

Assessing Asymmetrical Effects of Government Spending

A Two-Country DSGE Model for Scotland and the
Rest of the UK

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1. Introduction: 1000 words
2. Literature Review 2000 words
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6. Conclusion: 1000 words

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1 Introduction

In May, the Bank of England's Monetary Policy Committee (MPC) decided to raise the interest rate in response to the exceptionally high inflation of the consumer price index. The committee members were presented with the latest data and thorough analyses conducted by experienced teams of economists. However, the decision was not unanimous: a few members expressed concerns that the contractionary monetary policy shock had not yet been fully internalised, and further rate increases might lead to inflation falling below the target level. Despite this dissent, the majority of the committee members voted in favour of raising the interest rate, considering the potential negative impact of high inflation on households and businesses to outweigh the risks. The meeting minutes effectively demonstrate the informal aspects of the decision-making process. Modern macroeconomic frameworks provide insights into policy objectives and how they can be achieved through fiscal and monetary measures. This dissertation aims to develop a two-country New Keynesian Dynamic Stochastic General Equilibrium (NK DSGE) model for Scotland and the rest of the UK. The model also considers the existence of the world economy and international trading with other countries. The objective of the dissertation is not to build the most factually accurate model tailored to Scotland or the rest of the UK but build a model that would allow an assessment of regional variations in government spending responses under a predefined set of policy scenarios. More specifically, the dissertation considers: a unified fiscal authority scenario in which there exists a single government able to issue bonds and raise taxes and; two autonomous fiscal authorities scenario in which both Scotland and the rest of the UK can issue bonds and raise taxes. Moreover, the dissertation allows fiscal authorities to raise tax revenue by lump-sum or distortionary labour tax. Across all policy scenarios, the model assumes that one monetary authority exists - the Bank of England, and both countries adhere to a UK-wide interest rate. Most model parameters are calibrated in line with the academic literature on DSGE models in the UK. A few model parameters related to government

spending and taxes are estimated using Bayesian techniques. Finally, the estimated values are used to derive impulse response functions that are key to delivering the dissertation objectives.

A literature review is presented in the first section. It covers an overview of the evolution of economic frameworks from the 1940s to today and the application of DSGE models by central banks and international organisations. The literature review section is followed by the description of the model. It is split into four parts: households, the government, firms, and equilibrium or market clearing conditions. Given the high number of employed equations by DSGE models, most of the derivations and proofs were put in Appendix, but the accompanying interpretation and intuition are provided in the main text. The remaining two sections are Application and Results. The application section covers the calibration of the model, a brief introduction to the Bayesian estimation of the model parameters and estimation results. The application section also presents the data used in the estimation process. The results section analyses endogenous variables' dynamic responses to two types of shocks: monetary and fiscal. It discusses government spending when the economy is introduced with a distortionary labour tax, as well as the asymmetry in responses when Scotland is considered fiscally autonomous. Finally, the dissertation provides a parameter sensitivity analysis - a demonstration and a discussion on how the dynamic responses vary given different sets of parameters. The conclusion summarises all sections of a dissertation, overviews the results and discusses the limitations of the model.

1.1 Literature Review

On the 10th of May, 2023, the Monetary Policy Committee at the Bank of England gathered to discuss the latest international and domestic data on economic activity. Even though the Committee has a 2% CPI target, the UK's economy had undergone a sequence of very large and unexpected shocks and disturbances, resulting in twelve-month CPI inflation above 10%. The majority of the Committee members (78%) believed that an increase in interest rate “was warranted” (BoE, 2023: 4), while

the remaining members believed that the CPI inflation will “fall sharply in 2023” (BoE, 2023: 5) as a result of the economy naturally adjusting to the effects of the energy price shocks. They feared that the preceding increases in the interest rate have not yet been internalised and raising the interest rate any further could result in a reduction of inflation “well below the target” (BoE, 2023: 5). This is an excellent illustration of the “informal dimension of the monetary policy process”, that (Galsí and Gertler, 2007: 26) referred to in their work explaining modern macro models and new frameworks. According to them, while the informal dimension cannot be removed, we can build formal and rigorous models that would help the Committee and institutions-alike to understand “objectives of the monetary policy and how the latter should be conducted in order to attain those objectives” (Galí, 2015: 2). This task is not straightforward and has been central (albeit - fruitful) to most macroeconomic research in the past decades. The following section of the literature review will present a brief evolution of the study of business cycles and monetary theory. It will be followed by an overview of large macroeconomic models adopted by central banks and international organisations to illustrate the relevance of this research. The final part of the literature review will outline insitutional and fiscal policy differences in Scotland and the rest of the UK.

O. Blanchard (2000) offers a compact description of macroeconomic research in the twentieth century. In their panegyric and optimistic essay, the researcher argues that the century can be divided into three epochs based on the prevailing beliefs about the economy and frameworks of the time: Pre-1940, From 1940 to 1980, and Post-1980.

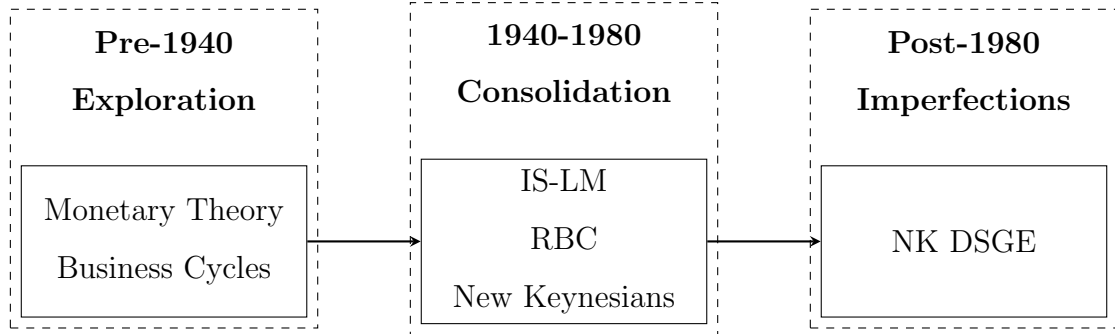


Figure 1: Timeline of macroeconomic research, according to Blanchard (2000). Authors own illustration.

According to them, the pre-1940 epoch was the epoch of exploration, with economists primarily concerned with the Monetary Theory (“Why does money affect output?”) and Business Cycles (“What are the major shocks that affect output?”). Even though both of those puzzles fall under the study of “macroeconomics”, the term did not even appear in the literature until the mid-1940s (O. Blanchard, 2000). The Monetary Theory ideas of that time did not differ drastically from what is believed today, e.g. short-run money non-neutrality and long-run neutrality, but the economic models were “incomplete and partial equilibrium in nature” (O. Blanchard, 2000: 1377). The business cycles were attributed to “real factors”, such as technological innovations (O. Blanchard, 2000), and this belief persisted until more data became available and more sophisticated time series methods were applied in the early 2000s (Galí, 2015: 3). The subsequent epoch was “the golden age of macroeconomics” (O. Blanchard, 2000: 1379). Hicks (1937) formalised the *IS-LM* framework TBC

In the 1980s, Kydland and Prescott (1982) and Prescott (1986) published seminal papers on the Real Business Cycles (RBC) theory. According to Galí (2015: 2), frameworks presented in the papers “provided the main reference” and firmly established the use of dynamic stochastic general equilibrium (DSGE) models as crucial tools for macroeconomic analysis. The models allow quantitative analysis and incorporation of data either via calibration or estimation of parameters. TBC

International Monetary Fund (IMF)

Government in DSGE literature review

To begin with, RBC models predict a negative response in consumption following an increase in government spending. More specifically, government spending is modelled to absorb resources, which makes households worse off and incentivises more hours worked. Greater labour supply for any given wage reduces firms' marginal cost and induces output (Baxter and King, 1993: 319). That is, consumption, conditional on shocks in government spending, is countercyclical. Keynesian models, in stark contrast, predict the opposite. The countercyclicality is why the DSGE models sometimes do not consider government spending.

Empirically, the findings of the Keynesian models are more in line with the observed macroeconomic patterns. For instance, Olivier Blanchard and Perotti (2002) performed a VAR analysis on the dynamics of consumption and government spending. They built six structural VAR models, one for each component of GDP: output, consumption, government spending, investment, export, and import. The two other variables were taxes and government spending/output¹. The key finding of the analysis is that government spending has a positive effect on consumption.

Galí, López-Salido, and Vallés (2005) show that NK DSGE models can be “recovered” by assuming that households have limited access to financial markets/saving technologies or are poor (they consume all of their labour income). Households that smooth their consumption by saving are often regarded as Ricardian households, while those that do not - non-Ricardian or *hand-to-mouth* households. Just like Galí, López-Salido, and Vallés (2005) did, some of the latest literature NK DSGE literature models both types of households explicitly² with their ratio determined by a time-invarying exogenous coefficient. Arguably, such modelling would allow an improved fit of data. However, the key focus of this dissertation is the four policy scenarios, all of which heavily depend on governments' abilities to issue bonds and borrow. Hand-to-mouth

¹That is, if the GDP component of interest is government spending, then the second variable is output; in all other cases, the second variable is government spending.

²In fact, there exists literature with more than two types of households. For instance, a recent paper by Eskelinen (2021) models poor hand-to-mouth, wealthy hand-to-mouth, and non-hand-to-mouth households.

households do not borrow/save, rendering bonds purposeless. While modelling hand-to-mouth households is even easier than the Ricardian households, modelling both types of households would drastically increase the complexity of the model, given the two-country setting. The absence of hand-to-mouth households is discussed in the limitations section.

2 Theoretical DSGE model

Ricci (2019) was the first to build a large-scale two-country DSGE model explicitly tailored to Scotland and the rest of the UK. In an attempt to retain the model's simplicity while still allowing policy analysis, this dissertation will primarily build on the work of Gali and Monacelli (2005) and Galí (2015). In contrast, Ricci (2019) model was based on the work of Rabanal and Tuesta (2010), who were among the first to build a medium-to-large two-country DSGE model. Neither Gali and Monacelli (2005) nor Galí (2015) models considered lump-sum or distortionary taxes, or government spending, more generally. While extensive literature covers government spending in DSGE models, few to none cover government spending in a small open economy (SOE) NK DSGE model, and even fewer apply it to a two-country setting. Therefore, most of the derivations had to be carried out using a pen and paper, and step-by-step derivations are provided in the Appendix. Many of Galí (2015) derivations relied on the assumption that steady state output is equal to the state consumption, i.e. $Y = C$. When the government term is introduced, then many of the expressions lose their inherent elegance and simplicity.

Moreover, the focus of this dissertation is not to build the most factually accurate model of Scotland or the United Kingdom but to assess the asymmetric responses in government spending under fiscal autonomy and fiscal union. The fiscal union scenario refers to the Westminster government collecting taxes from all four countries of the UK and distributing them according to the Barnett formula. The fiscal autonomy scenario refers to the Holyrood government's ability to collect tax revenue, issue bonds

(borrow), and spend it at its sole discretion. We further break down the scenarios by allowing public expenses to be funded by lump-sum and distortionary (labour) taxes. This brings the number of policy scenarios considered by the dissertation to four. In all four policy scenarios, the dissertation assumes that a single monetary authority sets one UK-wide interest rate.³

Finally, in line with most of the literature, variables referring to the home country (Scotland) will be denoted without an asterisk, i.e., Y_t , while foreign country (the rest of the UK) will be denoted with an asterisk, i.e., Y_t^* . UK-wide variables will be denoted as Y_t^{UK} . Given that Scotland and the rest of the UK are modelled as symmetrical,⁴ Sections 2.1-2.4 describe the model only for Scotland, but note that for each presented equation in the Sections, there exists a corresponding equation for the rest of the UK economy. In cases when this is not true, it will be stated explicitly.

2.1 Households

This model assumes that there is infinitely many households in the economy represented by a unit interval. All households are assumed to be symmetric, i.e. have the same preferences and behave identically. Below, we consider a representative household that wants to maximise their lifetime utility, represented by Equation (1):

$$\mathbb{E}_t \left\{ \sum_{t=0}^{\infty} \beta^t \mathcal{U}(C_t, N_t, Z_t) \right\} \quad (1)$$

$$\mathcal{U}(C_t, N_t, Z_t) = \begin{cases} \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right) Z_t & \text{if } \sigma \geq 0 \text{ and } \sigma \neq 1 \\ \left(\log(C_t) - \frac{N_t^{1+\varphi}}{1+\varphi} \right) Z_t & \text{if } \sigma = 1 \end{cases} \quad (2)$$

The household's utility depends on consumption C_t and hours worked N_t . As seen from the utility function (Equation (2)), the model assumes the household's utility

³The Dynare code complementing this dissertation allows the user to switch between the assumption of monetary independence and union, as well as choosing a policy scenario of interest. The results section does not consider impulse response functions under monetary independence, as it is outside the scope of the dissertation.

⁴Equations can differ in parameter values but not structurally.

to be (decreasingly) increasing in consumption C_t and (increasingly) decreasing in hours worked N_t . $\beta \in (0, 1)$ is the discount factor, which can be thought of as an opportunity cost or an impatience rate, i.e. a unit of consumption C today will be worth $\beta * C < C$ tomorrow. We also introduce a preference shifter Z_t (Galí, 2015: 225) **expand**. The shock is assumed to follow an autoregressive process of order 1:

$$\log(Z_t) = \rho_z \log(Z_{t-1}) + \epsilon_t^z \quad (3)$$

The parameter $\sigma \geq 0$ is the relative risk aversion coefficient and $\varphi \geq 0$ is the labour disutility parameter. Together, they determine the curvature of the utility of consumption and disutility of labour, respectively. Finally, $\mathbb{E}_t[*]$ and $\mathbb{E}_t\{*\}$ are the expectational operators, conditional on all information available at period t .

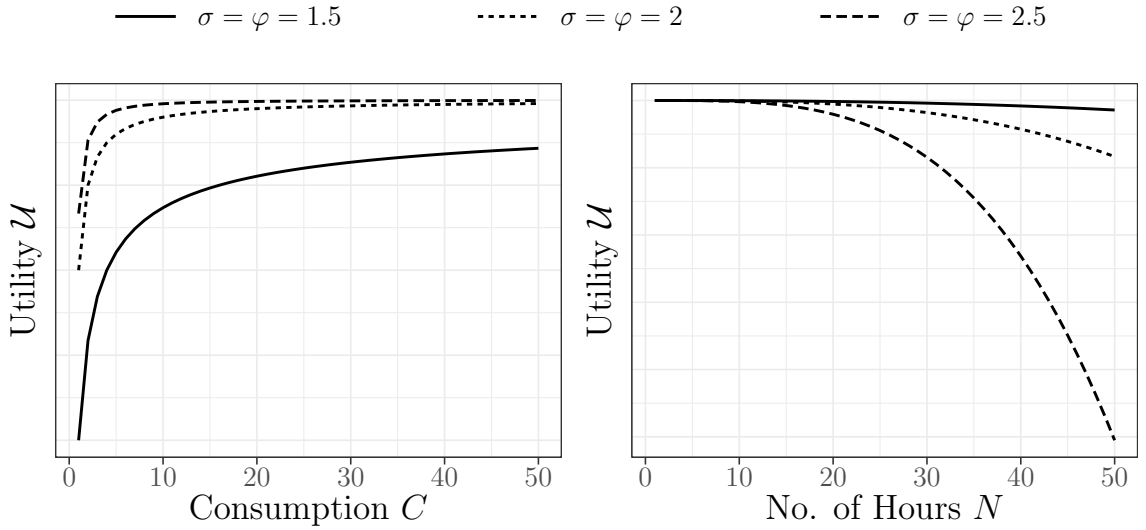


Figure 2: Illustration of how utility from consumption and disutility from labour vary given three sets of parameters: 1.5, 2, 2.5

To allow goods differentiation between domestic and foreign, the model assumes

that C_t is a composite consumption index defined by:

$$C_t = \begin{cases} \left[(1-v)^{\frac{1}{\eta}} (C_{H,t})^{\frac{\eta-1}{\eta}} + v^{\frac{1}{\eta}} (C_{F,t})^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}} & \text{if } \eta > 0 \text{ and } \eta \neq 1 \\ \frac{1}{(1-v)^{(1-v)v}} (C_{H,t})^{(1-v)} (C_{F,t})^v & \text{if } \eta = 1 \end{cases} \quad (4)$$

Where $C_{H,t}$ and $C_{F,t}$ are indices of consumption of home produced and imported goods, respectively. The parameter $v \in [0, 1]$ reflects economy's openness for trading, while $\eta > 0$ denotes household's willingness to substitute a domestic good with a foreign good, often referred to as 'home bias'. When $\eta = 1$, then the share of domestic and foreign consumption is determined by the country's willingness to trade. In an extreme case, $v = 0$ would imply that the economy is an autarky, while $v = 1$ would suggest that our households consume foreign goods only. Our economy is assumed to be small, in the sense that it takes the world output, consumption, and prices as given, and cannot influence them. This is a common assumption for the UK (**refs**) and even more so for Scotland. The world economy is assumed to be made of a continuum of infinitely many small economies i represented by a unit interval. Therefore, $C_{F,t}$ is a sum of indices of the quantity of goods imported from all countries i . In a similar fashion, if we denote j as a single variety of goods from a continuum of goods represented by a unit interval, we can express each consumption index as follows:

$$\begin{aligned} C_{H,t} &= \left(\int_0^1 C_{H,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} && \text{Index of consumption of home produced goods} \\ C_{i,t} &= \left(\int_0^1 C_{i,t}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} && \text{Index of consumption of country } i\text{'s produced goods} \\ C_{F,t} &= \left(\int_0^1 C_{i,t}^{\frac{\gamma-1}{\gamma}} di \right)^{\frac{\gamma}{\gamma-1}} && \text{Index of consumption of imported goods} \end{aligned}$$

Notice that all three indices take the form of *Constant Elasticity Substitution* (*CES*) form, with parameters ε (without subscripts) and γ representing the degree of substitutability between varieties of goods and countries, respectively. The following expressions note optimal allocation of each individual good (see Appendix A1 for

derivation):

$$C_{H,t}(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}} \right)^{-\varepsilon} C_{H,t}; \quad C_{i,t}(j) = \left(\frac{P_{i,t}(j)}{P_{i,t}} \right)^{-\varepsilon} C_{i,t}; \quad C_{i,t} = \left(\frac{P_{i,t}}{P_{F,t}} \right)^{-\gamma} C_{F,t} \quad (5)$$

where:

$$P_{H,t} = \left(\int_0^1 P_{H,t}(j)^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}} \quad \text{Domestic Price Index} \quad (6)$$

$$P_{i,t} = \left(\int_0^1 P_{i,t}(j)^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}} \quad \text{Price Index of goods produced by country } i \quad (7)$$

$$P_{F,t} = \left(\int_0^1 P_{i,t}^{1-\gamma} dj \right)^{\frac{1}{1-\gamma}} \quad \text{Price Index of Imported goods} \quad (8)$$

Intuitively, if $P_{H,t}(j) > P_{H,t}$, then that good is demanded less relative to an *average* good. To see this, note that $P_{H,t}(j)/P_{H,t} > 1$ when $P_{H,t}(j) > P_{H,t}$, and given that the term is to the power of a negative constant, the entire term decreases.

The representative household's choice of consumption and labour must satisfy the following budget constraint:

$$\underbrace{\int_0^1 P_{H,t}(j) C_{H,t}(j) dj + \int_0^1 \int_0^1 P_{i,t}(j) C_{i,t}(j) dj di}_{Expenses} + \mathbb{E}_t [R_{t+1}^{-1} B_{t+1}] \leq \underbrace{B_t + W_t N_t}_{Income} \quad (9)$$

where R_t is the gross nominal interest rate, B_t denotes bonds, W_t and N_t stand for nominal wage and hours worked, respectively. For intuition, the LHS of the budget constraint implies that the representative household needs to choose the quantity of good j produced domestically and in every country i , as well as the number of bonds at the expected nominal interest rate in period $t + 1$. The RHS implies that the only two sources of income are nominal payoffs from bonds and gross pay, which later will be different from the net pay. The expenses cannot exceed income.

Taking the three price indices (6)-(8), and plugging them into their respective

demand functions (5), yields:

$$\int_0^1 P_{H,t}(j)C_{H,t}(j) dj = P_{H,t}C_{H,t} \quad (10)$$

$$\int_0^1 P_{i,t}(j)C_{i,t}(j) dj = P_{i,t}C_{i,t} \quad (11)$$

$$\int_0^1 P_{i,t}C_{i,t} = P_{F,t}C_{F,t} \quad (12)$$

The following definitions are given:

$$C_{H,t} = (1 - v) \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} C_t \quad (13)$$

$$C_{F,t} = v \left(\frac{P_{F,t}}{P_t} \right)^{-\eta} C_t \quad (14)$$

$$P_t = \begin{cases} [(1 - v)(P_{H,t})^{1-\eta} + v(P_{F,t})^{1-\eta}]^{\frac{1}{1-\eta}} & \text{if } \eta > 0 \text{ and } \eta \neq 1 \\ (P_{H,t})^{1-v} \times (P_{F,t})^v & \text{if } \eta = 1 \end{cases} \quad (15)$$

Equations (13) and (14) are demand functions for domestic and foreign goods, respectively. Equation (15) is the Consumption Price Index (CPI). In the case when there is no home bias ($\eta = 1$), the log aggregate price level in the consumption is just a weighted sum of the two price indices, where weights are given by trade openness parameter v . Using (10) and (12), we can define the total consumption expenditures as:

$$P_{H,t}C_{H,t} + P_{F,t}C_{F,t} = P_tC_t \quad (16)$$

Which greatly simplifies the household's budget constraint:

$$P_tC_t + \mathbb{E}_t [R_{t+1}^{-1}B_{t+1}] \leq B_t + W_tN_t \quad (17)$$

Note that the budget constraint (as well as many other expressions introduced later) will vary depending on what policy scenario is considered. For instance, the house-

hold's budget constraints under each scenario is given below:

Scen. 1	Scot.	$P_t C_t + \mathbb{E}_t[R_{t+1}^{-1} B_{t+1}] = B_t + W_t N_t + T_t$
G: 2, $\tau : 0$	rUK	$P_t^* C_t^* + \mathbb{E}_t[R_{t+1}^{*-1} B_{t+1}^*] = B_t^* + W_t^* N_t^* + T_t^*$
Scen. 2	Scot.	$P_t C_t + \mathbb{E}_t[R_{t+1}^{UK-1} B_{t+1}^{UK}] = B_t^{UK} + W_t N_t + \varpi T_t^{UK}$
G: 1, $\tau : 0$	rUK	$P_t^* C_t^* + \mathbb{E}_t[R_{t+1}^{UK-1} B_{t+1}^{UK}] = B_t^{UK} + W_t^* N_t^* + (1 - \varpi) T_t^{UK}$
Scen. 3	Scot.	$P_t C_t + \mathbb{E}_t[R_{t+1}^{-1} B_{t+1}] = B_t + (1 - \tau_t) W_t N_t + T_t$
G: 2, $\tau : 1$	rUK	$P_t^* C_t^* + \mathbb{E}_t[R_{t+1}^{*-1} B_{t+1}^*] = B_t^* + (1 - \tau_t^*) W_t^* N_t^* + T_t^*$
Scen. 4	Scot.	$P_t C_t + \mathbb{E}_t[R_{t+1}^{UK-1} B_{t+1}^{UK}] = B_t^{UK} + (1 - \tau_t^{UK}) W_t N_t + \varpi T_t^{UK}$
G: 1, $\tau : 1$	rUK	$P_t^* C_t^* + \mathbb{E}_t[R_{t+1}^{UK-1} B_{t+1}^{UK}] = B_t^{UK} + (1 - \tau_t^{UK}) W_t^* N_t^* + (1 - \varpi) T_t^{UK}$

Here, ϖ denotes the Scotland's share of population in the United Kingdom. T_t denotes lump-sum transfers (subsidies or taxes), while τ_t denotes an income or labour tax rate. In the first column, G indicates the number of governments that can issue bonds (borrow) and set the labour tax rate. In the same column, τ indicates whether the government spending is funded by a labour tax.

