Assessing Asymmetrical Effects of Government Spending

A Two-Country DSGE Model for Scotland and the Rest of the UK

B204335

A dissertation presented for the degree of
MSc Economics (Econometrics)

School of Economics
University of Edinburgh
United Kingdom
The 15th of August, 2023

Structure. Max words: 10 000. Currently: 5441

1. Introduction: **1000 words**
2. Literature Review **2000 words**
   (a) From RBC to NK DSGE: **1200 words**
   (b) Why Scotland and the rest of the UK? (NIESR policy-related question): **800 words**
3. Theoretical DSGE model: **2000 words**
   (a) Households
   (b) Firms
   (c) Equilibrium
4. Application: **1500 words**
   (a) Data: **500 words**
   (b) Estimation (MCMC): **1000 words**
5. Results: **2500 words**
   (a) Analysis of IRFs: **1000 words**
   (b) Other insights (NIESR policy-related question): **1500 words**
6. Conclusion: **1000 words**

```
Sum count: 5441
Words in text: 4843
Words in headers: 80
Words outside text (captions, etc.): 251
Number of headers: 22
Number of floats/tables/figures: 8
Number of math inlines: 175
Number of math displayed: 92
Files: 25
Subcounts:
  text+headers+captions (#headers/#floats/#inlines/#displayed)
  1+7+0 (5/0/0/0) File: main.tex
  65+6+0 (1/0/0/0) Included file: ././structure.tex
  938+2+55 (1/0/0/0) Included file: ././Introduction/literature_review.tex
  324+0+0 (0/0/3/0) Included file: ././Theoretical DSGE model/model.tex
  1696+1+33 (1/1/82/27) Included file: ././Theoretical DSGE model/households.tex
  479+1+102 (1/4/25/1) Included file: ././Theoretical DSGE model/government.tex
  832+1+43 (1/0/44/17) Included file: ././Theoretical DSGE model/firms.tex
  390+1+7 (1/1/11/10) Included file: ././Theoretical DSGE model/equilibrium.tex
  0+1+0 (1/0/0/0) Included file: ././Appendix/appendix.tex
  6+0+0 (0/0/2/6) Included file: ././model.tex
  1+0+0 (0/0/0/0) Included file: ././final_wordcount.txt
  0+0+11 (0/1/0/0) Included file: ././Graphs/timeline.tex
  0+0+0 (0/1/0/0) Included file: ././Graphs/sigma_varphi.tex
  0+5+0 (1/0/0/1) Included file: ././Appendix/intratemporal_loglinear.tex
  0+6+0 (1/0/0/1) Included file: ././Appendix/log_aggregate_rc.tex
  66+8+0 (1/0/3/4) Included file: ././Appendix/euler_w_arrow.tex
  20+5+0 (1/0/0/8) Included file: ././Appendix/deriving_bilateral_exchange_rate.te
  0+5+0 (1/0/0/2) Included file: ././Appendix/loglinearised_tax_revenue.tex
  3+6+0 (1/0/3/2) Included file: ././Appendix/terms_of_trade_derivation.tex
  0+8+0 (1/0/0/2) Included file: ././Appendix/dynamic_is.tex
  4+5+0 (1/0/0/2) Included file: ././Appendix/final_dynamic_is.tex
  17+5+0 (1/0/2/4) Included file: ././Appendix/average_markup.tex
  1+7+0 (1/0/0/5) Included file: ././Appendix/natural_output.tex
```

1 Introduction

1.1 Literature Review

On the 10th of May, 2023, the Monetary Policy Committee at the Bank of England gathered to discuss the latest international and domestic data on economic activity. Even though the Committee has a 2% CPI target, the UK's economy had undergone a sequence of very large and unexpected shocks and disturbances, resulting in twelve-month CPI inflation above 10%. The majority of the Committee members (78%) believed that an increase in interest rate "was warranted" (BoE, 2023: 4), while the remaining members believed that the CPI inflation will "fall sharply in 2023" (BoE, 2023: 5) as a result of the economy naturally adjusting to the effects of the energy price shocks. They feared that the preceding increases in the interest rate have not yet been internalised and raising the interest rate any further could result in a reduction of inflation "well below the target" (BoE, 2023: 5). This is an excellent illustration of the "informal dimension of the monetary policy process", that (Galsí and Gertler, 2007: 26) referred to in their work explaining modern macro models and new frameworks. According to them, while the informal dimension cannot be removed, we can build formal and rigorous models that would help the Committee and institutions-alike to understand "objectives of the monetary policy and how the latter should be conducted in order to attain those objectives" (Galí, 2015: 2). This task is not straightforward and has been central (albeit - fruitful) to most macroeconomic research in the past decades. The following section of the literature review will present a brief evolution of the study of business cycles and monetary theory. It will be followed by an overview of large macroeconomic models adopted by central banks and international organisations to illustrate the relevance of this research. The final part of the literature review will discuss {NIESR policy question, also Scotland-UK deltas} and the latest research on the issue.

Blanchard (2000) offers a compact description of macroeconomic research in the twentieth century. In their panegyric and optimistic essay, the researcher argues that the century can be divided into three epochs based on the prevailing beliefs about the economy and frameworks of the time: Pre-1940, From 1940 to 1980, and Post-1980.

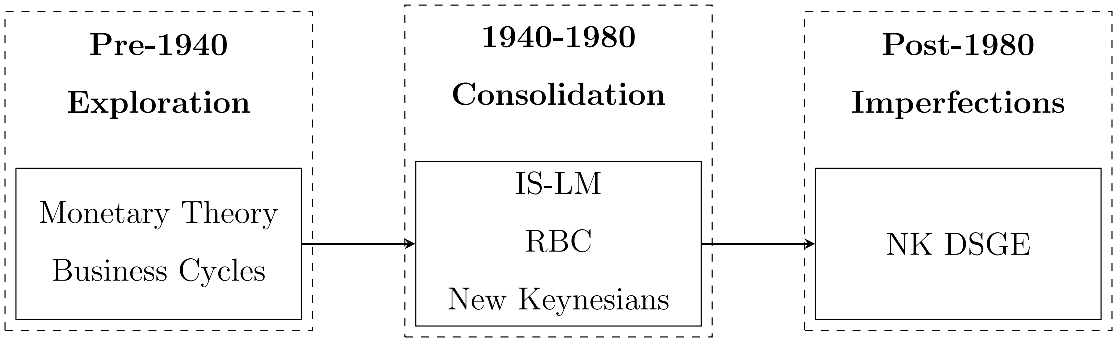


Figure 1: Timeline of macroeconomic research, according to Blanchard (2000). Authors own illustration.

According to them, the pre-1940 epoch was the epoch of exploration, with economists primarily concerned with the Monetary Theory ("Why does money affect output?") and Business Cycles ("What are the major shocks that affect output?"). Even though both of those puzzles fall under the study of "macroeconomics", the term did not appear in the literature until the mid-1940s (Blanchard, 2000). The Monetary Theory ideas of that time did not differ drastically from what is believed today, e.g. short-run money non-neutrality and long-run neutrality, but the economic models were "incomplete and partial equilibrium in nature" (Blanchard, 2000: 1377). The business cycles were attributed to "real factors", such as technological innovations (Blanchard, 2000), and this belief persisted until more data became available and more sophisticated time series methods were applied in the early 2000s (Galí, 2015: 3). The subsequent epoch was "the golden age of macroeconomics" (Blanchard, 2000: 1379). Hicks (1937) formalised the *IS-LM* framework TBC

In the 1980s, Kydland and Prescott (1982) and Prescott (1986) published seminal papers on the Real Business Cycles (RBC) theory. According to Galí (2015: 2), frameworks presented in the papers "provided the main reference" and firmly established the use of dynamic stochastic general equilibrium (DSGE) models as crucial tools for macroeconomic analysis. The models allow quantitative analysis and incorporation of data either via calibration or estimation of parameters. TBC

International Monetary Fund (IMF)

Government in DSGE literature review

To begin with, RBC models predict a negative response in consumption following an increase in government spending. More specifically, government spending is modelled to absorb resources, which makes households worse off and incentivises more hours worked. Greater labour supply for any given wage reduces firms' marginal cost and induces output (Baxter and King, 1993: 319). That is, consumption, conditional on shocks in government spending, is countercyclical. Keynesian models, in stark contrast, predict the opposite. The countercyclicality is why the DSGE models sometimes do not consider government spending.

Empirically, the findings of the Keynesian models are more in line with the observed macroeconomic patterns. For instance, Blanchard and Perotti (2002) performed a VAR analysis on the dynamics of consumption and government spending. They built six structural VAR models, one for each component of GDP: output, consumption, government spending, investment, export, and import. The two other variables were taxes and government spending/output[1]. The key finding of the analysis is that government spending has a positive effect on consumption.

Gali, Lopez-Salido, and Valles (2005) show that NK DSGE models can be "recovered" by assuming that households have limited access to financial markets/saving technologies or are poor (they consume all of their labour income). Households that smooth their consumption by saving are often regarded as Ricardian households, while those that do not - non-Ricardian or *hand-to-mouth* households. Just like Gali, Lopez-Salido, and Valles (2005) did, some of the latest literature NK DSGE literature models both types of households explicitly[2] with their ratio determined by a time-invarying exogenous coefficient. Arguably, such modelling would allow an improved fit of data.

[1] That is, if the GDP component of interest is government spending, then the second variable is output; in all other cases, the second variable is government spending.

[2] In fact, there exists literature with more than two types of households. For instance, a recent paper by Eskelinen (2021) models poor hand-to-mouth, wealthy hand-to-mouth, and non-hand-to-mouth households.

However, the key focus of this dissertation is the four policy scenarios, all of which heavily depend on governments' abilities to issue bonds and borrow. Hand-to-mouth households do not borrow/save, rendering bonds purposeless. While modelling hand-to-mouth households is even easier than the Ricardian households, modelling both types of households would drastically increase the complexity of the model, given the two-country setting. The absence of hand-to-mouth households is discussed in the limitations section.

2 Theoretical DSGE model

Ricci (2019) was the first to build a large-scale two-country DSGE model explicitly tailored to Scotland and the rest of the UK. In an attempt to retain the model's simplicity while still allowing policy analysis, this dissertation will primarily build on the work of Gali and Monacelli (2005) and Galí (2015). In contrast, Ricci (2019) model was based on the work of Rabanal and Tuesta (2010), who were among the first to build a medium-to-large two-country DSGE model. Neither Gali and Monacelli (2005) nor Galí (2015) models considered lump-sum or distortionary taxes, or government spending, more generally. While extensive literature covers government spending in DSGE models, few to none cover government spending in a small open economy (SOE) NK DSGE model, and even fewer apply it to a two-country setting. Therefore, most of the derivations had to be carried out using a pen and paper, and step-by-step derivations are provided in the Appendix.

Moreover, the focus of this dissertation is not to build the most factually accurate model of Scotland or the United Kingdom but to assess the asymmetric responses in government spending under factual and counterfactual policy scenarios. The factual scenario refers to the Westminister government collecting taxes from all four countries of the UK and distributing them according to the Barnett formula. The counterfactual scenario refers to the Holyrood government's ability to collect tax revenue, issue bonds (borrow), and spend it at its sole discretion. We further break down the scenarios by allowing public expenses to be funded by lump-sum and distortionary (labour) taxes. This brings the number of policy scenarios considered by the dissertation to four.

Finally, in line with most of the literature, variables referring to the home country (Scotland) will be denoted without an asterisk, i.e., Y_t, while foreign country (the Rest of the UK or **rUK**) will be denoted with an asterisk, i.e., Y_t^*. Population-weighted sums of these variables will be referred to as UK-wide variables and denoted as Y_t^{UK}.

2.1 Households

This model assumes that there is infinitely many households in the economy represented by a unit interval. All households are assumed to be be symmetric, i.e. have the same preferences and behave identically. Below, we consider a representative household that wants to maximise their lifetime utility, represented by Equation (1):

$$\mathbb{E}_t \left\{ \sum_{t=0}^{\infty} \beta^t \, \mathcal{U}(C_t, N_t, Z_t) \right\} \tag{1}$$

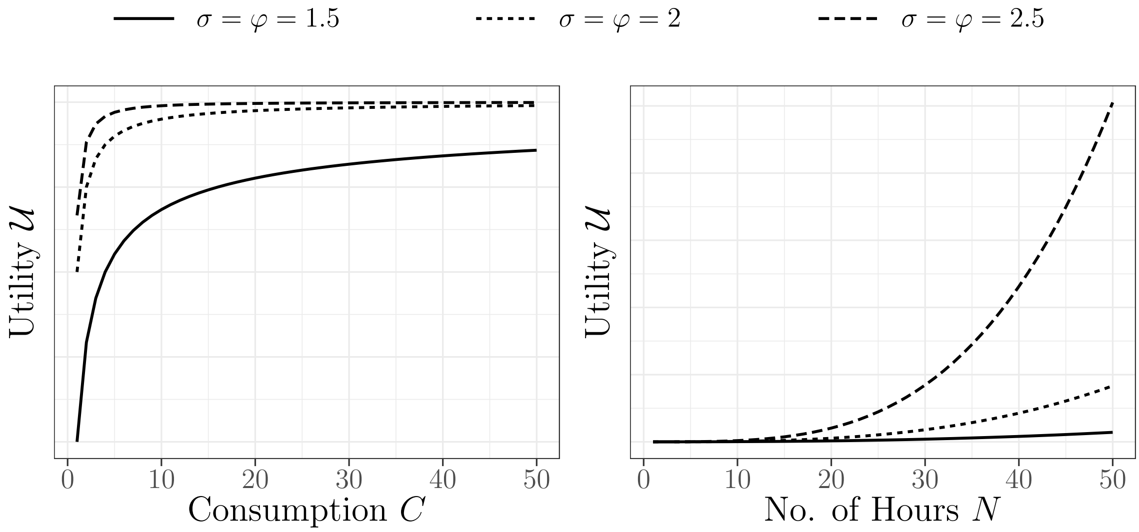
$$\mathcal{U}(C_t, N_t, Z_t) = \begin{cases} \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right) Z_t & \text{if } \sigma \geq 0 \text{ and } \sigma \neq 1 \\ \left(\log(C_t) - \frac{N_t^{1+\varphi}}{1+\varphi} \right) Z_t & \text{if } \sigma = 1 \end{cases} \tag{2}$$

The household's utility depends on consumption C_t and hours worked N_t. As seen from the utility function (Equation (2)), the model assumes the household's utility to be (decreasingly) increasing in consumption C_t and (increasingly) decreasing in hours worked N_t. $\beta \in (0, 1)$ is the discount factor, which can be thought of as an opportunity cost or an impatience rate, i.e. a unit of consumption C today will be worth $\beta * C < C$ tomorrow. We also introduce a preference shifter Z_t (Galí, 2015: 225) [3]. The shock is assumed to follow an autoregressive process of order 1:

$$\log(Z_t) = \rho_z \log(Z_{t-1}) + \epsilon_t^z \tag{3}$$

[3] While this specific shock is not relevant to the research question, it helps prevent stochastic singularity Pfeifer, 2021 and allows parameter estimation with a greater number of macroeconomic data series, see Section 3.

