cos 2\theta + \sin 2\theta]$ ...(2)

Step II : Differentiating Equation (2) w.r.t r and θ,

$$\frac{\partial u}{\partial r} + \frac{\partial v}{\partial r} = 2r[\cos 2\theta + \sin 2\theta] \qquad \text{...(3)}$$

and $$\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta} = r^2[-2\sin 2\theta + 2\cos 2\theta]$$

Step III : But by C – R equations from Equation (1)

$$-r\frac{\partial v}{\partial r} + r\frac{\partial u}{\partial r} = 2r^2[-\sin 2\theta + \cos 2\theta]$$

$$\Rightarrow -\frac{\partial v}{\partial r} + \frac{\partial u}{\partial r} = 2r[-\sin 2\theta + \cos 2\theta] \qquad \text{...(4)}$$

Step IV : By adding Equations (3) and (4)

$$\left(\frac{\partial u}{\partial r} + \cancel{\frac{\partial v}{\partial r}}\right) + \left(\cancel{-\frac{\partial v}{\partial r}} + \frac{\partial u}{\partial r}\right) = [2r(\cos 2\theta + \sin 2\theta)] + [2r(-\sin 2\theta + \cos 2\theta)]$$

$$2\frac{\partial u}{\partial r} = 4r\cos 2\theta$$

$$\frac{\partial u}{\partial r} = 2r\cos 2\theta \qquad \text{... (5)}$$

And subtracting Equations (3) and (4), we get

$$\left(\frac{\partial u}{\partial r} + \cancel{\frac{\partial v}{\partial r}}\right) - \left(\cancel{-\frac{\partial v}{\partial r}} + \frac{\partial u}{\partial r}\right) = [2r(\cos 2\theta + \sin 2\theta)] - [2r(-\sin 2\theta + \cos 2\theta)]$$

$$2\frac{\partial v}{\partial r} = 4r\sin 2\theta$$

$$\frac{\partial v}{\partial r} = 2r\sin 2\theta \qquad \text{...(6)}$$

Step VI : Since, $f(z) = u + iv$

Differentiate f(z) w.r.t. r

$$f'(z) = \left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]e^{-i\theta} \qquad \text{...polar form}$$

$$= 2[r\cos 2\theta + i\,r\sin 2\theta]\,e^{-i\theta}$$

Step VII : By Milne Thomson method,

Put $r = (z)$ and $\theta = 0$

$$f'(z) = 2[\underbrace{z\cos 0}_{(1)} + i\,\underbrace{r\sin 0}_{(0)}]\,\underbrace{e^0}_{(1)}$$

$$f'(z) = 2[z + 0]$$

$$f'(z) = 2z$$

Step VIII : Integrating both sides

$$f(z) = 2\int z\,dz + C$$

$$= \cancel{2}\frac{z^2}{\cancel{2}} + C$$

$$\mathbf{f(z) = z^2 + C}, \text{ is required analytic function } \checkmark$$

...Ans.

Ex. 5.5.43 : If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = (x - y)(x^2 + 4xy + y^2)$ then, find f (z).

Soln. :

Step I : Given, $u - v = (x - y)(x^2 + 4xy + y^2)$

$$= x^3 + 3x^2y - 3xy^2 - y^3 \qquad \text{...(1)}$$

Step II : Differentiating Equation (1) w.r.t. x, we get

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = 3x^2 + 6xy - 3y^2 \qquad \text{...(2)}$$

Step III : Differentiating Equation (1) w.r.t. y, we get

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = 3x^2 - 6xy - 3y^2 \qquad \text{...(3)}$$

Since, $f(z) = u + iv$ is an analytic function.

By C – R. equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Equation (3) becomes

$$-v_x - u_x = 3x^2 - 6xy - 3y^2$$

$$\Rightarrow\ v_x + u_x = -3x^2 + 6xy + 3y^2 \qquad \text{...(4)}$$

Step IV : Adding Equations (2) and (4), we get

$$(u_x - v_x) + (v_x + u_x) = (3x^2 + 6xy - 3y^2) + (-3x^2 + 6xy + 3y^2)$$

$$2u_x = 12xy \Rightarrow u_x = 6xy$$

Subtracting Equation (2) from Equation (4)

$$(v_x + u_x) - (u_x - v_x) = (-3x^2 + 6xy + 3y^2) - (3x^2 + 6xy - 3y^2)$$

and $$2v_x = -6x^2 + 6y^2$$

$$\Rightarrow v_x = -3x^2 + 3y^2$$

Since, $f(z) = u + iv$, Differentiating f(z) w.r.t. x

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 6xy + i3(y^2 - x^2)$$

Step V : By Milne Thomson method, put $x = z$, $y = 0$

$$f'(z) = 0 + i3(0 - z^2)$$

$$f'(z) = 0 - i3z^2$$

$\therefore$ Integrating w.r.t. z, we get,

$$f(z) = -i3\int z^2\,dz + C$$

$$= -i\cancel{3}\left(\frac{z^3}{\cancel{3}}\right) + C$$

$$\mathbf{f(z) = -iz^3 + C} \ \checkmark \qquad \text{...Ans.}$$

5.6 The Laplacian Operator

The operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called Laplacian operator.

$$\therefore\ \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \text{ is called Laplace equation}$$

Laplacian operator ∇^2 can be expressed in terms of z and $\bar{z}$ as follows.

$$4\frac{\partial^2}{\partial z\,\partial\bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \nabla^2$$

Since, $z = x + iy$; $\bar{z} = x - iy$

$$z + \bar{z} = 2x \quad\text{and}\quad z - \bar{z} = i2y$$

$$|z| = \sqrt{x^2 + y^2} \text{ and } |(\bar{z})| = \sqrt{x^2 + y^2}$$

$$z\cdot\bar{z} = x^2 + y^2 = |z|^2 \text{ or } |\bar{z}|^2$$

Let, $f(z) = u + iv$; and $f(\bar{z}) = u - iv$

$$f(z)\cdot f(\bar{z}) = (u + iv)(u - iv) = u^2 + v^2$$

$$|f(z)| = \sqrt{u^2 + v^2} = |f(\bar{z})|$$

$$\therefore\ f(z)\cdot f(\bar{z}) = |f(z)|^2 \quad\text{or}\quad |f(\bar{z})|^2$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \text{ and } f'(\bar{z}) = \frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}$$

$$f'(z)\,f'(\bar{z}) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

$$\therefore\ f'(z)\cdot f'(\bar{z}) = |f'(z)|^2 \quad\text{or}\quad |f'(\bar{z})|^2$$

Solved Example

Type 5

Ex. 5.6.1 : If f(z) is analytic function, then prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$

Soln. :

Step I : Since, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial\bar{z}}$

Step II : $\therefore$ L.H.S. $= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f'(z)|$

$$\ldots\left(\text{since, } |f'(z)| = \sqrt{f'(z)\cdot f'(\bar{z})}\right)$$

$$= 4\frac{\partial^2}{\partial z\,\partial\bar{z}}\log\sqrt{f'(z)\,f'(\bar{z})}$$

$$= 4\cdot\frac{\partial^2}{\partial z\,\partial\bar{z}}\log\left(f'(z)\cdot f'(\bar{z})^{\frac{1}{2}}\right)$$

$$= 4\cdot\frac{1}{z}\frac{\partial^2}{\partial z\,\partial\bar{z}}\cdot\log\left[f'(z)\cdot f'(\bar{z})\right]$$

$$= \frac{4}{2}\frac{\partial^2}{\partial z\,\partial\bar{z}}\left[\log f'(z) + \log f'(\bar{z})\right]$$

➔ **Using standard result :** ...[log(A.B) = logA + logB]

Step III : L.H.S. $= 2\cdot\frac{\partial}{\partial z}\cdot\frac{\partial}{\partial\bar{z}}\left[\log f'(z) + \log f'(\bar{z})\right]$

$$= 2\cdot\frac{\partial}{\partial z}\left[0 + \frac{1}{f'(\bar{z})}\cdot f''(\bar{z})\right]$$

↑

Here, $f'(z)$ is a constant function

Now derivative w. r. t. z, so function of $\bar{z}$ becomes constant.

$$= 0$$

Step IV : = R.H.S.

Hence, $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$

Ex. 5.6.2 : If f(z) is an analytic function then prove that, $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^n = n(n-1)\,u^{n-2}\,|f'(z)|^2$

Soln. :

Step I : Let, $f(z) = u + iv$ and $f(\bar{z}) = u - iv$

Adding, $f(z) + f(\bar{z}) = (u + iv) + (u - iv) = 2u$

$$\therefore\ u = \frac{1}{2}[f(z) + f(\bar{z})] \qquad \therefore\ u^n = \left(\frac{1}{2}\right)^n[f(z) + f(\bar{z})]^n \quad \text{...(1)}$$

Step II : $\therefore$ L.H.S. $= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^n = 4\frac{\partial^2}{\partial z\,\partial\bar{z}}\cdot u^n$

$$= 4\frac{\partial^2}{\partial z\,\partial\bar{z}}\left(\frac{1}{2}\right)^n[f(z) + f(\bar{z})]^n$$

Step III : Operating first $\frac{\partial}{\partial\bar{z}}$; here function of z is constant,

$$= 4\cdot\frac{\partial}{\partial z}\left[\frac{\partial}{\partial\bar{z}}\left(\frac{1}{2}\right)^n\left[f(z) + f(\bar{z})\right]^n\right]$$

$$= 4\frac{\partial}{\partial z}\left(\frac{1}{2}\right)^n \cdot n\,[f(z)+f(\bar{z})]^{n-1}\cdot f'(\bar{z})$$

$$\left(\because \frac{\partial}{\partial \bar{z}} f(z) = 0\right)$$

$$= 4n(n-1)\left(\frac{1}{2}\right)^n [f(z)+f(\bar{z})]^{n-2}\, f'(z)\, f'(\bar{z})$$

$$\ldots\left(\because \frac{\partial}{\partial z} f(\bar{z}) = 0\right)$$

Step IV : $= 4n(n-1)\left(\frac{1}{2}\right)^n (2u)^{n-2}\cdot |f'(z)|^2$

$$\left(\because \text{ since } |f'(z)|^2 = f'(z)\,f'(\bar{z}) \text{ and } u = \frac{1}{2}\left(f(z)+f(\bar{z})\right)\right)$$

$$= 4n(n-1)\frac{1}{2^n}\cdot 2^{n-2}\, u^{n-2}\cdot |f'(z)|^2$$

$$= \cancel{4}n\cdot(n-1)\frac{1}{\cancel{2^n}}\cdot \cancel{2^2}\cdot\frac{1}{\cancel{2^2}}\cdot u^{n-2}\cdot |f'(z)|^2$$

Step V : $\therefore \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u^n = n(n-1)\,u^{n-2}\cdot |f'(z)|^2$

✓ **Hence proved**

Ex. 5.6.3 : If (z) is analytic function, then prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2$$

Soln. :

Step I : Since, $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}$

Step II : $\therefore$ L.H.S. $= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2$

$$= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}|f(z)|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}} f(z)\,f(\bar{z})$$

$$\left(\because |f(z)|^2 = f(z)\cdot f(\bar{z})\right)$$

$$= 4\cdot\frac{\partial}{\partial z}\cdot\frac{\partial}{\partial \bar{z}}\Big[\underbrace{f(z)\,f(\bar{z})}_{\textit{constant}}\Big]$$

Step III : Differentiate first w.r.t. $\bar{z}$ keeping z as constant and then differentiate w.r.t. z

$$\therefore \text{ L.H.S.} = 4\frac{\partial}{\partial z}\underbrace{f(z)\cdot f'(\bar{z})}_{\textit{constant}} = 4\,f'(z)\,f'(\bar{z})$$

Since, $|f'(z)|^2 = f'(z)\cdot f'(\bar{z})$

$\therefore \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2$ ✓ **Hence proved**

Ex. 5.6.4 : If f (z) = u + iv is an analytic function then prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{Re } f(z)|^2 = 2\,|f'(z)|^2$$

Soln. :

Step I : Since, we know $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}$;

$f(z) = u + iv$ and $\text{Re } f(z) = u$

Step II : Differentiate f(z) w.r.t. x

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$|f'(z)| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2}$$

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2 \qquad \ldots(1)$$

Step III : $\therefore$ L.H.S. $= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\cdot|\text{Re } f(z)|^2$

$$= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\cdot u^2$$

Step IV : $= \frac{\partial^2}{\partial x^2}(u^2)+\frac{\partial^2}{\partial y^2}(u^2)$

$$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(u)^2\right]+\frac{\partial}{\partial y}\cdot\left[\frac{\partial}{\partial y}\cdot(u^2)\right]$$

$$= \frac{\partial}{\partial x}(2u)\frac{\partial u}{\partial x}+\frac{\partial}{\partial y}\left(2u\frac{\partial u}{\partial y}\right)$$

$$= 2\frac{\partial u}{\partial x}\cdot\left(\frac{\partial u}{\partial x}\right)+2\frac{\partial u}{\partial y}\cdot\left(\frac{\partial u}{\partial y}\right) = 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]$$

$= 2\,|f'(z)|^2$ ...(From Equation (1))

= R. H. S.

Hence, $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{Re } f(z)|^2 = 2\,|f'(z)|^2$ ✓ **Hence proved**

Ex. 5.6.5 : If f(z) = u + iv is an analytic function then prove that,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^4 = 16\,|f(z)|^2\cdot|f'(z)|^2$$

Soln. :

Step I : Since, $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}$

$f(z) = u + iv$; $f(\bar{z}) = u - iv$

$f(z)\cdot f(\bar{z}) = u^2 + v^2$ and

$|f(z)| = |f(\bar{z})| = \sqrt{u^2+v^2}$

$\therefore |f(z)|^2 = f(z)\cdot f(\bar{z})$ and $|f'(z)|^2 = f'(z)\cdot f'(\bar{z})$

and $|f(z)|^4 = f^2(z)\cdot f^2(\bar{z})$

Step II : L.H.S. $= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^4$

$$= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}[f(z)]^2\cdot[f(\bar{z})]^2$$

Step III : Differentiating w.r.t. $\bar{z}$ first keeping z is constant

$$\text{L.H.S.} = 4\frac{\partial}{\partial z}\cdot\frac{\partial}{\partial \bar{z}}\left[f^2(z)\cdot f^2(\bar{z})\right]$$

$$= 4\frac{\partial}{\partial z}\cdot f^2(z)\left[\frac{\partial}{\partial \bar{z}} f^2(\bar{z})\right]$$

$$\text{L.H.S} = 4\frac{\partial}{\partial z}[f(z)]^2\cdot(2)\cdot f(\bar{z})\cdot f'(\bar{z})$$

$$= 8\frac{\partial}{\partial z}[f(z)]^2\cdot f(\bar{z})\cdot f'(\bar{z})$$

$$= 8\cdot f(\bar{z})\cdot f'(\bar{z})\cdot\frac{\partial}{\partial z}\cdot f^2(z)$$

$$= 8\cdot f(\bar{z})\cdot f'(\bar{z})\cdot(2)\,f(z)\cdot f'(z)$$

$$= 16\,[f(z)\cdot f'(z)]\,f(\bar{z})\cdot f'(\bar{z})$$

$$= 16\,|f(z)\,f(\bar{z})|\times|f'(z)\,f'(\bar{z})|$$

$$= 16\,|f(z)|^2\cdot|f'(z)|^2$$

Hence, $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^4 = 16\,|f(z)|^2\cdot|f'(z)|^2$ ✓

Hence proved.

Exercise

Ex. 1 : Find the value of m such that the function $f(z) = r^3\cos 3\theta + i\,r^3\sin m\theta$ is an analytic function.

Ex. 2 : Find the value of m such that the function $f(z) = e^x\cos y + ie^x\sin my$ is an analytic function.

Ex. 3 : If f(z) = u + iv is analytic function, then show that u and v satisfy Laplace equation in polar form $\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$.

Ex. 4 : Show that v = x cosx sinh y + y sinx cosh y is harmonic. Find its harmonic conjugate u. Also find corresponding analytic function f(z) in terms of z.

Ex. 5 : If $u = \frac{1}{r}\log(x^2+y^2)$ Show that u is harmonic. If yes, find its harmonic conjugate and corresponding analytic function f(z) in terms of z.

Ex. 6 : Find the analytic function f(z) = u + iv whose imaginary part is $r^n\sin n\theta$.

Ex. 7 : Show that $f(z) = \sqrt{xy}$ satisfies C – R equations v at origin but not analytic at origin.

Ex. 8 : Show that following functions are harmonic. Find their harmonic conjugates. Also find corresponding analytic functions in terms of z.

(i) $u = x^4 - 6x^2y^2 + y^4$ (ii) $v = \frac{-y}{x^2+y^2+2x+1}$

(iii) $u = \cosh x\cos y$ (iv) $v = \frac{-y}{x^2+y^2}$

(v) $u = e^x[(x^2-y^2)\cos y - 2xy\sin y]$

Ex. 9 : An electrostatic field in the xy plane is given by the potential function $x^3 - 3xy^2$, find the stream function.

Ex. 10 : Find the complex conjugates of following functions. Also find the corresponding analytic functions in terms of z.

(i) $u = 3x^2 - 3y^2 + 2y$

(ii) $u = \left(r+\frac{1}{r}\right)\cos\theta\,;\, r\neq 0$ (iii) $v = r^n\sin n\theta$

Ex. 11 : If $w = \phi + i\psi$, represents, the complex potential for an electric field and $\phi = -2xy + \frac{y}{x^2+y^2}$, determine the function ψ.

Ans. : $(x^2-y^2)+\frac{x}{x^2+y^2}$

Ex. 12 : If $u = \frac{1}{2}\log(x^2+y^2)$, find v such that f(z) = u + iv is analytic. Determine f(z) in terms of z.

Ans. : $v = \tan^{-1}\left(\frac{y}{x}\right)+c$ and $f(z) = \log z + c$

Ex. 13 : If $v = 3x^2y - y^3$; find its harmonic conjugate u. Find f(z) = u + iv in terms of z.

Ans. : $u = x^3 - 3y^2x + c$ and $f(z) = z^3 + c$

Ex. 14 : Show that the functions are harmonic and if so, find their harmonic conjugate and the corresponding analytic functions, in terms of z

(i) $v = r^n\sin n\theta$

(ii) $u = e^{-2xy}\sin(x^2-y^2)$

(iii) $u = x^2 - y^2 - 2xy - 2x + 3y$

(iv) $\phi = x^2 - y^2$

(v) $\psi = \tan^{-1}\left(\frac{y}{x}\right)$

(vi) $v = \frac{(x-y)}{x^2+y^2}$

Ans. :

(i) $f(z)z^n$

(ii) $v = -e^{-2xy}\cdot\cos(x^2-y^2)+c$ and $f(z) = -ie^{iz^2}+c$

(iii) $v = x^2 - y^2 + 2xy - 2y - 3x + 6$

$f(z) = (1+i)z^2 - (2+3i)z + 6i$

(iv) $\psi = 2xy + c$

(v) $\phi = \frac{1}{2}\log(x^2+y^2)+c$

(vi) $u = \left(\frac{1+i}{2}\right)+c$

Ex. 15 : Find $f(z) = u + iv$ is an analytic function, $f(z)$ if

(i) $u - v = e^x(\cos y - \sin y)$

(ii) $u + v = \dfrac{x}{x^2 + y^2}$; when $f(1) = 1$

(iii) $u - v = \dfrac{\cos x + \sin x - e^{-4}}{2\cos x - e^y - e^{-y}}$, when $f(\pi/2) = 0$

(iv) $u - v = (x - y)(x^2 + 4xy + y^2)$

(v) $u + v = \dfrac{2\sin 2x}{e^{2y} - e^{-2y} - 2\cos(2x)}$

Ans. :

(i) $e^z + c$ (ii) $\dfrac{1}{(1-i)}\left(\dfrac{1}{Z} - 1\right)$

(iii) $\dfrac{1}{2}\left(1 - \cot + \dfrac{z}{2}\right)$ (iv) $-i z^3 + c$

(v) $\dfrac{\cot z}{1 + c} + c\left(\dfrac{1 + 3i}{5}\right)e^z + c$

Ex. 16 : Prove the analytic function whose real part is

(i) $\dfrac{y}{x^2 + y^2}$ (ii) $\cos x \cosh y$ (iii) $y + e^x . \cos y$

Ans. :

(i) $\log z$ (ii) $\cos z$ (iii) i/z

Ex. 17 : Prove that polar form of C.R. equation are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad ; \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$ and deduce that :

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Ex. 18 : Prove that $u = x^2 - y^2$ and $v = \dfrac{y}{x^2 + y^2}$ are harmonic functions of (x, y), but are not harmonic conjugate.

Ex. 19 : Find the analytic function $f(z) = u + iv$, where $u = \left(r + \dfrac{1}{r}\right)\cos\theta ; r \neq 0$.

Ex. 20 If $f(z) = u + iv$ is an analytic function ; then prove that

(ii) $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) | f(z) |^n = n^2 | f(z) |^{n-2} . | f'(z) |^2$

(v) $\left(\dfrac{\partial \phi}{\partial x}\right)^2 + \left(\dfrac{\partial \phi}{\partial y}\right)^2 = \left[\left(\dfrac{\partial \phi}{\partial u}\right)^2 + \left(\dfrac{\partial \phi}{\partial v}\right)^2\right] | f'(z) |^2$

; (where ϕ is a function of x and y.)

(vi) $\left(\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2}\right) = \left[\dfrac{\partial^2 \phi}{\partial u^2} + \dfrac{\partial^2 \phi}{\partial v^2}\right] | f'(z) |^2$

(where ϕ is a function of x and y.)

(vii) $\dfrac{\partial(u, v)}{\partial(x, y)} = | f'(z) |^2$

Ex. 21 : Find analytic function $f(z) = u + iv$ if $u + v = \dfrac{x}{x^2 + y^2}$ when $f(1) = 0$

Ex. 22 : Find analytic function $f(z) = u + iv$ if $u - v = \dfrac{\cos x + \sin x - e^{-\pi/4}}{2\cos x - e^y - e^{-y}}$ when $f'(\pi) = 0$

Ex. 23 : Find analytic function $f(z) = u + iv$ if $u - v = e^x(\cos y - \sin y)$

Ex. 24 : Find analytic function $f(z) = u + iv$ if $u - v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$.

Ex. 25 : Find analytic function $f(z) = u + iv$ if $u + v = e^{-x}(\cos y - \sin y)$

Ex. 26 : Find analytic function $f(z) = u + iv$ if $u + v = \sin x \cosh y + \cos x \sinh y$

Ex. 27 : If $f(z) = u + iv$ is analytic function, find $f(z)$, if $u + v = 3(x + y) + \dfrac{x - y}{x^2 + y^2}$

Ex. 28 : Find analytic function $f(z) = u + iv$ if $u + v = r^2(\cos 2\theta + \sin\theta)$ when $f(0) = 0$

Ex. 29 : Find analytic function $f(z) = u + iv$ if $u + v = e^x(\cos y + \sin y)$

Ex. 30 : Find analytic function $f(z) = u + iv$ if $u + v = \dfrac{1}{r}[(r^2 + 1)\cos\theta + (r^2 - 1)\sin\theta]$ when $f(1) = 2$

Syllabus Topic : Mapping : Conformal Mapping, Translation, Magnification and rotation, Inversion and Reflection

5.7 Conformal Mapping

Introduction

Let, $y = f(x)$ be a continuous function in xy – plane. The order pair (x, y) of real numbers can be plotted with the help of two co-ordinate axes OX and OY; we get the graph of function $f(x)$. This graph representation gives the visual representation of given function and better understanding of the properties of the given function.

Let,

$$w = f(z) = u + iv \quad \text{or}$$
$$w = f(x + iy) = u(x, y) + iv(x, y);$$

be a function of complex variable z. The complex variable $w = f(z)$ defines a mapping or transformation of z-plane into w-plane. The corresponding points, curves or regions of the two planes are called images of each other.

Let us consider,

$$w = f(z) = z + (3 - i)$$
$$u(x, y) + iv(x, y) = (x + iy) + 3 - i$$
$$u(x, y) + iv(x, y) = (x + 3) + i(y - 1)$$

By equating real and imaginary parts on both side, we get

$u(x, y) = x + 3$ and $v(x, y) = y - 1$

for $x = 0$, $u = 3$ and for $y = 0$; $v = -1$

$x = 1$; $u = 4$ and $y = 1$; $v = 0$

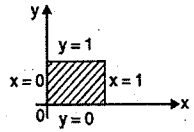


Fig. 5.7.1 : xy plane (z plane)

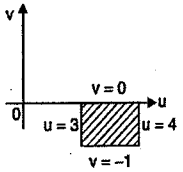


Fig. 5.7.2 : uv plane (w - plane)

∴ Corresponding to (0, 0) in z-plane, we get (3, – 1) in w-plane and at (1, 1) in z-plane it gives (4, 0) in w-plane. Therefore, the graphical representation of function of complex variable $w = f(z)$ is not possible in one plane as it involves four variables x, y, u, v which required four dimensional region. It require two separate complex planes (one z-plane and another w-plane).

For every point in $z (= x + iy)$ plane (or xy plane) the relation/mapping/transformation $w = f(z)$ defines a corresponding point (u, y) in w-plane (uv - plane).

i.e. a mapping of the domain of function f (in z plane), onto the range of f in the w-plane mapping $f : z \to w$.

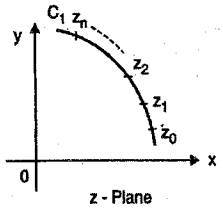


Fig. 5.7.3 : z-plane

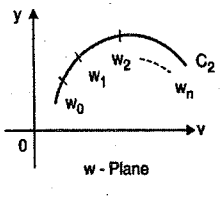


Fig. 5.7.4 : w-plane

i.e. If $z_0, z_1, z_2, \ldots z_n$ points lies on z-plane on curve C_1 and corresponding points $w_0, w_1, w_2, \ldots w_n$ lies on w-plane on curve C_2 then we say that the curve C_1 in z-plane mapped onto the curve C_2 in w-plane.

5.7.1 Definition of Conformal Mapping

Let us consider two curves C_1 and C_2 in the z-plane intersects at the point P and let the corresponding curves C_1' and C_2' in the w-plane intersects at the point P′ under the mapping / transformation / relation $w = f(z)$.

Definition :

If the angle of intersection of the curves in z-plane at P is equal to the angle of intersection between the corresponding curves in w-plane at P′, both in magnitude and sense, then the transformation (or mapping) is called conformal transformation (or mapping).

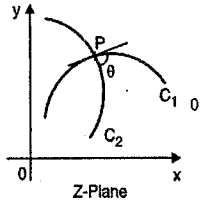


Fig. 5.7.5 : z – plane

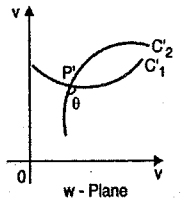


Fig. 5.7.6 : w – plane

1. **Properties of Conformal Mapping**

(a) Angle preservation : If the angle between two curves in z-plane is θ then the corresponding curves in w-plane intersects to each other at the same angle.

(b) Orthogonality preservation : Conformal mapping preserves orthogonality of curves i.e. If $\phi(x, y) = C_1$; $\psi(x, y) = C_2$ are mutually othogonal in z-plane then the corresponding image curves $\phi'(u, v) = C_1'$ and $\psi'(u, v) = C_2'$ in w-plane are also mutually orthogonal.

(c) The conjugate functions remains conjugate under conformal mapping

(d) Magnification : The ratio of linear element in z plane to corresponding linear element in w-plane is $| f'(z) |$; So, $f'(z) \neq 0$ in conformal mapping.

(e) Harmonic function remains harmonic under conformal mapping.

Note : 1. If a mapping preserves angles in magnitude only but not in sense then the mapping is called **isogonal mapping.**

2. A point at which $f'(z) = 0$, is called a critical point of the transformation.

2. **Conformal mapping theorem**

If $w = f(z)$ is analytic and $f'(z) \neq 0$ in a region R of the z-plane, then the mapping $w = f(z)$ is conformal at all points of R.

Some Basic conformal transformations

1. **Identity transformation : $w = z$**

Consider a general linear transformation $w = f(z)$

$= az + b$

For $a = 1$, and $b = 0$, we get $w = z$

is called identity transformation which maps a point z onto it self.

2. **Translation transformation**

Consider the general linear transformation

$$w = f(z) = az + b$$

For $a = 1$, and b is any complex constant we get, ($w = z + b$ translates or shifts z through a distance $|b|$ in the direction of b).

i.e. if $z = x + iy$, $w = u + iv$, $b = b_1 + ib_2$

Then, the transformation

$$w = z + b \quad \text{becomes}$$

$$u + iv = x + iy + b_1 + ib_2 = (x + b_1) + i(y + b_2)$$

i.e. In this transformation, mere translation of the axes and it preserves the shape and size of the image in z-plane to w-plane.

The rectangle in z-plane is transformed to rectangle in w-plane

The transformation changes circles into circles.

3. **Rotation Transformation**

In linear transformation if $a = e^{i\alpha}$ and $b = 0$, we get $w = e^{i\alpha} z$ is called rotation transformation which gives the rotation of radius vector of a point z through a scalar angle α anticlockwise if $\alpha > 0$ and clockwise if $\alpha < 0$.

4. **Magnification or scaling or stretching**

From linear transformation if $b = 0$ we get, $w = az$ magnifies (stretches) the radius vector by a constant factor 'a' if 'a' is real and $a > 1$ and contracts if 'a' is real and $0 < a < 1$. The corresponding regions on the z-plane and on w-plane will preserve the shape, size and orientation.

5. **Rotation and magnification : $w = cz$**

Where $c = re^{i\theta}$ and $z = r_1 e^{i\phi}$

$\therefore\ w = re^{i\theta} r_1 e^{i\phi} = R e^{i\psi}$, where, $R = r r_1$ and $\psi = \theta + \phi$

$\Rightarrow |w| = R$ and $\arg(w) = \psi$

Thus any image in z-plane is transformed into a geometrically similar figure in the w-plane.

6. **Inversion or Reflection transformation : $w = \frac{1}{z}$** is known as inversion transformation.

If $z = re^{i\theta}$ and $w = Re^{i\phi}$

$$Re^{i\phi} = \frac{1}{r} e^{-i\theta} \Rightarrow R = \frac{1}{r} \quad \text{and } \phi = -\theta$$

Thus under this transformation, the point Q (r, ϕ) in z-plane is mapped into the point Q' $\left(\frac{1}{r}, -\phi\right)$ in w-plane. Clearly the transformation, $w = \frac{1}{z}$ maps the interior of the unit circle $|z| = 1$ into the exterior of unit circle $|w| = 1$ and the exterior of $|z| = 1$ into the interior of $|w| = 1$,

However, the origin $z = 0$ is mapped to the point. $w = \infty$ in w-plane, called, **point at infinity.**

There are other special conformal transformations, $w = z^2$, $w = z^n$ $(n \in N)$, $w = e^z$, $w = \sin z$, $w = \cos z$, $w = \sinh z$, $w = \cosh z$, $w = \log z$, Joukowski Airfoil transformation etc.

Ex. 5.7.1 : Find the image of the circle $|z| = 1$ under the transformation $w = z + (2 + 2i)$

Soln. :

Step I : Given transformation is,

$$w = z + (2 + 2i)$$

Since, $w = u + iv$ and $z = x + iy$

$$\therefore\ (u + iv) = (x + iy) + (2 + 2i)$$

$$(u + iv) = (x + 2) + i(y + 2)$$

By comparing real and imaginary parts on both sides, we get

$$u = x + 2\ ; \qquad v = y + 2$$

$$\Rightarrow x = u - 2 \text{ and } \qquad y = v - 2$$

Step II : Given, $|z| = 1$, is a unit circle

i.e. $|x + iy| = 1$

$$\Rightarrow \sqrt{x^2 + y^2} = 1$$

$$\Rightarrow x^2 + y^2 = 1$$

By putting the values of x and y, we get

$$(u - 2)^2 + (v - 2)^2 = 1$$

Which is a circle having centre at (2, 2) and radius 1.