What follows is the derivation of the intratemporal and intertemporal optimality conditions for the policy scenario 3, i.e. when households face a labour tax:

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right) Z_t + \lambda_t \{ B_t + (1 - \tau_t) W_t N_t + T_t - P_t C_t - \mathbb{E}_t [R_{t+1}^{-1} B_{t+1}] \} \quad (18)$$

$$\frac{\partial \mathcal{L}}{\partial C_t} = \beta^t C_t^{-\sigma} Z_t - \lambda_t P_t = 0; \quad \Rightarrow \quad \beta^t C_t^{-\sigma} Z_t P_t^{-1} = \lambda_t \quad (19)$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = -\beta^t N_t^{\varphi} Z_t + \lambda_t (1 - \tau_t) W_t = 0; \quad \Rightarrow \quad \beta^t N_t^{\varphi} Z_t ((1 - \tau_t) W_t)^{-1} = \lambda_t \quad (20)$$

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = -\lambda_t \mathbb{E}_t [R_{t+1}^{-1}] + \mathbb{E}_t [\lambda_{t+1}] = 0; \quad \Rightarrow \quad \mathbb{E}_t [R_{t+1}^{-1}] = \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} \right] \quad (21)$$

Equating and rearranging Equations (19) and (20) yields *intratemporal optimality*

condition:

$$\begin{aligned}
\Rightarrow \quad C_t^\sigma N_t^\varphi &= \frac{W_t}{P_t}(1 - \tau_t) && \text{Scenario 3} \\
C_t^\sigma N_t^\varphi &= \frac{W_t}{P_t}(1 - \tau_t^{UK}) && \text{Scenario 4} \\
C_t^\sigma N_t^\varphi &= \frac{W_t}{P_t} && \text{Scenarios 1 \& 2}
\end{aligned}$$

The condition implies that the marginal utility of consumption and leisure is equal to the net real wage. As mentioned before, in the case of labour tax absence, the net real wage is equal to the gross real wage. The log-linearisation of Equation (22) around a steady state yields:

$$\begin{aligned}
C_t^\sigma N_t^\varphi &= \frac{W_t}{P_t}(1 - \tau_t) = \frac{W_t}{P_t} - \frac{W_t}{P_t}\tau_t \\
&\quad \vdots \quad (\text{see Appendix A.121 - A.129}) \\
(1 - \tau)(\sigma c_t + \varphi n_t) &= [(1 - \tau)(w_t - p_t) - \tau \tilde{\tau}_t] \\
\sigma c_t + \varphi n_t &= w_t - p_t - \frac{\tau}{1 - \tau} \tilde{\tau}_t \tag{22}
\end{aligned}$$

Where τ and $\tilde{\tau}_t$ denote steady state labour tax rate and deviation from the steady state, respectively. As it is common in the literature, we denote natural logs of corresponding variables in lowercase letters, i.e. $x_t = \ln(X_t)$, and use tildes to denote deviations from the steady state. While loglinearising, we widely make use of Uhlig (1995) proposed methods for multivariate equations with additive terms, i.e., $X_t Y_t \approx XY \mathbf{e}^{\tilde{X}_t + \tilde{Y}_t}$, $X_t + Y_t \approx X \mathbf{e}^{\tilde{X}_t} + Y \mathbf{e}^{\tilde{Y}_t}$ and $\mathbf{e}^{\tilde{X}_t} \approx (1 + \tilde{X}_t)$.

Iterating Equation (19) one period forward, yields:

$$\frac{\partial L}{\partial C_t} = \beta^t C_t^{-\sigma} Z_t P_t^{-1} = \lambda_t; \quad \Rightarrow \quad \mathbb{E}_t[\beta^{t+1} C_{t+1}^{-\sigma} Z_{t+1} P_{t+1}^{-1}] = \mathbb{E}_t[\lambda_{t+1}]$$

Dividing one by the other and rearranging yields *intertemporal optimality condition*:

$$\begin{aligned} \mathbb{E}_t \left[\frac{\beta^t C_t^{-\sigma} Z_t P_t^{-1}}{\beta^{t+1} C_{t+1}^{-\sigma} Z_{t+1} P_{t+1}^{-1}} \right] &= \mathbb{E}_t \left[\frac{\lambda_t}{\lambda_{t+1}} \right] \\ \beta^{t-(t+1)} \mathbb{E}_t \left[\frac{C_t^{-\sigma} Z_t P_t^{-1}}{C_{t+1}^{-\sigma} Z_{t+1} P_{t+1}^{-1}} \right] &= \mathbb{E}_t \left[\frac{1}{R_{t+1}} \right] \end{aligned} \quad (23)$$

: (see Appendix A.xx - A.xx)

$$\beta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \left(\frac{Z_{t+1}}{Z_t} \right) \left(\frac{P_t}{P_{t+1}} \right) \right] = \mathbb{E}_t \left[\frac{1}{R_{t+1}} \right] \quad (24)$$

where Equation (23) used Equation (21). $\mathbb{E}_t[R_{t+1}^{-1}]$ is the gross return on a risk-free one-period discount bond or a stochastic discount factor. More generally, Equation (24) is the Euler equation, and it determines the consumption path of a lifetime utility-maximising representative household. To state it in more intuitive terms, households choose consumption “today” and “tomorrow” and take all other terms as given. According to the equation, they choose consumption in the two periods in such a way so that the marginal utility “today” would be equal to the marginal consumption tomorrow while taking into account that saving consumption “today”, will result in $R_t > 1$ consumption “tomorrow”. Note that Galí (2015) uses a different approach to derive the Euler equation, which introduces Arrow securities. *As it adds little value to our research question, this dissertation only provides a step-by-step derivation and interpretation in Appendix A for an interested reader.* Also note, that Galí (2015) uses $Q_t = \mathbb{E}_t[\frac{1}{R_{t+1}}]$ to denote the stochastic discount factor and D_t to denote bonds or “portfolio” as they call it.

Log-linearising (24):

$$\begin{aligned}
\beta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \left(\frac{P_t}{P_{t+1}} \right) \right] &= \mathbb{E}_t \left[\frac{1}{R_{t+1}} \right] \\
\ln \beta - \mathbb{E}_t[\sigma c_{t+1}] + \sigma c_t + p_t - \mathbb{E}_t[p_{t+1}] &= -\ln R_{t+1} \\
\sigma c_t &= -\ln R_{t+1} - \ln \beta + \mathbb{E}_t[\sigma c_{t+1}] - p_t + \mathbb{E}_t[p_{t+1}] \\
c_t &= \mathbb{E}_t[c_{t+1}] - \frac{1}{\sigma} (\ln R_{t+1} - \rho - \mathbb{E}_t[\pi_{t+1}]) \\
c_t &= \mathbb{E}_t[c_{t+1}] - \frac{1}{\sigma} (i_t - \mathbb{E}_t[\pi_{t+1}] - \rho)
\end{aligned} \tag{25}$$

where $i_t = \ln R_{t+1}$ is the nominal interest rate, $\rho = -\log \beta$ is the log discount rate, and $\pi_t = p_t - p_{t-1}$ is the CPI inflation. The loglinearised Euler equation makes it clearer to see, that consumption “today” is increasing in expected inflation “tomorrow”, while the opposite is true for the nominal interest rate. The effect is scaled by σ^{-1} parameter.

Furthermore, OECD (2022) defines terms of trade as a ratio of import and export price indices, which in this model is denoted as S_t :

$$S_{i,t} = \frac{P_{i,t}}{P_{H,t}} \tag{26}$$

$$S_t = \frac{P_{F,t}}{P_{H,t}} = \left(\int_0^1 (S_{i,t} di)^{1-\gamma} \right)^{\frac{1}{1-\gamma}} \tag{27}$$

where Equation (26) marks *bilateral* terms of trade with a country i , while Equation (27) is for *effective* terms of trade, i.e. terms of trade with all countries in the unit interval defined earlier. The latter can be loglinearised to yield:

$$s_t = p_{F,t} - p_{H,t} = \left(\int_0^1 s_{i,t} di \right) \tag{28}$$

Recall that when $\eta = 1$, then CPI is $P_t = (P_{H,t})^{1-v} \times (P_{F,t})^v$. Using the previous definition (28) and loglinearised CPI, the price level can be expressed as a sum of

domestic price level and terms of trade (see Appendix A.xx-A.xx):

$$p_t = (1 - v)p_{H,t} + vp_{F,t} = p_{H,t} + vs_t \quad (29)$$

Note that Equations (28) and (29) hold *exactly* when $\gamma = 1$ and $\eta = 1$. Similarly, knowing that inflation as a difference of log prices in two consecutive periods, we can extend the previous definition to yield:

$$\pi_{H,t} = p_{H,t+1} - p_{H,t} \quad \text{Domestic Inflation} \quad (30)$$

$$\pi_t = \pi_{H,t} + v\Delta s_t \quad \text{CPI Inflation} \quad (31)$$

The gap between domestic inflation and CPI inflation is due to percentage change in the terms of trade and degree of openness. In the case of an autarky ($v = 0$), even if imported goods were much more expensive ($P_{F,t} \gg P_{H,t}$), domestic inflation will be equal to CPI inflation because the country simply does not trade.

Furthermore, we assume that the Law of One Price (LOP) holds for all goods j . That is, the price of a single good in country i is equal to the price of the same good in country $-i$ times the nominal exchange rate. It implies, that there are no opportunities for arbitrage.

$$P_{i,t}(j) = \mathcal{E}_{i,t}P_{i,t}^i(j) \quad (32)$$

$$P_{i,t} = \mathcal{E}_{i,t}P_{i,t}^i \quad \text{Law of One Price (LOP)} \quad (33)$$

where $\mathcal{E}_{i,t}$ is the nominal exchange rate between the home currency and the country's i currency, and the second equation is derived by integrating both sides with respect to j . Even though we do not model currencies explicitly, it is useful to think about $\mathcal{E}_{i,t}$ as the price of one unit of currency in terms of another currency, i.e. the home

currency. The two equations can be loglinearised to yield:

$$p_{i,t} = e_{i,t} + p_{i,t}^i \quad (\text{Log}) \text{ Law of One Price (LOP)} \quad (34)$$

$$p_{F,t} = \int_0^1 (e_{i,t} + p_{i,t}^i) di = e_t + p_t^* \quad (\text{Log}) \text{ Price index of Imported Goods} \quad (35)$$

Where e_t is (Log) Effective Nominal Exchange Rate, p_t^* is the World Price Index. This allows us to redefine log effective terms of trade in terms of the nominal exchange rate, domestic price, and the world price index:

$$s_t = p_{F,t} - p_{H,t} = e_t + p_t^* - p_{H,t} \quad (36)$$

In contrast, *real* exchange rate between two countries is the ratio between their CPI and home CPI, expressed in home currency:

$$\mathcal{Q}_{i,t} = \frac{\mathcal{E}_{i,t} P_t^i}{P_t} \quad \text{Bilateral Exchange Rate} \quad (37)$$

Integrating both sides with respect to i and using previous definitions yields:

$$q_t = \int_0^1 \log \left(\frac{\mathcal{E}_{i,t} P_t^i}{P_t} \right) di \quad (38)$$

$$= \int_0^1 (e_{i,t} + p_{i,t}^i - p_t) di \quad (39)$$

$$= e_t + p_t^* - p_t \quad \text{using (35)} \quad (40)$$

$$= s_t + p_{H,t} - p_t \quad \text{using (36)} \quad (41)$$

$$= (1 - v)s_t \quad \text{using (29)} \quad (42)$$

Finally, if we assume that all countries i have symmetrical preferences and their households maximise lifetime-utility in the same manner that the home country's do,

then maximising the Lagrangian function for country i , will yield:

$$\begin{aligned}\frac{\partial \mathcal{L}^i}{\partial C_t^i} &= \beta^t (C_t^i)^{-\sigma} Z_t^i (\mathcal{E}_{i,t} P_t^i)^{-1} = \lambda_t^i \\ \frac{\partial \mathcal{L}^i}{\partial D_{t+1}^i} &= -\lambda_t^i \mathbb{E}_t[R_{t,t+1}^{-1}] = -\mathbb{E}_t[\lambda_{t+1}^i] \\ &\vdots \quad (\text{see Appendix A.xx - A.xx})\end{aligned}\tag{43}$$

$$C_t = C_t^i Z_t^{i\frac{1}{\sigma}} \mathcal{Q}_{i,t}^{\frac{1}{\sigma}}\tag{44}$$

If we assume that there had been no shocks in the country's i preferences ($Z_t^i = 1$), then Equation (44) states that consumption in the home country is equal to the consumption in the country i , while taking into account bilateral real exchange rate. This can be generalised to derive a relationship between home consumption and world consumption by log linearising (for simplicity) and integrating both sides with respect to i :

$$c_t = c_t^i + \frac{1}{\sigma} z_{i,t} + \frac{1}{\sigma} q_{i,t}\tag{45}$$

$$\int_0^1 c_t di = \int_0^1 \left(c_t^i + \frac{1}{\sigma} z_{i,t} + \frac{1}{\sigma} q_{i,t} \right) di\tag{46}$$

$$\vdots \quad (\text{see Appendix A.xx - A.xx})\tag{47}$$

$$c_t = c_t^* + \frac{1}{\sigma} z_t + \left(\frac{1-v}{\sigma} \right) s_t \quad \text{using } q_t = (1-v)s_t\tag{48}$$

$$= y_t^* + \frac{1}{\sigma} z_t + \left(\frac{1-v}{\sigma} \right) s_t \quad \text{using } c_t^* = y_t^*\tag{49}$$

c_t^* is the log world consumption and the last Equation (49) follows by assuming that world consumption is equal to world output, i.e., there is no world government spending, or national government spending in any country i is infinitesimally small.

The next two parts will discuss government spending and firms, respectively. The final part will provide equilibrium (market clearing) conditions.

2.2 Government

As mentioned in the literature review, this dissertation makes two essential assumptions related to government spending: households are assumed to be Ricardian, and; government spending is non-productive. The latter follows from the fact that government spending does not enter the utility function nor the firm's production function. Therefore, government spending is equivalent to reducing the quantity of available resources. In line with most of the literature, deviations from the steady state government spending are assumed to be temporary, i.e. following an autoregressive process of order 1, as opposed to a permanent increase considered by Baxter and King (1993). Below is a typical budget constrained faced by the government every period:

$$\underbrace{\mathbb{E}_t[R_t^{-1}B_{t+1}] + P_tT_t}_{\text{Revenue}} = \underbrace{P_tG_t + B_t}_{\text{Spending}} \quad (50)$$

That is, government accrues revenue by collecting taxes in current prices P_tT_t and by issuing bonds at current gross return rate $\mathbb{E}_t[R_t^{-1}B_{t+1}]$. The government needs to pay one unit of consumption good for each mature bond issued previous period B_t and pay for its spending in current prices P_tG_t . The government budget constraint varies depending on the policy scenario considered:

Scen. 1 & Scen. 3	Scot.	$\mathbb{E}_t[R_t^{-1}B_{t+1}] + P_tT_t = P_tG_t + B_t$
G: 2, $\tau \in \{0, 1\}$	rUK	$\mathbb{E}_t[R_t^{*-1}B_{t+1}^*] + P_t^*T_t^* = P_t^*G_t^* + B_t^*$
Scen. 3 & Scen. 4	UK	$\mathbb{E}_t[R_t^{UK-1}B_{t+1}^{UK}] + P_t^{UK}T_t^{UK} = P_t^{UK}G_t^{UK} + B_t^{UK}$
G: 1, $\tau \in \{0, 1\}$		

Table 1: Government budget constraints under different policy scenarios

where P_t^{UK} is a weighted sum of price levels in Scotland and the rest of the UK, i.e., $P_t^{UK} = \varpi P_t + (1 - \varpi)P_t^*$. Similarly, a monetary union implies a single rate of gross return, which we define as $R_t^{UK} = \varpi R_t + (1 - \varpi)R_t^*$.⁵ Also note the log-

⁵Due to technical limitations, the dissertation modelled two countries as having individual nomi-

linearised government budget constraints in real terms (see Appendix A.131 - A.146 for derivation):

Scen. 1 & Scen. 3	Scot.	$b_{t+1} = (1 + \rho)(b_t + g_t - t_t)$
G: 2, $\tau \in \{0, 1\}$	rUK	$b_{t+1}^* = (1 + \rho^*)(b_t^* + g_t^* - t_t^*)$
Scen. 3 & Scen. 4	UK	$b_{t+1}^{UK} = (1 + \rho^{UK})(b_t^{UK} + g_t^{UK} - t_t^{UK})$
G: 1, $\tau \in \{0, 1\}$		

Table 2: (Log) Government budget constraints under different policy scenarios

where $\rho = \beta^{-1} - 1$ pins down the steady state interest rate. In each of the scenarios, the government tax revenue is accrued either by imposing lump-sum taxes or a labour tax:

Scen. 1	Scot.	$P_t T_t = P_t G_t$
G: 2, $\tau : 0$	rUK	$P_t^* T_t^* = P_t^* G_t^*$
Scen. 2	UK	$P_t^{UK} T_t^{UK} = P_t^{UK} G_t^{UK}$
G: 1, $\tau : 0$		
Scen. 3	Scot.	$P_t T_t = \tau_t W_t N_t$
G: 2, $\tau : 1$	rUK	$P_t^* T_t^* = \tau_t^* W_t^* N_t^*$
Scen. 4	UK	$P_t^{UK} T_t^{UK} = \varpi \tau_t^{UK} W_t N_t + (1 - \varpi) \tau_t^{UK} W_t^* N_t^*$
G: 1, $\tau : 1$		

Table 3: Tax revenue for each of the policy scenarios

That is, in the case of lump-sum taxes, tax revenue simply equals government spending; similarly, in the case of labour taxes, tax revenue is a share ($\tau_t \in [0, 1]$) of nominal labour income. However, given that the two sources of income are close

nal interest rates. However, in the budget constraints and debt-stabilising equations with one government, a weighted sum of the two interest rates was used. Conceptually, it is equivalent to having one UK-wide interest rate, where Scotland's interest rate "influences" UK-wide interest rate ($\varpi \approx 9\%$) but is primarily determined by the rest of the UK ($\varpi \approx 91\%$), so Scotland (almost) takes it as given.

substitutes for the government (it can raise revenue either by raising taxes or by issuing bonds), tax revenue and budget constraint alone do not lead to a stable equilibrium (there are many “solutions”). The governing of this relationship is captured by a fiscal rule of the following form:

Scen. 1 & Scen. 3	Scot.	$t_t = \phi_g g_t + \phi_b b_t$
G: 2, $\tau \in \{0, 1\}$	rUK	$t_t^* = \phi_g^* g_t^* + \phi_b^* b_t^*$
Scen. 3 & Scen. 4	UK	$t_t^{UK} = \phi_g^{UK} g_t^{UK} + \phi_b^{UK} b_t^{UK}$
G: 1, $\tau \in \{0, 1\}$		

Table 4: Fiscal (debt-stabilising) rules for each of the policy scenarios

where following Galí, López-Salido, and Vallés (2005), we define $g_t = \frac{G_t - G}{Y}$, $b_t = \frac{(B_t/P_{t-1}) - (B/P)}{Y}$, and $t_t = \frac{T_t - T}{Y}$. The model assumes that in the steady state, the government’s budget is balanced, i.e. $\frac{B}{Y} = 0$. Each fiscal policy rule states that tax revenue responds to changes in government spending and government debt, where the responses depend on $\phi_b \in [0, 1]$ and $\phi_g \in [0, 1]$. Intuitively, setting $\phi_b = 0$ would imply that the government cannot issue bonds/borrow, and taxpayers would have to absorb any shock in government spending. At the same time, $\phi_g = 0$ would make government fund its spending solely by issuing bonds. Plugging fiscal rules into the log linearised government budget constraints yields:

Scen. 1 & Scen. 3	Scot.	$b_{t+1} = (1 - \rho)(1 - \phi_g)g_t + (1 - \rho)(1 - \phi_b)b_t$
G: 2, $\tau \in \{0, 1\}$	rUK	$b_{t+1}^* = (1 - \rho^*)(1 - \phi_g^*)g_t^* + (1 - \rho^*)(1 - \phi_b^*)b_t^*$
Scen. 3 & Scen. 4	UK	$b_{t+1}^{UK} = (1 - \rho^{UK})(1 - \phi_g^{UK})g_t^{UK} + (1 - \rho^{UK})(1 - \phi_b^{UK})b_t^{UK}$
G: 1, $\tau \in \{0, 1\}$		

Table 5: Future bonds are determined by government spending and current government debt

These equations determine the equilibrium path for bond supply, with $(1 - \rho)(1 - \phi_b) < 1$ to ensure stability (non-explosiveness). **This is equivalent to**

imposing a No-Ponzi condition, i.e. the government cannot have outstanding debt in period T . Finally, note that B_t is a predetermined (state) variable, meaning its value is determined in the previous period, while the initial B_0 value is assumed to have been determined “historically”, a.k.a given.

2.3 Firms

In line with the literature, we assume a continuum of infinitesimally small firms, each producing a single good j over which they have monopolistic power. Given the monopolistic nature of the market for each good, the firms are allowed to adjust their prices to maximise profits but, following Calvo (1983), we assume that only a fraction $\theta \in [0, 1]$ of them actually do. The firms are assumed to be owned by households, implying that our budget constraint should have a Π_t (dividend) term. However, as all households own all firms and take profit/dividends as given, it does not affect first-order conditions/optimal behaviour and, as such, is not considered. The firm’s production function is given by:

$$Y_t(j) = A_t N_t(j)^{1-\alpha} \quad (51)$$

where A_t is the technology level shifter common to all firms and is assumed to evolve exogenously as an AR(1) process in log terms: $\ln A_t = \rho_a \ln A_{t-1} + \varepsilon_t^a$. Notice that the marginal product of labour is

$$\frac{\partial Y_t(j)}{\partial N_t(j)} = (1 - \alpha) A_t N_t(j)^{-\alpha} \quad (52)$$

which is increasing in technology level (increases labour productivity). Second order derivate with respect to hours worked indicates, that the production function exhibits decreasing returns to scale. Knowing this, the firm maximises their profit by choosing

the optimal amount of labour:

$$\max_{N_t(j)} \mathcal{F} = P_{H,t}(j)Y_t(j) - W_t(j)N_t(j) \quad (53)$$

$$\text{s.t. } Y_t(j) = A_t N_t(j)^{1-\alpha} \quad (54)$$

which can be solved (see Appendix A.xx-A.xx) to yield an optimality condition:

$$\frac{\partial \mathcal{F}}{\partial N_t(j)} \implies \frac{W_t}{P_t} = (1 - \alpha)A_t N_t^{-\alpha} \quad (55)$$

Suggesting that firm will hire workers up until the marginal product of labour is equal to the real wage. The optimality condition also acts as a link between real wage ($w_t - p_t$), labour and technology. Moreover, given that the marginal cost Ψ_t needs to equal domestic price level, rearranging Equation (55) yields:

$$\Psi_t = \frac{W_t}{(1 - \alpha)A_t N_t^{-\alpha}} \quad \text{Marginal cost} \quad (56)$$

$$\psi_t = w_t - (a_t - \alpha n_t + \log(1 - \alpha)) \quad (\text{Log}) \text{ Marginal cost} \quad (57)$$

Unsurprisingly, the marginal cost is increasing in wages and decreasing in marginal product of labour. It is important to emphasise that definition above is an *average* marginal cost across all firms producing goods j . The marginal cost varies across firms due to different existing levels of labour.