The parameter $\sigma \geq 0$ is the relative risk aversion coefficient and $\varphi \geq 0$ is the labour disutility parameter. Together, they determine the curvature of the utility of consumption and disutility of labour, respectively. Finally, $\mathbb{E}_t[*]$ is the expectational operator, conditional on all information available at period t (Gali, 2015: 20).



To allow goods differentiation between domestic and foreign, the model assumes that C_t is a composite consumption index defined by:

$$C_t = \begin{cases} \left[(1-\upsilon)^{\frac{1}{\eta}}(C_{H,t})^{\frac{\eta-1}{\eta}} + \upsilon^{\frac{1}{\eta}}(C_{F,t})^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} & \text{if } \eta > 0 \text{ and } \eta \neq 1 \\ \frac{1}{(1-\upsilon)^{(1-\upsilon)}\upsilon^{\upsilon}}(C_{H,t})^{(1-\upsilon)}(C_{F,t})^{\upsilon} & \text{if } \eta = 1 \end{cases} \tag{4}$$

Where $C_{H,t}$ and $C_{F,t}$ are indices of consumption of home produced and imported goods, respectively. The parameter $\upsilon \in [0, 1]$ reflects economy's openness for trading, while $\eta > 0$ denotes household's willingness to substitute a domestic good with a foreign good, often referred to as 'home bias'. When $\eta = 1$, then the share of domestic and foreign consumption is determined by the country's willingness to trade. In an extreme case, $\upsilon = 0$ would imply that the economy is an autarky, while $\upsilon = 1$ would suggest that our households consume foreign goods only. Our economy is assumed to be small, in the sense that it takes the world output, consumption, and prices as given, and cannot influence them. This is a common assumption for the UK (**refs**) and even more so for Scotland. The world economy is assumed to be made of a continuum of infinitely many small economies i represented by a unit interval. Therefore, $C_{F,t}$ is a sum of indices of the quantity of goods imported from all countries i. In a similar fashion, if we denote j as a single variety of goods from a continuum of goods represented by a unit interval, we can express each consumption index as follows:

$$C_{H,t} = \left(\int_0^1 C_{H,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} \, dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \qquad \text{Index of consumption of home produced goods}$$

$$C_{i,t} = \left(\int_0^1 C_{i,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} \, dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \qquad \text{Index of consumption of country } i\text{'s produced goods}$$

$$C_{F,t} = \left(\int_0^1 C_{i,t}^{\frac{\gamma-1}{\gamma}} \, di \right)^{\frac{\gamma}{\gamma-1}} \qquad \text{Index of consumption of imported goods}$$

Notice that all three indices take the form of *Constant Elasticity Substitution* (*CES*) form, with parameters ε (without subscripts) and γ representing the degree of substitutability between varieties of goods and countries, respectively. The following expresions note optimal allocation of each individual good (see Appendix A1 for derivation):

$$C_{H,t}(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}} \right)^{-\varepsilon} C_{H,t}; \quad C_{i,t}(j) = \left(\frac{P_{i,t}(j)}{P_{i,t}} \right)^{-\varepsilon} C_{i,t}; \quad C_{i,t} = \left(\frac{P_{i,t}}{P_{F,t}} \right)^{-\gamma} C_{F,t} \quad (5)$$

where:

$$P_{H,t} = \left(\int_0^1 P_{H,t}(j)^{1-\varepsilon} \, dj \right)^{\frac{1}{1-\varepsilon}} \qquad \text{Domestic Price Index} \quad (6)$$

$$P_{i,t} = \left(\int_0^1 P_{i,t}(j)^{1-\varepsilon} \, dj \right)^{\frac{1}{1-\varepsilon}} \qquad \text{Price Index of goods produced by country } i \quad (7)$$

$$P_{F,t} = \left(\int_0^1 P_{i,t}^{1-\gamma} \, dj \right)^{\frac{1}{1-\gamma}} \qquad \text{Price Index of Imported goods} \quad (8)$$

Intuitively, if $P_{H,t}(j) > P_{H,t}$, then that good is demanded less relative to an *average* good. To see this, note that $P_{H,t}(j)/P_{H,t} > 1$ when $P_{H,t}(j) > P_{H,t}$, and given that the term is to the power of a negative constant, the entire term decreases.

The representative household's choice of consumption and labour must satisfy the following budget constraint:

$$\underbrace{\int_0^1 P_{H,t}(j)C_{H,t}(j)\,dj + \int_0^1 \int_0^1 P_{i,t}(j)C_{i,t}(j)\,dj\,di + \mathbb{E}_t\left[R_{t+1}^{-1}B_{t+1}\right]}_{Expenses} \leq \underbrace{B_t + W_t N_t}_{Income} \quad (9)$$

where R_t is the gross nominal interest rate, B_t denotes bonds, W_t and N_t stand for nominal wage and hours worked, respectively. For intuition, the LHS of the budget constraint implies that the representative household needs to choose the quantity of good j produced domestically and in every country i, as well as the number of bonds at the expected nominal interest rate in period $t+1$. The RHS implies that the only two sources of income are nominal payoffs from bonds and gross pay, which later will be different from the same as net pay. The expenses cannot exceed income.

Taking the three price indices (6)-(8), and plugging them into their respective demand functions (5), yields:

$$\int_0^1 P_{H,t}(j)C_{H,t}(j)\,dj = P_{H,t}C_{H,t} \quad (10)$$

$$\int_0^1 P_{i,t}(j)C_{i,t}(j)\,dj = P_{i,t}C_{i,t} \quad (11)$$

$$\int_0^1 P_{i,t}C_{i,t} = P_{F,t}C_{F,t} \quad (12)$$

The following definitions are given:

$$C_{H,t} = (1-\upsilon)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t \quad (13)$$

$$C_{F,t} = \upsilon\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t \quad (14)$$

$$P_t = \begin{cases} [(1-\upsilon)(P_{H,t})^{1-\eta} + \upsilon(P_{F,t})^{1-\eta}]^{\frac{1}{1-\eta}} & \text{if } \eta > 0 \text{ and } \eta \neq 1 \\ (P_{H,t})^{1-\upsilon} \times (P_{F,t})^{\upsilon} & \text{if } \eta = 1 \end{cases} \quad (15)$$

Equations (13) and (14) are demand functions for domestic and foreign goods, respectively. Equation (15) is the Consumption Price Index (CPI). In the case when there is no home bias ($\eta = 1$), the log aggregate price level in the consumption is just a weighted sum of the two price indices, where weights are given by trade openness parameter υ. Using (10) and (12), we can define the total consumption expenditures as:

$$P_{H,t}C_{H,t} + P_{F,t}C_{F,t} = P_tC_t \tag{16}$$

Which greatly simplifies the household's budget constraint:

$$P_tC_t + \mathbb{E}_t\left[R_{t+1}^{-1}B_{t+1}\right] \leq B_t + W_tN_t \tag{17}$$

Note that the budget constraint (as well as many other expressions introduced later) will vary depending on what policy scenario is considered. For instance, the household's budget constraints under each scenario is given below:

Scen. 1	Scot.	$P_tC_t + \mathbb{E}_t[R_{t+1}^{-1}B_{t+1}] = B_t + W_tN_t + T_t$
G: 2, $\tau : 0$	rUK	$P_t^*C_t^* + \mathbb{E}_t[R_{t+1}^{*-1}B_{t+1}^*] = B_t^* + W_t^*N_t^* + T_t^*$
Scen. 2	Scot.	$P_tC_t + \mathbb{E}_t[R_{t+1}^{UK-1}B_{t+1}^{UK}] = B_t^{UK} + W_tN_t + \varpi T_t^{UK}$
G: 1, $\tau : 0$	rUK	$P_t^*C_t^* + \mathbb{E}_t[R_{t+1}^{UK-1}B_{t+1}^{UK}] = B_t^{UK} + W_t^*N_t^* + (1-\varpi)T_t^{UK}$
Scen. 3	Scot.	$P_tC_t + \mathbb{E}_t[R_{t+1}^{-1}B_{t+1}] = B_t + (1-\tau_t)W_tN_t + T_t$
G: 2, $\tau : 1$	rUK	$P_t^*C_t^* + \mathbb{E}_t[R_{t+1}^{*-1}B_{t+1}^*] = B_t^* + (1-\tau_t^*)W_t^*N_t^* + T_t^*$
Scen. 4	Scot.	$P_tC_t + \mathbb{E}_t[R_{t+1}^{UK-1}B_{t+1}^{UK}] = B_t^{UK} + (1-\tau_t^{UK})W_tN_t + \varpi T_t^{UK}$
G: 1, $\tau : 1$	rUK	$P_t^*C_t^* + \mathbb{E}_t[R_{t+1}^{UK-1}B_{t+1}^{UK}] = B_t^{UK} + (1-\tau_t^{UK})W_t^*N_t^* + (1-\varpi)T_t^{UK}$

Here, ϖ denotes the Scotland's share of population in the United Kingdom. T_t denotes lump-sum transfers (subsidies or taxes), while τ_t denotes an income or labour tax rate. In the first column, G indicates the number of governments that can issue bonds (borrow) and set the labour tax rate. In the same column, τ indicates whether the government spending is funded by a labour tax.

What follows is the derivation of the intratemporal and intertemporal optimality conditions for the policy scenario 3, i.e. when households face a labour tax:

$$
\begin{aligned}
\mathcal{L} = {} & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right) Z_t \\
& + \lambda_t \left\{ B_t + (1-\tau_t) W_t N_t + T_t - P_t C_t - \mathbb{E}_t \left[R_{t+1}^{-1} B_{t+1} \right] \right\} && (18) \\
\frac{\partial \mathcal{L}}{\partial C_t} & = \beta^t C_t^{-\sigma} Z_t - \lambda_t P_t = 0; \quad \Rightarrow \quad \beta^t C_t^{-\sigma} Z_t P_t^{-1} = \lambda_t && (19) \\
\frac{\partial \mathcal{L}}{\partial N_t} & = -\beta^t N_t^{\varphi} Z_t + \lambda_t (1-\tau_t) W_t = 0; \quad \Rightarrow \quad \beta^t N_t^{\varphi} Z_t ((1-\tau_t) W_t)^{-1} = \lambda_t && (20) \\
\frac{\partial \mathcal{L}}{\partial B_{t+1}} & = -\lambda_t \, \mathbb{E}_t \left[R_{t+1}^{-1} \right] + \mathbb{E}_t [\lambda_{t+1}] = 0; \quad \Rightarrow \quad \mathbb{E}_t [R_{t+1}^{-1}] = \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} \right] && (21)
\end{aligned}
$$

Equating and rearranging Equations (19) and (20) yields *intratemporal optimality condition*:

$$
\begin{aligned}
\Rightarrow \qquad & C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t}(1-\tau_t) && \text{Scenario 3} \\
& C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t}(1-\tau_t^{UK}) && \text{Scenario 4} \\
& C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t} && \text{Scenarios 1 \& 2}
\end{aligned}
$$

The condition implies that the marginal utility of consumption and leisure is equal to the net real wage. As mentioned before, in the case of labour tax absence, the net real wage is equal to the gross real wage. The log-linearisation of Equation (22) around a steady state yields:

$$
\begin{aligned}
C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t}(1-\tau_t) &= \frac{W_t}{P_t} - \frac{W_t}{P_t}\tau_t \\
&\vdots \quad \text{(see Appendix A.110 - A.121)} \\
(1-\tau)(\sigma c_t + \varphi n_t) &= [(1-\tau)(w_t - p_t) - \tau\tilde{\tau}_t] \\
\sigma c_t + \varphi n_t &= w_t - p_t - \frac{\tau}{1-\tau}\tilde{\tau}_t
\end{aligned} \tag{22}
$$

Where τ and $\tilde{\tau}_t$ denote steady state labour tax rate and deviation from the steady state, respectively. As it is common in the literature, we denote natural logs of corresponding variables in lowercase letters, i.e. $x_t = \ln(X_t)$, and use tildes to denote deviations from the steady state. While loglinearising, we widely make use of Uhlig's (1999) proposed methods for multivariate equations with additive terms, i.e., $X_t Y_t \approx XY\mathbf{e}^{\tilde{X}_t + \tilde{Y}_t}$, $X_t + Y_t \approx X\mathbf{e}^{\tilde{X}_t} + Y\mathbf{e}^{\tilde{Y}_t}$ and $\mathbf{e}^{\tilde{X}_t} \approx (1 + \tilde{X}_t)$.

