Assessing Asymmetrical Effects of Government Spending

A Two-Country DSGE Model for Scotland and the Rest of the UK

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1. Introduction: **1000 words**
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1 Introduction

1.1 Literature Review

On the 10th of May, 2023, the Monetary Policy Committee at the Bank of England gathered to discuss the latest international and domestic data on economic activity. Even though the Committee has a 2% CPI target, the UK's economy had undergone a sequence of very large and unexpected shocks and disturbances, resulting in twelve-month CPI inflation above 10%. The majority of the Committee members (78%) believed that an increase in interest rate "was warranted" (BoE, 2023: 4), while the remaining members believed that the CPI inflation will "fall sharply in 2023" (BoE, 2023: 5) as a result of the economy naturally adjusting to the effects of the energy price shocks. They feared that the preceding increases in the interest rate have not yet been internalised and raising the interest rate any further could result in a reduction of inflation "well below the target" (BoE, 2023: 5). This is an excellent illustration of the "informal dimension of the monetary policy process", that (Galsí and Gertler, 2007: 26) referred to in their work explaining modern macro models and new frameworks. According to them, while the informal dimension cannot be removed, we can build formal and rigorous models that would help the Committee and institutions-alike to understand "objectives of the monetary policy and how the latter should be conducted in order to attain those objectives" (Galí, 2015: 2). This task is not straightforward and has been central (albeit - fruitful) to most macroeconomic research in the past decades. The following section of the literature review will present a brief evolution of the study of business cycles and monetary theory. It will be followed by an overview of large macroeconomic models adopted by central banks and international organisations to illustrate the relevance of this research. The final part of the literature review will discuss {NIESR policy question, also Scotland-UK deltas} and the latest research on the issue.

Blanchard (2000) offers a compact description of macroeconomic research in the twentieth century. In their panegyric and optimistic essay, the researcher argues that the century can be divided into three epochs based on the prevailing beliefs about the economy and frameworks of the time: Pre-1940, From 1940 to 1980, and Post-1980.

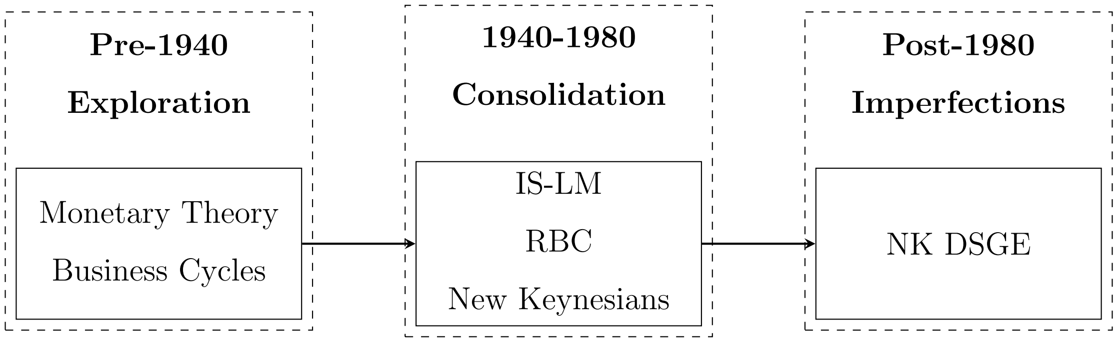


Figure 1: Timeline of macroeconomic research, according to Blanchard (2000). Authors own illustration.

According to them, the pre-1940 epoch was the epoch of exploration, with economists primarily concerned with the Monetary Theory ("Why does money affect output?") and Business Cycles ("What are the major shocks that affect output?"). Even though both of those puzzles fall under the study of "macroeconomics", the term did not appear in the literature until the mid-1940s (Blanchard, 2000). The Monetary Theory ideas of that time did not differ drastically from what is believed today, e.g. short-run money non-neutrality and long-run neutrality, but the economic models were "incomplete and partial equilibrium in nature" (Blanchard, 2000: 1377). The business cycles were attributed to "real factors", such as technological innovations (Blanchard, 2000), and this belief persisted until more data became available and more sophisticated time series methods were applied in the early 2000s (Galí, 2015: 3). The subsequent epoch was "the golden age of macroeconomics" (Blanchard, 2000: 1379). Hicks (1937) formalised the *IS-LM* framework TBC

In the 1980s, Kydland and Prescott (1982) and Prescott (1986) published seminal papers on the Real Business Cycles (RBC) theory. According to Galí (2015: 2), frameworks presented in the papers "provided the main reference" and firmly established the use of dynamic stochastic general equilibrium (DSGE) models as crucial tools for macroeconomic analysis. The models allow quantitative analysis and incorporation of data either via calibration or estimation of parameters. TBC

International Monetary Fund (IMF)

2 Theoretical DSGE model

Ricci (2019) was the first to build a large-scale two-country DSGE model explicitly tailored to Scotland and the rest of the UK. In an attempt to retain the model's simplicity while still allowing policy analysis, this dissertation will primarily build on the work of Gali and Monacelli (2005) and Galí (2015). In contrast, Ricci (2019) model was based on the work of Rabanal and Tuesta (2010), who were among the first to build a medium-to-large two-country DSGE model. Neither Gali and Monacelli (2005) nor Galí (2015) models considered lump-sum or distortionary taxes, or government spending, more generally. While extensive literature covers government spending in DSGE models, few to none cover government spending in a small open economy (SOE) NK DSGE model, and even fewer apply it to a two-country setting. Therefore, most of the derivations had to be carried out using a pen and paper, and step-by-step derivations are provided in the Appendix.

Moreover, the focus of this dissertation is not to build the most factually accurate model of Scotland or the United Kingdom but to assess the asymmetric responses in government spending under factual and counterfactual policy scenarios. The factual scenario refers to the Westminister government collecting taxes from all four countries of the UK and distributing them according to the Barnett formula. The counterfactual scenario refers to the Holyrood government's ability to collect tax revenue, issue bonds (borrow), and spend it at its sole discretion. We further break down the scenarios by allowing public expenses to be funded by lump-sum and distortionary (labour) taxes. This brings the number of policy scenarios considered by the dissertation to four.

Finally, in line with most of the literature, variables referring to the home country (Scotland) will be denoted without an asterisk, i.e., Y_t, while foreign country (the Rest of the UK or **rUK**) will be denoted with an asterisk, i.e., Y_t^*. Population-weighted sums of these variables will be referred to as UK-wide variables and denoted as Y_t^{UK}.

2.1 Households

This model assumes that there is infinitely many households in the economy represented by a unit interval. All households are assumed to be be symmetric, i.e. have the same preferences and behave identically. Below, we consider a representative household that wants to maximise their lifetime utility, represented by Equation (1):

$$\mathbb{E}_t \left\{ \sum_{t=0}^{\infty} \beta^t\, \mathcal{U}(C_t, N_t, Z_t) \right\} \tag{1}$$

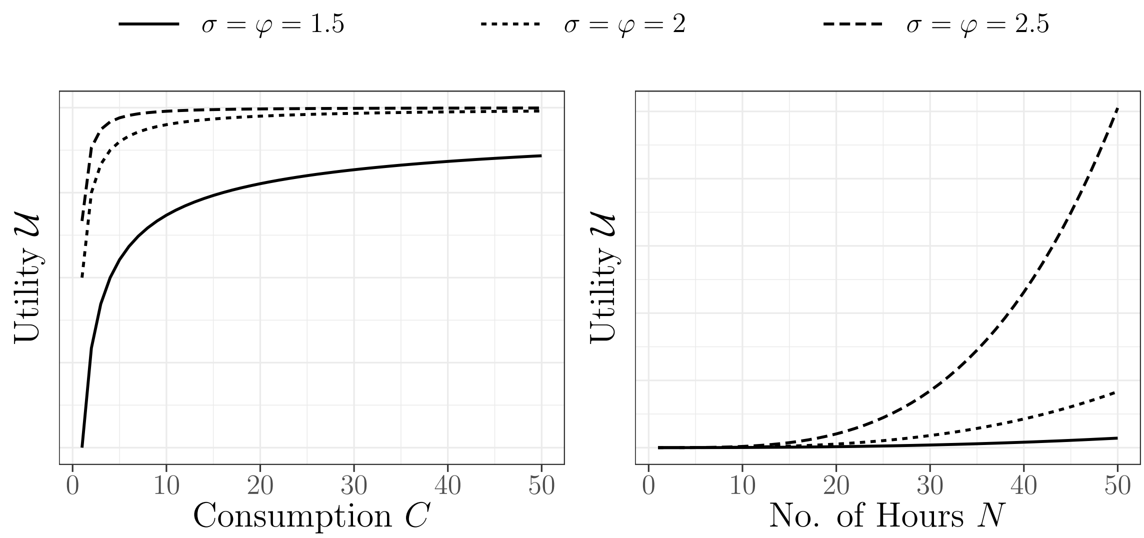
$$\mathcal{U}(C_t, N_t, Z_t) = \begin{cases} \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right) Z_t & \text{if } \sigma \geq 0 \text{ and } \sigma \neq 1 \\ \left(\log(C_t) - \frac{N_t^{1+\varphi}}{1+\varphi} \right) Z_t & \text{if } \sigma = 1 \end{cases} \tag{2}$$

The household's utility depends on consumption C_t and hours worked N_t. As seen from the utility function (Equation (2)), the model assumes the household's utility to be (decreasingly) increasing in consumption C_t and (increasingly) decreasing in hours worked N_t. $\beta \in (0, 1)$ is the discount factor, which can be thought of as an opportunity cost or an impatience rate, i.e. a unit of consumption C today will be worth $\beta {*} C < C$ tomorrow. We also introduce a preference shifter Z_t (Galí, 2015: 225) [1]. The shock is assumed to follow an autoregressive process of order 1:

$$\log(Z_t) = \rho_z \log(Z_{t-1}) + \epsilon_t^z \tag{3}$$

The parameter $\sigma \geq 0$ is the relative risk aversion coefficient and $\varphi \geq 0$ is the labour disutility parameter. Together, they determine the curvature of the utility of consumption and disutility of labour, respectively. Finally, $\mathbb{E}_t\left[*\right]$ is the expectational operator, conditional on all information available at period t (Gali, 2015: 20).

[1] While this specific shock is not relevant to the research question, it helps prevent stochastic singularity Pfeifer, 2021 and allows parameter estimation with a greater number of macroeconomic data series, see Section 3.



