Dynamic Responses to Government Spending in and out of a Fiscal Union

A Two-Country NK DSGE Model for Scotland and the Rest of the UK

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Contents

1 Introduction

In May, the Bank of England's Monetary Policy Committee (MPC) decided to raise the interest rate in response to the exceptionally high inflation of the consumer price index. The committee members were presented with the latest data and thorough analyses conducted by experienced teams of economists. However, the decision was not unanimous: a few members expressed concerns that the contractionary monetary policy shock had not yet been fully internalised, and further rate increases might lead to inflation falling below the target level. Despite this dissent, the majority of the committee members voted in favour of raising the interest rate, considering the potential negative impact of high inflation on households and businesses to outweigh the risks. The meeting minutes effectively demonstrate the informal aspects of the decision-making process. Modern macroeconomic frameworks provide insights into policy objectives and how they can be achieved through fiscal and monetary measures. This dissertation aims to develop a two-country New Keynesian Dynamic Stochastic General Equilibrium (NK DSGE) model for Scotland and the rest of the UK. The model also considers the existence of the world economy and international trading with other countries. The objective of the dissertation is not to build the most factually accurate model tailored to Scotland or the rest of the UK but build a model that would allow an assessment of regional variations in government spending responses under a predefined set of policy scenarios. More specifically, the dissertation considers a unified fiscal authority scenario in which there exists a single government able to issue bonds and raise taxes; and two autonomous fiscal authorities scenario in which both Scotland and the rest of the UK can issue bonds and raise taxes. Moreover, the dissertation allows fiscal authorities to raise tax revenue by lump-sum or distortionary labour tax. Across all policy scenarios, the model assumes that one monetary authority exists - the Bank of England, and both countries adhere to a UK-wide interest rate. Most model parameters are calibrated in line with the academic literature on DSGE models in the UK. A few model parameters related to government spending and taxes are estimated using Bayesian techniques. Finally, the estimated values are used to derive impulse response functions that are key to delivering the dissertation objectives.

A literature review is presented in the first section. It contrasts the ideas of Keynesian and Classical economists and how their century-long debate led to the development of the modern macroeconomic frameworks that are used by international organisations and national banks today. The last part of the literature review discusses fiscal policy in DSGE models. The literature review section is followed by a section describing the developed DSGE model. It is split into four parts: households, the government, firms, and equilibrium or market clearing conditions. Given the high number of employed equations by DSGE models, most derivations and proofs were put in Appendix, but the accompanying interpretation and intuition are provided in the main text. The fiscal policy parameters are estimated. Section 4 briefly introduces the Bayesian estimation of the model parameters, data used for estimating fiscal policy parameters, and estimation results. It also lists non-estimated parameters and strategies for calibrating them. The remaining sections are 'Dynamic Responses' and 'Limitations and Extensions'. The former section analyses the dynamic responses of the endogenous variables to two types of shocks: monetary and fiscal. It discusses the effect of government spending when the economy is introduced with a distortionary labour tax, as well as the asymmetry in responses when Scotland is considered fiscally autonomous. This section also provides a parameter sensitivity analysis - a demonstration and a discussion on how the dynamic responses vary given different sets of parameters. The limitations section discusses ways to extend the model to allow more sophisticated policy analysis or an improved data fit. The conclusion summarises all sections of the dissertation and reiterates key findings.

2 Literature Review

On the 10th of May, 2023, the Monetary Policy Committee at the Bank of England gathered to discuss the latest international and domestic data on economic activity. Even though the Committee has a 2% Consumer Price Index (CPI) inflation target, the UK's economy had undergone a sequence of very large and unexpected shocks and disturbances, resulting in twelve-month CPI inflation above 10%. The majority of the Committee members (78%) believed that an increase in interest rate "was warranted" (BoE, 2023:4), while the remaining members believed that the CPI inflation would "fall sharply in 2023" (*ibid.*, p. 5) as a result of the economy naturally adjusting to the effects of the energy price shocks. They feared that the preceding increases in the interest rate have not yet been internalised and raising the interest rate any further could result in a reduction of inflation "well below the target" (*ibid.*). This is an excellent illustration of the "informal dimension of the monetary policy process", that (Galsí and Gertler, 2007:26) referred to in their work explaining modern macroeconomic models and new frameworks. According to them, while the informal dimension cannot be removed, we can build formal and rigorous models that would help the Committee and institutions alike understand fiscal and monetary policy objectives and how these should be conducted. This task is not straightforward and has been central (albeit - fruitful) to most macroeconomic research in the past decades.

2.1 From Past Debates to Modern Frameworks

Snowdon, Vane, and Wynarczyk (1994) guide contrasts differences between competing schools of thought of the past century. It discusses the intellectual debate between John Maynard Keynes (1883-1946) and the "old" (as opposed to New) classical economists. According to Snowdon, Vane, and Wynarczyk (1994:42) the "Keynes v. Classics" debate started in the 1930s and "has continued in various forms ever since" (*ibid.*). Blanchard (2000) regards that period as the epoch of "exploration", as "all the right ingredients, and quite a few more were developed" (*ibid.*, p. 1376).

Indeed, the ideas of the time underpin mainstream schools of thought prevalent today: New Keynesianism and Real Business Cycle Theory (Galí, 2015:1; Snowdon, Vane, and Wynarczyk, 1994:42). Both schools of thought agree that any capitalist market economy does not always produce at its equilibrium level. They do disagree about the origin and persistence of such deviations (Snowdon, Vane, and Wynarczyk, 1994:43), as well as means for correcting them. For instance, classical economists believed such deviations are possible only in the short run because the market is efficient at restoring full employment equilibrium. The efficiency assumption stemmed from the belief that households and firms are rational and utility/profit maximisers, operating in a perfectly competitive environment with flexible prices and complete knowledge of market conditions. The classical economists did believe that persistent and long-run deviations are possible but only due to inefficient and undesirable government interference or monopolistic competition (*ibid.*). In stark contrast, Keynes questioned the "self-equilibrating properties of the economy" (*ibid.*, p. 89) and the role of the government. The central belief of the Keynesian school of thought is that the aggregate level of output is determined by the aggregate demand, which depends on unstable and inevitably uncertain expectations of investment profitability (*ibid.*, p. 65). That is, in the *General Theory*, Keynes (1936), cited in Snowdon, Vane, and Wynarczyk (1994:65), attributed output fluctuations to erratic shocks in investment. Keynes also "rejected the classical notion that interest was the reward for postponed current consumption" (*ibid.*, p. 66) and argued that the money liquidity preference[1] has much greater influence over the interest rate.[2] When the assumption of nominal rigidities in prices (such as the costs associated with updating prices, often referred to as price stickiness) is considered, monetary policy emerges as a viable tool that, according to Keynesians, can be utilised to stimulate aggregate demand. Keynesians believe that the government and central bank should counter-cyclically employ fiscal

[1] The demand for money holdings. A concept introduced by Keynes (Snowdon, Vane, and Wynarczyk, 1994:66).

[2] According to Blanchard (2000:1380), general equilibrium models suggest that both "loanable funds" and "liquidity preference" determine the interest rate, i.e. the "truth" is somewhere between the classical and Keynesian schools of thought.

and monetary policies, respectively, to ensure a more rapid return to full employment and stability.[3]

Intellectual debates and "exploration" was followed by an epoch of "consolidation" Blanchard (2000). In their panegyric and optimistic essay, the Blanchard (2000) argues that the period between 1940 and 1980 was the "golden age of macroeconomics" (*ibid.*, p. 1379) as the progress was "fast and visible". Hicks (1937) famously formalised ideas of the Keynesian economists and developed a widely known IS-LM model[4] that "integrates real and monetary factors in determining aggregate demand and therefore the level of output and employment" (Snowdon, Vane, and Wynarczyk, 1994:90). Over a few decades, the basic model or "skeleton apparatus" (Hicks, 1937:158) was developed in many directions. By the mid-1970s, it was able to accommodate a multitude of extensions, including "backward" and "forward" looking variables, rational expectations, and the Phillips curve linking inflation to unemployment (Blanchard, 2000:1382). However, according to Blanchard (2000:1382), the treatment of imperfections (deviations from the economy envisaged by the "old" classical school of thought economists) was too "casual" (superficial) and in the mid-1970s, Sargent, Fand, and Goldfeld (1973) prompted an "intellectual crisis" (Blanchard, 2000:1382) by showing that rational expectations of inflation, combined with Keynesian models, suggest the minimal effect of money on output. Following this, one camp of economists began exploring even "deeper" market imperfections and are aptly referred to as "New Keynesians". In contrast, the other camp explored models without any imperfections and developed models explaining business cycles primarily as fluctuations in the technology level. The former economists are referred to as "Real Business Cycle" (RBC) theorists. Since Kydland and Prescott (1982) and Prescott (1986), the role of monetary policy within the RBC framework became the primary reference and "to a large extent the core of macroeconomic theory" (Galí, 2015:2). The RBC theorists' devel-

[3]Counter-cyclical fiscal policy means increasing government spending, relaxing regulations, reducing taxation, etc. when the aggregate output decreases (and vice-versa). Similarly, counter-cyclical monetary policy means "raising" nominal interest rate when aggregate output increases and vice-versa.

[4]The abbreviation stands for "investment-saving" and "liquidity preference-money supply"

oped dynamic stochastic general equilibrium (DSGE) models proved essential and are widely adopted by researchers and economists.

Finally, ideas from both camps converged in less than twenty years, according to Blanchard (2000:1388), and most modern macroeconomic research focuses on imperfections in goods, labour, and financial markets using New Keynesian models (Galí, 2015). To summarise, according to Blanchard (2000), the macroeconomic research of the past century can be divided into three epochs based on the prevailing beliefs about the economy and frameworks of the time: Pre-1940, From 1940 to 1980, and Post-1980.

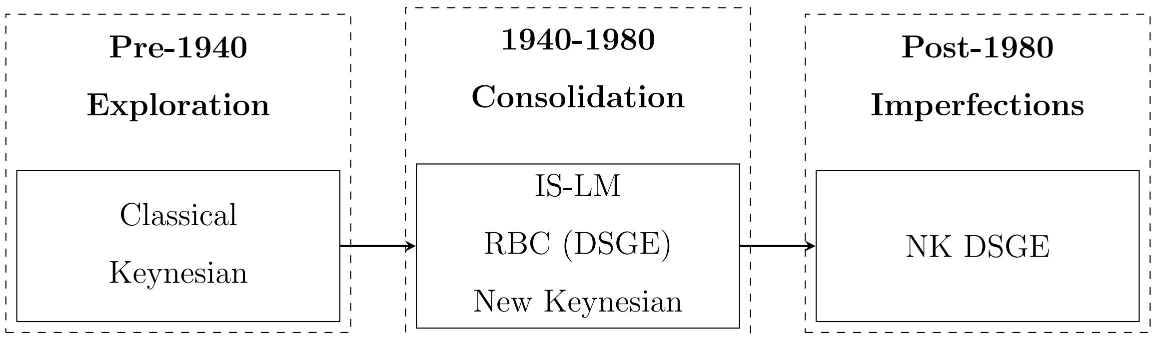

Figure 1: *Timeline of macroeconomic research, according to Blanchard (2000), combined with Snowdon, Vane, and Wynarczyk (1994) guide. Authors own illustration.*

2.2 Application of DSGE Models

As mentioned, DSGE models became the standard among researchers and macroeconomists in the public and private sectors. Smets and Wouters (2003) built a large-scale DSGE model tailored to the Euro area economy, and Smets and Wouters (2007:595) findings show that the theory-driven DSGE model exhibit forecasting performance akin to that of an a-theoretic (econometric) Bayesian VAR (with Minnesota-type prior) model. Following Adolfson et al. (2007), many recent models take into account the world economy and allow international trading. Two-country models were derived to allow examination of cross-country business cycles or the effect of business cycles originating in a foreign economy; examples include Kolasa (2009), who used a two-country DSGE model to analyse the potential effects of Poland joining the Euro

Area. Walque et al. (2017) and Gunter (2017) analysed exchange rate dynamics using a Euro Area-US model. International Monetary Fund (IMF) uses "GEM", a large-scale multi-country DSGE model for *World Economic Outlook* analyses (Tchakarov et al., 2004); Hervé et al. (2011) describes OECD's global model, while Albonico et al. (2019) presents *The Global Multi-Country Model* (GM), an estimated DSGE model for each Euro Area country, used by the European Central Bank. Albonico et al. (2019:3) lists DSGE models developed by national central banks tailored to their respective countries. At the same time, a recent survey by Yagihashi (2020) provides a quantitative and qualitative comparison of 84 DSGE models developed by policy institutions around the world.

2.3 Fiscal Policy in DSGE Models

To begin with, RBC models predict a negative response in consumption following an increase in government spending. More specifically, government spending is modelled to absorb resources, which makes households worse off and incentivises more hours worked. Greater labour supply for any given wage reduces firms' marginal cost and induces output (Baxter and King, 1993:319). That is, consumption, conditional on shocks in government spending, is countercyclical. Keynesian models, in stark contrast, predict the opposite.

Empirically, the findings of the Keynesian models are more in line with the observed macroeconomic patterns. For instance, Blanchard and Perotti (2002) performed a VAR analysis on the dynamics of consumption and government spending. They built six structural VAR models, one for each component of GDP: output, consumption, government spending, investment, export, and import. The key finding of the analysis is that government spending has a positive effect on consumption.

Galí, López-Salido, and Vallés (2005) show that NK DSGE models can be "recovered" by assuming that households have limited access to financial markets/saving technologies or are poor (they consume all of their labour income). Households that smooth their consumption by saving are often regarded as Ricardian households, while those that do not are referred to as non-Ricardian or *hand-to-mouth* households. Some of the latest NK DSGE literature models both types of households explicitly[5] with their ratio determined by a time-invarying exogenous coefficient. Arguably, such modelling would allow an improved fit of data. While modelling hand-to-mouth households is algebraically simpler than modelling the Ricardian households, modelling both types of households would drastically increase the complexity of the model, given the two-country and four policy scenarios setting. The absence of hand-to-mouth households is discussed in the limitations section.

[5] In fact, there exists literature with more than two types of households. For instance, a recent paper by Eskelinen (2021) models poor hand-to-mouth, wealthy hand-to-mouth, and non-hand-to-mouth households.

3 Presentation of the DSGE Model

Ricci (2019) was the first to build a large-scale two-country DSGE model explicitly tailored to Scotland and the rest of the UK. In an attempt to retain the model's simplicity while still allowing policy analysis, this dissertation will primarily build on the baseline model suggested by Gali and Monacelli (2005) and Galí (2015). In contrast, the model presented by Ricci (2019) was built upon the groundwork laid by Rabanal and Tuesta (2010), who were pioneers in the development of a medium-to-large two-country DSGE model. Neither Gali and Monacelli (2005) nor Galí (2015) models considered lump-sum or distortionary taxes, or government spending, more generally. While extensive literature covers government spending in DSGE models, few to none cover government spending in a small open economy (SOE) NK DSGE model, and even fewer apply it to a two-country setting. Therefore, most of the derivations had to be carried out using a pen and paper, and step-by-step derivations are provided in the Appendix. Many of Galí (2015) derivations relied on the assumption that steady-state output is equal to the steady-state consumption, i.e. $Y = C$. When the government term is introduced, many expressions lose their inherent elegance and simplicity. The dissertation also considered a zero government spending in the steady steady-state assumption to recover algebraic simplicity;[6] however, historically (over the sample period of twenty years), Scotland had significantly higher government spending-to-output ratio, compared to that of the rest of the UK (more on this in Section 4). Thus, the dissertation opted for not using a simplifying assumption because a major source of representative asymmetry in dynamic responses would have been forgone.

However, the focus of this dissertation is not to build the most factually accurate model of Scotland or the United Kingdom but to assess the asymmetric responses in government spending when Scotland is part of the fiscal union and when it is fiscally autonomous. The fiscal union scenario refers to the Westminister government collecting taxes from all four countries of the UK and distributing them according to the Barnett formula. The fiscal autonomy scenario refers to the Holyrood government's

[6] Log aggregate resource constraint $y_t = \left(1 - \frac{G}{Y}\right) c_t - \frac{G}{Y} g_t$ can be rewritten to $y_t = c_t - \hat{g}_t$ using

ability to collect tax revenue, issue bonds (borrow), and spend it at its sole discretion. We further break down the scenarios by allowing public expenses to be funded by lump-sum and distortionary (labour) taxes. This brings the number of policy scenarios considered by the dissertation to four. In all four policy scenarios, the dissertation assumes that a single monetary authority sets one UK-wide interest rate.[7]

Finally, in line with most of the literature, variables referring to the home country (Scotland) will be denoted without an asterisk, i.e., Y_t, while foreign country (the rest of the UK) will be denoted with an asterisk, i.e., Y_t^*. Y_t^{UK} and Y_t^W will denote UK- and world-wide variables, respectively. Given that Scotland and the rest of the UK are modelled as symmetrical,[8] Sections 2.1-2.4 describe the model only for Scotland, but for each presented equation in the Sections, there exists a corresponding equation for the rest of the UK economy. In cases when this is not true, it will be stated explicitly.

$\tilde{g}_t = G_t/Y$ instead of $g_t = (G_t - G)/G$.

[7] Due to technical limitations, the dissertation modelled two countries as having individual nominal interest rates. However, a population-weighted sum of the two interest rates was used in all applicable equations. Conceptually, it is equivalent to having one UK-wide interest rate, where Scotland's interest rate "influences" UK-wide interest rate ($\chi \approx 8.5\%$) but is primarily determined by the rest of the UK ($1 - \chi \approx 91.5\%$), so Scotland (almost) takes it as given.

[8] Equations can differ in parameter values but not structurally.

3.1 Households

This model assumes that there are infinitely many households in the economy. All households are assumed to be symmetric, i.e. have the same preferences and behave identically. Below, we consider a representative household that wants to maximise its lifetime utility:

$$\mathbb{E}_t \left\{ \sum_{t=0}^{\infty} \beta^t \, \mathcal{U}(C_t, N_t, Z_t) \right\} \tag{1}$$

$$\mathcal{U}(C_t, N_t, Z_t) = \begin{cases} \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right) Z_t & \text{if } \sigma \geq 0 \text{ and } \sigma \neq 1 \\ \left(\log(C_t) - \frac{N_t^{1+\varphi}}{1+\varphi} \right) Z_t & \text{if } \sigma = 1 \end{cases} \tag{2}$$

The household's utility depends on consumption C_t and hours worked N_t. As seen from the utility function (Equation 2), the model assumes the household's utility to be (decreasingly) increasing in consumption C_t and (increasingly) decreasing in hours worked N_t. $\beta \in (0, 1)$ is the discount factor, which can be thought of as an opportunity cost or an impatience rate, i.e. a unit of consumption C today will be worth $\beta * C < C$ tomorrow. We also introduce a preference shifter Z_t in line with Galí (2015:225) that captures changes in attitude towards consumption and leisure. The shock is assumed to follow an autoregressive process of order 1: $\log(Z_t) = \rho_z \log(Z_{t-1}) + \epsilon_t^z$.

The parameter $\sigma \geq 0$ is the relative risk aversion coefficient and $\varphi \geq 0$ is the labour disutility parameter. Together, they determine the curvature of the utility of consumption and the disutility of labour, respectively. Finally, $\mathbb{E}_t \{*\}$ are the expectational operators, conditional on all information available at period t.

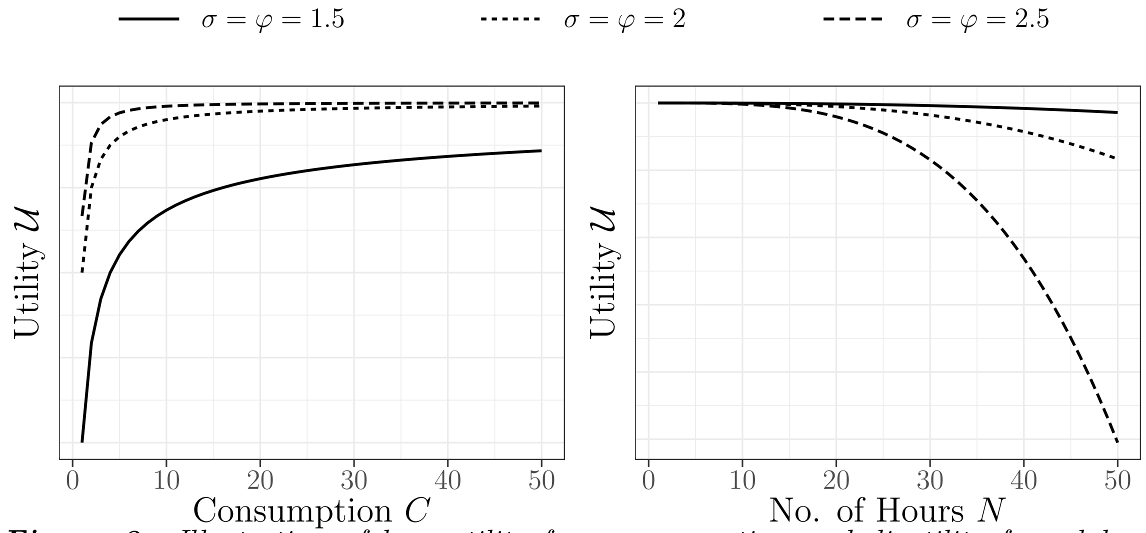

Figure 2: *Illustration of how utility from consumption and disutility from labour vary given three sets of parameters:* 1.5, 2, 2.5

To allow goods differentiation between domestic and foreign, the model assumes that C_t is a composite consumption index defined by:

$$C_t = \begin{cases} \left[(1-\upsilon)^{\frac{1}{\eta}}(C_{H,t})^{\frac{\eta-1}{\eta}} + \upsilon^{\frac{1}{\eta}}(C_{F,t})^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} & \text{if } \eta > 0 \text{ and } \eta \neq 1 \\ \frac{1}{(1-\upsilon)^{(1-\upsilon)}\upsilon^{\upsilon}}(C_{H,t})^{(1-\upsilon)}(C_{F,t})^{\upsilon} & \text{if } \eta = 1 \end{cases} \tag{3}$$

Where $C_{H,t}$ and $C_{F,t}$ are indices of consumption of home-produced and imported goods, respectively. The parameter $\upsilon \in [0,1]$ reflects the economy's openness for trading, while $\eta > 0$ denotes the household's willingness to substitute a domestic good with a foreign good, often referred to as 'home bias'. When $\eta = 1$, the share of domestic and foreign consumption is determined by the country's willingness to trade. In an extreme case, $\upsilon = 0$ would imply that the economy is an autarky, while $\upsilon = 1$ would suggest that our households consume foreign goods only. Our economy is assumed to be small in the sense that it takes the world's output, consumption, and prices as given and cannot influence them. This is a common assumption for the UK and even more so for Scotland. The world economy is assumed to be made of a continuum of infinitely many small economies i represented by a unit interval.

Therefore, $C_{F,t}$ is a sum of indices of the quantity of goods imported from all countries i. Similarly, if we denote j as a single variety of goods from a continuum of goods represented by a unit interval, we can express each consumption index as follows:

$$C_{H,t} = \left(\int_0^1 C_{H,t}(j)^{\frac{\varepsilon-1}{\varepsilon}}\,dj\right)^{\frac{\varepsilon}{\varepsilon-1}} \qquad \text{Home-produced goods consumption index} \qquad (4)$$

$$C_{i,t} = \left(\int_0^1 C_{i,t}(j)^{\frac{\varepsilon-1}{\varepsilon}}\,dj\right)^{\frac{\varepsilon}{\varepsilon-1}} \qquad i\text{-produced goods consumption index} \qquad (5)$$

$$C_{F,t} = \left(\int_0^1 C_{i,t}^{\frac{\gamma-1}{\gamma}}\,di\right)^{\frac{\gamma}{\gamma-1}} \qquad \text{Imported goods consumption index} \qquad (6)$$

Notice that all three indices take the form of *Constant Elasticity Substitution* (*CES*) form, with parameters ε (without subscripts) and γ representing the degree of substitutability between varieties of goods and countries, respectively.

The following expressions note the optimal allocation of each individual good (see Galí (2015:83) for derivation):

$$C_{H,t}(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon} C_{H,t} \qquad \text{Demand of home-produced good } j \qquad (7)$$

$$C_{i,t}(j) = \left(\frac{P_{i,t}(j)}{P_{i,t}}\right)^{-\varepsilon} C_{i,t} \qquad \text{Demand of } i\text{-produced good } j \qquad (8)$$

$$C_{i,t} = \left(\frac{P_{i,t}}{P_{F,t}}\right)^{-\gamma} C_{F,t} \qquad \text{Demand of } i\text{-produced goods} \qquad (9)$$

where:

$$P_{H,t} = \left(\int_0^1 P_{H,t}(j)^{1-\varepsilon}\,dj\right)^{\frac{1}{1-\varepsilon}} \qquad \text{Domestic Price Index} \qquad (10)$$

$$P_{i,t} = \left(\int_0^1 P_{i,t}(j)^{1-\varepsilon}\,dj\right)^{\frac{1}{1-\varepsilon}} \qquad \text{Price Index of } i\text{-produced goods} \qquad (11)$$

$$P_{F,t} = \left(\int_0^1 P_{i,t}^{1-\gamma}\,dj\right)^{\frac{1}{1-\gamma}} \qquad \text{Price Index of imported goods} \qquad (12)$$

Intuitively, Equations (7-8) show that if $P_{H,t}(j) > P_{H,t}$ (or $P_{i,t}(j) > P_{i,t}$), then that good is demanded less relative to an *average* home-produced (or i-produced) good.

The representative household's choice of consumption and labour must satisfy the following budget constraint:

$$\underbrace{\int_0^1 P_{H,t}(j)C_{H,t}(j)\,dj + \int_0^1\int_0^1 P_{i,t}(j)C_{i,t}(j)\,dj\,di + \mathbb{E}_t\left[R_{t+1}^{-1}B_{t+1}\right]}_{Expenses} \leq \underbrace{B_t + W_tN_t}_{Income} \quad (13)$$

where R_t is the gross nominal interest rate, B_t denotes bonds, W_t and N_t stand for nominal wage and hours worked, respectively. For intuition, the LHS of the budget constraint implies that the representative household needs to choose the quantity of good j produced domestically and in every country i, as well as the number of bonds at the expected nominal interest rate in period $t+1$. The RHS implies that the only two sources of income are nominal payoffs from bonds and gross pay, which will later differ from the net pay. The expenses cannot exceed income.