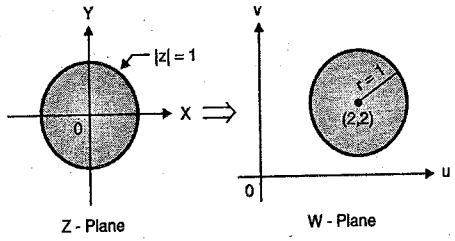


Fig. P. 5.7.1

Ex. 5.7.2 : Find the image of the region bounded by $x = 0$, $y = 0$, $x = 3$, $y = 2$ under the mapping $w = z + 2 - i$

Soln. :

Step I : Given transformation is :

$$w = z + 2 - i$$

Since, $w = u + iv$ and $z = x + iy$

$$\therefore\ u + iv = (x + iy) + (2 - i)$$

$$u + iv = (x + 2) + i(y - 1)$$

Comparing real and imaginary parts on both the sides, we get

$$u = x + 2 \qquad \ldots(1)$$

and

$$v = y - 1 \qquad \ldots(2)$$

Step II : Given, $x = 0 \Rightarrow$ from (1), we get $u = 2$

$y = 0 \Rightarrow$ from (2), we get $v = -1$

$x = 3 \Rightarrow$ from (1), we get $u = 5$

$y = 2 \Rightarrow$ from (2), we get $v = 1$

Hence,

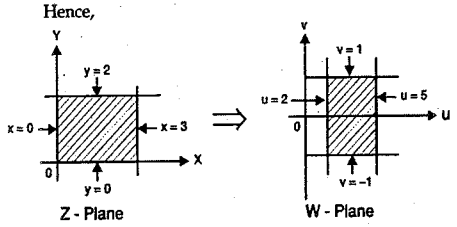


Fig. P. 5.7.2

Ex. 5.7.3 : Find the image of the region $x = 0$, $y = 0$, $x + y = 2$ under the mapping $w = 2z$

Soln. :

Step I : Given transformation is

$$w = 2z$$

Since, $w = u + iv$; $z = x + iy$

$$\therefore\ u + iv = 2(x + iy)$$

$$= 2x + i\,2y$$

Comparing real and imaginary parts on both sides,

$$u = 2x \text{ and } v = 2y$$

Step II : Given, $x = 0, y = 0$ and $x + y = 2$

$\therefore$ As $x = 0 \Rightarrow u = 0$

$y = 0 \Rightarrow v = 0$

and $x + y = 2 \Rightarrow \frac{u}{2} + \frac{v}{2} = 2$

$\Rightarrow u + v = 4$

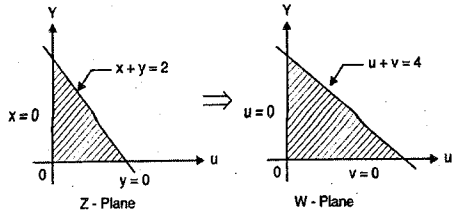


Fig. P. 5.7.3

Ex. 5.7.4 : Find the image of the half plane $y > 2$ under the mapping $w = (1 + i)z$

Soln. :

Step I : Given mapping is,

$$w = (1 + i)z$$

Since, $w = u + iv$ and $z = x + iy$

$$\therefore\ (u + iv) = (1 + i)(x + iy)$$

$$= x + iy + ix - y \qquad \ldots(\because i^2 = -1)$$

$$\Rightarrow (u + iv) = (x - y) + i(x + y)$$

Step II : Comparing real and imaginary parts on both sides,

$$u = x - y \text{ and } v = x + y$$

$\therefore$ Adding

$$u + v = (x - y) + (x + y) = 2x$$

$$\Rightarrow x = \frac{u + v}{2}$$

and subtracting $u - v = (x - y) - (x + y)$

$$= -2y$$

$$\Rightarrow u - v = -2y$$

$$\Rightarrow y = \frac{v - u}{2}$$

Step III : Given half plane $y > 2$

$$\Rightarrow \frac{v - u}{2} > 2$$

$$\Rightarrow v - u > 4 \Rightarrow u - v < -4$$

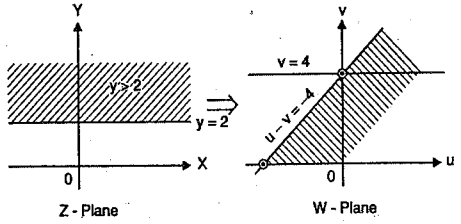


Fig. P. 5.7.4

Ex. 5.7.5 : Find the image of the rectangular hyperbola $xy = 1$ under $w = 2z$

Soln. :

Step I : Given mapping

$$w = 2z$$

Since, $w = u + iv$ and $z = x + iy$

$$\therefore\ (u + iv) = 2(x + iy)$$

$$= 2x + i\,2y$$

Step II : Comparing both sides

$$u = 2x \Rightarrow x = \frac{u}{2}$$

$$\text{and } v = 2y \Rightarrow y = \frac{v}{2}$$

Step III : Given $xy = 1$

$$\Rightarrow \left(\frac{u}{2}\right)\left(\frac{v}{2}\right) = 1$$

$$\Rightarrow uv = 4$$

The image is a rectangular hyperbola

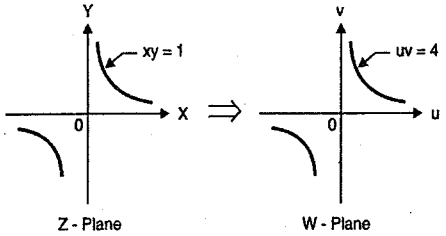


Fig. P. 5.7.5

Ex. 5.7.6 : Find the image of $x + y = 3$ under the mapping $w = z^2$

Soln. :

Step I : Given mapping is

$$w = z^2$$

Since, $w = u + iv$ and $z = (x + iy)$

$$\therefore u + iv = (x + iy)^2 = (x^2 - y^2) + i\,(2xy)$$

Comparing both sides, we get

$$u = (x^2 - y^2) \quad \ldots(1)$$

$$\text{and } v = 2xy \quad \ldots(2)$$

Step II : Given, $x + y = 3$

$$\Rightarrow y = 3 - x$$

Substitute the value of y in (1)

$$u = x^2 - (3 - x)^2$$

$$= \cancel{x^2} - (9 - 6x + \cancel{x^2})$$

$$u = -9 + 6x$$

$$\Rightarrow x = \frac{u + 9}{6} \quad \ldots(3)$$

Hence, $y = 3 - x$

$$= 3 - \left(\frac{u+9}{6}\right) = \frac{18 - u - 9}{6} = \left[\frac{6-u}{6}\right]$$

$$\therefore y = \left(\frac{6-u}{6}\right) \quad \ldots(4)$$

Step IV : By using Equation (3) and (4), Equation (2) becomes

$$V = 2xy$$

$$= 2\left(\frac{u+9}{6}\right)\left(\frac{6-u}{6}\right)$$

$$V = \frac{2\,(6\,u - u^2 + 54 - 9\,u)}{36}$$

$$V = \frac{1}{18}(-u^2 - 3\,u + 54)$$

$$\Rightarrow 18V = -u^2 - 3u + 54$$

$$\Rightarrow u^2 + 3u + 18v = 54$$

is a parabola.

Ex. 5.7.7 : Find the image of half plane $x > a$ where $a > 0$ under the transformation $w = \frac{1}{z}$

Soln. :

Step I : Given mapping is

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

Since, $w = u + iv$ and $z = x + iy$

$$\therefore (x + iy) = \frac{1}{(u + iv)}$$

Rationalisation

$$x + iy = \frac{1}{(u+iv)}\frac{(u-iv)}{(u-iv)} = \frac{u - iv}{u^2 + v^2}$$

$$\ldots(\because (a - b)(a + b) = (a^2 + b^2) \text{ and } i^2 = -1)$$

$$\therefore (x + iy) = \frac{u - iv}{u^2 + v^2} = \frac{u}{u^2 + v^2} - i\frac{v}{u^2 + v^2}$$

Comparing both sides, we get

$$x = \frac{u}{u^2 + v^2} \text{ and } y = -\frac{v}{u^2 + v^2}$$

Step II : Given, $x > a$ and $a > 0$

$\therefore$ for $x > a$

$$\therefore \frac{u}{u^2 + v^2} > a$$

$$\Rightarrow \frac{u}{a} > (u^2 + v^2)$$

Subtracting $\frac{u}{a}$ on both sides

$$\left[\frac{u}{a} - \frac{u}{a}\right] > \left[(u^2 + v^2) - \frac{u}{a}\right]$$

$$\therefore 0 > \left[(u^2 + v^2) - \frac{u}{a}\right]$$

$$\Rightarrow \left[(u^2 + v^2) - \frac{u}{a}\right] < 0$$

$$\Rightarrow \left(u - \frac{1}{2a}\right)^2 + (v - 0)^2 < \left(0 + \frac{1}{4\,a^2}\right)$$

$$\Rightarrow \left(u - \frac{1}{2a}\right)^2 + (v - 0)^2 < \frac{1}{4\,a^2}$$

is a circle having centre

at $\left(\frac{1}{2a}, 0\right)$ and radius $r = \frac{1}{2a}$

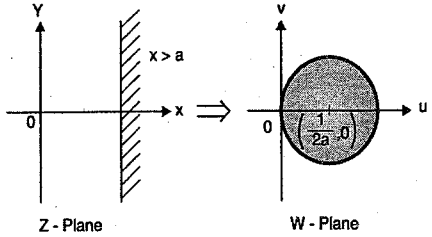


Fig. P. 5.7.7

Ex. 5.7.8 : Show that the transformation $w = \frac{1}{z}$ maps to, circle in Z-plane to circle in w-plane or a straight line.

OR

Find the image of circle in under the mapping $w = \frac{1}{z}$

Soln. :

Step I : Given transformation is

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

Since, $w = u + iv$ and $z = x + iy$

$$\therefore (x + iy) = \frac{1}{(u + iv)}$$

By rationalization,

$$(x + iy) = \frac{1}{(u+iv)}\left[\frac{u - iv}{u - iv}\right]$$

$$= \frac{u - iv}{(u^2 + v^2)}$$

$$= \frac{u}{u^2 + v^2} - i\frac{v}{u^2 + v^2}$$

Step II : Comparing both sides,

$$x = \frac{u}{u^2 + v^2} \text{ and } y = \frac{-v}{u^2 + v^2}$$

Since, the general equation of circle in Z-plane is

$$a\,(x^2 + y^2) + bx + cy + d = 0$$

$$\Rightarrow ax^2 + ay^2 + bx + cy + d = 0 \quad \ldots(1)$$

$$\Rightarrow a\left(\frac{u}{u^2 + v^2}\right)^2 + a\left(\frac{-v}{u^2 + v^2}\right)^2 + b\left(\frac{u}{u^2 + v^2}\right) + c\left(\frac{-v}{u^2 + v^2}\right) + d = 0$$

$$\Rightarrow \frac{au^2 + av^2}{(u^2 + v^2)^2} + \frac{bu}{(u^2 + v^2)} - \frac{cv}{(u^2 + v^2)} + d = 0$$

$$\Rightarrow \frac{a}{(u^2 + v^2)} + \frac{bu}{(u^2 + v^2)} - \frac{cv}{(u^2 + v^2)} + d = 0$$

$$\Rightarrow a + bu - cv + d\,(u^2 + v^2) = 0 \quad \ldots(2)$$

Step III :

If $a \neq 0$; (1) represent circle

If $a = 0$; (1) represent a straight line

If $d \neq 0$; (2) represent a circle

If $d = 0$; (2) represent a straight line.

Step IV :

Case (i) : If $a \neq 0, d = 0$

The transformation $w = \frac{1}{z}$ transforms circle (which is not passes through origin) into a straight line (which is also not passes through origin)

Case (ii) : $a \neq 0, d \neq 0$

$w = \frac{1}{z}$ transforms circle (which passes through origin) to a circle which is not passes through origin.

Case (iii) : $a = 0; d \neq 0$

$w = \frac{1}{z}$ transforming a straight line (not passing through origin) to circle passing through circle.

Case (iv) : $a = 0; d = 0$

$w = \frac{1}{z}$ transforming a straight line (Passing through origin) to a straight line (which also passing through origin).

Hence, the result.

Ex. 5.7.9 : Find the image or $|z - 2i| = 2$ under $w = \frac{1}{z}$

Soln. :

Step I : Given mapping is

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

Since $w = u + iv$; $z = x + iy$

$$(x + iy) = \frac{1}{u + iv} = \frac{u - iv}{(u + iv)\cdot(u - iv)}$$

$$= \frac{u - iv}{u^2 + v^2} = \frac{u}{u^2 + v^2} - i\frac{v}{u^2 + v^2}$$

Comparing both sides,

$$x = \frac{u}{u^2 + v^2}; \text{ and } y = \frac{-v}{u^2 + v^2} \quad \ldots(1)$$

Step II :

Given, $|z - 2i| = 2$

$$\Rightarrow |x + iy - 2i| = 2$$

$$|x + i\,(y - 2)| = 2$$

$$\Rightarrow \sqrt{x^2 + (y - 2)^2} = 2 \quad \ldots\text{Squaring both sides}$$

$$x^2 + (y - 2)^2 = 4$$

$$\Rightarrow x^2 + y^2 - 4y + 4 = 4$$

$$\text{i.e. } x^2 + y^2 - 4y = 0 \quad \ldots(2)$$

Step III :

From Equation (1), Equation (2) becomes

$$\left(\frac{u}{u^2 + v^2}\right)^2 + \left(\frac{-v}{u^2 + v^2}\right)^2 - 4\left(\frac{-v}{u^2 + v^2}\right) = 0$$

$$\Rightarrow \frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} + \frac{4v}{(u^2 + v^2)} = 0$$

$$\Rightarrow \frac{(u^2 + v^2)}{(u^2 + v^2)^2} + \frac{4v}{(u^2 + v^2)} = 0$$

$$\Rightarrow \frac{1}{(u^2 + v^2)} + \frac{4v}{(u^2 + v^2)} = 0$$

$$\Rightarrow \frac{1 + 4v}{(u^2 + v^2)} = 0$$

$$\Rightarrow 1 + 4v = 0$$

is a straight line in w-plane

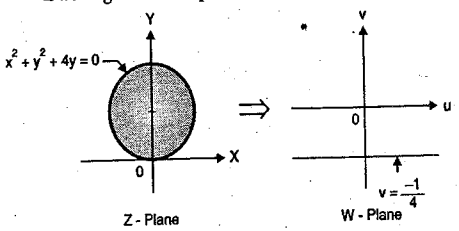


Fig. P. 5.7.9

Ex. 5.7.10 : Find the image of the rectangular region bounded by the lines x = 2; y = 2, x = 1; y = 3 under $w = z^2$

Soln. :

Step I : Given transformation :

$$w = z^2$$

since, $w = u + iz$; $z = x + iy$

$$\therefore (u + iv) = (x + iy)^2$$
$$= x^2 - y^2 + i\,2xy$$

Comparing real and imaginary parts an both sides, we get

$$u = x^2 - y^2 \quad ...(1)$$
$$v = 2xy \quad ...(2)$$

Step II :

(i) The image of x = 2

From Equation (1) ; $u = 4 - y^2$

From Equation (2) ; $v = 4y$

$$\Rightarrow y = \frac{v}{4}$$

Substitute the value of y in $u = 4 - y^2$

$$\therefore u = 4 - \left(\frac{v}{4}\right)^2 = 4 - \frac{v^2}{16}$$
$$\Rightarrow 16u = 16 - v^2$$
$$\Rightarrow v^2 = (16 - 16u) = 16(1 - u)$$

Which is the image of x = 2

$v^2 = 16(1 - u)$ is a parabola with vertex at (1, 0)

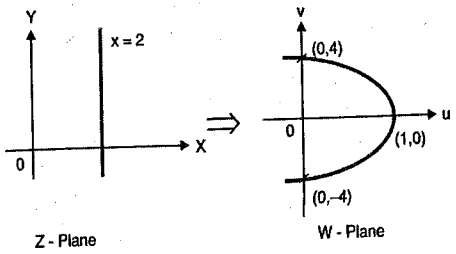


Fig. P. 5.7.10(a)

(ii) The image of y = 2

Put y = 2 in Equations (1) and (2), we get

$$u = x^2 - 4$$
$$v = 4x$$
$$\Rightarrow x = \frac{v}{4} \text{ put in } u = x^2 - 4$$
$$\therefore u = \left(\frac{v}{4}\right)^2 - 4 = \frac{v^2 - 16}{16}$$
$$\Rightarrow 16u = v^2 - 16$$
$$\Rightarrow v^2 = 16u + 16 = 16(u + 1)$$

is a parabola of vertex at (– 1, 0)

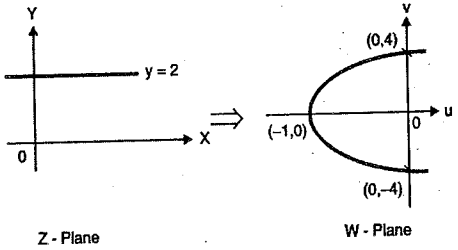


Fig. P. 5.7.10(b)

(iii) The image of x = 1

Put x = 1 in Equations (1) and (2)

$$u = 1 - y^2$$
$$v = 2y$$
$$\Rightarrow y = \frac{v}{2} \text{ put in } u = 1 - y^2$$
$$\Rightarrow u = 1 - \left(\frac{v}{2}\right)^2 \Rightarrow u = 1 - \frac{v^2}{4}$$
$$4u = 4 - v^2 \qquad \Rightarrow v^2 = 4 - 4u$$
$$v^2 = 4(1 - u)$$

is a parabola of vertex at (1, 0)

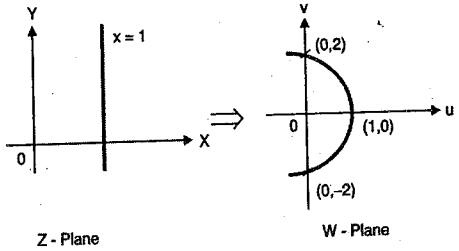


Fig. P. 5.7.10(c)

(iv) The image of y = 3

Put y = 3 in Equations (1) and (2)

$$\therefore u = x^2 - 9$$

and $v = 6x \Rightarrow x = \frac{v}{6}$

put in Equation $u = x^2 - 9$

$$\therefore u = \left(\frac{v}{6}\right)^2 - 9$$

$$36u = v^2 - 9 \times 36$$
$$\Rightarrow v^2 = 36u + 9 \times 36$$
$$= 36(u + 9)$$

is a parabola of vertex at (– 9, 0)

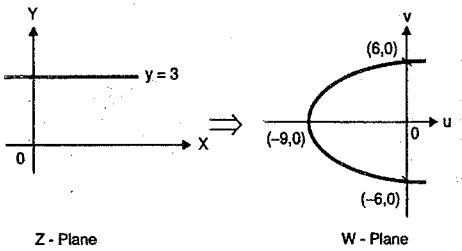


Fig. P. 5.7.10(d)

Hence, the image of x = 2; y = 2; x = 1 and y = 3 in w-plane are

For x = 2 ; $v^2 = 16(1 - u)$

For y = 2 ; $v^2 = 16(u + 1)$

For x = 1 ; $v^2 = 4(1 - u)$

For y = 3 ; $v^2 = 36(u + 9)$

The combined image is

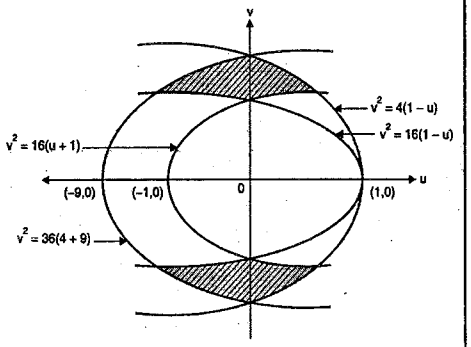


Fig. P. 5.7.10(e)

Ex. 5.7.11 : Find the mapping of the straight line y = x under the transformation $w = \frac{z-1}{z+1}$.

Soln. :

Step I : Given transformation is,

$$w = \frac{z-1}{z+1}$$
$$w(z + 1) = (z - 1)$$
$$\therefore wz + w = z - 1$$
$$wz - z = -1 - w$$
$$z(w - 1) = -(1 + w)$$

Step II :

$$z = \frac{-(1+w)}{-(1-w)} = \frac{(1+w)}{(1-w)}$$

Since, $z = x + iy$ and $w = u + iv$

$$\therefore (x + iy) = \frac{[1 + (u + iv)]}{[1 - (u + iv)]} = \frac{(1 + u) + iv}{(1 - u) - iv}$$

Step III : By rationalization

$$(x + iy) = \frac{[(1 + u) + iv][(1 - u) + iv]}{[(1 - u) - iv][(1 - u) + iv]}$$

➔ **Using standard formula :** $...\left[(a - b)(a + b) = a^2 - b^2\right]$

$$x + iy = \frac{[(1 + u) + iv][(1 - u) + iv]}{(1 - u)^2 + v^2} \quad ...(\because i^2 = -1)$$

$$(x + iy) = \frac{[(1 + u)(1 - u) - v^2] + i[v(1 + u) + v(1 - u)]}{(1 - u)^2 + v^2}$$

Step IV : By comparing real and imaginary parts on both sides

$$x = \frac{(1 + u)(1 - u) - v^2}{(1 - u)^2 + v^2} \quad \text{and}$$
$$y = \frac{v(1 + u) + v(1 - u)}{(1 - u)^2 + v^2}$$

Step V : But given, y = x

$$\therefore \frac{(1 + u)(1 - u) - v^2}{(1 - u)^2 + v^2} = \frac{v(1 + u) + v(1 - u)}{(1 - u)^2 + v^2}$$
$$\Rightarrow (1 + u)(1 - u) - v^2 = v(1 + u) + v(1 - u)$$

➔ **Using standard formula :** $...\left[(a + b)(a - b) = a^2 - b^2\right]$

$$1 - u^2 - v^2 = v + \cancel{vu} + v - \cancel{vu}$$
$$1 - u^2 - v^2 = 2v$$
$$1 - u^2 - v^2 - 2v = 0$$
$$u^2 + v^2 + 2v - 1 = 0$$
$$(u - 0)^2 + (v + 1)^2 = 2$$

is a circle of radius $\sqrt{2}$ and centre at (0, – 1) ✓ **...Ans.**

Ex. 5.7.12 : Show that the image of the hyperbola $x^2 - y^2 = 1$ under the transformation $w = \frac{1}{z}$ is the lemniscates $\rho^2 = \cos 2\theta$; where $w = \rho e^{i\phi}$.

Soln. :

Step I : Given, $w = \rho e^{i\phi}$.

Given transformation is, $w = \frac{1}{z}$ ...(1)

Step II : Since, $z = x + iy = r e^{i\theta}$ (polar form)

$$\therefore \rho e^{i\phi} = \frac{1}{r e^{i\theta}} = \frac{1}{r} e^{-i\theta}$$
$$\rho = \frac{1}{r} \text{ and } \phi = -\theta$$
$$\Rightarrow r = \frac{1}{\rho} \text{ and } \phi = -\theta \quad ...(2)$$

Step III : Since, equation of hyperbola is

$$x^2 - y^2 = 1$$

$$x = r\cos\theta;\ y = r\sin\theta$$

$$\therefore (r\cos\theta)^2 - (r\sin\theta)^2 = 1$$

$$r^2\underbrace{(\cos^2\theta - \sin^2\theta)}_{(\cos 2\theta)} = 1$$

$$\therefore r^2\cos 2\theta = 1$$

$$\therefore \frac{1}{\rho^2}\cos(-2\phi) = 1$$

by using Equation (2)

$$\Rightarrow \rho^2 = \cos 2\phi \checkmark \qquad (\because \cos(-2\phi) = \cos 2\phi)$$

Which is required lemniscates

Ex. 5.7.13 : Show that the mapping $w = \frac{2z+3}{z-4}$ transforms the circle $x^2 + y^2 - 4x = 0$ into the straight line $4u + 3 = 0$.

Soln. :

Step I :

Given : $w = \frac{2z+3}{z-4}$

$$\therefore (u+iv) = \frac{2(x+iy)+3}{x+iy-4} = \frac{(2x+3)+i2y}{(x-4)+iy}$$

$$= \frac{[(2x+3)+i2y][(x-4)-iy]}{(x-4)^2+y^2}$$

$$(u+iv) = \frac{(x-4)[(2x+3)+i2y]-iy[(2x+3)+i2y]}{(x-4)^2+y^2}$$

$$= \frac{[(2x+3)(x-4)+2y^2]+i[2y(x-4)-y(2x+3)]}{(x-4)^2+y^2}$$

Step II : Comparing real parts, on both sides, we get

$$\therefore u = \frac{(2x+3)(x-4)+2y^2}{(x-4)^2+y^2} = \frac{2x^2+2y^2-5x-12}{x^2+y^2-8x+16}$$

Step III :

$$4u+3 = 4\left[\frac{2x^2+2y^2-5x-12}{x^2+y^2-8x+16}\right]+3$$

$$= \frac{(8x^2+8y^2-20x-48)+3x^2+3y^2-24x+48}{(x^2+y^2-8x+16)}$$

$$= \frac{11x^2+11y^2-44x-48+48}{x^2+y^2-8x+16}$$

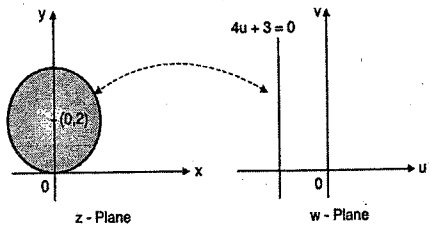


Fig. P. 5.7.13

Step IV : Given, $4u + 3 = 0$

$$11x^2 + 11y^2 - 44x = 0$$

$$x^2 + y^2 - 4x = 0$$

$$(x-0)^2 + (y-2)^2 = 2^2;$$

is a circle of radius 2 with centre at (0, 2) ✓

Ex. 5.7.14 : Find the mapping of the line $2y = x$ under the transformation $w = \frac{2z-1}{2z+1}$.

Soln. :

Step I : Given, $w = u + iv = \frac{2z-1}{2z+1}$

$$w(2z+1) = (2z-1)$$

Step II : $\Rightarrow 2zw + w = 2z - 1$

$$2zw - 2z = -1 - w$$

$$\Rightarrow 2z(w-1) = -1 - w$$

$$2z = \frac{-(1+w)}{(w-1)} = \frac{(1+w)}{(1-w)}$$

$$z = \frac{1}{2}\frac{(1+w)}{(1-w)}$$

$$= \frac{1}{2}\frac{(u+iv+1)}{(1-(u+iv))} = \frac{1}{2}\left[\underbrace{\frac{(u+1)+iv}{(1-u)-iv}}_{\text{Rationalisation}}\right]$$

$$= \frac{1}{2}\left[\frac{(u+1)+iv}{(1-u)-iv} \times \frac{(1-u)+iv}{(1-u)+iv}\right]$$

$$= \frac{1}{2}\left[\frac{(u+1)+iv}{(1-u)^2+v^2}\right][(1-u)+iv]$$

$$\ldots(\because (a-b)(a+b) = a^2 - b^2)$$

$$x + iy = \frac{1}{2}\left[\frac{[(1+u)(1-u)-v^2]+i[v(1-u)+v(u+1)]}{(1-u)^2+v^2}\right]$$

By equating real and imaginary parts, we get,

Step III : $\Rightarrow x = \frac{1}{2}\left[\frac{(1+u)(1-u)-v^2}{(1-u)^2+v^2}\right];$

$$y = \frac{v}{(1-u)^2+v^2}$$

Step IV : Given, $2y = x$

$$\therefore \frac{2v}{(1-u)^2+v^2} = \frac{1}{2}\left[\frac{1-u^2-v^2}{(1-u)^2+v^2}\right]$$

Step V : $\Rightarrow 4v = 1 - u^2 - v^2$

$$\Rightarrow u^2 + v^2 + 4v = 1 \Rightarrow (u-0)^2 + (v+2)^2 = (\sqrt{5})^2 \checkmark \quad \ldots\text{Ans.}$$

is a circle of radius $\sqrt{5}$ and centre of (0, – 2)

Ex. 5.7.15 : Find the map of the circle $|z - i| = 1$ under the transformation $w = \frac{1}{z}$ into w-plane.

Soln. :

Step I : Given, $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

Step II : $x + iy = \frac{1}{u+iv} = \underbrace{\frac{u-iv}{(u+iv)(u-iv)}}_{\text{Rationalisation}} = \frac{u-iv}{u^2+v^2}$

$$\ldots(\text{Since, } (a-b)(a+b) = a^2 - b^2)$$

$$\Rightarrow x + iy = \frac{u}{u^2+v^2} - \frac{iv}{u^2+v^2}$$

Step III : By equating real and imaginary parts on both sides, we get

Hence, $x = \frac{u}{u^2+v^2}$; $y = \frac{-v}{u^2+v^2}$

Step IV : Since, $|z - i| = 1$

$$\Rightarrow |x + iy - i| = 1$$

$$\Rightarrow |x + i(y-1)| = 1$$

$$\Rightarrow \sqrt{(x-0)^2 + (y-1)^2} = 1$$

$$x^2 + (y-1)^2 = 1$$

$$\Rightarrow x^2 + y^2 - 2y + \cancel{1} = \cancel{1}$$

$$\Rightarrow x^2 + y^2 = 2y$$

Step V : $\therefore \frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} = \left(2 \cdot \frac{-v}{u^2+v^2}\right)$

$$\Rightarrow \frac{u^2+v^2}{(u^2+v^2)^2} = \frac{-2v}{u^2+v^2}$$

$$\Rightarrow v = \frac{-1}{2}$$

Step VI : Thus, the circle $|z - i| = 1$ in z-plane maps to a straight line $v = -\frac{1}{2}$ in w-plane under the transformation, $w = \frac{1}{z}$

Ex. 5.7.16 : Show that the transformation $w = z + \frac{1}{z}$ transforms r = constant in the z-plane into a family of ellipses in the w-plane.

Soln. :

Step I : Let, $z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta)$

Given, $w = z + \frac{1}{z}$

Step II : $\therefore w = u + iv = z + \frac{1}{z}$

$$u + iv = re^{i\theta} + \frac{1}{re^{i\theta}}$$

$$u + iv = re^{i\theta} + \frac{1}{r}e^{-i\theta} = r[\cos\theta + i\sin\theta] + \frac{1}{r}[\cos\theta - i\sin\theta]$$

$$u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

Step III : Comparing the real and imaginary parts on the both the sides. we get

$$u = \left(r + \frac{1}{r}\right)\cos\theta;\ v = \left(r - \frac{1}{r}\right)\sin\theta \qquad \ldots(1)$$

Step IV : Let, $r = k$ (say) and $k \neq 1$

$$\left(\because \text{if } r = 1 \text{ then } r - \frac{1}{r} = 0\right)$$

$$\ldots(\text{Given } r = \text{constant})$$

From Equation (1),

$$u = \left(k + \frac{1}{k}\right)\cos\theta;\quad v = \left(k - \frac{1}{k}\right)\sin\theta$$

Step V : $\therefore \frac{u^2}{\left(k+\frac{1}{k}\right)^2} + \frac{v^2}{\left(k-\frac{1}{k}\right)^2} = \cos^2\theta + \sin^2\theta = 1$

i.e. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; Here $a = \left(k + \frac{1}{k}\right)$; $b = \left(k - \frac{1}{k}\right)$

[Equation of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ having $x = 2a$ is major axis and $y = 2b$ is minor axis]

[**Step VI :** If the eccentricity is e, then

$$e^2 = a^2 - b^2 = \left(k + \frac{1}{k}\right)^2 - \left(k - \frac{1}{k}\right)^2$$

$$= \left(k^2 + 2 + \frac{1}{k^2}\right) - \left(k^2 - 2 + \frac{1}{k^2}\right) = 4$$

$$\therefore \sqrt{a^2 - b^2} = 2$$

Step VII : The focii of an ellipse are at

$s = (e, 0)$ and $s = (-e, 0)$

$= \left(\sqrt{a^2 - b^2}, 0\right)$ $= \left(-\sqrt{a^2 - b^2}, 0\right)$

$= (2, 0)$ $= (-2, 0)$]

Ex. 5.7.17 : Under the transformation $w = \frac{1}{z}$, find the image of $|z - 2i| = 2$

Soln. :

Step I : Let $z = x + iy$ and $w = u + iv$

Step II : The given transformation is, $w = \frac{1}{z}$

i.e. $u + iv = \frac{1}{x+iy} = \underbrace{\frac{(x-iy)}{(x+iy)(x-iy)}}_{\text{Rationalisation}}$

➔ **Using standard formula :** $\ldots\left[(a-b)(a+b) = a^2 - b^2\right]$

$$u + iv = \frac{(x-iy)}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

Step III : Comparing real and imaginary parts, we get

$$\therefore\ u = \frac{x}{x^2+y^2}\ ;\ v = -\frac{y}{x^2+y^2} \qquad ...(1)$$

Step IV : Given,

$$|z-2i| = 2$$
$$\Rightarrow\ |x+iy-2i| = 2$$
$$\Rightarrow\ |x+i(y-2)| = 2$$
$$\therefore\ \sqrt{x^2+(y-2)^2} = 2$$

Squaring on both sides,

$$x^2+(y-2)^2 = 4$$
$$\Rightarrow\ x^2+y^2-4y = 0 \qquad ...(2)$$

Which is a circle with centre at (0, 2) with radius 2 in z-plane.