To allow goods differentiation between domestic and foreign, the model assumes that C_t is a composite consumption index defined by:

$$C_t = \begin{cases} \left[(1-\upsilon)^{\frac{1}{\eta}}(C_{H,t})^{\frac{\eta-1}{\eta}} + \upsilon^{\frac{1}{\eta}}(C_{F,t})^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} & \text{if } \eta > 0 \text{ and } \eta \neq 1 \\ \frac{1}{(1-\upsilon)^{(1-\upsilon)}\upsilon^{\upsilon}}(C_{H,t})^{(1-\upsilon)}(C_{F,t})^{\upsilon} & \text{if } \eta = 1 \end{cases} \tag{4}$$

Where $C_{H,t}$ and $C_{F,t}$ are indices of consumption of home produced and imported goods, respectively. The parameter $\upsilon \in [0,1]$ reflects economy's openness for trading, while $\eta > 0$ denotes household's willingness to substitute a domestic good with a foreign good, often referred to as 'home bias'. When $\eta = 1$, then the share of domestic and foreign consumption is determined by the country's willingness to trade. In an extreme case, $\upsilon = 0$ would imply that the economy is an autarky, while $\upsilon = 1$ would suggest that our households consume foreign goods only. Our economy is assumed to be small, in the sense that it takes the world output, consumption, and prices as given, and cannot influence them. This is a common assumption for the UK (**refs**) and even more so for Scotland. The world economy is assumed to be made of a continuum of infinitely many small economies i represented by a unit interval. Therefore, $C_{F,t}$ is a sum of indices of the quantity of goods imported from all countries i. In a similar fashion, if we denote j as a single variety of goods from a continuum of goods represented by a unit interval, we can express each consumption index as follows:

$$C_{H,t} = \left(\int_0^1 C_{H,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} \, dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \qquad \text{Index of consumption of home produced goods}$$

$$C_{i,t} = \left(\int_0^1 C_{i,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} \, dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \qquad \text{Index of consumption of country } i\text{'s produced goods}$$

$$C_{F,t} = \left(\int_0^1 C_{i,t}^{\frac{\gamma-1}{\gamma}} \, di \right)^{\frac{\gamma}{\gamma-1}} \qquad \text{Index of consumption of imported goods}$$

Notice that all three indices take the form of *Constant Elasticity Substitution* (*CES*) form, with parameters ε (without subscripts) and γ representing the degree of substitutability between varieties of goods and countries, respectively. The following expresions note optimal allocation of each individual good (see Appendix A1 for derivation):

$$C_{H,t}(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}} \right)^{-\varepsilon} C_{H,t}; \quad C_{i,t}(j) = \left(\frac{P_{i,t}(j)}{P_{i,t}} \right)^{-\varepsilon} C_{i,t}; \quad C_{i,t} = \left(\frac{P_{i,t}}{P_{F,t}} \right)^{-\gamma} C_{F,t} \quad (5)$$

where:

$$P_{H,t} = \left(\int_0^1 P_{H,t}(j)^{1-\varepsilon} \, dj \right)^{\frac{1}{1-\varepsilon}} \qquad \text{Domestic Price Index} \quad (6)$$

$$P_{i,t} = \left(\int_0^1 P_{i,t}(j)^{1-\varepsilon} \, dj \right)^{\frac{1}{1-\varepsilon}} \qquad \text{Price Index of goods produced by country } i \quad (7)$$

$$P_{F,t} = \left(\int_0^1 P_{i,t}^{1-\gamma} \, dj \right)^{\frac{1}{1-\gamma}} \qquad \text{Price Index of Imported goods} \quad (8)$$

Intuitively, if $P_{H,t}(j) > P_{H,t}$, then that good is demanded less relative to an *average* good. To see this, note that $P_{H,t}(j)/P_{H,t} > 1$ when $P_{H,t}(j) > P_{H,t}$, and given that the term is to the power of a negative constant, the entire term decreases.

The representative household's choice of consumption and labour must satisfy the following budget constraint:

$$\underbrace{\int_0^1 P_{H,t}(j)C_{H,t}(j)\,dj + \int_0^1\int_0^1 P_{i,t}(j)C_{i,t}(j)\,dj\,di + \mathbb{E}_t\left[R_{t+1}^{-1}B_{t+1}\right]}_{Expenses} \leq \underbrace{B_t + W_tN_t}_{Income} \quad (9)$$

where R_t is the gross nominal interest rate, B_t denotes bonds, W_t and N_t stand for nominal wage and hours worked, respectively. For intuition, the LHS of the budget constraint implies that the representative household needs to choose the quantity of good j produced domestically and in every country i, as well as the number of bonds at the expected nominal interest rate in period $t+1$. The RHS implies that the only two sources of income are nominal payoffs from bonds and gross pay, which later will be different from the same as net pay. The expenses cannot exceed income.

Taking the three price indices (6)-(8), and plugging them into their respective demand functions (5), yields:

$$\int_0^1 P_{H,t}(j)C_{H,t}(j)\,dj = P_{H,t}C_{H,t} \quad (10)$$

$$\int_0^1 P_{i,t}(j)C_{i,t}(j)\,dj = P_{i,t}C_{i,t} \quad (11)$$

$$\int_0^1 P_{i,t}C_{i,t} = P_{F,t}C_{F,t} \quad (12)$$

The following definitions are given:

$$C_{H,t} = (1-\upsilon)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t \quad (13)$$

$$C_{F,t} = \upsilon\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t \quad (14)$$

$$P_t = \begin{cases} [(1-\upsilon)(P_{H,t})^{1-\eta} + \upsilon(P_{F,t})^{1-\eta}]^{\frac{1}{1-\eta}} & \text{if } \eta > 0 \text{ and } \eta \neq 1 \\ (P_{H,t})^{1-\upsilon} \times (P_{F,t})^{\upsilon} & \text{if } \eta = 1 \end{cases} \quad (15)$$

Equations (13) and (14) are demand functions for domestic and foreign goods, respectively. Equation (15) is the Consumption Price Index (CPI). In the case when there is no home bias ($\eta = 1$), the log aggregate price level in the consumption is just a weighted sum of the two price indices, where weights are given by trade openness parameter υ. Using (10) and (12), we can define the total consumption expenditures as:

$$P_{H,t}C_{H,t} + P_{F,t}C_{F,t} = P_tC_t \tag{16}$$

Which greatly simplifies the household's budget constraint:

$$P_tC_t + \mathbb{E}_t\left[R_{t+1}^{-1}B_{t+1}\right] \leq B_t + W_tN_t \tag{17}$$

Note that the budget constraint (as well as many other expressions introduced later) will vary depending on what policy scenario is considered. For instance, the household's budget constraints under each scenario is given below:

Scen. 1	Scot.	$P_tC_t + \mathbb{E}_t[R_{t+1}^{-1}B_{t+1}] = B_t + W_tN_t + T_t$
G: 2, τ : 0	rUK	$P_t^*C_t^* + \mathbb{E}_t[R_{t+1}^{*-1}B_{t+1}^*] = B_t^* + W_t^*N_t^* + T_t^*$
Scen. 2	Scot.	$P_tC_t + \mathbb{E}_t[R_{t+1}^{UK-1}B_{t+1}^{UK}] = B_t^{UK} + W_tN_t + \varpi T_t^{UK}$
G: 1, τ : 0	rUK	$P_t^*C_t^* + \mathbb{E}_t[R_{t+1}^{UK-1}B_{t+1}^{UK}] = B_t^{UK} + W_t^*N_t^* + (1-\varpi)T_t^{UK}$
Scen. 3	Scot.	$P_tC_t + \mathbb{E}_t[R_{t+1}^{-1}B_{t+1}] = B_t + (1-\tau_t)W_tN_t + T_t$
G: 2, τ : 1	rUK	$P_t^*C_t^* + \mathbb{E}_t[R_{t+1}^{*-1}B_{t+1}^*] = B_t^* + (1-\tau_t^*)W_t^*N_t^* + T_t^*$
Scen. 4	Scot.	$P_tC_t + \mathbb{E}_t[R_{t+1}^{UK-1}B_{t+1}^{UK}] = B_t^{UK} + (1-\tau_t^{UK})W_tN_t + \varpi T_t^{UK}$
G: 1, τ : 1	rUK	$P_t^*C_t^* + \mathbb{E}_t[R_{t+1}^{UK-1}B_{t+1}^{UK}] = B_t^{UK} + (1-\tau_t^{UK})W_t^*N_t^* + (1-\varpi)T_t^{UK}$

Here, ϖ denotes the Scotland's share of population in the United Kingdom. T_t denotes lump-sum transfers (subsidies or taxes), while τ_t denotes an income or labour tax rate. In the first column, G indicates the number of governments that can issue bonds (borrow) and set the labour tax rate. In the same column, τ indicates whether the government spending is funded by a labour tax.

What follows is the derivation of the intratemporal and intertemporal optimality conditions for the policy scenario 3, i.e. when households face a labour tax:

$$
\begin{aligned}
\mathcal{L} = & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right) Z_t \\
& + \lambda_t \left\{ B_t + (1-\tau_t) W_t N_t + T_t - P_t C_t - \mathbb{E}_t \left[R_{t+1}^{-1} B_{t+1} \right] \right\} && (18) \\
\frac{\partial \mathcal{L}}{\partial C_t} & = \beta^t C_t^{-\sigma} Z_t - \lambda_t P_t = 0; \quad \Rightarrow \quad \beta^t C_t^{-\sigma} Z_t P_t^{-1} = \lambda_t && (19) \\
\frac{\partial \mathcal{L}}{\partial N_t} & = -\beta^t N_t^{\varphi} Z_t + \lambda_t (1-\tau_t) W_t = 0; \quad \Rightarrow \quad \beta^t N_t^{\varphi} Z_t ((1-\tau_t) W_t)^{-1} = \lambda_t && (20) \\
\frac{\partial \mathcal{L}}{\partial B_{t+1}} & = -\lambda_t \, \mathbb{E}_t \left[R_{t+1}^{-1} \right] + \mathbb{E}_t [\lambda_{t+1}] = 0; \quad \Rightarrow \quad \mathbb{E}_t [R_{t+1}^{-1}] = \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} \right] && (21)
\end{aligned}
$$

Equating and rearranging Equations (19) and (20) yields *intratemporal optimality condition*:

$$
\begin{aligned}
\Rightarrow \qquad & C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t}(1-\tau_t) && \text{Scenario 3} \\
& C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t}(1-\tau_t^{UK}) && \text{Scenario 4} \\
& C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t} && \text{Scenarios 1 \& 2}
\end{aligned}
$$

The condition implies that the marginal utility of consumption and leisure is equal to the net real wage. As mentioned before, in the case of labour tax absence, the net real wage is equal to the gross real wage. The log-linearisation of Equation (22) around a steady state yields:

$$
\begin{aligned}
C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t}(1-\tau_t) & = \frac{W_t}{P_t} - \frac{W_t}{P_t}\tau_t \\
& \vdots \quad \text{(see Appendix A.90 - A.101)} \\
(1-\tau)(\sigma c_t + \varphi n_t) & = [(1-\tau)(w_t - p_t) - \tau \tilde{\tau}_t] \\
\sigma c_t + \varphi n_t & = w_t - p_t - \frac{\tau}{1-\tau}\tilde{\tau}_t && (22)
\end{aligned}
$$

Where τ and $\tilde{\tau}_t$ denote steady state labour tax rate and deviation from the steady state, respectively. As it is common in the literature, we denote natural logs of corresponding variables in lowercase letters, i.e. $x_t = \ln(X_t)$, and use tildes to denote deviations from the steady state. While loglinearising, we widely make use of Uhlig's (1999) proposed methods for multivariate equations with additive terms, i.e., $X_t Y_t \approx XY\mathbf{e}^{\tilde{X}_t+\tilde{Y}_t}$, $X_t + Y_t \approx X\mathbf{e}^{\tilde{X}_t} + Y\mathbf{e}^{\tilde{Y}_t}$ and $\mathbf{e}^{\tilde{X}_t} \approx (1+\tilde{X}_t)$.