Iterating Equation (19) one period forward, yields:

$$
\frac{\partial L}{\partial C_t} = \beta^t C_t^{-\sigma} Z_t P_t^{-1} = \lambda_t; \qquad \Rightarrow \quad \mathbb{E}_t[\beta^{t+1} C_{t+1}^{-\sigma} Z_{t+1} P_{t+1}^{-1}] = \mathbb{E}_t[\lambda_{t+1}]
$$

Dividing one by the other and rearranging yields *intertemporal optimality condition*:

$$
\begin{aligned}
\mathbb{E}_t\left[\frac{\beta^t C_t^{-\sigma} Z_t P_t^{-1}}{\beta^{t+1} C_{t+1}^{-\sigma} Z_{t+1} P_{t+1}^{-1}}\right] &= \mathbb{E}_t\left[\frac{\lambda_t}{\lambda_{t+1}}\right] \\
\beta^{t-(t+1)}\,\mathbb{E}_t\left[\frac{C_t^{-\sigma} Z_t P_t^{-1}}{C_{t+1}^{-\sigma} Z_{t+1} P_{t+1}^{-1}}\right] &= \mathbb{E}_t\left[\frac{1}{R_{t+1}}\right]
\end{aligned} \tag{23}
$$

$$
\vdots \quad \text{(see Appendix A.xx - A.xx)}
$$

$$
\beta\,\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{Z_{t+1}}{Z_t}\right)\left(\frac{P_t}{P_{t+1}}\right)\right] = \mathbb{E}_t\left[\frac{1}{R_{t+1}}\right] \tag{24}
$$

where Equation (23) used Equation (21). $\mathbb{E}_t[R_{t+1}^{-1}]$ is the gross return on a risk-free one-period discount bond or a stochastic discount factor. More generally, Equation (A.138) is the Euler equation, and it determines the consumption path of a lifetime utility-maximising representative household. To state it in more intuitive terms, households choose consumption “today” and “tomorrow” and take all other terms as given. According to the equation, they choose consumption in the two periods in such a way so that the marginal utility “today” would be equal to the marginal consumption tomorrow while taking into account that saving consumption “today”, will result in $R_t > 1$ consumption “tomorrow”. Note that Galí (2015) uses a different approach to derive the Euler equation, which introduces Arrow securities. As it adds little value to our research question, this dissertation only provides a step-by-step derivation and interpretation in Appendix A for an interested reader. Also note, that Galí (2015) uses $Q_t = \mathbb{E}_t[\frac{1}{R_{t+1}}]$ to denote the stochastic discount factor and D_t to denote bonds or “portfolio” as they call it.

Log-linearising (A.138):

$$\begin{aligned}
\beta\,\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{P_t}{P_{t+1}}\right)\right] &= \mathbb{E}_t\left[\frac{1}{R_{t+1}}\right] \\
\ln\beta - \mathbb{E}_t[\sigma c_{t+1}] + \sigma c_t + p_t - \mathbb{E}_t[p_{t+1}] &= -\ln R_{t+1} \\
\sigma c_t &= -\ln R_{t+1} - \ln\beta + \mathbb{E}_t[\sigma c_{t+1}] - p_t + \mathbb{E}_t[p_{t+1}] \\
c_t &= \mathbb{E}_t[c_{t+1}] - \frac{1}{\sigma}(\ln R_{t+1} - \rho - \mathbb{E}_t[\pi_{t+1}]) \\
c_t &= \mathbb{E}_t[c_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho) \qquad (25)
\end{aligned}$$

where $i_t = \ln R_{t+1}$ is the nominal interest rate, $\rho = -\log\beta$ is the log discount rate, and $\pi_t = p_t - p_{t-1}$ is the CPI inflation. The loglinearised Euler equation makes it clearer to see, that consumption “today” is increasing in expected inflation “tomorrow”, while the opposite is true for the nominal interest rate. The effect is scaled by σ^{-1} parameter.

Furthermore, OECD (2023) defines terms of trade as a ratio of import and export price indices, which in this model is denoted as S_t:

$$S_{i,t} = \frac{P_{i,t}}{P_{H,t}} \tag{26}$$

$$S_t = \frac{P_{F,t}}{P_{H,t}} = \left(\int_0^1 (S_{i,t}\, di)^{1-\gamma} \right)^{\frac{1}{1-\gamma}} \tag{27}$$

where Equation (26) marks *bilateral* terms of trade with a country i, while Equation (27) is for *effective* terms of trade, i.e. terms of trade with all countries in the unit interval defined earlier. The latter can be loglinearised to yield:

$$s_t = p_{F,t} - p_{H,t} = \left(\int_0^1 s_{i,t}\, di \right) \tag{28}$$

Recall that when $\eta = 1$, then CPI is $P_t = (P_{H,t})^{1-\upsilon} \times (P_{F,t})^{\upsilon}$. Using the previous definition (28) and loglinearised CPI, the price level can be expressed as a sum of domestic price level and terms of trade (see Appendix A.xx-A.xx):

$$p_t = (1-\upsilon)p_{H,t} + \upsilon p_{F,t} = p_{H,t} + \upsilon s_t \tag{29}$$

Note that Equations (28) and (29) hold *exactly* when $\gamma = 1$ and $\eta = 1$. Similarly, knowing that inflation as a difference of log prices in two consecutive periods, we can extend the previous definition to yield:

$$\pi_{H,t} = p_{H,t+1} - p_{H,t} \qquad \text{Domestic Inflation} \tag{30}$$

$$\pi_t = \pi_{H,t} + \upsilon \Delta s_t \qquad \text{CPI Inflation} \tag{31}$$

The gap between domestic inflation and CPI inflation is due to percentage change in the terms of trade and degree of openness. In the case of an autarky ($\upsilon = 0$), even if imported goods were much more expensive ($P_{F,t} \gg P_{H,t}$), domestic inflation will be equal to CPI inflation because the country simply does not trade.

Furthermore, we assume that the Law of One Price (LOP) holds for all goods

j. That is, the price of a single good in country i is equal to the price of the same good in country $-i$ times the nominal exchange rate. It implies, that there are no opportunities for arbitrage.

$$P_{i,t}(j) = \mathcal{E}_{i,t} P^i_{i,t}(j) \qquad \text{Law of One Price (LOP)} \tag{32}$$

$$P_{i,t} = \mathcal{E}_{i,t} P^i_{i,t} \qquad \text{Law of One Price (LOP)} \tag{33}$$

where $\mathcal{E}_{i,t}$ is the nominal exchange rate between the home currency and the country's i currency, and the second equation is derived by integrating both sides with respect to j. Even though we do not model currencies explicitly, it is useful to think about $\mathcal{E}_{i,t}$ as the price of one unit of currency in terms of another currency, i.e. the home currency. The two equations can be loglinearised to yield:

$$p_{i,t} = e_{i,t} + p^i_{i,t} \qquad \text{(Log) Law of One Price (LOP)} \tag{34}$$

$$p_{F,t} = \int_0^1 (e_{i,t} + p^i_{i,t})\, di = e_t + p^\star_t \qquad \text{(Log)Price index of Imported Goods} \tag{35}$$

Where e_t is (Log) Effective Nominal Exchange Rate, $p^\star_t$ is the World Price Index. This allows us to redefine log effective terms of trade in terms of the nominal exchange rate, domestic price, and the world price index:

$$s_t = p_{F,t} - p_{H,t} = e_t + p^\star_t - p_{H,t} \tag{36}$$

In contrast, *real* exchange rate between two countries is the ratio between their CPI and home CPI, expressed in home currency:

$$\mathcal{Q}_{i,t} = \frac{\mathcal{E}_{i,t} P^i_t}{P_t} \qquad \text{Bilateral Exchange Rate} \tag{37}$$

Integrating both sides with respect to i and using previous definitions yields:

$$
\begin{aligned}
q_t &= \int_0^1 \log\left(\frac{\mathcal{E}_{i,t} P_t^i}{P_t}\right) di && && (38)\\
&= \int_0^1 (e_{i,t} + p_{i,t}^i - p_t)\, di && && (39)\\
&= e_t + p_t^\star - p_t && \text{using (35)} && (40)\\
&= s_t + p_{H_t} - p_t && \text{using (36)} && (41)\\
&= (1-\upsilon) s_t && \text{using (29)} && (42)
\end{aligned}
$$

Finally, if we assume that all countries i have symmetrical preferences and their households maximise lifetime-utility in the same manner that our home country's do, then maximising the Lagrangian function for country i, will yield:

$$
\begin{aligned}
\frac{\partial \mathcal{L}^i}{\partial C_t^i} &= \beta^t (C_t^i)^{-\sigma} Z_t^i (\mathcal{E}_{i,t} P_t^i)^{-1} = \lambda_t^i \\
\frac{\partial \mathcal{L}^i}{\partial D_{t+1}^i} &= -\lambda_t^i\, \mathbb{E}_t[R_{t,t+1}^{-1}] = -\,\mathbb{E}_t[\lambda_{t+1}^i] \\
&\vdots \quad \text{(see Appendix A.xx - A.xx)} && (43)\\
C_t &= C_t^i Z_t^{i\,\frac{1}{\sigma}} \mathcal{Q}_{i,t}^{\frac{1}{\sigma}} && (44)
\end{aligned}
$$

If we assume that there had been no shocks in the country's i preferences ($Z_t^i = 1$), then Equation (44) states that consumption in the home country is equal to the consumption in the country i, while taking into account bilateral real exchange rate. This can be generalised to derive a relationship between home consumption and world consumption by log linearising (for simplicity) and integrating both sides with respect to i:

$$c_t = c_t^i + \frac{1}{\sigma} z_{i,t} + \frac{1}{\sigma} q_{i,t} \tag{45}$$

$$\int_0^1 c_t \, di = \int_0^1 \left(c_t^\star + \frac{1}{\sigma} z_{i,t} + \frac{1}{\sigma} q_{i,t} \right) di \tag{46}$$

$$\vdots \quad \text{(see Appendix A.xx - A.xx)} \tag{47}$$

$$c_t = c_t^\star + \frac{1}{\sigma} z_t + \left(\frac{1-\upsilon}{\sigma} \right) s_t \qquad \text{using } q_t = (1-\upsilon) s_t \tag{48}$$

$$= y_t^\star + \frac{1}{\sigma} z_t + \left(\frac{1-\upsilon}{\sigma} \right) s_t \qquad \text{using } c_t^\star = y_t^\star \tag{49}$$

$c_t^\star$ is the log world consumption and the last Equation (49) follows by assuming that world consumption is equal to world output, i.e., there is no world government spending, or national government spending in any country i is infinitesimally small.

The next two parts will discuss government spending and firms, respectively. The final part will provide equilibrium (market clearing) conditions.

2.2 Government

As mentioned in the literature review, this dissertation makes two essential assumptions related to government spending: households are assumed to be Ricardian, and; government spending is non-productive. The latter follows from the fact that government spending does not enter the utility function nor the firm's production function. Therefore, government spending is equivalent to reducing the quantity of available resources. In line with most of the literature, deviations from the steady state government spending are assumed to be temporary, i.e. following an autoregressive process of order 1, as opposed to a permanent increase considered by Baxter and King (1993). Below is a typical budget constrained faced by the government every period:

$$\underbrace{\mathbb{E}_t[R_t^{-1} B_{t+1}] + P_t T_t}_{Revenue} = \underbrace{P_t G_t + B_t}_{Spending} \tag{50}$$

That is, government accrues revenue by collecting taxes in current prices P_tT_t and by issuing bonds at current gross return rate $\mathbb{E}_t[R_t^{-1}B_{t+1}]$. The government needs to pay one unit of consumption good for each mature bond issued previous period B_t and pay for its spending in current prices P_tG_t. The government budget constraint varies depending on the policy scenario considered:

Scen. 1	Scot.	$\mathbb{E}_t[R_t^{-1}B_{t+1}] + P_tT_t = P_tG_t + B_t$
G: 2, $\tau : 0$	rUK	$\mathbb{E}_t[R_t^{*-1}B_{t+1}^*] + P_t^*T_t^* = P_t^*G_t^* + B_t^*$
Scen. 2	Scot.	N/A
G: 1, $\tau : 0$	rUK	$\mathbb{E}_t[R_t^{UK-1}B_{t+1}^{UK}] + P_t^{UK}T_t^{UK} = P_t^{UK}G_t^{UK} + B_t^{UK}$
Scen. 3	Scot.	$\mathbb{E}_t[R_t^{-1}B_{t+1}] + P_tT_t = P_tG_t + B_t$
G: 2, $\tau : 1$	rUK	$\mathbb{E}_t[R_t^{*-1}B_{t+1}^*] + P_t^*T_t^* = P_t^*G_t^* + B_t^*$
Scen. 4	Scot.	N/A
G: 1, $\tau : 1$	rUK	$\mathbb{E}_t[R_t^{UK-1}B_{t+1}^{UK}] + P_t^{UK}T_t^{UK} = P_t^{UK}G_t^{UK} + B_t^{UK}$

where P_t^{UK} is a weighted sum of price levels in Scotland and the rest of the UK, i.e., $P_t^{UK} = \varpi P_t + (1-\varpi)P_t^*$. Similarly, a monetary union implies a single rate of gross return, which we define as $R_t^{UK} = \varpi R_t + (1-\varpi)R_t^*$.[4]. In each of the scenarios, the government tax revenue is accrued either by imposing lump-sum taxes or a labour tax:

[4]Due to technical limitations (see limitations section), the dissertation modelled two countries as having individual nominal interest rates. However, in the budget constraints and debt-stabilising equations with one government, a weighted sum of the two interests rates was used. Conceptually, it is equivalent to having one UK-wide interest rate, where Scotland's interest rate "influences" UK-wide interest rate ($\varpi \approx 9\%$) but is primarily determined by the rest of the UK ($\varpi \approx 91\%$), so Scotland (almost) takes it as given.