Step V : Now from Equation (1)

$$u^2+v^2 = \frac{x^2}{(x^2+y^2)^2}+\frac{y^2}{(x^2+y^2)^2} = \frac{x^2+y^2}{(x^2+y^2)^2}$$
$$= \frac{1}{(x^2+y^2)}$$
$$\therefore\ u^2+v^2 = \frac{1}{x^2+y^2}$$

i.e. $$x^2+y^2 = \frac{1}{u^2+v^2} \qquad ...(3)$$

Step VI : Also, from Equation (1)

$$x = u(x^2+y^2) = u\frac{1}{u^2+v^2} \text{ (from Equation (3))}$$

and $$y = -v(x^2+y^2) = -\frac{v}{u^2+v^2} \qquad ...(4)$$

(from Equation (3))

Step VII : Substitute values from Equations (3) and (4) in Equation (2), it gives,

$$\frac{1}{u^2+v^2}+\frac{4v}{u^2+v^2} = 0$$
$$\Rightarrow\ \frac{1+4v}{u^2+v^2} = 0$$
$$\Rightarrow 1+4v = 0,$$

Which is a straight line.

Another Method : Let $z = x+iy,\ w = u+iv$

Step I : Given : $w = \frac{1}{z}$

Step II : i.e. $z = \frac{1}{w}$

$$\therefore\ x+iy = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)}$$
$$= \frac{u-iv}{u^2+v^2}$$
$$x+iy = \frac{u}{u^2+v^2}-i\frac{v}{u^2+v^2}$$

Step III : Equating real and imaginary parts, we get,

$$x = \frac{u}{u^2+v^2};\quad y = -\frac{v}{u^2+v^2} \qquad ...(1)$$

Step IV : Since, $|z-2i| = 2 \Rightarrow |x+iy-2i| = 2$

$$\Rightarrow\ |x+i(y-2)| = 2 \Rightarrow \sqrt{x^2+(y-2)^2} = 2$$

Step V : Squaring, $x^2+(y-2)^2 = 4$

is a circle in z-plane having centre (0, 2) and radius r = 2.

$$x^2+y^2-4y = 0$$

Step VI : Put the values of x and y from Equation (1)

$$\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}+\frac{4v}{(u^2+v^2)^2} = 0$$
$$\therefore\ \frac{(u^2+v^2)}{(u^2+v^2)^2}+\frac{4v}{u^2+v^2} = 0$$
$$\therefore\ \frac{1}{u^2+v^2}+\frac{4v}{u^2+v^2} = 0$$
$$\Rightarrow\ \frac{1+4v}{u^2+v^2} = 0$$

$\therefore 1+4v = 0$ ✓ ...Ans.

is a straight line in w-plane.

Ex. 5.7.18 : Show that the transformation $w = z+\frac{1}{z}-2i$ maps the circle $|z| = 2$ into an ellipse in w-plane.

Soln. :

Step I : Given, $w = z+\frac{1}{z}-2i$

Step II : Since, $z = x+iy$, and $w = u+iv$

$$u+iv = (x+iy)+\frac{1}{(x+iy)}-2i$$
$$= (x+iy)+\frac{(x-iy)}{(x+iy)(x-iy)}-2i$$

➔ **Using standard formula :** $...\left[(a-b)(a+b) = a^2-b^2\right]$

$$= (x+iy)+\frac{(x-iy)}{(x^2+y^2)}-2i \qquad ...(i^2=-1)$$
$$u+iv = \left(x+\frac{x}{x^2+y^2}\right)+i\left(y-2-\frac{y}{x^2+y^2}\right)$$

Step III : Comparing real and imaginary parts on both sides, we get

$$u = x+\frac{x}{x^2+y^2}\ ;\quad v = y-2-\left(\frac{y}{x^2+y^2}\right)$$

Step IV : For given circle, $|z| = 2$

i.e. $x^2+y^2 = 4$

$$u = x+\frac{x}{4}\ ;\quad v = y-2-\frac{y}{4}$$
$$v = \frac{3y}{4}-2$$
$$u = \frac{5x}{4};\quad v = \frac{3y-8}{4}$$

$$\therefore\ x = \frac{4}{5}u;\quad y = (v+2)\frac{4}{3}$$

Step V : But $|z| = 2 \Rightarrow x^2+y^2 = 4$

$$\Rightarrow \left(\frac{4}{5}u\right)^2+\left[(v+z)\frac{4}{3}\right]^2 = 4$$
$$\Rightarrow \frac{u^2}{(5/2)^2}+\frac{(v+2)^2}{(3/2)^2} = 1 \checkmark$$

is an ellipse

Ex. 5.7.19 : Find the image in the w – plane of the circle $|z-3| = 2$ in the z-plane under the inverse mapping $w = \frac{1}{z}$.

Soln. :

Step I : Given, $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

Step II : Since, $z = x+iy,\quad w = u+iv$

$$\therefore x+iy = \frac{1}{u+iv}$$
$$= \underbrace{\frac{u-iv}{(u+iv)(u-iv)}}_{\text{Rationalisation}} = \frac{u-iv}{u^2+v^2} = \frac{u}{u^2+v^2}-i\frac{v}{u^2+v^2}$$

Comparing real and imaginary part on both sides

Hence, $$x = \frac{u}{u^2+v^2};\quad y = \frac{-v}{u^2+v^2}$$

Step III : Given circle,

$$|z-3| = 2$$
$$|x-3+iy| = 2$$
$$\sqrt{(x-3)^2+y^2} = 2$$
$$(x-3)^2+y^2 = 4$$
$$\Rightarrow\ x^2-6x+9+y^2 = 4$$
$$x^2+y^2-6x = -5$$
$$\Rightarrow\ x^2+y^2 = 6x-5$$

Step - IV : Substitute the values of x and y,

$$\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2} = \frac{6(u)}{(u^2+v^2)}-5$$
$$\frac{(u^2+v^2)}{(u^2+v^2)^2} = \frac{6u}{u^2+v^2}-5$$
$$\frac{1}{u^2+v^2} = \frac{6u}{u^2+v^2}-5$$
$$\frac{1}{u^2+v^2} = \frac{6u-5(u^2+v^2)}{u^2+v^2}$$
$$1 = 6u-5(u^2+v^2)$$
$$\therefore 5(u^2+v^2) = 6u-1$$
$$\Rightarrow\ (u^2+v^2) = \frac{6u-1}{5} = \frac{6u}{5}-\frac{1}{5}$$
$$u^2-\frac{6u}{5}+v^2 = \frac{-1}{5}$$

Step V : $$\left(u-\frac{3}{5}\right)^2+v^2 = -\frac{1}{5}+\frac{9}{25}$$

$$\left(u-\frac{3}{5}\right)^2+(v-0)^2 = \frac{20}{125} = \frac{4}{25} \checkmark \qquad ...\text{Ans.}$$

Is a circle having centre at $\left(\frac{3}{5}, 0\right)$ and radius $\frac{2}{5}$.

Ex. 5.7.20 : Show that the transformation $w = \sin z$ transforms the straight line $x = c$ of z-plane into hyperbola in the w-plane.

Soln. :

Step I : Given, $w = \sin z$

Since, $z = x+iy\ ;\ w = u+iv$

Step II : $u+iv = \sin(x+iy)$

➔ **Using standard formula**

$$...\left[\sin(A+B) = \sin A\cos B+\cos A\sin B\right]$$
$$u+iv = \sin x\underbrace{\cos iy}_{(\cosh y)}+\cos x\underbrace{\sin iy}_{(i\sinh y)}$$
$$= \sin x\cosh y+i\cos x\sinh y$$

Comparing both sides,

$$u = \sin x\cosh y\ ;\ v = \cos x\sinh y \qquad ...(1)$$
$$\Rightarrow \frac{u}{\sin x} = \cosh y\ ;\ \frac{v}{\cos x} = \sinh y$$

Step IV : Eliminate y from equation (1), we get

$$\frac{u^2}{\sin^2 x}-\frac{v^2}{\cos^2 x} = \cosh^2 y-\sinh^2 y = 1$$
$$(\because \cosh^2\theta-\sinh^2\theta = 1)$$

Thus, if $x = c$, it becomes $\frac{u^2}{\sin^2 c}-\frac{v^2}{\cos^2 c} = 1$; confocal hyperbolas in w-plane.

Syllabus Topic : Bilinear Transformation

5.7.2 Bilinear Transformation

A transformation of the form

$$w = f(z) = \frac{az+b}{cz+d};\ \text{provided } (ad-bc)\neq 0 \qquad ...(5.7.1)$$

Where a, b, c, d are complex or real constants is called bilinear or mobius transformation. It is also called linear fractional transformation.

It is also written as

$$w = f(z) = \frac{a+bz}{c+dz};\ \text{where } (bc-ad)\neq 0 \qquad ...(5.7.2)$$

Equation (5.7.1) can be written as

$$cwz+dw-(az+b) = 0$$

Which is linear in w as well as in z, so called bilinear transformation. Differentiating Equation (5.7.1) w.r.t. z, we have,

$$\frac{dw}{dz} = f'(z) = \frac{(cz+d)a-(az+b)c}{(cz+d)^2} = \frac{(ad-bc)}{(cz+d)^2}$$

If $ad - bc = 0$ then $\frac{dw}{dz} = 0$, so the transformation is not conformal at any z if $(ad - bc) = 0$.

So, we take a, b, c, d such that $(ad - bc) \neq 0$

By taking $a = 1, c = 0, d = 1$ Equation (5.7.1) gives the form, $w = z + b$ (Translation)

By taking $b = 0, c = 0, d = 1$, Equation (5.7.1) takes the form $w = az$ (Magnification or rotation)

By taking $c = 0, d = 1, w = az + b$ (linear)

By taking $a = 0, b = 1, c = 1, d = 0$; $w = \frac{1}{z}$ (inversion)

This shows that bilinear transformation is the combination of basic transformations. Also, corresponding to each z, bilinear transformation gives a unique value of w except when $cz + d = 0$.

(If $c \neq 0$, $z = -d/c$, then $w = \infty$; is called the "point at infinity" and the complex plane is called "extended complex plane").

Suppose $b \neq 0$ then $w = f(z)$ can be written as,

$$w = f(z) = \frac{\left(\frac{a}{b}\right)z + 1}{\left(\frac{c}{b}\right)z + \left(\frac{d}{b}\right)}$$

Hence, bilinear transformation consists of three independent constants $\frac{a}{b}, \frac{c}{b}, \frac{d}{b}$.

i.e. three distinct point (say) z_1, z_2, z_3 in z-plane and the corresponding point in w-plane w_1, w_2, w_3 are required for bilinear transformation.

5.7.2.1 Cross ratio of Four Points

Let z_1, z_2, z_3, z_4 be four distinct point in the extended complex plane then the cross ratio of these points denoted by (z_1, z_2, z_3, z_4) and is defined as

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}, \text{ If none of } z_1, z_2, z_3, z_4 \text{ are } \infty.$$

If $z_2 = \infty$; then $\frac{z_2 - z_4}{z_2 - z_3} = \left(\frac{1 - \frac{z_4}{z_2}}{1 - \frac{z_3}{z_2}}\right) \to 1$ as $z_2 \to \infty$

If $z_3 = \infty$; then $\frac{z_1 - z_3}{z_2 - z_3} = \left(\frac{\frac{z_1}{z_3} - 1}{\frac{z_2}{z_3} - 1}\right) \to 1$ as $z_3 \to \infty$

Similarly, If $z_1 \to \infty$; $\frac{z_1 - z_3}{z_1 - z_4} \to 1$ and if $z_4 \to \infty$; $\frac{z_2 - z_4}{z_1 - z_4} \to 1$

Similarly for w - points. The cross ratio is constant

i.e. If (z_1, z_2, z_3, z_4) and (w_1, w_2, w_3, w_4) are the corresponding points under bilinear transformation $w = f(z) = \frac{az + b}{cz + d}$ then $(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$

$$\frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_4)(w_2 - w_3)} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

As we know, for bilinear transformation, we require only three points in z-plane and three points in w-plane. For this purpose, we take one point as z and w.

Therefore, the cross ratio becomes,

$$\frac{(w - w_2)(w_1 - w_3)}{(w - w_3)(w_1 - w_2)} = \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)} \quad \text{...(5.7.3)}$$

OR

$$\frac{(w - w_2)(w_3 - w_1)}{(w - w_1)(w_3 - w_2)} = \frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}$$

OR

$$\frac{(w - w_3)(w_1 - w_2)}{(w - w_2)(w_1 - w_3)} = \frac{(z - z_3)(z_1 - z_3)}{(z - z_2)(z_1 - z_3)}$$

For finding bilinear transformation, we can use Equation (5.7.1) or Equation (5.7.3).

Bilinear transformation maps circles into circles; straight line onto straight lines in limiting cases

OR

Cross ratio can also defined as,

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

Therefore, we can write,

$$z_1 = z \quad \text{and} \quad w_1 = w$$

$$(z, z_1, z_2, z_3) = (w, w_1, w_2, w_3)$$

$$\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}$$

This formula also used for bilinear transformation

Note : Matrix method : Let $A_w = B$, be the system. Let, $w = \frac{az + b}{cz + d}$ be the bilinear transformations then associated matrix D is $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and inverse bilinear transformation is $D = \text{Adj}(A, B)$ and find a, b, c, d.

Applications of conformal mapping

1. It can be used to solve the problems arising in fluid mechanics, electrostatics, heat conduction, stream lines, electromagnetic potential problems.
2. It has various applications in medical physics. It is used to brain surface mapping.
3. Conformal mapping can be used in scattering and diffraction problems of plane electromagnetic waves.

Ex. 5.7.21 : Find the Bilinear transformation which maps the points $z = 0, 1, \infty$ in the z - plane onto the points $w = -1, -(2 + i), i$ in the w - plane.

Soln. :

Step I : Given , $z = (0, 1, \infty) \equiv (z_1, z_2, z_3)$ and

$$w = (-1, -(2 + i)), i \equiv (w_1, w_2, w_3)$$

Step II : Since we know,

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

Step III : $\Rightarrow \frac{(w + 1)(-2 - i - i)}{(w - i)(-2 - i + 1)} = \frac{(z - 0)}{(-1 - 0)}$

$$\left(\text{Here, } z_3 = \infty \text{ ; so, } \frac{(z_2 - z_3)}{(z - z_3)} = 1\right)$$

$$\Rightarrow \frac{(w + 1)\,2\,(1 \cancel{- i})}{(w - i)\,(1 \cancel{- i})} = \frac{z}{-1}$$

$$\Rightarrow \frac{2w + 2}{w - i} = -z$$

Step IV : $\therefore\ 2w + 2 = -z(w - i) = -zw + iz$

$$2w + zw = iz - 2$$

$$w(2 + z) = iz - 2$$

Step V : $\Rightarrow \quad w = \frac{iz - 2}{2 + z}$ ✓ ...Ans.

be the required bilinear transformation.

Ex. 5.7.22 : Find the bilinear transformation that maps the points $\infty, i, 0$ in z-plane into the points $(0, i, \infty)$ in w-plane.

Soln. :

Step I :

Given : $(z_1, z_2, z_3) \equiv (\infty, i, 0)$ and $(w_1, w_2, w_3) \equiv (0, i, \infty)$

Step II : By cross ratio, we can write

$$\frac{(w - w_2)(w_1 - w_3)}{(w - w_3)(w_1 - w_2)} = \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}$$

Step III : $\Rightarrow \frac{(w - i)(w_1 - \infty)}{(w - \infty)(0 - i)} = \frac{(z - i)(\infty - 0)}{(z - 0)(\infty - i)}$

As $w_3 \to \infty$ then $\frac{(w_1 - w_3)}{(w - w_3)} = \left(\frac{\frac{w_1}{w_3} - 1}{\frac{w_1}{w_3} - 1}\right) = 1$

And $z_1 \to \infty$, then $\frac{(z_1 - z_3)}{(z_1 - z_2)} = \left(\frac{1 - \frac{z_3}{z_1}}{1 - \frac{z_2}{z_4}}\right) = 1$

$$\therefore \frac{(w - i)}{(0 - i)} = \frac{(z - i)}{z}$$

Step IV : $\Rightarrow \frac{w - i}{-i} = 1 - \frac{i}{z}$

$$\Rightarrow w - i = -i - \frac{1}{z} \qquad (i^2 = -1)$$

$$w = -\frac{1}{z} \checkmark \qquad \text{...Ans.}$$

be the required bilinear transformation

Ex. 5.7.23 : Find the bilinear transformation which maps the points $z = 1, i, -1$ into the points $w = 0, 1, \infty$.

Soln. :

Step I :

Given : $z \equiv (z_1, z_2, z_3) \equiv (1, i, -1)$ and

$$w \equiv (w_1, w_2, w_3) \equiv (0, 1, \infty)$$

Step II : By cross ratio

$$\frac{(w - w_2)(w_1 - w_3)}{(w - w_3)(w_1 - w_2)} = \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}$$

Step III :

$$\Rightarrow \frac{(w - 1)(0 - \infty)}{(w - \infty)(0 - 1)} = \frac{(z - i)(1 + 1)}{(z + 1)(1 - i)}$$

As $w_3 \to 0$ then $\frac{(w_1 - w_3)}{(w - w_3)} = \left(\frac{\frac{w_1}{w_3} - 1}{\frac{w}{w_3} - 1}\right) = 1$

Take $\left(\frac{0 - \infty}{w - \infty}\right) \to 1$

Step IV : $\therefore \left(\frac{w - 1}{-1}\right) = \frac{2(z - i)}{(z + 1)(1 - i)}$

$$w - 1 = \frac{-2(z - i)}{(z + 1)(1 - i)}$$

$$w = 1 - \frac{2(z - i)}{(z + 1)(1 - i)} = \frac{(z + 1)(1 - i) - 2(z - i)}{(z + 1)(1 - i)}$$

$$= \frac{z - iz + 1 - i - 2z + 2i}{(z + 1)(1 - i)} = \frac{-z - iz + 1 + i}{(z + 1)(1 - i)}$$

Step V :

$$= \frac{-z(1 + i) + (1 + i)}{(z + 1)(1 - i)} = -\frac{(z - 1)(1 + i)}{(z + 1)(1 - i)}$$

$$= -\frac{(z - 1)}{(z + 1)} \times \underbrace{\left[\frac{(1 + i)}{(1 - i)} \times \frac{(1 + i)}{(1 + i)}\right]}_{\text{Rationalisation}}$$

$$= -\frac{(z - 1)}{(z + 1)}\left[\frac{(1 + i)^2}{2}\right]$$

$$= -\frac{(z - 1)}{(z + 1)}\left[\frac{\cancel{1} + 2i - \cancel{1}}{2}\right]$$

$$\text{...}(\because (1 + i)(1 - i) = 1 + 1 = 2)$$

$$w = -i\frac{(z - 1)}{(z + 1)} \checkmark \qquad \text{...Ans.}$$

be the required bilinear transformation

Ex. 5.7.24 : Find the bilinear transformation which maps the points $1, 0, i$ of the z-plane onto the points. $\infty, -2, -\frac{1}{2}(1 + i)$ of the w-plane.

Soln. :

Step I : The given points are

$$z \equiv (1, 0, i) \equiv (z_1, z_2, z_3) \text{ and}$$

$$w \equiv \left(\infty, -2, -\frac{1}{2}(1 + i)\right) \equiv (w_1, w_2, w_3)$$

Step II : Let, the bilinear transformation is,

$$w = \frac{az+b}{cz+d} \qquad ...(1)$$

Step III : Put $w = \infty$ and $z = 1$ in Equation (1)

$$\infty = \frac{a+b}{c+d} \Rightarrow c+d=0 \qquad c = -d$$

Step IV : Put $w = -2$ and $z = 0$ in Equation (1)

$$-2 = \frac{0+b}{0+d} \Rightarrow \mathbf{b = -2d}$$

Step V : Put $w = -\frac{1}{2}(1+i)$ and $z = i$ in Equation (1)

$$-\frac{1}{2}(1+i) = \frac{ai+b}{ci+d}$$

$$\Rightarrow -(1+i)(ci+d) = 2ai+2b$$

$$\Rightarrow -(1+i)(-di+d) = 2ai-4d$$

(By using values of c and b)

$$-\left[-\cancel{d}i+\cancel{d}i-\cancel{d}i+\cancel{i}d\right] = 2ai-4d$$

$$2ai = 4d$$

Step VI : $\Rightarrow a = \frac{d}{2i} = \frac{-id}{2}$

Step VII : By putting the values of a, b, c in Equation (1), we get

$$w = \frac{\frac{id}{2}z-2d}{-dz+d} = \frac{2+\frac{iz}{2}}{1-z}$$

$$\therefore \quad w = \frac{4+iz}{2(1-z)} \checkmark \qquad ...\text{Ans.}$$

be the required transformation.

Ex. 5.7.25 : Find the bilinear transformation that maps the points $-i, 0, i$ on to the points $-1, i, 1$ respectively.

Soln. :

Given :

Step I : $(z_1, z_2, z_3) \equiv (-i, 0, i)$ and $(w_1, w_2, w_3) \equiv (-1, i, 1)$

Step II : By cross ratio, we have

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w+1)(i-1)}{(w-1)(i+1)} = \frac{(z+i)(0-i)}{(z-i)(0+i)}$$

$$\left(\because \frac{i-1}{i+1} = i\left(\frac{1-\frac{1}{i}}{1+i}\right) = i\left(\frac{i+1}{i+1}\right)\right)$$

$$\Rightarrow \left(\frac{w+1}{w-1}\right)\times i = -\left(\frac{z+i}{z-i}\right)$$

Step III : $\Rightarrow \frac{w+1}{w-1} = \frac{-1}{i}\left(\frac{z+i}{z-i}\right) = i\left(\frac{z+i}{z-i}\right) = \frac{iz-1}{z-i}$

$$\left(\because \frac{1}{i} = -i\right)$$

Step IV : $\Rightarrow$ By componendo-dividendo method

$$\frac{(w+1)+(w-1)}{(w+1)-(w-1)} = \frac{(iz-1)+(z-i)}{(iz-1)-(z-i)}$$

$$\frac{\cancel{2}w}{\cancel{2}} = \frac{z(1+i)-(1+i)}{z(-1+i)+(i-1)}$$

$$w = \frac{(z-1)(1+i)}{(z+1)(i-1)}$$

Step V : $w = \left(\frac{z-1}{z+1}\right)\times(i) \qquad \left(\because \frac{1+i}{i-1} = i\right)$

$$w = i\left(\frac{1-z}{1+z}\right) \checkmark \qquad ...\text{Ans.}$$

is required bilinear transformation

Ex. 5.7.26 : Find the bilinear transformation which maps the points $1, i, -1$ from Z-plane into the points $i, 0, -i$ of the w-plane.

Soln. :

Step I :

Given : $(z_1, z_2, z_3) \equiv (1, i, -1)$ and $(w_1, w_2, w_3) \equiv (i, 0, -i)$

Step II : Let, the bilinear transformation be

$$w \equiv \frac{az+b}{cz+d} \qquad ...(1)$$

Step III : Put $z = z_1 = 1$ and $w = w_1 = i$ in Equation (1)

$$i = \frac{a+b}{c+d} \Rightarrow \mathbf{a+b = i(c+d)} \qquad ...(2)$$

Step IV : Put $z = z_2 = i$ and $w = w_2 = 0$ in Equation (1)

$$0 = \frac{ia+b}{ci+d} \Rightarrow ai+b=0 \Rightarrow \mathbf{b = -ai} \qquad ...(3)$$

Step V : Put $z = z_3 = -1$ and $w = w_3 = -i$ in Equation (1)

$$-i = \frac{-a+b}{-c+d} \Rightarrow +ic-id = -a+b$$

$$\Rightarrow \mathbf{i(c-d) = -a+b} \qquad ...(4)$$

Step VI : Adding Equations (2) and (4)

$$a+b = ic+id$$

$$-a+b = ic-id$$

$$2b = 2ic$$

$$\Rightarrow b = ic \text{ but } b = -ai$$

$$\therefore -ai = ic$$

$$\Rightarrow c = -a$$

Step VII : And subtracting Equations (4) from Equation (2)

$$(a+b) = (ic+id)$$

$$(-a+b) = (ic-id)$$

$$+ \quad - \qquad - \quad +$$

$$2a = i2d$$

$$a = id \Rightarrow d = \frac{a}{i} = -ai$$

Step VIII : Equation (1) becomes,

$$w = \frac{az-ai}{-az-ai} = \frac{z-i}{(-z-i)} = \frac{-(i-z)}{-(i+z)}$$

$$w = \frac{i-z}{i+z} \checkmark \qquad ...\text{Ans.}$$

is the required transformation.

Ex. 5.7.27 : Find the bilinear transformation which maps the points $0, -1, i$ of the z-plane on to the points $2, \infty, \frac{1}{2}(5+i)$ of w-plane.

Soln. :

Step I :

Given : $(z_1, z_2, z_3) \equiv (0, -1, i)$ and $(w_1, w_2, w_3) \equiv (2, \infty, \frac{1}{2}(5+i))$

Step II : By linear transformation, $w \equiv \frac{az+b}{cz+d} \qquad ...(1)$

Put z_1, z_2, z_3 and w_1, w_2, w_3 in Equation (1)

Step III : For $z = z_1 = 0$ and $w = w_1 = 2$

$$2 = \frac{0+b}{0+d} \Rightarrow \mathbf{b = 2d} \qquad ...(2)$$

Step IV : $z = z_2 = -1$ and $w = w_2 = \infty$ in Equation (1)

$$\infty = \frac{-a+b}{-c+d} \Rightarrow -c+d=0 \Rightarrow \mathbf{c = d} \qquad ...(3)$$

$$\left(\because \frac{1}{0} = \infty\right)$$

Step V : And $z = z_3 = i$ and $w = w_3 = \frac{1}{2}(5+i)$ in Equation (1)

$$\frac{1}{2}(5+i) = \frac{ia+b}{ic+d}$$

$$(5+i)(ic+d) = 2ai+2b$$

$$i5c+5d-c+id = 2ai+2b$$

but $c = d$ and $b = 2d$

$$\therefore \quad i5d+5d-d+id = 2ai+4d$$

$$i6d = 2ai \Rightarrow \mathbf{a = 3d} \qquad ...(4)$$

Step VI : By using Equations (2), (3) and (4) Equation (1) becomes

$$w = \frac{3dz+2d}{dz+d} = \frac{3z+2}{z+1}$$

be the required transformation.

Another method

Step I :

Given : $(w_1, w_2, w_3) \equiv (2, \infty, \frac{1}{2}(5+i))$

$(z_1, z_2, z_3) \equiv (0, -1, i)$

Step II : By cross ratio,

$$\frac{(w-w_2)(w_1-w_3)}{(w-w_3)(w_1-w_2)} = \frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)}$$

Step III : $\Rightarrow \frac{(w-\infty)\left(2-\frac{1}{2}(5+i)\right)}{\left(w-\frac{1}{2}(5+i)\right)(2-\infty)} = \frac{(z+1)(0-i)}{(z-i)(0+1)}$

Step IV : As $w_2 \to \infty$ then $\left(\frac{w-w_2}{w_1-w_2}\right) = \left(\frac{\frac{w}{w_2}-1}{\frac{w_1}{w_2}-1}\right) = 1$

$$\text{so, } \left(\frac{w-\infty}{2-\infty}\right) \to 1$$

$$\therefore \frac{2-\frac{1}{2}(5+i)}{w-\frac{1}{2}(5+i)} = (-i)\frac{(z+1)}{(z-i)}$$

$$\frac{4-(5+i)}{2w-(5+i)} = \frac{-iz-i}{(z-i)}$$

$$\frac{(-1-i)}{2w-(5+i)} = \frac{-iz-i}{z-i}$$

$$\frac{(1+i)}{(2w-5-i)} = \frac{iz+i}{z-i}$$

$$(1+i)(z-i) = (iz+i)(2w-5-i)$$

$$z-i+iz+1 = iz2w-i5z+z+i2w-5i+1 \qquad ...(i^2=-1)$$

$$\cancel{z}-i+iz+\cancel{z}+i5z-\cancel{z}+5i-\cancel{z} = iz2w+i2w$$

$$4i+6iz = i2w(z+1)$$

$$\cancel{2i}(2+3z) = \cancel{i2}w(z+1)$$

$$w = \frac{2+3z}{z+1} \checkmark \qquad ...\text{Ans.}$$

be the required transformation.

Ex. 5.7.28 : Find the bilinear transformation which maps the points $z = -1, 0, 1$ on the points. $w = 0, i, 3i$ of w-plane.

Soln. :

Step I : The given points :

$$z \equiv (-1, 0, 1) \equiv (z_1, z_2, z_3) \text{ and}$$

$$w = (0, i, 3i) \equiv (w_1, w_2, w_3)$$

Step II : Let, the bilinear transformation is,

$$w = \frac{az+b}{cz+d} \qquad ...(1)$$

Step III : Put $z = -1$, $w = 0$, in Equation (1)

$$0 = \frac{-a+b}{-c+d} \Rightarrow \mathbf{b = a}$$

Step IV : Put $z = 0$, $w = i$, in Equation (1)

$$i = \frac{0+b}{0+d} \Rightarrow \mathbf{b = di \Rightarrow d = -ib = -ia}$$

$$...\left(\because \frac{1}{i} = -i\right)$$

and put $z = 1$, $w = 3i$ in Equation (1)

Step V : $3i = \frac{a+b}{c+d}$

$$3i(c+d) = a+b$$

$$3i(c-ia) = a+a \qquad ...(\because \text{by the values of d, b})$$

$$3ic+3a = 2a$$

$$3ic = 2a-3a = -a$$

$$c = \frac{a}{3}i \qquad ...\left(\because \frac{1}{i} = -i\right)$$

Step VI : Put the values of b, c, d in Equation (1)

$$\therefore \quad w = \left[\frac{az+a}{\frac{a}{3}iz - ia}\right] = \frac{z+1}{\frac{iz}{3} - i} = \frac{z+1}{\frac{i}{3}(z-3)} = \frac{3}{i}\left(\frac{z+1}{z-3}\right)$$

$$= -3i\left[\frac{(z+1)}{(z-3)}\right] \checkmark \qquad \text{...Ans.}$$

(I) Find the fixed points of the Transformation

Note : Invariant or Fixed Points : If z-maps into itself in w-plane (i.e. w = z), then $w = \frac{az+b}{cz+d}$ becomes, $z = \frac{az+b}{cz+d}$ or $cz^2 + (d-a)z - b = 0$. The roots of this equation (say : z_1, z_2) are called invariant or fixed points of the bilinear transformation.