Taking the three price indices (10)-(12), and plugging them into their respective demand functions (9), yields:

$$\int_0^1 P_{H,t}(j)C_{H,t}(j)\,dj = P_{H,t}C_{H,t} \quad (14)$$

$$\int_0^1 P_{i,t}(j)C_{i,t}(j)\,dj = P_{i,t}C_{i,t} \quad (15)$$

$$\int_0^1 P_{i,t}C_{i,t} = P_{F,t}C_{F,t} \quad (16)$$

The following definitions are given:

$$C_{H,t} = (1-\upsilon)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t \quad (17)$$

$$C_{F,t} = \upsilon\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t \quad (18)$$

$$P_t = \begin{cases} [(1-\upsilon)(P_{H,t})^{1-\eta} + \upsilon(P_{F,t})^{1-\eta}]^{\frac{1}{1-\eta}} & \text{if } \eta > 0 \text{ and } \eta \neq 1 \\ (P_{H,t})^{1-\upsilon} \times (P_{F,t})^{\upsilon} & \text{if } \eta = 1 \end{cases} \quad (19)$$

Equations (17) and (18) are demand functions for domestic and foreign goods, respectively. Equation (19) is the CPI. When there is no home bias ($\eta = 1$), the *log* aggregate price level is just a weighted sum of the two price indices, where weights are given by trade openness parameter υ. Using (14) and (16), we can define the total consumption expenditures as:

$$P_{H,t}C_{H,t} + P_{F,t}C_{F,t} = P_tC_t \tag{20}$$

Which greatly simplifies the household's budget constraint:

$$P_tC_t + \mathbb{E}_t\left[R_{t+1}^{-1}B_{t+1}\right] \leq B_t + W_tN_t \tag{21}$$

Note that the budget constraint (as well as many other expressions introduced later) will vary depending on what policy scenario is considered. For instance, the household's budget constraints under each scenario are given below:

Scen. 1	Scot.	$P_tC_t + \mathbb{E}_t[R_{t+1}^{-1}B_{t+1}] = B_t + W_tN_t + T_t$
G: 2, $\tau : 0$	rUK	$P_t^*C_t^* + \mathbb{E}_t[R_{t+1}^{*-1}B_{t+1}^*] = B_t^* + W_t^*N_t^* + T_t^*$
Scen. 2	Scot.	$P_tC_t + \mathbb{E}_t[R_{t+1}^{UK-1}B_{t+1}^{UK}] = B_t^{UK} + W_tN_t + \chi T_t^{UK}$
G: 1, $\tau : 0$	rUK	$P_t^*C_t^* + \mathbb{E}_t[R_{t+1}^{UK-1}B_{t+1}^{UK}] = B_t^{UK} + W_t^*N_t^* + (1-\chi)T_t^{UK}$
Scen. 3	Scot.	$P_tC_t + \mathbb{E}_t[R_{t+1}^{-1}B_{t+1}] = B_t + (1-\tau_t)W_tN_t + T_t$
G: 2, $\tau : 1$	rUK	$P_t^*C_t^* + \mathbb{E}_t[R_{t+1}^{*-1}B_{t+1}^*] = B_t^* + (1-\tau_t^*)W_t^*N_t^* + T_t^*$
Scen. 4	Scot.	$P_tC_t + \mathbb{E}_t[R_{t+1}^{UK-1}B_{t+1}^{UK}] = B_t^{UK} + (1-\tau_t^{UK})W_tN_t + \chi T_t^{UK}$
G: 1, $\tau : 1$	rUK	$P_t^*C_t^* + \mathbb{E}_t[R_{t+1}^{UK-1}B_{t+1}^{UK}] = B_t^{UK} + (1-\tau_t^{UK})W_t^*N_t^* + (1-\chi)T_t^{UK}$

Table 1: *Household budget constraints for each of the policy scenarios.*

Here, χ denotes Scotland's share of the population in the United Kingdom. T_t denotes lump-sum transfers (subsidies or taxes), while τ_t denotes an income or labour tax rate. In the first column, G indicates the number of governments that can issue bonds (borrow) and set the labour tax rate. In the same column, τ indicates whether government spending is funded by a labour tax.

What follows is the derivation of the intratemporal and intertemporal optimality conditions for policy scenario 3, i.e. when households face a labour tax:

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right) Z_t$$
$$+ \lambda_t \left\{ B_t + (1-\tau_t) W_t N_t + T_t - P_t C_t - \mathbb{E}_t \left[R_{t+1}^{-1} B_{t+1} \right] \right\} \tag{22}$$

$$\frac{\partial \mathcal{L}}{\partial C_t} = \beta^t C_t^{-\sigma} Z_t - \lambda_t P_t = 0; \quad \Rightarrow \quad \beta^t C_t^{-\sigma} Z_t P_t^{-1} = \lambda_t \tag{23}$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = -\beta^t N_t^{\varphi} Z_t + \lambda_t (1-\tau_t) W_t = 0; \quad \Rightarrow \quad \beta^t N_t^{\varphi} Z_t ((1-\tau_t) W_t)^{-1} = \lambda_t \tag{24}$$

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = -\lambda_t \, \mathbb{E}_t \left[R_{t+1}^{-1} \right] + \mathbb{E}_t [\lambda_{t+1}] = 0; \quad \Rightarrow \quad \mathbb{E}_t [R_{t+1}^{-1}] = \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} \right] \tag{25}$$

Equating and rearranging Equations (23) and (24) yields *intratemporal optimality condition*:

$$\Rightarrow \qquad C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t}(1-\tau_t) \qquad \text{Scenario 3} \tag{26}$$

$$C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t}(1-\tau_t^{UK}) \qquad \text{Scenario 4} \tag{27}$$

$$C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t} \qquad \text{Scenarios 1 \& 2} \tag{28}$$

The condition implies that the marginal utility of consumption and leisure equals the net real wage. As mentioned before, in the case of labour tax absence, the net real wage is equal to the gross real wage. The log-linearisation of Equation (26) around a steady state yields:

$$C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t}(1-\tau_t) = \frac{W_t}{P_t} - \frac{W_t}{P_t}\tau_t$$

$$\vdots \quad \text{(see Appendix A.125 - A.133)}$$

$$(1-\tau)(\sigma c_t + \varphi n_t) = [(1-\tau)(w_t - p_t) - \tau \tilde{\tau}_t]$$

$$\sigma c_t + \varphi n_t = w_t - p_t - \frac{\tau}{1-\tau} \tilde{\tau}_t \tag{29}$$

Where τ and $\tilde{\tau}_t$ denote steady-state labour tax rate and deviation from the steady state, respectively. Note that under Scenarios 1 & 2, the tax term would disappear, while under Scenario 4, the term would become $\frac{\tau^{UK}}{1-\tau^{UK}}\tilde{\tau}_t^{UK}$. While log-linearising, we widely make use of Uhlig (1995) proposed methods for multivariate equations with additive terms, i.e., $X_tY_t \approx XY\mathbf{e}^{\tilde{X}_t+\tilde{Y}_t}$, $X_t + Y_t \approx X\mathbf{e}^{\tilde{X}_t} + Y\mathbf{e}^{\tilde{Y}_t}$ and $\mathbf{e}^{\tilde{X}_t} \approx (1 + \tilde{X}_t)$. Iterating Equation (23) one period forward, yields:

$$\frac{\partial L}{\partial C_t} = \beta^t C_t^{-\sigma} Z_t P_t^{-1} = \lambda_t \quad \Rightarrow \quad \mathbb{E}_t[\beta^{t+1} C_{t+1}^{-\sigma} Z_{t+1} P_{t+1}^{-1}] = \mathbb{E}_t[\lambda_{t+1}]$$

Dividing one by the other and rearranging yields *intertemporal optimality condition*:

$$\beta\,\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{Z_{t+1}}{Z_t}\right)\left(\frac{P_t}{P_{t+1}}\right)\right] = \mathbb{E}_t\left[\frac{1}{R_{t+1}}\right] \tag{30}$$

$$\beta^{t-(t+1)}\,\mathbb{E}_t\left[\frac{C_t^{-\sigma} Z_t P_t^{-1}}{C_{t+1}^{-\sigma} Z_{t+1} P_{t+1}^{-1}}\right] = \mathbb{E}_t\left[\frac{1}{R_{t+1}}\right] \tag{31}$$

$$\beta\,\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{Z_{t+1}}{Z_t}\right)\left(\frac{P_t}{P_{t+1}}\right)\right] = \mathbb{E}_t\left[\frac{1}{R_{t+1}}\right] \tag{32}$$

where Equation (31) used Equation (25). $\mathbb{E}_t[R_{t+1}^{-1}]$ is the gross return on a risk-free one-period discount bond or a stochastic discount factor. More generally, Equation (32) is the Euler equation, and it determines the consumption path of a lifetime utility-maximising representative household. To state it in more intuitive terms, households choose consumption "today" and "tomorrow" and take all other terms as given. According to the equation, they choose consumption in the two periods in such a way so that the marginal utility of consumption"today" would be equal to the marginal utility of consumption "tomorrow" while taking into account that saving consumption "today", will result in $R_t > 1$ consumption "tomorrow". Note that Galí (2015) uses a different approach to derive the Euler equation, which introduces Arrow securities (for step-by-step derivation and interpretation, see A.161-A.166). Also note, that Galí (2015) uses $Q_t = \mathbb{E}_t[\frac{1}{R_{t+1}}]$ to denote the stochastic discount factor and D_t to denote bonds or "portfolio" as they call it. Log-linearising (32):

$$\begin{aligned}
\beta\,\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{Z_{t+1}}{Z_t}\right)\left(\frac{P_t}{P_{t+1}}\right)\right] &= \mathbb{E}_t\left[\frac{1}{R_{t+1}}\right] \\
\ln\beta - \mathbb{E}_t[\sigma c_{t+1}] + \sigma c_t + \mathbb{E}_t[z_{t+1}] - z_t + p_t - \mathbb{E}_t[p_{t+1}] &= -\ln R_{t+1} \\
&\vdots \quad \text{(see Appendix A.135 - A.143)} \\
c_t = \mathbb{E}_t[c_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho) - \frac{1}{\sigma}&(1-\rho_z)z_t \qquad (33)
\end{aligned}$$

where $i_t = \ln R_{t+1}$ is the nominal interest rate, $\rho = -\log\beta$ is the log discount rate, and $\pi_t = p_t - p_{t-1}$ is the CPI inflation. The log-linearised Euler equation makes it clearer to see that consumption "today" is increasing in expected inflation "tomorrow" and decreasing in the nominal interest rate as saving becomes *relatively* more rewarding behaviour. The effect is scaled by the inverse of the risk aversion coefficient σ, i.e. risk-loving households (low σ) are confident about the *expected* rate of inflation and adjust their consumption "today" accordingly, while risk-averse households (high σ) do not.

Furthermore, OECD (2022) defines terms of trade as a ratio of import and export price indices, which in this model is denoted as S_t:

$$S_{i,t} = \frac{P_{i,t}}{P_{H,t}} \qquad (34)$$

$$S_t = \frac{P_{F,t}}{P_{H,t}} = \left(\int_0^1 (S_{i,t}\,di)^{1-\gamma}\right)^{\frac{1}{1-\gamma}} \qquad (35)$$

where Equation (34) marks *bilateral* terms of trade with a country i, while Equation (35) is for *effective* terms of trade, i.e. terms of trade with all countries in the unit interval defined earlier. The latter can be log-linearised to yield:

$$s_t = p_{F,t} - p_{H,t} = \left(\int_0^1 s_{i,t}\,di\right) \qquad (36)$$

Recall that when $\eta = 1$, then CPI is $P_t = (P_{H,t})^{1-\upsilon} \times (P_{F,t})^{\upsilon}$. Using the previous definition (36) and log-linearised CPI, the price level can be expressed as a sum of the domestic price level and terms of trade:

$$p_t = (1-\upsilon)p_{H,t} + \upsilon p_{F,t} \tag{37}$$

$$= (1-\upsilon)p_{H,t} + \upsilon(s_t + p_{H,t}) \tag{38}$$

$$= p_{H,t} + \upsilon s_t \tag{39}$$

Note that Equations (36) and (39) hold *exactly* when $\gamma = 1$ and $\eta = 1$. Similarly, knowing that inflation is a difference of log prices in two consecutive periods, we can extend the previous definition to yield:

$$\pi_{H,t} = p_{H,t+1} - p_{H,t} \qquad \text{Domestic Inflation} \tag{40}$$

$$\pi_t = \pi_{H,t} + \upsilon\Delta s_t \qquad \text{CPI Inflation} \tag{41}$$

The gap between domestic inflation and CPI inflation is due to percentage changes in terms of trade and degree of openness. In the case of an autarky ($\upsilon = 0$), even if imported goods were much more expensive ($P_{F,t} \gg P_{H,t}$), domestic inflation will be equal to CPI inflation because the country simply does not trade.

Furthermore, we assume that the Law of One Price (LOP) holds for all goods j. That is, the price of a single good in country i is equal to the price of the same good in country $-i$ times the nominal exchange rate.[9] It implies, that there are no opportunities for arbitrage.

$$P_{i,t}(j) = \mathcal{E}_{i,t}P^i_{i,t}(j)$$

$$P_{i,t} = \mathcal{E}_{i,t}P^i_{i,t} \qquad \text{Law of One Price (LOP)} \tag{42}$$

where $\mathcal{E}_{i,t}$ is the nominal exchange rate between the home currency and the country's i currency, and the second equation is derived by integrating both sides with respect

[9]Here and in later sections, "$-x$" index will refer to "all but x" indices

to j. Even though we do not model currencies explicitly, it is useful to think about $\mathcal{E}_{i,t}$ as the price of one unit of currency in terms of another, i.e. the home currency. The two equations can be log-linearised to yield:

$$p_{i,t} = e_{i,t} + p^i_{i,t} \qquad \text{(Log) Law of One Price (LOP)} \tag{43}$$

$$p_{F,t} = \int_0^1 (e_{i,t} + p^i_{i,t})\, di = e_t + p^W_t \qquad \text{(Log) Price index of Imported Goods} \tag{44}$$

Where e_t is (Log) Effective Nominal Exchange Rate, p^W_t is the World Price Index. This allows us to redefine log effective terms of trade in terms of the nominal exchange rate, domestic price, and the world price index:

$$s_t = p_{F,t} - p_{H,t} = e_t + p^W_t - p_{H,t} \tag{45}$$

In contrast, *real* exchange rate between two countries is the ratio between their CPI and home CPI, expressed in home currency:

$$\mathcal{Q}_{i,t} = \frac{\mathcal{E}_{i,t} P^i_t}{P_t} \qquad \text{Bilateral Exchange Rate} \tag{46}$$

Integrating both sides with respect to i and using previous definitions yields:

$$q_t = \int_0^1 \log\left(\frac{\mathcal{E}_{i,t} P^i_t}{P_t}\right) di \tag{47}$$

$$= \int_0^1 (e_{i,t} + p^i_{i,t} - p_t)\, di \tag{48}$$

$$= e_t + p^W_t - p_t \qquad \text{using (44)} \tag{49}$$

$$= s_t + p_{H_t} - p_t \qquad \text{using (45)} \tag{50}$$

$$= (1 - \upsilon) s_t \qquad \text{using (39)} \tag{51}$$

Finally, if we assume that all countries i have symmetrical preferences and their households maximise lifetime utility in the same manner that the home country's do, then maximising the Lagrangian function for country i will yield:

$$\frac{\partial \mathcal{L}^i}{\partial C_t^i} = \beta^t (C_t^i)^{-\sigma} Z_t^i (\mathcal{E}_{i,t} P_t^i)^{-1} = \lambda_t^i$$
$$\frac{\partial \mathcal{L}^i}{\partial D_{t+1}^i} = -\lambda_t^i \, \mathbb{E}_t[R_{t,t+1}^{-1}] = -\,\mathbb{E}_t[\lambda_{t+1}^i]$$
$$\vdots \quad \text{(see Appendix A.144 - A.160)} \tag{52}$$
$$C_t = C_t^i {Z_t^i}^{\frac{1}{\sigma}} \mathcal{Q}_{i,t}^{\frac{1}{\sigma}} \tag{53}$$

If we assume that there had been no shocks in the country's i preferences ($Z_t^i = 1$), then Equation (53) states that consumption in the home country is equal to the consumption in the country i, while taking into account bilateral real exchange rate. This can be generalised to derive a relationship between home consumption and world consumption by log-linearising (for simplicity) and integrating both sides with respect to i:

$$c_t = c_t^i + \frac{1}{\sigma} z_{i,t} + \frac{1}{\sigma} q_{i,t} \tag{54}$$
$$\int_0^1 c_t \, di = \int_0^1 \left(c_t^W + \frac{1}{\sigma} z_{i,t} + \frac{1}{\sigma} q_{i,t} \right) di \tag{55}$$
$$c_t = c_t^W + \frac{1}{\sigma} z_t + \left(\frac{1-\upsilon}{\sigma} \right) s_t \qquad \text{using } q_t = (1-\upsilon) s_t \tag{56}$$
$$= y_t^W + \frac{1}{\sigma} z_t + \left(\frac{1-\upsilon}{\sigma} \right) s_t \qquad \text{using } c_t^W = y_t^W \tag{57}$$

c_t^W is the log world consumption, and the last Equation (57) follows by assuming that world consumption is equal to the world output, i.e., there is no world government spending, or national government spending in any country i is infinitesimally small.

The following two parts will discuss government spending and firms, respectively. The final part will provide equilibrium (market clearing) conditions.

3.2 Government

As mentioned in the literature review, this dissertation makes two essential assumptions related to government spending: households are assumed to be Ricardian, and; government spending is non-productive. The latter follows from the fact that government spending does not enter the utility function nor the firm's production function. Therefore, government spending is equivalent to reducing the quantity of available resources. In line with most of the literature, deviations from the steady state government spending are assumed to be temporary, i.e. following an autoregressive process of order 1, as opposed to a permanent increase considered by Baxter and King (1993). Below is a typical budget constraint faced by the government every period:

$$\underbrace{\mathbb{E}_t[R_t^{-1}B_{t+1}] + P_tT_t}_{Revenue} = \underbrace{P_tG_t + B_t}_{Spending} \tag{58}$$

That is, government accrues revenue by collecting taxes in current prices P_tT_t and by issuing bonds at current gross return rate $\mathbb{E}_t[R_t^{-1}B_{t+1}]$. The government needs to pay one unit of consumption good for each mature bond issued previous period B_t and pay for its spending in current prices P_tG_t. The government budget constraint varies depending on the policy scenario considered:

Scen. 1 & Scen. 3	Scot.	$\mathbb{E}_t[R_t^{-1}B_{t+1}] + P_tT_t = P_tG_t + B_t$
G: 2, $\tau \in \{0, 1\}$	rUK	$\mathbb{E}_t[R_t^{*-1}B_{t+1}^*] + P_t^*T_t^* = P_t^*G_t^* + B_t^*$
Scen. 3 & Scen. 4 G: 1, $\tau \in \{0, 1\}$	UK	$\mathbb{E}_t[R_t^{UK-1}B_{t+1}^{UK}] + P_t^{UK}T_t^{UK} = P_t^{UK}G_t^{UK} + B_t^{UK}$

Table 2: *Government budget constraints under different policy scenarios*

where P_t^{UK} is a weighted sum of price levels in Scotland and the rest of the UK, i.e., $P_t^{UK} = \chi P_t + (1-\chi)P_t^*$. Similarly, a monetary union implies a single rate of gross return, which we define as $R_t^{UK} = \chi R_t + (1-\chi)R_t^*$. Also note the log-linearised government budget constraints in real terms (see Appendix A.167 - A.182 for derivation):

Scen. 1 & Scen. 3	Scot.	$b_{t+1} = (1+\rho)(b_t + g_t - t_t)$
G: 2, $\tau \in \{0,1\}$	rUK	$b^*_{t+1} = (1+\rho^*)(b^*_t + g^*_t - t^*_t)$
Scen. 3 & Scen. 4 G: 1, $\tau \in \{0,1\}$	UK	$b^{UK}_{t+1} = (1+\rho^{UK})(b^{UK}_t + g^{UK}_t - t^{UK}_t)$

Table 3: *(Log) Government budget constraints under different policy scenarios*

where $\rho = \beta^{-1} - 1$ pins down the steady state interest rate. In each of the scenarios, the government tax revenue is accrued either by imposing lump-sum taxes or a labour tax:

Scen. 1	Scot.	$P_t T_t = P_t G_t$
G: 2, $\tau : 0$	rUK	$P^*_t T^*_t = P^*_t G^*_t$
Scen. 2 G: 1, $\tau : 0$	UK	$P^{UK}_t T^{UK}_t = P^{UK}_t G^{UK}_t$
Scen. 3	Scot.	$P_t T_t = \tau_t W_t N_t$
G: 2, $\tau : 1$	rUK	$P^*_t T^*_t = \tau^*_t W^*_t N^*_t$
Scen. 4 G: 1, $\tau : 1$	UK	$P^{UK}_t T^{UK}_t = \chi \tau^{UK}_t W_t N_t + (1-\chi)\tau^{UK}_t W^*_t N^*_t$

Table 4: *Tax revenue under each of the policy scenarios*

That is, in the case of lump-sum taxes, tax revenue equals government spending; similarly, in the case of labour taxes, tax revenue is a share ($\tau_t \in [0,1]$) of nominal labour income. However, given that the two sources of income are close substitutes for the government (it can raise revenue either by raising taxes or by issuing bonds), tax revenue and budget constraint alone do not lead to a stable equilibrium (there are many "solutions"). The governing of this relationship is captured by a fiscal rule of the following form:

Scen. 1 & Scen. 3 G: 2, $\tau \in \{0,1\}$	Scot. rUK	$t_t = \phi_g g_t + \phi_b b_t$ $t_t^* = \phi_g^* g_t^* + \phi_b^* b_t^*$
Scen. 3 & Scen. 4 G: 1, $\tau \in \{0,1\}$	UK	$t_t^{UK} = \phi_g^{UK} g_t^{UK} + \phi_b^{UK} b_t^{UK}$

Table 5: *Fiscal rules under each of the policy scenarios*

where following Galí, López-Salido, and Vallés (2005), we define $g_t = \frac{G_t - G}{Y}$, $b_t = \frac{(B_t/P_{t-1})-(B/P)}{Y}$, and $t_t = \frac{T_t - T}{Y}$. The model assumes that the government's budget is balanced in the steady state, i.e. $\frac{B}{Y} = 0$. Each fiscal policy rule states that tax revenue responds to changes in government spending and government debt, where the responses depend on $\phi_b \in [0,1]$ and $\phi_g \in [0,1]$. Intuitively, setting ϕ_b to a value close to 1 would imply that tax revenue (and tax rate) are highly responsive to debt (b_t). Similarly, a high ϕ_g value suggests high responsiveness of tax revenue (and tax rate) to government spending (g_t). Plugging fiscal rules into the log-linearised government budget constraints yields:

Scen. 1 & Scen. 3 G: 2, $\tau \in \{0,1\}$	Scot. rUK	$b_{t+1} = (1-\rho)(1-\phi_g)g_t + (1-\rho)(1-\phi_b)b_t$ $b_{t+1}^* = (1-\rho^*)(1-\phi_g^*)g_t^* + (1-\rho^*)(1-\phi_b^*)b_t^*$
Scen. 3 & Scen. 4 G: 1, $\tau \in \{0,1\}$	UK	$b_{t+1}^{UK} = (1-\rho^{UK})(1-\phi_g^{UK})g_t^{UK} + (1-\rho^{UK})(1-\phi_b^{UK})b_t^{UK}$

Table 6: *Future bonds are determined by government spending and current government debt*

These equations determine the equilibrium path for bond supply, with $(1-\rho)(1-\phi_b) < 1$ to ensure stability (non-explosiveness). Also, setting $\phi_g = 1$ would result in $b_t = 0$ for all t and ϕ_b, implying that government spending would have to be solely funded by lump-sum or labour taxes. Finally, note that B_t is a predetermined (state) variable, meaning its value is determined in the previous period, while the initial B_0 value is assumed to have been determined "historically", a.k.a given.

3.3 Firms

In line with the literature, we assume a continuum of infinitesimally small firms, each producing a single good j over which they have monopolistic power. Given the monopolistic nature of the market for each good, the firms are allowed to adjust their prices to maximise profits but, following Calvo (1983), we assume that only a fraction $\theta \in [0, 1]$ of them actually do. The firms are assumed to be owned by households, implying that our budget constraint should have a Π_t (dividend) term. However, as all households own all firms and take profit/dividends as given, it does not affect first-order conditions/optimal behaviour and, as such, is not considered. The firm's production function is given by:

$$Y_t(j) = A_t N_t(j)^{1-\alpha} \tag{59}$$

where A_t is the technology level shifter common to all firms and is assumed to evolve exogenously as an AR(1) process in log terms: $\ln A_t = \rho_a \ln A_{t-1} + \varepsilon_t^a$. Notice that the marginal product of labour is

$$\frac{\partial Y_t(j)}{\partial N_t(j)} = (1-\alpha) A_t N_t(j)^{-\alpha} \tag{60}$$

which is increasing in technology level (increases labour productivity). Second-order derivate with respect to hours worked indicates that the production function exhibits decreasing returns to scale. Knowing this, the firm maximises its profit by choosing the optimal amount of labour:

$$\max_{N_t(j)} \quad \mathcal{F} = P_{H,t}(j) Y_t(j) - W_t(j) N_t(j) \tag{61}$$

$$\text{s.t.} \quad Y_t(j) = A_t N_t(j)^{1-\alpha} \tag{62}$$

which can be solved to yield an optimality condition:

$$\frac{\partial \mathcal{F}}{\partial N_t(j)} \implies \frac{W_t}{P_t} = (1-\alpha)A_t N_t^{-\alpha} \tag{63}$$

Suggesting that the firm will hire workers until the marginal product of labour is equal to the real wage. The optimality condition also acts as a link between real wage $(w_t - p_t)$, labour and technology. Moreover, given that the marginal cost Ψ_t needs to equal domestic price level, rearranging Equation (63) yields:

$$\Psi_t = \frac{W_t}{(1-\alpha)A_t N_t^{-\alpha}} \qquad \text{Marginal cost} \tag{64}$$

$$\psi_t = w_t - (a_t - \alpha n_t + log(1-\alpha)) \qquad \text{(Log) Marginal cost} \tag{65}$$

Unsurprisingly, the marginal cost is increasing in wages and decreasing in the marginal product of labour. It is important to emphasise that definition above is an *average* marginal cost across all firms producing goods j. The marginal cost varies across firms due to different existing levels of labour.