Iterating Equation (19) one period forward, yields:

$$\frac{\partial L}{\partial C_t} = \beta^t C_t^{-\sigma} Z_t P_t^{-1} = \lambda_t; \qquad\qquad \Rightarrow \quad \mathbb{E}_t[\beta^{t+1} C_{t+1}^{-\sigma} Z_{t+1} P_{t+1}^{-1}] = \mathbb{E}_t[\lambda_{t+1}]$$

Dividing one by the other and rearranging yields *intertemporal optimality condition*:

$$\begin{aligned}
\mathbb{E}_t\left[\frac{\beta^t C_t^{-\sigma} Z_t P_t^{-1}}{\beta^{t+1} C_{t+1}^{-\sigma} Z_{t+1} P_{t+1}^{-1}}\right] &= \mathbb{E}_t\left[\frac{\lambda_t}{\lambda_{t+1}}\right] \\
\beta^{t-(t+1)}\,\mathbb{E}_t\left[\frac{C_t^{-\sigma} Z_t P_t^{-1}}{C_{t+1}^{-\sigma} Z_{t+1} P_{t+1}^{-1}}\right] &= \mathbb{E}_t\left[\frac{1}{R_{t+1}}\right] && (23)\\
&\;\vdots \;\; \text{(see Appendix A.xx - A.xx)} \\
\beta\,\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{Z_{t+1}}{Z_t}\right)\left(\frac{P_t}{P_{t+1}}\right)\right] &= \mathbb{E}_t\left[\frac{1}{R_{t+1}}\right] && (24)
\end{aligned}$$

where Equation (23) used Equation (21). $\mathbb{E}_t[R_{t+1}^{-1}]$ is the gross return on a risk-free one-period discount bond or a stochastic discount factor. More generally, Equation (24) is the Euler equation, and it determines the consumption path of a lifetime utility-maximising representative household. To state it in more intuitive terms, households choose consumption "today" and "tomorrow" and take all other terms as given. According to the equation, they choose consumption in the two periods in such a way so that the marginal utility "today" would be equal to the marginal consumption tomorrow while taking into account that saving consumption "today", will result in $R_t > 1$ consumption "tomorrow". Note that Galí (2015) uses a different approach to derive the Euler equation, which introduces Arrow securities. As it adds little value to our research question, this dissertation only provides a step-by-step derivation and interpretation in Appendix A for an interested reader. Also note, that Galí (2015) uses $Q_t = \mathbb{E}_t[\frac{1}{R_{t+1}}]$ to denote the stochastic discount factor and D_t to denote bonds or "portfolio" as they call it.

Log-linearising (24):

$$\begin{aligned}
\beta\,\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{P_t}{P_{t+1}}\right)\right] &= \mathbb{E}_t\left[\frac{1}{R_{t+1}}\right] \\
\ln\beta - \mathbb{E}_t[\sigma c_{t+1}] + \sigma c_t + p_t - \mathbb{E}_t[p_{t+1}] &= -\ln R_{t+1} \\
\sigma c_t = -\ln R_{t+1} - \ln\beta + \mathbb{E}_t[\sigma c_{t+1}] - p_t + \mathbb{E}_t[p_{t+1}] & \\
c_t = \mathbb{E}_t[c_{t+1}] - \frac{1}{\sigma}(\ln R_{t+1} - \rho - \mathbb{E}_t[\pi_{t+1}]) & \\
c_t = \mathbb{E}_t[c_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho) & \qquad (25)
\end{aligned}$$

where $i_t = \ln R_{t+1}$ is the nominal interest rate, $\rho = -\log\beta$ is the log discount rate, and $\pi_t = p_t - p_{t-1}$ is the CPI inflation. The loglinearised Euler equation makes it clearer to see, that consumption "today" is increasing in expected inflation "tomorrow", while the opposite is true for the nominal interest rate. The effect is scaled by σ^{-1} parameter.

Furthermore, OECD (2023) defines terms of trade as a ratio of import and export price indices, which in this model is denoted as S_t:

$$S_{i,t} = \frac{P_{i,t}}{P_{H,t}} \qquad (26)$$

$$S_t = \frac{P_{F,t}}{P_{H,t}} = \left(\int_0^1 (S_{i,t}\,di)^{1-\gamma}\right)^{\frac{1}{1-\gamma}} \qquad (27)$$

where Equation (26) marks *bilateral* terms of trade with a country i, while Equation (27) is for *effective* terms of trade, i.e. terms of trade with all countries in the unit interval defined earlier. The latter can be loglinearised to yield:

$$s_t = p_{F,t} - p_{H,t} = \left(\int_0^1 s_{i,t}\,di\right) \qquad (28)$$

Recall that when $\eta = 1$, then CPI is $P_t = (P_{H,t})^{1-\upsilon} \times (P_{F,t})^{\upsilon}$. Using the previous definition (28) and loglinearised CPI, the price level can be expressed as a sum of domestic price level and terms of trade (see Appendix A.xx-A.xx):

$$p_t = (1 - \upsilon)p_{H,t} + \upsilon p_{F,t} = p_{H,t} + \upsilon s_t \tag{29}$$

Note that Equations (28) and (29) hold *exactly* when $\gamma = 1$ and $\eta = 1$. Similarly, knowing that inflation as a difference of log prices in two consecutive periods, we can extend the previous definition to yield:

$$\pi_{H,t} = p_{H,t+1} - p_{H,t} \qquad \text{Domestic Inflation} \tag{30}$$

$$\pi_t = \pi_{H,t} + \upsilon \Delta s_t \qquad \text{CPI Inflation} \tag{31}$$

The gap between domestic inflation and CPI inflation is due to percentage change in the terms of trade and degree of openness. In the case of an autarky ($\upsilon = 0$), even if imported goods were much more expensive ($P_{F,t} \gg P_{H,t}$), domestic inflation will be equal to CPI inflation because the country simply does not trade.

Furthermore, we assume that the Law of One Price (LOP) holds for all goods j. That is, the price of a single good in country i is equal to the price of the same good in country $-i$ times the nominal exchange rate. It implies, that there are no opportunities for arbitrage.

$$P_{i,t}(j) = \mathcal{E}_{i,t} P^i_{i,t}(j) \qquad \text{Law of One Price (LOP)} \tag{32}$$

$$P_{i,t} = \mathcal{E}_{i,t} P^i_{i,t} \qquad \text{Law of One Price (LOP)} \tag{33}$$

where $\mathcal{E}_{i,t}$ is the nominal exchange rate between the home currency and the country's i currency, and the second equation is derived by integrating both sides with respect to j. Even though we do not model currencies explicitly, it is useful to think about $\mathcal{E}_{i,t}$ as the price of one unit of currency in terms of another currency, i.e. the home currency. The two equations can be loglinearised to yield:

$$p_{i,t} = e_{i,t} + p^i_{i,t} \qquad \text{(Log) Law of One Price (LOP)} \tag{34}$$

$$p_{F,t} = \int_0^1 (e_{i,t} + p^i_{i,t})\, di = e_t + p^\star_t \qquad \text{(Log)Price index of Imported Goods} \tag{35}$$

Where e_t is (Log) Effective Nominal Exchange Rate, $p^\star_t$ is the World Price Index. This allows us to redefine log effective terms of trade in terms of the nominal exchange rate, domestic price, and the world price index:

$$s_t = p_{F,t} - p_{H,t} = e_t + p^\star_t - p_{H,t} \tag{36}$$

In contrast, *real* exchange rate between two countries is the ratio between their CPI and home CPI, expressed in home currency:

$$\mathcal{Q}_{i,t} = \frac{\mathcal{E}_{i,t} P^i_t}{P_t} \qquad \text{Bilateral Exchange Rate} \tag{37}$$

Integrating both sides with respect to i and using previous definitions yields:

$$q_t = \int_0^1 \log\left(\frac{\mathcal{E}_{i,t} P^i_t}{P_t}\right)\, di \tag{38}$$

$$= \int_0^1 (e_{i,t} + p^i_{i,t} - p_t)\, di \tag{39}$$

$$= e_t + p^\star_t - p_t \qquad \text{using (35)} \tag{40}$$

$$= s_t + p_{H_t} - p_t \qquad \text{using (36)} \tag{41}$$

$$= (1 - \upsilon) s_t \qquad \text{using (29)} \tag{42}$$

Finally, if we assume that all countries i have symmetrical preferences and their households maximise lifetime-utility in the same manner that our home country's do, then maximising the Lagrangian function for country i, will yield:

$$
\begin{aligned}
\frac{\partial L^i}{\partial C_t^i} &= \beta^t (C_t^i)^{-\sigma} Z_t^i (\mathcal{E}_{i,t} P_t^i)^{-1} = \lambda_t^i \\
\frac{\partial L^i}{\partial D_{t+1}^i} &= -\lambda_t^i \, \mathbb{E}_t[R_{t,t+1}^{-1}] = -\, \mathbb{E}_t[\lambda_{t+1}^i] \\
&\vdots \quad \text{(see Appendix A.xx - A.xx)} && (43) \\
C_t &= C_t^i Z_t^{i\,\frac{1}{\sigma}} \mathcal{Q}_{i,t}^{\frac{1}{\sigma}} && (44)
\end{aligned}
$$

If we assume that there had been no shocks in the country's i preferences ($Z_t^i = 1$), then Equation (44) states that consumption in the home country is equal to the consumption in the country i, while taking into account bilateral real exchange rate. This can be generalised to derive a relationship between home consumption and world consumption by log linearising (for simplicity) and integrating both sides with respect to i:

$$
\begin{aligned}
c_t &= c_t^i + \frac{1}{\sigma} z_{i,t} + \frac{1}{\sigma} q_{i,t} && (45) \\
\int_0^1 c_t \, di &= \int_0^1 \left(c_t^\star + \frac{1}{\sigma} z_{i,t} + \frac{1}{\sigma} q_{i,t} \right) di && (46) \\
&\vdots \quad \text{(see Appendix A.xx - A.xx)} && (47) \\
c_t &= c_t^\star + \frac{1}{\sigma} z_t + \left(\frac{1-\upsilon}{\sigma} \right) s_t && \text{using } q_t = (1-\upsilon) s_t \quad (48) \\
&= y_t^\star + \frac{1}{\sigma} z_t + \left(\frac{1-\upsilon}{\sigma} \right) s_t && \text{using } c_t^\star = y_t^\star \quad (49)
\end{aligned}
$$

$c_t^\star$ is the log world consumption and the last Equation (49) follows by assuming that world consumption is equal to world output, i.e., there is no world government spending, or national government spending in any country i is infinitesimally small.

The next two parts will discuss government spending and firms, respectively. The final part will provide equilibrium (market clearing) conditions.

2.2 Firms

$$Y_t(j) = A_t N_t(j) \tag{50}$$

$$\log A_t = \alpha_t \tag{51}$$

$$\alpha_t = \rho_a \alpha_{t-1} + \varepsilon_t \tag{52}$$

$$L = P_t(j)Y_t(j) - W_t(j)N_t(j) \tag{53}$$

$$\Rightarrow L = P_t Y_t - W_t N_t \tag{54}$$

$$\Rightarrow L = P_t A_t N_t - W_t N_t \tag{55}$$

$$\tag{56}$$

$$\frac{\partial L}{\partial N_t} = P_t A_t - W_t = 0 \qquad \Rightarrow W_t - P_t A_t = 0 \tag{57}$$

$$MC_t = W_t - P_t A_t \tag{58}$$

$$mc_t = w_t - p_t - a_t \tag{59}$$

$$mc_t = -\nu + w_t - p_t - a_t \tag{60}$$

$$mc_t = -\nu + w_t - p_{H,t} - a_t \tag{61}$$

$$\tag{62}$$

$\nu = -(\log(1-\tau))$, where τ is the employment subsidy, introduced later. $p_{H,t}$ because this is for domestic firms.

