Two-country small open economy NK DSGE model for Scotland and the Rest of the UK.

1. Home country: Scotland
2. Foreign country: RUK
3. Scotland and RUK are small and open economies
   (a) They take world output, inflation, and consumption as given and cannot influence it
4. Scotland and RUK are assumed to be symmetrical in market structure and preferences
5. Shocks in Scotland, RUK, and in the World Economy are assumed to be correlated
6. Sticky nominal prices: Calvo Fairy
   (a) Here, I would prefer to use Rotemburg as it is more intuitive and less popular than Calvo Fairy
7. For simplicity:
   (a) No nontradeable goods
   (b) No trading costs
   (c) No possibility of international policy coordination
   (d) No cost-push shocks
   (e) No nominal wage rigidities
   (f) No international financial assets

1 Households

Consumer's problem

A representative household wants to maximise the lifetime discounted utility:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right) \tag{1}$$

Utility is is decreasingly increasing in consumption C_t and decreasingly decreasing in hours worked N_t.

C_t is a composite consumption index defined by

$$C_t = \left[(1-\alpha)^{\frac{1}{\eta}} (C_{H,t})^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} (C_{F,t})^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}} \tag{2}$$

$$C_{H,t} = \left(\int_0^1 C_{H,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} \, dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \qquad \text{Index of consumption of home produced goods} \tag{3}$$

$$C_{i,t} = \left(\int_0^1 C_{i,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} \, dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \qquad \text{Index of consumption of country } i\text{'s produced goods} \tag{4}$$

$$C_{F,t} = \left(\int_0^1 C_{i,t}{}^{\frac{\gamma-1}{\gamma}} \, di \right)^{\frac{\gamma}{\gamma-1}} \qquad \text{Index of consumption of imported goods} \tag{5}$$

Optimal allocation of each variety of goods:

$$C_{H,t}(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}} \right)^{-\varepsilon} C_{H,t}; \quad C_{i,t}(j) = \left(\frac{P_{i,t}(j)}{P_{i,t}} \right)^{-\varepsilon} C_{i,t}; \quad C_{i,t} = \left(\frac{P_{i,t}}{P_{F,t}} \right)^{-\gamma} C_{F,t} \tag{6}$$

$$P_{H,t} = \left(\int_0^1 P_{H,t}(j)^{1-\varepsilon} \, dj \right)^{\frac{1}{1-\varepsilon}} \qquad \text{Domestic Price Index} \tag{7}$$

$$P_{i,t} = \left(\int_0^1 P_{i,t}(j)^{1-\varepsilon} \, dj \right)^{\frac{1}{1-\varepsilon}} \qquad \text{Price Index of goods produced by country } i \tag{8}$$

$$P_{F,t} = \left(\int_0^1 P_{i,t}^{1-\gamma} \, dj \right)^{\frac{1}{1-\gamma}} \qquad \text{Price Index of Imported goods} \tag{9}$$

$$\int_0^1 P_{H,t}(j) C_{H,t}(j) \, dj = P_{H,t} C_{H,t} \qquad \int_0^1 P_{i,t}(j) C_{i,t}(j) \, dj = P_{i,t} C_{i,t} \tag{10}$$

Using (6) and (9) implies

$$\int_0^1 P_{i,t} C_{i,t} = P_{F,t} C_{F,t} \tag{11}$$

$$C_{H,t} = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t \qquad\qquad C_{F,t} = \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t \tag{12}$$

$$P_t = \left[(1-\alpha)(P_{H,t})^{1-\eta} + \alpha(P_{F,t})^{1-\eta}\right]^{\frac{1}{1-\eta}} \qquad \text{Consumption Price Index} \tag{13}$$

A special case $\eta = 1$:

$$P_t = (P_{H,t})^{1-\alpha} \times (P_{F,t})^{\alpha} \qquad\qquad C_t = \frac{1}{(1-\alpha)^{(1-\alpha)}\alpha^{\alpha}}(C_{H,t})^{(1-\alpha)}(C_{F,t})^{\alpha} \tag{14}$$