As mentioned earlier, price stickiness is introduced by assuming that firms update their prices with probability $(1-\theta)$. The newly set price is denoted as $\overline{P}_{H,t}(j)$. Following a similar proof offered by Galí (2015) for a closed-economy NK DSGE, note that if all firms are symmetrical, then they will choose the same price, i.e. $\overline{P}_{H,t}(j) = \overline{P}_{H,t}$. Thus, the domestic price index from Equation (10) can be rewritten as:

$$P_{H,t} = \left[\int_0^1 (P_{H,t}(j))^{1-\epsilon}\, dj\right]^{\frac{1}{1-\epsilon}} \tag{66}$$

$$= \left[\int_{S(t)}^1 P_{H,t}(j)^{1-\epsilon}\, dj + \int_0^{S(t)} P_{H,t}(j)^{1-\epsilon}\, dj\right]^{\frac{1}{1-\epsilon}} \tag{67}$$

$$= \left[\theta(P_{H,t-1})^{1-\epsilon} + (1-\theta)(\overline{P}_{H,t})^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}} \tag{68}$$

where $S(t)$ is a subset of firms that do not update their prices, and Equation (68) "follows from the fact that the distribution of prices among firms not adjusting in period t corresponds to the distribution of effective prices in period $t-1$, though with total mass reduced to θ" (Galí, 2015:84); similarly, the total mass of firms with prices $\overline{P}_{H,t}$ will be equal to $(1-\theta)$. Dividing (68) by $(P_{H,t-1})^{1-\epsilon}$ yields:

$$\Pi_{H,t}^{1-\epsilon} = \theta + (1-\theta)\left(\frac{\overline{P}_{H,t}}{P_{H,t-1}}\right)^{1-\epsilon} \tag{69}$$

Log-linearising (69) around zero inflation steady state yields:

$$\Pi_{H,t}^{1-\epsilon} = \theta + (1-\theta)\left(\frac{\overline{P}_{H,t}}{P_{H,t-1}}\right)^{1-\epsilon} \tag{70}$$

$$\Pi_H \mathbf{e}^{(1-\epsilon)\pi_{H,t}} = \theta + (1-\theta)\frac{\overline{P_H}}{P_H}\mathbf{e}^{(1-\epsilon)(\bar{p}_{H,t}-p_{H,t})} \tag{71}$$

$$\vdots \quad \text{(see Appendix A.302 - A.305)}$$

$$\pi_{H,t} = (1-\theta)(\bar{p}_{H,t} - p_{H,t}) \tag{72}$$

Intuitively, this means that log domestic inflation depends on two elements: the difference between the current and new domestic price levels, and the price stickiness parameter θ. Consider two extreme cases when $\theta = 1$ and $\theta = 0$: when $\theta = 1$, then no firms would be permitted to update their prices and the domestic inflation would always be equal to zero (CPI would still vary due to terms of trade, assuming $\upsilon \neq 0$). When $\theta = 0$, then all firms *immediately* react to any changes in the marginal cost of production in line with Classical/RBC models. Firms that get to update their prices do so by maximising their *discounted lifetime cash flow*:

$$\max_{\overline{P}_{H,t}} \sum_{k=0}^{\infty} \theta^k \, \mathbb{E}_t \left[\Lambda_{t,t+k}\left(\underbrace{\overline{P}_{H,t}Y_{t+k|t}}_{Revenue} - \underbrace{\mathcal{C}_{t+k}(Y_{t+k|t})}_{Cost}\right)/P_{H,t+k}\right] \tag{73}$$

$$\text{s.t.} \quad Y_{t+k|t} = \left(\frac{\overline{P}_{H,t}}{P_{H,t+k}}\right)^{-\epsilon} C_{t+k} \tag{74}$$

where $\mathbb{E}_t\left[\Lambda_{t,t+k}\right]\ \forall k \geq 0$ is the expected stochastic discount factor used to discount profit (revenue less cost) in every period starting from current. $Y_{t+k|t}$ is the expected output in periods $t+k$ given output in period t, and $\mathcal{C}_{t+k}$ is the nominal cost of producting the expected output. The maximisation problem is subject to k number of demand constraints (74). Substituting the constraint into (73), taking first-order conditions, and log-linearising around perfect-foresight zero-inflation steady state yields:[10]

$$\bar{p}_{H,t} = \mu + (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k\,\mathbb{E}_t\left[\psi_{t+k|t}\right] \tag{75}$$

where $\mathbb{E}_t\left[\psi_{t+k|t}\right]$ and μ are the expected log marginal cost and desired[11] markup, respectively. Equation (75) is known as the (log) *optimal price setting condition* and presents firms as forward-looking discounted profit maximisers. Note that it is consistent with the previous exposition of the two extreme cases when $\theta = 0$ and $\theta = 1$, i.e. when $\theta = 1$, then $\pi_{H,t} = \bar{p}_{H,t} - \bar{p}_{H,t-1} = \mu - \mu = 0$. Furthermore, the earlier discussion on optimality condition noted that firms would hire labour until the real wage equals the marginal product of labour. Given that the price (and, subsequently, real wage) changed, firms might choose a different level of labour compared to that of other firms. This allows us to derive a relationship between the average marginal cost and a marginal cost of a firm that just updated its price:

$$\psi_{t+k|t} = \psi_{t+k} + \alpha(n_{t+k|t} - n_{t+k}) \tag{76}$$

$$= \psi_{t+k} + \frac{\alpha}{1-\alpha}(n_{t+k|t} - n_{t+k}) \tag{77}$$

$$= \psi_{t+k} - \frac{\alpha\epsilon}{1-\alpha}(\bar{p}_{H,t} - p_{t+k}) \tag{78}$$

Plugging (78) to (75) and rewritting as a recursive equation yields:

$$\bar{p}_{H,t} = \beta\theta\,\mathbb{E}_t\left[\bar{p}_{H,t+1}\right] + (1-\beta\theta)(p_{H,t} - \Theta\hat{\mu}_t) \tag{79}$$

[10] As none of the fiscal terms (g_t, τ_t, t_t, b_t) enter optimal price setting equation, derivations are standard and as such are not provided in the Appendix.

[11] Markup that occurs under flexible or frictionless prices

where $\mu_t = p_{H,t} - \psi_t$ is the average markup, $\hat{\mu}_t$ is the gap between the average and desired markups, and $\Theta = \frac{1-\alpha}{1-\alpha+\alpha\epsilon}$. Using (72), it is easy to transform (79) into a version of the New Keynesian Phillips Curve (NKPC):

$$\pi_{H,t} = \beta\, \mathbb{E}_t \left[\pi_{H,t+1}\right] - \lambda \hat{\mu}_t \tag{80}$$

where $\lambda = \frac{(1-\theta)(1-\beta\theta)}{\theta}\Theta$. Equation (80) suggests that inflation will be increasing as long as $\mu > \mu_t$. That is, price-adjusting firms will continue raising prices until the difference between the marginal cost and the domestic price level is smaller than the desired markup, i.e. $p_{H,t} - \psi_t < \mu$. At $p_{H,t} - \psi_t = \mu$, the domestic inflation will (slowly) tend towards the steady state level of zero due to the discount factor $\beta \in (0, 1)$. The next section will show that the markup gap can be expressed as a function of the natural output gap, which itself is a function of technology level, taxes, government spending, world output, and the preference shifter. Therefore, the resulting expression (NKPC) will pin the relationship between domestic inflation and all other real variables. This is why NKPC is regarded as one of the key (non-policy) equations in NK DSGE models.

3.4 Equilibrium

The goods market for a specific good j clears when domestic firms produce just enough of the good to satisfy the demand of home households, foreign households, and the home government. In line with Galí (2015), the demand for exports of good j is taken to be given as:

$$X_t(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\epsilon} X_t \tag{81}$$

$$\text{where} \quad X_t = \left(\int_0^1 X_t(j)^{\frac{\epsilon-1}{\epsilon}}\, dj\right)^{\frac{\epsilon}{\epsilon-1}} \tag{82}$$

$$= \upsilon \left(\frac{P_{H,t}}{\mathcal{E}_t \overline{P}_{H,t}}\right)^{-\eta} Y_t^{\star} \tag{83}$$

$$= \upsilon \mathcal{S}_t^{\eta} Y_t^{\star} \tag{84}$$

where (82) is the index of aggregate exports, (83) determines aggregate exports as a function of world output (the relationship is assumed to be given), and (84) is derived by substituting the definition of the effective terms of trade. Differently than Galí (2015), we introduce index of government-purchased goods:

$$G_t = \left(\int_0^1 G_t(j)^{\frac{\epsilon-1}{\epsilon}}\, dj\right)^{\frac{\epsilon}{\epsilon-1}} \tag{85}$$

so that the government demand of any home-produced good j is defined as:[12]

$$G_t(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\epsilon} G_t \tag{86}$$

[12] Derivation *steps* are identical to those of Equation 7

Therefore, the total demand for good j is:

$$\begin{aligned} Y_t(j) &= C_{H,t}(j) + X_t(j) + G_t(j) \qquad (87)\\ &= \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon} C_{H,t} + \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\epsilon} X_t + \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\epsilon} G_t \\ &= \left(\frac{P_{H,t}(j)}{P_{H,t}}^{-\varepsilon}\right) [C_{H,t} + X_t + G_t] \\ &= \left(\frac{P_{H,t}(j)}{P_{H,t}}^{-\varepsilon}\right) \left[(1-\upsilon)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \upsilon \mathcal{S}^{\eta} Y_t^{\star} + G_t\right] \qquad (88) \end{aligned}$$

which can be plugged in to definition of aggregate output $Y_t = \left(\int_0^1 Y_t(j)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}$ to yield:

$$Y_t = (1-\upsilon)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \upsilon \mathcal{S}^{\eta} Y_t^{\star} + G_t \qquad (89)$$

Note that Equation (87) and all subsequent derivations (marginally) vary depending on the policy scenario in question:

Scen. 1 & Scen. 3	Scot.	$Y_t(j) = C_t(j) + X_t(j) + G_t(j)$
G: 2, $\tau \in \{0,1\}$	rUK	$Y_t^*(j) = C_t^*(j) + X_t^*(j) + G_t^*(j)$
Scen. 3 & Scen. 4	Scot.	$Y_t(j) = C_t(j) + X_t(j) + \chi G_t^{UK}(j)$
G: 1, $\tau \in \{0,1\}$	rUK	$Y_t^*(j) = C_t^*(j) + X_t^*(j) + (1-\chi) G_t^{UK}(j)$

Table 7: *Demand for good j under different policy scenarios*

That is when there is a single government (Westminister), then we assume that Scotland's contribution towards covering government spending is equal to the population share in the UK. Equation (87) can be log-linearised around a symmetric steady state to yield:

$$Y_t = (1-\upsilon)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \upsilon \mathcal{S}_t^{\eta} Y_t^* + G_t \tag{90}$$

$$Y\mathbf{e}^{y_t} = (1-\upsilon)\left(\frac{P}{P_H}\right)^{\eta} C\mathbf{e}^{-\eta p_{H,t}+\eta p_t + c_t} + \upsilon S^{\eta} Y^* \mathbf{e}^{\eta s_t + y_t^*} + G\mathbf{e}^{g_t} \tag{91}$$

$$\vdots \quad \text{(see Appendix A.183 - A.190)}$$

$$y_t = C_Y\left[(1-\upsilon)c_t + \upsilon(2-\upsilon)\eta s_t + \upsilon y_t^*\right] + G_Y g_t \tag{92}$$

where $C_Y = \frac{C}{Y}$ and $G_Y = \frac{G}{Y}$ are the consumption-to-output and the government spending-to-output ratios in the steady state, respectively. We can use Equation (57) that links domestic consumption to world output and the previous equation (92) to express terms of trade as a function of domestic output, world output, preference shifter, and government spending:

$$y_t = C_Y\left[(1-\upsilon)\left(y_t^* + \frac{1}{\sigma}z_t + \frac{1-\upsilon}{\sigma}s_t\right) + \upsilon(2-\upsilon)\eta s_t + \upsilon y_t^*\right] + G_Y g_t \tag{93}$$

$$\vdots \quad \text{(see Appendix A.191 - A.207)}$$

$$s_t = \sigma_\upsilon(C_Y^{-1}y_t - y_t^* - C_G^{-1}g_t) - (1-\upsilon)\Phi z_t \tag{94}$$

where $\varpi = \sigma\eta + (1-\upsilon)(\sigma\eta - 1)$, $\Phi = \frac{1}{1+\upsilon(\varpi-1)}$, $\sigma_\upsilon = \sigma\Phi$, and C_G is the consumption-to-government spending ratio in the steady state. Notice that when $G = 0,\ Y = C$, then $s_t = \sigma_\upsilon(y_t - y_t^*) - (1-\upsilon)\Phi z_t$, i.e. identical to Galí (2015:238). Negative government term implies that (effective) trade of terms is decreasing in government spending. This is intuitive: the government is modelled to demand exclusively domestic goods, which induces inflationary pressure and makes home goods less competitive internationally. The opposite is true for the domestic-world output gap: if our firms produce relatively more than the rest of the world and are able to export relatively more, then terms of trade increase (the first term of (94) is positive). However, $y_t > y_t^W$ does not *automatically* imply an increase in exports; the terms need to be explicitly linked. Following a similar approach to Galí (2015), the net exports are denoted in terms of each GDP component as a share of their respective steady-state values. The resulting expression can be combined with the aggregate resource constraint (92) and (57) to yield a link between terms of trade and net exports NX_t:

$$NX_t = \frac{1}{Y}Y_t - \frac{1}{C}\left(\frac{P_t}{P_{H,t}}C_t\right) - \frac{1}{G}G_t \tag{95}$$

$$nx_t = y_t - c_t - \upsilon s_t - g_t \tag{96}$$

$$\vdots \quad \text{(see Appendix A.292-A.301)}$$

$$nx_t = \upsilon\left(\frac{\varpi}{\sigma} - 1\right)s_t - \frac{\upsilon}{\sigma}z_t \tag{97}$$

Note that in scenarios 2 & 4, nx_t and nx_t^* are scaled by χ and $(1-\chi)$, respectively, so that aggregate resources would be proportionally affected by government spending. Furthermore, the Euler equation in (32) is a function of CPI, but using (41), it can be rewritten to be a function of domestic inflation and terms of trade:

$$c_t = \mathbb{E}\{c_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}\{\pi_{H,t+1}\} - \rho) + \frac{\upsilon}{\sigma}\mathbb{E}\{\Delta s_{t+1}\} + \frac{1}{\sigma}(1-\rho_z)z_t \tag{98}$$

Finally, aggregate resource constraint (92), terms of trade definition (94), and new Euler equation (98) can be combined to derive a version of dynamic IS equation:

$$c_t = \mathbb{E}\{c_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}\{\pi_{H,t+1}\} - \rho) + \frac{\upsilon}{\sigma}\mathbb{E}\{\Delta s_{t+1}\} + \frac{1}{\sigma}(1-\rho_z)z_t \tag{99}$$

$$\vdots \quad \text{(see Appendix A.208-A.221)}$$

$$y_t = \mathbb{E}_t\{y_{t+1}\} - C_Y\left[\frac{1}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t + \upsilon(\varpi - 1)\mathbb{E}\{\Delta y^*_{t+1}\}\right]$$
$$- G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\} \tag{100}$$

which implies that output in the current period depends not only on expected output, inflation, and change in world output, but it also depends on expected changes in government spending. Equation (100) can be expressed in terms of output and real interest rate gaps:

$$
\begin{aligned}
y_t^n &= \mathbb{E}_t\{y_{t+1}^n\} - C_Y \left[\frac{1}{\sigma_\upsilon}(r_t^n - \rho) + \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} + \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t\right] \\
&\quad - G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\} \qquad (101) \\
&\vdots \quad \text{(see Appendix A.222 - A.226)} \\
r_t^n &= \sigma_\upsilon C_Y^{-1}\,\mathbb{E}_t\{\Delta y_{t+1}^n\} + \rho + \sigma_\upsilon \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} + \Phi(1-\upsilon)(1-\rho_z)z_t \\
&\quad + \sigma_\upsilon C_G^{-1}\,\mathbb{E}_t\{\Delta g_{t+1}\} \qquad (102) \\
&\vdots \quad \text{(see Appendix A.226 - A.229)} \\
\tilde{y}_t &= \tilde{y}_{t+1} - \frac{1}{\sigma_\upsilon} C_Y(i_t - \mathbb{E}\{\pi_{H,+1}\} - r_t^n) \qquad (103)
\end{aligned}
$$

where y_t^n is the natural output, $\tilde{y}_t$ is the output gap, and Equation (102) defines the natural real rate of interest r_t^n. Equation (103) is called Dynamic IS equation. It shows that for every period (hence, *dynamic*), the output gap is determined by the output gap "tomorrow" the real interest rate gap "today". It is one of the key equations in NK DSGE models. It sets a path for the output gap, given a path for the real interest rate. The real interest rate gap path depends on the inflation path, which was earlier given to be a function of the markup gap. However, for the model to yield a solution, the inflation path needs to be directly linked to the output gap. Firstly, the average markup can be rewritten to yield:

$$
\begin{aligned}
\mu_t &= p_{H,t} - \psi_t \qquad (104) \\
&= p_{H,t} - (w_t - a_t + \alpha n_t) \\
&\vdots \quad \text{(see Appendix A.230-A.251)} \\
\mu_t &= -\left(\sigma C_Y^{-1} + \frac{\varphi + \alpha}{1-\alpha}\right) y_t + \upsilon(\varpi - 1)s_t - \frac{\tau}{1-\tau}\tau_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right) a_t - \upsilon z_t \\
&\quad + \sigma C_G^{-1} g_t \qquad (105)
\end{aligned}
$$

Evaluating (105) at flexible prices $\theta = 0$ and solving for the output term yields the expression for the natural level of output y_t^n:

$$\mu_t = -\left(\sigma C_Y^{-1} + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n + \upsilon(\varpi - 1)s_t^n - \frac{\tau}{1-\tau}\tau_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right) a_t - \upsilon z_t$$
$$+ \sigma C_G^{-1} g_t \tag{106}$$

$\vdots$ (see Appendix A.252 - A.278)

$$y_t^n = \Gamma_* y_t^* + \Gamma_z z_t + \Gamma_a a_t + \Gamma_\tau \tau_t + \Gamma_g g_t \tag{107}$$

where:

G ≠ 0, Y = C + G	G = 0, Y = C (Galí, 2015:238)	
$\Gamma_* = -\frac{\upsilon(\varpi-1)\sigma_\upsilon(1-\alpha)}{\sigma_\upsilon C_Y^{-1}(1-\alpha)+\varphi+\alpha}$	$\Gamma_* - \frac{\upsilon(\varpi-1)\sigma_\upsilon(1-\alpha)}{\sigma_\upsilon(1-\alpha)+\varphi+\alpha}$	(108)
$\Gamma_z = -\frac{\upsilon\varpi\Phi(1-\alpha)}{\sigma_\upsilon C_Y^{-1}(1-\alpha)+\varphi+\alpha}$	$\Gamma_z = -\frac{\upsilon\varpi\Phi(1-\alpha)}{\sigma_\upsilon(1-\alpha)+\varphi+\alpha}$	(109)
$\Gamma_a = \frac{1+\varpi}{\sigma_\upsilon C_Y^{-1}(1-\alpha)+\varphi+\alpha}$	$\Gamma_a = \frac{1+\varpi}{\sigma_\upsilon(1-\alpha)+\varphi+\alpha}$	(110)
$\Gamma_g = \frac{\sigma_\upsilon C_G^{-1}(1-\alpha)}{\sigma_\upsilon C_Y^{-1}(1-\alpha)+\varphi+\alpha}$	$\Gamma_g = 0$	(111)
Scenarios 3 & 4	Scenarios 1 & 2	
$\Gamma_\tau = -\frac{\tau}{1-\tau}\frac{1-\alpha}{\sigma_\upsilon C_Y^{-1}(1-\alpha)+\varphi+\alpha}$	$\Gamma_\tau = 0$	(112)

Notice that the natural output is increasing in technology level and government spending but decreasing in labour taxes, preference shifter, and world output. The expression for natural output can be used to update the expression for the natural rate of interest:

$$\begin{aligned}
r_t^n &= \sigma_\upsilon C_Y^{-1}\,\mathbb{E}_t\{\Delta y_{t+1}^n\} + \rho + \sigma_\upsilon \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} + \Phi(1-\upsilon)(1-\rho_z)z_t \\
&\quad + \sigma_\upsilon C_G^{-1}\,\mathbb{E}_t\{\Delta g_{t+1}\} \qquad (113) \\
&\;\vdots \quad \text{see Appendix A.279-A.291} \\
r_t^n &= \rho - C_Y^{-1}\sigma_\upsilon \Gamma_a(1-\rho_a)a_t + \Psi_*\,\mathbb{E}_t\{\Delta y_{t+1}^*\} + \Psi_z(1-\rho_z)z_t \\
&\quad - \Psi_g(1-\rho_g)g_t + \Psi_\tau\,\mathbb{E}_t\{\Delta \tau_{t+1}\} \qquad (114)
\end{aligned}$$

where:

$$\begin{array}{lll}
\text{G} \neq 0\text{, Y = C + G} & \text{G = 0, Y = C (Galí, 2015:239)} & \\
\Psi_* = \sigma_\upsilon(C_Y^{-1}\Gamma_* + \upsilon(\varpi-1)) & \Psi_* = \sigma_\upsilon(\Gamma_* + \upsilon(\varpi-1)) & (115) \\
\Psi_z = (1-\upsilon)\Phi - C_Y^{-1}\sigma_\upsilon\Gamma_z & \Psi_z = (1-\upsilon)\Phi - \sigma_\upsilon\Gamma_z & (116) \\
\Psi_g = \sigma_\upsilon\left(C_Y^{-1}\Gamma_g + C_G^{-1}\right) & \Psi_g = 0 & (117) \\
\text{Scenarios 3 \& 4} & \text{Scenarios 1 \& 2} & \\
\Psi_\tau = \sigma_\upsilon C_Y^{-1}\Gamma_\tau & \Psi_\tau = 0 & (118)
\end{array}$$

When the prices are flexible, then each firm will choose the same optimal level of labour, implying constant average markup $\mu_t = \mu$. Subtracting μ_t definition (105) from μ definition (106) yields:

$$\begin{aligned}
\hat{\mu}_t = \mu_t - \mu &= -\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)\tilde{y}_t + \upsilon(\varpi-1)\tilde{s}_t \qquad (119) \\
&= -\left(\sigma_\upsilon C_Y^{-1} + \frac{\varphi+\alpha}{1-\alpha}\right)\tilde{y}_t \qquad (120)
\end{aligned}$$

where $\tilde{s}_t$ denote terms of trade gap, and Equation (120) uses $\tilde{s}_t = \sigma_\upsilon C_Y^{-1}\tilde{y}_t$ relationship implied by (94). Finally, plugging (120) to a *version* of NKPC defined by Equation

(80), yields the *final* NKPC equation:

$$\pi_{H,t} = \beta\,\mathbb{E}_t\left[\pi_{H,t+1}\right] + \kappa\tilde{y}_t \tag{121}$$

where $\kappa = \lambda\left(\sigma_v C_Y^{-1} + \frac{\varphi+\alpha}{1-\alpha}\right)$. That is, whenever output is greater than that implied by the natural level of output ($\tilde{y}_t > 0$), inflation increases, which subsequently reduces demand and brings output closer to the natural level of output. The opposite is also true, i.e. whenever the economy is under-producing goods ($\tilde{y}_t < 0$), then inflation decreases to induce demand and bring output level to that under flexible prices. If prices were fully flexible ($\theta = 0$), then the economy would always be producing $y_t = y_t^n$, suggesting a zero inflation steady state.

Finally, we have a path for the output gap (implied by DIS) and a path for inflation (implied by NKPC); the only remaining path to "close" the model is that for the interest rate. Here, the monetary authority is assumed to set the interest rate following a Taylor rule:

$$i_t = \phi_\pi \pi_{H,t} + \phi_y \hat{y}_t + m_t \tag{122}$$

where m_t is an exogenous monetary policy shock, which is assumed to follow an AR(1) process $m_t = \rho_m m_{t-1} + \varepsilon_t^m,\ \varepsilon_t^m \sim \mathcal{N}(0, \sigma_m^2)$, and $\hat{y}_t$ is the deviation from the steady state output. The Taylor rule suggests that monetary authority will raise the interest rate whenever inflation increases or output exceeds the steady-state output.

The equations provided in this section outline the dynamic behaviour of the endogenous variables. For the model to be fully specified, each parameter within these equations must be estimated or calibrated. The following section describes these processes.

4 Estimating and Calibrating the Model

There is more than one method to estimate DSGE model parameters, and their theoretical comparisons are provided in An and Schorfheide (2007:123–130), as well as Ricci (2019:109–110). Ever since Schorfheide (2000) applied Random-Walk Metropolis (RMW) algorithm to estimate DSGE models, the method gained popularity and is the most prevalent estimation method in recent literature.

The algorithm aims to approximate the posterior density function using a large number of parameter draws (sets of values). The domain of the candidate values is pre-specified using prior distributions and combined with a vector of time series to produce *likelihood*. Likelihood values of two sequential draws are used to calculate the acceptance probability for the candidate value. A large number of accepted draws form the posterior density function. Due to the limitations of the scope of this dissertation, the specifics of the RMW algorithm will not be delved into, but an interested reader should see An and Schorfheide (2007:131) for a formal treatment of the algorithm and Blake and Mumtaz (2017) handbook of applied Bayesian econometrics that has a dedicated section for implementing the algorithm for DSGE models and even includes a complementary Matlab code repository. This dissertation will utilise out-of-the-box solutions offered by the Dynare computer package (Adjemian et al., 2021) to obtain parameter draws.

While estimating all parameters is an option, in theory, most parameter values will be calibrated in line with Harrison and Oomen (2010:41), who built a DSGE model for the United Kingdom and provides detailed reasoning behind calibrated values, and Galí (2015:67–75) and Gali and Monacelli (2005:3839) for less important parameters. The reason behind the decision to calibrate the majority of the parameters is the simplicity of our model. The model does not consider capital, habit formation, sticky wages, limited access to financial instruments, and many other extensions in the literature that improve data fit (Yagihashi, 2020). This makes our data uninformative for some parameters, and estimation results heavily rely on priors. Table 8 below lists calibrated (as opposed to estimated) model parameters for Scotland and the rest of the UK. All but fiscal policy parameters and steady-state ratios are assumed to be symmetrical.