Ex. 5.7.29 : Find invariant points of : $w = \frac{2z+3}{z+4}$

Soln. :

Step I :

Given, $w = \frac{2z+3}{z+4}$

The fixed points of the transformation, w = f(z) are such f(z) = z

$\therefore$ Put f(z) = z, we obtain

$$z = w = \frac{2z+3}{z+4}$$

$$\Rightarrow \quad z^2 + 4z = 2z + 3 \Rightarrow z^2 + 2z - 3 = 0$$

$$\therefore z^2 + 2z + 1 - 1 - 3 = 0$$

Step II : $\Rightarrow \quad (z+1)^2 - 2^2 = 0 \Rightarrow \quad (z+1)^2 = 4$

$$\therefore \quad z + 1 = \pm 2$$

$$z = (2-1), (-2-1)$$

$$\Rightarrow \quad z = 1; -3 \checkmark \qquad \text{...Ans.}$$

are the invariant points.

Ex. 5.7.30 : Find the invariant points of the transformation $w = \frac{2z-6}{z-2}$

Soln. :

Given : $w = \frac{2z-6}{z-2}$

Step I : To find invariant points, put w = f(z) = z, we obtain

$$z = \frac{2z-6}{z-2}$$

Step II : $\Rightarrow z^2 - 2z - 2z + 6 = 0 \Rightarrow z^2 - 4z + 6 = 0$

$$\Rightarrow (z-2)^2 - 4 + 6 = 0$$

$$\Rightarrow (z-2)^2 = -2$$

$$\Rightarrow z - 2 = \pm i\sqrt{2};$$

$$z = (2 + \sqrt{2}i); (2 - \sqrt{2}i) \checkmark \qquad \text{...Ans.}$$

are the invariant points.

Ex. 5.7.31 : Show that : $w = \frac{z-i}{1-iz}$ maps upper half of z-plane onto interior of unit circle in w-plane.

Soln. :

Step I : Given, $w = \frac{z-i}{1-iz}$

$$w - izw - z = -i$$

$$-z(iw+1) = -i - w = -(i+w)$$

$$z = \frac{i+w}{1+iw}$$

Step II : $x + iy = \frac{i+(u+iv)}{1+i(u+iv)} = \frac{u+i(1+v)}{1+iu-v} = \frac{u+i(1+v)}{(1-v)+iu}$

$$= \left[\frac{u+i(1+v)}{(1-v)+iu}\right]\left[\frac{(-1-v)-iu}{(1-v)-iu}\right]$$

$$= \frac{u(1-v) - iu^2 + i(1+v)(1-v) + u(1+v)}{(1-v)^2 + u^2}$$

$$x + iy = \frac{[u(1-v) + u(1+v)] + i[-u^2 + (1+v)(1-v)]}{(1-v^2) + u^2}$$

Step III : Comparing real and imaginary parts, we get

$$y = -u^2 + (1+v)(1-v)$$

$$y = -u^2 + (1-v^2)$$

Step IV : Given, upper half part in z-plane i.e. y > 0

$$\Rightarrow -u^2 + (1-v^2) > 0$$

$$\Rightarrow -u^2 - v^2 > -1 \Rightarrow u^2 + v^2 > 1$$

$$|w| > 1 \checkmark$$

Which is unit circle in w – plane.

Hence, the upper half of z-plane maps onto interior of unit circle in w-plane.

Exercise

Find the image of the following in w-plane.

Ex. 1 : $|z| = 1$, under $w = z + (3 + 4i)$

Ans. : Circle

Ex. 2 : $|z| = \alpha$, under $w = (3 + 2i)$

Ans. : Circle

Ex. 3 : Square of vertices at (0, 0), (0, 1), (1, 0), (1, 1) in Z-plane under w = (1 + i) z

Ans. : Square : A (0, 0); B (1, 1); C (0, 2) and D (– 1, 1)

Ex. 4 : $y > \alpha$ where $\alpha < 0$ under $w = \frac{1}{z}$

Ans. : Circle having center at $\left(0, -\frac{1}{2\alpha}\right)$; $r = \frac{1}{2\alpha}$

Ex. 5 : The region bounded by x = 1, y = 1, x + y = 1 under the mapping $w = z^2$

Ans. : For line $x = 1 \Rightarrow v^2 = -4(u-1)$

For line $y = 1 \Rightarrow v^2 = -4(u+1)$

For line $x + y = 1 \Rightarrow u^2 = -2\left(v - \frac{1}{2}\right)$

Ex. 6 : Find the image of

(i) $|z| = \alpha$ under w = 3z

(ii) $|z| = 3$ under $w = \sqrt{2}\, e^{i\frac{\pi}{4}z}$

Ex. 7 : Find the image of the infinite strip $0 \le x \le 2$ under w = i z

Ex. 8 : Show that under the transformation w= cosh z , lines parallel to x-axis are mapped on to the hyperbolas in w-plane.

Ex. 9 : Show that the transformation $w = \frac{z-b}{z+b}$ maps the right half of the z-plane into the unit circle $|w| < 1$ (b is a real positive number).

Ex. 10 : Find the image of the triangular region bounded by x = 0, y = 0, x + y = 1 under the transformation $w = z^2$.

Ex. 11 : Show that the transformation : $w = z + \frac{1}{z} - 2i$ maps the circle $|z| = 2$ into an ellipse. Find the centre of the ellipse and its semi major and minor axes.

Ex. 12 : Find the image of the circle $(x-3)^2 + y^2 = 2$ under the transformation : $w = \frac{1}{z}$.

Ans. : $\left\{\text{Circle} : u^2 + v^2 - \left(\frac{6}{7}\right)u - \left(\frac{1}{7}\right) = 0\right\}$

Ex. 13 : Discuss the mapping $w = x + \frac{1}{z}$.

Ex. 14 : Show the under transformation w = sinh z, family to straight lines parallel to y-axis and x-axis are mapped onto be family of ellipse and hyperbolae respectively.

Ex. 15 : Show that the transformation under $w = \left(\frac{z-i}{z+i}\right)$, real axis in the z - plane is mapped into the circle $|w| = 1$.

Ex. 16 : Show that the image of the hyperbola $x^2 - y^2 = 1$, under the transformation ; $w = \frac{1}{z}$ is the leminscate $r^2 = \cos 2\theta$ when, $w = re^{i\theta}$

Ex. 17 : Find the image of on infinite strip : $\frac{1}{n} \le y \le \frac{1}{2}$ under transformation $w = \frac{1}{z}$.

Ex. 18 : Find the bilinear transformation which maps the prints z = 1 – i, – 1 and points w = 0, 1, ∞ respectively.

Ex. 19 : Find the bilinear transformation which maps the points 1, i, – 1 from the z-plane onto the points i, 0, – 1 of the w-plane

Ans. : $\left\{w = \frac{i-z}{i+z}\right\}$

Ex. 20 : Find bilinear transformation which maps the points 0, 1, 2 from z-plane onto the points 1, $\frac{1}{2}$, $\frac{1}{3}$ of w-plane.

Ex. 21 : Find the bilinear transformation which maps the points 1, i, 2i from the z-plane onto the points – 2i, 0, 1 of the w-plane.

Ex. 22 : Find the bilinear transformation which maps the points – 2, 0, 2 from the z-plane onto the points 0, i, – i of the w-plane.

Ex. 23 : Find the bilinear transformation which maps the points z = 0, – 1, ∞ from the z-plane into the points w = – 1, – 2 – i, i of the w-plane.

Ans. : $\left\{w = \frac{iz-2}{z+2}\right\}$

Ex. 24 : Find the bilinear transformation which maps the points – i, 0, 2 + i of the z-plane onto the points 0, – 2i, 4 of the w-plane.

Ans. : $\left\{w = \frac{2(z+i)}{z-1}\right\}$

Ex. 25 : Find the bilinear transformation which maps the points 0, $\frac{1}{2}$, 1 + i from z-plane into the points – 4, ∞, $\frac{(13-12i)}{5}$ of the w-plane.

Ans. : $\left\{w = \frac{(17-14i)z + 20(i+1)}{10(1+i)z - 5(1+i)}\right\}$

Ex. 26 : Find the bilinear transformation which maps the points 0, i, – 1 of z-plane upto the points 0, 1, ∞ of w-plane.

Ex. 27 : Find bilinear transformation which maps the points z = 1, i, 2i on the points –2i, 0, 1 respectively.

Ex. 28 : Find the bilinear transformation which maps the points 2, i, – 2 from z-plane into the points1, i, – 1 of the w-plane.

Ans. : $\left\{w = \frac{3z+2i}{iz+6}\right\}$

Ex. 29 : Find the bilinear transformation which maps the points 0, 1, 2 from z-plane onto the points1, $\frac{1}{2}$, $\frac{1}{3}$ of the w-plane.

Ans. : $\left\{w = \frac{1}{z+1}\right\}$

Ex. 30 : Find the bilinear transformation which maps the points – i, 0, 2 + i of the z-plane onto the points u, 2i, 4 of the w - plane.

Ans. : $W = \frac{2(z+i)}{z-1}$

Ex. 31 : Find the invariant points of the transformation $W = \frac{2z-6}{z-2}$;

Ans. : $2 \pm i\sqrt{2}$

Ex. 32 : Show that $w = \frac{z-i}{1-iz}$ mapped upper half of z-plane onto interior of the unit circle in w - plane.

Ex. 33 : Find the Bilinear transformation which maps the points 0, 1, 2 from z -plane on to the points 1, 1/3, 1/3 of the w-plane.

Ans. : $W = \frac{1}{z+1}$

Ex. 34 : Show that, under the transformation $w = z + \frac{4}{z}$, the circle $|z| = 2$ is mapped onto the straight line.

Chapter Ends...

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Note

UNIT 5

Functions of Complex Variables (Differential calculus)

Syllabus

Limit and continuity of f(z); Derivative of f(z) ; Analytic functions; Cauchy- Riemann equations in Cartesian and polar forms; Harmonic functions in Cartesian form;Mapping: Translation, magnification and rotation, inversion and reflection , bilinear transformation; Conformal mapping.

➢ **Chapter 5 : Functions of Complex Variables (Differential calculus)**

CHAPTER 5

UNIT V

Functions of Complex Variables (Differential Calculus)

Multiple Choice Questions

Q. 1 Analytic function $W = f(z)$ is defined as

(a) A single valued function $W = f(z) = u + iv$ possesses a unique derivative with respect to z at all points in a region R, then it is said to be analytic function.

(b) The function which is not differentiated at any point, called analytic function..

(c) A single valued function $W = f(z) = u + iv$ not possesses a unique derivative with respect to z at all points in a region R, then it is said to be analytic function

(d) None of the above.

Ans. : (a)

Q. 2 For the function f(z) , the point a is said to be singular point if

(a) f(z) is analytic at $z = a$

(b) f(z) is not analytic is at $z = a$

(c) $f(z) = 0$ at $z = a$

(d) $f(z) = 1$ at $z = a$

Ans. : (b)

Q. 3 Cauchy-Riemann Equation in Cartesian form.

(a) $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$; $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

(b) $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$; $\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x}$

(c) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(d) $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Ans. : (c)

Q. 4 Cauchy-Riemann Equation in polar form is

(a) $\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta}$; $\frac{\partial v}{\partial r} = \frac{1}{r} \frac{\partial u}{\partial \theta}$

(b) $\frac{\partial u}{\partial r} = r \cdot \frac{\partial v}{\partial \theta}$; $\frac{\partial v}{\partial r} = -r \frac{\partial u}{\partial \theta}$

(c) $\frac{\partial u}{\partial r} = r \cdot \frac{\partial v}{\partial \theta}$; $\frac{\partial v}{\partial r} = \frac{\partial u}{\partial \theta}$

(d) $\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta}$; $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$

Ans. : (d)

Q. 5 Harmonic Function is defined as

(a) Any function ϕ satisfying the Laplace equation $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ then it is called a Harmonic Function

(b) Any function ϕ satisfying the equation $\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = 0$ is called a Harmonic Function

(c) Any function ϕ satisfying the equation $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2u$ is called a Harmonic Function

(d) For any function ϕ if $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \neq 0$ then it is called a Harmonic Function

Ans. : (a)

Q. 6 For the function $v = 3x^2 y - y^3$, if $\frac{\partial v}{\partial x} = 6xy$; $\frac{\partial v}{\partial y} = 3x^2 - 3y^2$

and $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$, then by using Cauchy-Riemann Equations, the analytic function f(z) in terms of z is

(a) $f(z) = -z^3 + C$

(b) $f(z) = z^3 + C$

(c) $f(z) = z^2 + C$

(d) $f(z) = -z^2 + C$

Ans. : (b)

Q. 7 For the function $v = 3x^2 y - y^3$, if $\frac{\partial v}{\partial x} = 6xy$; $\frac{\partial v}{\partial y} = 3x^2 - 3y^2$ and $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$

$du = (3x^2 - 3y^2)\, dx - (6xy)\, dy$, then the complex conjugate u of v is

(a) $u = x^3 + 3y^2 x + C$

(b) $u = -x^3 - 3y^2 x + C$

(c) $u = x^3 - 3y^2 x + C$

(d) $u = -x^3 + 3y^2 x + C$

Ans. : (c)

Q. 8 If $u = x^3 - 3y^2x$; $v = 3x^2y - y^3$ and then the analytic function f(z) in terms of z is

(a) $f(z) = -z^2$

(b) $f(z) = z^2$

(c) $f(z) = z$

(d) $f(z) = z^3$

Ans. : (d)

Q. 9 If f(z) = u + iv is analytic function, where, $u = r^2 \cos 2\theta$; $v = r^2 \sin a\theta$, then the value of a is

(a) $a = 2$ (b) $a = -2$

(c) $a = -3$ (d) $a = 1$

Ans. : (a)

Q. 10 If $f(z) = (x^2 + axy + by^2) + i(px^2 + qxy + y^2)$ is analytic function where, $\frac{\partial u}{\partial x} = 2x + ay$; $\frac{\partial v}{\partial x} = 2px + qy$ and $\frac{\partial u}{\partial y} = ax + 2by$ and $\frac{\partial v}{\partial y} = qx + 2y$, then by C-R equations the value of a, b, p and q.

(a) $a = -2,\ b = 1;\ p = -1$ and $q = 2$

(b) $a = 2,\ b = -1;\ p = -1$ and $q = 2$

(c) $a = 2,\ b = 1;\ p = 1$ and $q = 2$

(d) $a = -2,\ b = -1;\ p = -1$ and $q = -2$

Ans. : (b)

Q. 11 If f (z) = u + iv is an analytic function then the value of $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\text{Re } f(z)|^2$ is equal to

(a) $4|f'(z)|^2$

(b) $|f'(z)|^2$

(c) $2|f'(z)|^2$

(d) $-4|f'(z)|^2$

Ans. : (c)

Q. 12 The image of the rectangular hyperbola xy = 1 under w = 2 z, with u = 2x and v = 2y is

(a) $uv = -4$

(b) $uv = 2$

(c) $uv = 7$

(d) $uv = 4$

Ans. : (d)

Q. 13 A transformation of the form $w = f(z) = \frac{az + b}{cz + d}$, where a, b, c, d are complex or real constants is called bilinear transformation if

(a) $(ad - bc) \neq 0$

(b) $(ad - bc) = 0$

(c) $ad + bc = 0$

(d) $(d - c) \neq 0$

Ans. : (a)

Q. 14 For the bilinear transformation $w = \frac{az + b}{cz + d}$, the point z = 1 of the z-plane maps onto the point w = ∞ of the w-plane, then the relation between c and d is

(a) c = d

(b) c = −d

(c) c = 2 d

(d) c = −3 d

Ans. : (b)

Q. 15 The invariant points of $w = \frac{2z + 3}{z + 4}$ are

(a) z = −1 , −3

(b) z = 1 , 3

(c) z = 1 , −3

(d) z = 0 , −1

Ans. : (c)

□□□

CHAPTER 6

UNIT 6

Function of Complex Variable (Integral Calculus)

Syllabus

Cauchy's integral theorem; Cauchy's integral formula; Residues; Cauchy's residue theorem (All theorems without proofs).

6.1 Complex Integration (Line Integral)

Let, f(z) be a continuous function of the complex variable (z = x + iy) defined at all points on curve C having end points a and b. Divide the curve C into n parts at the points.

$a = P_0(z_0), P_1(z_1), P_2(z_2), \ldots P_i(z_i), \ldots P_{r-1}(z_{r-1}), P_r(z_r), \ldots P_n(z_n) = b.$

Let, $\delta z_r = z_r - z_{r-1}$ and ϵ_r be any point on the arc $[P_{r-1}, P_r]$ then the limit of the sum $\sum_{r=1}^{n} f(\epsilon_r)\,\delta z_r$ as $n \to \infty$ in such a way that $\delta z_r \to 0$, is called the 'line integral' of the function f (z) along the curve C.

Line integral is denoted by $\int_C f(z)\,dz$.

If the end points a and b coincide, so that C is a closed curve then this integral is called contour integral and is denoted by $\oint f(z)\,dz$

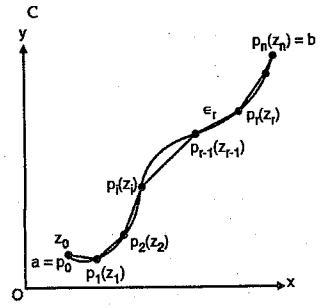


Fig. 6.1.1

Since, z = x + iy then dz = dx + i dy and f (z) = u (x, y) + i v (x, y) then,

$$\int_C f(z)\,dz = \int_C (u + iv)\cdot dz = \int_C (u + iv)(dx + i\,dy)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

This shows that, the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions. If C_1 be a point on the arc joining a and b then,

$$\int_a^b f(z)\,dz = \int_a^{C_1} f(z)\,dz + \int_{C_1}^{b} f(z)\,dz$$

Type - 1

Ex. 6.1.1 : Evaluate : $\int_0^{1+i} (x - y + ix^2)\,dz$

(a) Along the straight line from z = 0 to z = 1 + i.

(b) Along the real axis from z = 0 to z = 1 and then along a line parallel to imaginary axis from z = 1 to z = 1 + i.

(c) Along the imaginary axis from z = 0 to z = i and then along a line parallel to real axis from z = i to z = 1 + i.

Soln. :

Step I :

Let, P be a fixed point at z = 1 + i

Step II :

(I) Along a straight line OP joining O ($z = 0$) (0, 0) and P ($z = 1 + i$) (1, 1)

The equation of the straight line is $y = x$

$\therefore\ z = x + iy$ becomes $dz = dx + idy$

$\therefore\ z = x + ix \qquad \therefore\ dz = dx + idx$

and x varies from 0 to 1

Given,

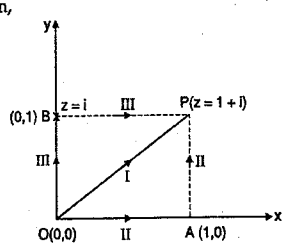


Fig. P. 6.1.1

$$\therefore \int_0^{1+i} (x - y + ix^2)\, dz = \int_0^{1+i} (x - y + ix^2)\,(dx + idy) \qquad \ldots(\because y = x)$$

$$= \int_0^1 (\cancel{x} - \cancel{x} + ix^2)\,(dx + idx)$$

$$= \int_0^1 ix^2(1+i)\,dx = \int_0^1 (i-1)\,x^2\,dx$$

$$= (i-1)\left[\frac{x^3}{3}\right]_0^1 = (i-1)\cdot\left(\frac{1}{3} - 0\right)$$

$$= -\frac{1}{3} + i\frac{1}{3}$$

Step III :

(II) Along the path OAP, where A is (1, 0).

$$\int_0^{1+i} (x - y + ix^2)\, dz = \int_{OA} (x - y + ix^2)\, dz + \int_{AP} (x - y + ix^2)\, dz \qquad \ldots(1)$$

Step IV :

Now along OA, $y = 0$, $dy = 0$ and x varies from $x = 0$ to $x = 1$

$$\therefore dz = dx + idy = dx$$

$$\therefore \int_{OA} (x - y + ix^2)\, dz = \int_0^1 (x + ix^2)\, dx$$

$$= \left[\frac{x^2}{2} + i\frac{x^3}{3}\right]_0^1 = \frac{1}{2} + i\frac{1}{3}$$

Step V :

And along AP ; $x = 1 \therefore dx = 0$ and y varies from $y = 0$ to $y = 1$

$$\therefore\ dz = dx + idy = idy$$

$$\therefore \int_{AP} (x - y + ix^2)\, dz = \int_0^1 (1 - y + i)\, idy = \int_0^1 [(i-1) - iy]\, dy$$

$$= \left[(-1+i)y - i\frac{y^2}{2}\right]_0^1 = (-1+i) - i\frac{1}{2}$$

$$= -1 + i\frac{1}{2}$$

Hence, from Equation (1)

$$\int_0^{1+i} (x - y + ix^2)\, dz = \frac{1}{2} + i\frac{1}{3} - 1 + i\frac{1}{2}$$

$$= \left(-\frac{1}{2} + \frac{5}{6}i\right)$$

Step VI :

(III) Along path OBP, where B(0, 1)

$$\int_0^{1+i} (x - y + ix^2)\, dz = \int_{OB} (x - y + ix^2)\, dz + \int_{BP} (x - y + ix^2)\, dz \qquad \ldots(2)$$

Now, along OB, $x = 0$, $dx = 0$ and y varies from 0 to 1

$$\therefore dz = dx + idy = i\,dy$$

$$\therefore \int_{OB} (x - y + ix^2)\, dz = \int_0^1 (-y)\, idy = \left[-i\frac{y^2}{2}\right]_0^1 = -i\frac{1}{2}$$

Step VII :

Along BP, $y = 1 \Rightarrow dy = 0$ and x varies from 0 to 1.

$dz = dx + idy = dx$

$$\therefore \int_{BP} (x - y + ix^2)\, dz = \int_0^1 (x - 1 + ix^2)\, dx$$

$$= \left[\frac{x^2}{2} - x + i\frac{x^3}{3}\right]_0^1 = \frac{1}{2} - 1 + i\frac{1}{3} = \frac{-1}{2} + i\frac{1}{3}$$

$\therefore$ from Equation (2)

$$\int_0^{1+i} (x - y + ix^2)\, dz = -i\frac{1}{2} - \frac{1}{2} + i\frac{1}{3} = -\frac{1}{2} - \frac{1}{6}i \checkmark \qquad \ldots\text{Ans.}$$

Note :

1. The values of the integral are different along three different paths.
2. The line integration depends on not only terminal points but also on the paths joining the points.

3. If f (z) is analytic in the region R, then $\int_{z_1}^{z_2} f(z)\, dz$ is independent of path joining z_1 and z_2 in the region R.

Ex. 6.1.2 : Evaluate : $\int_0^{1+i} (x^2 - iy)\, dz$ along the paths,

(a) $y = x$, (b) $y = x^2$.

Soln. :

Step I :

$$\text{Let,}\quad I = \int_0^{1+i} (x^2 - iy)\, dz$$

Step II :

(a) Along the straight line $y = x$ (along OCB)

$$\therefore\ z = x + iy = x + ix$$

$$dz = dx + idx \qquad \therefore x \text{ varies from } x = 0 \text{ to } x = 1$$

$$\therefore \int_0^{1+i} (x^2 - iy)\, dz = \int_0^1 (x^2 - ix)\,(dx + idx)$$

$$= (1+i)\int_0^1 (x^2 - ix)\, dx$$

$$= (1+i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1 = (1+i)\left[\frac{1}{3} - \frac{1}{2}i\right]$$

$$= \frac{1}{3} + i\frac{1}{3} - \frac{1}{2}i + \frac{1}{2}$$

$$= \left(\frac{1}{3} + \frac{1}{2}\right) + i\left(\frac{1}{3} - \frac{1}{2}\right) = \left(\frac{2+3}{6}\right) + i\left(\frac{2-3}{6}\right)$$

$$= \frac{5}{6} - \frac{1}{6}i$$

(b) Along the parabola $y = x^2$ (along OAB)

$$\therefore\ dy = 2x\,dx\ ;\ z = x + iy = x + ix^2$$

$$\Rightarrow dz = dx + i\,2xdx = (1 + i\,2x)\, dx$$

and x varies from 0 to 1.

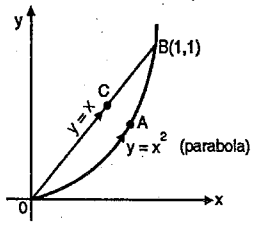


Fig. P. 6.1.2

$$\therefore \int_0^{1+i} (x^2 - iy)\, dz = \int_0^1 (x^2 - ix^2)\,(1 + i2x)\, dx$$

$$= \int_0^1 (x^2 + i2x^3 - ix^2 + 2x^3)\cdot dx$$

$$= \left[\frac{x^3}{3} + i\frac{2x^4}{4} - i\frac{x^3}{3} + \frac{2x^4}{4}\right]_0^1$$

$$= \left[\frac{1}{3} + i\frac{1}{2} - \frac{1}{3}i + \frac{1}{2}\right]$$

$$= \left(\frac{1}{3} + \frac{1}{2}\right) + i\left(\frac{1}{2} - \frac{1}{3}\right)$$

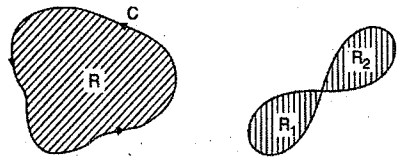
$$= \left(\frac{2+3}{6}\right) + i\left(\frac{3-2}{6}\right)$$

$$\int_0^{1+i} (x^2 - iy)\, dz = \frac{5}{6} + i\frac{1}{6} \checkmark \qquad \ldots\text{Ans.}$$

6.2 Contour Integral

6.2.1 Simple Closed Curve

- A curve is called simple closed curve if it does not crosses itself.
- A curve which crosses itself is called a multiple curve.



(a) Simple closed curve (b) Multiple curve

Fig. 6.2.1

6.2.2 Connected Region

A region is said to be connected region if any two points of the region R can be connected by a curve which lies entirely within the region R.

6.2.3 Simply Connected

A connected region 'R' is called simple connected if every closed curve in the region 'R' encloses points of the region 'R' only. i.e. every closed curve lying in region R can be contracted indefinitely without passing out of region R.

(OR A simply connected region is one which has no holes). A region which is not simply connected is called multiply connected R.

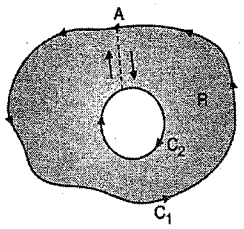


Fig. 6.2.2

We can convert multiply connected region into a simply connected one, by giving it one or more cuts (or cross).

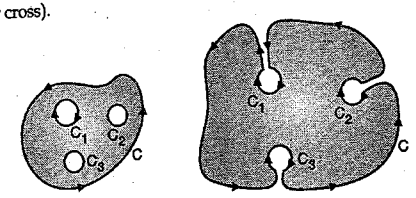


Fig. 6.2.3

Syllabus Topic : Cauchy's Integral Theorem

6.3 Cauchy's Integral Theorem

Statement : If f (z) is an analytic function and $f'(z)$ is continuous at each point within and on a simple closed curve C, then

$$\oint_C f(z)\,dz = 0$$

Proof : Let, R be the region bounded by the curve C.

Let $f(z) = u(x, y) + iv(x, y)$

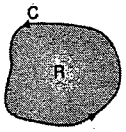


Fig. 6.3.1

Then,

$$\oint_C f'(z)\,dz = \oint_C (u+iv)(dx+idy)$$

$$= \oint_C (udx - vdy) + i\oint_C (vdx + udy) \qquad ...(6.3.1)$$

Since, $f'(z)$ is continuous, the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in R.

[Green's theorem in plane : If $\phi(x, y)$, $\psi(x, y)$, $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}$ are continuous in a region R of the xy plane bounded by a closed curve C then,

$$\oint_C (\phi\,dx + \psi dy) = \iint_E \left[\left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dxdy\right]$$

Then by Greens theorem, we have

$$\oint_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy \qquad ...(6.3.2)$$

Now, f (z) is an analytic function at each point of the region R, by C – R equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus Equation (6.3.2) becomes

$$\oint_C f(z)\,dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dxdy$$

$$= 0$$

$$\therefore \oint_C f(z)\,dz = 0$$

Corollary 1

If f (z) is analytic in a region R, P and Q are two points in R then $\int_P^Q f(z)\,dz$ is independent of the path joining P and Q and lying entirely in R.

Proof : Let, PAQ and PBQ be any two paths joining P and Q.

By Cauchy's theorem,

$$\oint_{PAQBP} f(z)\,dz = 0$$

$$\Rightarrow \oint_{PAQ} f(z)\,dz + \oint_{QBP} f(z)\,dz = 0$$

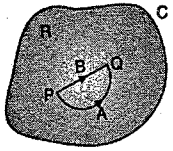


Fig. 6.3.2

$$\Rightarrow \oint_{PAQ} f(z)\,dz - \oint_{PBQ} f(z)\,dz = 0$$

$$\Rightarrow \oint_{PAQ} f(z)\,dz = \oint_{PBQ} f(z)\,dz$$

Corollary 2

If f (z) is analytic in the region bounded by two simple closed curve C and C_1 then,

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz$$

Proof : Let AB be a cross cut joining the curves C and C_1 then the double bounded region becomes simply connected region.

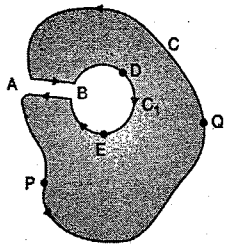


Fig. 6.3.3

By Cauchy's theorem,

$$\oint_{APQ\,ABD\,EBA} f(z)\,dz = 0$$

$$\Rightarrow \oint_{APQA} f(z)\,dz + \oint_{AB} f(z)\,dz + \oint_{BDEB} f(z)\,dz + \oint_{BA} f(z)\,dz = 0$$

$$\Rightarrow \oint_{APQA} f(z)\,dz + \oint_{AB} f(z)\,dz - \oint_{BEDB} f(z)\,dz - \oint_{AB} f(z)\,dz = 0$$

$$\therefore \oint_C f(z)\,dz - \oint_{C_1} f(z)\,dz = 0$$

(Integral around a closed curve are taken positive when the curves is traversed in contour clock-wise direction)

$$\therefore \oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz$$

The theorem can be extended for n curves lies inside region r

If a closed curve C contains non-intersecting closed curves $C_1, C_2, \ldots C_n$ then by

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \ldots + \oint_{C_n} f(z)\,dz$$

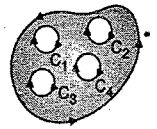


Fig. 6.3.4

Syllabus Topic : Cauchy's Integral Formula

6.4 Cauchy's Integral Formula

Statement

If f (z) is analytic within and on a simple closed curve C and 'a' is any point within C, then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

Proof : Consider the function $\frac{f(z)}{z-a}$, which is analytic at every point within C except at z = a.

Draw a circle C_1 with 'a' as centre and radius ρ such that C_1 lies entirely inside C

Thus, $\frac{f(z)}{z-a}$ is analytic in the region between C and C_1.