Firms that get to reset their price, do it using the following problem:

$$p_{\bar{H},t} = \mu + (1-\beta\theta) \sum_{k=0}^{\infty} (\beta\theta)^k \, \mathbb{E}_t[mc_{t+k} + p_{H,t+k}] \tag{63}$$

$$p_{\bar{H},t} \qquad \text{Is the (log) new price} \tag{64}$$

$$\mu \qquad \text{Is the (log) markup in the steady state} \tag{65}$$

2.3 Equilibrium

$$Y_t(j) = C_{H,t}(j) + \int_0^1 C^i_{H,t}(j)\, di \tag{66}$$

$$= \left(\frac{P_{H,t}(j)}{P_{H,t}}^{-\varepsilon}\right)\left[(1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \alpha \int_0^1 \left(\frac{P_{H_t}}{\mathcal{E}_{i,t}P^i_{F,t}}\right)^{-\gamma}\left(\frac{P^i_{F,t}}{P^i_t}\right)^{-\eta} C^i_t\, di\right] \tag{67}$$

Given that

$$Y_t = \left(\int_0^1 (Y_t(j))^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{68}$$

$$Y_t = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \alpha \int_0^1 \left(\frac{P_{H_t}}{\mathcal{E}_{i,t}P^i_{F,t}}\right)^{-\gamma}\left(\frac{P^i_{F,t}}{P^i_t}\right)^{-\eta} C^i_t\, di \tag{69}$$

$$= \left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t\left[(1-\alpha) + \alpha\int_0^1 (S^i_t S_{i,t})^{\gamma-\eta}\mathcal{Q}_{i,t}^{\eta-\frac{1}{\sigma}}\, di\right] \tag{70}$$

Which can be log-linearised to:

$$y_t = c_t + \alpha\gamma s_t + \alpha(\eta - \frac{1}{\sigma})q_t \tag{71}$$

$$= c_t + \frac{\alpha w}{\sigma} s_t \tag{72}$$

$$w_t = \sigma\gamma + (1-\alpha)(\sigma\eta - 1) \tag{73}$$

Assuming that country i is symmetric:

$$y^i_t = c^i_t + \frac{\alpha w}{\sigma} s^i_t \tag{74}$$

$$\int_0^1 y^i_t = \int_0^1 c^i_t + 0 = c^\star_t \qquad \text{World Consumption} \tag{75}$$

Using equations (), (), ():

$$
\begin{aligned}
y &= c_t + \frac{\alpha w}{\sigma} s_t && (76)\\
y_t &= c_t^\star + \frac{1-\alpha}{\sigma} s_t + \frac{\alpha w}{\sigma} s_t && (77)\\
y_t &= y_t^\star + \frac{1-\alpha}{\sigma} s_t + \frac{\alpha w}{\sigma} s_t && (78)\\
y_t &= y_t^\star + \frac{1-\alpha+\alpha w}{\sigma} s_t && (79)\\
y_t &= y_t^\star + \frac{1+\alpha(w-1)}{\sigma} s_t && (80)\\
\sigma_\alpha &= \frac{1+\alpha(w-1)}{\sigma} && (81)\\
\Rightarrow \quad y_t &= y_t^\star + \frac{1}{\sigma_\alpha} s_t && (82)
\end{aligned}
$$

Combining Euler equation () and (82) gives :

$$
\begin{aligned}
c_t &= \mathbb{E}_t[c_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho) \\
y_t - \frac{\alpha w}{\sigma} s_t &= \mathbb{E}_t\left[y_{t-1} - \frac{\alpha w}{\sigma} s_{t+1}\right] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho) \\
y_t &= \mathbb{E}_t\left[y_{t-1}\right] - \frac{\alpha w}{\sigma}\,\mathbb{E}_t[(s_{t+1} - s_t)] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho) \\
&= \mathbb{E}_t\left[y_{t-1}\right] - \frac{\alpha w}{\sigma}\,\mathbb{E}_t[\Delta s_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho) \\
&= \mathbb{E}_t\left[y_{t-1}\right] - \frac{\alpha w}{\sigma}\,\mathbb{E}_t[\Delta s_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \alpha\,\mathbb{E}_t[\Delta s_{t+1}] - \rho) \\
&= \mathbb{E}_t\left[y_{t-1}\right] - \frac{\alpha w}{\sigma}\,\mathbb{E}_t[\Delta s_{t+1}] + \frac{\alpha}{\sigma}\,\mathbb{E}_t[\Delta s_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \rho) \\
&= \mathbb{E}_t\left[y_{t-1}\right] - \frac{\alpha w + \alpha}{\sigma}\,\mathbb{E}_t[\Delta s_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \rho) \\
&= \mathbb{E}_t\left[y_{t-1}\right] + \frac{\alpha - \alpha w}{\sigma}\,\mathbb{E}_t[\Delta s_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \rho) \\
&= \mathbb{E}_t\left[y_{t-1}\right] + \frac{-\alpha(-1 + w)}{\sigma}\,\mathbb{E}_t[\Delta s_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \rho) \\
&= \mathbb{E}_t\left[y_{t-1}\right] - \frac{\alpha(w - 1)}{\sigma}\,\mathbb{E}_t[\Delta s_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \rho) \\
&= \mathbb{E}_t\left[y_{t-1}\right] - \frac{\alpha\Theta}{\sigma}\,\mathbb{E}_t[\Delta s_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \rho) \\
\mathbb{E}_t[\Delta s_{t+1}] &= \sigma_\alpha\,\mathbb{E}_t[\Delta y_{t+1} - \Delta y^\star_{t+1}] \quad \text{from Equation():} \\
&= \mathbb{E}_t\left[y_{t-1}\right] - \frac{\alpha\Theta}{\sigma}\sigma_\alpha\,\mathbb{E}_t[\Delta y_{t+1} - \Delta y^\star_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \rho) \\
&= \mathbb{E}_t\left[y_{t-1}\right] - \frac{\alpha\Theta}{1 + \alpha\Theta}\,\mathbb{E}_t[\Delta y_{t+1} - \Delta y^\star_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \rho) \\
&= \mathbb{E}_t\left[y_{t-1}\right] - \frac{\alpha\Theta}{1 + \alpha\Theta}\,\mathbb{E}_t[\Delta y_{t+1}] + \frac{\alpha\Theta}{1 + \alpha\Theta}\,\mathbb{E}_t[\Delta y^\star_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \rho) \\
&= \mathbb{E}_t\left[y_{t-1}\right] - \frac{\alpha\Theta}{1 + \alpha\Theta}\,\mathbb{E}_t[\Delta y_{t+1}] + \frac{\alpha\Theta}{1 + \alpha\Theta}\,\mathbb{E}_t[\Delta y^\star_{t+1}] - \frac{1}{(1 + \alpha\Theta)\sigma_\alpha}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \rho)
\end{aligned}
$$

$$
\begin{aligned}
(1+\alpha\Theta)y_t &= (1+\alpha\Theta)\,\mathbb{E}_t\left[y_{t-1}\right] - \alpha\Theta\,\mathbb{E}_t[\Delta y_{t+1}] + \alpha\Theta\,\mathbb{E}_t[\Delta y^{\star}_{t+1}] - \frac{1}{\sigma_\alpha}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \rho) \\
(1+\alpha\Theta)\,\mathbb{E}_t\left[y_t - y_{t-1}\right] &= -\alpha\Theta\,\mathbb{E}_t[\Delta y_{t+1}] + \alpha\Theta\,\mathbb{E}_t[\Delta y^{\star}_{t+1}] - \frac{1}{\sigma_\alpha}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \rho) \\
-(1+\alpha\Theta)\,\mathbb{E}_t\left[y_{t-1} - y_t\right] &= -\alpha\Theta\,\mathbb{E}_t[\Delta y_{t+1}] + \alpha\Theta\,\mathbb{E}_t[\Delta y^{\star}_{t+1}] - \frac{1}{\sigma_\alpha}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \rho) \\
(1+\alpha\Theta)\,\mathbb{E}_t\left[\Delta y_{t+1}\right] &= \alpha\Theta\,\mathbb{E}_t[\Delta y_{t+1}] - \alpha\Theta\,\mathbb{E}_t[\Delta y^{\star}_{t+1}] + \frac{1}{\sigma_\alpha}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \rho) \\
(1+\alpha\Theta)\,\mathbb{E}_t\left[\Delta y_{t+1}\right] - \alpha\Theta\,\mathbb{E}_t[\Delta y_{t+1}] &= -\alpha\Theta\,\mathbb{E}_t[\Delta y^{\star}_{t+1}] + \frac{1}{\sigma_\alpha}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \rho) \\
\mathbb{E}_t\left[\Delta y_{t+1}\right] + \alpha\Theta\,\mathbb{E}_t\left[\Delta y_{t+1}\right] - \alpha\Theta\,\mathbb{E}_t[\Delta y_{t+1}] &= -\alpha\Theta\,\mathbb{E}_t[\Delta y^{\star}_{t+1}] + \frac{1}{\sigma_\alpha}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \rho) \\
\mathbb{E}_t\left[\Delta y_{t+1}\right] &= -\alpha\Theta\,\mathbb{E}_t[\Delta y^{\star}_{t+1}] + \frac{1}{\sigma_\alpha}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \rho) \\
\mathbb{E}_t\left[y_{t+1} - y_t\right] &= -\alpha\Theta\,\mathbb{E}_t[\Delta y^{\star}_{t+1}] + \frac{1}{\sigma_\alpha}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \rho) \\
-y_t &= -\,\mathbb{E}_t\left[y_{t+1}\right] - \alpha\Theta\,\mathbb{E}_t[\Delta y^{\star}_{t+1}] + \frac{1}{\sigma_\alpha}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \rho) \\
\Rightarrow y_t &= \mathbb{E}_t\left[y_{t+1}\right] - \frac{1}{\sigma_\alpha}(i_t - \mathbb{E}_t[\pi_{H,t+1}] - \rho) + \alpha\Theta\,\mathbb{E}_t[\Delta y^{\star}_{t+1}] \qquad (83)
\end{aligned}
$$

$$(84)$$

$$\pi_{avg,t} = 0.0816\pi_{H,t}^{scot} + 0.9184\pi_{H,t}^{ruk} \tag{85}$$

$$\tilde{y}_{avg,t} = 0.0816\tilde{y}_{scot,t} + 0.9184\tilde{y}_{ruk,t} \tag{86}$$

$$i_t = \rho_\pi \pi_{avg,t-1} + \rho_{\tilde{y}} \tilde{y}_{avg,t-1} + \Delta\pi_{avg,t-1} + \Delta\tilde{y}_{avg,t-1} + \nu_t \tag{87}$$