Parameter	Value	Parameter	Value	Strategy
β	0.99	β^*	β	Calculated so that $R_t = 1.01$, *
σ	0.44	σ^*	σ	HO41
φ	2.2	φ^*	φ	GM3839
α	1/3	α^*	α	HO41
ϵ	9.67	ϵ^*	ϵ	HO41
θ	0.75	θ^*	θ	G67, *
υ	0.40	υ^*	υ	GM3839
ρ_a	0.89	ρ_a^*	ρ_a	HO41
η	1	η^*	η	GM3839
ρ_z	0.50	ρ_z^*	ρ_z	G70, *
ρ_ν	0.50	ρ_ν^*	ρ_ν	G67, *
ϕ_π	1.50	ϕ_π^*	ϕ_π	G67, *
ϕ_y	0.125	ϕ_y^*	ϕ_y	G67, *
G_Y	0.256313	G_Y^*	0.179667	QNAS/QNA, 1998Q1-2021Q4 arithmetic mean
C_Y	1.00 - G_Y	C_Y^*	1.00 - G_Y^*	Definition
Y_C	$1/C_Y$	Y_C^*	$1/C_Y^*$	Definition
G_C	Y_C - 1	G_C^*	Y_C^* - 1	Definition
For policy scenarios with multiple governments:				
τ	G_Y	τ^*	G_Y^*	By assumption
For policy scenarios with a single government:				
		τ^{UK}	$\chi G_Y + (1-\chi)G_Y^*$	By assumption

Table 8: *Model parameters, their values, and the strategy for obtaining the values. HO - Harrison and Oomen (2010), G - Galí (2015), GM - Gali and Monacelli (2005), * - common value in the literature. Numbers indicate page number.*

The dissertation, however, does estimate three sensitive parameters key to the research question: ϕ_b, ϕ_g, and ρ_g (and their rUK / UK counterparts). The acceptance rate per chain was c. 22%, close to the optimal acceptance rate suggested by Roberts and Rosenthal (2001). The acceptance rate in the range of 23% ensures that the variance of candidate parameter values is neither too large nor too small (*ibid.*), meaning that all sets of parameters had a reasonable probability of being drawn. The number of MH draws and initial ("burn-in") draws was set to 300,000 and 100,000, respectively. Figures (3, 16, and 17) display how the 300,000 draws varied. The variance of the draws suggests that the serial correlation of lagged draws was not persistent (fading), which is also indicative of successful estimation (*ibid.*). The posterior means of all estimated parameters are close to their prior means, which can be primarily attributed to tight prior variance. However, the two means do not coincide, suggesting (to some degree) an informative estimation of the parameters. In fact, estimation of ϕ_b and ϕ_g are bound to be difficult given that the fiscal rule, while being reasonable and intuitive, is imposed arbitrarily to "close" the model. The true rule followed by the decision-makers, however, might be more complex and considerate of other metrics not captured by the model. All parameter values were drawn from the Beta distribution due to its values being bounded by zero and one. Table 9 lists shape parameters, implied prior means and standard deviations, as well as posterior means of all estimated parameters.

Variable	Distribution	Shape	Prior SD	Prior Mean	Post. Mean
For policy scenarios with multiple governments:					
ϕ_b	$\mathcal{B}e(\alpha,\ \beta)$	$\alpha = 1.49$ $\beta = 3.03$	0.15	0.33	**0.362**
ϕ_g	$\mathcal{B}e(\alpha,\ \beta)$	$\alpha = 3.50$ $\beta = 31.50$	0.05	0.10	**0.101**
ρ_g	$\mathcal{B}e(\alpha,\ \beta)$	$\alpha = 809.10$ $\beta = 89.90$	0.01	0.90	**0.901**
ϕ_b^*	$\mathcal{B}e(\alpha,\ \beta)$	$\alpha = 1.49$ $\beta = 3.03$	0.15	0.33	**0.410**
ϕ_g^*	$\mathcal{B}e(\alpha,\ \beta)$	$\alpha = 3.50$ $\beta = 31.50$	0.05	0.10	**0.098**
ρ_g^*	$\mathcal{B}e(\alpha,\ \beta)$	$\alpha = 809.10$ $\beta = 89.90$	0.01	0.90	**0.904**
For policy scenarios with a single government:					
ϕ_b^{UK}	$\mathcal{B}e(\alpha,\ \beta)$	$\alpha = 1.49$ $\beta = 3.03$	0.15	0.33	**0.335**
ϕ_g^{UK}	$\mathcal{B}e(\alpha,\ \beta)$	$\alpha = 3.50$ $\beta = 31.50$	0.05	0.10	**0.100**
ρ_g^{UK}	$\mathcal{B}e(\alpha,\ \beta)$	$\alpha = 809.10$ $\beta = 89.90$	0.01	0.90	**0.900**

Table 9: *Model parameter estimation set-up and results.*

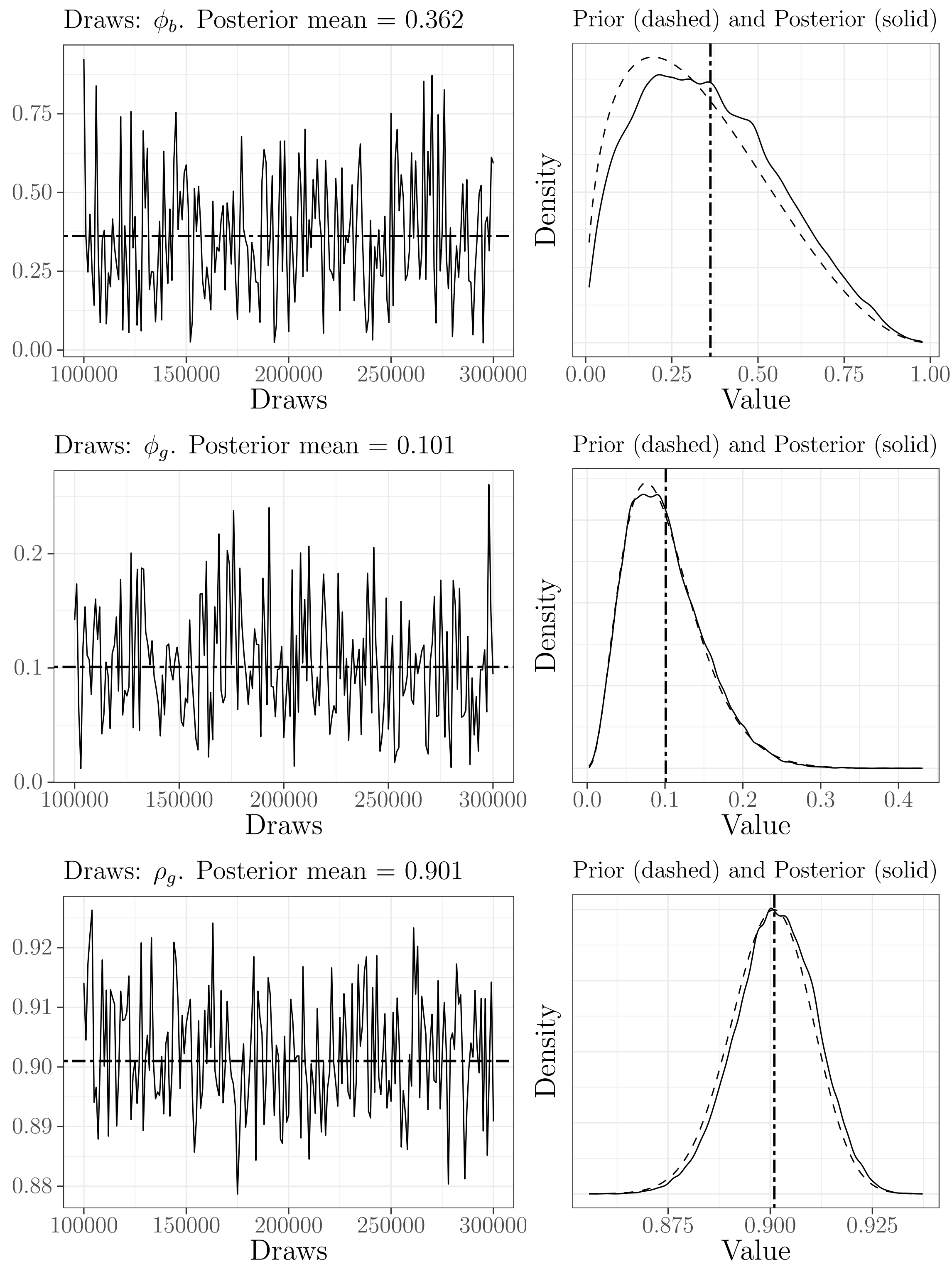

Figure 3: *Estimation results: parameter draws, prior and posterior density functions (Scotland).*

Figures (16 and 17) displaying draws of corresponding variables for the rest of the UK and UK can be found in Appendix. In terms of data, most time series for Scotland and the rest of the UK were acquired from the Quarterly National Accounts of Scotland (QNAS) 2022 and the Quarterly Nationals Account (QNA) 2023, respectively. In contrast to Ricci (2019), who used the 1998Q1-2007Q4 dataset, this dissertation utilised the full QNA/QNAS dataset (1998Q1-2021Q4) for the variables of interest.[13] Each time series was processed in line with Pfeifer (2018). That is, the time series for Scotland were adjusted to 2023 price levels and divided by the number of (estimated) working-age residents in Scotland. Then, the time series were expressed in natural logarithms and first-differenced to induce stationarity. Finally, the series was demeaned in line with the model specification. Identical data processing steps were taken for the rest of the UK time series. Note that data was not available for all regions of the UK. Therefore, in most cases, the rest of the UK time series were calculated by taking Scotland's time series and subtracting them from the UK-wide ones. The resulting values should be approximately equal to the rest of the UK ones, as long as QNAS and QNA use symmetrical accounting/data processing methods for each pair of time series. Figures 14 and 15 display the pre- and post-transformed time series line charts, respectively.

[13] Ricci (2019) argued that their DSGE model is not "equipped to account for"(*ibid.*, p. 123) the global financial crisis of 2008. However, using a full dataset was found to improve estimation results for this model and is a more transparent choice. Therefore, the dissertation opted for the 1998Q1-2021Q4 sample date range.

Time Series	**Source**
Scot. Output	QNAS: Table A, column D
Scot. Consumption	QNAS: Supplementary Tables, Table 12, Column P
Scot. Compensation of Employees	QNAS: Table I, column C
Scot. Working Population	Population Estimates Time Series Data, National Records of Scotland
Scot. Government Spending	Table G1: GDP Expenditure Approach at Current Prices (onshore): Government Spending
UK Deflator (2023=100)	QNA: Implied GDP deflator at prices: market SA Index
UK Output	QNA: Gross Domestic Product at market prices: Current price: Seasonally adjusted £m
UK Consumption	QNA: 0 Household final consumption expenditure: Domestic concept CP SA £m
UK Compensation of Employees	QNA: UK (S.1): Compensation of employees (D.1) Current prices: £m: SA
UK Government Spending	QNA: General Government: Final consumption expenditure (P3): CPSA £m
rUK Working Population	NOMIS Population Estimates

Table 10: *Time series used for estimation.*

5 Dynamic Responses

This section will consider the responses of endogenous variables, given two types of shocks: monetary and fiscal. Although the examination of monetary shocks falls beyond the scope of this dissertation, it serves as a valuable benchmark to assess the accuracy and validity of our model. The Dynare file accompanying this dissertation also allows for deriving dynamic responses to shocks in world output, price level, and preferences (z_t). Still, these fall outside the scope of the dissertation and are not discussed. For the monetary policy shock, we use the policy scenario four model, i.e. a single fiscal government that uses labour tax and bond issuance as fiscal instruments to raise budget revenue.

Figure 4 presents the responses of fourteen endogenous variables, where two variables, namely the labour tax rate and the interest rate itself, apply across the entire UK. Additionally, ten variables encompass each country's consumption, output, real wage ($wp_t = w_t - p_t$), domestic inflation, and employment (hours worked). The last two variables represent country-specific interest rates, which, although not directly pertinent to the main focus (being included purely for technical reasons), display a co-movement in line with the model's imposition of the monetary union. Generally, dynamic responses are very much in line with textbook literature (Galí, 2015:243): An increase in interest rate makes saving more attractive, making households incentivised to consume less "today" in hopes to consume more "tomorrow". A decline in demand for home goods results in a decline in output, which creates an abundance of labour, leading to an increase in unemployment. An increase in interest rate also leads to downward inflationary pressure because goods become relatively less attractive to households compared to saving, and making goods cheaper becomes the best strategy for price-setting, profit-maximising firms. Finally, given that employment decreases, bringing down the tax revenue, the government responds with an increase in the labour tax rate (τ_t^{UK}) to clear the government budget. All effects fade as the economy returns to its steady state equilibrium.

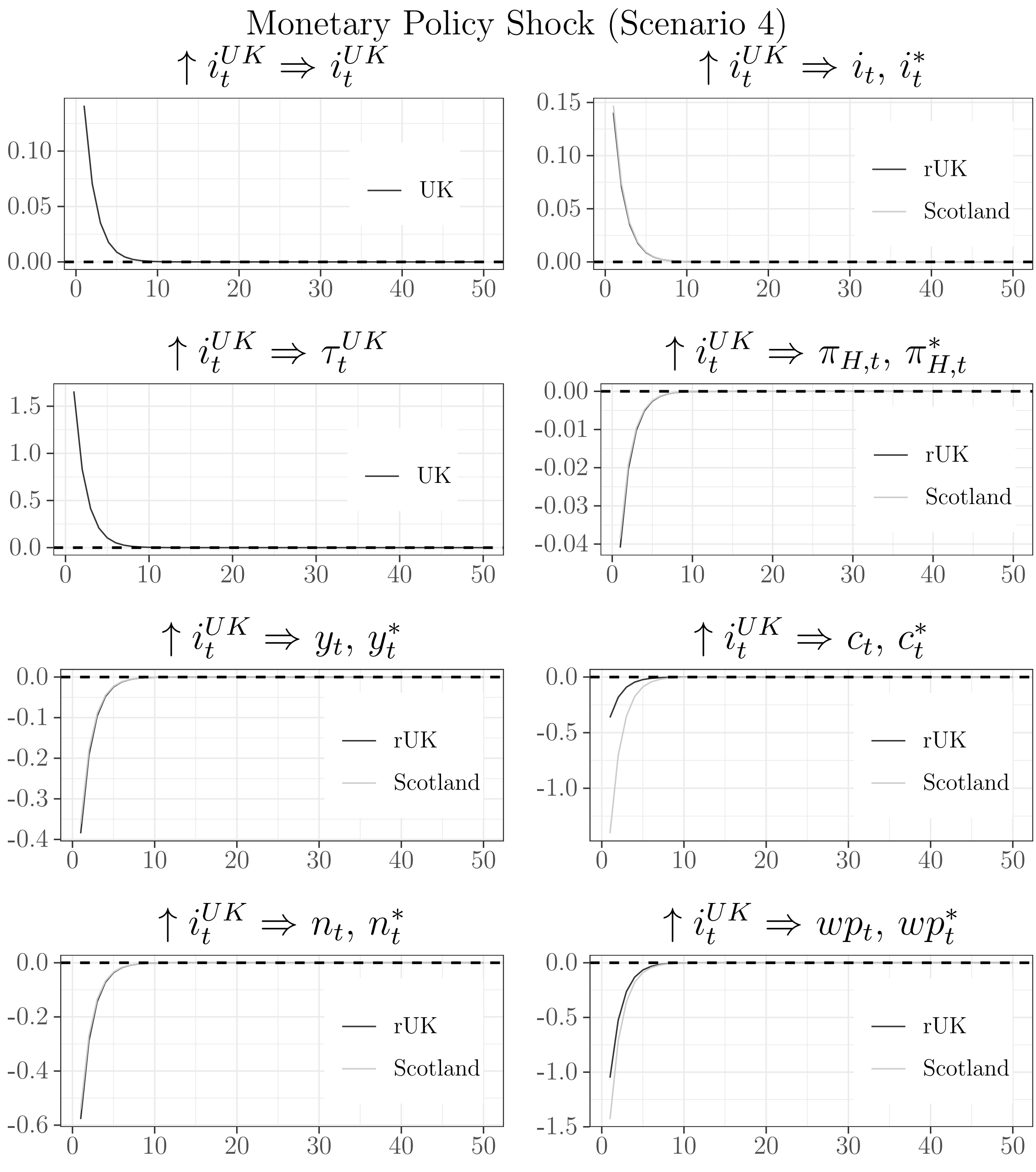

Figure 4: *Dynamic responses in Scotland and rUK following a contractionary monetary policy schock*

Next, dynamic responses to an increase in government spending are considered. Across all policy scenarios, the responses exhibit consistent behaviour. That is, the government reduces the quantity of available resources to the household, leading to decreased consumption. The negative wealth effect becomes more significant than the substitution effect between consumption and leisure, leading to increased employment. An increase in labour supply allows firms to produce more (output increases) while paying less in real terms (real wage decreases). In all scenarios, the supply of government-issued bonds increases as the government raises funds to cover its spending. In the scenarios where labour tax exists, the government raises the tax rate, as well as issues bonds with the intensity of the two financial instruments determined by the fiscal rule and fiscal policy parameters ϕ_b and ϕ_g. The model considered in this dissertation assumes that the government "consumes" domestic goods. Therefore, increasing government spending raises demand for these goods and creates upward inflationary pressure, as shown below in Figures 5-10. Domestic goods become less competitive internationally, leading to depreciation in terms of trade (displayed only in Figures 5 and 6 in the interest of space). In all IRFs, responses of *domestic* inflation will be displayed as it allows easier comparison with more common closed economy DSGE models found in the literature; however, CPI inflation can be easily inferred from Equation (41), which shows that CPI is a sum of domestic inflation and period-to-period changes in terms of trade times the openness parameter υ.

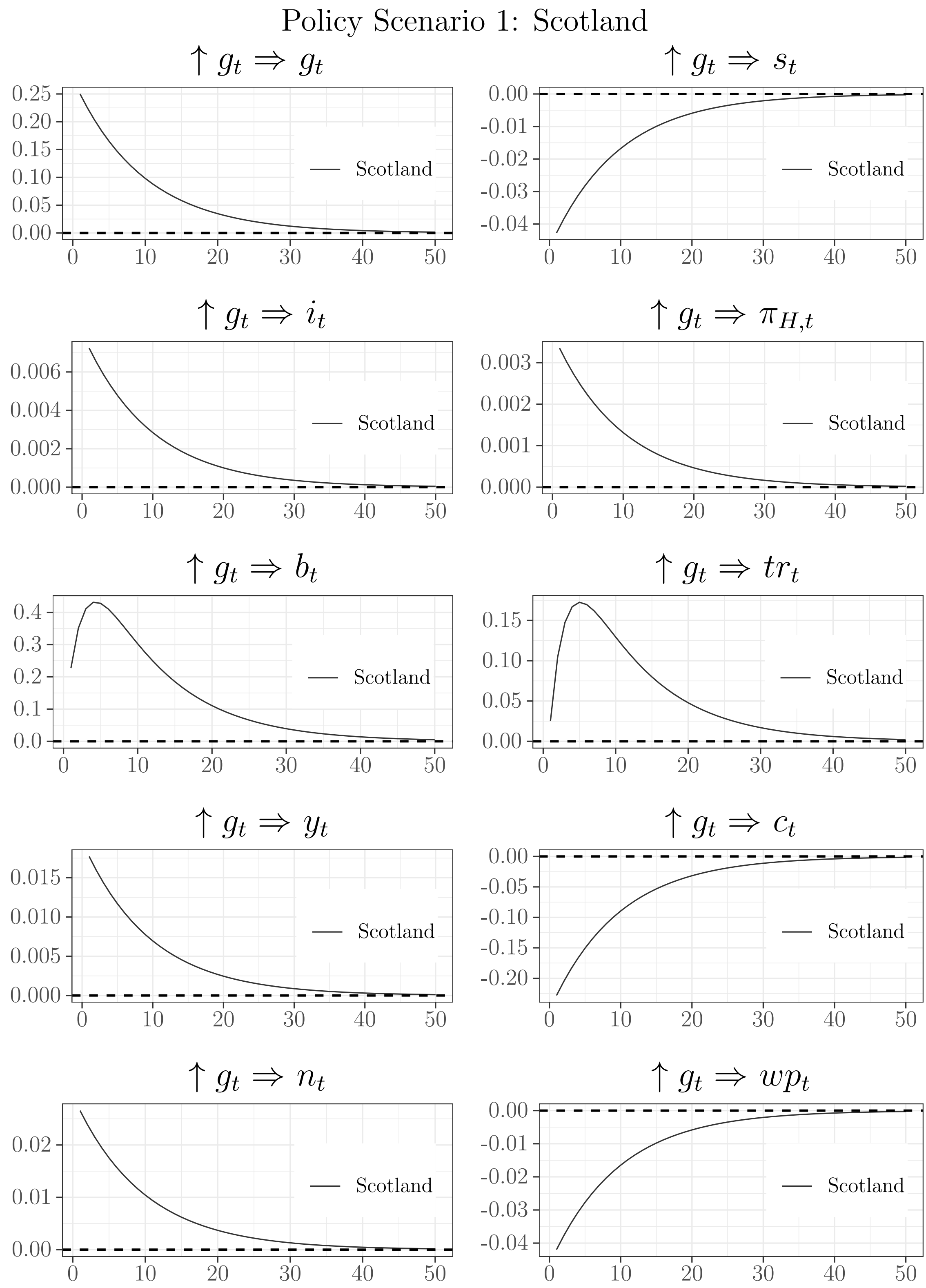

Figure 5: *Dynamic responses in Scotland following an expansionary fiscal policy shock under policy scenario 1.*

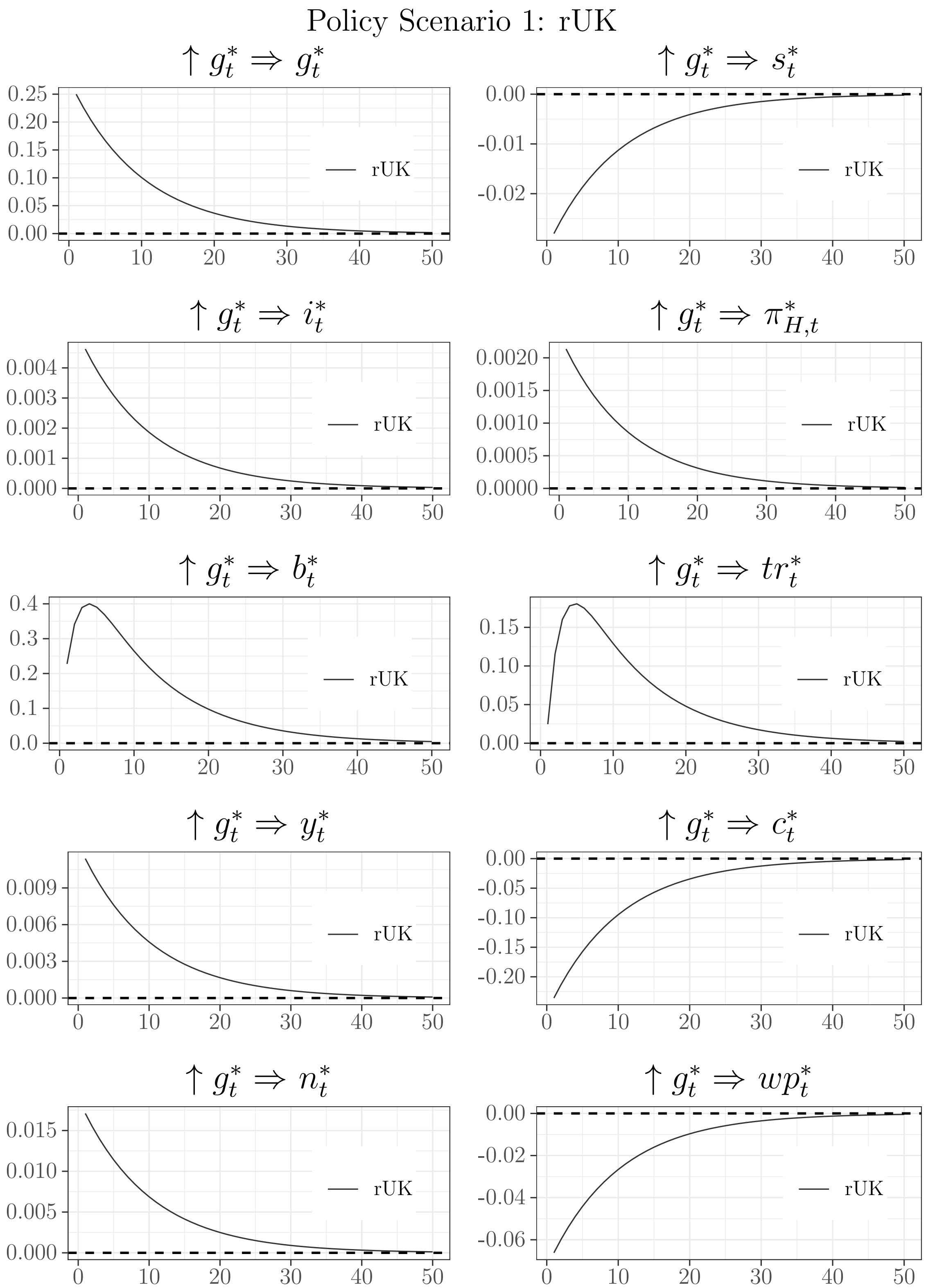

Figure 6: *Dynamic responses in rUK following an expansionary fiscal policy shock under policy scenario 1.*

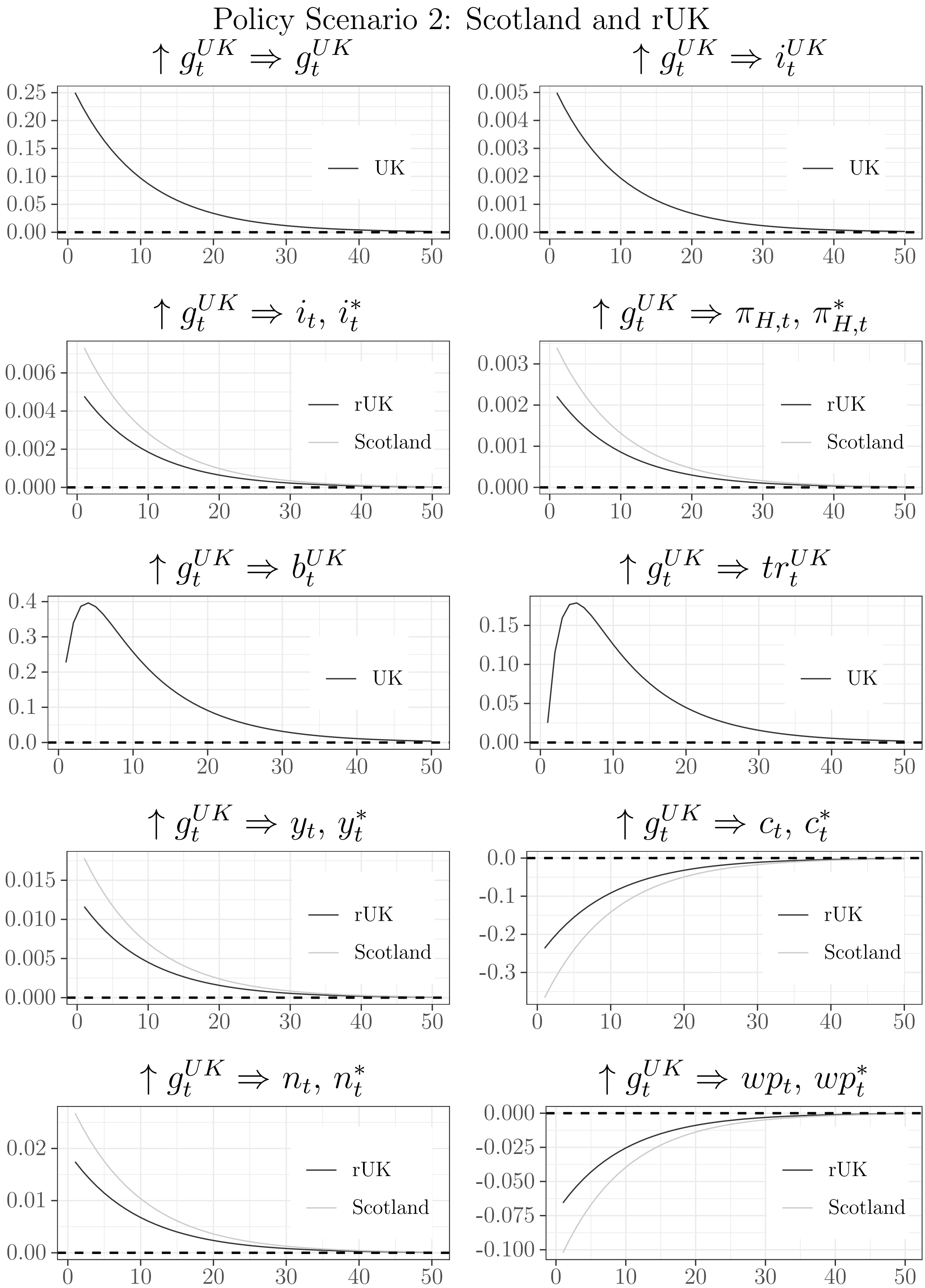

Figure 7: *Dynamic responses in Scotland and rUK following an expansionary fiscal policy shock under policy scenario 2.*