∴ By Cauchy's theorem, we have (i.e. corollary - 2)

$$\oint_C \frac{f(z)}{z-a}\,dz = \oint_{C_1} \frac{f(z)}{z-a}\,dz \qquad ...(6.4.1)$$

Now, the equation of the circle C_1 is

$$|z-a| = \rho \text{ or } z-a = \rho\cos\theta + i\rho\sin\theta$$

$$\Rightarrow z = a + \rho e^{i\theta}$$

$$\therefore dz = i\rho e^{i\theta}\,d\theta$$

$$\therefore \oint_{C_1} \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+\rho e^{i\theta})}{\rho e^{i\theta}}\, i\rho e^{i\theta}\,d\theta$$

$$= \int_0^{2\pi} f(a+\rho e^{i\theta})\,d\theta$$

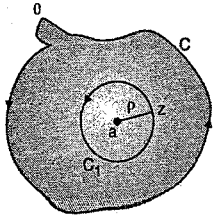


Fig. 6.4.1

Equation (6.4.1) becomes,

$$\oint_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a+\rho e^{i\theta})\,d\theta$$

In the limiting form i.e. $\rho \to 0$ i.e. circle C_1 becomes a point 'a' then above result becomes,

$$\oint_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a)\,d\theta = i\,f(a)\,[\theta]_0^{2\pi}$$

$$= i\,f(a)\cdot 2\pi = 2\pi i\,f(a)$$

$$\Rightarrow f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \quad \text{OR}$$

$$\oint_C \frac{f(z)}{z-a}\,da = 2\pi i\,f(a)$$

Which is required Cauchy's integral formula.

Corollary

By Cauchy's integral formula, we have

$$\frac{2\pi i}{n!} f^{(n)}(a) = \oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \quad \text{OR}$$

$$\oint_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(a)$$

Thus, if a function of a complex variable has exists first derivative in a simply connected region, all its higher order derivatives exists in that region "but this is not necessarily true for a function of real variables".

Note : 1. $|z + a + ib| = r$ gives circle with centre $(-a, -b)$ and radius r.
2. When the point lies inside the circle then use Cauchy's Integral formula.
3. When the point lies outside the circle then use Cauchys Integral Theorem.

Type - 2

Ex. 6.4.1 : Show that the integration does not depends on path, for the analytic function $f(z) = \frac{1}{z}$, bounded by the circle $C_1 : |z| = 1, C_2 : |z| = 2$.

Soln. :

Step I : Given function $f(z) = \frac{1}{z}$ is analytic function everywhere except at $z = 0$

($\because$ At $z = 0$, $f(z) = \frac{1}{0}$ undefined)

Step II :

$$z = x + iy$$

$$\therefore z = re^{i\theta} \quad \text{(Polar form)}$$

For $C_1 : |z| = 1$; is a circleof radius 1 and centre (0, 0)

$$\Rightarrow \quad \therefore z = e^{i\theta}$$

$$dz = e^{i\theta}\,i d\theta \text{ and limits for circle are,}$$

$$\theta = 0 \text{ to } \theta = 2\pi$$

$$\therefore \oint_0^{2\pi} f(z)\,dz = \int_0^{2\pi}\frac{1}{z}dz = \oint_0^{2\pi}\frac{1}{e^{i\theta}}\,i e^{i\theta}d\theta$$

$$= i\int_0^{2\pi} d\theta = i\,[\theta]_0^{2\pi} = 2\pi i$$

Step III :

and for $C_2 : |z| = 2$; is a circle of radius 2 and centre at (0, 0)

$$\because z = re^{i\theta} \quad \text{...(Polar form of z)}$$

$$\therefore z = 2e^{i\theta} \Rightarrow dz = 2e^{i\theta}\,i\,d\theta$$

$$\therefore \oint_0^{2\pi} f(z)\,dz = \int_0^{2\pi}\frac{1}{z}dz = \oint_0^{2\pi}\frac{1}{2e^{i\theta}}\cdot 2e^{i\theta}\,i d\theta$$

$$= i\int_0^{2\pi} d\theta = i\,[\theta]_0^{2\pi} = 2\pi i$$

Step IV :

$$\text{Thus, } \oint_{C_1} f(z)\,dz = \oint_{C_1} f(z)\,dz = 2\pi i$$

This shows that, when f (z) is analytic function, integration is independent of paths.

Type - 3

Ex. 6.4.2 : Evaluate : $\oint_C \frac{z}{z-(2+2i)}\,dz$; where C is simple closed curve.

Soln. :

Step I :

Since the function, $f(z) = \frac{z}{z-(2+2i)}$, is analytic everywhere within and on the curve except at $z = 2 + 2i$

... ($\because$ By equating denominator term / terms to zero, we get the points where function is not analytic. That points are called singular points.)

Step II :

$\therefore$ Draw a circle with centre at $z = 2 + 2i$ and radius 1

($\because$ $|z-(2+2i)| = 1$)

Step III :

Then the function f (z) is analytic in the bounded region C and C_1.

$\therefore$ By theorem " If f (z) is analytic in a region bounded by two simple closed curve C_1 and C_2 then

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz"$$

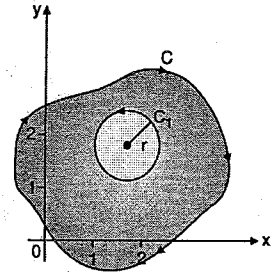


Fig. P. 6.4.2

Step IV :

$$\oint_C \frac{z}{z-(2+2i)}\,dz = \oint_{C_1} \frac{z}{z-(2+2i)}\,dz$$

Now, $C_1 : |z-(2+2i)| = 1$

$\therefore$ Here, $1 = |e^{i\theta}|$

$$z-(2+2i) = re^{i\theta} = e^{i\theta}$$

...($\because$ Let $z-(2+2i)$ is a complex number like $z = re^{i\theta}$)

$$\Rightarrow z = (2+2i) + e^{i\theta}$$

$$\Rightarrow dz = ie^{i\theta}d\theta \text{and limits for } \theta = 0 \text{ to } \theta = 2\pi$$

Step V :

$$\therefore \oint_C \frac{z}{z-(2+2i)}\,dz = \int_0^{2\pi}\left(\frac{(2+2i)+e^{i\theta}}{e^{i\theta}}\right) i e^{i\theta}d\theta$$

$$= i\left[(2+2i)\,\theta + e^{i\theta}\right]_0^{2\pi}$$

$$= i\left[(2+2i)\,2\pi + e^{i2\pi} - 0 - e^0\right]$$

$$= i\left[(2+2i)\,2\pi + e^{i2\pi} - 1\right]$$

$$= i\left[(2+2i)\,2\pi + \underbrace{\cos 2\pi}_{(1)} + i\underbrace{\sin 2\pi}_{(0)} - 1\right]$$

$$\oint_C \frac{z}{z-(2+2i)}\,dz = (2+2i)\,2\pi i \checkmark \quad \text{...Ans.}$$

Note : The above example can be solved directly by Cauchy's Integral formula.

Ex. 6.4.3 : Evaluate : $\int_c (z - z^2)\,dz$, where c is the upper half of the unit circle $|z| = 1$.

Soln. :

Step I : Given, C : $|z| = 1$ is a unit circle

i.e. $r = 1$

since, $z = re^{i\theta} = e^{i\theta}$

$dz = ie^{i\theta}\,d\theta$

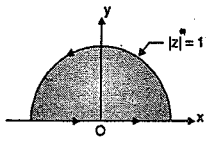


Fig. P. 6.4.3

Limits : $\theta = 0$ to π ($\because$ upper half of circle)

Step II :

$$\int_c (z - z^2)\,dz = \int_{\theta=0}^{\pi} [e^{i\theta} - (e^{i\theta})^2]\,ie^{i\theta}\,d\theta$$

$$= \int_{\theta=0}^{\pi} i\,[e^{i2\theta} - e^{i3\theta}]\,d\theta = i\left[\frac{e^{i2\theta}}{2i} - \frac{e^{i3\theta}}{3i}\right]_0^{\pi}$$

$$= i\left[\left(\frac{e^{i2\pi}}{2} - \frac{e^{i3\pi}}{3}\right) - \left(\frac{1}{2} - \frac{1}{3}\right)\right]$$

Step III :

Since, $e^{i\theta} = \cos\theta + i\sin\theta$

$$= \frac{1}{2}\,[\underbrace{\cos 2\pi}_{(1)} + i\underbrace{\sin 2\pi}_{(0)}] - \frac{1}{3}\,[\underbrace{\cos 3\pi}_{(-1)} + i\underbrace{\sin 3\pi}_{(0)}] - \frac{1}{6}$$

$$= \frac{1}{2}\,[1 + 0] - \frac{1}{3}\,[-1 + 0] - \frac{1}{6}$$

$$= \left(\frac{1}{2} + \frac{1}{3}\right) - \frac{1}{6} = \left(\frac{3+2}{6}\right) - \frac{1}{6} = \frac{5}{6} - \frac{1}{6}$$

$$= \left(\frac{5-1}{6}\right) = \frac{4}{6} = \frac{2}{3}$$

$$\int_c (z - z^2)\,dz = \frac{2}{3}\checkmark \quad \text{...Ans.}$$

Ex. 6.4.4 : Evaluate : $\oint_C (x^2 - y^2 + i2xy)\,dz$; where C is contour $|z| = 1$

Soln. :

Step I : Given,

$$\Rightarrow f(z) = (x^2 - y^2 + i\,2xy) = u + iv$$

$$u = x^2 - y^2;\ v = 2xy$$

Step II :

Differentiate u and v partially, w.r.t. x and y

$$u_x = 2x;\ v_x = 2y$$

$$\text{And, } u_y = -2y;\ v_y = 2x$$

Step III :

By C-R equations,

$$\therefore u_x = v_y = 2x \text{ and } u_y = -v_x = -2y$$

This shows that f (z) is analytic everywhere.

Step IV :

By Cauchy's theorem, "If f (z) is analytic within and on the closed curve C then $\oint_C f(z)\,dz = 0$".

$$\therefore \oint_C f(z)\,dz = \oint_0^{2\pi} (x^2 - y^2 + i\,2xy)\,dz = 0 \checkmark \qquad \text{...Ans.}$$

Ex. 6.4.5 : Evaluate : $\oint_C \frac{e^{-z}}{z+1}dz$, where C is a circle :

(a) $|z| = 2$ (b) $|z| = \frac{1}{2}$

Soln. :

Step I : Given : $f(z) = \frac{e^{-z}}{z+1}$ is analytic function everywhere except at $z + 1 = 0$ i.e. except at $z = -1$.

Step II :

(a) For C_1 : $|z| = 2$ is a circle having centre at (0, 0) and radius 2

For $z = -1$

$\therefore |z| = |-1| = 1 < 2$

This shows that the point $z = -1$ lies inside the circle $|z| = 2$

Step III :

➔ **Using Cauchy's Integral formula :**

$$\ldots\left[\oint_C \frac{f(z)}{(z-a)}\,dz = 2\pi i\,[f(z)]_{z=a}\right]$$

Here, $f(z) = e^{-z}$ and $a = -1$

$f(a) = f(-1) = e^{-(-1)} = e$

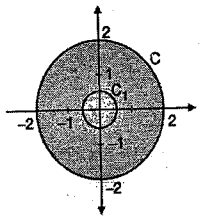


Fig. P. 6.4.5

$$\therefore \oint_C \left(\frac{e^{-z}}{z+1}\right)dz = 2\pi i\,f(a) = 2\pi i\,f(-1) = 2\pi i e$$

$$= i\,2\pi e$$

Step IV :

(b) For C_2 : $|z| = \frac{1}{2}$ is circle having centre at (0,0) and radius $\frac{1}{2}$

For $z = -1$

$|z| = |-1| = 1 > \frac{1}{2}$

$\therefore$ The point $z = -1$ lies outside of the circle $|z| = \frac{1}{2}$

$\ldots(\because |z| = |x + iy| = \sqrt{x^2 + y^2})$

Step V : $\therefore$ By Cauchy's Integral theorem, we have,

$$\oint_C \frac{e^z}{z+1}\,dz = 0 \checkmark \qquad \text{...Ans.}$$

Ex. 6.4.6 : Evaluate, By Cauchy's integral formula :

(i) $\oint_C \frac{e^{2z}}{(z-1)(z-2)}\,dz$;

where C is the circle $|z| = 3$

(ii) $\oint_C \frac{\sin^2 z}{(z-\pi/6)^3}\,dz$;

Where, C is the circle $|z| = 1$

Soln. :

Step I :

(i) The function $\frac{e^{2z}}{(z-1)(z-2)}$ is analytic everywhere, except, $z = 1$; $z = 2$

$(\because z - 1 = 0$ and $z - 2 = 0)$

Step II :

The integral has two singularities at $z = 1$ and $z = 2$

(i) For $z = 1$;
Given C : $|z| = 3$
$\therefore |z| = |1| = 1 < 3$

(ii) For $z = 2$
Given : C : $|z| = 3$
$\therefore |z| = |2| = 2 < 3$

This shows that both points $z = 1, 2$ lies inside the given circle.

Step III :

$$\therefore \oint_C \frac{e^{2z}}{(z-1)(z-2)}\,dz = \oint_C e^{2z}\cdot\left[\frac{A}{z-1} + \frac{B}{z-2}\right]dz$$

$\because 1 = A(z-2) + B(z-1)$

Put, $z = 2 \Rightarrow 1 = B(2-1) = B$

$\Rightarrow B = 1$

and put, $z = 1 \Rightarrow 1 = A(1-2)$

$\Rightarrow 1 = A(-1) \Rightarrow A = -1$

$$\therefore \quad = \oint_C e^{2z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right)dz$$

$$= \underbrace{\oint_C \frac{e^{2z}}{z-2}\,dz}_{\text{Here } a = 2} - \underbrace{\oint_C \frac{e^{2z}}{(z-1)}\,dz}_{\text{Here, } a = 1}$$

➔ **Using Cauchy's Integral formula,**

$$\ldots\left[\oint_C \frac{f(z)}{(z-a)}\,dz = 2\pi i\,[f(z)]_{z=a}\right]$$

$$= 2\pi i\,[e^{2z}]_{z=2} - 2\pi i\,[e^{2z}]_{z=1}$$

$$= 2\pi i\,[e^4] - 2\pi i\,e^2$$

$$= 2\pi i\,e^4 - 2\pi i\,e^2 = 2\pi i\,e^2(e^2 - 1)$$

Step IV :

(ii) The function $\frac{\sin^2 z}{(z-\pi/6)^3}$ is analytic everywhere except at $z = \pi/6$.

i.e., $z = \pi/6$ is a singularity of order 3.

For given curve C : $|z| = 1$

$z = \pi/6 \Rightarrow |z| = |\pi/6| = \frac{\pi}{6} = (0.5) < 1$

this shows that, $z = \frac{\pi}{6}$ lies inside the given circle.

Step V :

➔ **Using Cauchy's Integral formula,**

$$\ldots\left[\oint_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}\cdot f(z)\right]_{z=a}\right]$$

Here, $f(z) = \sin^2 z$; $a = \pi/6$ and $n = 3$

$$\frac{d}{dz}f(z) = 2\sin z\cos z = \sin 2z$$

and $$\frac{d^2 f(z)}{dz^2} = 2\cos 2z$$

We get,

$$\oint_C \frac{\sin^2 z}{\left(z - \frac{\pi}{6}\right)^3}\,dz = \frac{2\pi i}{(3-1)!}\left[\frac{d^2}{dz^2}\sin^2 z\right]_{z=\pi/6}$$

$$= \frac{2\pi i}{2!}\left[\frac{d}{dz}(2\sin z\cos z)\right]_{z=\pi/6} \quad \ldots(\because 2! = 2)$$

$$= \pi i\left[\frac{d}{dz}(\sin 2z)\right]_{z=\pi/6} = \pi i\,[2\cos 2z]_{z=\pi/6}$$

$$= \pi i\left[2\cos\left(2\frac{\pi}{6}\right)\right] = 2\pi i\underbrace{\left[\cos\frac{\pi}{3}\right]}_{1/2} = 2\pi i \times \frac{1}{2}$$

$$\oint_C \frac{\sin^2 z}{\left(z - \frac{\pi}{6}\right)^3}\,dz = \pi i \checkmark \qquad \text{...Ans.}$$

Ex. 6.4.7 : Evaluate : $\oint_C |z|\,dz$; over circle $|z + 1| = 1$

Soln. :

Step I :

Given $|z + 1| = 1$

Means a circle of radius $r = 1$.

$(z + 1)$ is a complex number $\therefore z + 1 = re^{i\theta} = |e^{i\theta}|$

$\Rightarrow z = e^{i\theta} - 1 \Rightarrow dz = ie^{i\theta}\,d\theta$

Step II :

Also, if $z + 1 = e^{i\theta}$

$z = e^{i\theta} - 1 = \cos\theta + i\sin\theta - 1$

$$z = \underbrace{(\cos\theta - 1)}_{x} + i\underbrace{\sin\theta}_{y}$$

$$|z| = \sqrt{x^2 + y^2}$$

$$\therefore |z| = \sqrt{(\cos\theta - 1)^2 + \sin^2\theta}$$

$$= \sqrt{\cos^2\theta + \sin^2\theta - 2\cos\theta + 1}$$

$$= \sqrt{1 - 2\cos\theta + 1} = \sqrt{2 - 2\cos\theta}$$

$$= \sqrt{2(1 - \cos\theta)}$$

$$= \sqrt{2\cdot 2\sin^2\frac{\theta}{2}} \quad \ldots\left(\because 1 - \cos\theta = 2\sin^2\frac{\theta}{2}\right)$$

$$|z| = 2\sin\frac{\theta}{2}$$

Limits are θ = 0 to 2π

Step IV : $$\therefore \oint_C |z|\,dz = \int_0^{2\pi} 2\sin\frac{\theta}{2}\,i\,e^{i\theta}\,d\theta$$

$$= 2i\int_0^{2\pi} e^{i\theta}\cdot\sin\frac{\theta}{2}\,d\theta$$

➔ **Using Standard result**

$$\ldots\left[\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx)\right]$$

Here, $a = i$ and $b = \frac{1}{2}$

$$= 2i\left[\frac{e^{i\theta}}{\left(-1 + \frac{1}{4}\right)}\left\{i\sin\frac{\theta}{2}\right\} - \frac{1}{2}\cos\frac{\theta}{2}\right]_0^{2\pi}$$

$$= 2i\left[\frac{e^{i2\pi}}{-3/4}\left(\underbrace{i\sin\frac{2\pi}{2}}_{(0)} - \frac{1}{2}\underbrace{\cos\frac{2\pi}{2}}_{(-1)}\right) - \frac{e^0}{-3/4}\left(\underbrace{i\sin\frac{0}{2}}_{(0)} - \frac{1}{2}\underbrace{\cos\frac{0}{2}}_{(1)}\right)\right]$$

$$= 2i\left[\frac{-4}{3}e^{i2\pi}\left(0 + \frac{1}{2}\right) + \frac{4}{3}\left(0 - \frac{1}{2}\right)\right]$$

$$= 2i\left[-\frac{4}{6} - \frac{4}{6}\right] = 2i\left[-2 \times \frac{4}{6}\right]$$

$$= 2i\,[-4/3]$$

$(\because e^0 = 1$ and $e^{i2\pi} = \underbrace{\cos 2\pi}_{(1)} + i\underbrace{\sin 2\pi}_{(0)})$

$$\oint_C |z|\,dz = -\frac{8}{3}i \checkmark \qquad \text{...Ans.}$$

Ex. 6.4.8 : Evaluate : $\oint_C \frac{e^{-2z}}{(z-i)^2}\,dz$,

where C is the square whose vertices are $(-2, 2), (-2, -2), (2, -2)$ and $(2, 2)$

Soln. :

Step I : Given, A (– 2, 2), B (– 2, – 2), C(2, – 2), D (2, 2) are the vertices of the square.

Step II : The given function $\frac{e^{-2z}}{(z-i)^2}$ is analytic everywhere except at z = i i.e. z = i is a singularity.Since, all points are at distance 2 from origin. So, the point i is inside the bounded region.

$(\therefore\ z = i \Rightarrow |z| = |i| = 1 < 2)$

Step III :

➔ **Using Cauchy Integral formula,**

$$\ldots\left[\oint_C \frac{f(z)}{(z-a)^n}dz = \frac{2\pi i}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}\cdot f(z)\right]_{z=a}\right]$$

Here $f(z) = e^{-2z}$; $a = i$ and $n = 2$

$$\frac{d}{dz}f(z) = \frac{d}{dz}(e^{-2z}) = (-2)\,e^{-2z}$$

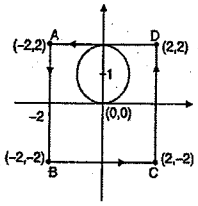


Fig. P. 6.4.8

Step IV :

$$\therefore \oint_C \frac{e^{-2z}}{(z-i)^2}dz = \frac{2\pi i}{(2-1)!}\left[\frac{d}{dz}e^{-2z}\right]_{z=i} = \frac{2\pi i}{1!}\left[-2e^{-2z}\right]_{z=i}$$

$$= \frac{2\pi i}{1!}\left[-2e^{-2i}\right]$$

$$\therefore \oint_C \frac{e^{-2z}}{(z-i)^2}dz = -4\pi i\ [\cos 2 - i\sin 2]\ ✓ \qquad \text{...Ans.}$$

$$\ldots(\because e^{-i\theta} = \cos\theta - i\sin\theta)$$

Ex. 6.4.9 : Evaluate : $\oint_C \frac{e^{2z}}{(z+1)^4}dz$, where, C : |z| = 2

Soln. :

Step I : The given function $\frac{e^{2z}}{(z+1)^4}$ is analytic everywhere except at z = – 1.

$\because (z+1)^4 = 0 \Rightarrow z = -1$

$\therefore$ z = – 1 is a singularity of order n = 4

Since, |z| = 2

$\therefore$ |z| = |–1| = 1 < 2

z = – 1 lies inside the given circle

Step II :

➔ **Using Cauchy's integral formula**

$$\ldots\left[\oint_C \frac{f(z)}{(z-a)^n}dz = \frac{2\pi i}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}\cdot f(z)\right]_{z=a}\right]$$

Here, n = 4; a = –1 ;

$\therefore f(z) = e^{2z}$; $f'(z) = 2e^{2z}$;

$f''(z) = 4e^{2z}$; $f'''(z) = 8e^{2z}$

and $f'''(z=-1) = 8e^{-2}$

$$\therefore \oint_C \frac{e^{2z}}{(z+1)^4}dz = \frac{2\pi i}{(4-1)!}\left[\frac{d^3}{dz^3}\cdot e^{2z}\right]_{z=-1}$$

$$= \frac{2\pi i}{3!}\left[8e^{2z}\right]_{z=-1}$$

$$= \frac{2\pi i}{3!}\cdot(8e^{-2}) \qquad \ldots(3! = 3\times2\times1 = 6)$$

$$\therefore \oint_C \frac{e^{2z}}{(z+1)^4}dz = \frac{8}{3}\,i\,e^{-2}\ ✓ \qquad \text{...Ans.}$$

Ex. 6.4.10 : Evaluate : $\oint_C \frac{\cos\pi z^2}{(z-1)(z-2)}dz$, where,

C : |z| = 3

Soln. :

Step I : The given function $\frac{\cos\pi z^2}{(z-1)(z-2)}$ is analytic everywhere except at z = 1, z = 2

$(\because z-1 = 0$ and $z-2 = 0)$

$\therefore$ z = 1, 2 are the singularities of order n = 1

Step II :

Given, C : |z| = 3

(i) For z = 1 ⇒ |z| = |1| = 1 < 3

(ii) For z = 2 ⇒ |z| = |2| = 2 < 3

Both the points z = 1, z = 2 are lies inside the circle C : | z | = 3

Step III :

➔ **Using Standard formula**

$$\ldots\left[\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left[\frac{1}{z-a} - \frac{1}{z-b}\right]\right]$$

$$\therefore \frac{1}{(z-1)(z-2)} = \underbrace{\frac{1}{(1-2)}}_{(-1)}\left[\frac{1}{z-1} - \frac{1}{z-2}\right]$$

$$= \left(\frac{1}{z-2}\right) - \left(\frac{1}{z-1}\right)$$

Step IV :

➔ **Using Cauchy's integral formula,**

$$\ldots\left[\oint_C \frac{f(z)}{(z-a)}dz = 2\pi i\,[f(z)]_{z=a}\right]$$

$$\therefore \oint_C \frac{\cos\pi z^2}{(z-1)(z-2)}dz = \oint_C \cos\pi z^2\left(\frac{1}{z-2} - \frac{1}{z-1}\right)dz$$

$$= \oint_C \frac{\cos\pi z^2}{(z-2)}dz - \oint_C \frac{\cos\pi z^2}{(z-1)}dz$$

$$= 2\pi i\,[\cos\pi z^2]_{z=2} - 2\pi i\,[\cos\pi z^2]_{z=1}$$

$$= 2\pi i\underbrace{\cos 4\pi}_{(1)} - 2\pi i\underbrace{\cos\pi}_{(-1)} = 2\pi i\,[1+1]$$

$$\therefore \oint_C \frac{\cos\pi z^2}{(z-1)(z-2)}dz = 4\pi i ✓ \qquad \text{...Ans.}$$

Ex. 6.4.11 : Evaluate : $\oint_C \frac{z}{(z^2+1)}dz$, where

(I) C : $\left|z + \frac{1}{z}\right| = 2$ (II) C : | z + i | = 1

Soln. :

Step I : The given function $\left(\frac{z}{z^2+1}\right)$ is analytic everywhere except at $z^2 = -1 \Rightarrow z = \pm i$.

$\therefore$ z = ± i are the singularities of order 1.

Step II :

(I) Given circle C : $\left|z + \frac{1}{z}\right| = 2$

For z = i ⇒ $\left|z + \frac{1}{z}\right| = \left|i + \frac{1}{i}\right| = |i - i| = |0| = 0 < 2$

And For z = – i ⇒ $\left|z + \frac{1}{z}\right| = \left|-i - \frac{1}{i}\right| = |i - i|$

= |0| = 0 < 2

This shows that both points lies inside the given circle.

$$\therefore \oint_C \frac{z}{z^2+1}dz = \oint_C \frac{z}{(z+i)(z-i)}dz$$

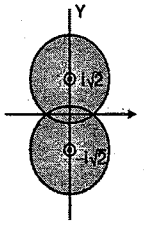


Fig. P. 6.4.11

Step III :

➔ **Using Cauchy's integral formula,**

$$\left[\oint_C \frac{f(z)}{z-a}dz = 2\pi i\,[f(z)]_{z=a}\right]$$

$$\oint_C \frac{z}{z^2+1}dz = \oint_{C_1} \frac{\left(\frac{z}{z+i}\right)}{(z-i)}dz + \oint_{C_2} \frac{\frac{z}{(z-i)}}{(z+i)}dz$$

$f(z) = \frac{z}{z+i}$	$f(z) = \frac{z}{z+i}$
and a = –1	and a = –1

$$= 2\pi i\left[\frac{z}{z+i}\right]_{z=i} + 2\pi i\left[\frac{z}{z-i}\right]_{z=-i}$$

$$= 2\pi i\left[\frac{i}{i+i} + \frac{-i}{-i-i}\right] = 2\pi i\left[\frac{1}{2} + \frac{1}{2}\right] = 2\pi i$$

Step IV :

(II) Given circle C : | z + i | = 1

For z = i ⇒ |z + i| = |i + i| = |2i| = 2 > 1

and for z = – i, | z + i | = | – i + i | = 0 < 1

The point z = i lies outside and the point z = – i lies inside

Step V :

Since, $\oint_C \frac{z}{z^2+1}dz = \oint_C \frac{z}{(z+i)(z-i)}dz$

$$= \underbrace{\oint_C \frac{\left(\frac{z}{z-i}\right)}{(z+i)}dz}_{\text{Cauchy's integral Formula}} + \underbrace{\oint_C \frac{\left(\frac{z}{z+i}\right)}{(z-i)}dz}_{\text{Cauchy's integral theorem}}$$

$$\oint_C \frac{f(z)}{z-a}dz = 2\pi i\,f(a) \qquad \oint_C f(z)\,dz = 0$$

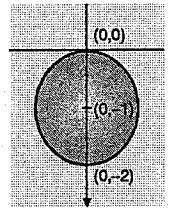


Fig. P. 6.4.11(a)

$$\therefore \oint_C \frac{z}{z^2+1}dz = 2\pi i\left[\frac{z}{z-i}\right]_{z=-i} + \underbrace{0}$$

Point z = i lies outside

$$= 2\pi i\left[\frac{z}{z-i}\right]_{z=-i} = 2\pi i\left[\frac{-i}{-i-i}\right]$$

$$= 2\pi i\left(\frac{-i}{-2i}\right)$$

$$\therefore \oint_C \frac{z}{z^2+1}dz = \pi i ✓ \qquad \text{...Ans.}$$

Ex. 6.4.12 : Evaluate : $\oint_C \frac{z^2+1}{z^2-1}dz$, where C is the circle :

|z – 1| = 1

Soln. :

Step I : The given function, $\left(\frac{z^2+1}{z^2-1}\right)$ is an analytic function everywhere except at z = 1, – 1

Therefore, z = 1 , – 1 are singularities of order 1

Step II :

Given curve, c : | z – 1 | = 1 :

(i) For z = 1, | z – 1 | = | 1 – 1 | = 0 < 1

This shows that, z = 1 lies inside the given circle.

and

(ii) For z = – 1 ⇒ | z – 1 | = | – 1 – 1 | = | – 2 | = 2 > 1

This shows that, $z=-1$ lies outside the given circle

Step III :

Hence

$$\oint_C \left(\frac{z^2+1}{z^2-1}\right) dz = \oint_C \frac{z^2+1}{(z-1)(z+1)} dz$$

➔ **Using standard result**

$$\cdots\left[\frac{1}{(z-a)(z-b)} = \frac{1}{(a-b)}\left[\frac{1}{z-a} - \frac{1}{z-b}\right]\right]$$

$$= \oint_C (z^2+1)\cdot\frac{1}{(1-(-1))}\left[\frac{1}{z-1} - \frac{1}{z+1}\right] dz$$

$$= \oint_C (z^2+1)\frac{1}{(2)}\left[\frac{1}{z-1} - \frac{1}{z+1}\right] dz$$

$$= \frac{1}{2}\oint_C \left[\frac{z^2+1}{z-1} - \frac{z^2+1}{z+1}\right] dz$$

$$= \underbrace{\frac{1}{2}\oint \frac{z^2+1}{z-1} dz}_{\text{By Cauchy's Integral formula}} - \underbrace{\frac{1}{2}\oint \frac{z^2+1}{z+1} dz}_{\text{By Cauchy's Integral theorem}}$$

By Cauchy's Integral formula : $\oint_C \frac{f(z)}{z-a} dz = 2\pi i\,[f(z)]_{z=a}$

By Cauchy's Integral theorem : $\oint_C f(z)\,dz = 0$

$$= \frac{1}{2}\cdot 2\pi i\,[z^2+1]_{z=} - \frac{1}{2}(0) = \pi i\,[z^2+1]_{z=1} - 0 = \pi i\,[1+1]$$

$$= \pi i(2)$$

$$\oint_C \left(\frac{z^2+1}{z^2-1}\right) dz = 2\pi i \checkmark \qquad \text{...Ans.}$$

Ex. 6.4.13 : By using Cauchy's integral formula, evaluate

$$\oint_C \left(\frac{2z^2+z}{z^2-1}\right) dz, \text{ where, C is the circle } |z-1| = 1.$$

Soln. :

Step I : Given, $f(z) = \frac{2z^2+z}{z^2-1} = \frac{2z^2+z}{(z-1)(z+1)}$ is analytic function everywhere except at $z=1, -1$

$z=1, -1$ are the singularities of order 1

Step II :

Given, $C: |z-1| = 1$

Now, $z = 1$,

$\Rightarrow |z-1| = |1-1| = |0| = 0 < 1$

and, $z = -1 \Rightarrow |-1-1| = |-2| = 2 > 1$

Hence, $z = 1$ point lies inside the circle and $z=-1$ lies outside the circle.