As mentioned earlier, the price stickiness is introduced by assuming that firms update their prices with probability $(1 - \theta)$. The newly set price is denoted as $\bar{P}_{H,t}(j)$. Following a similar proof offered by Galí (2015) for a closed-economy NK DSGE, note that if all firms are symmetrical, then they will choose the same price, i.e. $\bar{P}_{H,t}(j) = \bar{P}_{H,t}$. Thus, the domestic price index from Equation (6) can be rewritten

as:

$$P_{H,t} = \left[\int_0^1 (P_{H,t}(j))^{1-\epsilon} dj \right]^{\frac{1}{1-\epsilon}} \quad (58)$$

$$= \left[\int_{S(t)}^1 P_{H,t}(j)^{1-\epsilon} dj + \int_0^{S(t)} P_{H,t}(j)^{1-\epsilon} dj \right]^{\frac{1}{1-\epsilon}} \quad (59)$$

$$= \left[\theta (P_{H,t-1})^{1-\epsilon} + (1-\theta) (\bar{P}_{H,t})^{1-\epsilon} \right]^{\frac{1}{1-\epsilon}} \quad (60)$$

where $S(t)$ is a subset of firms that do not update their prices, and Equation (60) “follows from the fact that the distribution of prices among firms not adjusting in period t corresponds to the distribution of effective prices in period $t-1$, though with total mass reduced to θ ” (Galí, 2015: 84); similarly, the total mass of firms with prices $\bar{P}_{H,t}$ will be equal to $(1-\theta)$. Dividing (60) by $(P_{H,t-1})^{1-\epsilon}$ yields:

$$\Pi_{H,t}^{1-\epsilon} = \theta + (1-\theta) \left(\frac{\bar{P}_{H,t}}{P_{H,t-1}} \right)^{1-\epsilon} \quad (61)$$

Log-linearising (61) around zero inflation steady state yields:

$$\Pi_{H,t}^{1-\epsilon} = \theta + (1-\theta) \left(\frac{\bar{P}_{H,t}}{P_{H,t-1}} \right)^{1-\epsilon} \quad (62)$$

$$\Pi_H \mathbf{e}^{(1-\epsilon)\pi_{H,t}} = \theta + (1-\theta) \frac{\bar{P}_H}{P_H} \mathbf{e}^{(1-\epsilon)(\bar{p}_{H,t} - p_{H,t})} \quad (63)$$

$$\vdots \quad (\text{see Appendix A.xx - A.xx}) \quad (64)$$

$$\pi_{H,t} = (1-\theta)(\bar{p}_{H,t} - p_{H,t}) \quad (65)$$

Intuitively, this means that log domestic inflation depends on two elements: the difference between the current and new domestic price levels, and the price stickiness parameter θ . Consider two extreme cases when $\theta = 1$ and $\theta = 0$: when $\theta = 1$, then no firms would be permitted to update their prices and the domestic inflation would always be equal to zero (CPI would still vary due to terms of trade, assuming $v \neq 0$).

When $\theta = 0$, then all firms immediately react to any changes in marginal cost of production and traditional RBC “ineffective-money” results would follow. Firms that get to update their prices, do so by maximising their *discounted lifetime cash flow*:

$$\max_{\bar{P}_{H,t}} \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[\Lambda_{t,t+k} \left(\underbrace{\bar{P}_{H,t} Y_{t+k|t}}_{\text{Revenue}} - \underbrace{\mathcal{C}_{t+k}(Y_{t+k|t})}_{\text{Cost}} \right) / P_{H,t+k} \right] \quad (66)$$

$$\text{s.t. } Y_{t+k|t} = \left(\frac{\bar{P}_{H,t}}{P_{H,t+k}} \right)^{-\epsilon} C_{t+k} \quad (67)$$

where $\mathbb{E}_t [\Lambda_{t,t+k}] \forall k \geq 0$ is the expected stochastic discount factor used to discount profit (revenue less cost) in every period starting from current⁶. $Y_{t+k|t}$ is the expected output in periods $t+k$ given output in period t , and \mathcal{C}_{t+k} is the nominal cost of producing the expected output. The maximisation problem is subject to k number of demand constraints (67). Substituting the constraint into (66), taking first-order conditions, and log-linearising around zero-inflation steady state yields:

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[\Lambda_{t,t+k} \left(\bar{P}_{H,t} \left(\frac{\bar{P}_{H,t}}{P_{H,t+k}} \right)^{-\epsilon} C_{t+k} - \mathcal{C}_{t+k}(Y_{t+k|t}) \right) / P_{H,t+k} \right] \quad (68)$$

∴ (see Appendix A.xx-A.xx)

$$\bar{p}_{H,t} = \mu + (1 - \beta\theta) \sum_{k=0}^{\infty} (\beta\theta)^k \mathbb{E}_t [\psi_{t+k|t}] \quad (69)$$

where $\mathbb{E}_t [\psi_{t+k|t}]$ and μ are the expected log marginal cost and desired⁷ markup, respectively. Equation (69) is known as the (log) *optimal price setting condition* and presents firms as forward looking discounted profit maximisers. Note, that it is consistent with the previous exposition of the two extreme cases when $\theta = 0$ and $\theta = 1$, i.e. when $\theta = 1$, then $\pi_{H,t} = \bar{p}_{H,t} - \bar{p}_{H,t-1} = \mu - \mu = 0$. Furthermore, the earlier discussion on optimality condition noted that firms will hire labour up

⁶Notice that we could write this as $\mathbb{E}_t [\Lambda_{t,t+k}] = \mathbb{E}_t \left[\frac{1}{R_{t,t+k}} \right]$ derived from the households' optimisation problem.

⁷Markup that occurs under flexible or frictionless prices

until the real wage is equal to the marginal product of labour. Given that the price (and, subsequently, real wage) changed, firms might choose a different level of labour compared to that of other firms. This allows us to derive a relationship between the average marginal cost and a marginal cost of a firm that just updated their price:

$$\psi_{t+k|t} = \psi_{t+k} + \alpha(n_{t+k|t} - n_{t+k}) \quad (70)$$

\vdots (see Appendix A.xx-A.xx)

$$= \psi_{t+k} - \frac{\alpha\epsilon}{1-\alpha} \quad (71)$$

Plugging (71) to (69) and rewritting as a recursive equation yields:

$$\bar{p}_{H,t} = \mu + (1-\beta\theta) \sum_{k=0}^{\infty} (\beta\theta)^k \mathbb{E}_t \left[\psi_{t+k} - \frac{\alpha\epsilon}{1-\alpha} \right] \quad (72)$$

\vdots (see Appendix A.xx - A.xx)

$$\bar{p}_{H,t} = \beta\theta \mathbb{E}_t [\bar{p}_{H,t+1}] + (1-\beta\theta)(p_{H,t} - \Theta\hat{\mu}_t) \quad (73)$$

where $\mu_t = p_{H,t} - \psi_t$ is the average markup, $\hat{\mu}_t$ is the gap between the average and desired markups, and $\Theta = \frac{1-\alpha}{1-\alpha+\alpha\epsilon}$. Using (65), it is easy to transform (73) into a version of New Keynesian Phillips Curve (NKPC):

$$\bar{p}_{H,t} = \beta\theta \mathbb{E}_t [\bar{p}_{H,t+1}] + (1-\beta\theta)(p_{H,t} - \Theta\hat{\mu}_t) \quad (74)$$

\vdots (see Appendix A.xx-A.xx)

$$\pi_{H,t} = \beta \mathbb{E}_t [\pi_{H,t+1}] - \lambda\hat{\mu}_t \quad (75)$$

where $\lambda = \frac{(1-\theta)(1-\beta\theta)}{\theta}\Theta$. Equation (75) suggests that inflation will be increasing as long as $\mu > \mu_t$. That is, price-adjusting firms will continue raising prices until the difference between the marginal cost and the domestic price level is smaller than the desired markup, i.e. $p_{H,t} - \psi_t < \mu$. At $p_{H,t} - \psi_t = \mu$, the domestic inflation will (slowly) tend towards the steady state level of zero due to the discount factor $\beta \in (0,1)$. The next section will show that the markup gap can be expressed as

a function of the natural output gap, which itself is a function of technology level, taxes, government spending, world output, and the preference shifter. Therefore, the resulting expression (NKPC) will pin the relationship between domestic inflation and all other real variables. This is why NKPC is regarded as one of the key (non-policy) equations in NK DSGE models.

2.4 Equilibrium

The goods market for a specific good j clears when domestic firms produce just enough of the good to satisfy the demand of home households, foreign households, and the home government. In line with Galí (2015), the demand for exports of good j is taken to be given as:

$$X_t(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}} \right)^{-\epsilon} X_t \quad (76)$$

$$\text{where } X_t = \left(\int_0^1 X_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}} \quad (77)$$

$$= v \left(\frac{P_{H,t}}{\mathcal{E}_t \bar{P}_{H,t}} \right)^{-\eta} Y_t^* \quad (78)$$

$$= v \mathcal{S}_t^\eta Y_t^* \quad (79)$$

where (77) is the index of aggregate exports, (78) determines aggregate exports as a function of world output (the relationship is assumed to be given), and (79) is derived by substituting definition of the effective terms of trade. Differently than Galí (2015), we introduce index of government purchasing:

$$G_t = \left(\int_0^1 G_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}} \quad (80)$$

so that the government demand of any good j is defined as (derivation provided by Appendix A.xx-A.xx):

$$G_t(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}} \right)^{-\epsilon} G_t \quad (81)$$

Therefore, total demand for good j is:

$$Y_t(j) = C_t(j) + X_t(j) + G_t(j) \quad (82)$$

$$\vdots \quad (\text{see Appendix A.xx-A.xx}) \quad (83)$$

$$Y_t(j) = \left(\frac{P_{H,t}(j)^{-\varepsilon}}{P_{H,t}} \right) \left[(1-v) \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} C_t + v \mathcal{S}^\eta Y_t^* + G_t \right] \quad (84)$$

which can be plugged in to definition of aggregate output $Y_t = \left(\int_0^1 Y_t(j)^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}}$ to yield:

$$Y_t = (1-v) \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} C_t + v \mathcal{S}^\eta Y_t^* + G_t \quad (85)$$

Note that Equation (82) and all subsequent derivations (marginally) vary depending on the policy scenario in question:

Scen. 1 & Scen. 3	Scot.	$Y_t(j) = C_t(j) + X_t(j) + G_t(j)$
G: 2, $\tau \in \{0, 1\}$	rUK	$Y_t^*(j) = C_t^*(j) + X_t^*(j) + G_t^*(j)$
Scen. 3 & Scen. 4	Scot.	$Y_t(j) = C_t(j) + X_t(j) + \varpi G_t^{UK}(j)$
G: 1, $\tau \in \{0, 1\}$	rUK	$Y_t^*(j) = C_t^*(j) + X_t^*(j) + (1-\varpi)G_t^{UK}(j)$

Table 6: Demand for good j for different policy scenarios

That is, when there is a single government (Westminister), then we assume that Scotland's contribution towards covering government spending is equal to the population share in the UK. Equation (82) can be loglinearised around a symmetric steady state to yield:

$$Y_t = (1-v) \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} C_t + v \mathcal{S}_t^\eta Y_t^* + G_t \quad (86)$$

$$Y \mathbf{e}^{y_t} = (1-v) \left(\frac{P}{P_H} \right)^\eta C \mathbf{e}^{-\eta p_{H,t} + \eta p_t + c_t} + v \mathcal{S}^\eta Y^* \mathbf{e}^{\eta s_t + y_t^*} + G \mathbf{e}^{g_t} \quad (87)$$

$$\vdots \quad (\text{see Appendix A.147 - A.154})$$

$$y_t = C_Y [(1-v)c_t + v(2-v)\eta s_t + v y_t^*] + G_Y g_t \quad (88)$$

where $C_Y = \frac{C}{Y}$ and $G_Y = \frac{G}{Y}$ are the consumption-to-output and the government spending-to-output ratios in the steady state, respectively. We can use Equation (49) that links domestic consumption to world output and previous equation (88) to express terms of trade as a function of domestic output, world output, preference shifter, and government spending:

$$y_t = C_Y \left[(1-v) \left(y_t^* + \frac{1}{\sigma} z_t + \frac{1-v}{\sigma} s_t \right) + v(2-v)\eta s_t + v y_t^* \right] + G_Y g_t \quad (89)$$

∴ (see Appendix A.162 - A.178)

$$s_t = \sigma_v (C_Y^{-1} y_t - y_t^* - C_G^{-1} g_t) - (1-v)\Phi z_t \quad (90)$$

where $\varpi = \sigma\eta + (1-v)(\sigma\eta - 1)$, $\Phi = \frac{1}{1+v(\varpi-1)}$, $\sigma_v = \sigma\Phi$, and C_G is the consumption-to-government spending ratio in the steady state. Notice that when $G = 0$, $Y = C$, then $s_t = \sigma_v(y_t - y_t^*) - (1-v)\Phi z_t$, i.e. identical to Galí (2015). Negative government term implies that (effective) trade of terms are decreasing in government spending. This is intuitive: the government is modelled to demand exclusively domestic goods, which induces inflationary pressure and makes home goods less competitive internationally. The opposite is true for the domestic-world output gap: if our firms produce relatively more than the rest of the world and is able to export relatively more, then terms of trade increase (the first term of (90) is positive). However, $y_t > y_t^*$, does not *automatically* imply $\Delta x_{t+1} > 0$ (an increase in exports), the terms to be explicitly linked. Following a similar approach to Galí (2015), the net exports is denoted in terms of each GDP component as a share of their respective steady state values. The resulting expression can be combined with the aggregate resource constraint (88) and

(49) to yield a link between terms of trade and net exports NX_t :

$$NX_t = \frac{1}{Y}Y_t - \frac{1}{C} \left(\frac{P_t}{P_{H,t}} C_t \right) - \frac{1}{G} G_t \quad (91)$$

$$\vdots \quad (\text{see Appendix A.xx-A.xx})$$

$$nx_t = y_t - c_t - v s_t - g_t \quad (92)$$

$$\vdots \quad (\text{see Appendix A.xx-A.xx})$$

$$nx_t = C_Y \left[v \left(\frac{\varpi}{\sigma} - 1 \right) s_t - \frac{v}{\sigma} z_t \right] \quad (93)$$

Furthermore, the Euler equation in (24) is a function of CPI, but using (31), it can be rewritten to be a function of domestic inflation and terms of trade:

$$c_t = \mathbb{E}\{c_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}\{\pi_{H,t+1}\} - \rho) + \frac{v}{\sigma} \mathbb{E}\{\Delta s_{t+1}\} + \frac{1}{\sigma}(1 - \rho_z)z_t \quad (94)$$

Finally, aggregate resource constraint (88), terms of trade definition (90), and new Euler equation (94) can be combined to derive a version of dynamic IS equation:

$$c_t = \mathbb{E}\{c_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}\{\pi_{H,t+1}\} - \rho) + \frac{v}{\sigma} \mathbb{E}\{\Delta s_{t+1}\} + \frac{1}{\sigma}(1 - \rho_z)z_t \quad (95)$$

$$\vdots \quad (\text{see Appendix A.179-A.192})$$

$$y_t = \mathbb{E}_t\{y_{t+1}\} - C_Y \left[\frac{1}{\sigma_v}(i_t - \mathbb{E}\{\pi_{H,t+1}\} - \rho) + \frac{1-v}{\sigma}(1 - \rho_z)z_t + v(\varpi - 1) \mathbb{E}\{\Delta y_{t+1}^*\} \right] - G_Y \mathbb{E}_t\{\Delta g_{t+1}\} \quad (96)$$

which implies that output in current period depends not only on expected output, inflation, and change in world output, but it also depends on expected changes in government spending. Equation (96) can be expressed in terms of output and real

interest rate gaps:

$$y_t^n = \mathbb{E}_t\{y_{t+1}^n\} - C_Y \left[\frac{1}{\sigma_v}(r_t^n - \rho) + v(\varpi - 1) \mathbb{E}\{\Delta y_{t+1}^*\} + \frac{1-v}{\sigma}(1 - \rho_z)z_t \right] - G_Y \mathbb{E}_t\{\Delta g_{t+1}\} \quad (97)$$

∴ (see Appendix A.193 - A.197)

$$r_t^n = \sigma_v C_Y^{-1} \mathbb{E}_t\{\Delta y_{t+1}^n\} + \rho + \sigma_v v(\varpi - 1) \mathbb{E}\{\Delta y_{t+1}^*\} + \Phi(1 - v)(1 - \rho_z)z_t + \sigma_v C_G^{-1} \mathbb{E}_t\{\Delta g_{t+1}\} \quad (98)$$

∴ (see Appendix A.197 - A.200)

$$\tilde{y}_t = \tilde{y}_{t+1} - \frac{1}{\sigma_v} C_Y (i_t - \mathbb{E}\{\pi_{H,t+1}\} - r_t^n) \quad (99)$$

where y_t^n is the natural output, \tilde{y}_t is the output gap, and Equation (98) defines the natural real rate of interest r_t^n . Equation (99) is called Dynamic IS equation. It shows, that every period (hence, *dynamic*), output gap is determined by the output gap “tomorrow” the real interest rate gap “today”. It is one of the key equations in NK DSGE models. It sets a path for the output gap given a path for the real interest rate. The real interest rate gap path depends on inflation path, which was earlier given to be a function of the markup gap. However, for the model to yield a solution, the inflation path needs to be directly linked to the output gap. Firstly, average markup can be rewritten to yield:

$$\mu_t = p_{H,t} - \psi_t \quad (100)$$

$$= p_{H,t} - (w_t - a_t + \alpha n_t)$$

∴ (see Appendix A.201-A.222)

$$\mu_t = - \left(\sigma C_Y^{-1} + \frac{\varphi + \alpha}{1 - \alpha} \right) y_t + v(\varpi - 1)s_t - \frac{\tau}{1 - \tau} \tau_t + \left(1 + \frac{\varphi + \alpha}{1 - \alpha} \right) a_t - v z_t + \sigma C_G^{-1} g_t \quad (101)$$

Evaluating (101) at flexible prices $\theta = 0$ and solving for the output term yields the

expression for the natural level of output y_t^n :

$$\begin{aligned} \mu_t = & - \left(\sigma C_Y^{-1} + \frac{\varphi + \alpha}{1 - \alpha} \right) y_t^n + v(\varpi - 1)s_t^n - \frac{\tau}{1 - \tau} \tau_t + \left(1 + \frac{\varphi + \alpha}{1 - \alpha} \right) a_t - v z_t \\ & + \sigma C_G^{-1} g_t \end{aligned} \quad (102)$$

\vdots (see Appendix A.223 - A.249)

$$y_t^n = \Gamma_* y_t^* + \Gamma_z z_t + \Gamma_a a_t + \Gamma_\tau \tau_t + \Gamma_g g_t \quad (103)$$

where:

G \neq 0, Y = C + G

$$\Gamma_* = - \frac{v(\varpi - 1)\sigma_v(1 - \alpha)}{\sigma_v C_Y^{-1}(1 - \alpha) + \varphi + \alpha}$$

$$\Gamma_z = - \frac{v\varpi\Phi(1 - \alpha)}{\sigma_v C_Y^{-1}(1 - \alpha) + \varphi + \alpha}$$

$$\Gamma_a = \frac{1 + \varpi}{\sigma_v C_Y^{-1}(1 - \alpha) + \varphi + \alpha}$$

$$\Gamma_g = \frac{\sigma_v C_G^{-1}(1 - \alpha)}{\sigma_v C_Y^{-1}(1 - \alpha) + \varphi + \alpha}$$

Scenarios 3 & 4

$$\Gamma_\tau = - \frac{\tau}{1 - \tau} \frac{1 - \alpha}{\sigma_v C_Y^{-1}(1 - \alpha) + \varphi + \alpha}$$

G = 0, Y = C (Galí, 2015)

$$\Gamma_* = - \frac{v(\varpi - 1)\sigma_v(1 - \alpha)}{\sigma_v(1 - \alpha) + \varphi + \alpha} \quad (104)$$

$$\Gamma_z = - \frac{v\varpi\Phi(1 - \alpha)}{\sigma_v(1 - \alpha) + \varphi + \alpha} \quad (105)$$

$$\Gamma_a = \frac{1 + \varpi}{\sigma_v(1 - \alpha) + \varphi + \alpha} \quad (106)$$

$$\Gamma_g = 0 \quad (107)$$

Scenarios 1 & 2

$$\Gamma_\tau = 0 \quad (108)$$

Notice that the natural output is increasing in technology level and government spending, but decreasing in labour taxes, preference shifter, and world output. The expression for natural output can be used to update the expression for the natural rate of

interest:

$$\begin{aligned} r_t^n &= \sigma_v C_Y^{-1} \mathbb{E}_t \{\Delta y_{t+1}^n\} + \rho + \sigma_v v(\varpi - 1) \mathbb{E} \{\Delta y_{t+1}^*\} + \Phi(1 - v)(1 - \rho_z) z_t \\ &\quad + \sigma_v C_G^{-1} \mathbb{E}_t \{\Delta g_{t+1}\} \end{aligned} \quad (109)$$

∴ see Appendix A.250-A.262

$$\begin{aligned} r_t^n &= \rho - C_Y^{-1} \sigma_v \Gamma_a (1 - \rho_a) a_t + \Psi_* \mathbb{E}_t \{\Delta y_{t+1}^*\} + \Psi_z (1 - \rho_z) z_t \\ &\quad - \Psi_g (1 - \rho_g) g_t + \Psi_\tau \mathbb{E}_t \{\Delta \tau_{t+1}\} \end{aligned} \quad (110)$$

where:

$$\begin{array}{ll} G \neq 0, Y = C + G & G = 0, Y = C \text{ (Galí, 2015)} \end{array}$$

$$\Psi_* = \sigma_v (C_Y^{-1} \Gamma_* + v(\varpi - 1)) \quad \Psi_* = \sigma_v (\Gamma_* + v(\varpi - 1)) \quad (111)$$

$$\Psi_z = (1 - v)\Phi - C_Y^{-1} \sigma_v \Gamma_z \quad \Psi_z = (1 - v)\Phi - \sigma_v \Gamma_z \quad (112)$$

$$\Psi_g = \sigma_v (C_Y^{-1} \Gamma_g + C_G^{-1}) \quad \Psi_g = 0 \quad (113)$$

$$\begin{array}{ll} \text{Scenarios 3 \& 4} & \text{Scenarios 1 \& 2} \end{array}$$

$$\Psi_\tau = \sigma_v C_Y^{-1} \Gamma_\tau \quad \Psi_\tau = 0 \quad (114)$$

When the prices are flexible, then each firm will choose the same optimal level of labour, implying constant average markup $\mu_t = \mu$. Subtracting μ_t definition (101) from μ definition (102) yields:

$$\hat{\mu}_t = \mu_t - \mu = - \left(\sigma + \frac{\varphi + \alpha}{1 - \alpha} \right) \tilde{y}_t + v(\varpi - 1) \tilde{s}_t \quad (115)$$

$$= - \left(\sigma_v C_Y^{-1} + \frac{\varphi + \alpha}{1 - \alpha} \right) \tilde{y}_t \quad (116)$$

where \tilde{s}_t denote terms of trade gap, and Equation (116) uses $\tilde{s}_t = \sigma_v C_Y^{-1} \tilde{y}_t$ relationship implied by (90). Finally, plugging (116) in a version of NKPC defined by Equation (75), yields:

$$\pi_{H,t} = \beta \mathbb{E}_t [\pi_{H,t+1}] + \kappa \tilde{y}_t \quad (117)$$

where $\kappa = \lambda \left(\sigma_v C_Y^{-1} + \frac{\varphi + \alpha}{1 - \alpha} \right)$. That is, whenever output is greater than that implied by the natural level of output ($\tilde{y}_t > 0$), inflation increases, which subsequently reduces demand and brings output closer to the natural level of output. The opposite is also true, i.e. whenever economy is under-producing goods ($\tilde{y}_t < 0$), then inflation decreases to induce demand and bring output level to that under flexible prices. If prices were fully flexible ($\theta = 0$), then economy would always be producing $y_t = y_t^n$, suggesting a zero inflation steady state.