3 Application

4 Results

5 Conclusion

6 Appendix

$$C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t}(1-\tau_t) \tag{A.88}$$

$$C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t} - \frac{W_t}{P_t}\tau_t \tag{A.89}$$

$$C^{\sigma} N^{\varphi} = \frac{W}{P} - \frac{W}{P}\tau \quad \text{(Steady state)} \tag{A.90}$$

$$\text{Using Uhlig's (1999) method, } X_t Y_t \approx XY\mathbf{e}^{\tilde{X}_t + \tilde{Y}_t} \tag{A.91}$$

$$C^{\sigma} N^{\varphi} \mathbf{e}^{\sigma c_t + \varphi n_t} = \frac{W}{P}\mathbf{e}^{w_t - p_t} - \frac{W}{P}\tau\mathbf{e}^{w_t - p_t - \tilde{\tau}_t} \tag{A.92}$$

$$\text{Using } \mathbf{e}^{\tilde{X}_t} \approx 1 + \tilde{X}_t\text{:} \tag{A.93}$$

$$C^{\sigma} N^{\varphi}(1 + \sigma c_t + \varphi n_t) = \frac{W}{P}(1 + w_t - p_t) - \frac{W}{P}\tau(1 + w_t - p_t - \tilde{\tau}_t) \tag{A.94}$$

$$\text{Subtract (A.90):} \tag{A.95}$$

$$C^{\sigma} N^{\varphi}(\sigma c_t + \varphi n_t) = \frac{W}{P}(w_t - p_t) - \frac{W}{P}\tau(w_t - p_t - \tilde{\tau}_t) \tag{A.96}$$

$$C^{\sigma} N^{\varphi}(\sigma c_t + \varphi n_t) = \frac{W}{P}\left[(w_t - p_t) - \tau(w_t - p_t - \tilde{\tau}_t)\right] \tag{A.97}$$

$$C^{\sigma} N^{\varphi}(\sigma c_t + \varphi n_t) = \frac{W}{P}\left[(w_t - p_t) - \tau(w_t - p_t) - \tau\tilde{\tau}_t\right] \tag{A.98}$$

$$C^{\sigma} N^{\varphi}(\sigma c_t + \varphi n_t) = \frac{W}{P}\left[(1-\tau)(w_t - p_t) - \tau\tilde{\tau}_t\right] \tag{A.99}$$

$$C^{\sigma} N^{\varphi}\frac{P}{W}(\sigma c_t + \varphi n_t) = \left[(1-\tau)(w_t - p_t) - \tau\tilde{\tau}_t\right] \tag{A.100}$$

$$(1-\tau)(\sigma c_t + \varphi n_t) = \left[(1-\tau)(w_t - p_t) - \tau\tilde{\tau}_t\right] \tag{A.101}$$

$$\sigma c_t + \varphi n_t = w_t - p_t - \frac{\tau}{1-\tau}\tilde{\tau}_t \tag{A.102}$$

Two-country DSGE model for Scotland and the rest of the UK

Closely follows Galí (2015) and Gali and Monacelli (2005)

1. Home country: Scotland (notation - X)
2. Foreign country: rUK (notation - X^*)
3. Notation for UK-wide variables - X^{UK}
4. Scotland and rUK are SOEs
   (a) They trade, take world output, inflation, and consumption as given and cannot influence it
5. Calvo staggered prices, no capital/investment
6. Scotland and RUK are assumed to be symmetrical in market structure and preferences
7. Monetary Union: There is a population-weighted UK-wide interest rate in place, and all four nations within the UK purchase government-issued bonds at this rate
   (a) We also consider counterfactual scenarios, i.e. where both Holyrood and Westminister can issue bonds at country-specific interest rate
8. The government spending can be financed via a lump-sum tax, a labour (income) tax, and borrowing.
9. We consider 4 scenarios:
   (a) Two governments funded by lump-sum tax, <u>two</u> of which can issue bonds
   (b) Two governments funded by lump-sum tax, <u>one</u> of which can issue bonds

(c) Two governments funded by an income tax, two of which can issue bonds

(d) Two governments funded by an income tax, one of which can issue bonds

	S HBC:	$P_tC_t + \mathbb{E}_t[R_{t+1}^{-1}B_{t+1}] = B_t + W_tN_t + T_t$
	rUK HBC:	$P_t^*C_t^* + \mathbb{E}_t[R_{t+1}^{*-1}B_{t+1}^*] = B_t^* + W_t^*N_t^* + T_t^*$
	S GBC:	$\mathbb{E}_t[R_{t+1}^{-1}B_{t+1}] + T_t = P_tG_t + B_t$
	rUK GBC:	$\mathbb{E}_t[R_{t+1}^{*-1}B_{t+1}^*] + T_t^* = P_t^*G_t^* + B_t^*$
	S TR:	$T_t = P_tG_t$
Scenario 1	rUK TR:	$T_t^* = P_tG_t^*$
(G: 2, $\tau : 0$)	S Debt s.:	$T_t/P_t = \phi_g^*G_Yg_t + \phi_b^*(B_t/P_t); \quad G_Y = 0$
	rUK Debt s.:	$T_t^*/P_t^* = \phi_g^*G_Y^*g_t^* + \phi_b^*(B_t^*/P_t^*); \quad G_Y = 0$
	S Bonds:	$(B_{t+1}/P_t)/R_{t+1} = (1-\phi_g)G_Yg_t + (1-\phi_b)(B_t/P_t); \quad G_Y = 0$
	rUK Bonds:	$(B_{t+1}^*/P_t^*)/R_{t+1}^* = (1-\phi_g^*)G_Y^*g_t^* + (1-\phi_b^*)(B_t^*/P_t^*); \quad G_Y^* = 0$
	S RC:	$Y_t = C_t + G_t + X_t$
	rUK RC:	$Y_t^* = C_t^* + G_t^* + X_t^*$
	S HBC:	$P_tC_t + \mathbb{E}_t[R_{t+1}^{UK-1}B_{t+1}^{UK}] = B_t^{UK} + W_tN_t + \varpi T_t^{UK}$
	rUK HBC:	$P_t^*C_t^* + \mathbb{E}_t[R_{t+1}^{UK-1}B_{t+1}^{UK}] = B_t^{UK} + W_t^*N_t^* + (1-\varpi)T_t^{UK}$
	S GBC:	N/A
	rUK GBC:	$\mathbb{E}_t[R_{t+1}^{UK-1}B_{t+1}^{UK}] + T_t^{UK} = P_t^{UK}G_t^{UK} + B_t^{UK}$
	S TR:	N/A
Scenario 2	rUK TR:	$T_t^{UK} = P_t^{UK}G_t^{UK}$
(G: 1, $\tau : 0$)	S Debt s.:	N/A
	rUK Debt s.:	$T_t^{UK}/P_t^{UK} = \phi_g^{UK}G_Y^{UK}g_t^{UK} + \phi_b^{UK}(B_{t+1}^{UK}/P_t^{UK}); \quad G_Y^{UK} = 0$
	S Bonds:	N/A
	rUK Bonds:	$(B_t^{UK}/P_t^{UK})/R_{t+1}^{UK} = (1-\phi_g^{UK})G_Y^{UK}g_t^{UK} + (1-\phi_b^{UK})(B_t^{UK}/P_t^{UK}); \quad G_Y^{UK} = 0$
	S RC:	$Y_t = C_t + \varpi G_t^{UK} + X_t$
	rUK RC:	$Y_t^* = C_t^* + (1-\varpi)G_t^{UK} + X_t^*$
	S HBC:	$P_tC_t + \mathbb{E}_t[R_{t+1}^{-1}B_{t+1}] = B_t + (1-\tau_t)W_tN_t + T_t$
	rUK HBC:	$P_t^*C_t^* + \mathbb{E}_t[R_{t+1}^{*-1}B_{t+1}^*] = B_t^* + (1-\tau_t^*)W_t^*N_t^* + T_t^*$
	S GBC:	$\mathbb{E}_t[R_{t+1}^{-1}B_{t+1}] + T_t = P_tG_t + B_t$
	rUK GBC:	$\mathbb{E}_t[R_{t+1}^{*-1}B_{t+1}^*] + T_t^* = P_t^*G_t^* + B_t^*$
	S TR:	$T_t = \tau_tW_tN_t$
Scenario 3	rUK TR:	$T_t^* = \tau_t^*W_t^*N_t^*$
(G: 2, $\tau : 1$)	S Debt s.:	$T_t/P_t = \phi_g^*G_Yg_t + \phi_b^*(B_t/P_t); \quad G_Y = \tau$
	rUK Debt s.:	$T_t^*/P_t^* = \phi_g^*G_Y^*g_t^* + \phi_b^*(B_t^*/P_t^*); \quad G_Y = \tau^*$
	S Bonds:	$(B_{t+1}/P_t)/R_{t+1} = (1-\phi_g)G_Yg_t + (1-\phi_b)(B_t/P_t); \quad G_Y = \tau$
	rUK Bonds:	$(B_{t+1}^*/P_t^*)/R_{t+1}^* = (1-\phi_g^*)G_Y^*g_t^* + (1-\phi_b^*)(B_t^*/P_t^*); \quad G_Y^* = \tau^*$
	S RC:	$Y_t = C_t + G_t + X_t$

where:

S HBC	Household (Scotland) budget constraint
rUK HBC	Household (rUK) budget constraint
S GBC	Government (Holyrood) budget constraint
rUK GBC	Government (Westminister) budget constraint
S TR	Government (Holyrood) tax revenue
rUK TR	Government (Westminister) tax revenue
S RC	Resource (Scotland) constraint
rUK RC	Resource (rUK) constraint

Scenario 1	Scotland	$P_tC_t + \mathbb{E}_t[R_{t+1}^{-1}B_{t+1}] = B_t + W_tN_t + T_t$
(G: 2, $\tau : 0$)	rUK	$P_t^*C_t^* + \mathbb{E}_t[R_{t+1}^{*-1}B_{t+1}^*] = B_t^* + W_t^*N_t^* + T_t^*$
Scenario 2	Scotland	$P_tC_t + \mathbb{E}_t[R_{t+1}^{UK-1}B_{t+1}^{UK}] = B_t^{UK} + W_tN_t + \varpi T_t^{UK}$
(G: 1, $\tau : 0$)	rUK	$P_t^*C_t^* + \mathbb{E}_t[R_{t+1}^{UK-1}B_{t+1}^{UK}] = B_t^{UK} + W_t^*N_t^* + (1-\varpi)T_t^{UK}$
Scenario 3	Scotland	$P_tC_t + \mathbb{E}_t[R_{t+1}^{-1}B_{t+1}] = B_t + (1-\tau_t)W_tN_t + T_t$
(G: 2, $\tau : 1$)	rUK	$P_t^*C_t^* + \mathbb{E}_t[R_{t+1}^{*-1}B_{t+1}^*] = B_t^* + (1-\tau_t^*)W_t^*N_t^* + T_t^*$
Scenario 4	Scotland	$P_tC_t + \mathbb{E}_t[R_{t+1}^{UK-1}B_{t+1}^{UK}] = B_t^{UK} + (1-\tau_t^{UK})W_tN_t + \varpi T_t^{UK}$
(G: 1, $\tau : 1$)	rUK	$P_t^*C_t^* + \mathbb{E}_t[R_{t+1}^{UK-1}B_{t+1}^{UK}] = B_t^{UK} + (1-\tau_t^{UK})W_t^*N_t^* + (1-\varpi)T_t^{UK}$