Total consumption expenditures are:

$$P_{H,t}C_{H,t} + P_{F,t}C_{F,t} = P_tC_t \tag{15}$$

So the budget constraint is:

$$P_tC_t + \mathbb{E}_t[Q_{t,t+1}D_{t+1}] \leq D_t + W_tN_t + T_t \tag{16}$$

$$\begin{aligned}\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} &\left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}\right) \\ &+ \lambda_t \left\{D_t + W_tN_t + T_t - P_tC_t - \mathbb{E}_t[Q_{t,t+1}D_{t+1}]\right\}\end{aligned} \tag{17}$$

$$\begin{aligned}\frac{\partial L}{\partial C_t} &= \beta^t C_t^{-\sigma} - \lambda_t P_t = 0; \quad \Rightarrow \quad \beta^t C_t^{-\sigma} P_t^{-1} = \lambda_t \\ \frac{\partial L}{\partial N_t} &= -\beta^t N_t^{\varphi} + \lambda_t N_t = 0; \quad \Rightarrow \quad \beta^t N_t^{\varphi} W_t^{-1} = \lambda_t \\ \frac{\partial L}{\partial C_t} = \frac{\partial L}{\partial N_t} &: \beta^t C_t^{-\sigma} P_t^{-1} = \beta^t N_t^{\varphi} W_t^{-1} \\ &\Rightarrow C_t^{-\sigma} P_t^{-1} = N_t^{\varphi} W_t^{-1} \\ &\Rightarrow C_t^{-\sigma} N_t^{-\varphi} = W_t^{-1} P_t\end{aligned}$$

$$\Rightarrow \qquad C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t} \qquad \text{Intratemporal Optimality Condition} \tag{18}$$

$$\begin{aligned}\frac{\partial L}{\partial C_t} &= \beta^t C_t^{-\sigma} P_t^{-1} = \lambda_t; \quad \Rightarrow \quad \mathbb{E}_t[\beta^{t+1} C_{t+1}^{-\sigma} P_{t+1}^{-1}] = \mathbb{E}_t[\lambda_{t+1}] \\ \frac{\partial L}{\partial D_{t+1}} &= -\lambda_t \, \mathbb{E}_t[Q_{t,t+1}] + \mathbb{E}_t[\lambda_{t+1}] = 0; \quad \Rightarrow \quad \mathbb{E}_t[Q_{t,t+1}] = \frac{\lambda_{t+1}}{\lambda_t}\end{aligned}$$

$$\begin{aligned}\mathbb{E}_t\left[\frac{\beta^t C_t^{-\sigma} P_t^{-1}}{\beta^{t+1} C_{t+1}^{-\sigma} P_{t+1}^{-1}}\right] &= \mathbb{E}_t\left[\frac{\lambda_t}{\lambda_{t+1}}\right] \\ \mathbb{E}_t\left[\frac{\beta^t C_t^{-\sigma} P_t^{-1}}{\beta^{t+1} C_{t+1}^{-\sigma} P_{t+1}^{-1}}\right] &= \mathbb{E}_t\left[\frac{1}{Q_{t,t+1}}\right] \\ \mathbb{E}_t\left[\frac{1}{\beta}\left(\frac{C_t}{C_{t+1}}\right)^{-\sigma}\left(\frac{P_t}{P_{t+1}}\right)^{-1}\right] &= \mathbb{E}_t\left[\frac{1}{Q_{t,t+1}}\right]\end{aligned}$$

$$\beta\,\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{P_t}{P_{t+1}}\right)\right] = Q_t \qquad \text{Euler equation} \tag{19}$$

However, Gali uses a different approach to derive the Euler equation, which introduces Arrow securities:

$$\frac{V_{t,t+1}}{P_t}C_t^{-\sigma} = \xi_{t,t+1}\beta C_{t+1}^{-\sigma}\frac{1}{P_{t+1}} \tag{20}$$