Scen. 1	Scot.	$P_t T_t = P_t G_t$
G: 2, $\tau : 0$	rUK	$P_t^* T_t^* = P_t^* G_t^*$
Scen. 2	Scot.	N/A
G: 1, $\tau : 0$	rUK	$P_t^{UK} T_t^{UK} = P_t^{UK} G_t^{UK}$
Scen. 3	Scot.	$P_t T_t = \tau_t W_t N_t$
G: 2, $\tau : 1$	rUK	$P_t^* T_t^* = \tau_t^* W_t^* N_t^*$
Scen. 4	Scot.	N/A
G: 1, $\tau : 1$	rUK	$P_t^{UK} T_t^{UK} = \varpi \tau_t^{UK} W_t N_t + (1-\varpi)\tau_t^{UK} W_t^* N_t^*$

Table 1: Tax revenue for each of the policy scenarios

That is, in the case of lump-sum taxes, tax revenue simply equals government spending; similarly, in the case of labour taxes, tax revenue is a share ($\tau_t \in [0,1]$) of nominal labour income. However, given that the two sources of income are close substitutes for the government (it can raise revenue either by raising taxes or by issuing bonds), tax revenue and budget constraint alone do not lead to a stable equilibrium (there are many "solutions"). The governing of this relationship is captured by a fiscal rule of the following form:

Scen. 1	Scot.	$t_t = \phi_g G_Y g_t + \phi_b b_t, \quad G_Y = 0$
G: 2, $\tau : 0$	rUK	$t_t^* = \phi_g^* G_Y^* g_t^* + \phi_b^* b_t^*, \quad G_Y^* = 0$
Scen. 2	Scot.	N/A
G: 1, $\tau : 0$	rUK	$t_t^{UK} = \phi_g^{UK} G_Y^{UK} g_t^{UK} + \phi_b^{UK} b_t^{UK}, \quad G_Y^{UK} = 0$
Scen. 3	Scot.	$t_t = \phi_g G_Y g_t + \phi_b b_t, \quad G_Y = \tau$
G: 2, $\tau : 1$	rUK	$t_t^* = \phi_g^* G_Y^* g_t^* + \phi_b^* b_t^*, \quad G_Y^* = \tau^*$
Scen. 4	Scot.	N/A
G: 1, $\tau : 1$	rUK	$t_t^{UK} = \phi_g^{UK} G_Y^{UK} g_t^{UK} + \phi_b^{UK} b_t^{UK}, \quad G_Y^{UK} = \tau^{UK}$

Table 2: Fiscal (debt-stabilising) rules for each of the policy scenarios

where following Gali, Lopez-Salido, and Valles (2005), we define $g_t = \frac{G_t - G}{Y}$, $b_t = \frac{(B_t/P_{t-1}) - (B/P)}{Y}$, and $t_t = \frac{T_t - T}{Y}$. Moreover, we denote the government spending-to-GDP ratio as G_Y and assume that in the steady state, the government's budget is balanced, i.e. $\frac{B}{Y} = 0$ and $G_Y = \tau$ if the government is funded by a labour tax and $G_Y = 0$ otherwise. Each fiscal policy rule states that tax revenue responds to changes in government spending and government debt, where the responses depend on $\phi_b \in [0, 1]$ and $\phi_g \in [0, 1]$. Intuitively, setting $\phi_b = 0$ would imply that the government cannot issue bonds/borrow, and taxpayers would have to absorb any shock in government spending. At the same time, $\phi_g = 0$ would make government fund its spending solely by issuing bonds. Plugging fiscal rules into the log linearised government budget constraints yields:

Scen. 1	Scot.	$b_{t+1} = (1-\rho)(1-\phi_g)g_t + (1-\rho)(1-\phi_b)b_t$
G: 2, $\tau : 0$	rUK	$b^*_{t+1} = (1-\rho^*)(1-\phi^*_g)g^*_t + (1-\rho^*)(1-\phi^*_b)b^*_t$
Scen. 2	Scot.	N/A
G: 1, $\tau : 0$	rUK	$b^{UK}_{t+1} = (1-\rho^{UK})(1-\phi^{UK}_g)g^{UK}_t + (1-\rho^{UK})(1-\phi^{UK}_b)b^{UK}_t$
Scen. 3	Scot.	$b_{t+1} = (1-\rho)(1-\phi_g)g_t + (1-\rho)(1-\phi_b)b_t$
G: 2, $\tau : 1$	rUK	$b^*_{t+1} = (1-\rho^*)(1-\phi^*_g)g^*_t + (1-\rho^*)(1-\phi^*_b)b^*_t$
Scen. 4	Scot.	N/A
G: 1, $\tau : 1$	rUK	$b^{UK}_{t+1} = (1-\rho^{UK})(1-\phi^{UK}_g)g^{UK}_t + (1-\rho^{UK})(1-\phi^{UK}_b)b^{UK}_t$

Table 3: Future bonds are determined by government spending and current government debt

These equations determine the equilibrium path for bond supply, with $(1-\rho)(1-\phi_b) < 1$ to ensure stability. This is equivalent to imposing a No-Ponzi condition, i.e. the government cannot have outstanding debt in period T. Finally, note that B_t is a predetermined (state) variable, meaning its value is determined in the previous period, while the initial B_0 value is assumed to have been determined "historically", a.k.a given.

2.3 Firms

In line with the literature, we assume a continuum of infinitesimally small firms, each producing a single good j over which they have monopolistic power. Given the monopolistic nature of the market for each good, the firms are allowed to adjust their prices to maximise profits but, following Calvo (1983), we assume that only a fraction $\theta \in [0, 1]$ of them actually do. The firms are assumed to be owned by households, implying that our budget constraint should have a Π_t (dividend) term. However, as all households own all firms and take profit/dividends as given, it does not affect first-order conditions/optimal behaviour and, as such, is not considered. The firm's production function is given by:

$$Y_t(j) = A_t N_t(j)^{1-\alpha} \tag{51}$$

where A_t is the technology level shifter common to all firms and is assumed to evolve exogenously as an AR(1) process in log terms: $\ln A_t = \rho_a \ln A_{t-1} + \varepsilon_t^a$. Notice that the marginal product of labour is

$$\frac{\partial Y_t(j)}{\partial N_t(j)} = (1 - \alpha) A_t N_t^{-\alpha} \tag{52}$$

which is increasing in technology level (increases labour productivity). Second order derivate with respect to hours worked indicates, that the production function exhibits decreasing returns to scale. Knowing this, the firm maximises their profit by choosing the optimal amount of labour:

$$\max_{N_t(j)} \quad \mathcal{F} = P_{H,t}(j) Y_t(j) - W_t(j) N_t(j) \tag{53}$$

$$\text{s.t.} \quad Y_t(j) = A_t N_t(j)^{1-\alpha} \tag{54}$$

which can be solved (see Appendix A.xx-A.xx) to yield an optimality condition:

$$\frac{\partial \mathcal{F}}{\partial N_t(j)} \implies \frac{W_t}{P_t} = (1-\alpha)A_t N_t^{-\alpha} \tag{55}$$

Suggesting that firm will hire workers up until the marginal product of labour is equal to the real wage. The optimality condition also acts as a link between real wage ($w_t - p_t$), labour and technology. Moreover, given that the marginal cost Ψ_t needs to equal domestic price level, rearranging Equation (55) yields:

$$\Psi_t = \frac{W_t}{(1-\alpha)A_t N_t^{-\alpha}} \qquad \text{Marginal cost} \tag{56}$$

$$\psi_t = w_t - (a_t - \alpha n_t + log(1-\alpha)) \qquad \text{(Log) Marginal cost} \tag{57}$$

Unsurprisingly, the marginal cost is increasing in wages and decreasing in marginal product of labour. It is important to emphasise that definition above is an *average* marginal cost across all firms producing goods j. The marginal cost varies across firms due to different existing levels of labour.

As mentioned earlier, the price stickiness is introduced by assuming that firms update their prices with probability $(1-\theta)$. The newly set price is denoted as $\overline{P}_{H,t}(j)$. Following a similar proof offered by Galí (2015)[5], note that if all firms are symmetrical, then they will choose the same price, i.e. $\overline{P}_{H,t}(j) = \overline{P}_{H,t}$. Thus, the domestic price index from Equation (6) can be rewritten as:

$$P_{H,t} = \left[\int_0^1 (P_{H,t-1}(j))^{1-\epsilon}\, dj\right]^{\frac{1}{1-\epsilon}} \tag{58}$$

$$= \left[\int_{S(t)}^1 P_{H,t-1}(j)^{1-\epsilon}\, dj + \int_0^{S(t)} P_{H,t-1}(j)^{1-\epsilon}\, dj\right]^{\frac{1}{1-\epsilon}} \tag{59}$$

$$= \left[\theta(\overline{P}_{H,t-1})^{1-\epsilon} + (1-\theta)(\overline{P}_{H,t})^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}} \tag{60}$$

[5]In, Galí (2015) the proof is only available for a closed-economy NK DSGE (Chapter 3) and less detailed than presented here

where $S(t)$ is a subset of firms that do not update their prices, and Equation (60) "follows from the fact that the distribution of prices among firms not adjusting in period t corresponds to the distribution of effective prices in period $t-1$, though with total mass reduced to θ" (Galí, 2015: 84). Dividing (60) by $(P_{H,t-1})^{1-\epsilon}$ yields:

$$\Pi_{H,t}^{1-\epsilon} = \theta + (1-\theta)\left(\frac{\overline{P}_{H,t}}{P_{H,t-1}}\right)^{1-\epsilon} \tag{61}$$

Log-linearising (61) around zero inflation steady state yields:

$$\begin{aligned} \Pi_{H,t}^{1-\epsilon} &= \theta + (1-\theta)\left(\frac{\overline{P}_{H,t}}{P_{H,t-1}}\right)^{1-\epsilon} && (62)\\ \Pi_H \mathbf{e}^{(1-\epsilon)\pi_{H,t}} &= \theta + (1-\theta)\frac{\overline{P_H}}{P_H}\mathbf{e}^{(1-\epsilon)(\bar{p}_{H,t}-p_{H,t})} && (63)\\ &\vdots \quad \text{(see Appendix A.xx - A.xx)} && (64)\\ \pi_{H,t} &= (1-\theta)(\bar{p}_{H,t}-p_{H,t}) && (65) \end{aligned}$$

Intuitively, this means that log domestic inflation depends on two elements: the difference between the current and new domestic price levels, and the price stickiness parameter θ. Consider two extreme cases when $\theta = 1$ and $\theta = 0$: when $\theta = 1$, then no firms would be permitted to update their prices and the domestic inflation would always be equal to zero (CPI would still vary due to terms of trade, assuming $\upsilon \neq 0$). When $\theta = 0$, then all firms immediately react to any changes in marginal cost of production and traditional RBC "ineffective-money" results would follow. Firms that get to update their prices, do so by maximising their *discounted lifetime cash flow*:

$$\max_{\overline{P}_{H,t}} \sum_{k=0}^{\infty} \theta^k \, \mathbb{E}_t \left[\Lambda_{t,t+k}\left(\underbrace{\overline{P}_{H,t}Y_{t+k|t}}_{Revenue} - \underbrace{\mathcal{C}_{t+k}(Y_{t+k|t})}_{Cost}\right)/P_{H,t+k}\right] \tag{66}$$

$$\text{s.t.} \quad Y_{t+k|t} = \left(\frac{\overline{P}_{H,t}}{P_{H,t+k}}\right)^{-\epsilon} C_{t+k} \tag{67}$$

where $\mathbb{E}_t\left[\Lambda_{t,t+k}\right]\ \forall k \geq 0$ is the expected stochastic discount factor used to discount profit (revenue less cost) in every period starting from current[6]. $Y_{t+k|t}$ is the expected output in periods $t+k$ given output in period t, and $\mathcal{C}_{t+k}$ is the nominal cost of producting the expected output. The maximisation problem is subject to k number of demand constraints (67). Substituting the constraint into (66), taking first-order conditions, and log-linearising around zero-inflation steady state yields:

$$\sum_{k=0}^{\infty} \theta^k\, \mathbb{E}_t \left[\Lambda_{t,t+k} \left(\overline{P}_{H,t} \left(\frac{\overline{P}_{H,t}}{P_{H,t+k}} \right)^{-\epsilon} C_{t+k} - \mathcal{C}_{t+k}(Y_{t+k|t}) \right) / P_{H,t+k} \right] \tag{68}$$

$$\vdots \quad \text{(see Appendix A.xx-A.xx)}$$

$$\bar{p}_{H,t} = \mu + (1-\beta\theta) \sum_{k=0}^{\infty} (\beta\theta)^k\, \mathbb{E}_t \left[\psi_{t+k|t} \right] \tag{69}$$

where $\mathbb{E}_t\left[\psi_{t+k|t}\right]$ and μ are the expected log marginal cost and desired[7] markup, respectively. Equation (69) is known as the (log) *optimal price setting condition* and presents firms as forward looking discounted profit maximisers. Note, that it is consistent with the previous exposition of the two extreme cases when $\theta = 0$ and $\theta = 1$, i.e. when $\theta = 1$, then $\pi_{H,t} = \bar{p}_{H,t} - \bar{p}_{H,t-1} = \mu - \mu = 0$. Furthermore, the earlier discussion on optimality condition noted that firms will hire labour up until the real wage is equal to the marginal product of labour. Given that the price (and, subsequently, real wage) changed, firms might choose a different level of labour compared to that of other firms, i.e.:

$$\psi_{t+k|t} = \psi_{t+k} + \alpha(n_{t+k|t} - n_{t+k}) \tag{70}$$

$$\vdots \quad \text{(see Appendix A.xx-A.xx)}$$

$$= \psi_{t+k} - \frac{\alpha\epsilon}{1-\alpha} \tag{71}$$

[6]Notice that we could write this as $\mathbb{E}_t\left[\Lambda_{t,t+k}\right] = \mathbb{E}_t\left[\frac{1}{R_{t,t+k}}\right]$ derived from the households' optimisation problem.