Tax Revenue

$$T_t^* = \varpi\tau_t W_t N_t + (1-\varpi)\tau_t W_t^* N_t^*$$

$$T^* = \varpi\tau W N + (1-\varpi)\tau W^* N^* \quad \text{(Steady state)}$$

Using Uhlig's (1999) method, $X_t Y_t \approx XY\mathbf{e}^{\tilde{X}_t+\tilde{Y}_t}$:

$$T\mathbf{e}^{\tilde{T}_t^*} = \varpi\tau W N\mathbf{e}^{\tilde{\tau}_t+\tilde{W}_t+\tilde{N}_t} + (1-\varpi)\tau W^* N^*\mathbf{e}^{\tilde{\tau}_t+\tilde{W}_t^*+\tilde{N}_T^*}$$

Using $\mathbf{e}^{\tilde{X}_t} \approx 1+\tilde{X}_t$:

$$T^*(1+\tilde{T}_t^*) = \varpi\tau W N(1+\tilde{\tau}_t+\tilde{W}_t+\tilde{N}_t) + (1-\varpi)\tau W^* N^*(1+\tilde{\tau}_t+\tilde{W}_t^*+\tilde{N}_T^*)$$

Subtract (**??**):

$$T^*(1+\tilde{T}_t^*) - T^* = \varpi\tau W N(1+\tilde{\tau}_t+\tilde{W}_t+\tilde{N}_t) - \varpi\tau W N + (1-\varpi)\tau W^* N^*(1+\tilde{\tau}_t+\tilde{W}_t^*+\tilde{N}_T^*) - (1$$

$$T^*[(1+\tilde{T}_t^*) - 1] = \varpi\tau W N[(1+\tilde{\tau}_t+\tilde{W}_t+\tilde{N}_t) - 1] + (1-\varpi)\tau W^* N^*[(1+\tilde{\tau}_t+\tilde{W}_t^*+\tilde{N}_T^*) - 1]$$

$$T^*\tilde{T}_t^* = \varpi\tau W N(\tilde{\tau}_t+\tilde{W}_t+\tilde{N}_t) + (1-\varpi)\tau W^* N^*(\tilde{\tau}_t+\tilde{W}_t^*+\tilde{N}_T^*)$$

Divide by T^* :

$$\boxed{\tilde{T}_t^* = \tau\tfrac{WN}{T^*} \times \varpi(\tilde{\tau}_t+\tilde{W}_t+\tilde{N}_t) + \tau\tfrac{W^*N^*}{T^*} \times (1-\varpi)(\tilde{\tau}_t+\tilde{W}_t^*+\tilde{N}_t^*)} \tag{A.103}$$

$$y_t = a_t + (1-\alpha)n_t \tag{A.104}$$

$$n_t = \frac{1}{1-\alpha}(y_t - a_t) \tag{A.105}$$

$$\varpi = \sigma\eta + (1-\upsilon)(\sigma\eta - 1) \tag{A.106}$$

$$\sigma c_t + \varphi n_t = w_t - p_t - \frac{1}{1-\tau}\tau_t \tag{A.107}$$

$$y_t = (1-\upsilon)c_t + \upsilon_t(2-\upsilon)\eta s_t + \upsilon y_t^* \tag{A.108}$$

$$c_t = y_t^* + \frac{1}{\sigma}z_t + \frac{1-\upsilon}{\sigma}s_t \tag{A.109}$$

$$\implies c_t = y_t - \upsilon s_t\left((2-\upsilon)\eta + \frac{1-\upsilon}{\sigma}\right) + \frac{\upsilon}{\sigma}z_t \tag{A.110}$$

$$\mu_t = -(\sigma c_t + \varphi n_t + \frac{1}{1-\tau}\tau_t) - \upsilon s_t + a_t - \alpha n_t \tag{A.111}$$

$$= -\sigma c_t - \varphi n_t - \frac{1}{1-\tau}\tau_t - \upsilon s_t + a_t - \alpha n_t \tag{A.112}$$

$$= -\sigma c_t - \frac{1}{1-\tau}\tau_t - \upsilon s_t + a_t - n_t(\varphi + \alpha) \tag{A.113}$$

$$\tag{A.114}$$

$$-n_t(\varphi + \alpha) = -\frac{1}{1-\alpha}(y_t - a_t)(\varphi + \alpha) \tag{A.115}$$

$$-n_t(\varphi + \alpha) = -\frac{\varphi + \alpha}{1-\alpha}(y_t - a_t) \tag{A.116}$$

$$-n_t(\varphi + \alpha) = -\frac{\varphi + \alpha}{1-\alpha}y_t + \frac{\varphi + \alpha}{1-\alpha}a_t \tag{A.117}$$

$$\tag{A.118}$$

$$\mu_t = -\sigma c_t - \frac{1}{1-\tau}\tau_t - \upsilon s_t + a_t - \frac{\varphi+\alpha}{1-\alpha}y_t + \frac{\varphi+\alpha}{1-\alpha}a_t \tag{A.119}$$

$$\mu_t = -\sigma c_t - \frac{1}{1-\tau}\tau_t - \upsilon s_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right)a_t - \frac{\varphi+\alpha}{1-\alpha}y_t \tag{A.120}$$

$$\mu_t = -\sigma\left(y_t - \upsilon s_t\left((2-\upsilon)\eta + \frac{1-\upsilon}{\sigma}\right) + \frac{\upsilon}{\sigma}z_t\right) - \frac{1}{1-\tau}\tau_t - \upsilon s_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right)a_t - \frac{\varphi+\alpha}{1-\alpha}y_t \tag{A.121}$$

$$\mu_t = -\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)y_t + \sigma\upsilon s_t\left((2-\upsilon)\eta + \frac{1-\upsilon}{\sigma}\right) - \upsilon z_t - \frac{1}{1-\tau}\tau_t - \upsilon s_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right)a_t \tag{A.122}$$

$$\mu_t = -\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)y_t + \upsilon s_t\left(\sigma\left((2-\upsilon)\eta + \frac{1-\upsilon}{\sigma}\right) - 1\right) - \frac{1}{1-\tau}\tau_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t \tag{A.123}$$

$$\mu_t = -\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)y_t + \upsilon s_t\left((2-\upsilon)\sigma\eta + 1 - \upsilon\right) - 1 - \frac{1}{1-\tau}\tau_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t \tag{A.124}$$

$$\mu_t = -\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)y_t + \upsilon s_t\left((2-\upsilon)\sigma\eta - \upsilon\right) - \frac{1}{1-\tau}\tau_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t \tag{A.125}$$

$$\mu_t = -\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)y_t + \upsilon s_t\left(2\sigma\eta - \upsilon\sigma\eta - \upsilon\right) - \frac{1}{1-\tau}\tau_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t \tag{A.126}$$

$$\mu_t = -\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)y_t + \upsilon(\varpi - 1)s_t - \frac{1}{1-\tau}\tau_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t \tag{A.127}$$

$$\pi_{H,t} = \beta\,\mathbb{E}\{\pi_{H,t+1}\} + \lambda\hat{\mu}_t \implies \pi_{H,t} = \beta\,\mathbb{E}\{\pi_{H,t+1}\} - \lambda\mu_t \tag{A.128}$$

$$\pi_{H,t} = \beta\,\mathbb{E}\{\pi_{H,t+1}\} + \lambda\left(\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)y_t - \upsilon(\varpi-1)s_t + \frac{1}{1-\tau}\tau_t - \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t + \upsilon z_t\right) \tag{A.129}$$

$$\pi_{H,t} = \beta\,\mathbb{E}\{\pi_{H,t+1}\} + \kappa y_t + \lambda\left(-\upsilon(\varpi-1)s_t + \frac{1}{1-\tau}\tau_t - \left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t + \upsilon z_t\right) \tag{A.130}$$

$$\pi_{H,t} = \beta\,\mathbb{E}\{\pi_{H,t+1}\} + \kappa y_t - \lambda\upsilon(\varpi-1)s_t + \lambda\frac{1}{1-\tau}\tau_t - \lambda\left(1+\frac{\varphi+\alpha}{1-\alpha}\right)a_t + \lambda\upsilon z_t \tag{A.131}$$

$$\tag{A.132}$$

$$\mu_t = -\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)$$

$$\mu_t - \left(\upsilon(\varpi-1)s_t^n - \frac{1}{1-}\right.$$

$$\upsilon(\varpi-1)s_t^n - \frac{1}{1-\tau}$$

$$\upsilon(\varpi-1)s_t^n - \frac{1}{1-\tau}$$

$$\upsilon(\varpi-1)(\sigma_\upsilon(y_t^n - y_t^*) - (1-\upsilon)\Phi z_t - G_Y\sigma_\upsilon g_t) - \frac{1}{1-\tau}$$

$$(-1+1+\upsilon(\varpi-1))(\sigma_\upsilon(y_t^n - y_t^*) - (1-\upsilon)\Phi z_t - G_Y\sigma_\upsilon g_t) - \frac{1}{1-\tau}$$

$$(-1+\Phi^{-1})(\sigma_\upsilon(y_t^n - y_t^*) - (1-\upsilon)\Phi z_t - G_Y\sigma_\upsilon g_t) - \frac{1}{1-\tau}$$

$$(-1+\Phi^{-1})(\sigma_\upsilon y_t^n - \sigma_\upsilon y_t^* - (1-\upsilon)\Phi z_t - G_Y\sigma_\upsilon g_t) - \frac{1}{1-\tau}$$

$$(-1+\Phi^{-1})\sigma_\upsilon y_t^n + (1-\Phi^{-1})\sigma_\upsilon y_t^* + (1-\Phi^{-1})(1-\upsilon)\Phi z_t + (1-\Phi^{-1})G_Y\sigma_\upsilon g_t - \frac{1}{1-\tau}$$

$$(-1+\Phi^{-1})\sigma\Phi y_t^n + (1-\Phi^{-1})\sigma_\upsilon y_t^* + (1-\Phi^{-1})(1-\upsilon)\Phi z_t + (1-\Phi^{-1})G_Y\sigma_\upsilon g_t - \frac{1}{1-\tau}$$

$$-\sigma\Phi y_t^n + \sigma y_t^n + (1-\Phi^{-1})\sigma_\upsilon y_t^* + (1-\Phi^{-1})(1-\upsilon)\Phi z_t + (1-\Phi^{-1})G_Y\sigma_\upsilon g_t - \frac{1}{1-\tau}$$

$$(1-\Phi)\sigma y_t^n + (1-\Phi^{-1})\sigma_\upsilon y_t^* + (1-\Phi^{-1})(1-\upsilon)\Phi z_t + (1-\Phi^{-1})G_Y\sigma_\upsilon g_t - \frac{1}{1-\tau}$$

$$(1-\Phi^{-1})\sigma_\upsilon y_t^* + (1-\Phi^{-1})(1-\upsilon)\Phi z_t + (1-\Phi^{-1})G_Y\sigma_\upsilon g_t - \frac{1}{1-\tau}\tau_t + \left(1 + \frac{\varphi+}{1-}\right.$$

$$(1-\Phi^{-1})\sigma_\upsilon y_t^* + (1-\Phi^{-1})(1-\upsilon)\Phi z_t + (1-\Phi^{-1})G_Y\sigma_\upsilon g_t - \frac{1}{1-\tau}\tau_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z$$