Finally, we have a path for output gap (implied by DIS), a path for inflation (implied by NKPC), the only remaining path to “close” the model is that for the interest rate. Here, the monetary authority is assumed to set interest rate following a Taylor rule:

$$i_t = \phi_\pi \pi_{H,t} + \phi_y \hat{y}_t + m_t \quad (118)$$

where m_t is a contractionary shock, which is assumed to follow an AR(1) process $m_t = \rho_m m_{t-1} + \varepsilon_t^m$, $\varepsilon_t^m \sim \mathcal{N}(0, \sigma_m^2)$, and $\hat{y}_t = y_t - y$ is the deviation from the steady state output. The Taylor rule suggests that monetary authority will raise the interest rate whenever inflation increases or output exceeds the steady state output.

3 Application

Regarding the RBC family of models, the application typically refers to calibrating model parameters and analysing impulse response functions, given temporary shocks in interest rate, level of technology, world output, etc. In 20XX, Smith and Watson (??) applied Markov Chain Monte Carlo (MCMC) technique to estimate model parameters, which has become standard practice. However, due to the limitations of the word count and the scope of this dissertation, the specifics of Bayesian estimation will not be delved into. Blake and Mumtaz’s (2017) handbook of applied Bayesian econometrics has a dedicated section for implementing random walk Metropolis-Hastings algorithm for DSGE models and even includes a complementary Matlab code repository. For model parameters estimation and impulse response functions, this disser-

tation will utilise out-of-the-box solutions offered by the Dynare computer package (**refs**). This section will cover parameters, data, and estimation results. The following section will provide an analysis of impulse response functions under different policy scenarios and a parameter sensitivity analysis.

While estimating all parameters is an option, in theory, most parameter values will be calibrated in line with the literature. The reason behind the decision is the simplicity of our model. The model does not consider capital, habit formation, sticky wages, limited access to financial instruments, and many other extensions in the literature that improve data fit. This makes our data uninformative for some parameters, and estimation results heavily rely on priors. Table 3 below lists calibrated (as opposed to estimated) model parameters for Scotland and the rest of the UK. In the majority of the cases, parameters are assumed to be symmetrical. The asymmetry in responses to government spending is assumed to stem from different steady state ratios and estimated fiscal policy feedback parameters ϕ_b and ϕ_g , as well as government spending persistence parameter ρ_g . The last column provides the reasoning behind all model parameter values.

Parameter	Value	Parameter	Value	Strategy
β	0.99	β^*	β	Calculated so that $R_t = 1.01$
σ	1.00	σ^*	σ	(refs)
φ	6.00	φ^*	φ	(refs)
α	1/3	α^*	α	(refs)
ϵ	6.00	ϵ^*	ϵ	(refs)
θ	0.75	θ^*	θ	(refs)
v	0.60	v^*	v	(refs)
ρ_a	0.50	ρ_a^*	ρ_a	(refs)
η	1	η^*	η	(refs)
ρ_z	0.50	ρ_z^*	ρ_z	(refs)
ρ_ν	0.50	ρ_ν^*	ρ_ν	(refs)
ϕ_π	1.50	ϕ_π^*	ϕ_π	(refs)
ϕ_y	0.125	ϕ_y^*	ϕ_y	(refs)
G_Y	0.24825	G_Y^*	0.16374	QNAS/QNA, 1997Q1-2007Q4 arithmetic mean
C_Y	$1.00 - G_Y$	C_Y^*	$1 - G_Y^*$	Implied by $1 = \frac{C}{Y} + \frac{G}{Y}$
Y_C	$1/C_Y$	Y_C^*	$1/C_Y^*$	Implied
G_C	$Y_C - 1$	G_C^*	$Y_C^* - 1$	Implied
<i>For policy scenarios with multiple governments:</i>				
τ	G_Y	τ^*	G_Y^*	By assumption
<i>For policy scenarios with a single government:</i>				
		τ^{UK}	$\varpi G_Y + (1 - \varpi)G_Y^*$	Calculated

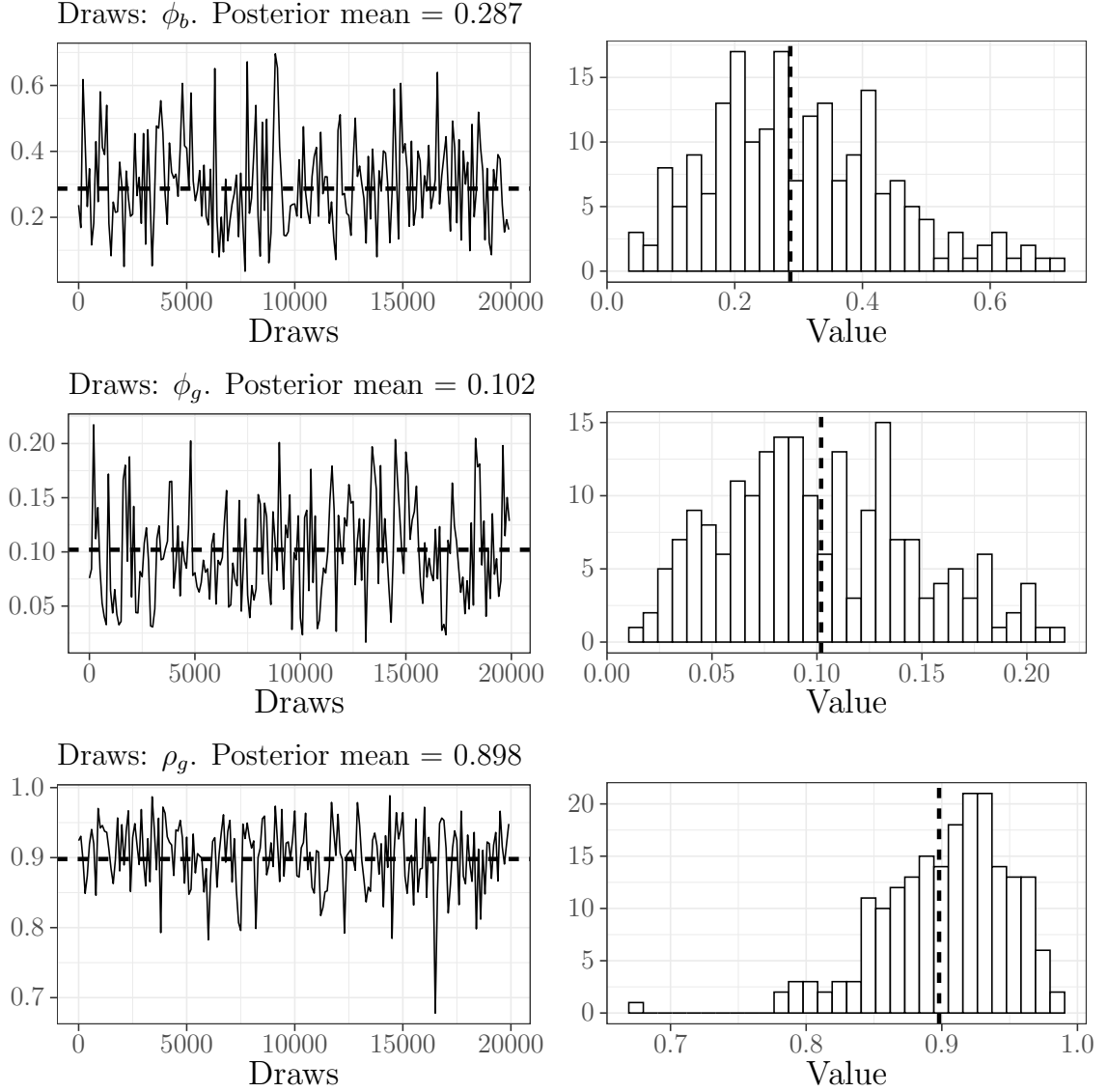
Table 7: Model parameters, their values, and the strategy for obtaining the values

The dissertation, however, does estimate three sensitive parameters key to the research question: ϕ_b , ϕ_g , and ρ_g (and their rUK / UK counterparts). The accep-

tance rate per chain was c. 22%, close to the optimal acceptance rate suggested by Roberts and Rosenthal (2001). The acceptance rate in the range of 23% ensures that the variance of candidate parameter values is neither too large nor too small (ibid), meaning that all sets of parameters had a reasonable probability of being drawn. The number of MH draws and initial (“burn-in”) draws was set to 300.000 and 100.000, respectively. Figures (??)-(??) display how the 300.000 draws varied. The variance of the draws suggests that the serial correlation of lagged draws was not persistent (fading), which is also indicative of successful estimation Roberts and Rosenthal (2001). The posterior means of all estimated parameters are close to their prior means, which can be primarily attributed to tight prior variance. However, the two means do not coincide, suggesting (to some degree) an informative estimation of the parameters. In fact, estimation of ϕ_b and ϕ_g are bound to be difficult given that the fiscal rule, while being reasonable and intuitive, is imposed arbitrarily to “close” the model. The true rule followed by the decision-makers, however, might be more complex and considerate of other metrics not captured by the model. All parameter values were drawn from Beta distribution due to it’s values being bounded by zero and one. Table 3 lists distributions, prior means and variances, and posterior means of all estimated parameters.

Variable	Distribution	Prior Variance	Prior Mean	Posterior Mean
<i>For policy scenarios with multiple governments:</i>				
ϕ_b	Beta	0.15	0.33	0.351
ϕ_g	Beta	0.05	0.10	0.102
ρ_g	Beta	0.05	0.90	0.898
ϕ_b^*	Beta	0.15	0.33	0.341
ϕ_g^*	Beta	0.05	0.10	0.104
ρ_g^*	Beta	0.05	0.90	0.902
<i>For policy scenarios with a single government:</i>				
ϕ_b^{UK}	Beta	0.15	0.33	0.341
ϕ_g^{UK}	Beta	0.05	0.10	0.104
ρ_g^{UK}	Beta	0.05	0.90	0.902

Table 8: Model parameter estimation set-up and results.



Appendix C displays draws of corresponding variables for the rest of the UK and UK. In terms of data, most time series for Scotland and the rest of the UK were acquired from the Quarterly National Accounts of Scotland (QNAS) and the Quarterly National Account (QNA), respectively. In contrast to Ricci (2019), who used 1998Q1-2007Q4 dataset, this dissertation utilised full QNA/QNAS dataset (1998Q1-2022Q4) for the variables of interest.⁸

⁸Ricci (2019) argued that their DSGE model is not “equipped to account for” (Ricci, 2019: 123) the global financial crisis of 2008. However, the use of a full dataset was found to improve estimation results for this model and is a more transparent choice. Therefore, the dissertation opted for 1998Q1-2022Q4 sample date range.

Time Series	Source
Scot. Deflator (2018=100)	QNAS 2022: Q2 Table A, column O
Scot. Output	QNAS 2022: Q2 Table A, column D
Scot. Consumption	QNAS 2022: Q2 Supplementary Tables, Table 12, Column P
Scot. Compensation of Employees	QNAS 2022: Q2 Table I, column C
Scot. Working Population	Population Estimates Time Series Data, National Records of Scotland
Scot. Government Spending	Table G1: GDP Expenditure Approach at Current Prices (onshore): Government Spending
UK Deflator (2023=100)	QNA: Implied GDP deflator at market prices: SA Index
UK Output	QNA: Gross Domestic Product at market prices: Current price: Seasonally adjusted £m
UK Consumption	QNA: 0 Household final consumption expenditure: Domestic concept CP SA £m
UK Compensation of Employees	QNA: UK (S.1): Compensation of employees (D.1) Uses: Current price: £m: SA
UK Government Spending	QNA: General Government: Final consumption expenditure (P3): CPSA £m
rUK Working Population	NOMIS Population Estimates

Table 9: Time series used for estimation

Each time series were then processed in line with Pfifer (2019). That is, the time series for Scotland were adjusted to 2018 price levels and divided by the number of (estimated) working-age residents in Scotland. Then, the time series were expressed in natural logarithms and first-differenced to induce stationarity. Finally, the series was demeaned. Identical data processing steps were taken for the rest of the UK time series, except they were adjusted to 2023 price levels. Note that the use of 2018 and

2023 deflators does not have an impact on variables' sample cross-moments but allows to be in line with QNA/QNAS. Also note, that data was not available for all regions of the UK. Therefore, in most cases, the rest of the UK time series were calculated by taking Scotland's time series and subtracting them from the UK-wide ones. The resulting values should be approximately equal to the rest of the UK ones, as long as QNAS and QNA use symmetrical accounting/data processing methods for each pair of time series. Appendix B displays plots of the pre- and post-transformed time series.

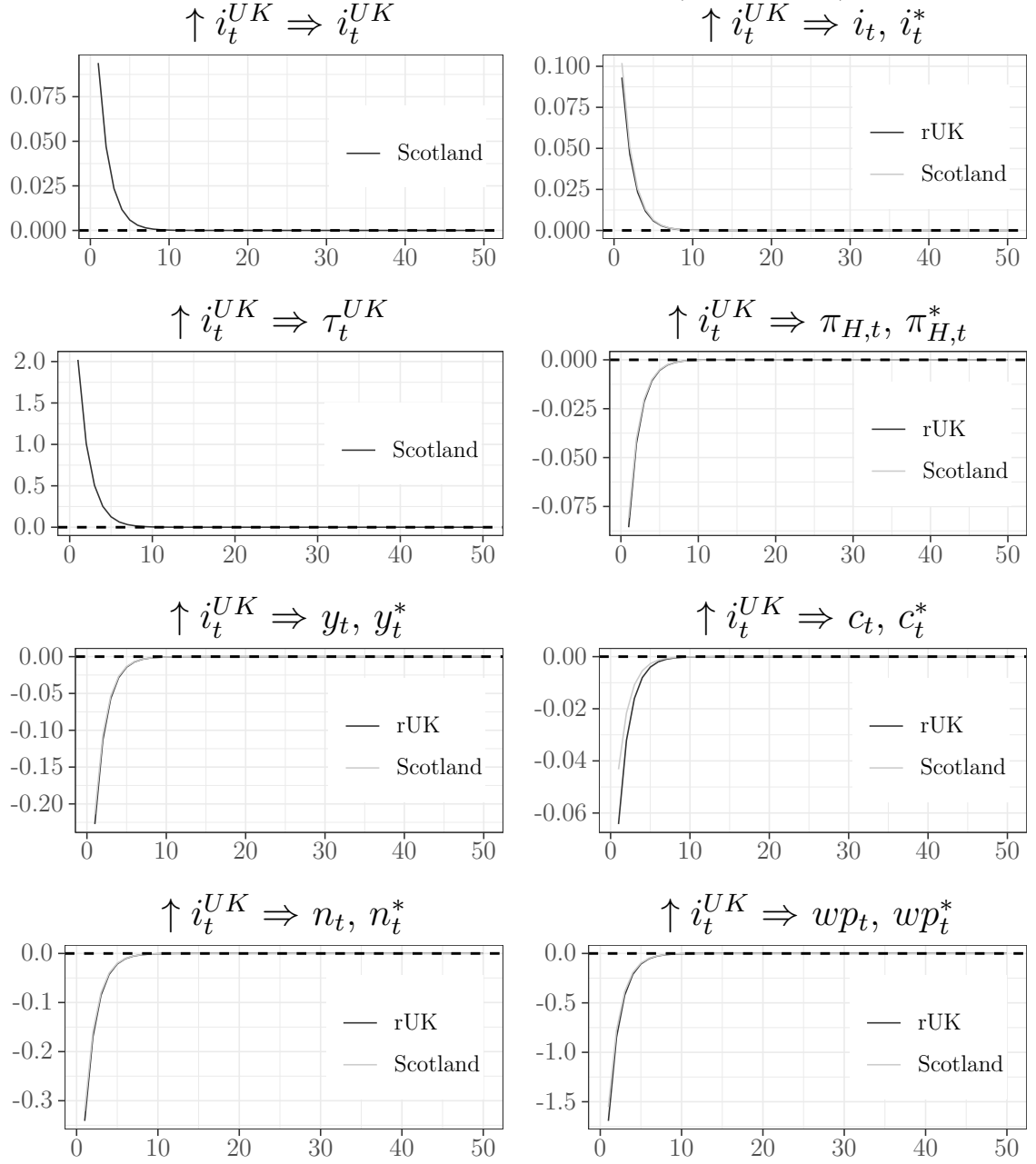
4 Results

This section will consider the responses of endogenous variables, given two types of shocks: monetary and fiscal. Although the examination of monetary shocks falls beyond the scope of this dissertation, it serves as a valuable benchmark to assess the accuracy and validity of our model. The Appendix Z also includes dynamic responses to world output and price level, as well as preferences (z_t). This dissertation does not provide a discussion on responses to these shocks. For the monetary policy shock, we use the policy scenario four model, i.e. a single fiscal government that uses labour tax and bond-issuance as fiscal instruments to raise budget revenue. In the interest of space, majority of the impulse response function (IRF) plots are placed in the Appendix Z.

Figure ? displays responses of fourteen endogenous variables. Two of the variables are UK-wide, namely labour tax rate and interest rate itself. Ten of the parameters are consumption, output, real wage ($wp_t = w_t - p_t$), domestic inflation and employment (hours worked) for each country. The remaining two parameters are country-specific interest rates. Whilst not of interest directly (they exist purely for technical reasons), it shows that the two interest rates co-move in line with our monetary union imposition. Other responses are very much in line with textbook literature (Galí, 2015). An increase in interest rate makes saving more attractive,

making households incentivised to consume less “today” in hopes to consume more “tomorrow” (see Euler equation). A decline in demand for home goods results in decline to output, which creates an abundance of labour, leading to an increase in unemployment. An increase in interest rate also leads to downward inflationary pressure, because goods become relatively less attractive to households compared to saving and making goods cheaper becomes the best strategy for price-setting, profit-maximising firms. Finally, given that employment decreases, bringing down the tax revenue, the government responds with an increase in labour tax rate (τ_t^{UK}) to clear the government budget. All effects fade as economy returns to its steady state equilibrium.

Contractionary Monetary Shock (Scenario 4)



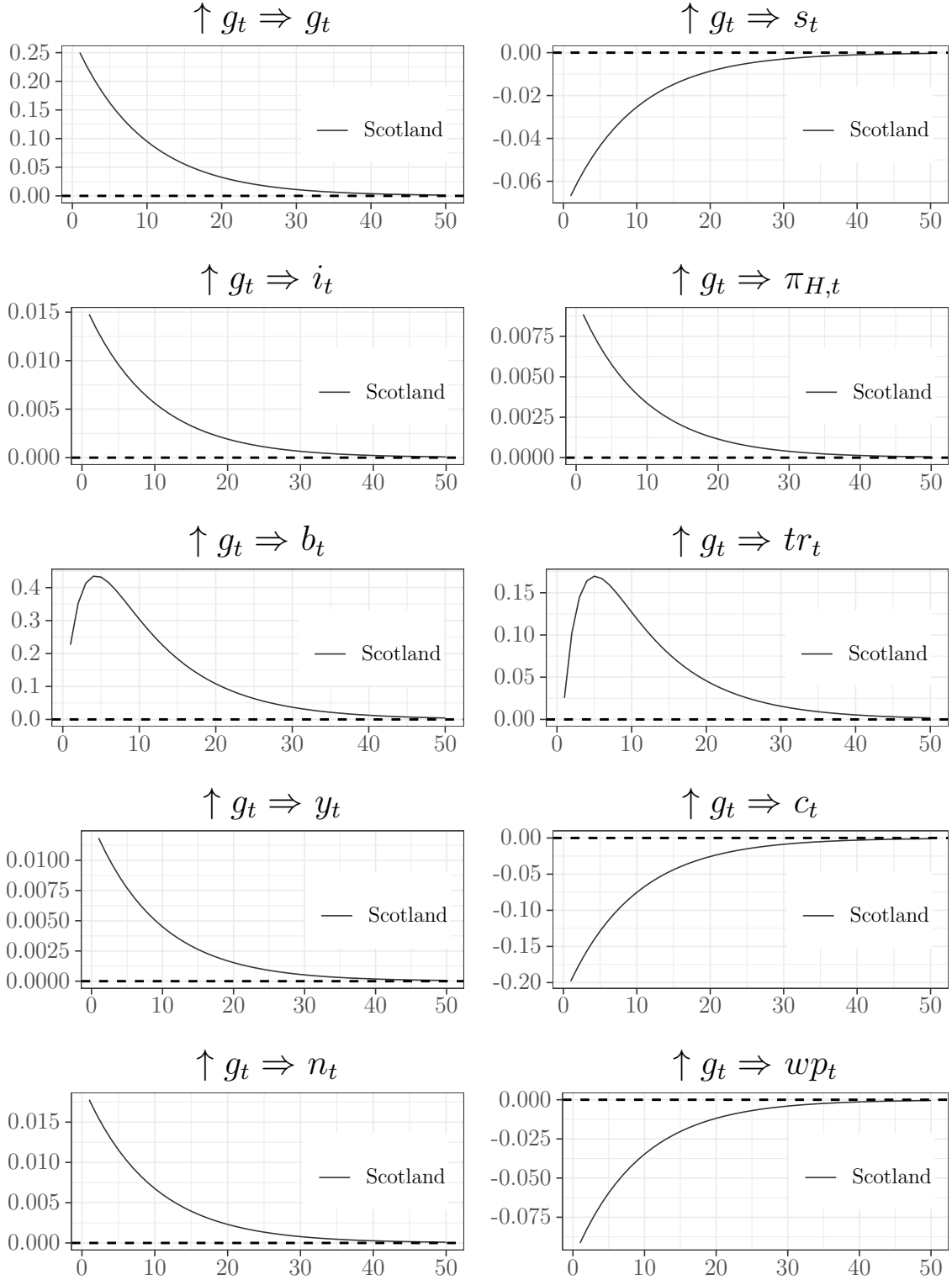
Next, dynamic responses to an increase in government spending are considered. Across all policy scenarios, the responses exhibit a consistent behaviour. That is, the government reduces the quantity of available resources to the household, leading to a decrease in consumption. The negative wealth effect becomes greater than the substi-

tution effect between consumption and leisure, leading to an increase in employment. An increase in labour supply allows firms to produce more (output increases) while paying less in real terms (real wage decreases). In all scenarios, the supply of government issued bonds increases as the government raises funds to cover its spending. In the scenarios where labour tax exists, the government raises the tax rate, as well as issues bonds. The model considered in this dissertation assumes that the government “consumes” domestic goods. Therefore, an increase in government spending raises demand for these goods and creates an upward inflationary pressure, as shown in the Figure ? below. Domestic goods become less competitive internationally, leading to depreciation in terms of trade (displayed only in Figure ? in the interest of space).