[7]Markup that occurs under flexible or frictionless prices

Plugging (71) to (69) and rewritting as a recursive equation yields:

$$\bar{p}_{H,t} = \mu + (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k\,\mathbb{E}_t\left[\psi_{t+k} - \frac{\alpha\epsilon}{1-\alpha}\right] \tag{72}$$

$$\vdots \quad \text{(see Appendix A.xx - A.xx)}$$

$$\bar{p}_{H,t} = \beta\theta\,\mathbb{E}_t\left[\bar{p}_{H,t+1}\right] + (1-\beta\theta)(p_{H,t} - \Theta\hat{\mu}_t) \tag{73}$$

where $\mu_t = p_{H,t} - \psi_t$ is the average markup, $\hat{\mu}_t$ is the gap between the average and desired markups, and $\Theta = \frac{1-\alpha}{1-\alpha+\alpha\epsilon}$. Using (65), it is easy to transform (73) into a version of New Keynesian Phillips Curve (NKPC):

$$\bar{p}_{H,t} = \beta\theta\,\mathbb{E}_t\left[\bar{p}_{H,t+1}\right] + (1-\beta\theta)(p_{H,t} - \Theta\hat{\mu}_t) \tag{74}$$

$$\vdots \quad \text{(see Appendix A.xx-A.xx)}$$

$$\pi_{H,t} = \beta\,\mathbb{E}_t\left[\pi_{H,t+1}\right] - \lambda\hat{\mu}_t \tag{75}$$

where $\lambda = \frac{(1-\theta)(1-\beta\theta)}{\theta}\Theta$. Average markup is derived using Equation (57) without the constant term:

$$\mu_t = p_{H,t} - \psi_t \tag{76}$$

$$= p_{H,t} - (w_t - a_t + \alpha n_t)$$

$$\vdots \quad \text{(see Appendix A.182-A.205)}$$

$$\mu_t = -\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)y_t + \upsilon(\varpi-1)s_t - \frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t \tag{77}$$

Evaluating (77) at flexible prices $\theta = 0$ and solving for the output term yields the expression for the natural level of output y_t^n:

$$\mu_t = -\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)y_t^n + \upsilon(\varpi-1)s_t^n - \frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t \tag{78}$$

$$\vdots \quad \text{(see Appendix A.206 - A.233)}$$

$$y_t^n = \Gamma_* y_t^* + \Gamma_z z_t + \Gamma_a a_t + \Gamma_\tau \tau_t + \Gamma_g g_t \tag{79}$$

where:

$$\Gamma_* = -\frac{\upsilon(\varpi - 1)\sigma_\upsilon(1-\alpha)}{\sigma_\upsilon(1-\alpha) + \varphi + \alpha} \tag{80}$$

$$\Gamma_z = -\frac{\upsilon\varpi\Phi(1-\alpha)}{\sigma_\upsilon(1-\alpha) + \varphi + \alpha} \tag{81}$$

$$\Gamma_a = \frac{1+\varpi}{\sigma_\upsilon(1-\alpha) + \varphi + \alpha} \tag{82}$$

$$\Gamma_g = -\frac{\upsilon(\varpi - 1)\sigma_\upsilon(1-\alpha)}{\sigma_\upsilon(1-\alpha) + \varphi + \alpha} \tag{83}$$

$$\Gamma_\tau = \frac{(1-\tau)(1-\alpha)}{\sigma_\upsilon(1-\alpha) + \varphi + \alpha} \tag{84}$$

When the prices are flexible, then each firm will choose the same optimal level of labour, implying constant average markup $\mu_t = \mu$. Subtracting μ_t definition (77) from μ definition (78) yields:

$$\hat{\mu}_t = \mu_t - \mu = -\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)\tilde{y}_t + \upsilon(\varpi - 1)\tilde{s}_t \tag{85}$$

$$= -\left(\sigma_\upsilon + \frac{\varphi+\alpha}{1-\alpha}\right)\tilde{y}_t \tag{86}$$

where the second $\tilde{y}_t$ and $\tilde{s}_t$ denote natural output and terms of trade gaps, and the (86) uses $\tilde{s}_t = \sigma_\upsilon \tilde{y}_t$ relationship introduced in the next section. Finally, plugging (86) in a version of NKPC defined by Equation (75), yields:

$$\pi_{H,t} = \beta\,\mathbb{E}_t\left[\pi_{H,t+1}\right] - \kappa\tilde{y}_t \tag{87}$$

where $\kappa = \lambda\left(\sigma_\upsilon + \frac{\varphi+\alpha}{1-\alpha}\right)$.

The next section is about the equilibrium or market clearing conditions, that will introduce NKPC as well as Dynamic IS curve, “natural” variables and the monetary policy rule.

This equation is arguably one of the most important equations of the model: it introduces anticipation, which is used to derive relationship between domestic inflation "today" and (expected) "tomorrow", better known as the *New Keynesian Phillips Curve* (NKPC).

2.4 Equilibrium

The goods market for a specific good j clears when domestic firms produce just enough of the good to satisfy the demand of home households, foreign households, and the home government. In line with Galí (2015), the demand for exports of good j is taken to be given as:

$$X_t(j) = \left(\frac{P_{H,t(i)}}{P_{H,t}}\right)^{-\epsilon} X_t \tag{88}$$

$$\text{where} \quad X_t = \left(\int_0^1 X_t(j)^{\frac{\epsilon-1}{\epsilon}} \, dj\right)^{\frac{\epsilon}{\epsilon-1}} \tag{89}$$

$$= \upsilon \left(\frac{P_{H,t}}{\mathcal{E}_t \overline{P}_{H,t}}\right)^{-\eta} Y_t^{\star} \tag{90}$$

$$= \upsilon \mathcal{S}_t^{\eta} Y_t^{\star} \tag{91}$$

where (89) is the index of aggregate exports, (90) determines aggregate exports as a function of world output (the relationship is assumed to be given), and (91) is derived by substituting definition of the effective terms of trade. Differently than Galí (2015), we introduce index of government purchasing:

$$G_t = \left(\int_0^1 G_t(j)^{\frac{\epsilon-1}{\epsilon}} \, dj\right)^{\frac{\epsilon}{\epsilon-1}} \tag{92}$$

so that the government demand of any good j is defined as (derivation provided by Appendix A.xx-A.xx):

$$G_t(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\epsilon} G_t \tag{93}$$

Therefore, total demand for good j is:

$$Y_t(j) = C_t(j) + X_t(j) + G_t(j) \tag{94}$$

$$\vdots \quad \text{(see Appendix A.xx-A.xx)} \tag{95}$$

$$Y_t(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}^{-\varepsilon}\right)\left[(1-\upsilon)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \upsilon \mathcal{S}^{\eta} Y_t^{\star} + G_t\right] \tag{96}$$

which can be plugged in to definition of aggregate output $Y_t = \left(\int_0^1 Y_t(j)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}$ to yield:

$$Y_t = (1-\upsilon)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \upsilon \mathcal{S}^{\eta} Y_t^{\star} + G_t \tag{97}$$

Note that Equation (94) and all subsequent derivations (marginally) vary depending on the policy scenario in question:

Scen. 1 & Scen. 3	Scot.	$Y_t(j) = C_t(j) + X_t(j) + G_t(j)$
G: 2, $\tau \in \{0,1\}$	rUK	$Y_t^*(j) = C_t^*(j) + X_t^*(j) + G_t^*(j)$
Scen. 3 & Scen. 4	Scot.	$Y_t(j) = C_t(j) + X_t(j) + \varpi G_t^{UK}(j)$
G: 1, $\tau \in \{0,1\}$	rUK	$Y_t^*(j) = C_t^*(j) + X_t^*(j) + (1-\varpi) G_t^{UK}(j)$

Table 4: Demand for good j for different policy scenarios

Equation (94) can be loglinearised around a symmetric steady state to yield:

$$Y_t = (1-\upsilon)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \upsilon \mathcal{S}_t^{\eta} Y_t^{*} + G_t \tag{98}$$

$$Y\mathbf{e}^{y_t} = (1-\upsilon)\left(\frac{P}{P_H}\right)^{\eta} C\mathbf{e}^{-\eta p_{H,t} + \eta p_t + c_t} + \upsilon S^{\eta} Y^{*} \mathbf{e}^{\eta s_t + y_t^*} + G\mathbf{e}^{g_t} \tag{99}$$

$$\vdots \quad \text{(see Appendix A.123 - A.132)}$$

$$y_t = (1-\upsilon)c_t + \upsilon(2-\upsilon)\eta s_t + \upsilon y_t^* + G_Y g_t \tag{100}$$

where the last term is equal to zero if the government is financed via lump-sum taxes. (**??**) We can use Equation (49) that links domestic consumption to world output and previous equation (100) to express terms of trade as a function of domestic output, world output, preference shifter, and government spending:

$$y_t = (1-\upsilon)\left(y_t^* + \frac{1}{\sigma}z_t + \frac{1-\upsilon}{\sigma}s_t\right) + \upsilon(2-\upsilon)\eta s_t + \upsilon y_t^* + G_Y g_t \tag{101}$$

$\vdots$ (see Appendix A.147 - A.162)

$$s_t = \sigma_\upsilon(y_t - y_t^*) - (1-\upsilon)\Phi z_t - \sigma_\upsilon G_Y g_t \tag{102}$$

where $\varpi = \sigma\eta + (1-\upsilon)(\sigma\eta - 1)$, $\Phi = \frac{1}{1+\upsilon(\varpi-1)}$ and $\sigma_\upsilon = \sigma\Phi$. Negative government term implies that (effective) trade of terms are decreasing in government spending. This is intuitive: the government is modelled to demand exclusively domestic goods, which induces inflationary pressure and makes home goods less competitive internationally. The opposite is true for the domestic-world output gap: if our firms produce relatively more than the rest of the world and is able to export relatively more, then terms of trade increase (the first term of (102) is positive).

Furthermore, the Euler equation in (A.138) is a function of CPI, but using CPI definition (31), it can be rewritten to be a function of domestic inflation and terms of trade:

$$c_t = \mathbb{E}\{c_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}\{\pi_{H,t+1}\} - \rho) + \frac{\upsilon}{\sigma}\mathbb{E}\{\Delta s_{t+1}\} + \frac{1}{\sigma}(1-\rho_z)z_t \tag{103}$$

Finally, aggregate resource constraint (100), terms of trade definition (102), and new Euler equation (103) can be combined and used to derive a version of dynamic IS equation:

$$c_t = \mathbb{E}\{c_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}\{\pi_{H,t+1}\} - \rho) + \frac{\upsilon}{\sigma}\mathbb{E}\{\Delta s_{t+1}\} + \frac{1}{\sigma}(1-\rho_z)z_t \tag{104}$$

$\vdots$ (see Appendix A.163-A.174)

$$\begin{aligned} y_t = {} & \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} \\ & + \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t + \frac{\upsilon\varpi - 1}{1-\upsilon}G_Y\{\Delta g_{t+1}\} \end{aligned} \tag{105}$$

which implies that output in current period depends not only on expected output, inflation, and change in world output, but it also depends on expected government spending. Equation (105) can be expressed in terms of output and real interest rate gaps:

$$
\begin{aligned}
y_t^n &= \mathbb{E}_t\{y_{t+1}^n\} - \frac{1}{\sigma_\upsilon}(r_t^n - \rho) + \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} + \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t \\
&+ \frac{\upsilon\varpi - 1}{1-\upsilon} G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\} \\
&\vdots \quad \text{(see Appendix A.175 - A.178)} \\
r_t^n &= \sigma_\upsilon\,\mathbb{E}_t\{\Delta y_{t+1}^n\} + \rho + \sigma_\upsilon\upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} + \Phi(1-\upsilon)(1-\rho_z)z_t \\
&+ \sigma_\upsilon\frac{\upsilon\varpi - 1}{1-\upsilon} G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\} \qquad (106) \\
&\vdots \quad \text{(see Appendix A.178 - A.181)} \\
\tilde{y}_t &= \mathbb{E}_t\{\tilde{y}_{t+1}\} - \frac{1}{\sigma_\upsilon}(i_t - \mathbb{E}_t\{\pi_{H,+1}\} - r_t^n) \qquad (107)
\end{aligned}
$$

where $\mathbb{E}_t\{y_{t+1}^n\}$ is the expected natural output and Equation (106) defines the natural real rate of interest r_t^n.

3 Application

4 Results

5 Conclusion

6 Appendix

Derivation of loglinearised intratemporal condition

$$C_t^\sigma N_t^\varphi = \frac{W_t}{P_t}(1-\tau_t) \tag{A.108}$$

$$C_t^\sigma N_t^\varphi = \frac{W_t}{P_t} - \frac{W_t}{P_t}\tau_t \tag{A.109}$$

$$C^\sigma N^\varphi = \frac{W}{P} - \frac{W}{P}\tau \quad \text{(Steady state)} \tag{A.110}$$

$$\text{Using Uhlig's (1999) method, } X_t Y_t \approx XY\mathbf{e}^{\tilde{X}_t+\tilde{Y}_t} \tag{A.111}$$

$$C^\sigma N^\varphi \mathbf{e}^{\sigma c_t+\varphi n_t} = \frac{W}{P}\mathbf{e}^{w_t-p_t} - \frac{W}{P}\tau\mathbf{e}^{w_t-p_t-\tilde{\tau}_t} \tag{A.112}$$

$$\text{Using } \mathbf{e}^{\tilde{X}_t} \approx 1+\tilde{X}_t\text{:} \tag{A.113}$$

$$C^\sigma N^\varphi(1+\sigma c_t+\varphi n_t) = \frac{W}{P}(1+w_t-p_t) - \frac{W}{P}\tau(1+w_t-p_t-\tilde{\tau}_t) \tag{A.114}$$

$$\text{Subtract (A.110):} \tag{A.115}$$

$$C^\sigma N^\varphi(\sigma c_t+\varphi n_t) = \frac{W}{P}(w_t-p_t) - \frac{W}{P}\tau(w_t-p_t-\tilde{\tau}_t) \tag{A.116}$$

$$C^\sigma N^\varphi(\sigma c_t+\varphi n_t) = \frac{W}{P}\left[(w_t-p_t) - \tau(w_t-p_t-\tilde{\tau}_t)\right] \tag{A.117}$$

$$C^\sigma N^\varphi(\sigma c_t+\varphi n_t) = \frac{W}{P}\left[(w_t-p_t) - \tau(w_t-p_t) - \tau\tilde{\tau}_t\right] \tag{A.118}$$

$$C^\sigma N^\varphi(\sigma c_t+\varphi n_t) = \frac{W}{P}\left[(1-\tau)(w_t-p_t) - \tau\tilde{\tau}_t\right] \tag{A.119}$$

$$C^\sigma N^\varphi \frac{P}{W}(\sigma c_t+\varphi n_t) = \left[(1-\tau)(w_t-p_t) - \tau\tilde{\tau}_t\right] \tag{A.120}$$

$$(1-\tau)(\sigma c_t+\varphi n_t) = \left[(1-\tau)(w_t-p_t) - \tau\tilde{\tau}_t\right] \tag{A.121}$$

$$\sigma c_t+\varphi n_t = w_t-p_t-\frac{\tau}{1-\tau}\tilde{\tau}_t \tag{A.122}$$