$$(1-\Phi^{-1})\sigma_\upsilon y_t^* + (1-\Phi^{-1})(1-\upsilon)\Phi z_t + (1-\Phi^{-1})G_Y\sigma_\upsilon g_t - \frac{1}{1-\tau}\tau_t + \left(1 + \frac{\varphi+\alpha}{1-\alpha}\right)a_t - \upsilon z_t - \mu_t =$$

$$(1-\Phi^{-1})\sigma_\upsilon y_t^* + (1-\Phi^{-1})(1-\upsilon)\Phi z_t + (1-\Phi^{-1})G_Y\sigma_\upsilon g_t - \frac{1}{1-\tau}\tau_t + \left(1+\right.$$

$$\left(\frac{-\sigma_\upsilon(1-\alpha)+\varphi+\alpha}{1-\alpha}\right)^{-1}\left((1-\Phi^{-1})\sigma_\upsilon y_t^* + (1-\Phi^{-1})(1-\upsilon\right.$$

$$\Gamma_* y_t^* + \left(\frac{-\sigma_\upsilon(1-\alpha)+\varphi+\alpha}{1-\alpha}\right)^{-1}\left((1-\Phi^{-1})(1-\upsilon)\Phi z_t + (1-\Phi^{-1})G_Y\right.$$

$$\Gamma_* y_t^* + \left(\frac{-\sigma_\upsilon(1-\alpha)+\varphi+\alpha}{1-\alpha}\right)^{-1}\left(\upsilon\varpi\Phi z_t + (1-\Phi\right.$$

$$\Gamma_* y_t^* + \Gamma_z z_t + \Gamma_a a_t + \left(\frac{-\sigma_\upsilon(1-\alpha)+}{1-\alpha}\right.$$

$$\Gamma_* y_t^* + \Gamma_z z_t + \Gamma_a a_t + $$

$$y_t^n = \Gamma_* y_t^* + \Gamma_z z_t + \Gamma_a a_t + \Gamma_\tau \tau_t \tag{A.133}$$

$$\Gamma_* = -\frac{\upsilon(\varpi - 1)\sigma_\upsilon(1-\alpha)}{\sigma_\upsilon(1-\alpha) + \varphi + \alpha} \tag{A.134}$$

$$\Gamma_z = -\frac{\upsilon\varpi\Phi(1-\alpha)}{\sigma_\upsilon(1-\alpha) + \varphi + \alpha} \tag{A.135}$$

$$\Gamma_a = \frac{1+\varpi}{\sigma_\upsilon(1-\alpha) + \varphi + \alpha} \tag{A.136}$$

$$\Gamma_g = -\frac{\upsilon(\varpi - 1)\sigma_\upsilon(1-\alpha)}{\sigma_\upsilon(1-\alpha) + \varphi + \alpha} \times \tag{A.137}$$

$$\Gamma_\tau = \frac{(1-\tau)(1-\alpha)}{\sigma_\upsilon(1-\alpha) + \varphi + \alpha} \tag{A.138}$$

$$c_t = \mathbb{E}\{c_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}\{\pi_{t+1}\} - \rho) + \frac{1}{\sigma}(1-\rho_z)z_t \tag{A.139}$$

$$\pi_t = \pi_{H,t} + \upsilon \Delta s_t \tag{A.140}$$

$$\implies c_t = \mathbb{E}\{c_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}\{\pi_{H,t+1}\} - \rho) + \frac{\upsilon}{\sigma}\mathbb{E}\{\Delta s_{t+1}\} + \frac{1}{\sigma}(1-\rho_z)z_t \tag{A.141}$$

$$c_t = y_t^* + \frac{1}{\sigma}z_t + \left(\frac{1-\upsilon}{\sigma}\right)s_t \tag{A.142}$$

$$Y_t(i) = C(i)_t + X(i)_t + G(i)_t \tag{A.143}$$

$$= \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\epsilon}\left[(1-\upsilon)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \upsilon \mathcal{S}_t^{\eta} Y_t^* + G_t\right] \tag{A.144}$$

Given that

$$Y_t = \left(\int_0^1 (Y_t(i))^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{A.145}$$

$$Y_t = (1-\upsilon)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \upsilon \mathcal{S}_t^{\eta} Y_t^* + G_t \tag{A.146}$$

$$Y\mathbf{e}^{y_t} = (1-\upsilon)\left(\frac{P}{P_H}\right)^{\eta} C\mathbf{e}^{-\eta p_{H,t} + \eta p_t + c_t} + \upsilon S^{\eta} Y^* \mathbf{e}^{\eta s_t + y_t^*} + G\mathbf{e}^{g_t} \tag{A.147}$$

$$Y(1+y_t) = (1-\upsilon)\left(\frac{P}{P_H}\right)^{\eta} C(1 - \eta p_{H,t} + \eta p_t + c_t) + \upsilon S^{\eta} Y^*(1 + \eta s_t + y_t^*) + G(1+g_t) \tag{A.148}$$

$$Y y_t = (1-\upsilon)\left(\frac{P}{P_H}\right)^{\eta} C(-\eta p_{H,t} + \eta p_t + c_t) + \upsilon S^{\eta} Y^*(\eta s_t + y_t^*) + G g_t \tag{A.149}$$

$$\text{Using } P/P_H = \upsilon\mathcal{S},\ \mathcal{S} = 1, \text{ and } C = Y^* : \tag{A.150}$$

$$y_t = (1-\upsilon)(-\eta p_{H,t} + \eta p_t + c_t) + \upsilon(\eta s_t + y_t^*) + G_Y g_t \tag{A.151}$$

$$\text{Using } p_t - p_{H,t} = \upsilon s_t : \tag{A.152}$$

$$y_t = (1-\upsilon)(\eta\upsilon s_t + c_t) + \upsilon(\eta s_t + y_t^*) + G_Y g_t \tag{A.153}$$

$$y_t = (1-\upsilon)c_t + (1-\upsilon)\eta\upsilon s_t + \upsilon\eta s_t + \upsilon y_t^* + G_Y g_t \tag{A.154}$$

$$y_t = (1-\upsilon)c_t + \upsilon(2-\upsilon)\eta s_t + \upsilon y_t^* + G_Y g_t \tag{A.155}$$

$$\tag{A.156}$$

$$c_t = y_t^* + \frac{1}{\sigma}z_t + \frac{1-\upsilon}{\sigma}s_t \tag{A.157}$$

$$y_t = (1-\upsilon)c_t + \upsilon(2-\upsilon)\eta s_t + \upsilon y_t^* + G_Y g_t \tag{A.158}$$

$$\implies y_t = (1-\upsilon)\left(y_t^* + \frac{1}{\sigma}z_t + \frac{1-\upsilon}{\sigma}s_t\right) + \upsilon(2-\upsilon)\eta s_t + \upsilon y_t^* + G_Y g_t \tag{A.159}$$

$$y_t = (1-\upsilon)y_t^* + \frac{1-\upsilon}{\sigma}z_t + (1-\upsilon)\frac{1-\upsilon}{\sigma}s_t + \upsilon(2-\upsilon)\eta s_t + \upsilon y_t^* + \tag{A.160}$$

$$y_t = y_t^* + \frac{1-\upsilon}{\sigma}z_t + (1-\upsilon)\frac{1-\upsilon}{\sigma}s_t + \upsilon(2-\upsilon)\eta s_t + G_Y g_t \tag{A.161}$$

$$y_t = y_t^* + \frac{1-\upsilon}{\sigma}z_t + \left((1-\upsilon)\frac{1-\upsilon}{\sigma} + \upsilon(2-\upsilon)\eta\right)s_t + G_Y g_t \tag{A.162}$$

$$\left((1-\upsilon)\frac{1-\upsilon}{\sigma} + \upsilon(2-\upsilon)\eta\right)s_t = y_t - y_t^* - \frac{1-\upsilon}{\sigma}z_t - G_Y g_t \tag{A.163}$$

$$\left(\frac{(1-\upsilon)(1-\upsilon) + \sigma\upsilon(2-\upsilon)\eta}{\sigma}\right)s_t = y_t - y_t^* - \frac{1-\upsilon}{\sigma}z_t - G_Y g_t \tag{A.164}$$

$$\left(\frac{1-2\upsilon+\upsilon^2 + \sigma\upsilon(2-\upsilon)\eta}{\sigma}\right)s_t = y_t - y_t^* - \frac{1-\upsilon}{\sigma}z_t - G_Y g_t \tag{A.165}$$

$$\left(\frac{1-\upsilon(2-\upsilon) + \sigma\upsilon(2-\upsilon)\eta}{\sigma}\right)s_t = y_t - y_t^* - \frac{1-\upsilon}{\sigma}z_t - G_Y g_t \tag{A.166}$$

$$\left(\frac{1-\upsilon(2-\upsilon)(1-\sigma\eta)}{\sigma}\right)s_t = y_t - y_t^* - \frac{1-\upsilon}{\sigma}z_t - G_Y g_t \tag{A.167}$$

$$\left(\frac{1+\upsilon(2-\upsilon)(\sigma\eta-1)}{\sigma}\right)s_t = y_t - y_t^* - \frac{1-\upsilon}{\sigma}z_t - G_Y g_t \tag{A.168}$$

$$\left(\frac{1+\upsilon(2\sigma\eta-\upsilon\sigma\eta-2+\upsilon)}{\sigma}\right)s_t = y_t - y_t^* - \frac{1-\upsilon}{\sigma}z_t - G_Y g_t \tag{A.169}$$

$$\left(\frac{1+\upsilon(\varpi-1)}{\sigma}\right)s_t = y_t - y_t^* - \frac{1-\upsilon}{\sigma}z_t - G_Y g_t \tag{A.170}$$

$$\sigma^{-1}\Phi^{-1}s_t = y_t - y_t^* - \frac{1-\upsilon}{\sigma}z_t - G_Y g_t \tag{A.171}$$

$$s_t = \sigma\Phi(y_t - y_t^*) - (1-\upsilon)\Phi z_t - G_Y\sigma\Phi g_t \tag{A.172}$$

$$s_t = \sigma_\upsilon(y_t - y_t^*) - (1-\upsilon)\Phi z_t - G_Y\sigma_\upsilon g_t \tag{A.173}$$

$$\tag{A.174}$$

$$\Phi = \frac{1}{1 + \upsilon(\varpi - 1)} \tag{A.175}$$

$$\varpi = \sigma\eta + (1 - \upsilon)(\sigma\eta - 1) = 2\sigma\eta - \upsilon\sigma\eta - 1 + \upsilon \tag{A.176}$$

$$\sigma_\upsilon = \sigma\Phi \tag{A.177}$$

$$y_t = (1-\upsilon)c_t + \upsilon(2-\upsilon)\eta s_t + \upsilon y_t^* + G_Y g_t \tag{A.178}$$

$$\implies c_t = (1-\upsilon)^{-1}(y_t - \upsilon(2-\upsilon)\eta s_t - \upsilon y_t^* - G_Y g_t) \tag{A.179}$$

$$c_t = \mathbb{E}\{c_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}\{\pi_{t+1}\} - \rho) + \frac{\upsilon}{\sigma}\mathbb{E}\{\Delta s_{t+1}\} + \frac{1}{\sigma}(1-\rho_z)z_t \tag{A.180}$$