In terms of asymmetric responses, the dissertation assumed symmetrical parameters across the two countries, with the only exceptions of government spending-to-output ratios (G_Y , G_Y^*) and steady state labour tax rates τ , τ^* . In the first two policy scenarios, there is no steady state labour tax, leaving G_Y and G_Y^* as the only sources of potential asymmetry. The impulse response functions indicate that endogenous variables across the two countries respond identically to an increase in lump-sum taxes, see Figures ? and ?. However, when labour taxes are introduced (scenarios three and four), the responses vary noticeably. More specifically, the model suggests that a higher steady state labour tax rate amplifies the upward effect on output and employment. The effect on consumption is close to identical, while the real wage decreases considerably more in an economy with lower long-run steady state labour tax rate (the rest of the UK).

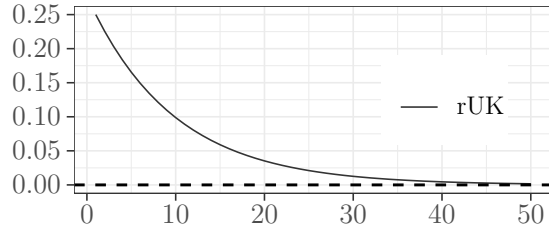
In terms of transmission of shock from one country to the other, the model does not consider migration, symmetrical wages, and significant price pass-through (low real exchange rate between the two economies). Therefore, in policy scenarios with two governments, endogenous variables in one country do not respond to shocks in government spending in another country. This is a clear limitation of the model.

Expansionary Fiscal Shock in Scotland (Scenario 1)

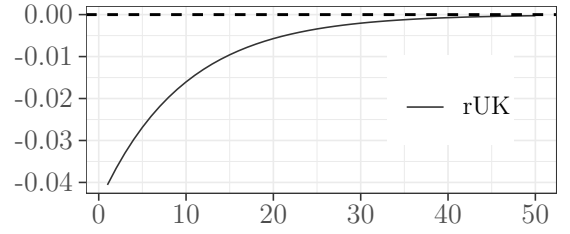


Expansionary Fiscal Shock in the rest of the UK (Scenario 1)

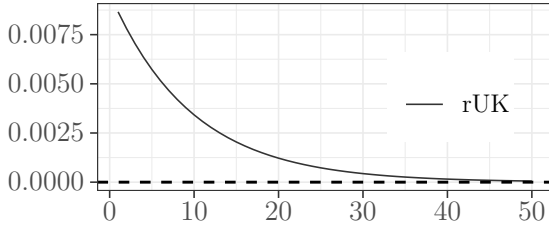
$$\uparrow g_t^* \Rightarrow g_t^*$$



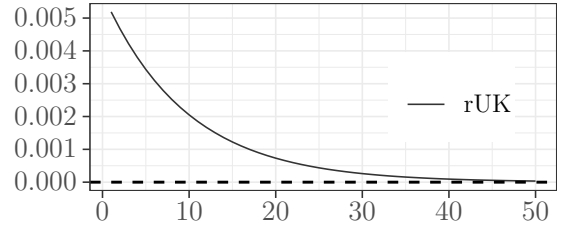
$$\uparrow g_t^* \Rightarrow s_t^*$$



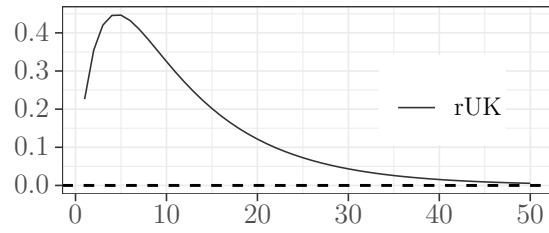
$$\uparrow g_t^* \Rightarrow i_t^*$$



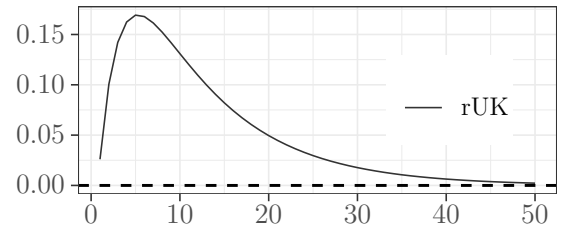
$$\uparrow g_t^* \Rightarrow \pi_{H,t}^*$$



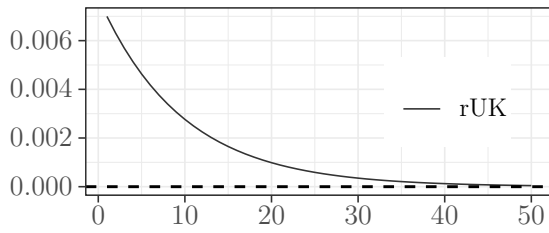
$$\uparrow g_t^* \Rightarrow b_t^*$$



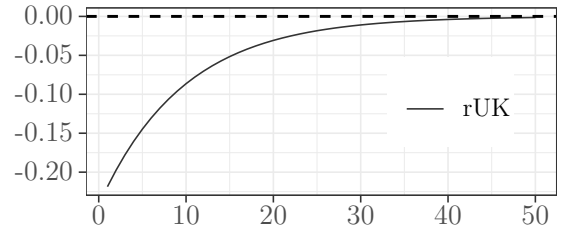
$$\uparrow g_t^* \Rightarrow tr_t^*$$



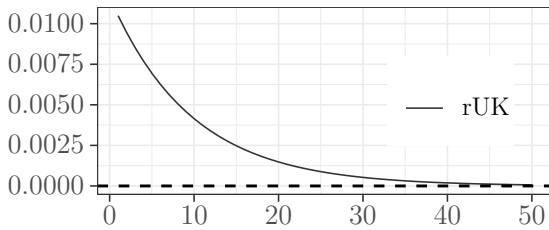
$$\uparrow g_t^* \Rightarrow y_t^*$$



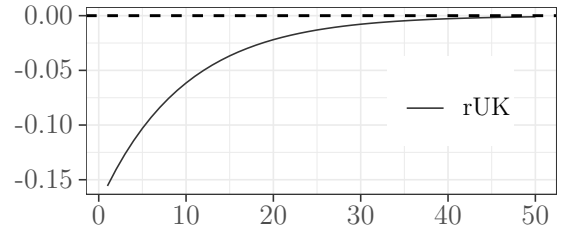
$$\uparrow g_t^* \Rightarrow c_t^*$$



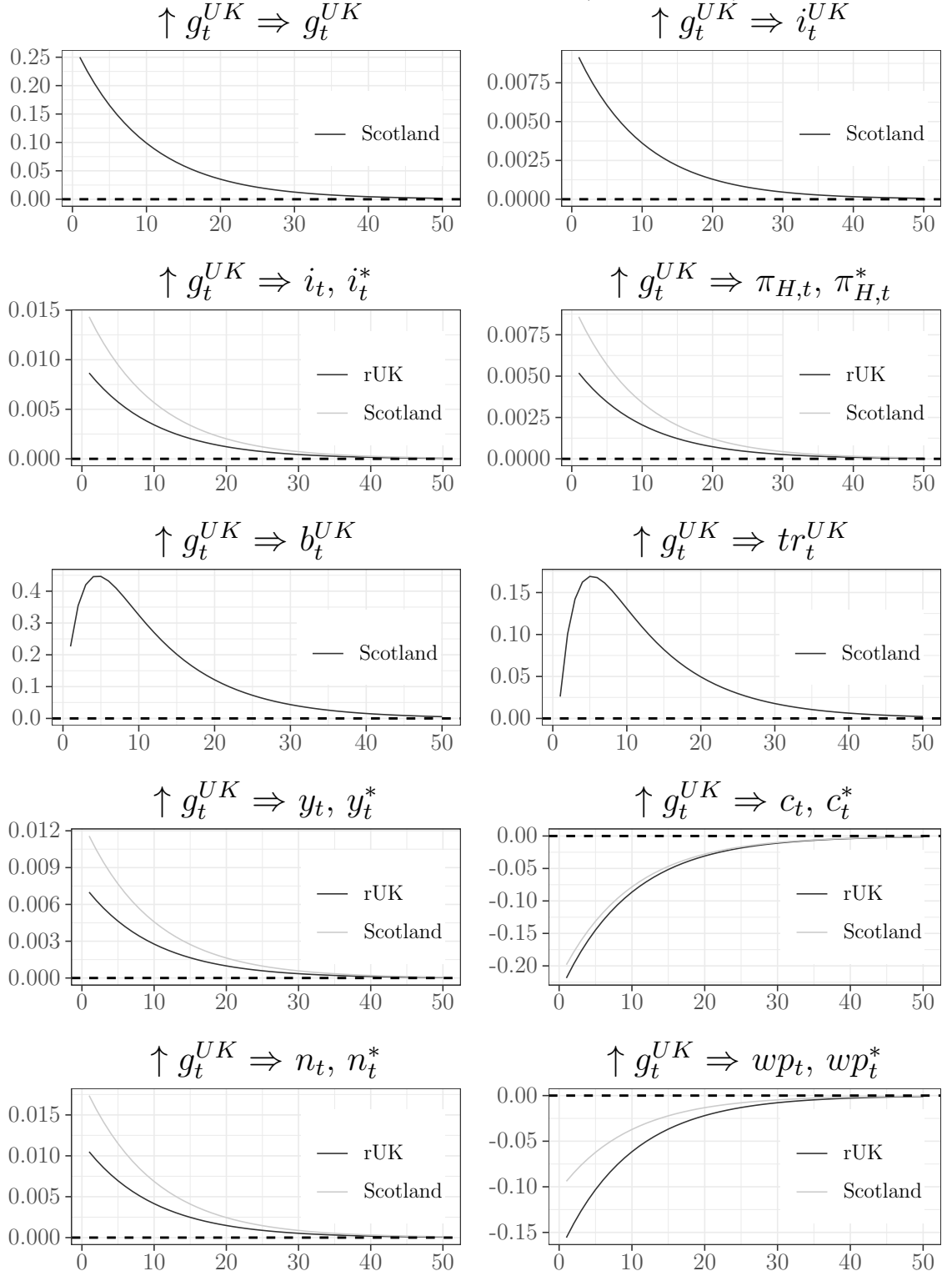
$$\uparrow g_t^* \Rightarrow n_t^*$$



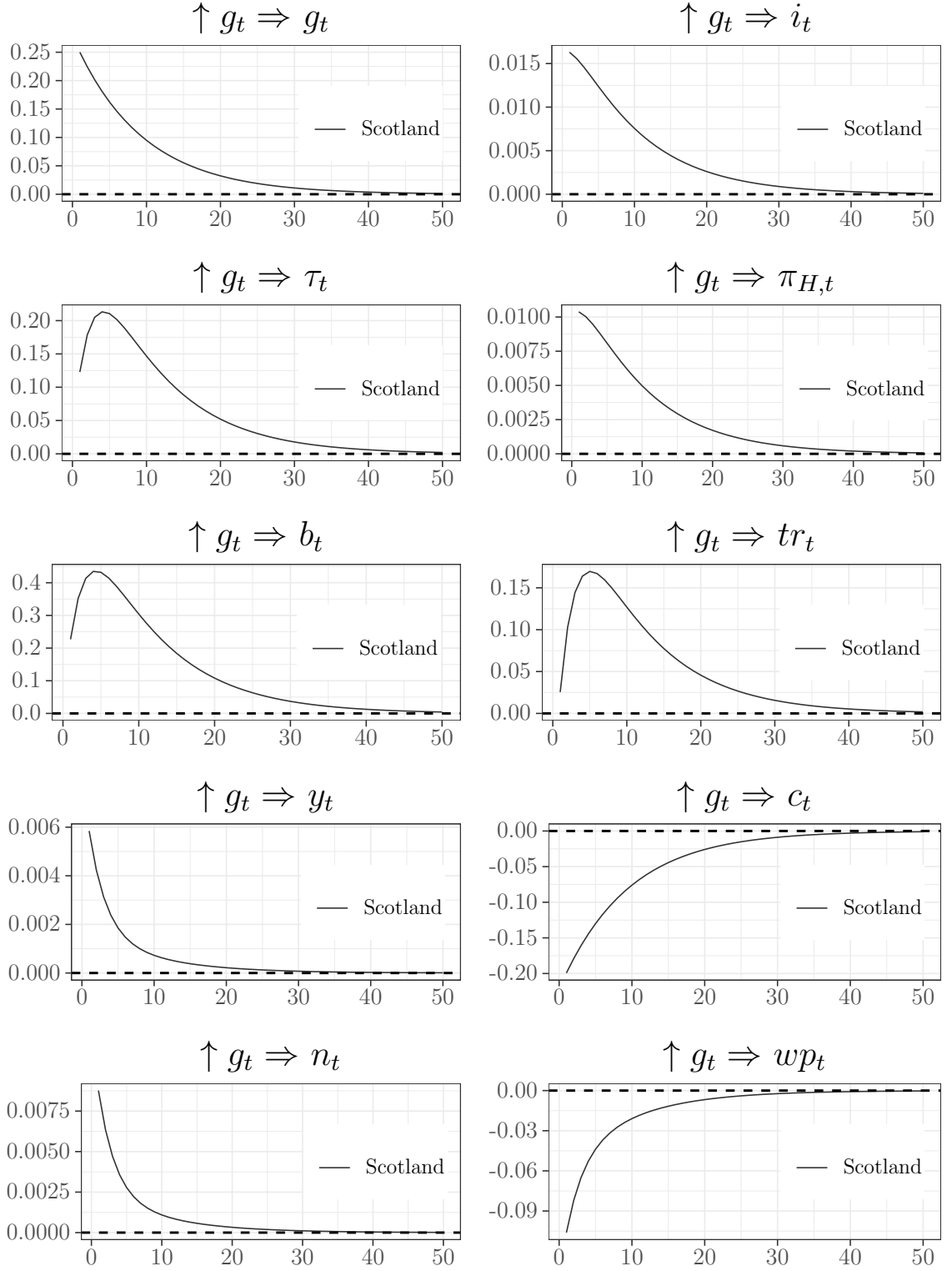
$$\uparrow g_t^* \Rightarrow wp_t^*$$



Expansionary Fiscal Shock (Scenario 2)

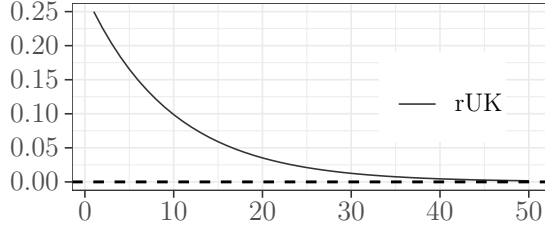


Expansionary Fiscal Shock in Scotland (Scenario 3)

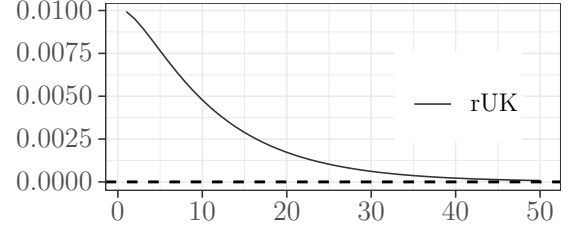


Expansionary Fiscal Shock in the rest of the UK (Scenario 3)

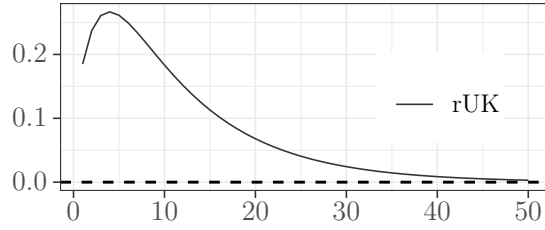
$$\uparrow g_t^* \Rightarrow g_t^*$$



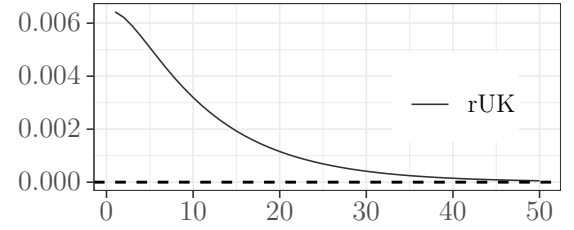
$$\uparrow g_t^* \Rightarrow i_t^*$$



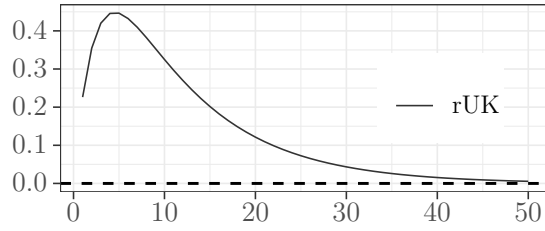
$$\uparrow g_t^* \Rightarrow \tau_t^*$$



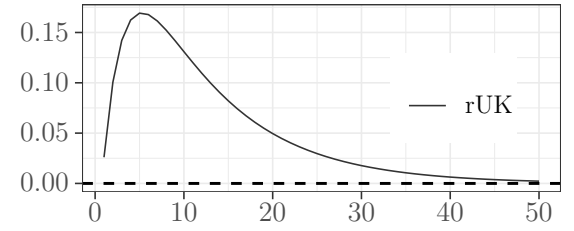
$$\uparrow g_t^* \Rightarrow \pi_{H,t}^*$$



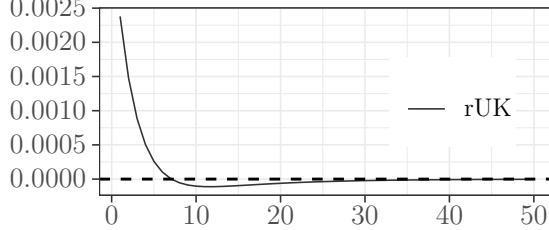
$$\uparrow g_t^* \Rightarrow b_t^*$$



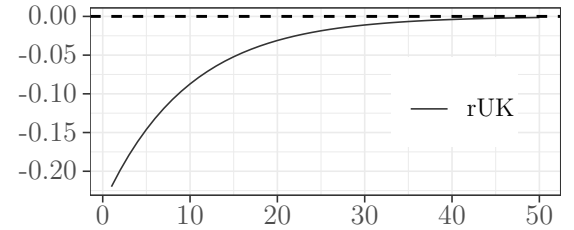
$$\uparrow g_t^* \Rightarrow tr_t^*$$



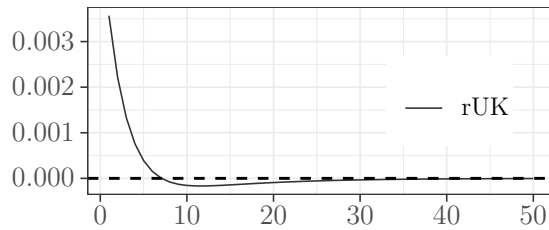
$$\uparrow g_t^* \Rightarrow y_t^*$$



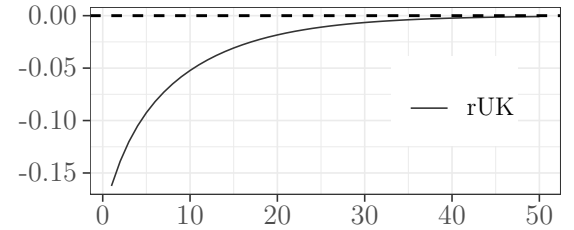
$$\uparrow g_t^* \Rightarrow c_t^*$$



$$\uparrow g_t^* \Rightarrow n_t^*$$



$$\uparrow g_t^* \Rightarrow wp_t^*$$



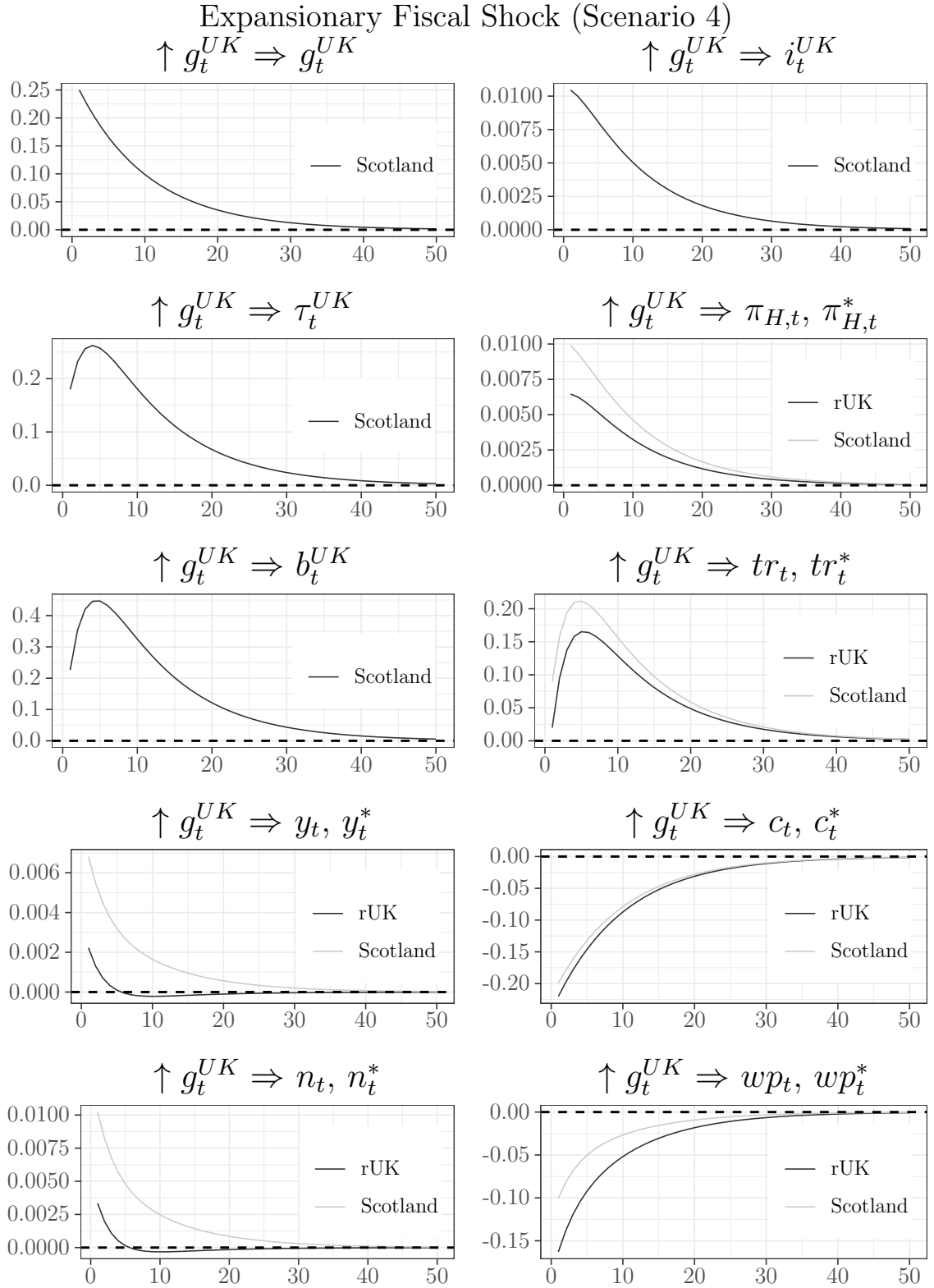
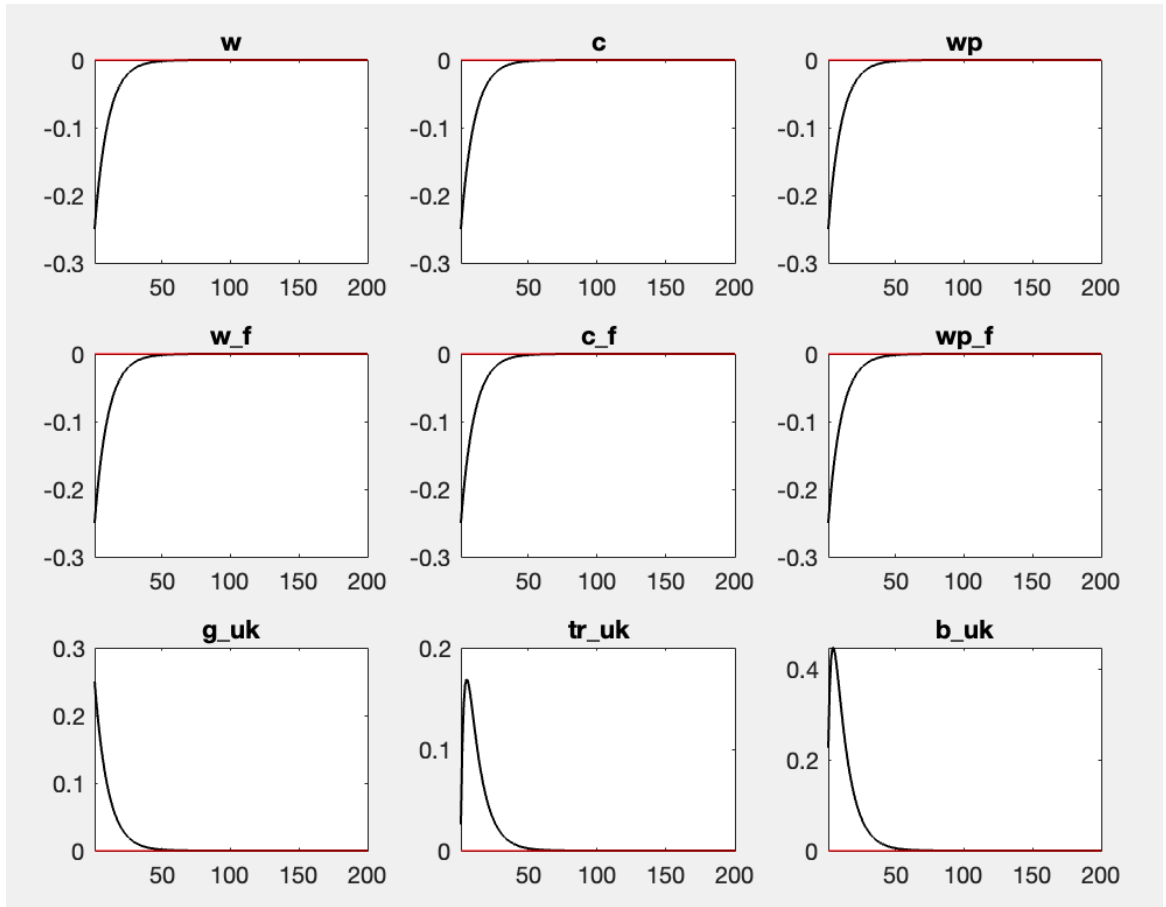