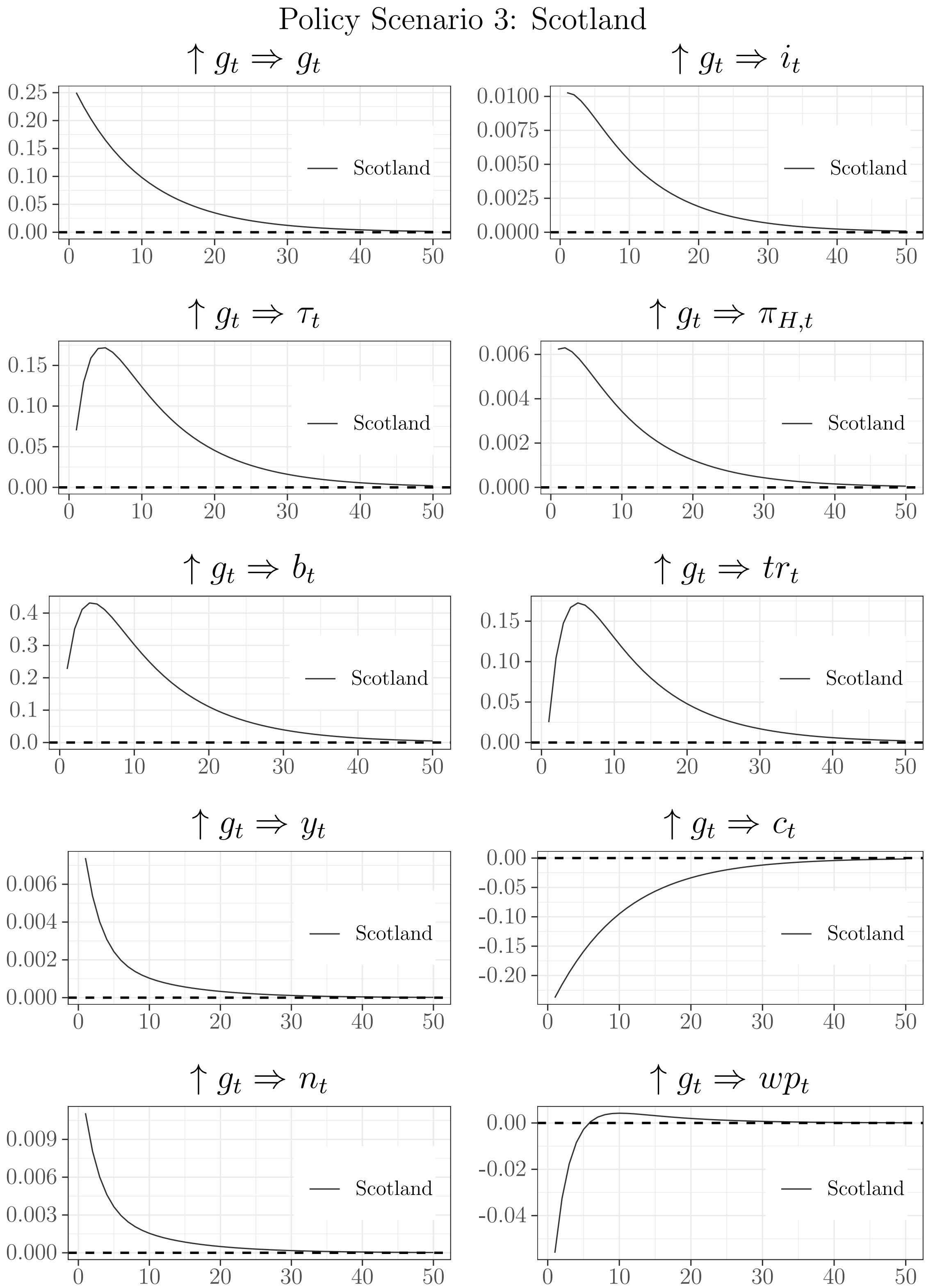

Figure 8: *Dynamic responses in Scotland following an expansionary fiscal policy shock under policy scenario 3.*

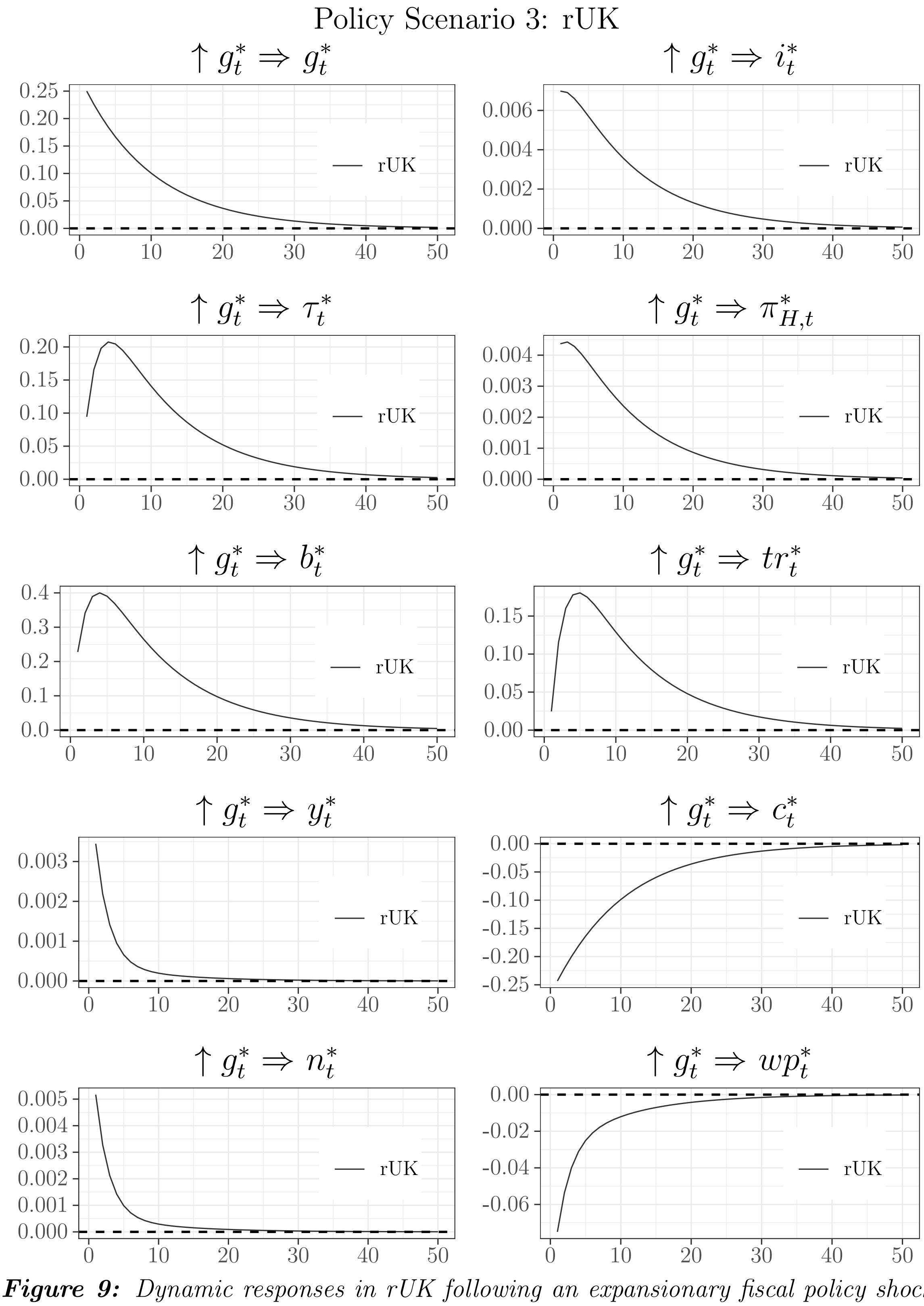

Figure 9: *Dynamic responses in rUK following an expansionary fiscal policy shock under policy scenario 3.*

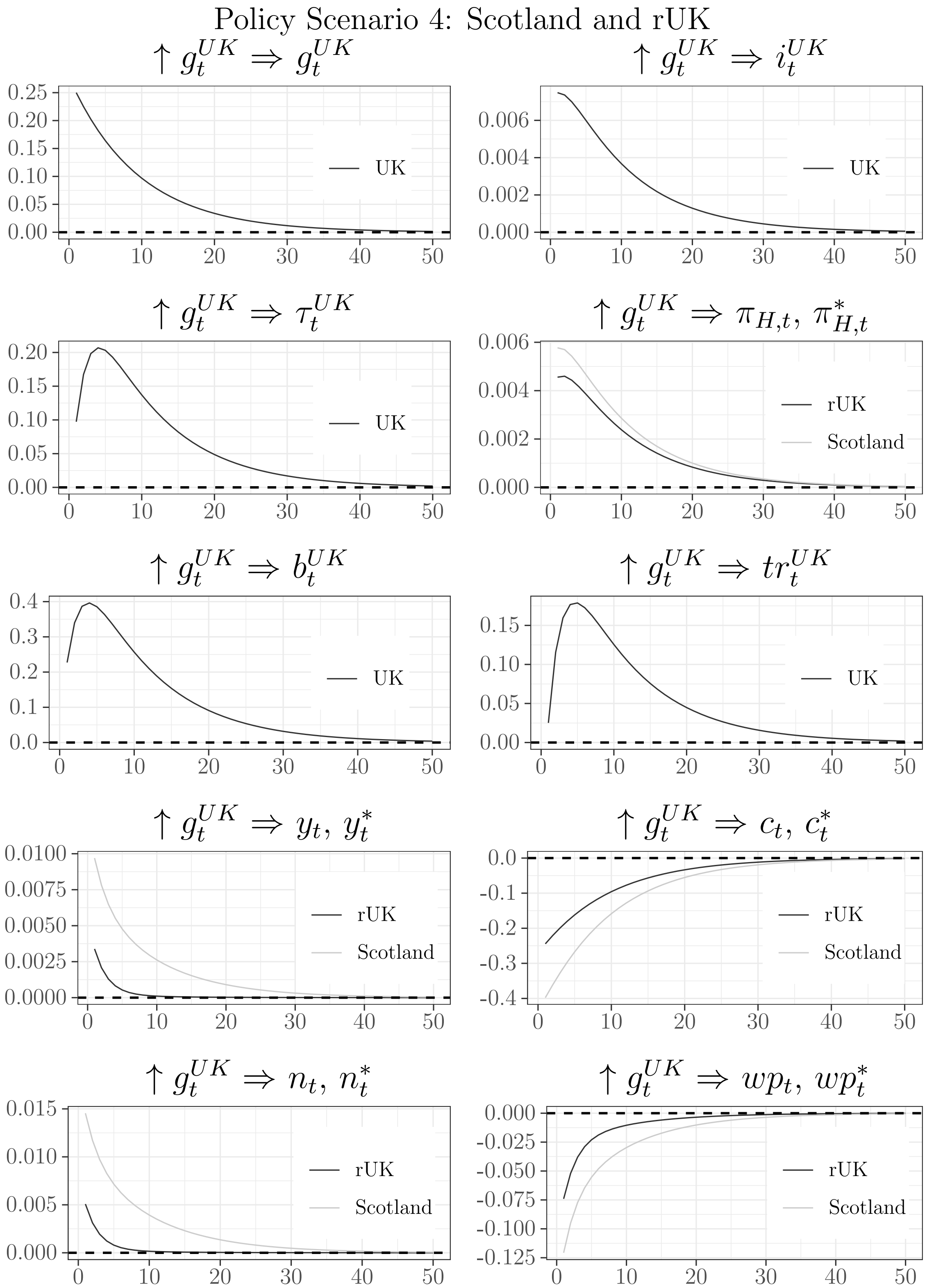

Figure 10: *Dynamic responses in Scotland and rUK following an expansionary fiscal policy shock under policy scenario 4.*

More specifically, under policy scenario four (G : 1, τ : 1), IRFs suggest that a 0.25% deviation from the steady state government spending in any given quarter results in 0.09% in Scotland and 0.02% in the rest of the UK deviation in output in the same quarter.[14] Consumption decreased by 0.4% and 0.25%, while employment increased by 0.15% and 0.05% in Scotland and the rest of the UK, respectively. A more significant deviation from the steady state of employment level in Scotland also resulted in a more significant decrease in real wages (0.125%) compared to that of the rest of the UK (0.075%). Finally, the domestic inflation increase was similar across both countries (0.04% in Scotland and 0.06% in the rest of the UK). Tables 11 and 12 contrast 200-period cumulative responses between countries for each of the policy scenarios (column "Diff.").[15] It also displays the changes in responses when the labour tax is introduced. In this model, the introduction of labour taxes is found to make government spending more distortionary: the cumulative increase in output and employment across both countries decreased by 0.248% and 0.372%, and consumption fell by an additional 0.227%. These findings apply to both sets of scenarios, 1 & 3 and 2 & 4, with slightly larger multipliers observed in the latter pair.

Tables 13 and 14 show how the cumulative effect changes when either country joins the fiscal union. When there are no labour taxes, being in or out of a fiscal union makes (almost) no difference for the rest of the UK. However, Scotland is found to be significantly more susceptible to such changes (with or without the existence of labour tax). When Scotland is modelled to join a fiscal union with the rest of the UK, fewer bonds (-0.85%) are issued, and households face a 0.398% higher (cumulative) labour tax rate. The result is a more significant decrease in consumption (-1.625%) and real wage (-0.814%). Output and employment remain almost unchanged.

[14]Note that the dissertation uses 25 basis points quarterly increase, so the annualised increase would be 1 percentage point. This makes the results more comparable to those found in the literature.

[15]Note that all variables return to their steady-state levels within the first 50 periods. Using 200 as an arbitrarily large number ensures that the full effect is accurately calculated even if extreme persistence parameters were chosen.

Variable	Scot., rUK (1)	Diff. (1)	Scot., rUK (3)	Diff. (3)	(τ: 1 − τ: 0)
i_t, i_t^*	0.073, 0.048	0.025	0.129, 0.088	0.041	0.095
τ_t, τ_t^*	0, 0	0	2.594, 3.052	-0.458	5.646
$\pi_{H,t}$, $\pi_{H,t}^*$	0.034, 0.022	0.012	0.083, 0.058	0.025	0.084
b_t, b_t^*	6.479, 5.852	0.627	6.479, 5.852	0.627	0
tr_t, tr_t^*	2.597, 2.655	-0.058	2.597, 2.655	-0.058	0
y_t, y_t^*	0.179, 0.118	0.061	0.038, 0.012	0.026	-0.248
c_t, c_t^*	-2.312, -2.451	0.139	-2.441, -2.549	0.108	-0.227
n_t, n_t^*	0.269, 0.178	0.092	0.057, 0.018	0.04	-0.372
wp_t, wp_t^*	-0.425, -0.688	0.263	-0.054, -0.414	0.36	0.645

Table 11: *Scenarios 1 and 3: Expansionary fiscal policy cumulative 200-period effect. The "(τ: 1 − τ: 0)" column shows the difference in the effect of government spending with and without labour tax across both countries, i.e.* $(y_t|_{\tau:1} - y_t|_{\tau:0}) + (y_t^*|_{\tau:1} - y_t^*|_{\tau:0})$.

Variable	Scot., rUK (2)	Diff. (2)	Scot., rUK (4)	Diff. (4)	(τ: 1 − τ: 0)
i_t^{UK}	0.05	0	0.09	0	0.04
τ_t^{UK}	0	0	2.952	0	2.952
$\pi_{H,t}$, $\pi_{H,t}^*$	0.034, 0.022	0.012	0.07, 0.058	0.012	0.072
b_t^{UK}	5.629	0	5.629	0	0
tr_t^{UK}	2.556	0	2.556	0	0
y_t, y_t^*	0.178, 0.116	0.062	0.074, 0.01	0.065	-0.21
c_t, c_t^*	-3.661, -2.365	-1.295	-4.066, -2.466	-1.6	-0.506
n_t, n_t^*	0.268, 0.175	0.093	0.112, 0.015	0.097	-0.316
wp_t, wp_t^*	-1.022, -0.657	-0.365	-0.868, -0.377	-0.491	0.435

Table 12: *Scenarios 2 and 4: Expansionary fiscal policy cumulative 200-period effect. A single value in "Scot., rUK column" indicate a UK-wide variable. The "(τ: 1 − τ: 0)" column shows the difference in the effect of government spending with and without labour tax across both countries, i.e.* $(y_t|_{\tau:1} - y_t|_{\tau:0}) + (y_t^*|_{\tau:1} - y_t^*|_{\tau:0})$.

Variable	Scot., rUK (1)	Diff. (1)	Scot., rUK (2)	Diff. (2)	(G: 1 − G: 2)
i_t, i_t^*	0.073, 0.048	0.025	0.073, 0.048	0.025	-0.001
$\pi_{H,t}$, $\pi_{H,t}^*$	0.034, 0.022	0.012	0.034, 0.022	0.012	0
b_t, b_t^*	6.479, 5.852	0.627	5.629	0	-1.073
tr_t, tr_t^*	2.597, 2.655	-0.058	2.556	0	-0.14
y_t, y_t^*	0.179, 0.118	0.061	0.178, 0.116	0.062	-0.003
c_t, c_t^*	-2.312, -2.451	0.139	-3.661, -2.365	-1.295	-1.263
n_t, n_t^*	0.269, 0.178	0.092	0.268, 0.175	0.093	-0.005
wp_t, wp_t^*	-0.425, -0.688	0.263	-1.022, -0.657	-0.365	-0.566

Table 13: *Scenarios 1 and 2: Expansionary fiscal policy cumulative 200-period effect. The "(G: 1 − G: 2)" column shows the difference in the effect of government spending in and out of a fiscal union across both countries, i.e.* $(y_t|_{G:1} - y_t|_{G:2}) + (y_t^*|_{G:1} - y_t^*|_{G:2})$.

Variable	Scot., rUK (3)	Diff. (3)	Scot., rUK (4)	Diff. (4)	(G: 1 − G: 2)
i_t, i_t^*	0.129, 0.088	0.041	0.115, 0.088	0.027	-0.014
τ_t, τ_t^*	2.594, 3.052	-0.458	2.952	0	0.258
$\pi_{H,t}$, $\pi_{H,t}^*$	0.083, 0.058	0.025	0.07, 0.058	0.012	-0.012
b_t, b_t^*	6.479, 5.852	0.627	5.629	0	-1.073
tr_t, tr_t^*	2.597, 2.655	-0.058	2.556	0	-0.14
y_t, y_t^*	0.038, 0.012	0.026	0.074, 0.01	0.065	0.034
c_t, c_t^*	-2.441, -2.549	0.108	-4.066, -2.466	-1.6	-1.541
n_t, n_t^*	0.057, 0.018	0.04	0.112, 0.015	0.097	0.051
wp_t, wp_t^*	-0.054, -0.414	0.36	-0.868, -0.377	-0.491	-0.776

Table 14: *Scenarios 3 and 4: Expansionary fiscal policy cumulative 200-period effect. The "(G: 1 − G: 2)" column shows the difference in the effect of government spending in and out of a fiscal union across both countries, i.e.* $(y_t|_{G:1} - y_t|_{G:2}) + (y_t^*|_{G:1} - y_t^*|_{G:2})$.

Moreover, Figures 11-13 illustrate the sensitivity of dynamic responses given changes in fiscal feedback and persistence parameters ϕ_b, ϕ_g, ρ_g (and their rUK/UK counterparts). For each figure, one parameter values are allowed to vary while the other two are fixed to prior mean values. The sensitivity analysis is carried out for policy scenario four only. Unsurprisingly, the fiscal policy persistence parameter ρ_g is found to have the most significant impact on the magnitude of responses, with some of the variables even responding in the opposite than expected direction when (extremely) small ρ_g values are considered. According to Galí, López-Salido, and Vallés (2005:27), the persistence parameter is associated with the strength of the wealth effect (the more persistent the government spending shock, the greater the wealth effect is imposed on the households). This is confirmed by Figure 13: as ρ_g decreases, the dynamic responses return to their steady-state values quicker. The positive responses of the real wage when $\rho_g \in \{0.10, 0.60\}$ require further attention, but it could be attributed to the price stickiness and flexible wages: output and employment initial responses are dramatically greater to their counterpart responses when $\rho_g = 0.99$, while the initial response of inflation (period-to-period log price level change) is almost unchanged for all ρ_g values. This suggests that nominal wages increase more than sticky prices, resulting in a positive real wage response. As discussed in Section 2: Government, tax revenue and labour tax rate are increasing in ϕ_b and ϕ_g. A high value of either parameter results in a more significant immediate response. However, that does not result in significant changes in responses of other variables as only the amount of bonds issued, the labour tax rate, and the tax revenue are *considerably* sensitive to changes in ϕ_b and ϕ_g.

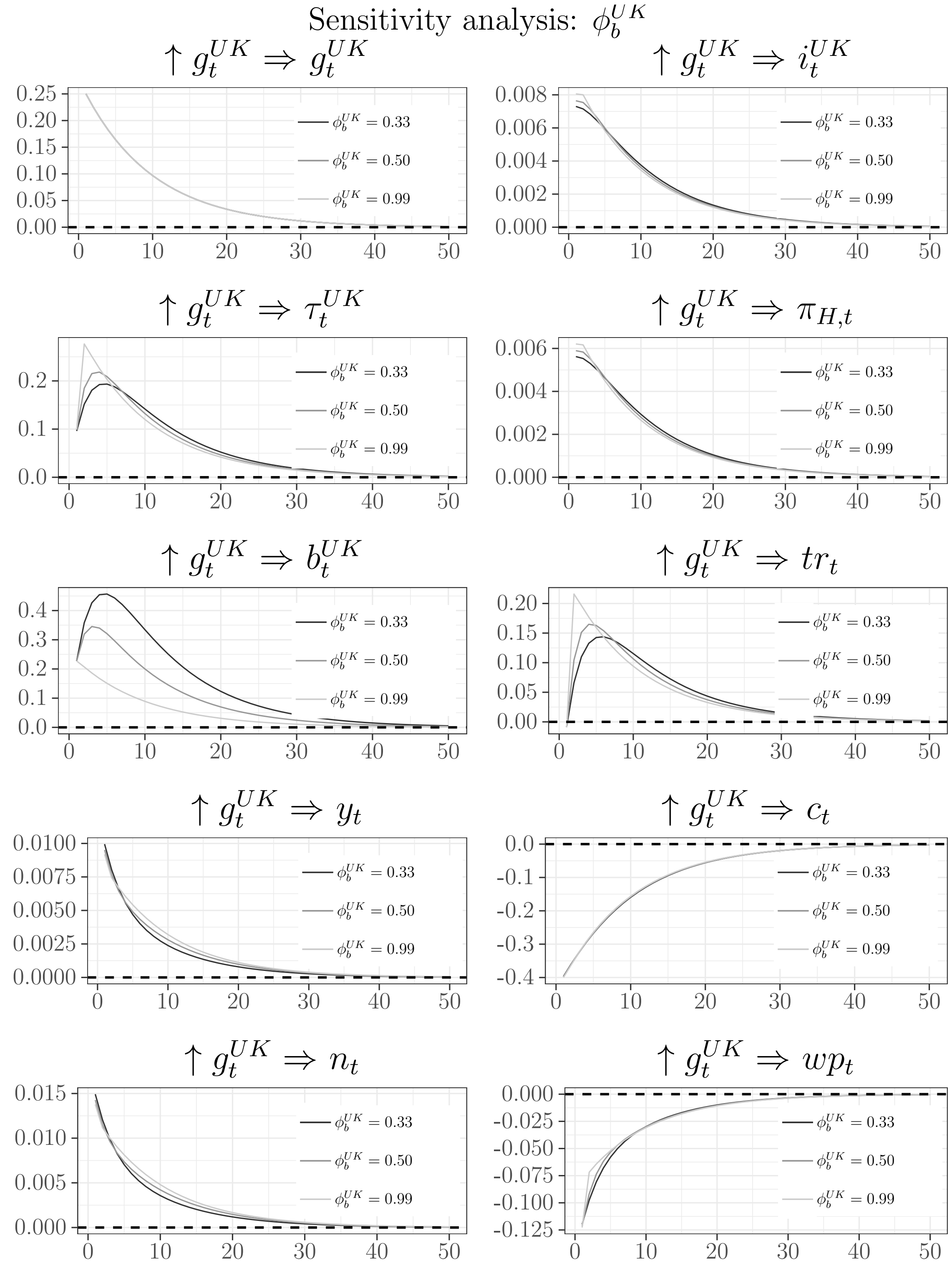

Figure 11: *Sensitivity analysis:* ϕ_b^{UK} $\forall\{0.33, 0.50, 0.99\}$, *when* $\phi_g^{UK} = 0.10$, $\rho_g^{UK} = 0.90$