Derivation of loglinearised aggregate resource constraint

$$Y_t = (1-\upsilon)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \upsilon \mathcal{S}_t^{\eta} Y_t^* + G_t \tag{A.123}$$

$$Y\mathbf{e}^{y_t} = (1-\upsilon)\left(\frac{P}{P_H}\right)^{\eta} C\mathbf{e}^{-\eta p_{H,t}+\eta p_t + c_t} + \upsilon S^{\eta} Y^* \mathbf{e}^{\eta s_t + y_t^*} + G\mathbf{e}^{g_t} \tag{A.124}$$

$$Y(1+y_t) = (1-\upsilon)\left(\frac{P}{P_H}\right)^{\eta} C(1-\eta p_{H,t}+\eta p_t + c_t) + \upsilon S^{\eta} Y^*(1+\eta s_t + y_t^*) + G(1+g_t) \tag{A.125}$$

$$Y y_t = (1-\upsilon)\left(\frac{P}{P_H}\right)^{\eta} C(-\eta p_{H,t}+\eta p_t + c_t) + \upsilon S^{\eta} Y^*(\eta s_t + y_t^*) + G g_t \tag{A.126}$$

$$\text{Using } P/P_H = \upsilon\mathcal{S},\ \mathcal{S} = 1, \text{ and } C = Y^* : \tag{A.127}$$

$$y_t = (1-\upsilon)(-\eta p_{H,t} + \eta p_t + c_t) + \upsilon(\eta s_t + y_t^*) + G_Y g_t \tag{A.128}$$

$$\text{Using } p_t - p_{H,t} = \upsilon s_t : \tag{A.129}$$

$$y_t = (1-\upsilon)(\eta\upsilon s_t + c_t) + \upsilon(\eta s_t + y_t^*) + G_Y g_t \tag{A.130}$$

$$y_t = (1-\upsilon)c_t + (1-\upsilon)\eta\upsilon s_t + \upsilon\eta s_t + \upsilon y_t^* + G_Y g_t \tag{A.131}$$

$$y_t = (1-\upsilon)c_t + \upsilon(2-\upsilon)\eta s_t + \upsilon y_t^* + G_Y g_t \tag{A.132}$$

Derivation of the Euler equation using Arrow securities

$$\frac{V_{t,t+1}}{P_t} C_t^{-\sigma} = \xi_{t,t+1} \beta C_{t+1}^{-\sigma} \frac{1}{P_{t+1}} \tag{A.133}$$

Where $V_{t,t+1}$ is an Arrow security and $\xi_{t,t+1}$ is the probability that the Arrow security will yield a payoff next period. Interpretation: LHS is paying for the security (expenses) in terms of consumption at P_t prices. RHS is the payoff if the Arrow security yields a payoff. The consumer will only be willing to pay LHS if the payoff is at least as big on the RHS. Given that:

$$Q_{t,t+1} = \frac{V_{t,t+1}}{\xi_{t,t+1}} \tag{A.134}$$

$$\frac{V_{t,t+1}}{P_t} C_t^{-\sigma} = \xi_{t,t+1} \beta C_{t+1}^{-\sigma} \frac{1}{P_{t+1}} \tag{A.135}$$

$$Q_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \tag{A.136}$$

$$\mathbb{E}_t[Q_{t,t+1}] = \beta \, \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right] \tag{A.137}$$

$$\beta \, \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \left(\frac{P_t}{P_{t+1}} \right) \right] = Q_t \qquad \text{Euler equation} \tag{A.138}$$

Derivation of bilateral exchange rate

International Risk-Sharing Equation for country i can be rewritten as:

$$\frac{V_{t,t+1}}{\mathcal{E}_t^i P_t^i}(C_t^i)^{-\sigma} = \xi_{t,t+1}\beta(C_{t+1}^i)^{-\sigma}\frac{1}{\mathcal{E}_{t+1}^i P_{t+1}^i} \tag{A.139}$$

Given that:

$$Q_{t,t+1} = \frac{V_{t,t+1}}{\xi_{t,t+1}} \tag{A.140}$$

$$\frac{V_{t,t+1}}{\mathcal{E}_t^i P_t^i}(C_t^i)^{-\sigma} = \xi_{t,t+1}\beta(C_{t+1}^i)^{-\sigma}\frac{1}{\mathcal{E}_{t+1}^i P_{t+1}^i} \tag{A.141}$$

$$Q_{t,t+1} = \beta\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\sigma}\left(\frac{P_t^i}{P_{t+1}^i}\right)\left(\frac{\mathcal{E}_t^i}{\mathcal{E}_{t+1}^i}\right) \tag{A.142}$$

Recall that:

$$\begin{aligned}\frac{\partial L}{\partial C_t} &= \beta^t C_t^{-\sigma} P_t^{-1} = \lambda_t \\ \frac{\partial L}{\partial D_{t+1}} &= -\lambda_t \,\mathbb{E}_t[Q_{t,t+1}] = -\,\mathbb{E}_t[\lambda_{t+1}] \\ &= -\beta^t C_t^{-\sigma} P_t^{-1}\,\mathbb{E}_t[Q_{t,t+1}] = -\,\mathbb{E}_t[\lambda_{t+1}]\end{aligned} \tag{A.143}$$

Which is symmetrical for country i:

$$\begin{aligned}\frac{\partial L^i}{\partial C_t^i} &= \beta^t (C_t^i)^{-\sigma}(\mathcal{E}_{i,t}P_t^i)^{-1} = \lambda_t^i \\ \frac{\partial L^i}{\partial D_{t+1}^i} &= -\lambda_t^i \,\mathbb{E}_t[Q_{t,t+1}] = -\,\mathbb{E}_t[\lambda_{t+1}^i] \\ &= -\beta^t (C_t^i)^{-\sigma}(\mathcal{E}_{i,t}P_t^i)^{-1}\,\mathbb{E}_t[Q_{t,t+1}] = -\,\mathbb{E}_t[\lambda_{t+1}^i]\end{aligned} \tag{A.144}$$

$$\frac{(\text{A.143})}{(\text{A.144})} : \frac{-\beta^t C_t^{-\sigma} P_t^{-1}\,\mathbb{E}_t[Q_{t,t+1}]}{-\beta^t (C_t^i)^{-\sigma}(\mathcal{E}_{i,t}P_t^i)^{-1}\,\mathbb{E}_t[Q_{t,t+1}]} = \frac{-\,\mathbb{E}_t[\lambda_{t+1}]}{-\,\mathbb{E}_t[\lambda_{t+1}^i]}$$

$$
\begin{aligned}
C_t^{-\sigma}(C_t^i)^{\sigma}\frac{\mathcal{E}_{i,t}P_t^i}{P_t} &= 1 \\
C_t^{-\sigma}(C_t^i)^{\sigma}\mathcal{Q}_{i,t} &= 1 \\
C_t^{-\sigma}(C_t^i)^{\sigma} &= \frac{1}{\mathcal{Q}_{i,t}} \\
C_t^{-\sigma} &= \frac{1}{\mathcal{Q}_{i,t}}(C_t^i)^{-\sigma} \\
C_t^{\sigma} &= \mathcal{Q}_{i,t}(C_t^i)^{\sigma}
\end{aligned}
$$

$$
\Rightarrow C_t = C_t^i \mathcal{Q}_{i,t}^{\frac{1}{\sigma}} \tag{A.145}
$$

Derivation of loglinearised tax revenue

$$T_t^* = \varpi \tau_t W_t N_t + (1-\varpi)\tau_t W_t^* N_t^*$$

$$T^* = \varpi \tau W N + (1-\varpi)\tau W^* N^* \quad \text{(Steady state)}$$

Using Uhlig's (1999) method, $X_t Y_t \approx XY\mathbf{e}^{\tilde{X}_t + \tilde{Y}_t}$:

$$T\mathbf{e}^{\tilde{T}_t^*} = \varpi \tau W N \mathbf{e}^{\tilde{\tau}_t + \tilde{W}_t + \tilde{N}_t} + (1-\varpi)\tau W^* N^* \mathbf{e}^{\tilde{\tau}_t + \tilde{W}_t^* + \tilde{N}_T^*}$$

Using $\mathbf{e}^{\tilde{X}_t} \approx 1 + \tilde{X}_t$:

$$T^*(1+\tilde{T}_t^*) = \varpi \tau W N (1 + \tilde{\tau}_t + \tilde{W}_t + \tilde{N}_t) + (1-\varpi)\tau W^* N^* (1 + \tilde{\tau}_t + \tilde{W}_t^* + \tilde{N}_T^*)$$

Subtract (**??**):

$$T^*(1+\tilde{T}_t^*) - T^* = \varpi \tau W N (1 + \tilde{\tau}_t + \tilde{W}_t + \tilde{N}_t) - \varpi \tau W N + (1-\varpi)\tau W^* N^* (1 + \tilde{\tau}_t + \tilde{W}_t^* + \tilde{N}_T^*) - (1$$

$$T^*[(1+\tilde{T}_t^*) - 1] = \varpi \tau W N [(1 + \tilde{\tau}_t + \tilde{W}_t + \tilde{N}_t) - 1] + (1-\varpi)\tau W^* N^* [(1 + \tilde{\tau}_t + \tilde{W}_t^* + \tilde{N}_T^*) - 1]$$

$$T^* \tilde{T}_t^* = \varpi \tau W N (\tilde{\tau}_t + \tilde{W}_t + \tilde{N}_t) + (1-\varpi)\tau W^* N^* (\tilde{\tau}_t + \tilde{W}_t^* + \tilde{N}_T^*)$$

Divide by T^* :

$$\boxed{\tilde{T}_t^* = \tau \tfrac{WN}{T^*} \times \varpi(\tilde{\tau}_t + \tilde{W}_t + \tilde{N}_t) + \tau \tfrac{W^* N^*}{T^*} \times (1-\varpi)(\tilde{\tau}_t + \tilde{W}_t^* + \tilde{N}_t^*)} \qquad \text{(A.146)}$$

Derivation of loglinearised terms of trade

$$c_t = y_t^* + \frac{1}{\sigma} z_t + \frac{1-\upsilon}{\sigma} s_t \tag{A.147}$$

$$y_t = (1-\upsilon)c_t + \upsilon(2-\upsilon)\eta s_t + \upsilon y_t^* + G_Y g_t \tag{A.148}$$

$$y_t = (1-\upsilon)\left(y_t^* + \frac{1}{\sigma} z_t + \frac{1-\upsilon}{\sigma} s_t\right) + \upsilon(2-\upsilon)\eta s_t + \upsilon y_t^* + G_Y g_t \tag{A.149}$$

$$y_t = (1-\upsilon)y_t^* + \frac{1-\upsilon}{\sigma} z_t + (1-\upsilon)\frac{1-\upsilon}{\sigma} s_t + \upsilon(2-\upsilon)\eta s_t + \upsilon y_t^* + G_Y g_t \tag{A.150}$$

$$y_t = y_t^* + \frac{1-\upsilon}{\sigma} z_t + (1-\upsilon)\frac{1-\upsilon}{\sigma} s_t + \upsilon(2-\upsilon)\eta s_t + G_Y g_t \tag{A.151}$$

$$y_t = y_t^* + \frac{1-\upsilon}{\sigma} z_t + \left((1-\upsilon)\frac{1-\upsilon}{\sigma} + \upsilon(2-\upsilon)\eta\right) s_t + G_Y g_t \tag{A.152}$$

Rearrange:

$$\left((1-\upsilon)\frac{1-\upsilon}{\sigma} + \upsilon(2-\upsilon)\eta\right) s_t = y_t - y_t^* - \frac{1-\upsilon}{\sigma} z_t - G_Y g_t \tag{A.153}$$

$$\left(\frac{(1-\upsilon)(1-\upsilon) + \sigma\upsilon(2-\upsilon)\eta}{\sigma}\right) s_t = y_t - y_t^* - \frac{1-\upsilon}{\sigma} z_t - G_Y g_t \tag{A.154}$$

$$\left(\frac{1-2\upsilon+\upsilon^2 + \sigma\upsilon(2-\upsilon)\eta}{\sigma}\right) s_t = y_t - y_t^* - \frac{1-\upsilon}{\sigma} z_t - G_Y g_t \tag{A.155}$$

$$\left(\frac{1-\upsilon(2-\upsilon) + \sigma\upsilon(2-\upsilon)\eta}{\sigma}\right) s_t = y_t - y_t^* - \frac{1-\upsilon}{\sigma} z_t - G_Y g_t \tag{A.156}$$

$$\left(\frac{1-\upsilon(2-\upsilon)(1-\sigma\eta)}{\sigma}\right) s_t = y_t - y_t^* - \frac{1-\upsilon}{\sigma} z_t - G_Y g_t \tag{A.157}$$

$$\left(\frac{1+\upsilon(2-\upsilon)(\sigma\eta-1)}{\sigma}\right) s_t = y_t - y_t^* - \frac{1-\upsilon}{\sigma} z_t - G_Y g_t \tag{A.158}$$

$$\left(\frac{1+\upsilon(2\sigma\eta-\upsilon\sigma\eta-2+\upsilon)}{\sigma}\right) s_t = y_t - y_t^* - \frac{1-\upsilon}{\sigma} z_t - G_Y g_t \tag{A.159}$$

$$\left(\frac{1+\upsilon(\varpi-1)}{\sigma}\right) s_t = y_t - y_t^* - \frac{1-\upsilon}{\sigma} z_t - G_Y g_t \tag{A.160}$$

$$\sigma^{-1}\Phi^{-1} s_t = y_t - y_t^* - \frac{1-\upsilon}{\sigma} z_t - G_Y g_t \tag{A.161}$$

$$s_t = \sigma_\upsilon(y_t - y_t^*) - (1-\upsilon)\Phi z_t - G_Y \sigma_\upsilon g_t \tag{A.162}$$

where $\varpi = \sigma\eta + (1-\upsilon)(\sigma\eta - 1)$, $\Phi = \frac{1}{1+\upsilon(\varpi-1)}$ and $\sigma_\upsilon = \sigma\Phi$.