Where $V_{t,t+1}$ is an Arrow security and $\xi_{t,t+1}$ is the probability that the Arrow security will yield a payoff next period. Interpretation: LHS is paying for the security (expenses) in terms of consumption at P_t prices. RHS is the payoff if the Arrow security yields a payoff. The consumer will only be willing to pay LHS if the payoff is at least as big on the RHS. Given that:

$$Q_{t,t+1} = \frac{V_{t,t+1}}{\xi_{t,t+1}} \tag{21}$$

$$\frac{V_{t,t+1}}{P_t}C_t^{-\sigma} = \xi_{t,t+1}\beta C_{t+1}^{-\sigma}\frac{1}{P_{t+1}} \tag{22}$$

$$Q_{t,t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}} \tag{23}$$

$$\mathbb{E}_t[Q_{t,t+1}] = \beta\,\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right] \tag{24}$$

$$\beta\,\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{P_t}{P_{t+1}}\right)\right] = Q_t \qquad \text{Euler equation} \tag{25}$$

Log-linearising (18):

$$C_t^{\sigma}N_t^{\varphi} = \frac{W_t}{P_t}; \quad \Rightarrow \quad w_t - p_t = \sigma c_t + \varphi n_t \tag{26}$$

Log-linearising (25):

$$\begin{aligned}
\beta\,\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{P_t}{P_{t+1}}\right)\right] &= Q_t\\
\ln\beta - \mathbb{E}_t[\sigma c_{t+1}] + \sigma c_t + p_t - \mathbb{E}_t[p_{t+1}] &= \ln Q_t\\
\sigma c_t &= \ln Q_t - \ln\beta + \mathbb{E}_t[\sigma c_{t+1}] - p_t + \mathbb{E}_t[p_{t+1}]\\
c_t &= \mathbb{E}_t[c_{t+1}] - \frac{1}{\sigma}(-\ln Q_t - \rho - \mathbb{E}_t[\pi_{t+1}])\\
c_t &= \mathbb{E}_t[c_{t+1}] - \frac{1}{\sigma}(i_t - \mathbb{E}_t[\pi_{t+1}] - \rho) \qquad (27)\\
\text{where } i_t = -\log Q_t,\ \rho &= -\log\beta,\ \pi_t = p_t - p_{t-1}
\end{aligned}$$

$$S_{i,t} = \frac{P_{i,t}}{P_{H,t}} \qquad \text{Bilateral terms of trade} \tag{28}$$

$$S_t = \frac{P_{F,t}}{P_{H,t}} = \left(\int_0^1 (S_{i,t}\,di)^{1-\gamma}\right)^{\frac{1}{1-\gamma}} \qquad \text{Effective terms of trade} \tag{29}$$

$$s_t = p_{F,t} - p_{H,t} = \left(\int_0^1 s_{i,t}\,di\right) \qquad \text{(log) Effective terms of trade} \tag{30}$$

Recall that when $\eta = 1$, then CPI is $P_t = (P_{H,t})^{1-\alpha} \times (P_{F,t})^{\alpha}$, which can be log-linearised to:

$$p_t = (1-\alpha)p_{H,t} + \alpha p_{F,t} = p_{H,t} + \alpha s_t \tag{31}$$

Equations (30) and (31) hold when $\gamma = 1$ and $\eta = 1$, respectively.

$$\pi_{H,t} = p_{H,t+1} - p_{H,t} \qquad \text{Domestic Inflation} \tag{32}$$

$$\pi_t = \pi_{H,t} + \alpha \Delta s_t \qquad \text{CPI Inflation} \tag{33}$$

The gap between domestic inflation and CPI inflation is only due to percentage change in the terms of trade.

$$P_{i,t}(j) = \mathcal{E}_{i,t} P^i_{i,t}(j) \qquad \text{Law of One Price (LOP)} \tag{34}$$

$$P_{i,t} = \mathcal{E}_{i,t} P^i_{i,t} \qquad \text{Law of One Price (LOP)} \tag{35}$$

$$p_{i,t} = e_{i,t} + p^i_{i,t} \qquad \text{(Log) Law of One Price (LOP)} \tag{36}$$

$$p_{F,t} = \int_0^1 (e_{i,t} + p^i_{i,t})\, di = e_t + p^\star_t \qquad \text{(Log)Price index of Imported Goods} \tag{37}$$

Where e_t is (Log) Effective Nominal Exchange Rate, $p^\star_t$ is the World Price Index.