Step III :

➔ **Using Cauchy's integral formula**

$$\cdots\left[\oint_C \frac{f(z)}{z-a} dz = 2\pi i\,[f(z)]_{z=a}\right]$$

$$\oint_C \frac{2z^2+z}{z^2-1} dz = \oint_C \frac{2z^2+z}{(z-1)(z+1)} dz = \oint_C \frac{\left[\frac{2z^2+z}{(z+1)}\right]}{(z-1)} dz$$

($\because$ The point which lies outside, the value of integration is equal to zero)

$$= 2\pi i\left[\frac{2z^2+z}{z+1}\right]_{z=1} = \cancel{2}\pi i\left(\frac{2+1}{\cancel{2}}\right)$$

$$\oint_C \frac{2z^2+z}{z^2-1} dz = 3\pi i \checkmark \qquad \text{...Ans.}$$

Ex. 6.4.14 : Using Cauchy's Integral formula, Evaluate :

$$\int_C \frac{(z+4)}{(z^2+2z+5)} dz, \text{ where, } C: |z+1-i| = 2.$$

Soln. :

Step I : Given function,

$$f(z) = \frac{z+4}{z^2+2z+5} = \frac{z+4}{z^2+2z+1+4}$$

$$= \frac{z+4}{(z+1)^2+2^2} = \frac{(z+4)}{(z+1+2i)(z+1-2i)}$$

is an analytic function everywhere except at.

$z = -(1+2i)$; $z = -(1-2i)$

These are the singularities of order 1 (simple poles)

Step II :

Given, circle $C: |z+1-i| = 2$

(i) For $z = -(1+2i)$, we have

$|z+1-i| = |-1-2i+1-i| = |-3i| = 3 > 2$

Hence, the point $z = -(1+2i)$ lies outside the circle.

(ii) For $z = -(1-2i)$, we heve

$|z+1-i| = |-1+2i+1-i| = |i| = 1 < 2$

Hence, the point, $z = -(1-2i)$ lies inside the circle.

(If singular point lies outside then the value of integration is equal to zero.)

➔ **Using Cauchy's integral formula**

$$\cdots\left[\int \frac{f(z)}{z-a} dz = 2\pi i\,[f(z)]_{z=a}\right]$$

Step III : $\therefore \oint_C \frac{z+4}{z^2+2z+5} dz = \oint_C \frac{\left(\frac{z+4}{z+1+2i}\right)}{(z+1-2i)} dz$

$$= 2\pi i\left[\frac{z+4}{z+1+2i}\right]_{z=-1+2i}$$

$$= 2\pi i\left[\frac{-1+2i+4}{-1+2i+1+2i}\right]$$

$$= 2\pi i\left[\frac{2i+3}{4i}\right]$$

$$= 2\pi\left[\frac{3+2i}{4}\right]$$

$$\therefore \oint_C \frac{z+4}{z^2+2z+5} dz = \frac{\pi}{2}(3+2i) \checkmark \qquad \text{...Ans.}$$

Ex. 6.4.15 : Evaluate : $\oint_C \frac{4z^2+z}{z^2-1} dz$, where $C: |z-1| = 3$

Soln. :

Step I : Given, $f(z) = \frac{4z^2+z}{z^2-1} = \frac{4z^2+z}{(z-1)(z+1)}$ is analytic everwhere except at $z = 1, -1$

Therefore singularities at $z=1$, $z=-1$ of order 1

Step II :

(i) For $z = 1$; $C: |z-1| = 3$

$\Rightarrow |1-1| = 0 < 3$

$\therefore z = 1$ lies inside the circle

(ii) For $z = -1$; Since $C: |z-1| = 3$

$\Rightarrow |-1-1| = |2| = 2 < 3$

$z = -1$ also lies inside the circle

Step III :

➔ **Using Cauchy's Integral formula**

$$\cdots\left[\oint_C \frac{f(z)}{z-a} dz = 2\pi i\,[f(z)]_{z=a}\right]$$

$$\oint_C \frac{4z^2+z}{z^2-1} dz = \oint_C \frac{\left(\frac{4z^2+z}{z-1}\right)}{z+1} dz + \oint_C \frac{\left(\frac{4z^2+z}{z+1}\right)}{z-1} dz$$

$$= 2\pi i\left[\left(\frac{4z^2+z}{z-1}\right)_{z=-1} + \left(\frac{4z^2+z}{z+1}\right)_{z=1}\right]$$

$$= 2\pi i\left[\frac{4-1}{-1-1}\right] + 2\pi i\left[\frac{4+1}{1+1}\right]$$

$$= 2\pi i\left[\frac{3}{-2} + \frac{5}{2}\right]$$

$$\oint_C \frac{4z^2+z}{z^2-1} dz = 2\pi i\left[\frac{2}{2}\right] = 2\pi i \checkmark \qquad \text{...Ans.}$$

Ex. 6.4.16 : Evaluate : $\oint_C \frac{\sin 2z}{(z+\pi/3)^4} dz$, where $C: |z| = 2$

Soln. :

Step I : The given function $\frac{\sin 2z}{(z+\pi/3)^4}$ is analytic function every where expect $z = -\pi/3$

Hence, $z = -\pi/3$ is singularity of order 4

Step II : Now, $z = -\pi/3$ for $C: |z| = 2$

$\Rightarrow |z| = |\pi/3| = |\pi/3| < 2$

Hence, $z = -\pi/3$ lies inside the circle.

Step III :

➔ **Using Cauchy's integral formula**

$$\cdots\left[\oint_C \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}\cdot f(z)\right]_{z=a}\right]$$

Here, $f(z) = \sin 2z$; $a = -\pi/3$, $n = 4$

$$\oint_C \frac{\sin 2z}{(z+\pi/3)^4} dz = 2\pi i\,\frac{1}{(4-1)!}\frac{d^3}{dz^3}[f(z)]_{z=-\pi/3}$$

$$= \frac{2\pi i}{6}[f'''(z)]_{z=-\pi/3} \qquad \text{...(1)}$$

Here, $f(z) = \sin 2z$

$f'(z) = 2\cos 2z$; $f''(z) = -4\sin 2z$;

$f'''(z) = -8\cos 2z$

Step V :

$$\therefore \oint \frac{\sin 2z}{(z+\pi/3)^4} dz = \frac{\pi i}{3}[-8\cos 2z]_{z=-\pi/3}$$

$$= \frac{-8\pi i}{3}[\cos(-2\pi/3)] = \frac{\pi}{3}8i\left[\frac{1}{2}\right]$$

$$\therefore \oint \frac{\sin 2z}{(z+\pi/3)^4} dz = \frac{4\pi}{3} i \checkmark \quad \text{...Ans.} \quad \left(\because \cos\left(\frac{2\pi}{3}\right) = \frac{-1}{2}\right)$$

Ex. 6.4.17 : Evaluate : $\oint_C \frac{z+2}{(z+1)^2(z-2)} dz$, where C is the circle $|z-i| = 2$.

Soln. :

Step I : The given function, $f(z) = \frac{z+2}{(z+1)^2(z-2)}$ is analytic everywhere except at $z = -1, 2$

$\therefore z = -1$ is a singularity of order $n = 2$ and $z = 2$ is a singularityof order one

Step II :

(i) for $z = -1$

Given curve $C: |z-i| = 2$

$\therefore |-1-i| = \sqrt{1+1} = \sqrt{2} < 2$

Therefore, $z = -1$ lies inside the given curve.

(ii) For $z = 2$: Given $C: |z-i| = 2$

$\therefore |z-i| = |2-i| = \sqrt{4+1} = \sqrt{5} > 2$

Therefore, $z = 2$ lies outside the given curve.

Step III :

$$\therefore \oint_C \frac{z+2}{(z+1)^2(z-2)}\,dz$$

$$= \underbrace{\oint_C \frac{\left(\frac{z+2}{z-2}\right)}{(z+1)^2}\,dz}_{\text{By Cauchy's Integral formula}} + \underbrace{\oint_C \frac{\left[\frac{z+2}{(z+1)^2}\right]}{(z-2)}\,dz}_{\text{By Cauchy's Integral Therom}}$$

By Cauchy's Integral formula :

$$\oint_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}\cdot f(z)\right]_{z=a}$$

By Cauchy's Integral Therom :

$$\oint_C f(z)\,dz = 0$$

$$= \frac{2\pi i}{(2-1)!}\left[\frac{d}{dz}\left(\frac{z+2}{z-2}\right)\right]_{z=-1} + 0$$

➔ **Using standard result :**

$$\ldots\left[\; d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}\right]$$

$$= 2\pi i\left[\frac{(z-2)\cdot 1-(z+2)\cdot 1}{(z-2)^2}\right]_{z=-1}$$

$$\oint_C \frac{z+2}{(z+1)^2(z-2)}\,dz = 2\pi i\left[\frac{-4}{9}\right] = \frac{-8\pi i}{9} \checkmark \qquad \text{...Ans.}$$

Ex. 6.4.18 : Evaluate : $\oint_C \frac{e^{-2z}}{(z-i)^2}\,dz$,

where C is the circle $|z+i| = 1$

Soln. :

Step I : The given function $\frac{e^{-2z}}{(z-i)^2}$ is an analytic function everywhere except at $z = i$.

Here, $z = i$ is a singularity of order $n = 2$

Step II : Given, $C : |z+i| = 1$

Put $z = i \Rightarrow |i+i| = |2i| = 2 > 1$

The point lies outside the circle

Step III :

$\therefore$ By Cauchy's Integral theorem, $\oint_C f(z)\,dz = 0$

$$\therefore \oint_C \frac{e^{-2z}}{(z-i)^2}\,dz = 0 \checkmark \qquad \text{...Ans.}$$

Ex. 6.4.19 : Evaluate : $\oint_C \frac{\cos \pi z}{z(z+2)}\,dz$,

where $C : |z+3-i| = 3$

Soln. :

Step I : The given function f(z) is analytic everywhere except at $z = 0$; $z = -2$.

The given function has singularity at $z = 0$ and $z = -2$ of order 1

Step II :

Now,

(i) for $z = 0$

$C : |z+3-i| = 3$

Put $z = 0$; $|0+3-i| = \sqrt{9+1} = \sqrt{10} > 3$

The point $z = 0$ lies outside the circle

and

(ii) for $z = -2$; $C : |z+3-i| = 3$

put $z = -2 \Rightarrow |-2+3-i| = |1-i| = \sqrt{2} < 3$

The point $z = -2$ lies inside the circle

Step III :

$$\therefore \oint_C \frac{\cos \pi z}{z(z+2)}\,dz = \underbrace{\oint_C \frac{\left(\frac{\cos \pi z}{z}\right)}{(z+2)}\,dz}_{\text{By Cauchy's Integral formula}} + \underbrace{\oint_C \frac{\left(\frac{\cos \pi z}{z+2}\right)}{z}\,dz}_{\text{By Cauchy's Integral Therom}}$$

By Cauchy's Integral formula

$$\oint_C \frac{f(z)}{(z-a)^n}\,dz = 2\pi i\,[f(z)]_{z=a}$$

By Cauchy's Integral Therom

$$\oint_C f(z)\,dz = 0$$

Step IV :

$$= 2\pi i\left(\frac{\cos \pi z}{z}\right)_{z=-2} + 0$$

$$= 2\pi i\left(\frac{\cos(-2\pi)}{-2}\right) = 2\pi i\left(\frac{\cos(2\pi)}{-2}\right)$$

($\because \cos z$ is an even function)

$$\therefore \oint_C \frac{\cos \pi z}{z(z+2)}\,dz = 2\pi i\left(\frac{1}{-2}\right) = -\pi i \checkmark \quad \text{...Ans. } (\because \cos 2\pi = 1)$$

Ex. 6.4.20 : Evaluate : $\oint_C \frac{z-3}{z^2+2z+5}\,dz$,

where $C : |z+1+i| = 2$

Soln. :

Step I : The given function $\left(\frac{z-3}{z^2+2z+5}\right)$ is analytic everywhere except at $z^2+2z+5 = 0$

$\therefore z^2+2z+1+4 = 0$

$\Rightarrow (z+1)^2 + (2)^2 = 0 \Rightarrow z+1 = 2i$ and $z+1 = -2i$

i.e. at $z = (-1+2i)$ and $(-1-2i)$ are the singularity of order 1

Step II : $z = (-1+2i)$ and $(-1-2i)$ are simple poles.

(i) for $z = -1+2i$

$C : |z+1+i| = 2$

Put $z = -1+2i$

$\therefore |-1+2i+1+i| = |3i|$

$= 3 > 2$

The point lies outside the circle

(ii) for $z = -1-2i$

$C : |z+1+i| = 2$

Put, $z = -1-2i$

$\therefore |-1-2i+1+i| = |-i|$

$= 1 < 2$

$\therefore$ The point lies inside the circle,

Step III :

$$\therefore \oint_C \frac{z-3}{z^2+2z+5}\,dz = \oint_C \frac{z-3}{(z+1-2i)(z+1+2i)}\,dz$$

$$= \underbrace{\oint_C \frac{\left(\frac{z-3}{z+1-2i}\right)}{(z+1+2i)}\,dz}_{\text{By Cauchy's Integral formula}} + \underbrace{\oint_C \frac{\left(\frac{z-3}{z+1+2i}\right)}{(z+1-2i)}\,dz}_{\text{By Cauchy's Integral Theorem}}$$

By Cauchy's Integral formula

$$\oint_C \frac{f(z)}{(z-a)^n}\,dz = 2\pi i\,[f(z)]_{z=a}$$

By Cauchy's Integral Theorem

$$\oint_C f(z)\,dz = 0$$

Step IV :

$$= 2\pi i\left(\frac{z-3}{z+1-2i}\right)_{z=(-1-2i)} + 0$$

$$= 2\pi i\left[\frac{-1-2i-3}{-\cancel{1}-2i+\cancel{1}-2i}\right] = 2\pi i\left(\frac{-4-2i}{-4i}\right)$$

$$\oint_C \frac{z-3}{z^2+2z+5}\,dz = \pi(2+i) \checkmark \qquad \text{...Ans.}$$

Ex. 6.4.21 : Evaluate : $\oint_C \frac{\sin\left(\frac{\pi}{2}z\right)}{(z-i-1)(z^2-1)}\,dz$;

where C : is a rectangle with vertices at $-i, 2+2i, 2i, 2-i$

Soln. :

Step I : Here, the given function is analytic everywhere except $z = i+1, \pm 1$.

These are singularity of order 1

Step II :

1. For $z = i+1$

C : is a rectangle having vertex at $(-i,\ 2+2i,\ 2i,\ 2-i)$

It is observed that,

The poles $z = i+1, 1$ lies inside the rectangle but the pole $z = -1$ lies outside the rectangle.

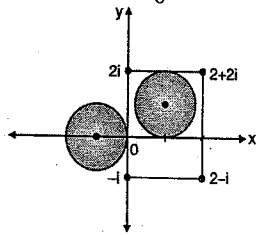


Fig. P. 6.4.21

Step III :

$$\therefore \oint_C \frac{\sin\frac{\pi}{2}z}{(z-i-1)(z+1)(z-1)}\,dz$$

➔ **Using Cauchy's integral formula**

$$\ldots\left[\oint \frac{f(z)}{z-a^n}\,dz = 2\pi i\,[f(z)]_{z=a} \text{ and } \oint f(z)\,dz = 0\right]$$

Use Cauchy's formula for first two terms and Cauchy's theorem for third term,

$$= \oint_C \frac{\left(\frac{\sin\frac{\pi}{2}z}{(z+1)(z-1)}\right)}{(z-i-1)}\,dz + \oint_C \frac{\left(\frac{\sin\frac{\pi}{2}z}{(z+1)(z-i-1)}\right)}{(z-1)}\,dz + \oint_C \frac{\left(\frac{\sin\frac{\pi}{2}z}{(z-1)(z-i-1)}\right)}{(z+1)}\,dz$$

$$= 2\pi i\left[\frac{\sin\frac{\pi}{2}z}{(z-1)(z+1)}\right]_{z=i+1} + 2\pi i\left[\frac{\sin\frac{\pi}{2}z}{(z+1)(z-i-1)}\right]_{z=1} + 0$$

Step IV :

$$= 2\pi I\left[\frac{\sin\frac{\pi}{2}(i+1)}{(i+1-1)(i+1+1)}\right] + 2\pi i\left[\frac{\sin\frac{\pi}{2}}{(1+1)(1-i-1)}\right]$$

$$= \frac{2\pi i}{(i)(i+2)}\left[\underbrace{\sin\frac{\pi}{2}}_{\left(i\sinh\frac{\pi}{2}\right)}\ \underbrace{i\cos\frac{\pi}{2}}_{(0)} + \underbrace{\cos\frac{\pi}{2}}_{\left(\cosh\frac{\pi}{2}\right)}\ \underbrace{i\sin\frac{\pi}{2}}_{(1)}\right] + \frac{2\pi i}{2(-i)}$$

$$\ldots\left(\because \sin\frac{\pi}{2} = 1\right)$$

$$= \frac{2\pi}{(i+2)}\left[0 + \cosh\frac{\pi}{2}\right] - \pi = \frac{2\pi(2-i)}{(2+i)(2-i)}\left(\cosh\frac{\pi}{2}\right)$$

$$= \frac{2\pi(2-i)}{(4+1)}\left[\cosh\frac{\pi}{2}\right] - \pi$$

$$\oint_C \frac{\sin\left(\frac{\pi}{2}z\right)}{(z-i-1)(z^2-1)}\,dz = \frac{1}{5}(4\pi - 2\pi i)\cosh\frac{\pi}{2} - \pi \checkmark \qquad \text{...Ans.}$$

Ex. 6.4.22 : Evaluate : $\oint_C \frac{2z^2+z+5}{\left(z-\frac{3}{2}\right)^2}\,dz$; where C is an ellipse : $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Soln. :

Step I : The given function is analytic everywhere except at $z = \frac{3}{2}$

$\therefore z = \frac{3}{2}$ is a singularity of order 2.

Given $C: \frac{x^2}{4} + \frac{y^2}{9} = 1$ with major axis $2a = 4$ and minor axis $2b = 6$

Hence, the point $z = 3/2$ lies inside.

Step III :

➔ **Using Cauchy's Integral formula**

$$\cdots\left[\oint_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}\cdot f(z)\right]_{z=a}\right]$$

$$\oint_C \frac{2z^2+z+5}{\left(z-\frac{3}{2}\right)^2}\,dz = \frac{2\pi i}{(2-1)!}\left[\frac{d}{dz}(2z^2+z+5)\right]_{z=3/2}$$

$$= \frac{2\pi i}{1}[4z+1]_{z=3/2} = 2\pi i\left[4\left(\frac{3}{2+1}\right)\right] = 14i$$

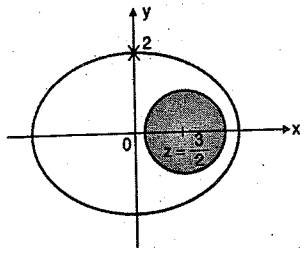


Fig. P. 6.4.22

Ex. 6.4.23 : Evaluate : $\oint_C \frac{(z^2+\cos^2 z)\,dz}{\left(z-\frac{\pi}{4}\right)^2}$;

where C is $|z| = 1$.

Soln. :

Step I : The given function is analytic everywhere except at $z = \frac{\pi}{4}$; Hence, $z = \frac{\pi}{4}$ is a singularity of order 2.

Step II : Given, $C: |z| = 1$;

for $z = \frac{\pi}{4}$

$\therefore |z| = \left|\frac{\pi}{4}\right| = \frac{3.142}{4} < 1$

$\therefore$ The point $z = \frac{\pi}{4}$ lies inside the circle, by Cauchy's Integral formula

$$\oint_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}\cdot f(z)\right]_{z=a}$$

Here, $f(z) = z^2 + \cos 2z$; $a = \pi/4, n = 2$

$$\oint_C \frac{(z^2+\cos^2 z)}{\left(z-\frac{\pi}{4}\right)^2}\,dz = \frac{2\pi i}{(2-1)!}\left[\frac{d}{dz}(z^2+\cos^2 z)\right]_{z=\pi/4}$$

$$= 2\pi i\,[2z - 2\cos z \sin z]_{z=\pi/4}$$

$$= 2\pi i\,[2z - \sin 2z]_{z=\pi/4}$$

$$= 2\pi i\left[2\frac{\pi}{4} - \underbrace{\sin 2\frac{\pi}{4}}_{(1)}\right]$$

$$\therefore \oint_C \frac{(z^2+\cos^2 z)}{\left(z-\frac{\pi}{4}\right)^2}\,dz = 2\pi i\left[2\frac{\pi}{4}-1\right] = 2\pi i\left[\frac{\pi}{2}-1\right] ✓ \quad \text{...Ans.}$$

Ex. 6.4.24 : Evaluate : $\oint_C \frac{e^z}{(z+1)(z+2)}\,dz$; where C is circle $|z+1| = \frac{1}{2}$

Soln. :

Step I : The given function is analytic everywhere except at $z = -1, -2$

$\therefore z = -1, -2$ are singularities of order 1.

Step II :

(i) for $z = -1$; $C: |z+1| = \frac{1}{2}$

$\Rightarrow |-1+1| = |0| = 0 < \frac{1}{2}$

$z = -1$ lies inside the curve

(ii) for $z = -2$

$C: |z+1| = \frac{1}{2}$

$|-2+1| = |-1| = 1 > \frac{1}{2}$

$z = -2$ lies outside the given curve

Step III :

$$\therefore \oint_C \frac{e^z}{(z+1)(z+2)}\,dz = \underbrace{\oint_C \frac{\left(\frac{e^z}{z+1}\right)}{(z+2)}\,dz}_{\text{By Cauchy's Integral Theorem}} + \underbrace{\oint_C \frac{\left(\frac{e^z}{z+2}\right)}{(z+1)}\,dz}_{\text{By Cauchy's Integral formula}}$$

By Cauchy's Integral Theorem: $\oint_C f(z)\,dz = 0$

By Cauchy's Integral formula: $\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\,[f(z)]_{z=a}$

$$= 0 + \left(\frac{e^z}{z+2}\right)_{z=-1} 2\pi i = 2\pi i\left(\frac{e^{-1}}{-1+2}\right)$$

$$\therefore \oint_C \frac{e^z}{(z+1)(z+2)}\,dz = \frac{2\pi i}{e} ✓ \quad \text{...Ans.}$$

Ex. 6.4.25 : Evaluate : $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$;

where $C: |z| = 3$

Soln. :

Step I : The given function $\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$ is analytic everywhere except at $z = 1, 2$.

$z = 1, 2$ are singularities of order 1

Step II : Given $C: |z| = 3$

(i) for $z = 1$
$C: |z| = 3$
$|1| = 1 < 3$

(ii) for $z = 2$
$C: |z| = 3$
$|2| = 2 < 3$

Step III : Here, both the points $z = 1, 2$ lies inside the circle.

➔ **Using Cauchy's integral formula,**

$$\cdots\left[\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\,[f(z)]_{z=a}\right]$$

$$\therefore \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$$

$$= \oint_C \frac{\left(\frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\right)}{(z-2)}\,dz + \oint_C \frac{\left(\frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\right)}{(z-1)}\,dz$$

$$= 2\pi i\left[\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)}\right]_{z=2} + 2\pi i\left[\frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)}\right]_{z=1}$$

$$= 2\pi i\left[\frac{\sin 4\pi + \cos 4\pi}{1}\right] + 2\pi i\left[\frac{\sin \pi + \cos \pi}{(-1)}\right]$$

...($\because \sin 4\pi = \sin \pi = 0$ and $\cos 4\pi = 1, \cos \pi = -1$)

$$= 2\pi i\,(0+1) + 2\pi i\left(\frac{0-1}{-1}\right) = (2\pi i + 2\pi i)$$

$$\therefore \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = 4\pi i ✓ \quad \text{...Ans.}$$

Exercise

Ex. 1 : Evaluate : $\oint_C \frac{dz}{z^3(z+4)}$; where C is the circle $|z| = 2$.

Ex. 2 : Evaluate : $\oint_C \frac{z^2}{\sin^3 z \cdot \cos z}\,dz$; where C is

(i) $C: |z| = 1$ (ii) $C: |z-3i| = 1$

Ex. 3 : Evaluate : $\oint_C (z^2+3z-2)\,dz$; where C is

(i) the circle $x^2 + y^2 = 1$

(ii) the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Ex. 4 : Evaluate : $\oint_C \frac{z-3}{(z^2+2z+5)}\,dz$;

where, C is $|z+1-i| = 2$

Ex. 5 : Evaluate : $\oint_C \frac{15z+9}{z^3-9z}\,dz$;

where, $C: |z-1| = 3$

Ex. 6 : Evaluate : $\oint_C \frac{2z^2+z}{(z^2+1)}\,dz$;

where, $C: |z-1| = 1$

Ex. 7 : Evaluate : $\oint_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)^2(z-2)}\,dz$;

where, $C: |z| = 3$.

Ex. 8 : Evaluate : $\oint_C \frac{e^{-3z}}{(z+1)^3}\,dz$;

where $C: |z| = 2, C: |z+i| = 1$

Ex. 9 : Evaluate : $\oint_C \frac{e^z}{(z-1)}\,dz$;

where $C: |z| = 2, C: |z| = \frac{1}{2}$

Ex. 10 : Evaluate : $\oint_C \frac{e^{3z}}{(z-\pi i)}\,dz$;

where $C: |z-2| + |z+2| = 6$

Ex. 11 : Evaluate : $\oint_C \frac{2z^2+z}{(z^2-1)}\,dz$;

where $C: |z-1| = 1$

Ex. 12 : Evaluate : $\int_0^{3+i} z^2\,dz$

(i) Along the line $x - 3y$

(ii) Along the real axis from 0 to 3 and then vertically to $3 + i$

(iii) Along the imaginary axis from 0 to i and then horizontally to $3 + i$

(iv) Along the parabola $x = 3y^2$

Examples on Cauchy 's Theorem

Ex. 1 : By using Cauchy's formula evaluate : $\oint_C \frac{e^z}{z^2+1}\,dz$, over $|z-1| = 1$

Ex. 2 : Evaluate : $\oint_C \frac{2z+2}{z^2+z+1}\,dz$,

where, $C: \left|\left(z+\frac{\sqrt{3}}{2}i\right)\right| = 1$

Ans. : $\frac{2\pi}{\sqrt{3}}(i\sqrt{3}-2)$

Ex. 3 : Evaluate : $\oint_C \frac{z^3-5}{(z+1)^2(z-1)}\,dz$;

where $c: |z| = 3$

Ans. : 0

Ex. 4: Evaluate : $\oint_C \frac{z^2+1}{z-2}\,dz$

where ; (i) C is the circle $|z-2|=1$

(ii) C is the circle $|z|=1$

Ans.: (i) $10\pi i$ (ii) 0

Ex. 5: Evaluate : $\oint_C \frac{4z^2+z}{z^2-1}\,dz$,

where, C ; $|z-1|=1/2$

Ans.: $5\pi i$

Ex. 6: Evaluate : $\oint_C \frac{e^z}{(z+1)^2(z-1)^2}\,dz$,

where C : $|z+1|=0.5$

Ans.: $\frac{9\pi}{8C}\,i$

Ex. 7: Evaluate : $\int_C \frac{e^{kz}}{z}\,dz$,

where K is real constant c : $|z|=1$

Ans.: π

Ex. 8: Evaluate : $\oint_C (z-z^2)\,dz$,

where C is lower half of the circle $|z|=1$

Ans.: $-2/3$

Ex. 9: Evaluate : $\oint_C \frac{z^2+4z}{(z-2)(z+|zi)}\,dz$,

where C : is (i) $|z-2|=2$ (ii) $|z+i|=2$

Ans.: (i) $\frac{12\pi i}{1+i}$ (ii) $\left(\frac{2+4i}{1+i}\right)$

Ex. 10: Evaluate : $\int_C \frac{\sinh z}{\left(Z-\frac{\pi}{2}\right)}\,dz$, where : c : $|z-3|=2$

Ans.: -2π

Ex. 11: Evaluate : $\oint_C \frac{\cos \pi z}{z^2-1}\,dz$;

around a rectangle with vertices $2 \pm i, -2 \pm i$

Ans.: 0

Ex. 12: Evaluate : $\oint_C \frac{e^{2z}}{z^2+1}\,dz$ where c : $|z|\,3$

Ex. 13: Evaluate : $\oint_C \frac{z^3+z+1}{z^2-7z+2}\,dz$,

where c is the ellipse $4x^2+9y^2=1$

Ans.: 0

Ex. 14: Evaluate : $\oint_C \frac{e^{3iz}}{(z+\pi)^3}\,dz$,

where c ; $|z-\pi|=3$

Ans.: 0

Ex. 15: Evaluate : $\oint_C \frac{z}{z^2-3z+2}\,dz$;

where c ; $|z-2|=1/2$

Ans.: $4\pi i$

Ex. 16: Evaluate : $\oint_C \frac{e^{-z}}{(z-1)(z-2)^2}\,dz$,

where c : $|z|=3$

Ans.: $2\pi ie$

Ex. 17: Evaluate : $\oint_C \frac{z^2+1}{z(2z+1)}\,dz$,

where c : $|z|=1$

Ans.: $\frac{\pi}{2}$

Ex. 18: Evaluate : $\oint_C \frac{z^3}{(z-1)^4(z-2)(z-3)}\,dz$,

where c : $|z|=2.5$

Ans.: $\frac{-27}{8}\pi i$

Ex. 19: Evaluate : $\oint_C \frac{4z^2+2}{z^2-1}\,dz$, where c is $|z-1|=\frac{1}{2}$

Ex. 20: By using Cauchy's Integral Formula, Evaluate :

$$\oint_C \frac{4z^2+2}{z^2-1}\,dz, \text{ where c : } |z-1|=\frac{1}{2}$$

Ex. 21: Evaluate : $\oint_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz$, where c : $|z|=1$

Ans.: $\frac{21\pi i}{16}$

Ex. 22: Evaluate : $\oint_C \frac{3z^2+2}{(z-1)(z^2+9)}\,dz$,

where c : $|z-2|=2$

Ans.: πi

Ex. 23: Evaluate : $\oint_C \frac{4z^2-4z+1}{(z-2)(z+4)^2}\,dz$,

where : c : $|z|=1$

Ans.: 0

Ex. 24: Evaluate : $\oint_C \frac{e^{2z}\,dz}{(z+2)(z+4)(z+7)}\,dz$,

where c : $|z|=3$

Ans.: $\frac{\pi i\, e^{-4}}{5}$

Ex. 25: Evaluate : $\oint_C \frac{3z^2+z+1}{(z^2-1)(z+3)}\,dz$, where $|z|=2$

Ans.: $\frac{\pi i}{4}$

Ex. 26: Evaluate : $\oint_C \frac{z+4}{z^2+2z+5}\,dz$,

where c : $|z+1-i|=2$

Ans.: $\frac{\pi}{2}(3+2i)$

6.5 Taylors Series

The series of the form

$$a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots\ldots + a_n(z-a)^n + \ldots\ldots$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n$$

is called a power series in $(z-a)$

If the series is converges for all values of z such that $|z-a|<R$ then this circle is called of convergence and the radius is called radius of convergence.

Binomial series

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!}z^2 + \frac{n(n-1)(n-2)}{3!}$$

$$\cdot\, z^3 + \ldots\ldots \text{ valid for } |z|<1$$

$$(1-z)^{-1} = 1 + z + z^2 + z^3 + z^4 + \ldots\ldots$$

$$(1+z)^{-1} = 1 - z + z^2 - z^3 + z^4 - z^5 \ldots\ldots$$

Taylors series

If f(z) is analytic everywhere inside a circle C with centre at z = a and radius R, then at each point inside C is,

$$f(z) = f(a) + \frac{f'(a)(z-a)}{1!} + \frac{f''(a)}{2!}(z-a)^2$$

$$+ \frac{f'''(a)}{3!}(z-a)^3 + \ldots\ldots$$

i.e. power series converges to f (z), when $|z-a|<R$.

This is known as Taylors series of f (z) about z = a.

This expansion is valid at all points interior to any circle having its centre at 'a' and circle is known as region of convergence.

Put a = 0, we get

$$f(z) = f(0) + f'(a)\frac{z}{1!} + f''(0)\frac{z^2}{2!} + f'''(0)\frac{z^3}{3!} + \ldots\ldots$$

This is called maclaurins series.

Examples :

Ex. 6.5.1: Expand $\frac{z-1}{z+1}$ in Taylor's series about z = 1.