Derivation of (a version of) dynamic IS equation

$$0 = \mathbb{E}\{\Delta c_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \frac{\upsilon}{\sigma}\,\mathbb{E}\{\Delta s_{t+1}\} + \frac{1}{\sigma}(1-\rho_z)z_t \tag{A.163}$$

$$\begin{aligned}0 &= \mathbb{E}\{\Delta y_{t+1}\} - \upsilon(2-\upsilon)\eta\,\mathbb{E}\{\Delta s_{t+1}\} - \upsilon\,\mathbb{E}\{\Delta y^*_{t+1}\} - G_Y\Delta g_{t+1} \\ &\quad - \frac{1-\upsilon}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \frac{(1-\upsilon)\upsilon}{\sigma}\,\mathbb{E}\{\Delta s_{t+1}\} + \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t\end{aligned} \tag{A.164}$$

$$\begin{aligned}0 &= \mathbb{E}\{\Delta y_{t+1}\} + \left(\frac{(1-\upsilon)\upsilon}{\sigma} - \upsilon(2-\upsilon)\eta\right)\mathbb{E}\{\Delta s_{t+1}\} - \upsilon\,\mathbb{E}\{\Delta y^*_{t+1}\} \\ &\quad - G_Y\Delta g_{t+1} - \frac{1-\upsilon}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t\end{aligned} \tag{A.165}$$

$$\begin{aligned}0 &= \mathbb{E}\{\Delta y_{t+1}\} - \left(\frac{\upsilon\varpi}{\sigma}\right)\mathbb{E}\{\Delta s_{t+1}\} - \upsilon\,\mathbb{E}\{\Delta y^*_{t+1}\} \\ &\quad - G_Y\Delta g_{t+1} - \frac{1-\upsilon}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t\end{aligned} \tag{A.166}$$

$$\begin{aligned}0 &= \mathbb{E}\{\Delta y_{t+1}\} - \left(\frac{\upsilon\varpi}{\sigma}\right)\mathbb{E}\left(\sigma_\upsilon(\Delta y_{t+1} - \Delta y^*_{t+1}) - (1-\upsilon)\Phi(\Delta z_{t+1}) - G_Y\sigma_\upsilon\Delta g_{t+1}\right) \\ &\quad - \upsilon\,\mathbb{E}\{\Delta y^*_{t+1}\} - \frac{1-\upsilon}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t\end{aligned} \tag{A.167}$$

$$\begin{aligned}0 &= \mathbb{E}\{\Delta y_{t+1}\} - \upsilon\varpi\Phi\Delta y_{t+1} + \upsilon\varpi\Phi\Delta y^*_{t+1} - \upsilon\,\mathbb{E}\{\Delta y^*_{t+1}\} + \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t \\ &\quad + \upsilon\varpi\Phi\frac{1-\upsilon}{\sigma}(\rho_z-1)z_t + \upsilon\varpi\Phi G_Y\Delta g_{t+1} - G_Y\Delta g_{t+1} \\ &\quad - \frac{1-\upsilon}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho)\end{aligned} \tag{A.168}$$

$$\begin{aligned}0 &= \Phi^{-1}\,\mathbb{E}\{\Delta y_{t+1}\} - \upsilon\varpi\Delta y_{t+1} + \upsilon\varpi\Delta y^*_{t+1} - \upsilon\Phi^{-1}\,\mathbb{E}\{\Delta y^*_{t+1}\} \\ &\quad + \Phi^{-1}\frac{1-\upsilon}{\sigma}(1-\rho_z)z_t + \upsilon\varpi\frac{1-\upsilon}{\sigma}(\rho_z-1)z_t + \upsilon\varpi G_Y\Delta g_{t+1} - G_Y\Delta g_{t+1} \\ &\quad - \frac{1-\upsilon}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho)\end{aligned} \tag{A.169}$$

$$\begin{aligned}0 &= (\Phi^{-1} - \upsilon\varpi)\Delta y_{t+1} + \upsilon(\varpi - \Phi^{-1})\,\mathbb{E}\{\Delta y^*_{t+1}\} \\ &\quad - (\Phi^{-1} - \upsilon\varpi)\frac{1-\upsilon}{\sigma}(\rho_z-1)z_t + (\upsilon\varpi - 1)G_Y\Delta g_{t+1} \\ &\quad - \frac{1-\upsilon}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho)\end{aligned} \tag{A.170}$$

$$\begin{aligned}0 &= \Delta y_{t+1} + \frac{\upsilon(\varpi - \Phi^{-1})}{\Phi^{-1} - \upsilon\varpi}\,\mathbb{E}\{\Delta y^*_{t+1}\} - \frac{1-\upsilon}{\sigma}(\rho_z-1)z_t + \frac{(\upsilon\varpi - 1)}{\Phi^{-1} - \upsilon\varpi}G_Y\Delta g_{t+1} \\ &\quad - \frac{1-\upsilon}{\Phi^{-1} - \upsilon\varpi}\frac{1}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho)\end{aligned} \tag{A.171}$$

$$
\begin{aligned}
y_t = {} & y_{t+1} - \frac{1-\upsilon}{\sigma}(\rho_z - 1)z_t + \frac{(\upsilon\varpi - 1)}{\Phi^{-1} - \upsilon\varpi} G_Y \Delta g_{t+1} + \frac{\upsilon(\varpi - \Phi^{-1})}{\Phi^{-1} - \upsilon\varpi}\,\mathbb{E}\{\Delta y^*_{t+1}\} \\
& - \frac{1-\upsilon}{\Phi^{-1} - \upsilon\varpi}\frac{1}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) && \text{(A.172)} \\
y_t = {} & y_{t+1} - \frac{1-\upsilon}{\sigma}(\rho_z - 1)z_t - \frac{1-\upsilon}{1-\upsilon}\frac{1}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) \\
& + \frac{(1-\upsilon)(\upsilon\varpi - \upsilon)}{1-\upsilon}\,\mathbb{E}\{\Delta y^*_{t+1}\} + \frac{\upsilon\varpi - 1}{1-\upsilon} G_Y \Delta g_{t+1} && \text{(A.173)} \\
y_t = {} & y_{t+1} - \frac{1}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y^*_{t+1}\} \\
& + \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t + \frac{\upsilon\varpi - 1}{1-\upsilon} G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\} && \text{(A.174)}
\end{aligned}
$$

Derivation of (final) Dynamic IS

$$y_t^n = y_{t+1}^n - \frac{1}{\sigma_\upsilon}(r_t^n - \rho) + \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} + \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t + \frac{\upsilon\varpi - 1}{1-\upsilon}G_Y \Delta g_{t+1} \tag{A.175}$$

$$0 = \Delta y_{t+1}^n - \frac{1}{\sigma_\upsilon}(r_t^n - \rho) + \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} + \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t + \frac{\upsilon\varpi - 1}{1-\upsilon}G_Y \Delta g_{t+1} \tag{A.176}$$

$$0 = \sigma_\upsilon \Delta y_{t+1}^n - (r_t^n - \rho) + \sigma_\upsilon \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} + \Phi(1-\upsilon)(1-\rho_z)z_t + \sigma_\upsilon \frac{\upsilon\varpi - 1}{1-\upsilon}G_Y \Delta g_{t+1} \tag{A.177}$$

$$r_t^n = \sigma_\upsilon \Delta y_{t+1}^n + \rho + \sigma_\upsilon \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} + \Phi(1-\upsilon)(1-\rho_z)z_t + \sigma_\upsilon \frac{\upsilon\varpi - 1}{1-\upsilon}G_Y \Delta g_{t+1} \tag{A.178}$$

Subtract (A.175) from (105) to yield:

$$\begin{aligned}
0 &= \sigma_\upsilon \Delta y_{t+1} - (i_t - \mathbb{E}\{\pi_{H,+1}\}) + \rho + \sigma_\upsilon \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} \\
&\quad + \Phi(1-\upsilon)(1-\rho_z)z_t + \sigma_\upsilon \frac{\upsilon\varpi - 1}{1-\upsilon}G_Y \Delta g_{t+1} \\
&\quad - (\sigma_\upsilon \Delta y_{t+1}^n - (r_t^n - \rho) + \sigma_\upsilon \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} \\
&\quad + \Phi(1-\upsilon)(1-\rho_z)z_t + \sigma_\upsilon \frac{\upsilon\varpi - 1}{1-\upsilon}G_Y\,\mathbb{E}_t\{\Delta g_{t+1}\}) && \text{(A.179)} \\
0 &= \sigma_\upsilon \Delta \tilde{y}_{t+1} - (i_t - \mathbb{E}\{\pi_{H,+1}\} - r_t^n) && \text{(A.180)} \\
\tilde{y}_t &= \tilde{y}_{t+1} - \frac{1}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - r_t^n) && \text{(A.181)}
\end{aligned}$$

Derivation of the average markup

We will make use of the following definitions

$$y_t = a_t + (1-\alpha)n_t \tag{A.182}$$
$$n_t = \frac{1}{1-\alpha}(y_t - a_t) \tag{A.183}$$
$$\varpi = \sigma\eta + (1-\upsilon)(\sigma\eta - 1) \tag{A.184}$$
$$\sigma c_t + \varphi n_t = w_t - p_t - \frac{\tau}{1-\tau}\tau_t \tag{A.185}$$
$$y_t = (1-\upsilon)c_t + \upsilon_t(2-\upsilon)\eta s_t + \upsilon y_t^* \tag{A.186}$$
$$c_t = y_t^* + \frac{1}{\sigma}z_t + \frac{1-\upsilon}{\sigma}s_t \tag{A.187}$$
$$\implies c_t = y_t - \upsilon s_t\left((2-\upsilon)\eta + \frac{1-\upsilon}{\sigma}\right) + \frac{\upsilon}{\sigma}z_t \tag{A.188}$$

$$\mu_t = p_{H,t} - \psi_t \tag{A.189}$$
$$= -(w_t - p_t) - (p_t - p_{H,t}) + a_t - \alpha n_t \tag{A.190}$$
$$= -(\sigma c_t + \varphi n_t + \frac{\tau}{1-\tau}\tau_t) - \upsilon s_t + a_t - \alpha n_t \tag{A.191}$$
$$= -\sigma c_t - \varphi n_t - \frac{\tau}{1-\tau}\tau_t - \upsilon s_t + a_t - \alpha n_t \tag{A.192}$$
$$= -\sigma c_t - \frac{\tau}{1-\tau}\tau_t - \upsilon s_t + a_t - n_t(\varphi + \alpha) \tag{A.193}$$

where (A.190) adds and subtracts p_t. Furthermore, expand $-n_t(\varphi+\alpha)$:

$$-n_t(\varphi+\alpha) = -\frac{1}{1-\alpha}(y_t - a_t)(\varphi+\alpha) \tag{A.194}$$
$$-n_t(\varphi+\alpha) = -\frac{\varphi+\alpha}{1-\alpha}(y_t - a_t) \tag{A.195}$$
$$-n_t(\varphi+\alpha) = -\frac{\varphi+\alpha}{1-\alpha}y_t + \frac{\varphi+\alpha}{1-\alpha}a_t \tag{A.196}$$

Substitute and rearrange:

$$\mu_t = -\sigma c_t - \frac{\tau}{1-\tau}\tau_t - \upsilon s_t + a_t - \frac{\varphi+\alpha}{1-\alpha}y_t + \frac{\varphi+\alpha}{1-\alpha}a_t \tag{A.197}$$

$$\mu_t = -\sigma c_t - \frac{\tau}{1-\tau}\tau_t - \upsilon s_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \frac{\varphi+\alpha}{1-\alpha}y_t \tag{A.198}$$

$$\mu_t = -\sigma\left(y_t - \upsilon s_t\left((2-\upsilon)\eta + \frac{1-\upsilon}{\sigma}\right) + \frac{\upsilon}{\sigma}z_t\right) - \frac{\tau}{1-\tau}\tau_t - \upsilon s_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \frac{\varphi+\alpha}{1-\alpha}y_t \tag{A.199}$$

$$\mu_t = -\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)y_t + \sigma\upsilon s_t\left((2-\upsilon)\eta + \frac{1-\upsilon}{\sigma}\right) - \upsilon z_t - \frac{\tau}{1-\tau}\tau_t - \upsilon s_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t \tag{A.200}$$

$$\mu_t = -\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)y_t + \upsilon s_t\left(\sigma\left((2-\upsilon)\eta + \frac{1-\upsilon}{\sigma}\right) - 1\right) - \frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t \tag{A.201}$$

$$\mu_t = -\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)y_t + \upsilon s_t\left((2-\upsilon)\sigma\eta + 1 - \upsilon\right) - 1 - \frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t \tag{A.202}$$

$$\mu_t = -\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)y_t + \upsilon s_t\left((2-\upsilon)\sigma\eta - \upsilon\right) - \frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t \tag{A.203}$$

$$\mu_t = -\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)y_t + \upsilon s_t\left(2\sigma\eta - \upsilon\sigma\eta - \upsilon\right) - \frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t \tag{A.204}$$

$$\mu_t = -\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)y_t + \upsilon(\varpi - 1)s_t - \frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t \tag{A.205}$$