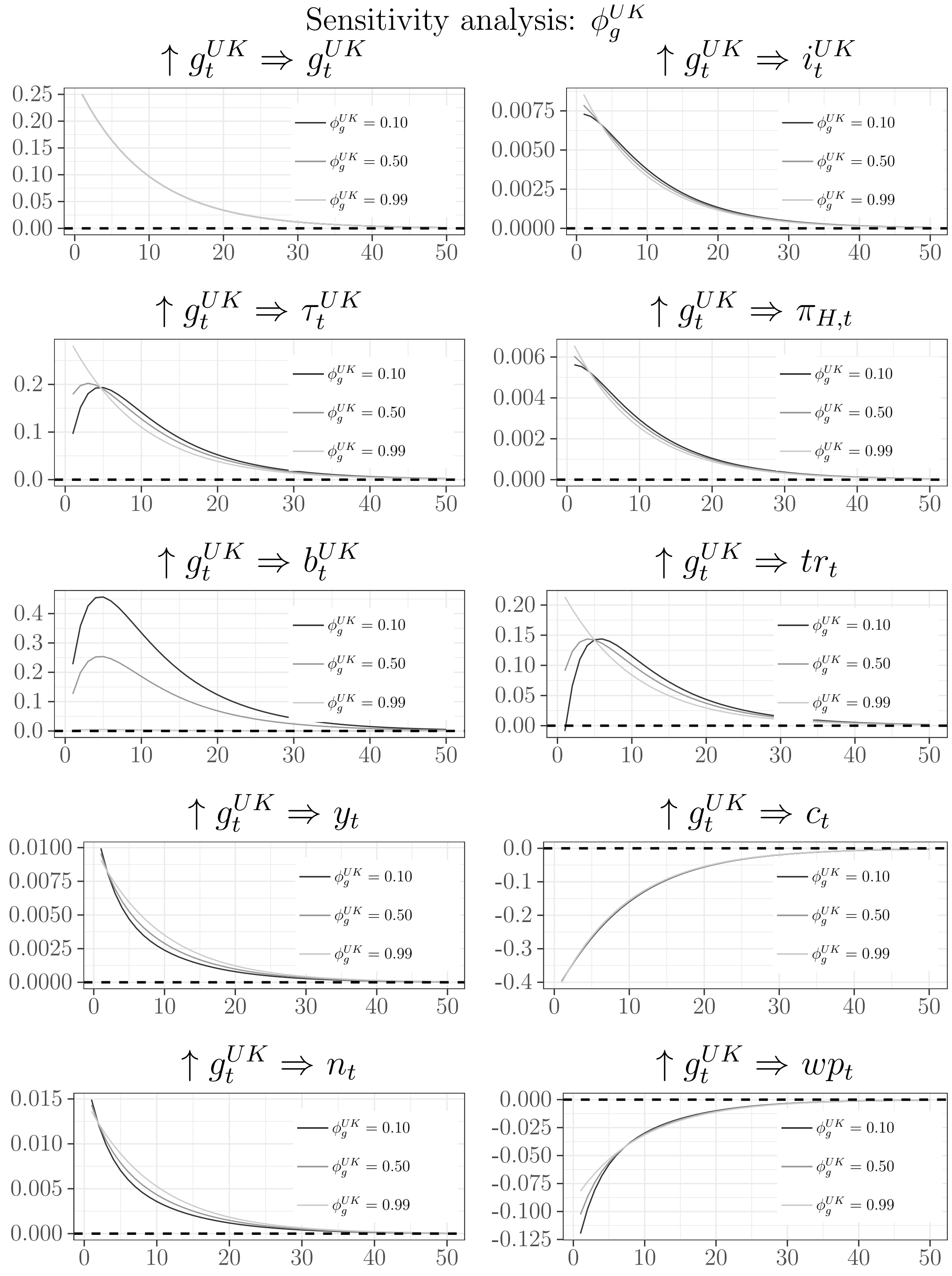

Figure 12: *Sensitivity analysis:* ϕ_g^{UK} $\forall\{0.10, 0.50, 0.99\}$, *when* $\phi_b^{UK} = 0.33$, $\rho_g^{UK} = 0.90$

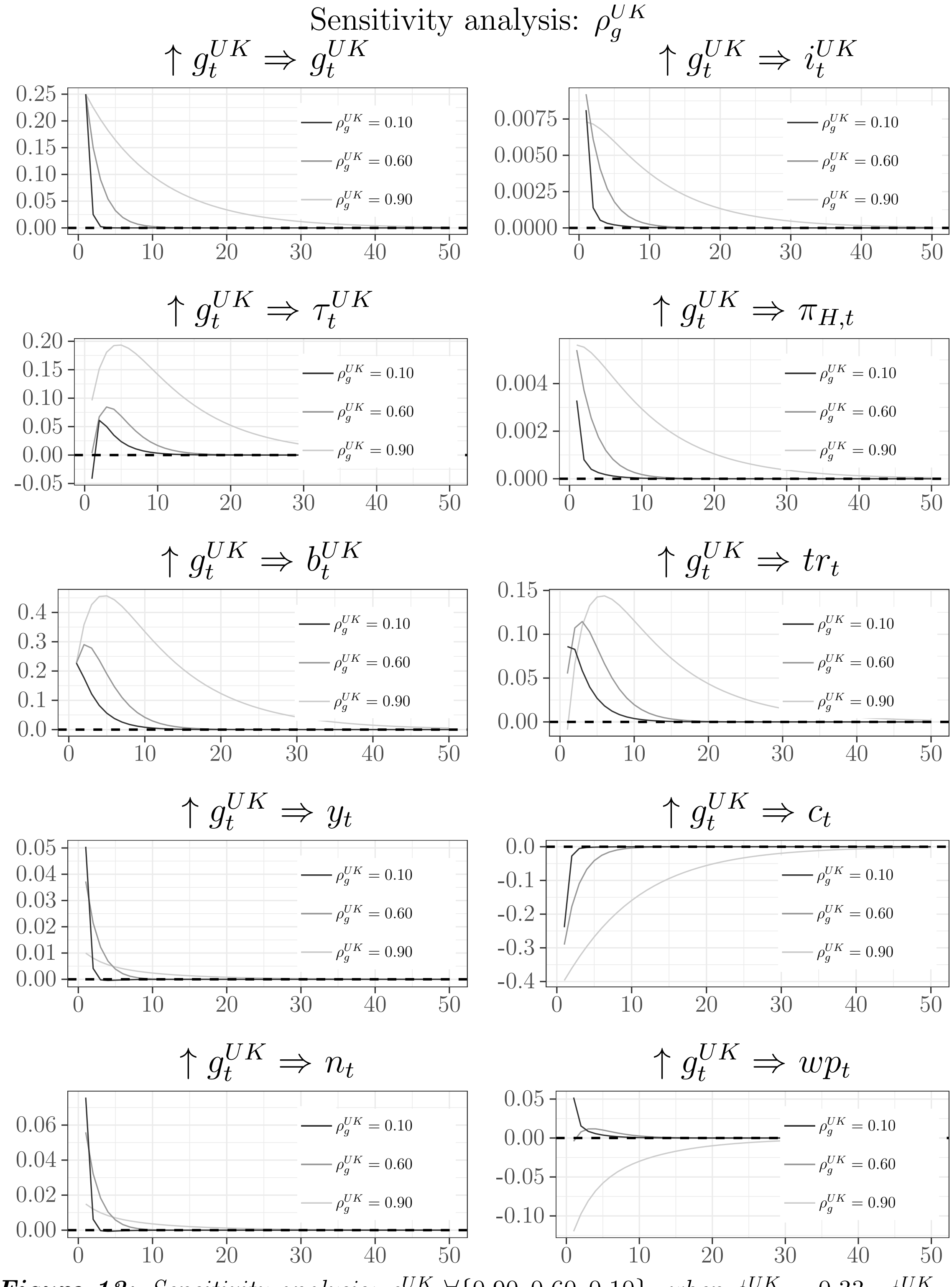

Figure 13: *Sensitivity analysis:* ρ_g^{UK} $\forall\{0.90, 0.60, 0.10\}$*, when* $\phi_b^{UK} = 0.33$*,* $\phi_g^{UK} = 0.10$

Finally, regarding the transmission of shock from one country to another, the model does not consider migration, national (UK-wide) wages, and significant price pass-through (low real exchange rate between the two economies). Therefore, in policy scenarios with two governments, endogenous variables in one country do not respond considerably to shocks in government spending in another country; the effect exhibits a magnitude on the order of 1e-13. This is a clear limitation of the model and calls for additional attention. The following section discusses other DSGE extensions that could improve the model's performance.

6 Limitations and Extensions

The greatest limitation of the model is its scale. For instance, the model assumes that a representative household derives utility from consumption and leisure only. In practice, government spending is not wasteful, and households directly benefit from the existence of public services. However, the government goods (services) did not enter the utility function, which could have enabled a more stimulating analysis. Moreover, the households were assumed to have frictionless access to financial instruments, allowing them to save a part of their consumption. When households are modelled as saving-constrained or poor (*hand-to-mouth*), DSGE results tend to align more with observed empirical data.

In terms of firms, the model made several simplifying assumptions. Firstly, it assumed that only domestic goods' price, not labour (wage), is sticky. In practice, wages are determined not only by labour supply but also by the existence of employment contracts, labour laws, and unions. Secondly, more than labour is needed to produce goods; firms also require private, public, and human capital, which the model ignored in an attempt to retain its small scale.

Ricci (2019) model is more tailored to Scotland and the rest of the UK economies. For instance, the government in Westminister administers funding to Scotland in line with the Barnett formula and there exists the oil industry. More careful consideration is given to the real exchange rate between the two economies to account for a significant price pass-through. The absence of these extensions are limitations of the model employed by this dissertation.

7 Conclusion

Modern macroeconomic frameworks aim to inform monetary and fiscal policy decision-makers about preferred policy objectives and ways they can be attained. The aim of the dissertation was to build a small-scale two-country NK DSGE SOE model for Scotland and the rest of the UK that would complement the informal dimension of the decision-making process.

More specifically, the first section provided a brief introduction to Keynesian and Classical schools of thought and how over time, they led to the development of modern macroeconomic frameworks actively employed by international organisations and central banks today. The second section presented the DSGE model: households, the government, firms, and equilibrium (market clearing) conditions. In the interest of space, most of the derivations were placed in the Appendix but the section explained the modelled behaviour of endogenous variables. The presentation of the model was followed by a section describing the processes of estimating and calibrating model parameters. Using more than twenty years of data from the Quarterly National Accounts (of Scotland), the estimated model parameters in Scotland were considerably different from their counterparts in the rest of the UK. Steady-state ratios in both economies were calculated using the 1998Q1-2021Q4 sample data.

While the two economies were modelled as symmetrical, the differences in model parameter values allowed the identification of asymmetric dynamic responses to shocks in government spending. The 'Dynamic Responses' section analysed impulse response functions and provided an intuition behind the source and direction of the response. Across all policy scenarios, the asymmetric parameter values did not alter the direction of dynamic responses. In fact, the parameter sensitivity analysis showed that only the government spending persistence parameter ρ_g can alter the effect's direction and only when extremely small values are considered. Overall, the findings suggest that the introduction of the labour tax makes government spending more distortionary. Being in or out of a fiscal union made almost no difference to the rest of the UK, but it significantly affected dynamic responses in Scotland. The parameter sensitivity analysis suggests that only the government spending persistence parameter has a considerable influence on dynamic responses. The fiscal feedback parameters influenced the amount of bonds issued, the labour tax rate, and tax revenue, but the changes had a limited effect on the dynamic responses of other endogenous variables. Finally, the limitations section discussed ways to extend the model to allow more sophisticated policy analysis or improved data fit.

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8 Appendix

8.1 Figures

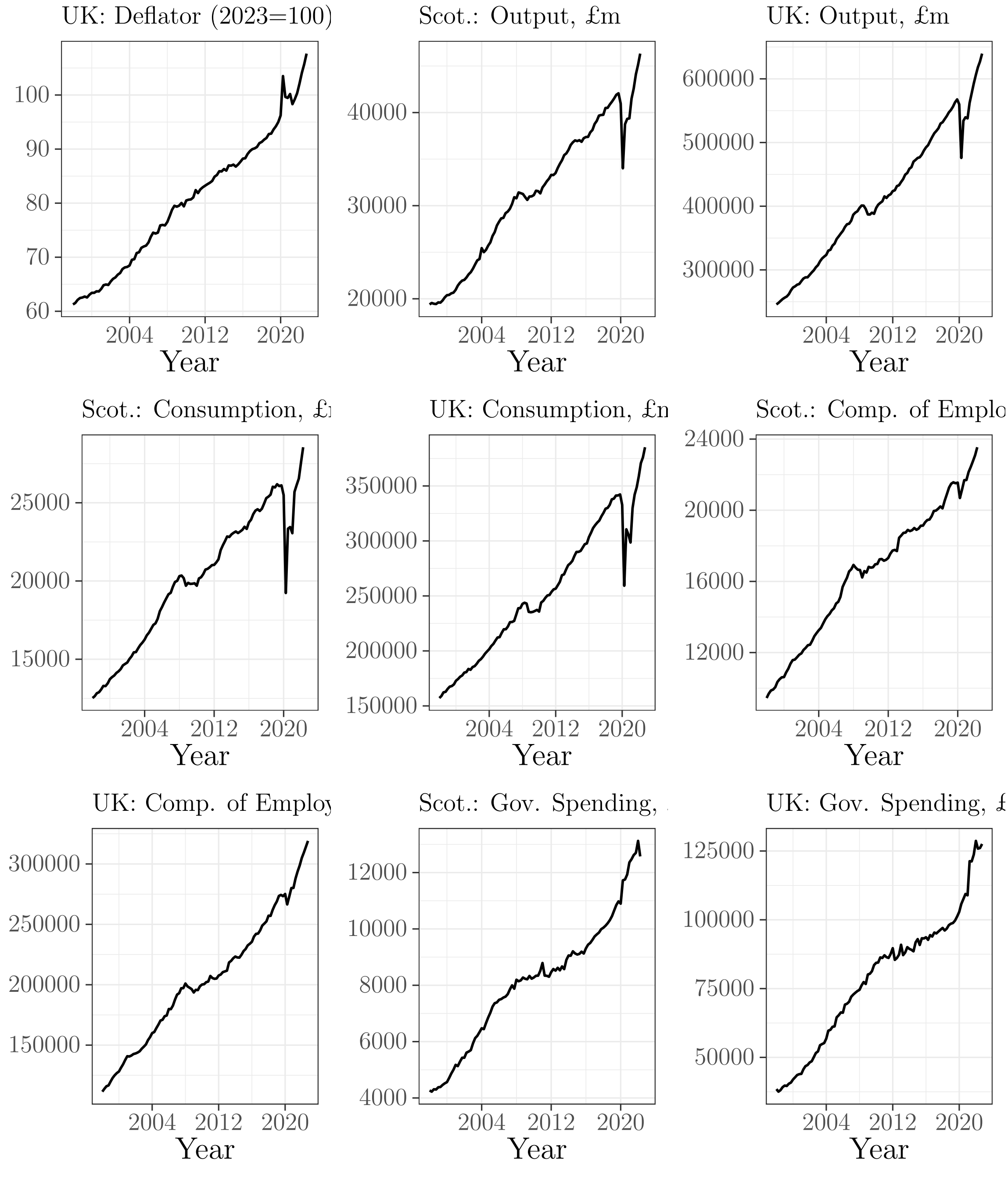

Figure 14: *Time series from QNA/QNAS (before transformations).*

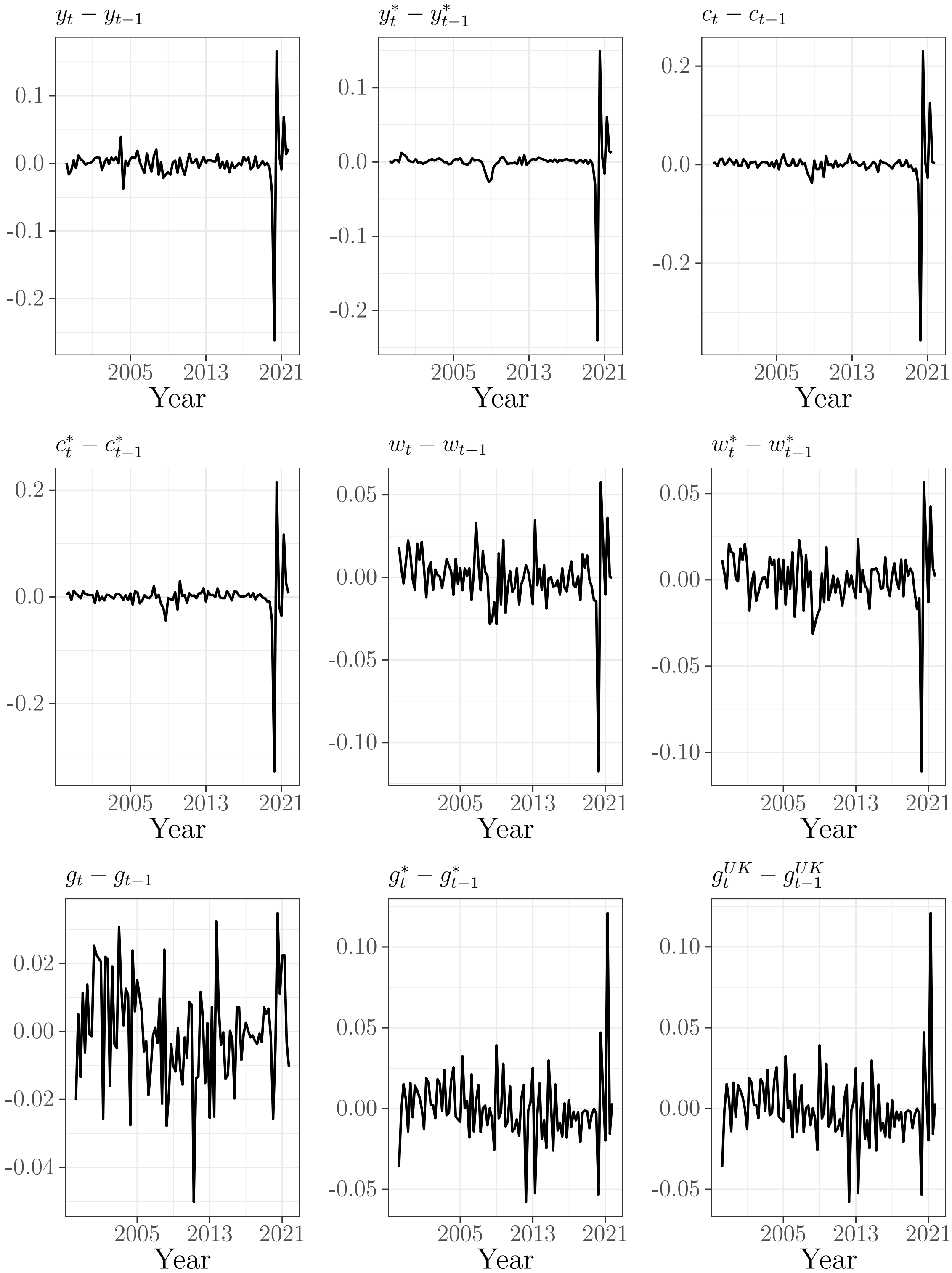

Figure 15: *Time series processed to work with Dynare and mapped to model variables*

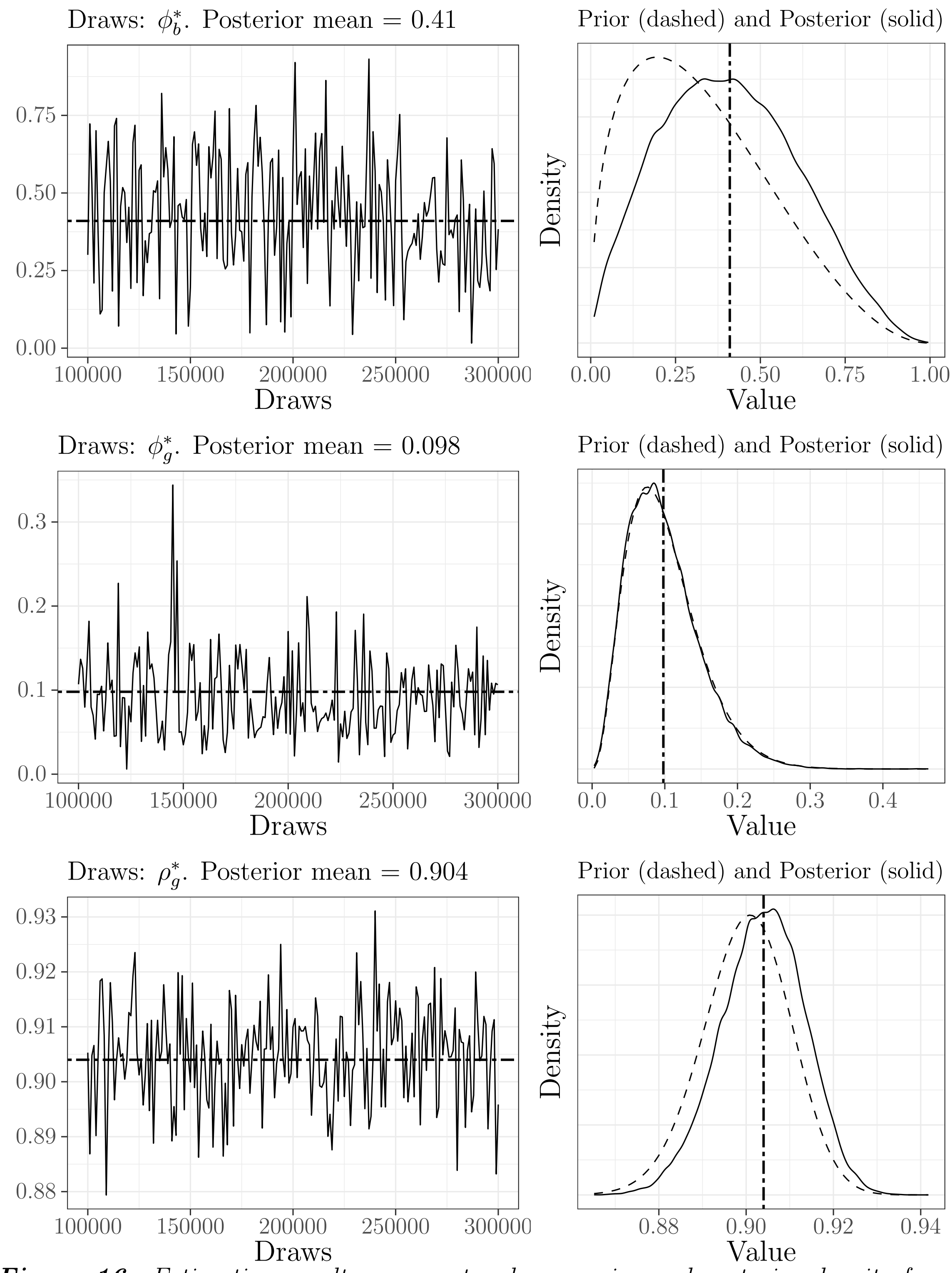

Figure 16: *Estimation results: parameter draws, prior and posterior density functions (rUK).*

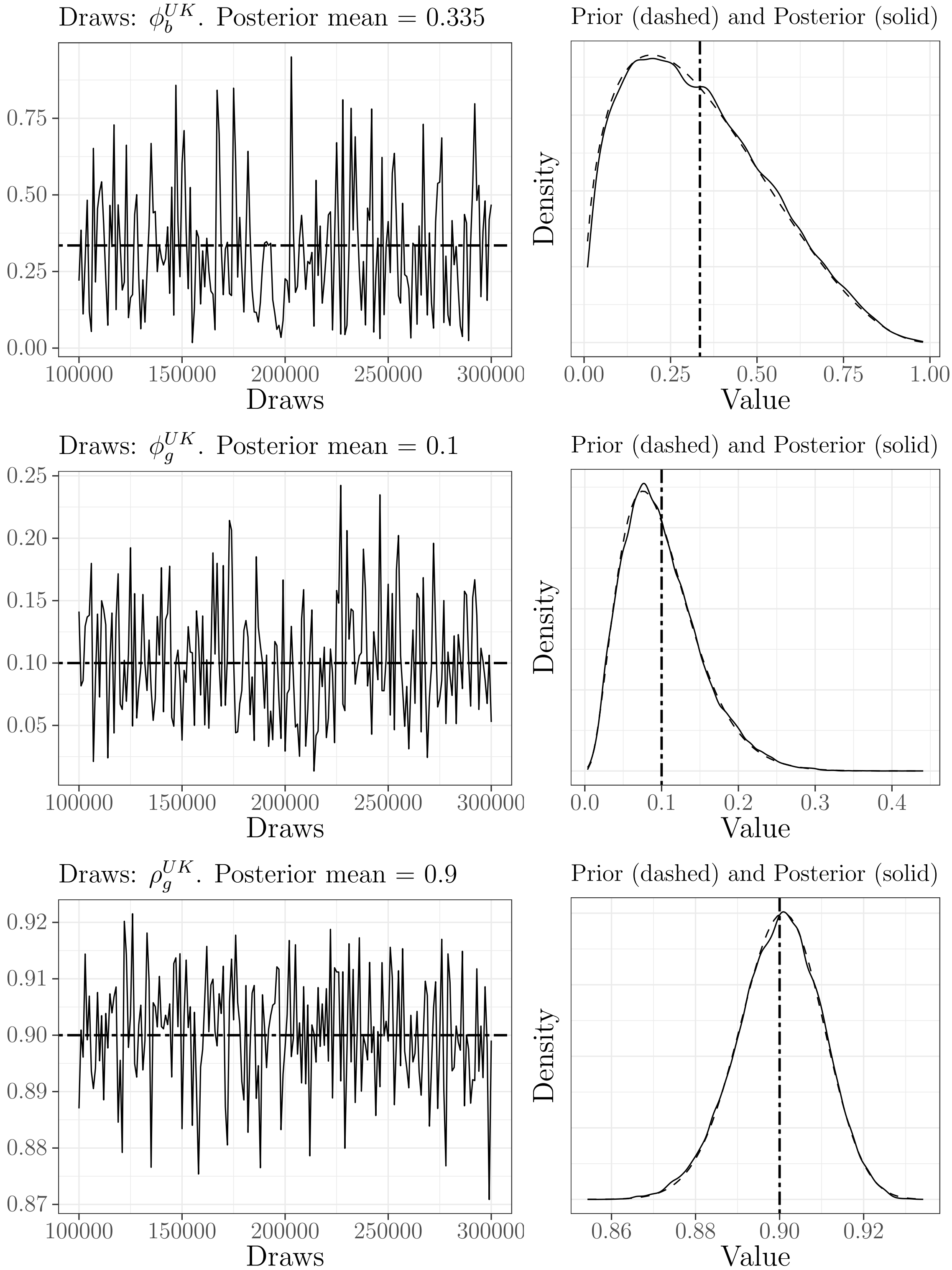

Figure 17: *Estimation results: parameter draws, prior and posterior density functions (UK).*

8.2 Derivations

8.2.1 Derivation of log-linearised intratemporal optimality condition

$$C_t^\sigma N_t^\varphi = \frac{W_t}{P_t}(1-\tau_t) \tag{A.123}$$

$$C_t^\sigma N_t^\varphi = \frac{W_t}{P_t} - \frac{W_t}{P_t}\tau_t \tag{A.124}$$

$$C^\sigma N^\varphi = \frac{W}{P} - \frac{W}{P}\tau \quad \text{(Steady state)} \tag{A.125}$$

Using Uhlig's (1999) method, $X_tY_t \approx XY\mathbf{e}^{\tilde{X}_t+\tilde{Y}_t}$

$$C^\sigma N^\varphi \mathbf{e}^{\sigma c_t+\varphi n_t} = \frac{W}{P}\mathbf{e}^{w_t-p_t} - \frac{W}{P}\tau\mathbf{e}^{w_t-p_t-\tilde{\tau}_t} \tag{A.126}$$

Using $\mathbf{e}^{\tilde{X}_t} \approx 1+\tilde{X}_t$:

$$C^\sigma N^\varphi(1+\sigma c_t+\varphi n_t) = \frac{W}{P}(1+w_t-p_t) - \frac{W}{P}\tau(1+w_t-p_t-\tilde{\tau}_t) \tag{A.127}$$