Soln.: We know,

$$f(z) = f(a) + \frac{(z-a)}{1!}f'(a) + \frac{(z-a)^2}{2!}f''(a)$$

$$+ \frac{(z-a)^3}{3!}f'''(a) + \ldots. \qquad \ldots(1)$$

$$|z-a|<R,\ |z|<1$$

$$f(z) = \frac{z-1}{z+1} \qquad \therefore\ f(1) = 0$$

$$f'(z) = \frac{2}{(z+1)^2}; \qquad f'(1) = 1/2$$

$$f''(z) = \frac{-4}{(z+1)^3} \qquad f''(1) = -1/2$$

$$f'''(z) = \frac{12}{(z+1)^4}; \qquad f'''(1) = 3/4$$

By Taylors series,

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(0)}{2!}(z-a)^2$$

$$+ \frac{f'''(a)(z-a)^3}{3!} + \ldots\ldots$$

a = 1

$$\frac{z-1}{z+1} = \frac{1}{z}(z-1) - \frac{1}{4}(z-1)^2 + \frac{3}{4}(z-1)^3 + \ldots..$$

$$\frac{z-1}{z+1} = \frac{z-1}{2} - \frac{(z-1)^2}{2^2} + \frac{(z-1)^3}{2^3} + \ldots\ldots \checkmark \qquad \text{...Ans.}$$

6.6 Laurent's Series

If f (z) is analytic inside R, then f (z) can always be represented by power series in the power of (z – a) i.e. about a point 'a'. But sometime we need to expand the given function about the point where function is not analytic. Taylors series fails in such case.

Statement

If f (z) is analytic inside and on the boundary of the ring shaped region R bounded by two concentric circles C_1 and C_2 with centre at 'a' and respective radii r_1 and r_2 ($r_1 > r_2$) then at each point z in R, f (z) can be represented by,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots\ldots$$

$$+ a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots..$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}$$

is called a power series in $(z-a)$

This series contains

1. Series consisting of positive power of $(z = z_0)$ is called analytic part / Taylors part of Laurent's series

2. Series consisting a negative powers of $(z = z_0)$ is called principle part of Laurent series.

where, $a_n = \frac{1}{2\pi i}\cdot\oint_C \frac{f(z)}{(z-a)^{n+1}}dz$;

$n = 0, 1, 2, \ldots\ldots$

$a_{-n} = \frac{1}{2\pi i}\cdot\oint_C \frac{f(z)}{(z-a)^{n+1}}dz$;

$n = 1, 2, 3, \ldots\ldots$

The part $\sum_{n=0}^{\infty} a_n (z-a)^n$ of Laurent series is called Taylors part and

The part $\sum_{n=1}^{\infty} a_{-n} (z-a)^n$ is called principle part of f (z).

In the Laurent series f (z) is not analytic inside C_1. If f (z) is analytic inside C_1 then $a_{-n} = 0$ and $a_n = \frac{f^n(a)}{n!}$ and the Laurent's series reduces to Taylor's series.

Ex. 6.6.1 : Find Laurent series $f(z) = \frac{1}{1-z^2}$ with centre at $z = 1$.

Soln. :

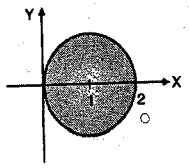
$$\frac{1}{1-z^2} = \frac{1}{(1-z)(1+z)} = -\frac{1}{2}\left(\frac{1}{z-1}\right) + \frac{1}{2}\left(\frac{1}{z+1}\right)$$

$$= \frac{1}{2(z-1)} + \frac{1}{2(z-1+2)}$$

$$= -\frac{1}{2(z-1)} + \frac{1}{4\left(1+\frac{z-1}{2}\right)}$$

$$= -\frac{1}{2(z-1)} + \frac{1}{4}\left[1 - \frac{z-1}{z} + \left(\frac{z-1}{z}\right)^2 - \left(\frac{z-1}{2}\right)^3 + \ldots\ldots\right]$$

Valid for $0 < |z-1| < 2$



$0 < |z-1| < 2$

Fig. P. 6.6.1

Ex. 6.6.2 : Find Laurent series for $\frac{z-1}{z^2}$ if $|z-1| > 1$.

Soln. :

$$f(z) = \frac{z-1}{z^2} = \frac{z-1}{[z-1+1]^2} \quad \text{for } |z-1| > 1$$

$$= \frac{z-1}{(z-1)^2\left[1+\frac{1}{z-1}\right]^2}$$

$$\ldots\left(\because 1 > \frac{1}{|z-1|}\right)$$

$$\frac{1}{\left[1+\frac{1}{z-1}\right]^2} = \frac{1}{(z-1)} - \frac{2}{(z-1)^2} + \frac{3}{(z-1)^3} - \frac{4}{(z-1)^4} + \ldots\ldots \checkmark \quad \ldots\text{Ans.}$$

Ex. 6.6.3 : Find Laurent's series for

$$f(z) = (z-3)\sin\left(\frac{1}{z+2}\right) \text{ about } z = -2.$$

Soln. : $f(z) = (z-3)\sin\left(\frac{1}{z+2}\right)$

Let, $z + 2 = x \Rightarrow z = x - 2$

$$f(z) = (x-5)\sin\left(\frac{1}{x}\right)$$

$$= (x-5)\left[\frac{1}{x} - \frac{1}{3!\,x^3} + \frac{1}{5!\,x^5} + \ldots\right]$$

$$= \frac{x-5}{x} - \frac{x-5}{3!\,(x^3)} + \ldots\ldots$$

$$f(z) = 1 - \frac{5}{x} - \frac{1}{6x^2} + \frac{5}{6x^3} + \ldots\ldots$$

$$f(z) = 1 - 5(z+2)^{-1} - \frac{1}{6}(z+2)^{-2} + \frac{5}{6}(z+2)^{-3} + \ldots\ldots \checkmark \ldots\text{Ans.}$$

Which is Laurent's series.

Exercise

Taylor's / Laurents

Ex. 1 : Expand the function $\frac{z}{(z^2-1)(z^2+4)}$ in the $1 < |z| < 2$

Ex. 2 : Expand $f(z) = \frac{1}{z(z+3)^3}$ by Laurent series about $z = -2$

Ex. 3 : Expand $f(z) = \cos z$ in Taylors series about $z = \frac{\pi}{4}$

Ex. 4 : Expand $f(z) = \frac{1}{(z+1)(z+3)}$ for $1 < |z| < 3$ in Laurents series.

Ex. 5 : Expand $\frac{z-1}{z+1}$ in Taylors series about $z = 1$ and $z = 0$.

Ex. 6 : Find Laurent's series of $f(z) = \frac{1}{1-z^2}$ with centre at $z = 1$.

Ex. 7 : Expand $\frac{7z-2}{z(z+1)(z-2)}$ in annulus $1 < |z+1| < 3$

Ex. 8 : Expand $\frac{1}{z(z-1)(z-2)}$ for $1 < |z-3| < z$ in Laurents series.

Syllabus Topic : Residues

6.7 Residues

1. **Singular point :** A point at which a function f (z) is not analytic is called as singular point.

e.g. $f(z) = \frac{1}{z-a} \Rightarrow z - a = 0 \Rightarrow z = a$ is a singular point.

2. **Isolated singularity :** If $z = a$ is a singularity of f (z) and if there is no other singularity within a small circle surrounding the point $z = a$, then $z = a$ is called isolated singularity of f (z); otherwise it is called non-isolated singularity.

e.g

1. $f(z) = \frac{1}{(z-1)(z-3)}$ has two isolated singularities at $z = 3$ and $z = 1$

2. For non-isolated singularity : $f(z) = \frac{1}{\sin \pi/z}$ this function is not analytic at $\sin\left(\frac{\pi}{2}\right) = 0$

i.e. $\frac{\pi}{z} = n\pi \Rightarrow z = \frac{1}{n}$ $(\because n = 1, 2, 3, \ldots)$

Thus, $z = 1, \frac{1}{2}, \frac{1}{3} \ldots$ are the points of singularities, $z = 0$ is an non-isolated singularity because there are infinite number of singularities $\left(n = 1, \frac{1}{2}, \frac{1}{3} \ldots\right)$ in the neighborhood of $z = 0$.

If f (z) has an isolated singularities then f (z) can be expanded by Laurent series around $z = a$, as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n} \quad \ldots(6.7.1)$$

3. **Essential singularity :** If the number of negative powers of $(z - a)$ in Laurent series (1) are infinite then $z = a$ is called essential singularity OR If f (z) is infinite as $z \to a$, then $z = a$ is an essential singularity.

e.g.

1. $f(z) = e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \ldots$

2. $f(z) = \sin\left(\frac{1}{z-a}\right) = \left(\frac{1}{z-a}\right) - \left(\frac{1}{z-a}\right)^3 \cdot \frac{1}{3!} + \frac{1}{(z-a)^5}\frac{1}{5!} + \ldots$

has infinite number of terms in negative powers of $(z - a)$

4. **Removable singularity :** If the principle part of the Laurent series of f (z) at $z = a$ is zero (absent) then $z = a$ is called removable singularity OR A singularity $z = a$ is called removable singularity of f (z) if $\lim_{z \to a} f(z)$ exists (finite)

e.g.

1. $f(z) = \frac{\sin(z-a)}{(z-a)}$ has removable singularity at $z = a$

$$= \frac{1}{z-a}\left[(z-a) - \frac{(z-a)^3}{3!} + \frac{(z-a)^5}{5!} - \frac{(z-a)^7}{7!} + \ldots\right]$$

$$= 1 - \frac{(z-a)^2}{3!} + \frac{(z-a)^4}{5!} - \frac{(z-a)^6}{7!} + \ldots$$

2. $f(z) = \frac{e^z - 1}{z}$ has removable singularity at $z = 0$

$$\ldots\left(\because \lim_{z \to 0}\left(\frac{e^z-1}{z}\right) = \lim_{z \to 0}\frac{e^z}{1} = 1\right)$$

5. **Pole :** The principle part of Laurent series (1) contains a finite number of terms

i.e. If the series $a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots + a_{-n}(z-a)^{-n}$ consists finite number of terms, then $z = a$ is called pole of order n of the function f (z).

Note :

- A point $z = a$ is a pole of a function f (z) if $\lim_{z \to a} f(z) \to \infty$ in any manner.
- A point $z = a$ is a pole of order m of f (z) if $\lim_{z \to a} (z-a)^m \cdot f(z) = A$ (finite).
- For a rational function ; $f(z) = \frac{\phi(z)}{\psi(z)}$; f (z) has a pole where, $\psi(z) = 0$
- If $f(z) = \frac{\phi(z)}{\psi(z)}$ and let $z = a$ is a singularity of f (z), then if $\phi(z)$ is analytic at $z = a$ but $\psi(z)$ is not analytic at $z = a$ then $z = a$ is a pole of f (z).

ex. $f(z) = \frac{e^z}{z(z-1)^2} = \frac{\phi(z)}{\psi(z)}$

f (z) has singularity at $z = 0$, $z = 1$.

The numerator function e^z is analytic everywhere at $z = 0$, $z = 1$ and so on. But these are the poles of $\psi(z)$ of order 1 and 2 respectively. Therefore, $z = 0$, $z = 1$ are the poles of f (z) of order 1 and 2 respectively.

6. **Simple pole :** If the principle part of Laurent series (1) contains only $a_{-1}(z-a)^{-1}$ then $z = a$ is called simple pole. (i.e. when $n = 1$; then $z = a$ is simple pole).

Ex. $f(z) = \frac{e^z}{z(z-1)^2}$

In this example $z = 0$ is a simple pole and $z = 1$ is a pole of order 2.

7. **Residue :** The co-efficient of $\frac{1}{(z-a)}$ i.e. a_{-1} is called residue of $f(z)$ at $z = a$.

6.7.1 Calculation of Residues

1. If $f(z)$ has a simple pole (order one) at $z = a$ then

$$\text{Res}(f(a)) = \lim_{z \to a} [(z-a) \cdot f(z)]$$
$$= \{(z-a) f(z)\}_{z=a}$$

2. If $f(z)$ has a pole of order $n > 1$ at $z = a$ then

$$\text{Res } f(a) = \lim_{z \to a} \left\{ \frac{1}{(n-1)!} \left[\frac{d^{n-1}}{dz^{n-1}} (z-a)^n \cdot f(z) \right] \right\}$$
$$= \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} [(z-a)^n f(z)]_{z=a}$$

3. The value of residue at a removable singularity is zero i.e. if $z = a$ is a removable singularity of $f(z)$ then $\text{Res } f(a) = 0$.

4. If $z = a$ is a simple pole of $f(z)$ and if $f(z) = \frac{\phi(z)}{\psi(z)}$, where $\phi(a) \neq 0$, but $\psi(a) = 0$

then $\text{Res } f(a) = \frac{\phi(a)}{\psi'(a)}$ where $\psi'(a) \neq 0$.

5. If $z = a$ is an essential singularity of $f(z)$ then we must find the coefficient of $(z-a)^{-1}$ in the Laurent series about $z = a$.

Syllabus Topic : Cauchy's Residue Theorem

6.8 Cauchy's Residue Theorem

If $f(z)$ is an analytic function within and on a simple closed curve C except at a finite number of singular points $a_1, a_2, a_3, \ldots a_n$ (Say) inside C, then

$$\int_C f(z)\, dz = 2\pi i\, [\text{Res } f(a_1) + \text{Res } f(a_2) + \ldots + \text{Res } f(a_n)]$$
$$= 2\pi i\, [\text{Sum of residues at } z = a_1, a_2, a_3, \ldots a_n]$$

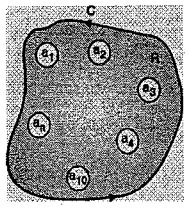


Fig. 6.8.1

Remember

1. If the pole lies outside the given closed curve then use Cauchy's Integral theorem
2. If the pole $z = a$ lies inside C then use Cauchy's integral formula or Cauchy's residue theorem.
3. If there are finite number of poles, $z = a_1, a_2, a_3, \ldots a_n$ lies inside C then use Cauchy's Residue theorem.
4. Find Residues, according to pole (simple or pole of order n)

Examples :

Type - I

Ex. 6.8.1 : Find residue of $\frac{z^3}{(z-1)(z-2)(z-3)}$ at each pole.

Soln. :

Step I :

The given function $\frac{z^3}{(z-1)(z-2)(z-3)}$ is analytic everywhere except at $z = 1, 2, 3$

The given function has simple poles at $z = 1, 2, 3$

Step II :

➔ **Using residues by the theorem**

$$\ldots \left[\text{Residue at simple pole } z = a \text{ Res } (z = a) = [(z-a) \cdot f(z)]_{z=a} \right]$$

(i) Residue at $z = 1$:

$$\therefore \text{Res } (z = 1) = \text{Res } f(1)$$
$$= \lim_{z \to 1} (\cancel{z-1}) \cdot \frac{z^3}{(\cancel{z-1})(z-2)(z-3)}$$
$$= \left[\frac{z^3}{(z-2)(z-3)} \right]_{z=1}$$
$$= \left[\frac{1}{(1-2)(1-3)} \right] = \frac{1}{(-1)(-2)} = \frac{1}{2}$$

(ii) Residue at $z = 2$;

$$\text{Res } (z = 2) = \text{Res } f(2)$$
$$= \lim_{z \to 2} \left[(\cancel{z-2}) \cdot \frac{z^3}{(z-1)(\cancel{z-2})(z-3)} \right]$$
$$= \left[\frac{z^3}{(z-1)(z-3)} \right]_{z=2}$$
$$= \left[\frac{8}{(2-1)(2-3)} \right] = \frac{8}{(1)(-1)} = -8$$

(iii) Residue at $z = 3$

$$\text{Res } (z = 3) = \text{Res } f(2)$$
$$= \lim_{z \to 3} \left[(\cancel{z-3}) \cdot \frac{z^3}{(z-1)(z-2)(\cancel{z-3})} \right]$$
$$= \left[\frac{z^3}{(z-1)(z-2)} \right]_{z=3}$$
$$= \left[\frac{27}{(3-1)(3-2)} \right] = \left[\frac{27}{(2)(1)} \right]$$

$$\frac{z^3}{(z-1)(z-2)(z-3)} = \frac{27}{2} \checkmark \quad \ldots\text{Ans.}$$

Ex. 6.8.2 : Find residue of the function and determine the type of singular points.

(i) $f(z) = z \sinh\left(\frac{1}{z}\right)$

(ii) $f(z) = \frac{\sin z}{z^2}$

(iii) $f(z) = \frac{\sin z}{z \cos z}$ inside the circle $|z| = 2$

(iv) $\frac{1 - \sin z}{\cos z}$; for $|z| = 2$

(v) $\cos\left(\frac{1}{z}\right) = f(z)$

Soln. :

Step I :

(i) $f(z) = z \sinh\left(\frac{1}{z}\right)$

Step II :

The given function $f(z) = z \sinh\left(\frac{1}{z}\right)$

Since, $\lim_{z \to 0} z \sinh\left(\frac{1}{z}\right)$ does not exists.

Hence, $z = 0$ is an essential singularity.

OR

Step III :

➔ **Using standard result**

(i) $\sinh \theta = \frac{e^{\theta} - e^{-\theta}}{2}$ (ii) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

(iii) $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$

$$f(z) = z \cdot \sinh\left(\frac{1}{z}\right) = \frac{z}{2}\left[e^{\frac{1}{z}} - e^{-\frac{1}{z}}\right]$$
$$= \frac{z}{2}\left[\left(1 + \frac{1}{z} + \frac{1}{z^2\, 2!} + \ldots\right) - \left(1 - \frac{1}{z} + \frac{1}{z^2\, 2!} + \ldots\right)\right]$$
$$= \frac{z}{2}\left[\cancel{1} + \frac{1}{z} + \frac{1}{2z^2} + \ldots - \cancel{1} + \frac{1}{z} - \frac{1}{2z^2} + \ldots\right] + \ldots$$
$$= \frac{z}{2}\left[\frac{2}{z} - \frac{1}{3 \cdot z^3} - \ldots\right] = 1 - \frac{1}{3!\, z^2} - \frac{1}{5!\, z^4} + \ldots$$

Step IV : This series contains an infinite number of terms with negative powers of z.

$\therefore z = 0$ is an essential singularity.

Step V : $z = 0$ is not pole because in series, the principle part contains **infinite terms.**

$\therefore \text{Res } f(a) = \text{Res } f(0) = \text{Coefficient of } z^{-1} = 0$

– Every Singularity need not be pole but every pole is a singularity.

– The residue of a function can be evaluated only if the pole is an isolated singularity.

(ii) $f(z) = \frac{\sin z}{z^2}$

Step I : The given function $f(z) = \frac{\sin z}{z^2}$ has singularity at $z = 0$

Since, $\lim_{z \to 0} \frac{\sin z}{z^2} = \lim_{z \to 0} \frac{1}{z} \frac{\sin z}{z} = \infty$

(limit does not exists)

$z = 0$ is an essential singularity.

$z = 0$ is a pole.

Step II : To find order : $f(z) = \frac{\sin z}{z^2} = \frac{\phi(z)}{\psi(z)}$

Since, $z = 0$ is a singularity

$\phi(z) = \sin z$; $\psi(z) = z^2$

$\phi'(z) = \cos z$; $\psi'(z) = 2z$

$\psi'''(z) = 2$

$\therefore \phi(0) = \sin(0) = 0$	$\psi(0) = 0$
$\phi'(0) = \cos(0) \neq 0$	$\psi'(0) = 0$
	$\psi''(0) \neq 0$

$\therefore z = 0$ is a pole of order 1 ($\because 2 - 1$)

OR

$$\frac{\sin z}{z^2} = \frac{1}{z^2}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots\right] = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} + \ldots$$

the power of $\frac{1}{z}$ is one. $\therefore z = 0$ is a pole of order 1.

$\therefore z = 0$ is a simple pole.

$\therefore \text{Res } (z = 0) = \text{co-efficient of } z^{-1} = 1.$

(iii) $f(z) = \frac{\sin z}{z \cos z}$ **for the circle** $|z| = 2$

Step I :

The given function $\left(\frac{\sin z}{z \cos z}\right)$ is analytic everywhere except at $z \cos z = 0$

The function has singularity at $z = 0, \pm\frac{\pi}{2}, \pm 3\frac{\pi}{2}, \ldots$ out of these singularities only $z = 0, \pm\frac{\pi}{2}$ lies inside $|z| = 2$.

($\because$ All $z = \pm 3\frac{\pi}{2}, \pm 5\frac{\pi}{2}, \ldots$ have modulus greater than 2)

(i) At $z = 0$

$$\lim_{z\to 0}\left(\frac{\sin z}{z\cdot\cos z}\right) = \lim_{z\to 0}\left(\frac{\tan z}{z}\right) = \lim_{z\to 0}\left[\frac{\sec^2 z}{1}\right] = 1$$

OR

$$f(z) = \frac{\sin z}{z\cdot\cos z} = \frac{\tan z}{z} = \frac{1}{z}\left[z + \frac{z^3}{3} + 2\frac{z^5}{15} + \ldots\right]$$

$$= 1 + \frac{z^2}{3} + 2\frac{z^4}{15} + \ldots$$

Principle part is absent.

$\therefore$ $z = 0$ is a removable singularity of $f(z)$.

$\therefore$ Res (at $z = 0$) $= 0$

...(See from **calculation of residue (4)**)

OR

By definition,

Res (at $z = 0$) = co-efficient of $z^{-1} = 0$

(ii) At $z = \frac{\pi}{2}$ is a simple pole ($\because$ isolated singularity)

$$\lim_{z\to\frac{\pi}{2}}\left(\frac{\sin z}{z\cdot\cos z}\right) = \infty \qquad \ldots(\because \text{Pole Note (1)})$$

$\therefore$ $z = \frac{\pi}{2}$ is a pole of $f(z)$ of order 1.

$$\text{Res}\left(\text{at } z = \frac{\pi}{2}\right) = \lim_{z\to\frac{\pi}{2}}\left[\left(z - \frac{\pi}{2}\right)\cdot\frac{\sin z}{z\cdot\cos z}\right] \quad \ldots\left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{z\to\frac{\pi}{2}}\frac{\left(z - \frac{\pi}{2}\right)\cdot\cos z + \sin z}{\cos z - z\sin z}$$

...by L' Hospital Rule

$$= \frac{0+1}{0-\frac{\pi}{2}} = -\frac{2}{\pi}$$

$$\text{Res}\left(\text{at } z = \frac{\pi}{2}\right) = -\frac{2}{\pi}$$

(iii) Similarly at $z = -\frac{\pi}{2}$ $\qquad$ $\text{Res}\left(\text{at } z = \frac{\pi}{2}\right) = \frac{2}{\pi}$

(iv) $\frac{1-\sin z}{\cos z}$; for $|z| = 2$

The function $f(z)$ has singularity at $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \pm \ldots\ldots$

only $z = \pm\frac{\pi}{2}$ lies inside the circle $|z| = 2$.

$$\lim_{z\to\frac{\pi}{2}}\cdot\left(\frac{1-\sin z}{\cos z}\right) = \lim_{z\to\frac{\pi}{2}}\left[\frac{+\cos z}{-\sin z}\right] = 0$$

$\therefore$ $z = \frac{\pi}{2}$ is a removable singularity.

$\therefore$ $\text{Res}\left(\text{at } z = \frac{\pi}{2}\right) = 0$

$\therefore$ $z = -\frac{\pi}{2}$ is a isolated singularity.

$$\because \lim_{z\to\frac{\pi}{2}}\cdot\frac{1-\sin z}{\cos z} = \infty$$

$\therefore$ $z = -\frac{\pi}{2}$ is a pole of order 1.

$$\therefore \text{Res}\left(\text{at } z = -\frac{\pi}{2}\right) = \lim_{z\to-\frac{\pi}{2}}\cdot\left(z + \frac{\pi}{2}\right)\left(\frac{1-\sin z}{\cos z}\right) = 2$$

(v) $\cos\left(\frac{1}{z}\right) = f(z)$

The given function $f(z)$ has singularity at $z = 0$.

$$\lim_{z\to 0}\left(\cos\left(\frac{1}{z}\right)\right) = \infty$$

$\therefore z = 0$ is an essential singularity.

OR

$$f(z) = \cos\left(\frac{1}{z}\right) = 1 - \frac{1}{z^2\,2!} + \frac{1}{z^4\,4!} - \frac{1}{z^6\,6!} + \ldots\ldots$$

This expansion contain infinite number of terms in principle part.

$\therefore z = 0$ is an essential singularity.

Res (at $z = 0$) = Co-efficient of $z^{-1} = 0$

Type- II

Ex. 6.8.3 : Evaluate $\oint_C \frac{4-3z}{z(z-1)(z-2)}dz$; by residue theorem, where c : $|z| = \frac{3}{2}$

Soln. :

Step I Given function, $\frac{4-3z}{z(z-1)(z-2)}$ is an analytic function everywhere except at $z = 0, 1, 2$

Therefore, $z = 0, 1, 2$ are simple poles.

(We get, poles by equating denominator to zero.)

Step II :

Now, given curve is : C : $|z| = \frac{3}{2}$

(i) For $z = 0$, $|z| = |0| < \frac{3}{2}$ and $z = 1$, $|z| = |1| < \frac{3}{2}$

These two points lies inside the circle.

(ii) For $z = 2$, $|z| = |2| > \frac{3}{2}$. This shows that, $z = 2$ lies outside the circle.

➔ **Using residue theorem we get**

$$\ldots\left[\oint f(z)\,dz = 2\pi i\,[\text{sum of residues}]\right]$$

Step III :

➔ **Using Residue Standard Result**

$$\ldots\left[\text{Residue at simple pole } z = a \quad \text{Res}(z = a) = [(z-a)\cdot f(z)]_{z=a}\right]$$

(i) Residue at $(z = 0) = \left[z\cdot\frac{(4-3z)}{z(z-1)(z-2)}\right]_{z=0}$

$$= \left[\frac{(4-3z)}{(z-1)(z-2)}\right]_{z=0} = \frac{4}{(-1)(-2)} = 2$$

And

(ii) Residue at $(z = 1)$

$$= \left[\frac{(z-1)\cdot(4-3z)}{z(z-1)(z-2)}\right]_{z=1} = \left[\frac{(4-3z)}{z(z-2)}\right]_{z=1}$$

$$= \frac{(4-3)}{1(1-2)} = \frac{1}{1(-1)} = -1$$

Step V :

Therefore, $\oint\frac{(4-3z)}{z(z-1)(z-2)}dz = 2\pi i$ (sum of residues)

$= 2\pi i\,[2-1] = 2\pi i$ ✓ ...Ans.

Ex. 6.8.4 : Find residue and evaluate : $\oint_C \tan z\,dz$; where C is circle $|z| = 2$.

Soln. :

Step I : Since, $f(z) = \tan z = \frac{\sin z}{\cos z}$ is analytic everywhere except at $\cos z = 0 \Rightarrow z = \pm\frac{\pi}{2}, \pm 3\frac{\pi}{2} \pm 5\frac{\pi}{2} \pm \ldots$

$\therefore$ $z = \pm\frac{\pi}{2}, \pm 3\frac{\pi}{2}, \pm 5\frac{\pi}{2} \ldots$ are the simple poles of $f(z)$.

Only, $z = \pm\frac{\pi}{2}$ lies inside the circle $|z| = 2$

$\left(\because z = \frac{\pi}{2} \Rightarrow |z| = \left|\frac{\pi}{2}\right| < 2 \text{ and at other value of } z; |z| > 2\right)$

Step II : Since, $\tan z = \left[z + \frac{z^3}{3} + \frac{2}{15}z^5 + \ldots\right]$

By definition, in this series principle part is absent.

$\therefore$ $z = \pm\frac{\pi}{2}$ are removable singularity.

$\therefore$ $\text{Res}\left(\text{at } z = \pm\frac{\pi}{2}\right) = 0$

Step III : $\therefore$ By Residue theorem we get,

$$\oint_C \tan z\,dz = 2\pi i\,(0) = 0$$

Alternate method

Step IV :

But, $z = \frac{\pi}{2}$ is a singularity

➔ **Using Residue theorem,**

$$\ldots\left[\text{Residue at simple pole } z = a \quad \text{Res}(z = a) = [(z-a)\cdot f(z)]_{z=a}\right]$$

(i) $$\text{Res}\left(\text{at } z = \frac{\pi}{2}\right) = \lim_{z\to\frac{\pi}{2}}\left[\frac{\left(z - \frac{\pi}{2}\right)\cdot\sin z}{\cos z}\right] \quad \ldots\left(\frac{0}{0}\text{form}\right)$$

By L'Hospital rule, differentiate numerator and denominator seperately

$$= \lim_{z\to\frac{\pi}{2}}\left[\frac{\left(z - \frac{\pi}{2}\right)\cdot\cos z + 1\cdot\sin z}{-\sin z}\right]$$

$$= \frac{+1}{-1} = -1 \qquad \left(\because \sin\left(\frac{-\pi}{2}\right) = -\sin\frac{\pi}{2}\right)$$

$$\text{Res}\left(\text{at } z = -\frac{\pi}{2}\right) = \lim_{z\to-\frac{\pi}{2}}\left[\left(z + \frac{\pi}{2}\right)\left(\frac{\sin z}{\cos z}\right)\right] \quad \ldots\left(\frac{0}{0}\text{form}\right)$$

By L'Hospital rule, differentiate numerator and denominator separately,

$$= \lim_{z\to-\frac{\pi}{2}}\left[\frac{\left(z + \frac{\pi}{2}\right)\cdot\cos z + \sin z}{-\sin z}\right]$$

$$= \frac{-1}{1} = -1 \qquad \left(\because \sin\left(\frac{-\pi}{2}\right) = -\sin\frac{\pi}{2}\right)$$

Step VI : $\oint_C \tan z\,dz = 2\pi i$ [sum of Residues]

$= 2\pi i\,(-1-1)$

$\oint_C \tan z\,dz = -4\pi i$ ✓ ...Ans.

Ex. 6.8.5 : Evaluate the following integral using Residue theorem :

$$\oint_C \frac{1+z}{z(2-z)}\,dz\,;\ C:\ |z| = 1$$

Soln. :

Step I :

The given function $\frac{1+z}{z(2-z)}$ is analytic everywhere except at $z = 0, z = 2$

The poles of the function are at $z = 0,\ z = 2$.

$\therefore$ $z = 0, 2$ are the simple poles of f(z).

Step II :

Given, c : $|z| = 1$

(i) For $z = 0 \Rightarrow |z| = |0| = 0 < 1$

and (ii) For $z = 2 \Rightarrow |z| = |2| = 2 > 1$

This shows that $z = 0$ lies inside and $z = 2$ lies outside

Step III :

➔ **Using Residue theorem**

$$\ldots\left[\text{Residue at simple pole } z = a \quad \text{Res}(z = a) = [(z-a)\cdot f(z)]_{z=a}\right]$$

$$\therefore \text{Res}(\text{at } z = 0) = \lim_{z\to 0}\ [z\,f(z)]$$

$$= \lim_{z\to 0}\left[\cancel{z}\cdot\frac{1+z}{\cancel{z}(2-z)}\right]_{z=0} = \frac{1}{2}$$

By Residue theorem,

Step IV :

$$\oint_C \frac{1+z}{z(2-z)}\,dz = 2\pi i\ (\text{Res}(\text{at } z = 0)) = 2\pi\, i\,\frac{1}{2}$$

$$\oint_C \frac{1+z}{z(2-z)}\,dz = \pi i \checkmark \qquad \ldots\text{Ans.}$$

Ex. 6.8.6 : Evaluate the following integral using Residue theorem : $\oint_C \frac{\cot h\, z}{(z-i)}\,dz\,;\ C:\ |z| = 2$

Soln. :

Step I :

$$\text{Given : } \oint_C \frac{\cot h\, z}{(z-i)}\,dz = \oint_C \frac{\cosh z}{\sinh z\,(z-i)}\,dz$$

$$= \oint_C \frac{e^z + e^{-z}}{(e^z - e^{-z})(z-i)}\,dz$$

$$\left(\because \cosh z = \frac{e^z + e^{-z}}{z} \text{ and } \sinh z = \frac{e^z - e^{-z}}{z}\right)$$

The given function is analytic everywhere except at $(e^z - e^{-z})(z-i) = 0$

Step II : For poles, put, $e^z - e^{-z} = 0$ and $z - i = 0$

$\Rightarrow e^z - \frac{1}{e^z} = 0\,;\ e^{2z} - 1 = 0$ and $z - i = 0$

$\therefore$ $e^{2z} = 1$ and $z = i$

$\therefore$ $z = 0$ and $z = i$ are the simple poles of f (z).