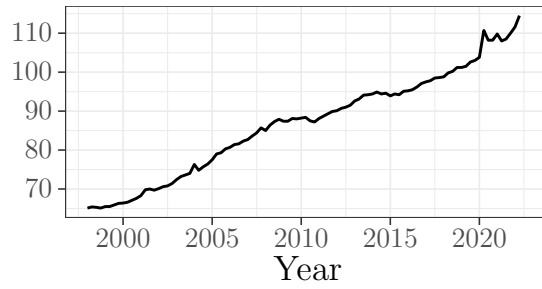
Figure 3: test



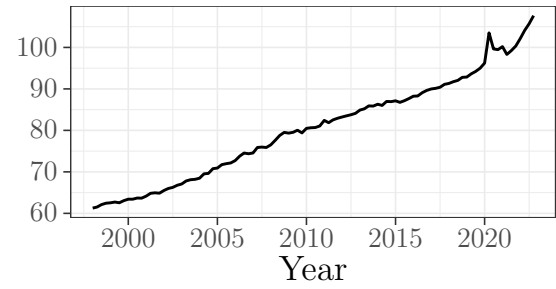
5 Conclusion

6 Appendix

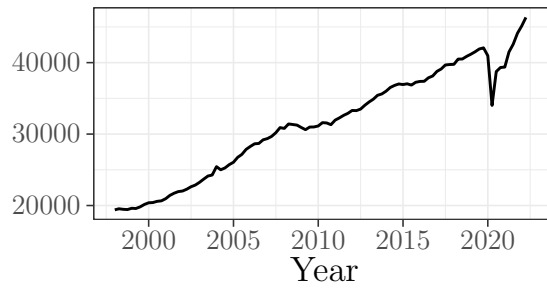
Scotland: Deflator (2018=100)



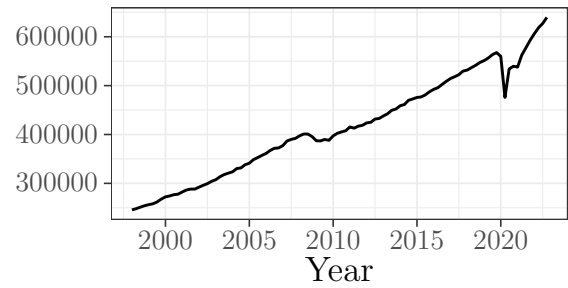
UK: Deflator (2023=100)



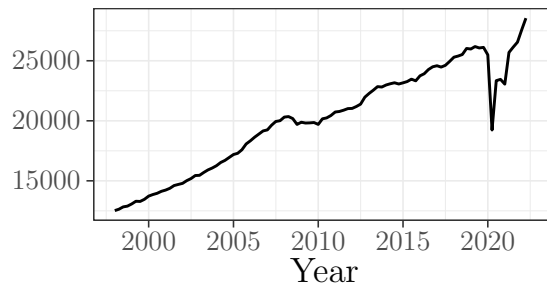
Scotland: Output, £m



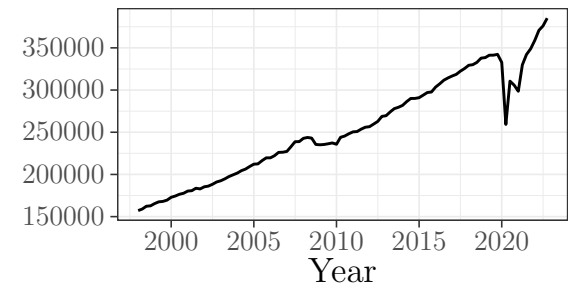
UK: Output, £m



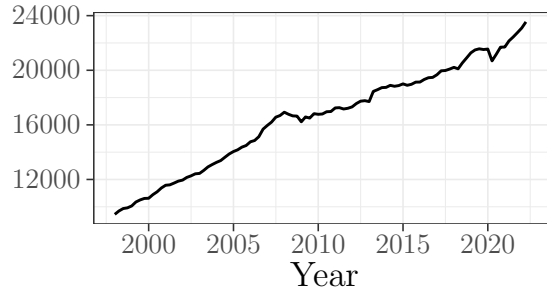
Scotland: Consumption, £m



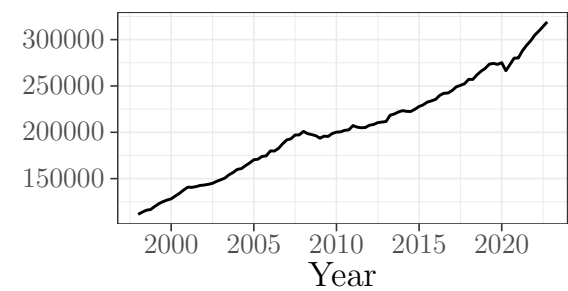
UK: Consumption, £m

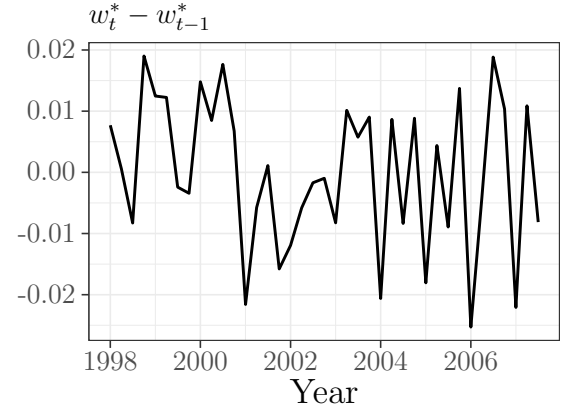
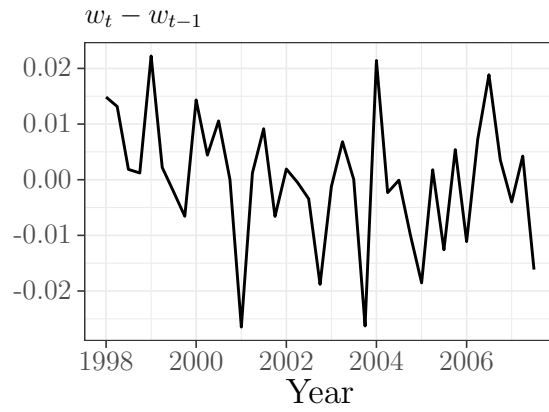
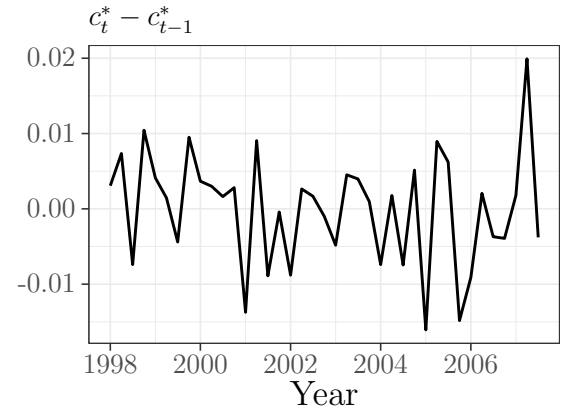
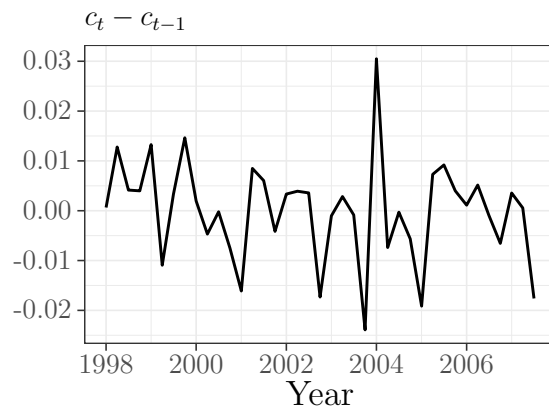
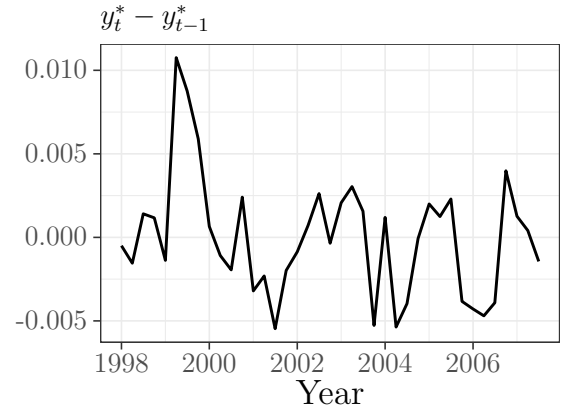
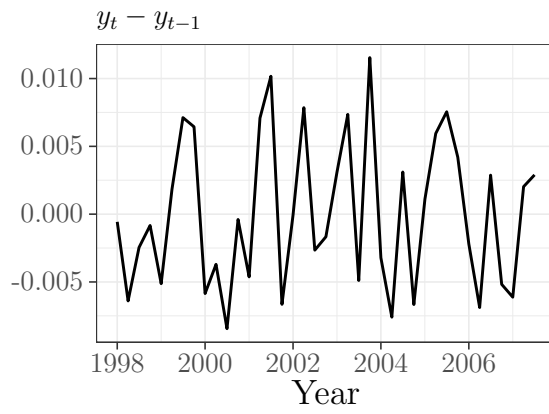


Scotland: Comp. of Employees, £m

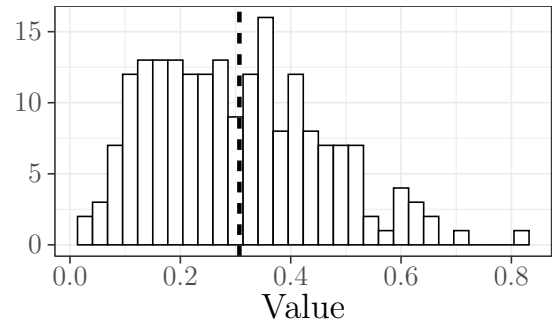
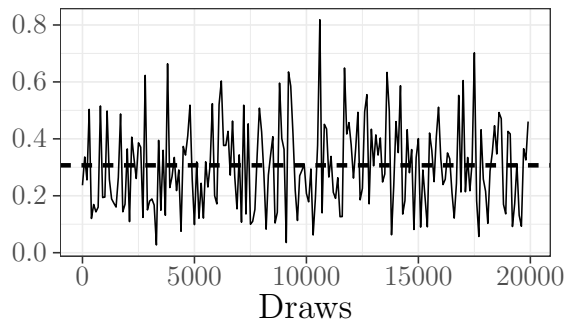


UK: Comp. of Employees, £m

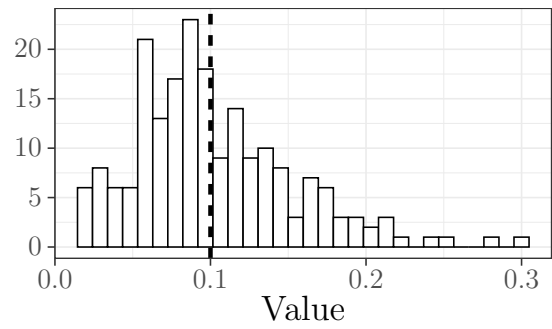
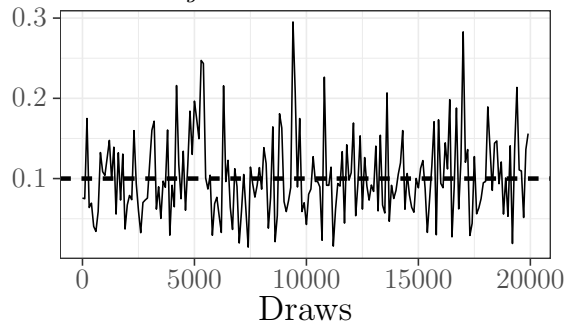




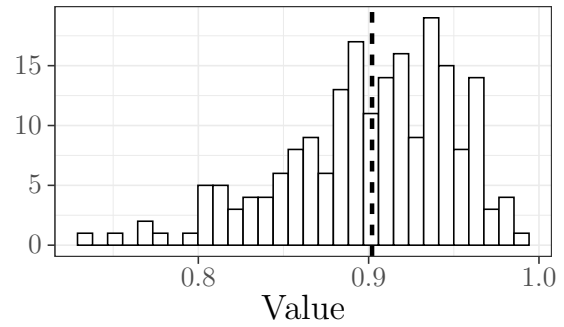
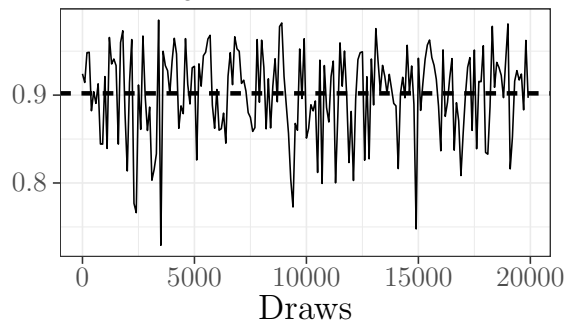
Draws: ϕ_b^* . Posterior mean = 0.307



Draws: ϕ_g^* . Posterior mean = 0.1



Draws: ρ_g^* . Posterior mean = 0.902



Derivation of loglinearised intratemporal condition

$$C_t^\sigma N_t^\varphi = \frac{W_t}{P_t}(1 - \tau_t) \quad (\text{A.119})$$

$$C_t^\sigma N_t^\varphi = \frac{W_t}{P_t} - \frac{W_t}{P_t}\tau_t \quad (\text{A.120})$$

$$C^\sigma N^\varphi = \frac{W}{P} - \frac{W}{P}\tau \quad (\text{Steady state}) \quad (\text{A.121})$$

Using Uhlig's (1999) method, $X_t Y_t \approx XY \mathbf{e}^{\tilde{X}_t + \tilde{Y}_t}$

$$C^\sigma N^\varphi \mathbf{e}^{\sigma c_t + \varphi n_t} = \frac{W}{P} \mathbf{e}^{w_t - p_t} - \frac{W}{P} \tau \mathbf{e}^{w_t - p_t - \tilde{\tau}_t} \quad (\text{A.122})$$

Using $\mathbf{e}^{\tilde{X}_t} \approx 1 + \tilde{X}_t$:

$$C^\sigma N^\varphi (1 + \sigma c_t + \varphi n_t) = \frac{W}{P} (1 + w_t - p_t) - \frac{W}{P} \tau (1 + w_t - p_t - \tilde{\tau}_t) \quad (\text{A.123})$$

Subtract (A.121):

$$C^\sigma N^\varphi (\sigma c_t + \varphi n_t) = \frac{W}{P} (w_t - p_t) - \frac{W}{P} \tau (w_t - p_t - \tilde{\tau}_t) \quad (\text{A.124})$$

$$C^\sigma N^\varphi (\sigma c_t + \varphi n_t) = \frac{W}{P} [(w_t - p_t) - \tau (w_t - p_t - \tilde{\tau}_t)] \quad (\text{A.125})$$

$$C^\sigma N^\varphi (\sigma c_t + \varphi n_t) = \frac{W}{P} [(w_t - p_t) - \tau (w_t - p_t) - \tau \tilde{\tau}_t] \quad (\text{A.126})$$

$$C^\sigma N^\varphi (\sigma c_t + \varphi n_t) = \frac{W}{P} [(1 - \tau)(w_t - p_t) - \tau \tilde{\tau}_t] \quad (\text{A.127})$$

$$C^\sigma N^\varphi \frac{P}{W} (\sigma c_t + \varphi n_t) = [(1 - \tau)(w_t - p_t) - \tau \tilde{\tau}_t] \quad (\text{A.128})$$

$$(1 - \tau)(\sigma c_t + \varphi n_t) = [(1 - \tau)(w_t - p_t) - \tau \tilde{\tau}_t] \quad (\text{A.129})$$

$$\sigma c_t + \varphi n_t = w_t - p_t - \frac{\tau}{1 - \tau} \tilde{\tau}_t \quad (\text{A.130})$$

Log linearising government budget constraint

$$\mathbb{E}_t[R_t^{-1}B_{t+1}] + P_t T_t = P_t G_t + B_t \quad (\text{A.131})$$

Divide by P_t

$$\mathbb{E}_t[R_t^{-1}B_{t+1}P_t^{-1}] + T_t = G_t + B_t P_t P_{t-1} P_{t-1}^{-1} \quad (\text{A.132})$$

$$\begin{aligned} & \frac{1}{R} \frac{B}{P} \left[1 + \frac{B_{t+1} - B}{B} - \frac{P_t - P}{P} - \frac{R_t - R}{R} \right] + T \left[1 + \frac{T_t - T}{T} \right] \\ &= G \left[1 + \frac{G_t - G}{G} \right] + \frac{B}{P} \frac{P}{P} \left[1 + \frac{B_t - B}{B} - \frac{P_{t-1} - P}{P} + \underbrace{\frac{P_t - P}{P} - \frac{P_{t-1} - P}{P}}_{\pi=0} \right] \end{aligned} \quad (\text{A.133})$$

Subtract steady state (A.131)

$$\frac{1}{R} \frac{B}{P} \left[\frac{B_{t+1} - B}{B} - \frac{P_t - P}{P} - \frac{R_t - R}{R} \right] + [T_t - T] = [G_t - G] + \frac{B}{P} \left[\frac{B_t - B}{B} - \frac{P_{t-1} - P}{P} \right] \quad (\text{A.134})$$

Divide by Y

$$\frac{1}{R} \frac{1}{Y} \frac{B}{P} \left[\frac{B_{t+1} - B}{B} - \frac{P_t - P}{P} - \frac{R_t - R}{R} \right] + t_t = g_t + \frac{1}{Y} \frac{B}{P} \left[\frac{B_t - B}{B} - \frac{P_{t-1} - P}{P} \right] \quad (\text{A.135})$$

$$\frac{1}{R} \frac{1}{Y} \left[\frac{B_{t+1} - B}{P} - \frac{B(P_t - P)}{P^2} - \frac{B(R_t - R)}{R} \right] + t_t = g_t + \frac{1}{Y} \left[\frac{B_t - B}{P} - \frac{B(P_{t-1} - P)}{P^2} \right] \quad (\text{A.136})$$

$$\frac{1}{R} \frac{1}{Y} \left[\frac{B_{t+1} - B}{P} - \frac{(B/P)(P_t - P)}{P} - \frac{B(R_t - R)}{R} \right] + t_t = g_t + \frac{1}{Y} \left[\frac{B_t - B}{P} - \frac{(B/P)(P_{t-1} - P)}{P} \right] \quad (\text{A.137})$$

$$\frac{1}{R} \frac{1}{Y} \left[\frac{B_{t+1} - B - (B/P)(P_t - P)}{P} - \frac{B(R_t - R)}{R} \right] + t_t = g_t + \frac{1}{Y} \left[\frac{B_t - B - (B/P)(P_{t-1} - P)}{P} \right] \quad (\text{A.138})$$

$$\frac{1}{R} \frac{1}{Y} \left[\frac{B_{t+1} - B - (B/P)P_t + B}{P} - \frac{B(R_t - R)}{R} \right] + t_t = g_t + \frac{1}{Y} \left[\frac{B_t - B - (B/P)P_{t-1} + B}{P} \right] \quad (\text{A.139})$$

$$\frac{1}{R} \frac{1}{Y} \left[\frac{B_{t+1} - (B/P)P_t}{P} - \frac{B(R_t - R)}{R} \right] + t_t = g_t + \frac{1}{Y} \left[\frac{B_t - (B/P)P_{t-1}}{P} \right] \quad (\text{A.140})$$

$$\frac{1}{R} \frac{1}{Y} \left[\frac{(B_{t+1}/P_t) - (B/P)}{P/P_t} - \frac{B(R_t - R)}{R} \right] + t_t = g_t + \frac{1}{Y} \left[\frac{(B_t/P_{t-1}) - (B/P)}{P/P_{t-1}} \right] \quad (\text{A.141})$$

$$\frac{1}{R} b_{t+1} \frac{1}{P/P_t} + t_t = g_t + b_t \left[\frac{1}{P/P_{t-1}} \right] \quad (\text{A.142})$$

$$\frac{1}{R} b_{t+1} \left(\frac{P_t}{P} - 1 + 1 \right) + t_t = g_t + b_t \left(\frac{P_{t-1}}{P} - 1 + 1 \right) \quad (\text{A.143})$$

$$\frac{1}{R} b_{t+1} (p_t + 1) + t_t = g_t + b_t (p_{t-1} + 1) \quad (\text{A.144})$$