Subtract (A.125):

$$C^\sigma N^\varphi(\sigma c_t+\varphi n_t) = \frac{W}{P}(w_t-p_t) - \frac{W}{P}\tau(w_t-p_t-\tilde{\tau}_t) \tag{A.128}$$

$$C^\sigma N^\varphi(\sigma c_t+\varphi n_t) = \frac{W}{P}\left[(w_t-p_t) - \tau(w_t-p_t-\tilde{\tau}_t)\right] \tag{A.129}$$

$$C^\sigma N^\varphi(\sigma c_t+\varphi n_t) = \frac{W}{P}\left[(w_t-p_t) - \tau(w_t-p_t) - \tau\tilde{\tau}_t\right] \tag{A.130}$$

$$C^\sigma N^\varphi(\sigma c_t+\varphi n_t) = \frac{W}{P}\left[(1-\tau)(w_t-p_t) - \tau\tilde{\tau}_t\right] \tag{A.131}$$

$$C^\sigma N^\varphi\frac{P}{W}(\sigma c_t+\varphi n_t) = \left[(1-\tau)(w_t-p_t) - \tau\tilde{\tau}_t\right] \tag{A.132}$$

$$(1-\tau)(\sigma c_t+\varphi n_t) = \left[(1-\tau)(w_t-p_t) - \tau\tilde{\tau}_t\right] \tag{A.133}$$

$$\sigma c_t+\varphi n_t = w_t-p_t - \frac{\tau}{1-\tau}\tilde{\tau}_t \tag{A.134}$$

8.2.2 Derivation of log-linearised intertemporal optimality condition

$$\beta\,\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{Z_{t+1}}{Z_t}\right)\left(\frac{P_t}{P_{t+1}}\right)\right]=\mathbb{E}_t\left[\frac{1}{R_{t+1}}\right] \tag{A.135}$$

$$\ln\beta-\mathbb{E}_t[\sigma c_{t+1}]+\sigma c_t+\mathbb{E}_t[z_{t+1}]-z_t+p_t-\mathbb{E}_t[p_{t+1}]=-\ln R_{t+1} \tag{A.136}$$

$$\ln\beta-\mathbb{E}_t[\sigma c_{t+1}]+\sigma c_t+\mathbb{E}_t[\rho_z z_t+\varepsilon_t^z]-z_t+p_t-\mathbb{E}_t[p_{t+1}]=-\ln R_{t+1} \tag{A.137}$$

$$\ln\beta-\mathbb{E}_t[\sigma c_{t+1}]+\sigma c_t+\rho_z z_t-z_t+\mathbb{E}_t[\varepsilon_t^z]+p_t-\mathbb{E}_t[p_{t+1}]=-\ln R_{t+1} \tag{A.138}$$

$$\ln\beta-\mathbb{E}_t[\sigma c_{t+1}]+\sigma c_t+(\rho_z-1)z_t+p_t-\mathbb{E}_t[p_{t+1}]=-\ln R_{t+1} \tag{A.139}$$

$$\ln\beta-\mathbb{E}_t[\sigma c_{t+1}]+\sigma c_t-(1-\rho_z)z_t+p_t-\mathbb{E}_t[p_{t+1}]=-\ln R_{t+1} \tag{A.140}$$

$$\sigma c_t=-\ln R_{t+1}-\ln\beta+\mathbb{E}_t[\sigma c_{t+1}]-p_t+\mathbb{E}_t[p_{t+1}]-(1-\rho_z)z_t \tag{A.141}$$

$$c_t=\mathbb{E}_t[c_{t+1}]-\frac{1}{\sigma}(\ln R_{t+1}-\rho-\mathbb{E}_t[\pi_{t+1}])-\frac{1}{\sigma}(1-\rho_z)z_t \tag{A.142}$$

$$c_t=\mathbb{E}_t[c_{t+1}]-\frac{1}{\sigma}(i_t-\mathbb{E}_t[\pi_{t+1}]-\rho)-\frac{1}{\sigma}(1-\rho_z)z_t \tag{A.143}$$

8.2.3 Derivation of bilateral exchange rate

International Risk-Sharing Equation for country i can be rewritten as:

$$\frac{V_{t,t+1}}{\mathcal{E}_t^i P_t^i}(C_t^i)^{-\sigma} = \xi_{t,t+1}\beta(C_{t+1}^i)^{-\sigma}\frac{1}{\mathcal{E}_{t+1}^i P_{t+1}^i} \tag{A.144}$$

Given that:

$$Q_{t,t+1} = \frac{V_{t,t+1}}{\xi_{t,t+1}} \tag{A.145}$$

$$\frac{V_{t,t+1}}{\mathcal{E}_t^i P_t^i}(C_t^i)^{-\sigma} = \xi_{t,t+1}\beta(C_{t+1}^i)^{-\sigma}\frac{1}{\mathcal{E}_{t+1}^i P_{t+1}^i} \tag{A.146}$$

$$Q_{t,t+1} = \beta\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\sigma}\left(\frac{P_t^i}{P_{t+1}^i}\right)\left(\frac{\mathcal{E}_t^i}{\mathcal{E}_{t+1}^i}\right) \tag{A.147}$$

Recall that:

$$\frac{\partial L}{\partial C_t} = \beta^t C_t^{-\sigma}P_t^{-1} = \lambda_t \tag{A.148}$$

$$\frac{\partial L}{\partial D_{t+1}} = -\lambda_t\,\mathbb{E}_t[Q_{t,t+1}] = -\,\mathbb{E}_t[\lambda_{t+1}] \tag{A.149}$$

$$= -\beta^t C_t^{-\sigma}P_t^{-1}\,\mathbb{E}_t[Q_{t,t+1}] = -\,\mathbb{E}_t[\lambda_{t+1}] \tag{A.150}$$

Which is symmetrical for country i:

$$\frac{\partial L^i}{\partial C_t^i} = \beta^t (C_t^i)^{-\sigma}(\mathcal{E}_{i,t}P_t^i)^{-1} = \lambda_t^i \tag{A.151}$$

$$\frac{\partial L^i}{\partial D_{t+1}^i} = -\lambda_t^i\,\mathbb{E}_t[Q_{t,t+1}] = -\,\mathbb{E}_t[\lambda_{t+1}^i] \tag{A.152}$$

$$= -\beta^t (C_t^i)^{-\sigma}(\mathcal{E}_{i,t}P_t^i)^{-1}\,\mathbb{E}_t[Q_{t,t+1}] = -\,\mathbb{E}_t[\lambda_{t+1}^i] \tag{A.153}$$

$$\frac{(A.150)}{(A.153)} : \frac{-\beta^t C_t^{-\sigma}P_t^{-1}\,\mathbb{E}_t[Q_{t,t+1}]}{-\beta^t (C_t^i)^{-\sigma}(\mathcal{E}_{i,t}P_t^i)^{-1}\,\mathbb{E}_t[Q_{t,t+1}]} = \frac{-\,\mathbb{E}_t[\lambda_{t+1}]}{-\,\mathbb{E}_t[\lambda_{t+1}^i]} \tag{A.154}$$

$$C_t^{-\sigma}(C_t^i)^\sigma \frac{\mathcal{E}_{i,t}P_t^i}{P_t} = 1 \tag{A.155}$$

$$C_t^{-\sigma}(C_t^i)^\sigma \mathcal{Q}_{i,t} = 1 \tag{A.156}$$

$$C_t^{-\sigma}(C_t^i)^\sigma = \frac{1}{\mathcal{Q}_{i,t}} \tag{A.157}$$

$$C_t^{-\sigma} = \frac{1}{\mathcal{Q}_{i,t}}(C_t^i)^{-\sigma} \tag{A.158}$$

$$C_t^{\sigma} = \mathcal{Q}_{i,t}(C_t^i)^{\sigma} \tag{A.159}$$

$$\Rightarrow C_t = C_t^i \mathcal{Q}_{i,t}^{\frac{1}{\sigma}} \tag{A.160}$$

8.2.4 Derivation of the Euler equation using Arrow securities

$$\frac{V_{t,t+1}}{P_t} C_t^{-\sigma} = \xi_{t,t+1} \beta C_{t+1}^{-\sigma} \frac{1}{P_{t+1}} \tag{A.161}$$

Where $V_{t,t+1}$ is an Arrow security and $\xi_{t,t+1}$ is the probability that the Arrow security will yield a payoff next period. Interpretation: LHS is paying for the security (expenses) in terms of consumption at P_t prices. RHS is the payoff if the Arrow security yields a payoff. The consumer will only be willing to pay LHS if the payoff is at least as big on the RHS. Given that:

$$Q_{t,t+1} = \frac{V_{t,t+1}}{\xi_{t,t+1}} \tag{A.162}$$

$$\frac{V_{t,t+1}}{P_t} C_t^{-\sigma} = \xi_{t,t+1} \beta C_{t+1}^{-\sigma} \frac{1}{P_{t+1}} \tag{A.163}$$

$$Q_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \tag{A.164}$$

$$\mathbb{E}_t[Q_{t,t+1}] = \beta \, \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right] \tag{A.165}$$

$$\beta \, \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \left(\frac{P_t}{P_{t+1}} \right) \right] = Q_t \tag{A.166}$$

8.2.5 Derivation of log-linearised government budget constraint

$$\mathbb{E}_t[R_t^{-1}B_{t+1}] + P_tT_t = P_tG_t + B_t \tag{A.167}$$

Divide by P_t

$$\mathbb{E}_t[R_t^{-1}B_{t+1}P_t^{-1}] + T_t = G_t + B_tP_tP_{t-1}P_{t-1}^{-1} \tag{A.168}$$

$$\begin{aligned}&\frac{1}{R}\frac{B}{P}\left[1 + \frac{B_{t+1}-B}{B} - \frac{P_t-P}{P} - \frac{R_t-R}{R}\right] + T\left[1 + \frac{T_t-T}{T}\right] \\ &= G\left[1 + \frac{G_t-G}{G}\right] + \frac{B}{P}\frac{P}{P}\left[1 + \frac{B_t-B}{B} - \frac{P_{t-1}-P}{P} + \underbrace{\frac{P_t-P}{P} - \frac{P_{t-1}-P}{P}}_{\pi=0}\right]\end{aligned} \tag{A.169}$$

Subtract steady state (A.167)

$$\frac{1}{R}\frac{B}{P}\left[\frac{B_{t+1}-B}{B} - \frac{P_t-P}{P} - \frac{R_t-R}{R}\right] + [T_t-T] = [G_t-G] + \frac{B}{P}\left[\frac{B_t-B}{B} - \frac{P_{t-1}-P}{P}\right] \tag{A.170}$$

Divide by Y

$$\frac{1}{R}\frac{1}{Y}\frac{B}{P}\left[\frac{B_{t+1}-B}{B} - \frac{P_t-P}{P} - \frac{R_t-R}{R}\right] + t_t = g_t + \frac{1}{Y}\frac{B}{P}\left[\frac{B_t-B}{B} - \frac{P_{t-1}-P}{P}\right] \tag{A.171}$$

$$\frac{1}{R}\frac{1}{Y}\left[\frac{B_{t+1}-B}{P} - \frac{B(P_t-P)}{P^2} - \frac{B(R_t-R)}{R}\right] + t_t = g_t + \frac{1}{Y}\left[\frac{B_t-B}{P} - \frac{B(P_{t-1}-P)}{P^2}\right] \tag{A.172}$$

$$\frac{1}{R}\frac{1}{Y}\left[\frac{B_{t+1}-B}{P} - \frac{(B/P)(P_t-P)}{P} - \frac{B(R_t-R)}{R}\right] + t_t = g_t + \frac{1}{Y}\left[\frac{B_t-B}{P} - \frac{(B/P)(P_{t-1}-P)}{P}\right] \tag{A.173}$$

$$\frac{1}{R}\frac{1}{Y}\left[\frac{B_{t+1}-B-(B/P)(P_t-P)}{P} - \frac{B(R_t-R)}{R}\right] + t_t = g_t + \frac{1}{Y}\left[\frac{B_t-B-(B/P)(P_{t-1}-P)}{P}\right] \tag{A.174}$$

$$\frac{1}{R}\frac{1}{Y}\left[\frac{B_{t+1}-B-(B/P)P_t+B}{P}-\frac{B(R_t-R)}{R}\right]+t_t=g_t+\frac{1}{Y}\left[\frac{B_t-B-(B/P)P_{t-1}+B}{P}\right] \tag{A.175}$$

$$\frac{1}{R}\frac{1}{Y}\left[\frac{B_{t+1}-(B/P)P_t}{P}-\frac{B(R_t-R)}{R}\right]+t_t=g_t+\frac{1}{Y}\left[\frac{B_t-(B/P)P_{t-1}}{P}\right] \tag{A.176}$$

$$\frac{1}{R}\frac{1}{Y}\left[\frac{(B_{t+1}/P_t)-(B/P)}{P/P_t}-\frac{B(R_t-R)}{R}\right]+t_t=g_t+\frac{1}{Y}\left[\frac{(B_t/P_{t-1})-(B/P)}{P/P_{t-1}}\right] \tag{A.177}$$

$$\frac{1}{R}b_{t+1}\frac{1}{P/P_t}+t_t=g_t+b_t\left[\frac{1}{P/P_{t-1}}\right] \tag{A.178}$$

$$\frac{1}{R}b_{t+1}\left(\frac{P_t}{P}-1+1\right)+t_t=g_t+b_t\left(\frac{P_{t-1}}{P}-1+1\right) \tag{A.179}$$

$$\frac{1}{R}b_{t+1}\left(p_t+1\right)+t_t=g_t+b_t\left(p_{t-1}+1\right) \tag{A.180}$$

Following Uhlig (1995), $x_t y_t \approx 0$

$$\frac{1}{R}b_{t+1}+t_t=g_t+b_t \tag{A.181}$$

$$b_{t+1}=(1+\rho)(b_t+g_t-t_t) \tag{A.182}$$

where: $\rho=\beta^{-1}-1$

8.2.6 Derivation of log-linearised aggregate resource constraint

$$Y_t = (1-\upsilon)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \upsilon \mathcal{S}_t^{\eta} Y_t^* + G_t \tag{A.183}$$

$$Y\mathbf{e}^{y_t} = (1-\upsilon)\left(\frac{P}{P_H}\right)^{\eta} C\mathbf{e}^{-\eta p_{H,t}+\eta p_t+c_t} + \upsilon S^{\eta} Y^* \mathbf{e}^{\eta s_t + y_t^*} + G\mathbf{e}^{g_t} \tag{A.184}$$

$$Y(1+y_t) = (1-\upsilon)\left(\frac{P}{P_H}\right)^{\eta} C(1-\eta p_{H,t}+\eta p_t+c_t) + \upsilon S^{\eta} Y^*(1+\eta s_t + y_t^*) + G(1+g_t) \tag{A.185}$$

$$Yy_t = (1-\upsilon)\left(\frac{P}{P_H}\right)^{\eta} C(-\eta p_{H,t}+\eta p_t+c_t) + \upsilon S^{\eta} Y^*(\eta s_t + y_t^*) + Gg_t \tag{A.186}$$

Using $P/P_H = \mathcal{S}^{\upsilon}$, $\mathcal{S} = 1$, and $C = Y^*$:

$$y_t = C_Y\left[(1-\upsilon)(-\eta p_{H,t}+\eta p_t+c_t) + \upsilon(\eta s_t + y_t^*)\right] + G_Y g_t \tag{A.187}$$

Using $p_t - p_{H,t} = \upsilon s_t$:

$$y_t = C_Y\left[(1-\upsilon)(\eta\upsilon s_t + c_t) + \upsilon(\eta s_t + y_t^*)\right] + G_Y g_t \tag{A.188}$$

$$y_t = C_Y\left[(1-\upsilon)c_t + (1-\upsilon)\eta\upsilon s_t + \upsilon\eta s_t + \upsilon y_t^*\right] + G_Y g_t \tag{A.189}$$

$$y_t = C_Y\left[(1-\upsilon)c_t + \upsilon(2-\upsilon)\eta s_t + \upsilon y_t^*\right] + G_Y g_t \tag{A.190}$$

8.2.7 Derivation of log-linearised terms of trade

We use two definitions:

$$c_t = y_t^* + \frac{1}{\sigma} z_t + \frac{1-\upsilon}{\sigma} s_t \tag{A.191}$$

$$y_t = C_Y \left[(1-\upsilon)c_t + \upsilon(2-\upsilon)\eta s_t + \upsilon y_t^*\right] + G_Y g_t \tag{A.192}$$

$$y_t = C_Y \left[(1-\upsilon)\left(y_t^* + \frac{1}{\sigma} z_t + \frac{1-\upsilon}{\sigma} s_t\right) + \upsilon(2-\upsilon)\eta s_t + \upsilon y_t^*\right] + G_Y g_t \tag{A.193}$$

$$y_t = C_Y \left[(1-\upsilon)y_t^* + \frac{1-\upsilon}{\sigma} z_t + (1-\upsilon)\frac{1-\upsilon}{\sigma} s_t + \upsilon(2-\upsilon)\eta s_t + \upsilon y_t^*\right] + G_Y g_t \tag{A.194}$$

Notice $(1-\upsilon)y_t^* + \upsilon y_t^* = y_t^*$

$$y_t = C_Y \left[y_t^* + \frac{1-\upsilon}{\sigma} z_t + (1-\upsilon)\frac{1-\upsilon}{\sigma} s_t + \upsilon(2-\upsilon)\eta s_t\right] + G_Y g_t \tag{A.195}$$

$$y_t = C_Y \left[y_t^* + \frac{1-\upsilon}{\sigma} z_t + \left((1-\upsilon)\frac{1-\upsilon}{\sigma} + \upsilon(2-\upsilon)\eta\right) s_t\right] + G_Y g_t \tag{A.196}$$

Rearrange:

$$C_Y\left((1-\upsilon)\frac{1-\upsilon}{\sigma}+\upsilon(2-\upsilon)\eta\right)s_t = y_t - C_Y\left[y_t^* - \frac{1-\upsilon}{\sigma}z_t\right] - G_Y g_t \tag{A.197}$$

$$C_Y\left(\frac{(1-\upsilon)(1-\upsilon)+\sigma\upsilon(2-\upsilon)\eta}{\sigma}\right)s_t = y_t - C_Y\left[y_t^* - \frac{1-\upsilon}{\sigma}z_t\right] - G_Y g_t \tag{A.198}$$

$$C_Y\left(\frac{1-2\upsilon+\upsilon^2+\sigma\upsilon(2-\upsilon)\eta}{\sigma}\right)s_t = y_t - C_Y\left[y_t^* - \frac{1-\upsilon}{\sigma}z_t\right] - G_Y g_t \tag{A.199}$$

$$C_Y\left(\frac{1-\upsilon(2-\upsilon)+\sigma\upsilon(2-\upsilon)\eta}{\sigma}\right)s_t = y_t - C_Y\left[y_t^* - \frac{1-\upsilon}{\sigma}z_t\right] - G_Y g_t \tag{A.200}$$

$$C_Y\left(\frac{1-\upsilon(2-\upsilon)(1-\sigma\eta)}{\sigma}\right)s_t = y_t - C_Y\left[y_t^* - \frac{1-\upsilon}{\sigma}z_t\right] - G_Y g_t \tag{A.201}$$

$$C_Y\left(\frac{1+\upsilon(2-\upsilon)(\sigma\eta-1)}{\sigma}\right)s_t = y_t - C_Y\left[y_t^* - \frac{1-\upsilon}{\sigma}z_t\right] - G_Y g_t \tag{A.202}$$

$$C_Y\left(\frac{1+\upsilon(2\sigma\eta-\upsilon\sigma\eta-2+\upsilon)}{\sigma}\right)s_t = y_t - C_Y\left[y_t^* - \frac{1-\upsilon}{\sigma}z_t\right] - G_Y g_t \tag{A.203}$$

$$C_Y\left(\frac{1+\upsilon(\varpi-1)}{\sigma}\right)s_t = y_t - C_Y\left[y_t^* - \frac{1-\upsilon}{\sigma}z_t\right] - G_Y g_t \tag{A.204}$$

$$C_Y\sigma^{-1}\Phi^{-1}s_t = y_t - C_Y\left[y_t^* - \frac{1-\upsilon}{\sigma}z_t\right] - G_Y g_t \tag{A.205}$$

$$s_t = Y_C\sigma_\upsilon y_t - \sigma_\upsilon\left[y_t^* - \frac{1-\upsilon}{\sigma}z_t\right] - G_C\sigma_\upsilon g_t \tag{A.206}$$

$$s_t = \sigma_\upsilon(C_Y^{-1}y_t - y_t^* - C_G^{-1}g_t) - (1-\upsilon)\Phi z_t \tag{A.207}$$

where $\varpi = \sigma\eta + (1-\upsilon)(\sigma\eta-1)$, $\Phi = \frac{1}{1+\upsilon(\varpi-1)}$ and $\sigma_\upsilon = \sigma\Phi$.

8.2.8 Derivation of (a version of) Dynamic IS equation

$$0 = \mathbb{E}\{\Delta c_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \frac{\upsilon}{\sigma}\mathbb{E}\{\Delta s_{t+1}\} + \frac{1}{\sigma}(1-\rho_z)z_t \tag{A.208}$$

From aggregate RC: $c_t = [y_t - C_Y[\upsilon(2-\upsilon)\eta s_t + \upsilon y_t^*] - G_Y g_t]\, C_Y^{-1}(1-\upsilon)^{-1}$

Substitute and multiply by $C_Y(1-\upsilon)$

$$\begin{aligned} 0 &= \mathbb{E}\{\Delta y_{t+1}\} - C_Y[\upsilon(2-\upsilon)\eta\,\mathbb{E}\{\Delta s_{t+1}\} + \upsilon\,\mathbb{E}\{\Delta y_{t+1}^*\}] - G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\} + \\ &C_Y\left[-\frac{1-\upsilon}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \frac{(1-\upsilon)\upsilon}{\sigma}\mathbb{E}\{\Delta s_{t+1}\} + \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t\right] \end{aligned} \tag{A.209}$$

$$\begin{aligned} 0 &= \mathbb{E}\{\Delta y_{t+1}\} + C_Y\left(\frac{(1-\upsilon)\upsilon}{\sigma} - \upsilon(2-\upsilon)\eta\right)\mathbb{E}\{\Delta s_{t+1}\} + C_Y\upsilon\,\mathbb{E}\{\Delta y_{t+1}^*\} \\ &- G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\} + C_Y\left[-\frac{1-\upsilon}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t\right] \end{aligned} \tag{A.210}$$

$$\begin{aligned} 0 &= \mathbb{E}\{\Delta y_{t+1}\} - C_Y\left(\frac{\upsilon\varpi}{\sigma}\right)\mathbb{E}\{\Delta s_{t+1}\} + C_Y\upsilon\,\mathbb{E}\{\Delta y_{t+1}^*\} \\ &- G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\} + C_Y\left[-\frac{1-\upsilon}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t\right] \end{aligned} \tag{A.211}$$

Recall $\mathbb{E}_t\{\Delta s_{t+1}\} = \mathbb{E}_t\{\sigma_\upsilon(C_Y^{-1}\Delta y_{t+1} - \Delta y_{t+1}^* - C_G^{-1}\Delta g_{t+1}) - (1-\upsilon)\Phi\Delta z_{t+1}\}$

$$\begin{aligned} 0 &= \mathbb{E}\{\Delta y_{t+1}\} - C_Y\left(\frac{\upsilon\varpi}{\sigma}\right)\mathbb{E}_t\{\sigma_\upsilon(C_Y^{-1}\Delta y_{t+1} - \Delta y_{t+1}^* - C_G^{-1}\Delta g_{t+1}) - (1-\upsilon)\Phi\Delta z_{t+1}\} \\ &+ C_Y\upsilon\,\mathbb{E}\{\Delta y_{t+1}^*\} - G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\} + C_Y\left[-\frac{1-\upsilon}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t\right] \end{aligned} \tag{A.212}$$

$$\begin{aligned} 0 &= \mathbb{E}\{\Delta y_{t+1} - \upsilon\varpi\Phi\Delta y_{t+1}\} + \mathbb{E}\{C_Y\upsilon\varpi\Phi\Delta y_{t+1}^* - C_Y\upsilon\Delta y_{t+1}^*\} + C_Y\frac{1-\upsilon}{\sigma}(1-\rho_z)z_t \\ &+ C_Y\upsilon\varpi\Phi\frac{1-\upsilon}{\sigma}(\rho_z - 1)z_t + \mathbb{E}_t\{\upsilon\varpi\Phi G_Y\Delta g_{t+1} - G_Y\Delta g_{t+1}\} \\ &- C_Y\frac{1-\upsilon}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) \end{aligned} \tag{A.213}$$

Divide by Φ^{-1}

$$\begin{aligned} 0 &= \mathbb{E}\{\Phi^{-1}\Delta y_{t+1} - \upsilon\varpi\Delta y_{t+1}\} + \mathbb{E}\{C_Y\upsilon\varpi\Delta y_{t+1}^* - C_Y\upsilon\Phi^{-1}\Delta y_{t+1}^*\} \\ &+ C_Y\Phi^{-1}\frac{1-\upsilon}{\sigma}(1-\rho_z)z_t + C_Y\upsilon\varpi\frac{1-\upsilon}{\sigma}(\rho_z - 1)z_t + \mathbb{E}_t\{\upsilon\varpi G_Y\Delta g_{t+1} - G_Y\Phi^{-1}\Delta g_{t+1}\} \\ &- C_Y\frac{1-\upsilon}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) \end{aligned} \tag{A.214}$$

$$
\begin{aligned}
0 &= (\Phi^{-1} - \upsilon\varpi)\,\mathbb{E}\{\Delta y_{t+1}\} + C_Y \upsilon(\varpi - \Phi^{-1})\,\mathbb{E}\{\Delta y^*_{t+1}\} \\
&- C_Y(\Phi^{-1} - \upsilon\varpi)\frac{1-\upsilon}{\sigma}(\rho_z - 1)z_t + (\upsilon\varpi - \Phi^{-1})G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\} \\
&- C_Y\frac{1-\upsilon}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) \qquad \text{(A.215)}
\end{aligned}
$$

$$\text{Divide by } (\Phi^{-1} - \upsilon\varpi) \qquad \text{(A.216)}$$

$$
\begin{aligned}
0 &= \mathbb{E}\{\Delta y_{t+1}\} + C_Y\frac{\upsilon(\varpi - \Phi^{-1})}{\Phi^{-1} - \upsilon\varpi}\,\mathbb{E}\{\Delta y^*_{t+1}\} - C_Y\frac{1-\upsilon}{\sigma}(\rho_z - 1)z_t \\
&+ \frac{(\upsilon\varpi - \Phi^{-1})}{\Phi^{-1} - \upsilon\varpi}G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\} - \frac{1-\upsilon}{\Phi^{-1} - \upsilon\varpi}C_Y\frac{1}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) \qquad \text{(A.217)}
\end{aligned}
$$

$$
\begin{aligned}
y_t &= \mathbb{E}_t\{y_{t+1}\} - C_Y\frac{1-\upsilon}{\sigma}(\rho_z - 1)z_t - G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\} + C_Y\frac{\upsilon(\varpi - \Phi^{-1})}{\Phi^{-1} - \upsilon\varpi}\,\mathbb{E}\{\Delta y^*_{t+1}\} \\
&- \frac{1-\upsilon}{\Phi^{-1} - \upsilon\varpi}C_Y\frac{1}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) \qquad \text{(A.218)}
\end{aligned}
$$

$$\text{Note that } (\Phi^{-1} - \upsilon\varpi) \text{ can be simplified to } (1-\upsilon) \qquad \text{(A.219)}$$

$$
\begin{aligned}
y_t &= \mathbb{E}_t\{y_{t+1}\} - C_Y\frac{1-\upsilon}{\sigma}(\rho_z - 1)z_t - \frac{1-\upsilon}{1-\upsilon}C_Y\frac{1}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) \\
&+ C_Y\frac{(1-\upsilon)(\upsilon\varpi - \upsilon)}{1-\upsilon}\,\mathbb{E}\{\Delta y^*_{t+1}\} - G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\} \qquad \text{(A.220)}
\end{aligned}
$$