Derivation of the natural level of output

$$\mu_t = -\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n + \upsilon(\varpi - 1)s_t^n - \frac{\tau}{1-\tau}\tau_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right) a_t - \upsilon z_t \quad \text{(A.206)}$$

$$\mu_t - \left(\upsilon(\varpi - 1)s_t^n - \frac{\tau}{1-\tau}\tau_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right) a_t - \upsilon z_t\right) = -\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n \quad \text{(A.207)}$$

$$\upsilon(\varpi - 1)s_t^n - \frac{\tau}{1-\tau}\tau_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right) a_t - \upsilon z_t - \mu_t = \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n \quad \text{(A.208)}$$

$$\upsilon(\varpi - 1)s_t^n - \frac{\tau}{1-\tau}\tau_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right) a_t - \upsilon z_t - \mu_t = \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n \quad \text{(A.209)}$$

$$\upsilon(\varpi - 1)(\sigma_\upsilon(y_t^n - y_t^*) - (1-\upsilon)\Phi z_t - G_Y \sigma_\upsilon g_t) - \frac{\tau}{1-\tau}\tau_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right) a_t$$
$$- \upsilon z_t - \mu_t = \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n \quad \text{(A.210)}$$

$$(-1 + 1 + \upsilon(\varpi - 1))(\sigma_\upsilon(y_t^n - y_t^*) - (1-\upsilon)\Phi z_t - G_Y \sigma_\upsilon g_t) - \frac{\tau}{1-\tau}\tau_t$$
$$+ \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right) a_t - \upsilon z_t - \mu_t = \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n \quad \text{(A.211)}$$

$$(-1 + \Phi^{-1})(\sigma_\upsilon(y_t^n - y_t^*) - (1-\upsilon)\Phi z_t - G_Y \sigma_\upsilon g_t) - \frac{\tau}{1-\tau}\tau_t$$
$$+ \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right) a_t - \upsilon z_t - \mu_t = \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n \quad \text{(A.212)}$$

$$(-1 + \Phi^{-1})(\sigma_\upsilon y_t^n - \sigma_\upsilon y_t^* - (1-\upsilon)\Phi z_t - G_Y \sigma_\upsilon g_t) - \frac{\tau}{1-\tau}\tau_t$$
$$+ \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right) a_t - \upsilon z_t - \mu_t = \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n \quad \text{(A.213)}$$

$$(-1 + \Phi^{-1})\sigma_\upsilon y_t^n + (1 - \Phi^{-1})\sigma_\upsilon y_t^* + (1 - \Phi^{-1})(1-\upsilon)\Phi z_t + (1 - \Phi^{-1}) G_Y \sigma_\upsilon g_t$$
$$- \frac{\tau}{1-\tau}\tau_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right) a_t - \upsilon z_t - \mu_t = \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n \quad \text{(A.214)}$$

$$(-1 + \Phi^{-1})\sigma\Phi y_t^n + (1 - \Phi^{-1})\sigma_\upsilon y_t^* + (1 - \Phi^{-1})(1-\upsilon)\Phi z_t + (1 - \Phi^{-1}) G_Y \sigma_\upsilon g_t$$
$$- \frac{\tau}{1-\tau}\tau_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right) a_t - \upsilon z_t - \mu_t = \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n \quad \text{(A.215)}$$

$$- \sigma\Phi y_t^n + \sigma y_t^n + (1 - \Phi^{-1})\sigma_\upsilon y_t^* + (1 - \Phi^{-1})(1-\upsilon)\Phi z_t + (1 - \Phi^{-1}) G_Y \sigma_\upsilon g_t$$
$$- \frac{\tau}{1-\tau}\tau_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right) a_t - \upsilon z_t - \mu_t = \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) y_t^n \quad \text{(A.216)}$$

$$(1-\Phi)\sigma y_t^n + (1-\Phi^{-1})\sigma_\upsilon y_t^* + (1-\Phi^{-1})(1-\upsilon)\Phi z_t + (1-\Phi^{-1})G_Y\sigma_\upsilon g_t$$
$$-\frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t - \mu_t = \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)y_t^n \tag{A.217}$$

$$(1-\Phi^{-1})\sigma_\upsilon y_t^* + (1-\Phi^{-1})(1-\upsilon)\Phi z_t + (1-\Phi^{-1})G_Y\sigma_\upsilon g_t$$
$$-\frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t - \mu_t = \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)y_t^n - (1-\Phi)\sigma y_t^n \tag{A.218}$$

$$(1-\Phi^{-1})\sigma_\upsilon y_t^* + (1-\Phi^{-1})(1-\upsilon)\Phi z_t + (1-\Phi^{-1})G_Y\sigma_\upsilon g_t$$
$$-\frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t - \mu_t = \left(\frac{\sigma(1-\alpha)+\varphi+\alpha-(1-\Phi)\sigma(1-\alpha)}{1-\alpha}\right)y_t^n \tag{A.219}$$

$$(1-\Phi^{-1})\sigma_\upsilon y_t^* + (1-\Phi^{-1})(1-\upsilon)\Phi z_t + (1-\Phi^{-1})G_Y\sigma_\upsilon g_t$$
$$-\frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t - \mu_t = \left(\frac{\sigma(1-\alpha)+\varphi+\alpha-(\sigma(1-\alpha)-\sigma\Phi(1-\alpha))}{1-\alpha}\right)y_t^n \tag{A.220}$$

$$(1-\Phi^{-1})\sigma_\upsilon y_t^* + (1-\Phi^{-1})(1-\upsilon)\Phi z_t + (1-\Phi^{-1})G_Y\sigma_\upsilon g_t$$
$$-\frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t - \mu_t = \left(\frac{-\sigma_\upsilon(1-\alpha)+\varphi+\alpha}{1-\alpha}\right)y_t^n \tag{A.221}$$

$$\left(\frac{-\sigma_\upsilon(1-\alpha)+\varphi+\alpha}{1-\alpha}\right)^{-1}\times$$
$$\left((1-\Phi^{-1})\sigma_\upsilon y_t^* + (1-\Phi^{-1})(1-\upsilon)\Phi z_t - \frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t - \mu_t\right) = y_t^n \tag{A.222}$$

$$\Gamma_* y_t^* + \left(\frac{-\sigma_\upsilon(1-\alpha)+\varphi+\alpha}{1-\alpha}\right)^{-1}\times$$
$$\left((1-\Phi^{-1})(1-\upsilon)\Phi z_t + (1-\Phi^{-1})G_Y\sigma_\upsilon g_t - \frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t - \mu_t\right) = y_t^n \tag{A.223}$$

$$\Gamma_* y_t^* + \left(\frac{-\sigma_\upsilon(1-\alpha)+\varphi+\alpha}{1-\alpha}\right)^{-1}\times$$
$$\left(\upsilon\varpi\Phi z_t + (1-\Phi^{-1})G_Y\sigma_\upsilon g_t - \frac{\tau}{1-\tau}\tau_t + \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t - \mu_t\right) = y_t^n \tag{A.224}$$

$$\Gamma_* y_t^* + \Gamma_z z_t + \Gamma_a a_t + \left(\frac{-\sigma_\upsilon(1-\alpha)+\varphi+\alpha}{1-\alpha}\right)^{-1}\left((1-\Phi^{-1})G_Y\sigma_\upsilon g_t - \frac{\tau}{1-\tau}\tau_t - \mu_t\right) = y_t^n \tag{A.225}$$

$$\Gamma_* y_t^* + \Gamma_z z_t + \Gamma_a a_t + \Gamma_g g_t + \left(\frac{-\sigma_\upsilon(1-\alpha) + \varphi + \alpha}{1-\alpha}\right)^{-1}\left(-\frac{\tau}{1-\tau}\tau_t\right) = y_t^n \quad \text{(A.226)}$$

$$\Gamma_* y_t^* + \Gamma_z z_t + \Gamma_a a_t + \Gamma_g g_t + \Gamma_\tau \tau_t = y_t^n \quad \text{(A.227)}$$

$$y_t^n = \Gamma_* y_t^* + \Gamma_z z_t + \Gamma_a a_t + \Gamma_\tau \tau_t + \Gamma_g g_t \quad \text{(A.228)}$$

where:

$$\Gamma_* = -\frac{\upsilon(\varpi - 1)\sigma_\upsilon(1-\alpha)}{\sigma_\upsilon(1-\alpha) + \varphi + \alpha} \quad \text{(A.229)}$$

$$\Gamma_z = -\frac{\upsilon\varpi\Phi(1-\alpha)}{\sigma_\upsilon(1-\alpha) + \varphi + \alpha} \quad \text{(A.230)}$$

$$\Gamma_a = \frac{1+\varpi}{\sigma_\upsilon(1-\alpha) + \varphi + \alpha} \quad \text{(A.231)}$$

$$\Gamma_g = -\frac{\upsilon(\varpi - 1)\sigma_\upsilon(1-\alpha)}{\sigma_\upsilon(1-\alpha) + \varphi + \alpha} \times \quad \text{(A.232)}$$

$$\Gamma_\tau = \frac{(1-\tau)(1-\alpha)}{\sigma_\upsilon(1-\alpha) + \varphi + \alpha} \quad \text{(A.233)}$$

$$\pi_{H,t} = \beta\,\mathbb{E}\{\pi_{H,t+1}\} + \lambda\hat{\mu}_t \implies \pi_{H,t} = \beta\,\mathbb{E}\{\pi_{H,t+1}\} - \lambda\mu_t \tag{A.234}$$

$$\pi_{H,t} = \beta\,\mathbb{E}\{\pi_{H,t+1}\} + \lambda\left(\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)y_t - \upsilon(\varpi-1)s_t + \frac{\tau}{1-\tau}\tau_t - \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right)a_t + \upsilon z_t\right) \tag{A.235}$$

$$\pi_{H,t} = \beta\,\mathbb{E}\{\pi_{H,t+1}\} + \kappa y_t + \lambda\left(-\upsilon(\varpi-1)s_t + \frac{\tau}{1-\tau}\tau_t - \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right)a_t + \upsilon z_t\right) \tag{A.236}$$

$$\pi_{H,t} = \beta\,\mathbb{E}\{\pi_{H,t+1}\} + \kappa y_t - \lambda\upsilon(\varpi-1)s_t + \lambda\frac{\tau}{1-\tau}\tau_t - \lambda\left(1 + \frac{\varphi+\alpha}{1-\alpha}\right)a_t + \lambda\upsilon z_t \tag{A.237}$$

$$\tag{A.238}$$

$$c_t = \mathbb{E}\{c_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}\{\pi_{t+1}\} - \rho) + \frac{1}{\sigma}(1-\rho_z)z_t \tag{A.239}$$

$$\pi_t = \pi_{H,t} + \upsilon \Delta s_t \tag{A.240}$$

$$\implies c_t = \mathbb{E}\{c_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}\{\pi_{H,t+1}\} - \rho) + \frac{\upsilon}{\sigma}\mathbb{E}\{\Delta s_{t+1}\} + \frac{1}{\sigma}(1-\rho_z)z_t \tag{A.241}$$

$$c_t = y_t^* + \frac{1}{\sigma}z_t + \left(\frac{1-\upsilon}{\sigma}\right)s_t \tag{A.242}$$

$$Y_t(i) = C(i)_t + X(i)_t + G(i)_t \tag{A.243}$$

$$= \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\epsilon}\left[(1-\upsilon)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \upsilon \mathcal{S}_t^{\eta} Y_t^* + G_t\right] \tag{A.244}$$

Given that

$$Y_t = \left(\int_0^1 (Y_t(i))^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{A.245}$$

$$y_t = (1-\upsilon)c_t + \upsilon(2-\upsilon)\eta s_t + \upsilon y_t^* + G_Y g_t \tag{A.246}$$

$$\implies c_t = (1-\upsilon)^{-1}(y_t - \upsilon(2-\upsilon)\eta s_t - \upsilon y_t^* - G_Y g_t) \tag{A.247}$$

$$c_t = \mathbb{E}\{c_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}\{\pi_{t+1}\} - \rho) + \frac{\upsilon}{\sigma}\mathbb{E}\{\Delta s_{t+1}\} + \frac{1}{\sigma}(1-\rho_z)z_t \tag{A.248}$$

$$s_t = \sigma_\upsilon(y_t - y_t^*) - (1-\upsilon)\Phi z_t - G_Y \sigma_\upsilon g_t \tag{A.249}$$

$$\tag{A.250}$$

natural rate of interest

$$r_t^n = \sigma_\upsilon \Delta y_{t+1}^n + \rho + \sigma_\upsilon \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} + \Phi(1-\upsilon)(1-\rho_z)z_t + \sigma_\upsilon \frac{\upsilon\varpi - 1}{1-\upsilon} G_Y \Delta g_{t+1} \tag{A.251}$$

$$\sigma_\upsilon \Delta y_{t+1}^n = \sigma_\upsilon(\Gamma_* \Delta y_{t+1}^* + \Gamma_z \Delta z_{t+1} + \Gamma_a \Delta a_{t+1} + \Gamma_g \Delta g_{t+1} + \Gamma_\tau \Delta \tau_{t+1}) \tag{A.252}$$

$$y_t^* : \sigma_\upsilon \Gamma_* \Delta y_{t+1}^* + \sigma_\upsilon \upsilon(\varpi - 1)\Delta y_{t+1}^* = \sigma_\upsilon(\Gamma_* + \upsilon(\varpi - 1))\Delta y_{t+1}^* = \Psi_* y_{t+1}^* \tag{A.253}$$

$$z_t : \sigma_\upsilon \Gamma_z \Delta z_{t+1} + \Phi(1-\rho_z)z_t = \sigma_\upsilon \Gamma_z(\rho_z - 1)z_t + \Phi(1-\upsilon)(1-\rho_z)z_t \tag{A.254}$$

$$= \sigma_\upsilon \Gamma_z(\rho_z - 1)z_t + \Phi(1-\upsilon)(1-\rho_z)z_t = (\Phi(1-\upsilon) - \sigma_\upsilon \Gamma_z)(1-\rho_z)z_t = \Psi_z(1-\rho_z)z_t \tag{A.255}$$

$$a_t : \sigma_\upsilon \Gamma_a \Delta a_{t+1} = \sigma_\upsilon \Gamma_a(\rho_a - 1)a_t = -\sigma_\upsilon \Gamma_a(1-\rho_a)a_t \tag{A.256}$$

$$g_t : \sigma_\upsilon \Gamma_g G_Y \Delta g_{t+1} + \sigma_\upsilon \frac{\upsilon\varpi - 1}{1-\upsilon} G_Y \Delta g_{t+1} = \left(\sigma_\upsilon \left(\Gamma_g + \frac{\upsilon\varpi - 1}{1-\upsilon}\right)\right) G_Y \Delta g_{t+1} = -\Psi_g G_Y(1 - \tag{A.257}$$

$$\tau_t : \sigma_\upsilon \Gamma_\tau \Delta \tau_{t+1} \tag{A.258}$$

$$r_t^n = \rho + \Psi_* y_{t+1}^* + \Psi_z(1-\rho_z)z_t - \sigma_\upsilon \Gamma_a(1-\rho_a)a_t - \Psi_g G_Y(1-\rho_g)g_t + \sigma_\upsilon \Gamma_\tau \Delta \tau_{t+1} \tag{A.259}$$

υ v