Following Uhlig (1995), $x_t y_t \approx 0$

$$\frac{1}{R} b_{t+1} + t_t = g_t + b_t \quad (\text{A.145})$$

$$b_{t+1} = (1 + \rho)(b_t + g_t - t_t) \quad (\text{A.146})$$

where: $\rho = \beta^{-1} - 1$

Derivation of loglinearised aggregate resource constraint

$$Y_t = (1 - v) \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} C_t + v \mathcal{S}_t^\eta Y_t^* + G_t \quad (\text{A.147})$$

$$Y \mathbf{e}^{y_t} = (1 - v) \left(\frac{P}{P_H} \right)^\eta C \mathbf{e}^{-\eta p_{H,t} + \eta p_t + c_t} + v S^\eta Y^* \mathbf{e}^{\eta s_t + y_t^*} + G \mathbf{e}^{g_t} \quad (\text{A.148})$$

$$Y(1 + y_t) = (1 - v) \left(\frac{P}{P_H} \right)^\eta C(1 - \eta p_{H,t} + \eta p_t + c_t) + v S^\eta Y^*(1 + \eta s_t + y_t^*) + G(1 + g_t) \quad (\text{A.149})$$

$$Y y_t = (1 - v) \left(\frac{P}{P_H} \right)^\eta C(-\eta p_{H,t} + \eta p_t + c_t) + v S^\eta Y^*(\eta s_t + y_t^*) + G g_t \quad (\text{A.150})$$

Using $P/P_H = \mathcal{S}^v$, $\mathcal{S} = 1$, and $C = Y^*$:

$$y_t = C_Y [(1 - v)(-\eta p_{H,t} + \eta p_t + c_t) + v(\eta s_t + y_t^*)] + G_Y g_t \quad (\text{A.151})$$

Using $p_t - p_{H,t} = v s_t$:

$$y_t = C_Y [(1 - v)(\eta v s_t + c_t) + v(\eta s_t + y_t^*)] + G_Y g_t \quad (\text{A.152})$$

$$y_t = C_Y [(1 - v)c_t + (1 - v)\eta v s_t + v\eta s_t + v y_t^*] + G_Y g_t \quad (\text{A.153})$$

$$y_t = C_Y [(1 - v)c_t + v(2 - v)\eta s_t + v y_t^*] + G_Y g_t \quad (\text{A.154})$$

Derivation of bilateral exchange rate

International Risk-Sharing Equation for country i can be rewritten as:

$$\frac{V_{t,t+1}}{\mathcal{E}_t^i P_t^i} (C_t^i)^{-\sigma} = \xi_{t,t+1} \beta (C_{t+1}^i)^{-\sigma} \frac{1}{\mathcal{E}_{t+1}^i P_{t+1}^i} \quad (\text{A.155})$$

Given that:

$$Q_{t,t+1} = \frac{V_{t,t+1}}{\xi_{t,t+1}} \quad (\text{A.156})$$

$$\frac{V_{t,t+1}}{\mathcal{E}_t^i P_t^i} (C_t^i)^{-\sigma} = \xi_{t,t+1} \beta (C_{t+1}^i)^{-\sigma} \frac{1}{\mathcal{E}_{t+1}^i P_{t+1}^i} \quad (\text{A.157})$$

$$Q_{t,t+1} = \beta \left(\frac{C_{t+1}^i}{C_t^i} \right)^{-\sigma} \left(\frac{P_t^i}{P_{t+1}^i} \right) \left(\frac{\mathcal{E}_t^i}{\mathcal{E}_{t+1}^i} \right) \quad (\text{A.158})$$

Recall that:

$$\begin{aligned} \frac{\partial L}{\partial C_t} &= \beta^t C_t^{-\sigma} P_t^{-1} = \lambda_t \\ \frac{\partial L}{\partial D_{t+1}} &= -\lambda_t \mathbb{E}_t[Q_{t,t+1}] = -\mathbb{E}_t[\lambda_{t+1}] \\ &= -\beta^t C_t^{-\sigma} P_t^{-1} \mathbb{E}_t[Q_{t,t+1}] = -\mathbb{E}_t[\lambda_{t+1}] \end{aligned} \quad (\text{A.159})$$

Which is symmetrical for country i :

$$\begin{aligned} \frac{\partial L^i}{\partial C_t^i} &= \beta^t (C_t^i)^{-\sigma} (\mathcal{E}_{i,t} P_t^i)^{-1} = \lambda_t^i \\ \frac{\partial L^i}{\partial D_{t+1}^i} &= -\lambda_t^i \mathbb{E}_t[Q_{t,t+1}] = -\mathbb{E}_t[\lambda_{t+1}^i] \\ &= -\beta^t (C_t^i)^{-\sigma} (\mathcal{E}_{i,t} P_t^i)^{-1} \mathbb{E}_t[Q_{t,t+1}] = -\mathbb{E}_t[\lambda_{t+1}^i] \end{aligned} \quad (\text{A.160})$$

$$\frac{(\text{A.159})}{(\text{A.160})} : \frac{-\beta^t C_t^{-\sigma} P_t^{-1} \mathbb{E}_t[Q_{t,t+1}]}{-\beta^t (C_t^i)^{-\sigma} (\mathcal{E}_{i,t} P_t^i)^{-1} \mathbb{E}_t[Q_{t,t+1}]} = \frac{-\mathbb{E}_t[\lambda_{t+1}]}{-\mathbb{E}_t[\lambda_{t+1}^i]}$$

$$\begin{aligned}
C_t^{-\sigma}(C_t^i)^\sigma \frac{\mathcal{E}_{i,t} P_t^i}{P_t} &= 1 \\
C_t^{-\sigma}(C_t^i)^\sigma \mathcal{Q}_{i,t} &= 1 \\
C_t^{-\sigma}(C_t^i)^\sigma &= \frac{1}{\mathcal{Q}_{i,t}} \\
C_t^{-\sigma} &= \frac{1}{\mathcal{Q}_{i,t}} (C_t^i)^{-\sigma} \\
C_t^\sigma &= \mathcal{Q}_{i,t} (C_t^i)^\sigma \\
\Rightarrow C_t &= C_t^i \mathcal{Q}_{i,t}^{\frac{1}{\sigma}}
\end{aligned} \tag{A.161}$$

Derivation of loglinearised terms of trade

We use two definitions:

$$c_t = y_t^* + \frac{1}{\sigma} z_t + \frac{1-v}{\sigma} s_t \quad (\text{A.162})$$

$$y_t = C_Y [(1-v)c_t + v(2-v)\eta s_t + v y_t^*] + G_Y g_t \quad (\text{A.163})$$

$$y_t = C_Y \left[(1-v) \left(y_t^* + \frac{1}{\sigma} z_t + \frac{1-v}{\sigma} s_t \right) + v(2-v)\eta s_t + v y_t^* \right] + G_Y g_t \quad (\text{A.164})$$

$$y_t = C_Y \left[(1-v)y_t^* + \frac{1-v}{\sigma} z_t + (1-v)\frac{1-v}{\sigma} s_t + v(2-v)\eta s_t + v y_t^* \right] + G_Y g_t \quad (\text{A.165})$$

Notice $(1-v)y_t^* + v y_t^* = y_t^*$

$$y_t = C_Y \left[y_t^* + \frac{1-v}{\sigma} z_t + (1-v)\frac{1-v}{\sigma} s_t + v(2-v)\eta s_t \right] + G_Y g_t \quad (\text{A.166})$$

$$y_t = C_Y \left[y_t^* + \frac{1-v}{\sigma} z_t + \left((1-v)\frac{1-v}{\sigma} + v(2-v)\eta \right) s_t \right] + G_Y g_t \quad (\text{A.167})$$

Rearrange:

$$C_Y \left((1-v) \frac{1-v}{\sigma} + v(2-v)\eta \right) s_t = y_t - C_Y \left[y_t^* - \frac{1-v}{\sigma} z_t \right] - G_Y g_t \quad (\text{A.168})$$

$$C_Y \left(\frac{(1-v)(1-v) + \sigma v(2-v)\eta}{\sigma} \right) s_t = y_t - C_Y \left[y_t^* - \frac{1-v}{\sigma} z_t \right] - G_Y g_t \quad (\text{A.169})$$

$$C_Y \left(\frac{1-2v+v^2+\sigma v(2-v)\eta}{\sigma} \right) s_t = y_t - C_Y \left[y_t^* - \frac{1-v}{\sigma} z_t \right] - G_Y g_t \quad (\text{A.170})$$

$$C_Y \left(\frac{1-v(2-v) + \sigma v(2-v)\eta}{\sigma} \right) s_t = y_t - C_Y \left[y_t^* - \frac{1-v}{\sigma} z_t \right] - G_Y g_t \quad (\text{A.171})$$

$$C_Y \left(\frac{1-v(2-v)(1-\sigma\eta)}{\sigma} \right) s_t = y_t - C_Y \left[y_t^* - \frac{1-v}{\sigma} z_t \right] - G_Y g_t \quad (\text{A.172})$$

$$C_Y \left(\frac{1+v(2-v)(\sigma\eta-1)}{\sigma} \right) s_t = y_t - C_Y \left[y_t^* - \frac{1-v}{\sigma} z_t \right] - G_Y g_t \quad (\text{A.173})$$

$$C_Y \left(\frac{1+v(2\sigma\eta-v\sigma\eta-2+v)}{\sigma} \right) s_t = y_t - C_Y \left[y_t^* - \frac{1-v}{\sigma} z_t \right] - G_Y g_t \quad (\text{A.174})$$

$$C_Y \left(\frac{1+v(\varpi-1)}{\sigma} \right) s_t = y_t - C_Y \left[y_t^* - \frac{1-v}{\sigma} z_t \right] - G_Y g_t \quad (\text{A.175})$$

$$C_Y \sigma^{-1} \Phi^{-1} s_t = y_t - C_Y \left[y_t^* - \frac{1-v}{\sigma} z_t \right] - G_Y g_t \quad (\text{A.176})$$

$$s_t = Y_C \sigma_v y_t - \sigma_v \left[y_t^* - \frac{1-v}{\sigma} z_t \right] - G_C \sigma_v g_t \quad (\text{A.177})$$

$$s_t = \sigma_v (C_Y^{-1} y_t - y_t^* - C_G^{-1} g_t) - (1-v) \Phi z_t \quad (\text{A.178})$$

where $\varpi = \sigma\eta + (1-v)(\sigma\eta-1)$, $\Phi = \frac{1}{1+v(\varpi-1)}$ and $\sigma_v = \sigma\Phi$.

Derivation of (a version of) dynamic IS equation

$$0 = \mathbb{E}\{\Delta c_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \frac{v}{\sigma} \mathbb{E}\{\Delta s_{t+1}\} + \frac{1}{\sigma}(1 - \rho_z)z_t \quad (\text{A.179})$$

From aggregate RC: $c_t = [y_t - C_Y[v(2-v)\eta s_t + v y_t^*] - G_Y g_t] C_Y^{-1}(1-v)^{-1}$

Substitute and multiply by $C_Y(1-v)$

$$0 = \mathbb{E}\{\Delta y_{t+1}\} - C_Y[v(2-v)\eta \mathbb{E}\{\Delta s_{t+1}\} + v \mathbb{E}\{\Delta y_{t+1}^*\}] - G_Y \mathbb{E}\{\Delta g_{t+1}\} + C_Y \left[-\frac{1-v}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \frac{(1-v)v}{\sigma} \mathbb{E}\{\Delta s_{t+1}\} + \frac{1-v}{\sigma}(1 - \rho_z)z_t \right] \quad (\text{A.180})$$

$$0 = \mathbb{E}\{\Delta y_{t+1}\} + C_Y \left(\frac{(1-v)v}{\sigma} - v(2-v)\eta \right) \mathbb{E}\{\Delta s_{t+1}\} + C_Y v \mathbb{E}\{\Delta y_{t+1}^*\} - G_Y \mathbb{E}\{\Delta g_{t+1}\} + C_Y \left[-\frac{1-v}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \frac{1-v}{\sigma}(1 - \rho_z)z_t \right] \quad (\text{A.181})$$

$$0 = \mathbb{E}\{\Delta y_{t+1}\} - C_Y \left(\frac{v\varpi}{\sigma} \right) \mathbb{E}\{\Delta s_{t+1}\} + C_Y v \mathbb{E}\{\Delta y_{t+1}^*\} - G_Y \mathbb{E}\{\Delta g_{t+1}\} + C_Y \left[-\frac{1-v}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \frac{1-v}{\sigma}(1 - \rho_z)z_t \right] \quad (\text{A.182})$$

Recall $\mathbb{E}_t\{\Delta s_{t+1}\} = \mathbb{E}_t\{\sigma_v(C_Y^{-1}\Delta y_{t+1} - \Delta y_{t+1}^* - C_G^{-1}\Delta g_{t+1}) - (1-v)\Phi\Delta z_{t+1}\}$

$$0 = \mathbb{E}\{\Delta y_{t+1}\} - C_Y \left(\frac{v\varpi}{\sigma} \right) \mathbb{E}_t\{\sigma_v(C_Y^{-1}\Delta y_{t+1} - \Delta y_{t+1}^* - C_G^{-1}\Delta g_{t+1}) - (1-v)\Phi\Delta z_{t+1}\} + C_Y v \mathbb{E}\{\Delta y_{t+1}^*\} - G_Y \mathbb{E}\{\Delta g_{t+1}\} + C_Y \left[-\frac{1-v}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) + \frac{1-v}{\sigma}(1 - \rho_z)z_t \right] \quad (\text{A.183})$$

$$0 = \mathbb{E}\{\Delta y_{t+1} - v\varpi\Phi\Delta y_{t+1}\} + \mathbb{E}\{C_Y v\varpi\Phi\Delta y_{t+1}^* - C_Y v\Delta y_{t+1}^*\} + C_Y \frac{1-v}{\sigma}(1 - \rho_z)z_t + C_Y v\varpi\Phi \frac{1-v}{\sigma}(\rho_z - 1)z_t + \mathbb{E}_t\{v\varpi\Phi G_Y \Delta g_{t+1} - G_Y \Delta g_{t+1}\} - C_Y \frac{1-v}{\sigma}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) \quad (\text{A.184})$$

Divide by Φ^{-1}

$$0 = \mathbb{E}\{\Phi^{-1}\Delta y_{t+1} - v\varpi\Delta y_{t+1}\} + \mathbb{E}\{C_Y v\varpi\Delta y_{t+1}^* - C_Y v\Phi^{-1}\Delta y_{t+1}^*\} + C_Y \Phi^{-1} \frac{1-v}{\sigma}(1 - \rho_z)z_t + C_Y v\varpi \frac{1-v}{\sigma}(\rho_z - 1)z_t + \mathbb{E}_t\{v\varpi G_Y \Delta g_{t+1} - G_Y \Phi^{-1}\Delta g_{t+1}\} - C_Y \frac{1-v}{\sigma_v}(i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) \quad (\text{A.185})$$

$$\begin{aligned}
0 &= (\Phi^{-1} - v\varpi) \mathbb{E}\{\Delta y_{t+1}\} + C_Y v(\varpi - \Phi^{-1}) \mathbb{E}\{\Delta y_{t+1}^*\} \\
&\quad - C_Y(\Phi^{-1} - v\varpi) \frac{1-v}{\sigma} (\rho_z - 1) z_t + (v\varpi - \Phi^{-1}) G_Y \mathbb{E}_t\{\Delta g_{t+1}\} \\
&\quad - C_Y \frac{1-v}{\sigma_v} (i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho)
\end{aligned} \tag{A.186}$$

$$\text{Divide by } (\Phi^{-1} - v\varpi) \tag{A.187}$$

$$\begin{aligned}
0 &= \mathbb{E}\{\Delta y_{t+1}\} + C_Y \frac{v(\varpi - \Phi^{-1})}{\Phi^{-1} - v\varpi} \mathbb{E}\{\Delta y_{t+1}^*\} - C_Y \frac{1-v}{\sigma} (\rho_z - 1) z_t \\
&\quad + \frac{(v\varpi - \Phi^{-1})}{\Phi^{-1} - v\varpi} G_Y \mathbb{E}_t\{\Delta g_{t+1}\} - \frac{1-v}{\Phi^{-1} - v\varpi} C_Y \frac{1}{\sigma_v} (i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho)
\end{aligned} \tag{A.188}$$

$$\begin{aligned}
y_t &= \mathbb{E}_t\{y_{t+1}\} - C_Y \frac{1-v}{\sigma} (\rho_z - 1) z_t - G_Y \mathbb{E}_t\{\Delta g_{t+1}\} + C_Y \frac{v(\varpi - \Phi^{-1})}{\Phi^{-1} - v\varpi} \mathbb{E}\{\Delta y_{t+1}^*\} \\
&\quad - \frac{1-v}{\Phi^{-1} - v\varpi} C_Y \frac{1}{\sigma_v} (i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho)
\end{aligned} \tag{A.189}$$

$$\text{Note that } (\Phi^{-1} - v\varpi) \text{ can be simplified to } (1-v) \tag{A.190}$$

$$\begin{aligned}
y_t &= \mathbb{E}_t\{y_{t+1}\} - C_Y \frac{1-v}{\sigma} (\rho_z - 1) z_t - \frac{1-v}{1-v} C_Y \frac{1}{\sigma_v} (i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) \\
&\quad + C_Y \frac{(1-v)(v\varpi - v)}{1-v} \mathbb{E}\{\Delta y_{t+1}^*\} - G_Y \mathbb{E}_t\{\Delta g_{t+1}\}
\end{aligned} \tag{A.191}$$

A version of Dynamic IS equation:

$$\begin{aligned}
y_t &= \mathbb{E}_t\{y_{t+1}\} \\
&\quad - C_Y \left[\frac{1}{\sigma_v} (i_t - \mathbb{E}\{\pi_{H,+1}\} - \rho) - \frac{1-v}{\sigma} (1 - \rho_z) z_t - v(\varpi - 1) \mathbb{E}\{\Delta y_{t+1}^*\} \right] - G_Y \mathbb{E}_t\{\Delta g_{t+1}\}
\end{aligned} \tag{A.192}$$

Derivation of (final) Dynamic IS

$$y_t^n = \mathbb{E}_t\{y_{t+1}^n\} - C_Y \left[\frac{1}{\sigma_v}(r_t^n - \rho) - v(\varpi - 1) \mathbb{E}\{\Delta y_{t+1}^*\} - \frac{1-v}{\sigma}(1 - \rho_z)z_t \right] - G_Y \mathbb{E}_t\{\Delta g_{t+1}\} \quad (\text{A.193})$$

$$0 = \mathbb{E}_t\{\Delta y_{t+1}^n\} - C_Y \left[\frac{1}{\sigma_v}(r_t^n - \rho) - v(\varpi - 1) \mathbb{E}\{\Delta y_{t+1}^*\} - \frac{1-v}{\sigma}(1 - \rho_z)z_t \right] - G_Y \mathbb{E}_t\{\Delta g_{t+1}\} \quad (\text{A.194})$$

$$0 = \sigma_v \mathbb{E}_t\{\Delta y_{t+1}^n\} - C_Y [(r_t^n - \rho) - \sigma_v v(\varpi - 1) \mathbb{E}\{\Delta y_{t+1}^*\} - \Phi(1 - v)(1 - \rho_z)z_t] - \sigma_v G_Y \mathbb{E}_t\{\Delta g_{t+1}\} \quad (\text{A.195})$$

$$\begin{aligned} 0 &= \sigma_v \mathbb{E}_t\{\Delta y_{t+1}^n\} \\ &\quad - C_Y r_t^n - C_Y [-\rho - \sigma_v v(\varpi - 1) \mathbb{E}\{\Delta y_{t+1}^*\} - \Phi(1 - v)(1 - \rho_z)z_t] - \sigma_v G_Y \mathbb{E}_t\{\Delta g_{t+1}\} \end{aligned} \quad (\text{A.196})$$

$$r_t^n = C_Y^{-1} \sigma_v \mathbb{E}_t\{\Delta y_{t+1}^n\} + \rho + \sigma_v v(\varpi - 1) \mathbb{E}\{\Delta y_{t+1}^*\} + \Phi(1 - v)(1 - \rho_z)z_t - \sigma_v C_G^{-1} \mathbb{E}_t\{\Delta g_{t+1}\} \quad (\text{A.197})$$

Subtract (A.193) from (96) to yield:

$$\begin{aligned} 0 &= \sigma_v \mathbb{E}\{\Delta y_{t+1}\} - C_Y [(i_t - \mathbb{E}\{\pi_{H,+1}\}) - \rho - \sigma_v v(\varpi - 1) \mathbb{E}\{\Delta y_{t+1}^*\} \\ &\quad - \Phi(1 - v)(1 - \rho_z)z_t] - \sigma_v G_Y \mathbb{E}_t\{\Delta g_{t+1}\} \\ &\quad - (\sigma_v \mathbb{E}\{\Delta y_{t+1}^n\} - C_Y [(r_t^n - \rho - \sigma_v v(\varpi - 1) \mathbb{E}\{\Delta y_{t+1}^*\} \\ &\quad - \Phi(1 - v)(1 - \rho_z)z_t] - \sigma_v G_Y \mathbb{E}_t\{\Delta g_{t+1}\}) \end{aligned} \quad (\text{A.198})$$

$$0 = \sigma_v \mathbb{E}_t\{\Delta \tilde{y}_{t+1}\} - C_Y (i_t - \mathbb{E}\{\pi_{H,+1}\} - r_t^n) \quad (\text{A.199})$$

$$\tilde{y}_t = \mathbb{E}_t\{\tilde{y}_{t+1}\} - \frac{1}{\sigma_v} C_Y (i_t - \mathbb{E}\{\pi_{H,+1}\} - r_t^n) \quad (\text{A.200})$$

Derivation of the average markup

We will make use of the following definitions

$$y_t = a_t + (1 - \alpha)n_t \quad (\text{A.201})$$

$$n_t = \frac{1}{1 - \alpha}(y_t - a_t) \quad (\text{A.202})$$

$$\varpi = \sigma\eta + (1 - v)(\sigma\eta - 1) \quad (\text{A.203})$$

$$\sigma c_t + \varphi n_t = w_t - p_t - \frac{\tau}{1 - \tau}\tau_t \quad (\text{A.204})$$

$$y_t = C_Y [(1 - v)c_t + v(2 - v)\eta s_t + v y_t^*] + G_Y g_t \quad (\text{A.205})$$

$$c_t = y_t^* + \frac{1}{\sigma}z_t + \frac{1 - v}{\sigma}s_t \quad (\text{A.206})$$

$$= [C_Y^{-1}y_t - (1 - v)c_t - v(2 - v)\eta s_t - C_G^{-1}g_t] v^{-1} + \frac{1}{\sigma}z_t + \frac{1 - v}{\sigma}s_t \quad (\text{A.207})$$

$$\implies c_t = C_Y^{-1}y_t + v s_t \left(-(2 - v)\eta + \frac{1 - v}{\sigma} \right) + \frac{v}{\sigma}z_t - C_G^{-1}g_t \quad (\text{A.208})$$

$$= C_Y^{-1}y_t - v \frac{\varpi}{\sigma}s_t + \frac{v}{\sigma}z_t - C_G^{-1}g_t \quad (\text{A.209})$$

$$\mu_t = p_{H,t} - \psi_t \quad (\text{A.210})$$

Ignoring constant term: $\psi_t = w_t - a_t + \alpha n_t$

$$= p_{H,t} - (w_t - a_t + \alpha n_t) \quad (\text{A.211})$$

add subtract p_t

$$= -(w_t - p_t) - (p_t - p_{H,t}) + a_t - \alpha n_t \quad (\text{A.212})$$

subst. with household intratemporal condition and terms of trade def.