A version of Dynamic IS equation:

$$
\begin{aligned}
y_t &= \mathbb{E}_t\{y_{t+1}\} \\
&- C_Y\left[\frac{1}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) - \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t - \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y^*_{t+1}\}\right] - G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\}
\end{aligned}
\qquad \text{(A.221)}
$$

8.2.9 Derivation of (the final) Dynamic IS

$$y_t^n = \mathbb{E}_t\{y_{t+1}^n\} - C_Y\left[\frac{1}{\sigma_\upsilon}(r_t^n - \rho) - \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} - \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t\right] - G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\} \tag{A.222}$$

$$0 = \mathbb{E}_t\{\Delta y_{t+1}^n\} - C_Y\left[\frac{1}{\sigma_\upsilon}(r_t^n - \rho) - \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} - \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t\right] - G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\} \tag{A.223}$$

$$0 = \sigma_\upsilon\,\mathbb{E}_t\{\Delta y_{t+1}^n\} - C_Y\left[(r_t^n - \rho) - \sigma_\upsilon\upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} - \Phi(1-\upsilon)(1-\rho_z)z_t\right] - \sigma_\upsilon G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\} \tag{A.224}$$

$$\begin{aligned} 0 &= \sigma_\upsilon\,\mathbb{E}_t\{\Delta y_{t+1}^n\} \\ &\quad - C_Y r_t^n - C_Y\left[-\rho - \sigma_\upsilon\upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} - \Phi(1-\upsilon)(1-\rho_z)z_t\right] - \sigma_\upsilon G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\} \end{aligned} \tag{A.225}$$

$$r_t^n = C_Y^{-1}\sigma_\upsilon\,\mathbb{E}_t\{\Delta y_{t+1}^n\} + \rho + \sigma_\upsilon\upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} + \Phi(1-\upsilon)(1-\rho_z)z_t - \sigma_\upsilon C_G^{-1}\,\mathbb{E}_t\{\Delta g_{t+1}\} \tag{A.226}$$

Subtract (A.222) from (100) to yield:

$$\begin{aligned} 0 &= \sigma_\upsilon\,\mathbb{E}\{\Delta y_{t+1}\} - C_Y\big[(i_t - \mathbb{E}\{\pi_{H,+1}\}) - \rho - \sigma_\upsilon\upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} \\ &\quad - \Phi(1-\upsilon)(1-\rho_z)z_t\big] - \sigma_\upsilon G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\} \\ &\quad - \big(\sigma_\upsilon\,\mathbb{E}\{\Delta y_{t+1}^n\} - C_Y\big[(r_t^n - \rho - \sigma_\upsilon\upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} \\ &\quad - \Phi(1-\upsilon)(1-\rho_z)z_t\big] - \sigma_\upsilon G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\}\big) \end{aligned} \tag{A.227}$$

$$0 = \sigma_\upsilon\,\mathbb{E}_t\{\Delta\tilde{y}_{t+1}\} - C_Y(i_t - \mathbb{E}\{\pi_{H,+1}\} - r_t^n) \tag{A.228}$$

$$\tilde{y}_t = \mathbb{E}_t\{\tilde{y}_{t+1}\} - \frac{1}{\sigma_\upsilon}C_Y(i_t - \mathbb{E}\{\pi_{H,+1}\} - r_t^n) \tag{A.229}$$

8.2.10 Derivation of the average markup

We will make use of the following definitions

$$y_t = a_t + (1-\alpha)n_t \tag{A.230}$$

$$n_t = \frac{1}{1-\alpha}(y_t - a_t) \tag{A.231}$$

$$\varpi = \sigma\eta + (1-\upsilon)(\sigma\eta - 1) \tag{A.232}$$

$$\sigma c_t + \varphi n_t = w_t - p_t - \frac{\tau}{1-\tau}\tau_t \tag{A.233}$$

$$y_t = C_Y\left[(1-\upsilon)c_t + \upsilon(2-\upsilon)\eta s_t + \upsilon y_t^*\right] + G_Y g_t \tag{A.234}$$

$$c_t = y_t^* + \frac{1}{\sigma}z_t + \frac{1-\upsilon}{\sigma}s_t \tag{A.235}$$

$$= \left[C_Y^{-1}y_t - (1-\upsilon)c_t - \upsilon(2-\upsilon)\eta s_t - C_G^{-1}g_t\right]\upsilon^{-1} + \frac{1}{\sigma}z_t + \frac{1-\upsilon}{\sigma}s_t \tag{A.236}$$

$$\implies c_t = C_Y^{-1}y_t + \upsilon s_t\left(-(2-\upsilon)\eta + \frac{1-\upsilon}{\sigma}\right) + \frac{\upsilon}{\sigma}z_t - C_G^{-1}g_t \tag{A.237}$$

$$= C_Y^{-1}y_t - \upsilon\frac{\varpi}{\sigma}s_t + \frac{\upsilon}{\sigma}z_t - C_G^{-1}g_t \tag{A.238}$$

$$\mu_t = p_{H,t} - \psi_t \tag{A.239}$$

Ignoring constant term: $\psi_t = w_t - a_t + \alpha n_t$

$$= p_{H,t} - (w_t - a_t + \alpha n_t) \tag{A.240}$$

add subtract p_t

$$= -(w_t - p_t) - (p_t - p_{H,t}) + a_t - \alpha n_t \tag{A.241}$$

subst. with household intratemporal condition and terms of trade def.

$$= -(\sigma c_t + \varphi n_t + \frac{\tau}{1-\tau}\tau_t) - \upsilon s_t + a_t - \alpha n_t \tag{A.242}$$

$$= -\sigma c_t - \varphi n_t - \frac{\tau}{1-\tau}\tau_t - \upsilon s_t + a_t - \alpha n_t \tag{A.243}$$

$$= -\sigma c_t - \frac{\tau}{1-\tau}\tau_t - \upsilon s_t + a_t - n_t(\varphi + \alpha) \tag{A.244}$$

Expand $-n_t(\varphi+\alpha)$:

$$-n_t(\varphi+\alpha) = -\frac{1}{1-\alpha}(y_t - a_t)(\varphi+\alpha) \tag{A.245}$$

$$= -\frac{\varphi+\alpha}{1-\alpha}(y_t - a_t) \tag{A.246}$$

$$= -\frac{\varphi+\alpha}{1-\alpha}y_t + \frac{\varphi+\alpha}{1-\alpha}a_t \tag{A.247}$$

Substitute and rearrange:

$$\mu_t = -\sigma c_t - \frac{\tau}{1-\tau}\tau_t - \upsilon s_t + a_t - \frac{\varphi+\alpha}{1-\alpha}y_t + \frac{\varphi+\alpha}{1-\alpha}a_t \tag{A.248}$$

$$\mu_t = -\sigma c_t - \frac{\tau}{1-\tau}\tau_t - \upsilon s_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right)a_t - \frac{\varphi+\alpha}{1-\alpha}y_t \tag{A.249}$$

$$\mu_t = -\sigma\left(C_Y^{-1}y_t - \upsilon\frac{\varpi}{\sigma}s_t + \frac{\upsilon}{\sigma}z_t - C_G^{-1}g_t\right) - \frac{\tau}{1-\tau}\tau_t - \upsilon s_t$$
$$+ \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right)a_t - \frac{\varphi+\alpha}{1-\alpha}y_t \tag{A.250}$$

$$\mu_t = -\left(\sigma C_Y^{-1} + \frac{\varphi+\alpha}{1-\alpha}\right)y_t + \upsilon\left(\varpi - 1\right)s_t - \upsilon z_t - \frac{\tau}{1-\tau}\tau_t$$
$$+ \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right)a_t + \sigma C_G^{-1}g_t \tag{A.251}$$

8.2.11 Derivation of the natural level of output

$$\mu = -\left(\sigma C_Y^{-1} + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n + \upsilon(\varpi-1)s_t^n - \frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t + \sigma C_G^{-1} g_t \tag{A.252}$$

$$\mu - \left(\upsilon(\varpi-1)s_t^n - \frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t + \sigma C_G^{-1} g_t\right) = -\left(\sigma C_Y^{-1} + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n \tag{A.253}$$

$$\upsilon(\varpi-1)s_t^n - \frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t + \sigma C_G^{-1} g_t - \mu = \left(\sigma C_Y^{-1} + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n \tag{A.254}$$

Using $s_t^n = \sigma_\upsilon(C_Y^{-1}y_t^n - y_t^* - C_G^{-1}g_t) - (1-\upsilon)\Phi z_t$

$$\begin{aligned}&\upsilon(\varpi-1)(\sigma_\upsilon(C_Y^{-1}y_t^n - y_t^* - C_G^{-1}g_t) - (1-\upsilon)\Phi z_t) - \frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t\\ &- \upsilon z_t + \sigma C_G^{-1} g_t - \mu = \left(\sigma C_Y^{-1} + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n\end{aligned} \tag{A.255}$$

Add and subtract 1

$$\begin{aligned}&(-1+1+\upsilon(\varpi-1))(\sigma_\upsilon(C_Y^{-1}y_t^n - y_t^* - C_G^{-1}g_t) - (1-\upsilon)\Phi z_t) - \frac{\tau}{1-\tau}\tau_t\\ &+ \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t + \sigma C_G^{-1} g_t - \mu = \left(\sigma C_Y^{-1} + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n\end{aligned} \tag{A.256}$$

$$\begin{aligned}&(-1+\Phi^{-1})(\sigma_\upsilon(C_Y^{-1}y_t^n - y_t^* - C_G^{-1}g_t) - (1-\upsilon)\Phi z_t) - \frac{\tau}{1-\tau}\tau_t\\ &+ \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t + \sigma C_G^{-1} g_t - \mu = \left(\sigma C_Y^{-1} + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n\end{aligned} \tag{A.257}$$

$$\begin{aligned}&(-1+\Phi^{-1})(\sigma_\upsilon C_Y^{-1}y_t^n - \sigma_\upsilon y_t^* - (1-\upsilon)\Phi z_t - \sigma_\upsilon C_G^{-1}g_t) - \frac{\tau}{1-\tau}\tau_t\\ &+ \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t + \sigma C_G^{-1} g_t - \mu = \left(\sigma C_Y^{-1} + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n\end{aligned} \tag{A.258}$$

$$\begin{aligned}&(-1+\Phi^{-1})\sigma_\upsilon C_Y^{-1}y_t^n + (1-\Phi^{-1})\sigma_\upsilon y_t^* + (1-\Phi^{-1})(1-\upsilon)\Phi z_t + (1-\Phi^{-1})\sigma_\upsilon C_G^{-1}g_t\\ &- \frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t + \sigma C_G^{-1} g_t - \mu = \left(\sigma C_Y^{-1} + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n\end{aligned} \tag{A.259}$$

Rearrange the two g_t terms to yield: $\sigma_\upsilon C_G^{-1} g_t$

$$\begin{aligned}&(-1+\Phi^{-1})\sigma\Phi C_Y^{-1}y_t^n + (1-\Phi^{-1})\sigma_\upsilon y_t^* + (1-\Phi^{-1})(1-\upsilon)\Phi z_t + \sigma_\upsilon C_G^{-1}g_t\\ &- \frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t - \mu = \left(\sigma C_Y^{-1} + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n\end{aligned} \tag{A.260}$$

$$\begin{aligned}&-\sigma\Phi C_Y^{-1}y_t^n + \sigma C_Y^{-1}y_t^n + (1-\Phi^{-1})\sigma_\upsilon y_t^* + (1-\Phi^{-1})(1-\upsilon)\Phi z_t + \sigma_\upsilon C_G^{-1}g_t\\ &- \frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t - \mu = \left(\sigma C_Y^{-1} + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n\end{aligned} \tag{A.261}$$

$$(1-\Phi)\sigma C_Y^{-1}y_t^n+(1-\Phi^{-1})\sigma_\upsilon y_t^*+(1-\Phi^{-1})(1-\upsilon)\Phi z_t+\sigma_\upsilon C_G^{-1}g_t$$
$$-\frac{\tau}{1-\tau}\tau_t+\left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t-\upsilon z_t-\mu=\left(\sigma C_Y^{-1}+\frac{\varphi+\alpha}{1-\alpha}\right)y_t^n \tag{A.262}$$

$$(1-\Phi^{-1})\sigma_\upsilon y_t^*+(1-\Phi^{-1})(1-\upsilon)\Phi z_t+\sigma_\upsilon C_G^{-1}g_t$$
$$-\frac{\tau}{1-\tau}\tau_t+\left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t-\upsilon z_t-\mu=\left(\sigma C_Y^{-1}+\frac{\varphi+\alpha}{1-\alpha}\right)y_t^n-(1-\Phi)\sigma C_Y^{-1}y_t^n \tag{A.263}$$

$$(1-\Phi^{-1})\sigma_\upsilon y_t^*+(1-\Phi^{-1})(1-\upsilon)\Phi z_t+\sigma_\upsilon C_G^{-1}g_t$$
$$-\frac{\tau}{1-\tau}\tau_t+\left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t-\upsilon z_t-\mu=\left(\frac{\sigma C_Y^{-1}(1-\alpha)+\varphi+\alpha-(1-\Phi)\sigma C_Y^{-1}(1-\alpha)}{1-\alpha}\right)y_t^n \tag{A.264}$$

$$(1-\Phi^{-1})\sigma_\upsilon y_t^*+(1-\Phi^{-1})(1-\upsilon)\Phi z_t+\sigma_\upsilon C_G^{-1}g_t$$
$$-\frac{\tau}{1-\tau}\tau_t+\left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t-\upsilon z_t-\mu=\left(\frac{\sigma C_Y^{-1}(1-\alpha)(1-1+\Phi)+\varphi}{1-\alpha}\right)y_t^n \tag{A.265}$$

$$(1-\Phi^{-1})\sigma_\upsilon y_t^*+(1-\Phi^{-1})(1-\upsilon)\Phi z_t+\sigma_\upsilon C_G^{-1}g_t$$
$$-\frac{\tau}{1-\tau}\tau_t+\left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t-\upsilon z_t-\mu=\left(\frac{\sigma_\upsilon C_Y^{-1}(1-\alpha)+\varphi+\alpha}{1-\alpha}\right)y_t^n \tag{A.266}$$

$$\left(\frac{\sigma_\upsilon C_Y^{-1}(1-\alpha)+\varphi+\alpha}{1-\alpha}\right)^{-1}\times$$
$$\left((1-\Phi^{-1})\sigma_\upsilon y_t^*+(1-\Phi^{-1})(1-\upsilon)\Phi z_t+\sigma_\upsilon C_G^{-1}g_t-\frac{\tau}{1-\tau}\tau_t+\left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t-\upsilon z_t-\mu\right)=y_t^n \tag{A.267}$$

Note that $(1-\Phi^{-1})=-\upsilon(\varpi-1)$

$$\Gamma_* y_t^*+\left(\frac{\sigma_\upsilon C_Y^{-1}(1-\alpha)+\varphi+\alpha}{1-\alpha}\right)^{-1}\times$$
$$\left((1-\Phi^{-1})(1-\upsilon)\Phi z_t+\sigma_\upsilon C_G^{-1}g_t-\frac{\tau}{1-\tau}\tau_t+\left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t-\upsilon z_t-\mu\right)=y_t^n \tag{A.268}$$

$$\Gamma_* y_t^*+\left(\frac{\sigma_\upsilon C_Y^{-1}(1-\alpha)+\varphi+\alpha}{1-\alpha}\right)^{-1}\times$$
$$\left(\upsilon\varpi\Phi z_t+\sigma_\upsilon C_G^{-1}g_t-\frac{\tau}{1-\tau}\tau_t+\left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t-\mu\right)=y_t^n \tag{A.269}$$

$$\Gamma_* y_t^* + \Gamma_z z_t + \Gamma_a a_t + \left(\frac{\sigma_\upsilon C_Y^{-1}(1-\alpha) + \varphi + \alpha}{1-\alpha}\right)^{-1} \left(\sigma_\upsilon C_G^{-1} g_t - \frac{\tau}{1-\tau}\tau_t - \mu\right) = y_t^n \tag{A.270}$$

$$\Gamma_* y_t^* + \Gamma_z z_t + \Gamma_a a_t + \Gamma_g g_t + \left(\frac{\sigma_\upsilon C_Y^{-1}(1-\alpha) + \varphi + \alpha}{1-\alpha}\right)^{-1} \left(-\frac{\tau}{1-\tau}\tau_t\right) = y_t^n \tag{A.271}$$

$$\Gamma_* y_t^* + \Gamma_z z_t + \Gamma_a a_t + \Gamma_g g_t + \Gamma_\tau \tau_t = y_t^n \tag{A.272}$$

$$y_t^n = \Gamma_* y_t^* + \Gamma_z z_t + \Gamma_a a_t + \Gamma_\tau \tau_t + \Gamma_g g_t \tag{A.273}$$

where:

$$\Gamma_* = -\frac{\upsilon(\varpi - 1)\sigma_\upsilon(1-\alpha)}{\sigma_\upsilon C_Y^{-1}(1-\alpha) + \varphi + \alpha} \tag{A.274}$$

$$\Gamma_z = -\frac{\upsilon\varpi\Phi(1-\alpha)}{\sigma_\upsilon C_Y^{-1}(1-\alpha) + \varphi + \alpha} \tag{A.275}$$

$$\Gamma_a = \frac{1+\varpi}{\sigma_\upsilon C_Y^{-1}(1-\alpha) + \varphi + \alpha} \tag{A.276}$$

$$\Gamma_g = \frac{\sigma_\upsilon C_G^{-1}(1-\alpha)}{\sigma_\upsilon C_Y^{-1}(1-\alpha) + \varphi + \alpha} \tag{A.277}$$

$$\Gamma_\tau = -\frac{\tau}{1-\tau}\frac{1-\alpha}{\sigma_\upsilon C_Y^{-1}(1-\alpha) + \varphi + \alpha} \tag{A.278}$$

8.2.12 Derivation of the natural real rate of interest

$$r_t^n = C_Y^{-1}\sigma_\upsilon \,\mathbb{E}_t\{\Delta y_{t+1}^n\} + \rho + \sigma_\upsilon \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} + \Phi(1-\upsilon)(1-\rho_z)z_t - \sigma_\upsilon C_G^{-1}\,\mathbb{E}_t\{\Delta g_{t+1}\} \tag{A.279}$$

Expand $C_Y^{-1}\sigma_\upsilon \,\mathbb{E}_t\{\Delta y_{t+1}^n\}$:

$$C_Y^{-1}\sigma_\upsilon \,\mathbb{E}_t\{\Delta y_{t+1}^n\} = C_Y^{-1}\sigma_\upsilon \,\mathbb{E}_t\{\Gamma_*\Delta y_{t+1}^* + \Gamma_z\Delta z_{t+1} + \Gamma_a\Delta a_{t+1} + \Gamma_g\Delta g_{t+1} + \Gamma_\tau\Delta\tau_{t+1}\} \tag{A.280}$$

Consider each term in turn:

$$y_t^* : C_Y^{-1}\sigma_\upsilon\Gamma_*\Delta y_{t+1}^* + \sigma_\upsilon\upsilon(\varpi-1)\Delta y_{t+1}^* = \sigma_\upsilon(C_Y^{-1}\Gamma_* + \upsilon(\varpi-1))\Delta y_{t+1}^* = \Psi_*\,\mathbb{E}_t\{\Delta y_{t+1}^*\} \tag{A.281}$$

$$z_t : C_Y^{-1}\sigma_\upsilon\Gamma_z\Delta z_{t+1} + \Phi(1-\upsilon)(1-\rho_z)z_t = C_Y^{-1}\sigma_\upsilon\Gamma_z(\rho_z-1)z_t + \Phi(1-\upsilon)(1-\rho_z)z_t \tag{A.282}$$

$$= C_Y^{-1}\sigma_\upsilon\Gamma_z(\rho_z-1)z_t + \Phi(1-\upsilon)(1-\rho_z)z_t = ((1-\upsilon)\Phi - C_Y^{-1}\sigma_\upsilon\Gamma_z)(1-\rho_z)z_t = \Psi_z(1-\rho_z)z_t \tag{A.283}$$

$$a_t : C_Y^{-1}\sigma_\upsilon\Gamma_a\Delta a_{t+1} = C_Y^{-1}\sigma_\upsilon\Gamma_a(\rho_a-1)a_t = -C_Y^{-1}\sigma_\upsilon\Gamma_a(1-\rho_a)a_t \tag{A.284}$$

$$g_t : C_Y^{-1}\sigma_\upsilon\Gamma_g\Delta g_{t+1} - \sigma_\upsilon C_G^{-1}\,\mathbb{E}_t\{\Delta g_{t+1}\} = \left(\sigma_\upsilon\left(C_Y^{-1}\Gamma_g - C_G^{-1}\right)\right)\mathbb{E}_t\{\Delta g_{t+1}\} = -\Psi_g(1-\rho_g)g_t \tag{A.285}$$

$$\tau_t : \sigma_\upsilon C_Y^{-1}\Gamma_\tau\Delta\tau_{t+1} = \Psi_\tau\,\mathbb{E}_t\{\Delta\tau_{t+1}\} \tag{A.286}$$

Finally:

$$r_t^n = \rho - C_Y^{-1}\sigma_\upsilon\Gamma_a(1-\rho_a)a_t + \Psi_*\,\mathbb{E}_t\{\Delta y_{t+1}^*\} + \Psi_z(1-\rho_z)z_t - \Psi_g(1-\rho_g)g_t + \Psi_\tau\,\mathbb{E}_t\{\Delta\tau_{t+1}\} \tag{A.287}$$

where:

$$\Psi_* = \sigma_\upsilon(C_Y^{-1}\Gamma_* + \upsilon(\varpi - 1)) \tag{A.288}$$

$$\Psi_z = (1 - \upsilon)\Phi - C_Y^{-1}\sigma_\upsilon\Gamma_z \tag{A.289}$$

$$\Psi_g = \sigma_\upsilon\left(C_Y^{-1}\Gamma_g - C_G^{-1}\right) \tag{A.290}$$

$$\Psi_\tau = \sigma_\upsilon C_Y^{-1}\Gamma_\tau \tag{A.291}$$

8.2.13 Derivation of the trade balance

$$y_t = C_Y\left[(1-\upsilon)c_t + \upsilon(2-\upsilon)\eta s_t + \upsilon y_t^*\right] + G_Y g_t \tag{A.292}$$

$$c_t = y_t^* + \frac{1}{\sigma}z_t + \frac{1-\upsilon}{\sigma}s_t \tag{A.293}$$

$$= \left[C_Y^{-1}y_t - (1-\upsilon)c_t - \upsilon(2-\upsilon)\eta s_t - C_G^{-1}g_t\right]\upsilon^{-1} + \frac{1}{\sigma}z_t + \frac{1-\upsilon}{\sigma}s_t \tag{A.294}$$

$$\implies c_t = C_Y^{-1}y_t + \upsilon s_t\left(-(2-\upsilon)\eta + \frac{1-\upsilon}{\sigma}\right) + \frac{\upsilon}{\sigma}z_t - C_G^{-1}g_t \tag{A.295}$$

$$= C_Y^{-1}y_t - \upsilon\frac{\varpi}{\sigma}s_t + \frac{\upsilon}{\sigma}z_t - C_G^{-1}g_t \tag{A.296}$$

Substitute (A.296) to $nx_t = y_t - C_Y[c_t + \upsilon s_t] - G_Y g_t$:

$$nx_t = y_t - C_Y\left[C_Y^{-1}y_t - \upsilon\frac{\varpi}{\sigma}s_t + \frac{\upsilon}{\sigma}z_t - C_G^{-1}g_t + \upsilon s_t\right] - G_Y g_t \tag{A.297}$$

$$nx_t = y_t - C_Y\left[C_Y^{-1}y_t + \upsilon\left(-\frac{\varpi}{\sigma} + 1\right)s_t + \frac{\upsilon}{\sigma}z_t - C_G^{-1}g_t\right] - G_Y g_t \tag{A.298}$$

y_t terms cancel out:

$$nx_t = -C_Y\left[\upsilon\left(-\frac{\varpi}{\sigma} + 1\right)s_t + \frac{\upsilon}{\sigma}z_t - C_G^{-1}g_t\right] - G_Y g_t \tag{A.299}$$

g_t terms cancel out because $C_Y \times C_G^{-1} = G_Y$

$$nx_t = C_Y\left[\upsilon\left(\frac{\varpi}{\sigma} - 1\right)s_t - \frac{\upsilon}{\sigma}z_t\right] \tag{A.300}$$

Because trade is balanced in the steaty state ($nx_t = 0$), we can multiply by C_Y

$$nx_t = \upsilon\left(\frac{\varpi}{\sigma} - 1\right)s_t - \frac{\upsilon}{\sigma}z_t \tag{A.301}$$

8.2.14 Derivation of domestic inflation as a function of updated and non-updated price levels

$$\Pi_{H,t}^{1-\epsilon} = \theta + (1-\theta)\left(\frac{\overline{P}_{H,t}}{P_{H,t-1}}\right)^{1-\epsilon} \tag{A.302}$$

$$\Pi_H \mathbf{e}^{(1-\epsilon)\pi_{H,t}} = \theta + (1-\theta)\frac{\overline{P_H}}{P_H}\mathbf{e}^{(1-\epsilon)(\bar{p}_{H,t}-p_{H,t})} \tag{A.303}$$

$\Pi_H = 1$ because the ratio of steady-state

price levels in two consecutive periods is 1

$$(1-\epsilon)\pi_{H,t} = (1-\theta)(1-\epsilon)(\bar{p}_{H,t} - p_{H,t}) \tag{A.304}$$

$$\pi_{H,t} = (1-\theta)(\bar{p}_{H,t} - p_{H,t}) \tag{A.305}$$