Given : c : $|z| = 2$

(i) For $z = 0 \Rightarrow |z| = |0| = 0 < 2$

(ii) For $z = i \Rightarrow |z| = |i| = 1 < 2$

Both poles lies inside the circle $|z| = 2$.

Step IV :

➔ **Using Residue theorem**

$$\ldots\left[\text{Residue at simple pole } z = a \quad \text{Res}(z = a) = [(z-a)\cdot f(z)]_{z=a}\right]$$

$$\therefore \text{Res}(z = 0) = \frac{\phi(z)}{\psi(z)} = \left[\left(\frac{\frac{e^z + e^{-z}}{z-i}}{e^z - e^{-z}}\right)\right]_{z=0} = \left(\frac{\phi(z)}{\psi'(z)}\right)_{z=.0}$$

$$= \left[\left(\frac{\frac{e^z + e^{-z}}{z-i}}{e^z + e^{-z}}\right)\right]_{z=0} = \frac{\frac{1+1}{0-i}}{1+1}$$

$$\left(\because \text{ If } z = a \text{ is a simple pole of f(z) and if } f(z) = \frac{\phi(z)}{\psi(z)}\right.$$

$$\left.\text{and } \phi(a) \neq 0 \text{ but } \psi(a) = 0 \text{ then Res}(z = a) = \left[\frac{\phi(z)}{\psi'(z)}\right]_{z=0}\right)$$

$$= \frac{\cancel{2}}{-i}\times\frac{1}{\cancel{2}} = -\frac{1}{i} = i$$

and,

Step IV : $\text{Res}(z = i) = \lim_{z\to i}\left[\cancel{z-i}\,\frac{\cot hz}{\cancel{z-i}}\right]dz = \coth i$

Step VI : $\therefore$ By Residue theorem,

$$\int_C \frac{\cot hz}{(1-i)}\,dz = 2\pi i\ (\text{Sum of Residue})$$

$$\int_C \frac{\cot hz}{(1-i)}\,dz = 2\pi i\,[\,i + \cot hi\,] \checkmark \qquad \ldots\text{Ans.}$$

Ex. 6.8.7 : Evaluate : $\oint_C \frac{dz}{\cosh(z)}$, where C : $|z| = 2$

Soln. :

Step I : Given :

$$f(z) = \frac{1}{\cosh(z)} = \frac{1}{\left(\frac{e^z + e^{-z}}{2}\right)} = \frac{2}{e^z + e^{-z}}$$

$$= \frac{2}{e^{-z}(e^{2z}+1)} = \frac{2e^z}{e^{2z}+1}$$

The given function is analytic everywhere except at $e^{2z} + 1 = 0$

Poles of f (z); where, $e^{2z} + 1 = 0$

$(\because$ by using, $x^2 + 1 = 0 \Rightarrow (x+i)(x-i) = 0)$

$\Rightarrow$ $(e^z + i)(e^z - i) = 0 \Rightarrow e^z = i, -i$

$$\left(\because i = e^{\frac{i\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$e^z = e^{\frac{\pi}{2}i}$; $e^z = e^{\frac{-\pi}{2}i}$

$\therefore z = \frac{\pi}{2}i$; $z = -\frac{\pi}{2}i$

Given : c : $|z| = 2$

(i) For $z = \frac{\pi}{2}i \Rightarrow |z| = \left|\frac{\pi}{2}i\right| = \frac{\pi}{2} < 2$

And (ii) For $z = -\frac{\pi}{2}i \Rightarrow |z| = \left|-\frac{\pi}{2}i\right| = \frac{\pi}{2} < 2$

This shows that, both poles lies inside the given circle.

Step II : By Residue Theorem:

$$\text{Res of } f(z)\left(\text{at } z = \frac{\pi}{2}i\right) = \left[\frac{\phi(z)}{\psi(z)}\right]_{z=\frac{\pi}{2}i} = \left[\frac{2e^z}{e^{2z}+1}\right]_{z=\frac{\pi}{2}i}$$

$$\left(\because \text{Here } z = \frac{\pi}{2}i \text{ is a pole of f(z) and } f(z) = \frac{\phi(z)}{\psi(z)} \text{ but } \phi(\pi i/2)\right.$$

$$\left.\neq 0 \text{ but } \psi(\pi i/2) = 0 \text{ then Res}\left(\frac{\pi}{2}i\right) = \left[\frac{\phi(a)}{\psi(a)}\right]_{z=\frac{\pi}{2}i}\right)$$

$$= \left[\frac{2e^z}{2e^{2z}}\right]_{z=\frac{\pi}{2}i} = [e^{-z}]_{z=\frac{\pi}{2}i} = e^{-\frac{\pi}{2}i} = -i$$

$$\ldots\left(\because e^{-\frac{\pi}{2}i} = \underbrace{\cos\frac{\pi}{2}}_{0} - i\underbrace{\sin\frac{\pi}{2}}_{(1)} = -i\right)$$

$$\text{And Res of } f(z)\left(\text{at } z = -\frac{\pi}{2}i\right) = \left[\frac{\phi(z)}{\psi(z)}\right]_{z=-\frac{\pi}{2}i}$$

$$= \left[\frac{2e^z}{e^{2z}+1}\right]_{z=-\frac{\pi}{2}i}$$

$$\left(\because \text{Here, } z = -\frac{\pi}{2}i \text{ is a pole of f(z) and } f(z) = \frac{\phi(z)}{\psi(z)} \text{ but}\right.$$

$$\phi\left(-\frac{\pi i}{2}\right) \neq 0 \text{ but } \Psi\left(-\frac{\pi i}{2}\right) = 0 \text{ then Res}\left(-\frac{\pi}{2}i\right) =$$

$$\left.\left[\frac{\phi(z)}{\psi'(z)}\right]_{z=-\frac{\pi}{2}i}\right)$$

$$= \left[\frac{2e^z}{2e^{2z}}\right]_{z=-\frac{\pi}{2}i} = [e^{-z}]_{z=-\frac{\pi}{2}i} = e^{\frac{\pi}{2}i} = i$$

Step III :

$$\therefore \oint_C f(z)\,dz = \oint_C \frac{dz}{\cosh z} = 2\pi i\ [\text{Sum of residue}] = 2\pi i\,(i - i)$$

$$\oint_C f(z)\,dz = 0 \checkmark \qquad \ldots\text{Ans.}$$

Ex. 6.8.8 : Evaluate : $\oint_C \frac{dz}{z\sin z}$;

C is the unit circle about origin.

Soln. :

Step I : Since, $\frac{1}{z\sin z} = \frac{1}{z\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots\right]}$

$$= \frac{1}{z^2\left[1 - \frac{z^2}{\lfloor 3} + \frac{z^4}{\lfloor 5} - \frac{z^6}{\lfloor 7} + \ldots\right]}$$

$$= \frac{1}{z^2}\left[1 - \left(\frac{z^2}{\lfloor 3} - \frac{z^4}{\lfloor 5} + \ldots\right)\right]^{-1}$$

$$= \frac{1}{z^2}\left[1 + \left(\frac{z^2}{8} - \frac{z^4}{120}\right) + \left(\frac{2^2}{6} - \frac{2^4}{120} + \ldots\right)^2 \ldots\right]$$

$$= \frac{1}{z^2}\left[1 + \frac{z^2}{6} - \frac{z^4}{120} + \frac{z^4}{36} + \ldots\right]$$

$$= \frac{1}{z^2} + \frac{1}{6} - \frac{z^2}{120} + \frac{z^2}{36} = \frac{1}{z^2} + \frac{1}{6} + \frac{7}{360}z^2 \ldots$$

Step II :

This shows that z = 0 is a pole of order 2 for the function $\frac{1}{z\sin z}$ and residue at the pole is zero (co-efficient of z^{-1}), the pole $z = 0$ lies inside circle $|z| = 1$

$(\because |z| = |0| = 0 < 1)$

Step III : $\therefore$ By Residue theorem

$$\oint_C \frac{1}{z\sin z}\,dz = 2\pi i\ (\text{Sum of residue}) = 0 \checkmark \qquad \ldots\text{Ans.}$$

Ex. 6.8.9 : Using residue theorem evaluate ;

$$\oint_C \frac{z}{z^4 + 13z^2 + 36}\,dz;\ \text{Where, } c:\ |z| = \frac{5}{2}$$

Soln. :

Step I : Given function :

$$\frac{z}{z^4 + 13z^2 + 36} = \frac{z}{(z^2+9)(z^2+4)}$$

$$= \frac{z}{(z+3i)(z-3i)(z+2i)(z-2i)}$$

The given functionis analytic everywhere except $z = 3i, -3i, 2i, -2i$

$z = 3i, -3i, 2i, -2i$ are simple poles

Step II : Given, Curve $|z| = \frac{5}{2}$

(i) For $z = \pm 3i \Rightarrow |z| = |\pm 3i| = 3 > \frac{5}{2}$

(ii) For $z = \pm 2i \Rightarrow |z| = |\pm 2i| = 2 < 5/2$

$\Rightarrow$ $z = \pm 3i$ lies outside the circle $|z| = \frac{5}{2}$ and $z = \pm 2i$ lies inside the circle $|z| = \frac{5}{2}$

Step III :

➔ **Using ResidueResult**

$$\ldots\left[\text{Residue at simple pole } z = a \quad \text{Res}(z = a) = [(z-a)\cdot f(z)]_{z=a}\right]$$

Step IV :

$$\text{And Res}(z = -2i) = (z+2i)\left[\frac{z}{(z^2+9)(z-2i)(z+2i)}\right]_{z=-2i}$$

$$= \left[\frac{z}{(z^2+9)(z-2i)}\right]_{z=-2i}$$

$$= \frac{-2i}{[(-2i)^2+9][-2i-2i]}$$

$$= \frac{-2i}{(-4+9)(-4i)} = \frac{2i}{(4i)(5)}$$

$$\text{Res}(z = -2i) = \frac{1}{10}$$

$$\text{And Res}(z = 2i) = (z-2i)\left[\frac{z}{(z^2+9)(z+2i)(z-2i)}\right]_{z=2i}$$

$$= \left[\frac{z}{(z^2+9)(z+2i)}\right]_{z=2i} = \frac{2i}{[(2i)^2+9][2i+2i]}$$

$$= \frac{2i}{(-4+9)(4i)} = \frac{2i}{20i} = \frac{1}{10}$$

Step V : By Residue theorem :

$$\oint \frac{z}{z^4+13z^2+36}\,dz = 2\pi i\,(\text{Sum of Residues at } z = 2i, -2i)$$

$$= 2\pi i\,\{\text{Res}(z = 2i) + \text{Res}(z = -2i)\}$$

$$\therefore \oint \frac{z}{z^4+13z^2+36}\,dz = 2\pi i\left[\frac{1}{10}+\frac{1}{10}\right]$$

$$= \frac{4}{10}\pi i$$

$$\therefore \oint \frac{z}{z^4+13z^2+36}\,dz = \frac{2}{5}\pi i \checkmark \qquad \text{...Ans.}$$

Ex. 6.8.10 : Evaluate : $\oint_C \frac{e^{\frac{1}{z}}}{1-z}\,dz$; where $C : |z| = 2$

Soln. :

Step I :

(The given function is analytic everywhere except at $z = 1, 0$)

➔ **Using standard result :** $\ldots\left[e^x = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right]$

Since,

$$f(z) = \frac{e^{\frac{1}{z}}}{1-z} = \frac{1}{(1-z)}\cdot\left[1+\frac{1}{z}+\frac{1}{2!\,z^2}+\frac{1}{3!\,z^3}+\ldots\right]$$

$$= (1-z)^{-1}\left[1+\frac{1}{z}+\frac{1}{2!\,z^2}+\ldots\right]$$

$$= (1+z+z^2+z^3+\ldots)\left(1+\frac{1}{z}+\frac{1}{2!\,z^2}+\frac{1}{3!\,z^3}+\ldots\right)$$

$$= \left[1+\frac{1}{z}+\frac{1}{2!\,z^2}+\frac{1}{3!\,z^3}+\ldots\; z+1+\frac{1}{2!\,z}+\frac{1}{3!\,z^2}+\ldots + z^2+\frac{1}{2!}+\frac{1}{3!z}+\frac{1}{4!z^2}+\ldots + z^3+z^2+\frac{z}{2!}+\frac{1}{3!}+\frac{1}{4!z}+\frac{1}{5!z^2}+\ldots\right]$$

$$= \left(1+1+\frac{1}{2}+\frac{1}{3!}+\ldots\right)+(2+2z+\ldots)\,z + \left(1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots\right)\frac{1}{z}+\ldots+\left(\frac{1}{2!}+\frac{1}{3!}+\ldots\right)\frac{1}{z^2}+\ldots$$

Step II : $z = 0, z = 1$ are the singularity

$z = 0, z = 1$ are the essential singularity and

...($\because$ Series containing infinitenumber of negative poles)

$z = 0, 1$ are the simple pole.

Step III :

$\therefore$ Res [at $(z = 0)$] = Co-efficient of $\frac{1}{z}$

$$= \left(1+\frac{1}{2!}+\frac{1}{3!}+\ldots\right) = e^1$$

➔ **Using Residue Standard result**

$$\ldots\left[\text{Residue at simple pole } z = a \quad \text{Res}(z = a) = [(z-a)\cdot f(z)]_{z=a}\right]$$

$$\text{Res [at } (z = 1)] = (1-z)\left[\frac{e^{1/z}}{(1-z)}\right]_{z=1} = e^1$$

Step V : By Residue theorem,

$$\therefore \int_C \frac{e^{\frac{1}{z}}}{(1-z)}\,dz = 2\pi i\,[\text{Sum of residue}] = 2\pi i\,[e^1 - e^1]$$

$$= 0 \checkmark \qquad \text{...Ans.}$$

Ex. 6.8.11 : Evaluate : $\oint_C \frac{\sin \pi z^2 + 2z}{(z-1)^2(z-2)}\,dz$ where,

$C : |Z| = 4$

Soln. :

Step I : Given, $f(z) = \frac{\sin \pi z^2 + 2z}{(z-1)^2(z-2)}$

is analytic function everywhere expect at $z = 1$ and $z = 2$.

$z = 1$ is pole of order two and $z = 2$ is a pole of order one.

Since, $C : |z| = 4$

(i) for $z = 1 \Rightarrow |z| = |1| = 1 < 4$

(ii) for $z = 2 \Rightarrow |z| = |2| = 2 < 4$

This shows that both poles lies inside the given circle

Step II : By Residue theorem,

$$\therefore I = \int_C f(z)\,dz = 2\pi i\,[\text{sum or residue}]$$

Using Residue Standard result

Residue at pole of order n at z = b

$$\text{Res}(z = b) = \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}(z-b)^n\cdot f(z)\right]_{z=b}$$

Step III : $r_1 = R_e(\text{at } z = 1) = \frac{1}{(2-1)!} = \left[\frac{d}{dz}(z-1)^2 f(z)\right]_{z=1}$

$$= \frac{d}{dz}\left[\cancel{(z-1)^2}\cdot\left(\frac{\sin \pi z^2 + 2z}{\cancel{(z-1)^2}}(z-2)\right)\right]_{z=1}$$

$$= \frac{d}{dz}\left[\frac{\sin \pi z^2 + 2z}{(z-2)}\right]_{z=1}$$

➔ **Using standard results :** $\ldots\left[d\left(\frac{u}{v}\right) = \left(\frac{v\,du - u\,dv}{v^2}\right)\right]$

$$= \left[\frac{[2\pi z\cdot\cos \pi z^2 + 2](z-2) - [\sin \pi z^2 + 2z]}{(z-2)^2}\right]_{z=1}$$

$$= \left[\frac{(1-2)\cdot\overbrace{(2\pi\cdot\cos\pi)}^{(-1)}\overbrace{(\sin\pi+2)}^{(0)}}{(1-2)^2}\right]$$

$$= \frac{(2\pi\cdot(-1)+2)(-1)-(0+2)}{1}$$

$$= (2\pi - 2 - 2) = (2\pi - 4)$$

➔ **Using Residue Standard result :**

$$\ldots\left[\text{Residue at simple pole } z = a \quad \text{Res}(z = a) = [(z-a)\cdot f(z)]_{z=a}\right]$$

$$r_2 = \text{res}(z = 2) = [(z-2)\cdot f(z)]_{z=2}$$

$$= \left[\cancel{(z-2)}\cdot\left\{\frac{\sin \pi z^2 + 2z}{(z-1)^2\cancel{(z-2)}}\right\}\right]_{z=2}$$

$$= \frac{\overbrace{\sin 4\pi}^{(0)} + 4}{(2-1)^2} = \frac{4}{1} = 4$$

Step IV :

$$\therefore \oint_C \left[\frac{\sin \pi z^2 + 2z}{(z-1)^2(z-2)}\right] dz = 2\pi i\,[2\pi - 4 + 4] = 4\pi^2 i \checkmark \qquad \text{...Ans.}$$

Ex. 6.8.12 : Evaluate : $\oint_C \frac{12z-7}{(z-1)^2(2z+3)}\,dz$, where, C is the circle $|z| = 2$ using. Cauchy's residue theorem.

Soln. :

Step I : Given,

$$f(z) = \frac{12z-7}{(z-1)^2(2z+3)}$$

is analytic everywhere except at $z = 1, z = \frac{-3}{2}$

$z = 1$ is a pole of order 2 and $z = -\frac{3}{2}$ is a simple pole

Step II :

(i) For $z = 1$; $C : |z| = 2$

$\Rightarrow |1| = 1 < 2$

$z = 1$ lies inside the given circle

(ii) For $z = -\frac{3}{2}$; $C : |z| = 2$

$$\Rightarrow \left|-\frac{3}{2}\right| = \frac{3}{2} < 2$$

$z = -\frac{3}{2}$ also lies inside the circle

Step III :

➔ **Using residue standard formula**

$$\ldots\left[\text{Residue at pole of order n at } z = b \quad \text{Res}(z = b) = \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}(z-b)^n\cdot f(z)\right]_{z=b}\right]$$

$$\text{(i)} \therefore \text{Res}(z = 1) = \frac{1}{(2-1)!}\frac{d}{dz}\left[\cancel{(z-1)^2}\,\frac{(12z-7)}{\cancel{(z-1)^2}(2z+3)}\right]_{at\,z=1}$$

$$= \frac{d}{dz}\left[\frac{12z-7}{2z+3}\right]_{z=1}$$

➔ **Using Standard result :** $\ldots\left[d\left(\frac{u}{v}\right) = \left[\frac{v\,du - u\,dv}{v^2}\right]\right]$

$$= \left[\frac{(2z+3)\cdot(12)-(12z-7)\cdot 2}{(2z+3)^2}\right]_{z=1} = \frac{60-10}{25} = 2$$

And by

Residue at simple pole z = a

$$\text{Res}(z = a) = [(z-a)\cdot f(z)]_{z=a}$$

$$\text{Res}(z = -3/2) = \left[(z+3/2)\frac{(12z-7)}{2\left(z+\frac{3}{2}\right)(z-1)^2}\right]_{at\,z=-3/2}$$

$$= \left[\frac{12z-7}{2(z-1)^2}\right]_{z=3/2} = \frac{12\left(-\frac{3}{2}\right)-7}{2\left(-\frac{3}{2}-1\right)^2}$$

$$= \frac{-18-7}{\left(\frac{25}{4}\right)\cdot 2} = \frac{-25}{\frac{25}{2}} = -2$$

Step IV :

➔ **Using Residue theorem,**

$$\ldots\left[\oint f(z)\,dz = 2\pi i\,[\text{sum of residues}]\right]$$

$$\therefore \oint \frac{12z-7}{(z-1)^2(2z+3)}\,dz = 2\pi i\,[2-2]$$

$$\therefore \oint \frac{12z-7}{(z-1)^2(2z+3)}\,dz = 0 \checkmark \qquad \text{...Ans.}$$

Exercise

Cauchy's Theorem/Formula

Ex. 1 : Evaluate : $\oint_C \frac{dz}{z^3(z+4)}$ where C is the circle $|z| = 2$.

Ex. 2 : Evaluate : $\oint_C \frac{z+6}{z^3-4}\,dz$ where C : $|z-2| = 1$.

Ex. 3 : Evaluate : $\oint_C \frac{3z^2+2}{(z-1)}\,dz$; where C : $|z-1| = 1$.

Ex. 4 : Evaluate : $\oint_C \frac{15z+9}{z(z^2-9)}\,dz$; C : $|z-1| = 3$.

Ex. 5 : Evaluate : State Cauchy's integral formula and use it to evaluate : $\oint \frac{e^{2z}}{(z+1)^4}\,dz$; C : $|z| = 2$.

Ex. 6 : Evaluate : $\oint_C \frac{e^z}{(z+1)^4}\,dz$; C : $|z-1| = 2$.

Ex. 7 : Evaluate : $\oint_C \frac{e^{3z}}{(z-pi)}\,dz$; C : $|z-2| + |z+2| = 6$

Ex. 8 : Evaluate : $\oint_C \frac{2z^2+2}{z^2-1}\,dz$: C $|z-1| = 1$

Ex. 9 : Evaluate : $\oint_C \frac{e^z}{(z+1)^3}\,dz$; C : $|z-1| = 2$

Ex. 10 : Evaluate : $\oint_C \frac{z-3}{z^2+2z+5}\,dz$; C : $|z+1-i| = 2$

Ex. 11 : Evaluate : $\oint_C \frac{e^{-3z}}{(z+1)^3}\,dz$; C : $|z| = 2$ and C : $|z+i| = 1$

Ex. 12 : Evaluate : $\oint \frac{e^z}{(z-1)}\,dz$; C : $|z| = 2$; $|z| = \frac{1}{2}$

Ex. 13 : Evaluate : $\oint_C \frac{e^{-2z}}{(z-i)^2}\,dz$; C :

(i) the circle $|z+i| = 1$

(ii) the square whose vertices (– 2, 2), (– 2, – 2), (2, – 2), (2, 2)

Ex. 14 : Evaluate : $\oint_C \frac{2z^2+2}{z^2-i}\,dz$; C : $|z-1| = 1$

Ex. 15 : Evaluate : $\oint (z^2+3z-2)\,dz$;

where C : $x^2+y^2 = 1$

C : $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Ex. 16 : Evaluate : $\oint_C \frac{15z+9}{z^3-9z}\,dz$;

where C : $|z-1| = 3$

Ex. 17 : Evaluate : $\oint \frac{\sin z}{z^2}\,dz$

where C : $|z| = \frac{1}{2}$

Ex. 18 : (i) Evaluate : $\oint_C z^2\,dz$

(ii) Evaluate : $\oint_C \frac{e^{2z}}{(z+1)^3}\,dz$; C : $|z| = 2$

Ex. 19 : Evaluate : $\oint_C \frac{z^3-5}{(z+1)^2(z-2)}\,dz$;

Where c is the circle $|z| = 3$

Ex. 20 : Evaluate $\oint_C \frac{z^2+2}{(z+1)(z^2-9)}\,dz$;

Where, c : $|z-3| = 5$

Ex. 21 : Evaluate : $\int_1^{1+i} (z^2+1)\,dz$ along a straight line

Ex. 22 : Evaluate : $\oint_C \frac{3z^2+z}{z^2-1}\,dz$: where c is the circle $|z-1| = 1$

Ex. 23 : Evaluate : $\int_{2+4i}^{5-5i} (x+iy+1)\,dz$, along the path $x = t^2+1$; $y = 3t+1$

Ex. 24 : By using Cauchy's integral formula :

evaluate $\oint_C \frac{e^z}{(z+1)^2(z+2)^2}\,dz$;over $|z+1| = \frac{1}{2}$

Ex. 25 : Evaluate : $\oint_C \frac{z^2+2z}{(z+1)(z^2-9)}\,dz$; over $|z-3| = 5$

Cauchy's Residue Theorem

Ex. 1 : State Residue theorem and Evaluate :

$$\int \frac{\sin z}{z^2}\,dz \text{ ; C : } |z| = \frac{1}{2}$$

Ex. 2 : By residue theorem evaluate :

$$\oint_C z^3 \sin\left(\frac{1}{z}\right) dz \text{ ; C : } |z| = 1$$

Ex. 3 : Evaluate : $\int \frac{e^{\frac{1}{z}}}{1-z}\,dz$; C : $|z| = 2$

Ex. 4 : Evaluate : $\oint \frac{2z^2+z}{z^2-1}\,dz$; $|z-1| = 1$

Ex. 5 : Evaluate : $\oint \frac{1}{z \sin z}\,dz$ where $|z| = \frac{1}{2}$

Ex. 6 : Evaluate : Evaluate the following integral using residue theorem : $\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$; $|z| = \frac{3}{2}$

Ex. 7 : Evaluate : $\oint_C \frac{e^z}{(z+1)^2}\,dz$ where C : $|z-3| = 3$

Here $z = -1$ is a pole of order 2.

Ex. 8 : Evaluate : $\oint_C \frac{2z-1}{z(z+1)(z-3)}\,dz$; C : $|z| = 2$

Ex. 9 : Evaluate : $\oint_C \frac{(z-3)}{(z^2+2z+5)}\,dz$; C :

(i) $|z| = 1$

(ii) $|z+1-i| = 2$

(iii) $|z+1+i| = 2$

Ex. 10 : Use residue theorem to evaluate : $\oint_C \frac{e^{2z}}{\cos \pi z}\,dz$; where C is the circle $|z| = 1$.

Ex. 11 : Evaluate by residue theorem : $\oint_C \frac{z^2}{(z-1)^2(z+1)}\,dz$; C : $|z| = 2$

Ex. 12 : State Cauchy's Residue theorem and use it to evaluate : $\oint_C \frac{\sin^2}{z^2}\,dz$; C : $|z| = \frac{1}{2}$

Ex. 13 : Evaluate : $\oint_C \frac{z^2}{(z-1)^2(z+2)}\,dz$; C : $|z| = 3$

Ex. 14 : Evaluate : $\oint_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)^2(z-2)}\,dz$; C : $|z| = 3$ by Residue theorem.

Chapter Ends...

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Note

UNIT 6

Functions of Complex Variables (Integral Calculus)

Syllabus

Cauchy's integral theorem; Cauchy's integral formula; Residues; Cauchy's residue theorem (All theorems without proofs).

➢ **Chapter 6 : Functions of Complex Variables (Integral Calculus)**

CHAPTER 6

UNIT VI

Functions of Complex Variables (Integral Calculus)

Multiple Choice Questions

Q. 1 If f (z) is an analytic function and $f'(z)$ is continuous at each point within and on a simple closed curve C, then

(a) $\oint_C f(z)\,dz = 0$ (b) $\oint_C f(z)\,dz = 0$

(c) $\oint_C f(z)\,dz = 0$ (d) $\oint_C f(z)\,dz = 0$

Ans. : (a)

Q. 2 If f (z) is analytic within and on a simple closed curve C and 'a' is any point within C, then

(a) $f(a) = \oint_C \frac{f(z)}{z-a}\,dz$

(b) $f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$

(c) $f(a) = \frac{1}{2}\oint_C \frac{f(z)}{z-a}\,dz$

(d) $f(a) = \frac{1}{2\pi i}\oint_C \frac{1}{z+a}\,dz$

Ans. : (b)

Q. 3 If f (z) is analytic within and on a simple closed curve C and 'a' is any point within C, then

(a) $\oint_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{(n)!} f^{(n-1)}(a)$

(b) $\oint_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n)}(a)$

(c) $\oint_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(a)$

(d) $\oint_C \frac{f(z)}{(z-a)^n}\,dz = \frac{1}{(n-1)!} f^{(n+1)}(a)$

Ans. : (c)

Q. 4 If the integrand of the integral $\oint_C (x^2 - y^2 + i2xy)\,dz$ is analytic inside contour C : |z| = 1 then the value of integral is

(a) $2\pi i$ (b) π (c) i (d) 0

Ans. : (d)

Q. 5 If the integrand of the integral $\oint_C \frac{e^{-z}}{z+1}\,dz$, where C is a circle : |z| = 2 , is analytic function everywhere except at z = – 1, then by Cauchy's Integral formula, the value of integral is

(a) $2\pi ei$ (b) πe (c) $2\pi I$ (d) 0

Ans. : (a)

Q. 6 If the integrand of the integral $\oint_C \frac{\sin^2 z}{(z-\pi/6)^3}\,dz$, where C is the circle |z| = 1, is analytic function everywhere except at singularity $z = \pi/6$ of order 3, and $\frac{d^2 f(z)}{dz^2} = 2\cos 2z$, then by Cauchy's Integral formula, the value of integral is

(a) $2\pi i$ (b) πi (c) 2π (d) 0

Ans. : (b)

Q. 7 If f (z) has a simple pole (order one) at z = a then residue of f(z) at z = a is equal to

(a) $\text{Res}(f(a)) = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]_{z=a}$

(b) $\text{Res}(f(a)) = \frac{d^{n-1}}{dz^{n-1}}[(z+a)^n f(z)]_{z=a}$

(c) $\text{Res}(f(a)) = \{(z-a) f(z)\}_{z=a}$

(d) $\text{Res}(f(a)) = \{(z+a) f(z)\}_{z=a}$

Ans. : (c)

Q. 8 If f (z) has a pole of order n > 1 at z = a then residue of f(z) at z = a is equal to

(a) $\text{Res}(f(a)) = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]_{z=a}$

(b) $\text{Res}(f(a)) = \frac{d^{n-1}}{dz^{n-1}}[(z+a)^n f(z)]_{z=a}$

(c) $\text{Res}(f(a)) = \{(z-a) f(z)\}_{z=a}$

(d) $\text{Res}(f(a)) = \{(z+a) f(z)\}_{z=a}$

Ans. : (a)

Q. 9 If f (z) is an analytic function within and on a simple closed curve C except at a finite number of singular points $a_1, a_2, a_3, \ldots a_n$ (Say) inside C, then

(a) $\int_C f(z)dz = [\text{Res } f(a_1) + \text{Res } f(a_2) + \ldots + \text{Res } f(a_n)]$

(b) $\int_C f(z)dz = 2\pi i[\text{Res } f(a_1) + \text{Res } f(a_2) + \ldots + \text{Res } f(a_n)]$

(c) $\int_C f(z)\,dz = 2\pi i\,[\text{Res } f(a_1).\ \text{Res } f(a_2) \ldots \text{Res } f(a_n)]$

(d) $\int_C f(z)\,dz = 2\pi i[\text{Res } f(a_1) - \text{Res } f(a_2) - \ldots - \text{Res } f(a_n)]$

Ans. : (b)

Q. 10 If the integrand of the integral $\oint_C \frac{4-3z}{z(z-1)(z-2)}dz$, where $C : |z| = \frac{3}{2}$, is an analytic function everywhere except at z = 0, 1, 2 and z = 0 lies inside the given circle then the residue at z = 0 is

(a) i

(b) πi

(c) 0

(d) 2

Ans. : (d)

Q. 11 If the integrand of the integral $\oint_C \frac{12z-7}{(z-1)^2(2z+3)}dz$, where, C is the circle $|z| = 2$, an analytic function everywhere except at z = 1, $z = \frac{-3}{2}$ with residue at z = 1 is equal to 2 and residue at $z = \frac{-3}{2}$ is equal to − 2 then by using Cauchy's residue theorem the value of integral $\oint_C \frac{12z-7}{(z-1)^2(2z+3)}dz$, is

(a) i (b) πi (c) 0 (d) 2

Ans. : (d)

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