$$s_t = p_{F,t} - p_{H,t} = e_t + p^\star_t - p_{H,t} \quad \text{Terms of trade but with the World Price Index} \tag{38}$$

$$\mathcal{Q}_{i,t} = \frac{\mathcal{E}_{i,t} P^i_t}{P_t} \qquad \text{Bilateral Exchange Rate} \tag{39}$$

$$q_t = \int_0^1 \log\left(\frac{\mathcal{E}_{i,t} P^i_t}{P_t}\right) di \tag{40}$$

$$= \int_0^1 (e_{i,t} + p^i_{i,t} - p_t)\, di \tag{41}$$

$$= e_t + p^\star_t - p_t \qquad \text{using (37)} \tag{42}$$

$$= s_t + p_{H_t} - p_t \qquad \text{using (38)} \tag{43}$$

$$= (1-\alpha)s_t \qquad \text{using (31)} \tag{44}$$

International Risk-Sharing Equation (20) for country i can be rewritten as:

$$\frac{V_{t,t+1}}{\mathcal{E}^i_t P^i_t}(C^i_t)^{-\sigma} = \xi_{t,t+1}\beta(C^i_{t+1})^{-\sigma}\frac{1}{\mathcal{E}^i_{t+1} P^i_{t+1}} \tag{45}$$

Given that:

$$Q_{t,t+1} = \frac{V_{t,t+1}}{\xi_{t,t+1}} \tag{46}$$

$$\frac{V_{t,t+1}}{\mathcal{E}_t^i P_t^i}(C_t^i)^{-\sigma} = \xi_{t,t+1}\beta(C_{t+1}^i)^{-\sigma}\frac{1}{\mathcal{E}_{t+1}^i P_{t+1}^i} \tag{47}$$

$$Q_{t,t+1} = \beta\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\sigma}\left(\frac{P_t^i}{P_{t+1}^i}\right)\left(\frac{\mathcal{E}_t^i}{\mathcal{E}_{t+1}^i}\right) \tag{48}$$

$$Q_{t,t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}} \text{ using Equation (XX)} \tag{49}$$

$$\beta\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\sigma}\left(\frac{P_t^i}{P_{t+1}^i}\right)\left(\frac{\mathcal{E}_t^i}{\mathcal{E}_{t+1}^i}\right) = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}} \tag{50}$$

$$\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\sigma}\left(\frac{P_t^i}{P_{t+1}^i}\right)\left(\frac{\mathcal{E}_t^i}{\mathcal{E}_{t+1}^i}\right) = \left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}} \tag{51}$$

$$\left(\frac{C_t^i}{C_{t+1}^i}\right)^{\sigma}\left(\frac{P_t^i}{P_{t+1}^i}\right)\left(\frac{\mathcal{E}_t^i}{\mathcal{E}_{t+1}^i}\right) = \left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\frac{P_t}{P_{t+1}} \tag{52}$$

$$C_t^{i\sigma} P_t^i \mathcal{E}_t^i = C_t^\sigma P_t \tag{53}$$

$$C_t^{i\sigma} = C_t^\sigma \frac{1}{\mathcal{Q}_{i,t}} \tag{54}$$

$$C_t = C_t^i \mathcal{Q}_{i,t}^{\frac{1}{\sigma}} \tag{55}$$

Log-linearising (55) yields:

$$c_t = c_t^i + \frac{1}{\sigma}q_{i,t} \tag{56}$$

Integrating both sides over i:

$$c_t = c_t^\star + \frac{1}{\sigma}q_t \tag{57}$$

$$= c_t^\star + \left(\frac{1-\alpha}{\sigma}\right)s_t \qquad \text{using } q_t = (1-\alpha)s_t \tag{58}$$

$c_t^\star$ is the log world consumption. Equation (58) is the link between the domestic consumption and the world consumption.