$$s_t = \sigma_\upsilon(y_t - y_t^*) - (1-\upsilon)\Phi z_t - G_Y \sigma_\upsilon g_t \tag{A.181}$$

$$\tag{A.182}$$

$$0 = \mathbb{E}\{\Delta c_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \frac{\upsilon}{\sigma}\,\mathbb{E}\{\Delta s_{t+1}\} + \frac{1}{\sigma}(1-\rho_z)z_t$$

$$0 = \mathbb{E}\{\Delta y_{t+1}\} - \upsilon(2-\upsilon)\eta\,\mathbb{E}\{\Delta s_{t+1}\} - \upsilon\,\mathbb{E}\{\Delta y^*_{t+1}\} - G_Y\Delta g_{t+1} - \frac{1-\upsilon}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \frac{(1-}{\sigma}$$

$$0 = \mathbb{E}\{\Delta y_{t+1}\} + \left(\frac{(1-\upsilon)\upsilon}{\sigma} - \upsilon(2-\upsilon)\eta\right)\mathbb{E}\{\Delta s_{t+1}\} - \upsilon\,\mathbb{E}\{\Delta y^*_{t+1}\} - G_Y\Delta g_{t+1} - \frac{1-\upsilon}{\sigma}(i_t - \mathbb{E}\{\pi_{H,}$$

$$0 = \mathbb{E}\{\Delta y_{t+1}\} - \left(\frac{\upsilon\varpi}{\sigma}\right)\mathbb{E}\{\Delta s_{t+1}\} - \upsilon\,\mathbb{E}\{\Delta y^*_{t+1}\} - G_Y\Delta g_{t+1} - \frac{1-\upsilon}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \frac{1-\upsilon}{\sigma}(1$$

$$0 = \mathbb{E}\{\Delta y_{t+1}\} - \left(\frac{\upsilon\varpi}{\sigma}\right)\mathbb{E}\left(\sigma_\upsilon(\Delta y_{t+1} - \Delta y^*_{t+1}) - (1-\upsilon)\Phi(\Delta z_{t+1}) - G_Y\sigma_\upsilon\Delta g_{t+1}\right) - \upsilon\,\mathbb{E}\{\Delta y^*_{t+1}\} - \frac{1}{}$$

$$0 = \mathbb{E}\{\Delta y_{t+1}\} - \upsilon\varpi\Phi\Delta y_{t+1} + \upsilon\varpi\Phi\Delta y^*_{t+1} - \upsilon\,\mathbb{E}\{\Delta y^*_{t+1}\} + \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t + \upsilon\varpi\Phi\frac{1-\upsilon}{\sigma}(\rho_z-1)z_t -$$

$$\begin{aligned}
0 &= \Phi^{-1}\,\mathbb{E}\{\Delta y_{t+1}\} - \upsilon\varpi\Delta y_{t+1} + \upsilon\varpi\Delta y^*_{t+1} - \upsilon\Phi^{-1}\,\mathbb{E}\{\Delta y^*_{t+1}\} \\
&+ \Phi^{-1}\frac{1-\upsilon}{\sigma}(1-\rho_z)z_t + \upsilon\varpi\frac{1-\upsilon}{\sigma}(\rho_z-1)z_t + \upsilon\varpi G_Y\Delta g_{t+1} - G_Y\Delta g_{t+1} \\
&- \frac{1-\upsilon}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho)
\end{aligned}$$

$$\begin{aligned}
0 &= (\Phi^{-1} - \upsilon\varpi)\Delta y_{t+1} + \upsilon(\varpi - \Phi^{-1})\,\mathbb{E}\{\Delta y^*_{t+1}\} \\
&- (\Phi^{-1} - \upsilon\varpi)\frac{1-\upsilon}{\sigma}(\rho_z-1)z_t + (\upsilon\varpi - 1)G_Y\Delta g_{t+1} \\
&- \frac{1-\upsilon}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho)
\end{aligned}$$

$$\begin{aligned}
0 &= \Delta y_{t+1} + \frac{\upsilon(\varpi - \Phi^{-1})}{\Phi^{-1} - \upsilon\varpi}\,\mathbb{E}\{\Delta y^*_{t+1}\} \\
&- \frac{1-\upsilon}{\sigma}(\rho_z-1)z_t + \frac{(\upsilon\varpi - 1)}{\Phi^{-1} - \upsilon\varpi}G_Y\Delta g_{t+1} \\
&- \frac{1-\upsilon}{\Phi^{-1} - \upsilon\varpi}\frac{1}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho)
\end{aligned}$$

$$\begin{aligned}
y_t &= y_{t+1} - \frac{1-\upsilon}{\sigma}(\rho_z-1)z_t \\
&+ \frac{(\upsilon\varpi - 1)}{\Phi^{-1} - \upsilon\varpi}G_Y\Delta g_{t+1} + \frac{\upsilon(\varpi - \Phi^{-1})}{\Phi^{-1} - \upsilon\varpi}\,\mathbb{E}\{\Delta y^*_{t+1}\} - \frac{1-\upsilon}{\Phi^{-1} - \upsilon\varpi}\frac{1}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho)
\end{aligned}$$

$$y_t = y_{t+1} - \frac{1-\upsilon}{\sigma}(\rho_z-1)z_t - \frac{1-\upsilon}{1-\upsilon}\frac{1}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \frac{(1-\upsilon)(\upsilon\varpi - \upsilon)}{1-\upsilon}\,\mathbb{E}\{\Delta y^*_{t+1}\} + \frac{\upsilon\varpi - 1}{1-\upsilon}G$$

$$y_t = y_{t+1} - \frac{1}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y^*_{t+1}\} + \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t + \frac{\upsilon\varpi - 1}{1-\upsilon}G_Y\Delta g_{t+1}$$

$$y_t^n = y_{t+1}^n - \frac{1}{\sigma_\upsilon}(r_t^n - \rho) + \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} + \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t + \frac{\upsilon\varpi - 1}{1-\upsilon}G_Y\Delta g_{t+1} \tag{A.183}$$

$$0 = \Delta y_{t+1}^n - \frac{1}{\sigma_\upsilon}(r_t^n - \rho) + \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} + \frac{1-\upsilon}{\sigma}(1-\rho_z)z_t + \frac{\upsilon\varpi - 1}{1-\upsilon}G_Y\Delta g_{t+1} \tag{A.184}$$

$$0 = \sigma_\upsilon\Delta y_{t+1}^n - (r_t^n - \rho) + \sigma_\upsilon\upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} + \Phi(1-\upsilon)(1-\rho_z)z_t + \sigma_\upsilon\frac{\upsilon\varpi - 1}{1-\upsilon}G_Y\Delta g_{t+1} \tag{A.185}$$

$$r_t^n = \sigma_\upsilon\Delta y_{t+1}^n + \rho + \sigma_\upsilon\upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} + \Phi(1-\upsilon)(1-\rho_z)z_t + \sigma_\upsilon\frac{\upsilon\varpi - 1}{1-\upsilon}G_Y\Delta g_{t+1} \tag{A.186}$$

$$\tag{A.187}$$

$$0 = \sigma_\upsilon\Delta y_{t+1} - (i_t - \mathbb{E}\{\pi_{H,+1}\}) + \rho + \sigma_\upsilon\upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} + \Phi(1-\upsilon)(1-\rho_z)z_t + \sigma_\upsilon\frac{\upsilon\varpi - 1}{1-\upsilon}G_Y\Delta \tag{A.188}$$

$$-(\sigma_\upsilon\Delta y_{t+1}^n - (r_t^n - \rho) + \sigma_\upsilon\upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} + \Phi(1-\upsilon)(1-\rho_z)z_t + \sigma_\upsilon\frac{\upsilon\varpi - 1}{1-\upsilon}G_Y\Delta g_{t+1}) \tag{A.189}$$

$$0 = \sigma_\upsilon\Delta\tilde{y}_{t+1} - (i_t - \mathbb{E}\{\pi_{H,+1}\} - r_t^n) \tag{A.190}$$

$$\tilde{y}_t = \tilde{y}_{t+1} - \frac{1}{\sigma_\upsilon}(i_t - \mathbb{E}\{\pi_{H,+1}\} - r_t^n) \quad \text{Dynamic IS Curve} \tag{A.191}$$

$$\tag{A.192}$$

$$r_t^n = \sigma_\upsilon \Delta y_{t+1}^n + \rho + \sigma_\upsilon \upsilon(\varpi - 1)\,\mathbb{E}\{\Delta y_{t+1}^*\} + \Phi(1-\upsilon)(1-\rho_z)z_t + \sigma_\upsilon \frac{\upsilon\varpi - 1}{1-\upsilon} G_Y \Delta g_{t+1} \tag{A.193}$$

$$\sigma_\upsilon \Delta y_{t+1}^n = \sigma_\upsilon(\Gamma_* \Delta y_{t+1}^* + \Gamma_z \Delta z_{t+1} + \Gamma_a \Delta a_{t+1} + \Gamma_g \Delta g_{t+1} + \Gamma_\tau \Delta \tau_{t+1}) \tag{A.194}$$

$$y_t^* : \sigma_\upsilon \Gamma_* \Delta y_{t+1}^* + \sigma_\upsilon \upsilon(\varpi - 1)\Delta y_{t+1}^* = \sigma_\upsilon(\Gamma_* + \upsilon(\varpi - 1))\Delta y_{t+1}^* = \Psi_* y_{t+1}^* \tag{A.195}$$

$$z_t : \sigma_\upsilon \Gamma_z \Delta z_{t+1} + \Phi(1-\rho_z)z_t = \sigma_\upsilon \Gamma_z(\rho_z - 1)z_t + \Phi(1-\upsilon)(1-\rho_z)z_t \tag{A.196}$$

$$= \sigma_\upsilon \Gamma_z(\rho_z - 1)z_t + \Phi(1-\upsilon)(1-\rho_z)z_t = (\Phi(1-\upsilon) - \sigma_\upsilon \Gamma_z)(1-\rho_z)z_t = \Psi_z(1-\rho_z)z_t \tag{A.197}$$

$$a_t : \sigma_\upsilon \Gamma_a \Delta a_{t+1} = \sigma_\upsilon \Gamma_a(\rho_a - 1)a_t = -\sigma_\upsilon \Gamma_a(1-\rho_a)a_t \tag{A.198}$$

$$g_t : \sigma_\upsilon \Gamma_g G_Y \Delta g_{t+1} + \sigma_\upsilon \frac{\upsilon\varpi - 1}{1-\upsilon} G_Y \Delta g_{t+1} = \left(\sigma_\upsilon \left(\Gamma_g + \frac{\upsilon\varpi - 1}{1-\upsilon}\right)\right) G_Y \Delta g_{t+1} = -\Psi_g G_Y(1 - \tag{A.199}$$

$$\tau_t : \sigma_\upsilon \Gamma_\tau \Delta \tau_{t+1} \tag{A.200}$$

$$r_t^n = \rho + \Psi_* y_{t+1}^* + \Psi_z(1-\rho_z)z_t - \sigma_\upsilon \Gamma_a(1-\rho_a)a_t - \Psi_g G_Y(1-\rho_g)g_t + \sigma_\upsilon \Gamma_\tau \Delta \tau_{t+1} \tag{A.201}$$

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