2 Firms

$$Y_t(j) = A_t N_t(j) \tag{59}$$

$$\log A_t = \alpha_t \tag{60}$$

$$\alpha_t = \rho_a \alpha_{t-1} + \varepsilon_t \tag{61}$$

$$L = P_t(j)Y_t(j) - W_t(j)N_t(j) \tag{62}$$

$$\Rightarrow L = P_t Y_t - W_t N_t \tag{63}$$

$$\Rightarrow L = P_t A_t N_t - W_t N_t \tag{64}$$

$$\tag{65}$$

$$
\begin{aligned}
\frac{\partial L}{\partial N_t} &= P_t A_t - W_t = 0 && \Rightarrow W_t - P_t A_t = 0 && (66)\\
MC_t &= W_t - P_t A_t && && (67)\\
mc_t &= w_t - p_t - a_t && && (68)\\
mc_t &= -\nu + w_t - p_t - a_t && && (69)\\
mc_t &= -\nu + w_t - p_{H,t} - a_t && && (70)\\
& && && (71)
\end{aligned}
$$

$\nu = -(\log(1-\tau))$, where τ is the employment subsidy, introduced later. $p_{H,t}$ because this is for domestic firms.

Firms that get to reset their price, do it using the following problem:

$$
p_{\bar{H},t} = \mu + (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k \, \mathbb{E}_t[mc_{t+k} + p_{H,t+k}] \tag{72}
$$

$$
\begin{aligned}
p_{\bar{H},t} &\quad \text{Is the (log) new price} && (73)\\
\mu &\quad \text{Is the (log) markup in the steady state} && (74)
\end{aligned}
$$

3 Equilibrium

$$
\begin{aligned}
Y_t(j) &= C_{H,t}(j) + \int_0^1 C^i_{H,t}(j)\,di && (75)\\
&= \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon}\left[(1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \alpha\int_0^1 \left(\frac{P_{H_t}}{\mathcal{E}_{i,t}P^i_{F,t}}\right)^{-\gamma}\left(\frac{P^i_{F,t}}{P^i_t}\right)^{-\eta} C^i_t\,di\right] && (76)
\end{aligned}
$$

Given that

$$
Y_t = \left(\int_0^1 (Y_t(j))^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{77}
$$

$$
\begin{aligned}
Y_t &= (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \alpha\int_0^1 \left(\frac{P_{H_t}}{\mathcal{E}_{i,t}P^i_{F,t}}\right)^{-\gamma}\left(\frac{P^i_{F,t}}{P^i_t}\right)^{-\eta} C^i_t\,di && (78)\\
&= \left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t\left[(1-\alpha) + \alpha\int_0^1 (S^i_t S_{i,t})^{\gamma-\eta}\mathcal{Q}_{i,t}^{\eta-\frac{1}{\sigma}}\,di\right] && (79)
\end{aligned}
$$

Which can be log-linearised to:

$$
\begin{aligned}
y_t &= c_t + \alpha\gamma s_t + \alpha(\eta - \frac{1}{\sigma})q_t && (80)\\
&= c_t + \frac{\alpha w}{\sigma}s_t && (81)\\
w_t &= \sigma\gamma + (1-\alpha)(\sigma\eta - 1) && (82)
\end{aligned}
$$

Assuming that country i is symmetric:

$$
\begin{aligned}
y^i_t &= c^i_t + \frac{\alpha w}{\sigma}s^i_t && && (83)\\
\int_0^1 y^i_t &= \int_0^1 c^i_t + 0 = c^\star_t && \text{World Consumption} && (84)
\end{aligned}
$$

Using equations (), (), ():

$$
\begin{aligned}
y &= c_t + \frac{\alpha w}{\sigma} s_t && (85)\\
y_t &= c_t^\star + \frac{1-\alpha}{\sigma} s_t + \frac{\alpha w}{\sigma} s_t && (86)\\
y_t &= y_t^\star + \frac{1-\alpha}{\sigma} s_t + \frac{\alpha w}{\sigma} s_t && (87)\\
y_t &= y_t^\star + \frac{1-\alpha+\alpha w}{\sigma} s_t && (88)\\
y_t &= y_t^\star + \frac{1+\alpha(w-1)}{\sigma} s_t && (89)\\
\sigma_\alpha &= \frac{1+\alpha(w-1)}{\sigma} && (90)\\
\Rightarrow \quad y_t &= y_t^\star + \frac{1}{\sigma_\alpha} s_t && (91)
\end{aligned}
$$