$$= -(\sigma c_t + \varphi n_t + \frac{\tau}{1 - \tau}\tau_t) - v s_t + a_t - \alpha n_t \quad (\text{A.213})$$

$$= -\sigma c_t - \varphi n_t - \frac{\tau}{1 - \tau}\tau_t - v s_t + a_t - \alpha n_t \quad (\text{A.214})$$

$$= -\sigma c_t - \frac{\tau}{1 - \tau}\tau_t - v s_t + a_t - n_t(\varphi + \alpha) \quad (\text{A.215})$$

Expand $-n_t(\varphi + \alpha)$:

$$-n_t(\varphi + \alpha) = -\frac{1}{1-\alpha}(y_t - a_t)(\varphi + \alpha) \quad (\text{A.216})$$

$$= -\frac{\varphi + \alpha}{1-\alpha}(y_t - a_t) \quad (\text{A.217})$$

$$= -\frac{\varphi + \alpha}{1-\alpha}y_t + \frac{\varphi + \alpha}{1-\alpha}a_t \quad (\text{A.218})$$

Substitute and rearrange:

$$\mu_t = -\sigma c_t - \frac{\tau}{1-\tau}\tau_t - v s_t + a_t - \frac{\varphi + \alpha}{1-\alpha}y_t + \frac{\varphi + \alpha}{1-\alpha}a_t \quad (\text{A.219})$$

$$\mu_t = -\sigma c_t - \frac{\tau}{1-\tau}\tau_t - v s_t + \left(1 + \frac{\varphi + \alpha}{1-\alpha}\right)a_t - \frac{\varphi + \alpha}{1-\alpha}y_t \quad (\text{A.220})$$

$$\begin{aligned} \mu_t = & -\sigma \left(C_Y^{-1}y_t - v\frac{\varpi}{\sigma}s_t + \frac{v}{\sigma}z_t - C_G^{-1}g_t \right) - \frac{\tau}{1-\tau}\tau_t - v s_t \\ & + \left(1 + \frac{\varphi + \alpha}{1-\alpha}\right)a_t - \frac{\varphi + \alpha}{1-\alpha}y_t \end{aligned} \quad (\text{A.221})$$

$$\begin{aligned} \mu_t = & -\left(\sigma C_Y^{-1} + \frac{\varphi + \alpha}{1-\alpha} \right) y_t + v(\varpi - 1)s_t - v z_t - \frac{\tau}{1-\tau}\tau_t \\ & + \left(1 + \frac{\varphi + \alpha}{1-\alpha}\right)a_t + \sigma C_G^{-1}g_t \end{aligned} \quad (\text{A.222})$$

Derivation of the natural level of output

$$\mu = - \left(\sigma C_Y^{-1} + \frac{\varphi + \alpha}{1 - \alpha} \right) y_t^n + v(\varpi - 1)s_t^n - \frac{\tau}{1 - \tau} \tau_t + \left(1 + \frac{\varphi + \alpha}{1 - \alpha} \right) a_t - v z_t + \sigma C_G^{-1} g_t \quad (\text{A.223})$$

$$\mu - \left(v(\varpi - 1)s_t^n - \frac{\tau}{1 - \tau} \tau_t + \left(1 + \frac{\varphi + \alpha}{1 - \alpha} \right) a_t - v z_t + \sigma C_G^{-1} g_t \right) = - \left(\sigma C_Y^{-1} + \frac{\varphi + \alpha}{1 - \alpha} \right) y_t^n \quad (\text{A.224})$$

$$v(\varpi - 1)s_t^n - \frac{\tau}{1 - \tau} \tau_t + \left(1 + \frac{\varphi + \alpha}{1 - \alpha} \right) a_t - v z_t + \sigma C_G^{-1} g_t - \mu = \left(\sigma C_Y^{-1} + \frac{\varphi + \alpha}{1 - \alpha} \right) y_t^n \quad (\text{A.225})$$

Using $s_t^n = \sigma_v(C_Y^{-1}y_t^n - y_t^* - C_G^{-1}g_t) - (1 - v)\Phi z_t$

$$\begin{aligned} & v(\varpi - 1)(\sigma_v(C_Y^{-1}y_t^n - y_t^* - C_G^{-1}g_t) - (1 - v)\Phi z_t) - \frac{\tau}{1 - \tau} \tau_t + \left(1 + \frac{\varphi + \alpha}{1 - \alpha} \right) a_t \\ & - v z_t + \sigma C_G^{-1} g_t - \mu = \left(\sigma C_Y^{-1} + \frac{\varphi + \alpha}{1 - \alpha} \right) y_t^n \end{aligned} \quad (\text{A.226})$$

Add and subtract 1

$$\begin{aligned} & (-1 + 1 + v(\varpi - 1))(\sigma_v(C_Y^{-1}y_t^n - y_t^* - C_G^{-1}g_t) - (1 - v)\Phi z_t) - \frac{\tau}{1 - \tau} \tau_t \\ & + \left(1 + \frac{\varphi + \alpha}{1 - \alpha} \right) a_t - v z_t + \sigma C_G^{-1} g_t - \mu = \left(\sigma C_Y^{-1} + \frac{\varphi + \alpha}{1 - \alpha} \right) y_t^n \end{aligned} \quad (\text{A.227})$$

$$\begin{aligned} & (-1 + \Phi^{-1})(\sigma_v(C_Y^{-1}y_t^n - y_t^* - C_G^{-1}g_t) - (1 - v)\Phi z_t) - \frac{\tau}{1 - \tau} \tau_t \\ & + \left(1 + \frac{\varphi + \alpha}{1 - \alpha} \right) a_t - v z_t + \sigma C_G^{-1} g_t - \mu = \left(\sigma C_Y^{-1} + \frac{\varphi + \alpha}{1 - \alpha} \right) y_t^n \end{aligned} \quad (\text{A.228})$$

$$\begin{aligned} & (-1 + \Phi^{-1})(\sigma_v C_Y^{-1} y_t^n - \sigma_v y_t^* - (1 - v)\Phi z_t - \sigma_v C_G^{-1} g_t) - \frac{\tau}{1 - \tau} \tau_t \\ & + \left(1 + \frac{\varphi + \alpha}{1 - \alpha} \right) a_t - v z_t + \sigma C_G^{-1} g_t - \mu = \left(\sigma C_Y^{-1} + \frac{\varphi + \alpha}{1 - \alpha} \right) y_t^n \end{aligned} \quad (\text{A.229})$$

$$\begin{aligned} & (-1 + \Phi^{-1})\sigma_v C_Y^{-1} y_t^n + (1 - \Phi^{-1})\sigma_v y_t^* + (1 - \Phi^{-1})(1 - v)\Phi z_t + (1 - \Phi^{-1})\sigma_v C_G^{-1} g_t \\ & - \frac{\tau}{1 - \tau} \tau_t + \left(1 + \frac{\varphi + \alpha}{1 - \alpha} \right) a_t - v z_t + \sigma C_G^{-1} g_t - \mu = \left(\sigma C_Y^{-1} + \frac{\varphi + \alpha}{1 - \alpha} \right) y_t^n \end{aligned} \quad (\text{A.230})$$

Rearrange the two g_t terms to yield: $\sigma_v C_G^{-1} g_t$

$$\begin{aligned} & (-1 + \Phi^{-1})\sigma_v C_Y^{-1} y_t^n + (1 - \Phi^{-1})\sigma_v y_t^* + (1 - \Phi^{-1})(1 - v)\Phi z_t + \sigma_v C_G^{-1} g_t \\ & - \frac{\tau}{1 - \tau} \tau_t + \left(1 + \frac{\varphi + \alpha}{1 - \alpha} \right) a_t - v z_t - \mu = \left(\sigma C_Y^{-1} + \frac{\varphi + \alpha}{1 - \alpha} \right) y_t^n \end{aligned} \quad (\text{A.231})$$

$$\begin{aligned} & - \sigma_v C_Y^{-1} y_t^n + \sigma_v C_Y^{-1} y_t^n + (1 - \Phi^{-1})\sigma_v y_t^* + (1 - \Phi^{-1})(1 - v)\Phi z_t + \sigma_v C_G^{-1} g_t \\ & - \frac{\tau}{1 - \tau} \tau_t + \left(1 + \frac{\varphi + \alpha}{1 - \alpha} \right) a_t - v z_t - \mu = \left(\sigma C_Y^{-1} + \frac{\varphi + \alpha}{1 - \alpha} \right) y_t^n \end{aligned} \quad (\text{A.232})$$

$$\begin{aligned}
& (1 - \Phi)\sigma C_Y^{-1}y_t^n + (1 - \Phi^{-1})\sigma_v y_t^* + (1 - \Phi^{-1})(1 - v)\Phi z_t + \sigma_v C_G^{-1}g_t \\
& - \frac{\tau}{1 - \tau}\tau_t + \left(1 + \frac{\varphi + \alpha}{1 - \alpha}\right) a_t - vz_t - \mu = \left(\sigma C_Y^{-1} + \frac{\varphi + \alpha}{1 - \alpha}\right) y_t^n
\end{aligned} \tag{A.233}$$

$$\begin{aligned}
& (1 - \Phi^{-1})\sigma_v y_t^* + (1 - \Phi^{-1})(1 - v)\Phi z_t + \sigma_v C_G^{-1}g_t \\
& - \frac{\tau}{1 - \tau}\tau_t + \left(1 + \frac{\varphi + \alpha}{1 - \alpha}\right) a_t - vz_t - \mu = \left(\sigma C_Y^{-1} + \frac{\varphi + \alpha}{1 - \alpha}\right) y_t^n - (1 - \Phi)\sigma C_Y^{-1}y_t^n
\end{aligned} \tag{A.234}$$

$$\begin{aligned}
& (1 - \Phi^{-1})\sigma_v y_t^* + (1 - \Phi^{-1})(1 - v)\Phi z_t + \sigma_v C_G^{-1}g_t \\
& - \frac{\tau}{1 - \tau}\tau_t + \left(1 + \frac{\varphi + \alpha}{1 - \alpha}\right) a_t - vz_t - \mu = \left(\frac{\sigma C_Y^{-1}(1 - \alpha) + \varphi + \alpha - (1 - \Phi)\sigma C_Y^{-1}(1 - \alpha)}{1 - \alpha}\right) y_t^n
\end{aligned} \tag{A.235}$$

$$\begin{aligned}
& (1 - \Phi^{-1})\sigma_v y_t^* + (1 - \Phi^{-1})(1 - v)\Phi z_t + \sigma_v C_G^{-1}g_t \\
& - \frac{\tau}{1 - \tau}\tau_t + \left(1 + \frac{\varphi + \alpha}{1 - \alpha}\right) a_t - vz_t - \mu = \left(\frac{\sigma C_Y^{-1}(1 - \alpha)(1 - 1 + \Phi) + \varphi}{1 - \alpha}\right) y_t^n
\end{aligned} \tag{A.236}$$

$$\begin{aligned}
& (1 - \Phi^{-1})\sigma_v y_t^* + (1 - \Phi^{-1})(1 - v)\Phi z_t + \sigma_v C_G^{-1}g_t \\
& - \frac{\tau}{1 - \tau}\tau_t + \left(1 + \frac{\varphi + \alpha}{1 - \alpha}\right) a_t - vz_t - \mu = \left(\frac{\sigma_v C_Y^{-1}(1 - \alpha) + \varphi + \alpha}{1 - \alpha}\right) y_t^n
\end{aligned} \tag{A.237}$$

$$\begin{aligned}
& \left(\frac{\sigma_v C_Y^{-1}(1 - \alpha) + \varphi + \alpha}{1 - \alpha}\right)^{-1} \times \\
& \left((1 - \Phi^{-1})\sigma_v y_t^* + (1 - \Phi^{-1})(1 - v)\Phi z_t + \sigma_v C_G^{-1}g_t - \frac{\tau}{1 - \tau}\tau_t + \left(1 + \frac{\varphi + \alpha}{1 - \alpha}\right) a_t - vz_t - \mu\right) = y_t^n
\end{aligned} \tag{A.238}$$

Note that $(1 - \Phi^{-1}) = -v(\varpi - 1)$

$$\begin{aligned}
& \Gamma_* y_t^* + \left(\frac{\sigma_v C_Y^{-1}(1 - \alpha) + \varphi + \alpha}{1 - \alpha}\right)^{-1} \times \\
& \left((1 - \Phi^{-1})(1 - v)\Phi z_t + \sigma_v C_G^{-1}g_t - \frac{\tau}{1 - \tau}\tau_t + \left(1 + \frac{\varphi + \alpha}{1 - \alpha}\right) a_t - vz_t - \mu\right) = y_t^n
\end{aligned} \tag{A.239}$$

$$\begin{aligned}
& \Gamma_* y_t^* + \left(\frac{\sigma_v C_Y^{-1}(1 - \alpha) + \varphi + \alpha}{1 - \alpha}\right)^{-1} \times \\
& \left(v\varpi\Phi z_t + \sigma_v C_G^{-1}g_t - \frac{\tau}{1 - \tau}\tau_t + \left(1 + \frac{\varphi + \alpha}{1 - \alpha}\right) a_t - \mu\right) = y_t^n
\end{aligned} \tag{A.240}$$

$$\Gamma_* y_t^* + \Gamma_z z_t + \Gamma_a a_t + \left(\frac{\sigma_v C_Y^{-1}(1 - \alpha) + \varphi + \alpha}{1 - \alpha} \right)^{-1} \left(\sigma_v C_G^{-1} g_t - \frac{\tau}{1 - \tau} \tau_t - \mu \right) = y_t^n \quad (\text{A.241})$$

$$\Gamma_* y_t^* + \Gamma_z z_t + \Gamma_a a_t + \Gamma_g g_t + \left(\frac{\sigma_v C_Y^{-1}(1 - \alpha) + \varphi + \alpha}{1 - \alpha} \right)^{-1} \left(-\frac{\tau}{1 - \tau} \tau_t \right) = y_t^n \quad (\text{A.242})$$

$$\Gamma_* y_t^* + \Gamma_z z_t + \Gamma_a a_t + \Gamma_g g_t + \Gamma_\tau \tau_t = y_t^n \quad (\text{A.243})$$

$$y_t^n = \Gamma_* y_t^* + \Gamma_z z_t + \Gamma_a a_t + \Gamma_\tau \tau_t + \Gamma_g g_t \quad (\text{A.244})$$

where:

$$\Gamma_* = -\frac{v(\varpi - 1)\sigma_v(1 - \alpha)}{\sigma_v C_Y^{-1}(1 - \alpha) + \varphi + \alpha} \quad (\text{A.245})$$

$$\Gamma_z = -\frac{v\varpi\Phi(1 - \alpha)}{\sigma_v C_Y^{-1}(1 - \alpha) + \varphi + \alpha} \quad (\text{A.246})$$

$$\Gamma_a = \frac{1 + \varpi}{\sigma_v C_Y^{-1}(1 - \alpha) + \varphi + \alpha} \quad (\text{A.247})$$

$$\Gamma_g = \frac{\sigma_v C_G^{-1}(1 - \alpha)}{\sigma_v C_Y^{-1}(1 - \alpha) + \varphi + \alpha} \quad (\text{A.248})$$

$$\Gamma_\tau = -\frac{\tau}{1 - \tau} \frac{1 - \alpha}{\sigma_v C_Y^{-1}(1 - \alpha) + \varphi + \alpha} \quad (\text{A.249})$$

Derivation of natural real interest rate

$$r_t^n = C_Y^{-1} \sigma_v \mathbb{E}_t \{\Delta y_{t+1}^n\} + \rho + \sigma_v v(\varpi - 1) \mathbb{E} \{\Delta y_{t+1}^*\} + \Phi(1 - v)(1 - \rho_z) z_t - \sigma_v C_G^{-1} \mathbb{E}_t \{\Delta g_{t+1}\} \quad (\text{A.250})$$

$$C_Y^{-1} \sigma_v \mathbb{E}_t \{\Delta y_{t+1}^n\} = C_Y^{-1} \sigma_v \mathbb{E}_t \{\Gamma_* \Delta y_{t+1}^* + \Gamma_z \Delta z_{t+1} + \Gamma_a \Delta a_{t+1} + \Gamma_g \Delta g_{t+1} + \Gamma_\tau \Delta \tau_{t+1}\} \quad (\text{A.251})$$

Consider each term in turn:

$$y_t^* : C_Y^{-1} \sigma_v \Gamma_* \Delta y_{t+1}^* + \sigma_v v(\varpi - 1) \Delta y_{t+1}^* = \sigma_v (C_Y^{-1} \Gamma_* + v(\varpi - 1)) \Delta y_{t+1}^* = \Psi_* \mathbb{E}_t \{\Delta y_{t+1}^*\} \quad (\text{A.252})$$

$$z_t : C_Y^{-1} \sigma_v \Gamma_z \Delta z_{t+1} + \Phi(1 - v)(1 - \rho_z) z_t = C_Y^{-1} \sigma_v \Gamma_z (\rho_z - 1) z_t + \Phi(1 - v)(1 - \rho_z) z_t \quad (\text{A.253})$$

$$= C_Y^{-1} \sigma_v \Gamma_z (\rho_z - 1) z_t + \Phi(1 - v)(1 - \rho_z) z_t = ((1 - v)\Phi - C_Y^{-1} \sigma_v \Gamma_z)(1 - \rho_z) z_t = \Psi_z (1 - \rho_z) z_t \quad (\text{A.254})$$

$$a_t : C_Y^{-1} \sigma_v \Gamma_a \Delta a_{t+1} = C_Y^{-1} \sigma_v \Gamma_a (\rho_a - 1) a_t = -C_Y^{-1} \sigma_v \Gamma_a (1 - \rho_a) a_t \quad (\text{A.255})$$

$$g_t : C_Y^{-1} \sigma_v \Gamma_g \Delta g_{t+1} - \sigma_v C_G^{-1} \mathbb{E}_t \{\Delta g_{t+1}\} = \left(\sigma_v (C_Y^{-1} \Gamma_g - C_G^{-1}) \right) \mathbb{E}_t \{\Delta g_{t+1}\} = -\Psi_g (1 - \rho_g) g_t \quad (\text{A.256})$$

$$\tau_t : \sigma_v C_Y^{-1} \Gamma_\tau \Delta \tau_{t+1} = \Psi_\tau \mathbb{E}_t \{\Delta \tau_{t+1}\} \quad (\text{A.257})$$

Finally:

$$r_t^n = \rho - C_Y^{-1} \sigma_v \Gamma_a (1 - \rho_a) a_t + \Psi_* \mathbb{E}_t \{\Delta y_{t+1}^*\} + \Psi_z (1 - \rho_z) z_t - \Psi_g (1 - \rho_g) g_t + \Psi_\tau \mathbb{E}_t \{\Delta \tau_{t+1}\} \quad (\text{A.258})$$

where:

$$\Psi_* = \sigma_v (C_Y^{-1} \Gamma_* + v(\varpi - 1)) \quad (\text{A.259})$$

$$\Psi_z = (1 - v)\Phi - C_Y^{-1} \sigma_v \Gamma_z \quad (\text{A.260})$$

$$\Psi_g = \sigma_v (C_Y^{-1} \Gamma_g - C_G^{-1}) \quad (\text{A.261})$$

$$\Psi_\tau = \sigma_v C_Y^{-1} \Gamma_\tau \quad (\text{A.262})$$

Derivation of the trade balance

$$y_t = C_Y [(1-v)c_t + v(2-v)\eta s_t + v y_t^*] + G_Y g_t \quad (\text{A.263})$$

$$c_t = y_t^* + \frac{1}{\sigma} z_t + \frac{1-v}{\sigma} s_t \quad (\text{A.264})$$

$$= \left[C_Y^{-1} y_t - (1-v)c_t - v(2-v)\eta s_t - C_G^{-1} g_t \right] v^{-1} + \frac{1}{\sigma} z_t + \frac{1-v}{\sigma} s_t \quad (\text{A.265})$$

$$\Rightarrow c_t = C_Y^{-1} y_t + v s_t \left(-(2-v)\eta + \frac{1-v}{\sigma} \right) + \frac{v}{\sigma} z_t - C_G^{-1} g_t \quad (\text{A.266})$$

$$= C_Y^{-1} y_t - v \frac{\varpi}{\sigma} s_t + \frac{v}{\sigma} z_t - C_G^{-1} g_t \quad (\text{A.267})$$

Substitute (A.267) to $n x_t = y_t - C_Y [c_t + v s_t] - G_Y g_t$:

$$n x_t = y_t - C_Y \left[C_Y^{-1} y_t - v \frac{\varpi}{\sigma} s_t + \frac{v}{\sigma} z_t - C_G^{-1} g_t + v s_t \right] - G_Y g_t \quad (\text{A.268})$$

$$n x_t = y_t - C_Y \left[C_Y^{-1} y_t + v \left(-\frac{\varpi}{\sigma} + 1 \right) s_t + \frac{v}{\sigma} z_t - C_G^{-1} g_t \right] - G_Y g_t \quad (\text{A.269})$$

y_t terms cancel out:

$$n x_t = -C_Y \left[v \left(-\frac{\varpi}{\sigma} + 1 \right) s_t + \frac{v}{\sigma} z_t - C_G^{-1} g_t \right] - G_Y g_t \quad (\text{A.270})$$

g_t terms cancel out because $C_Y \times C_G^{-1} = G_Y$

$$n x_t = C_Y \left[v \left(\frac{\varpi}{\sigma} - 1 \right) s_t - \frac{v}{\sigma} z_t \right] \quad (\text{A.271})$$

Because trade is balanced in the steady state ($n x_t = 0$), we can multiply by C_Y

$$n x_t = v \left(\frac{\varpi}{\sigma} - 1 \right) s_t - \frac{v}{\sigma} z_t \quad (\text{A.272})$$

Derivation of the Euler equation using Arrow securities

$$\frac{V_{t,t+1}}{P_t} C_t^{-\sigma} = \xi_{t,t+1} \beta C_{t+1}^{-\sigma} \frac{1}{P_{t+1}} \quad (\text{A.273})$$

Where $V_{t,t+1}$ is an Arrow security and $\xi_{t,t+1}$ is the probability that the Arrow security will yield a payoff next period. Interpretation: LHS is paying for the security (expenses) in terms of consumption at P_t prices. RHS is the payoff if the Arrow security yields a payoff. The consumer will only be willing to pay LHS if the payoff is at least as big on the RHS. Given that:

$$Q_{t,t+1} = \frac{V_{t,t+1}}{\xi_{t,t+1}} \quad (\text{A.274})$$

$$\frac{V_{t,t+1}}{P_t} C_t^{-\sigma} = \xi_{t,t+1} \beta C_{t+1}^{-\sigma} \frac{1}{P_{t+1}} \quad (\text{A.275})$$

$$Q_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \quad (\text{A.276})$$

$$\mathbb{E}_t[Q_{t,t+1}] = \beta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right] \quad (\text{A.277})$$

$$\beta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \left(\frac{P_t}{P_{t+1}} \right) \right] = Q_t \quad \text{Euler equation} \quad (\text{A.278})$$

Derivation of loglinearised tax revenue

$$T_t^* = \varpi \tau_t W_t N_t + (1 - \varpi) \tau_t W_t^* N_t^*$$

$$T^* = \varpi \tau W N + (1 - \varpi) \tau W^* N^* \quad (\text{Steady state})$$

Using Uhlig's (1999) method, $X_t Y_t \approx X Y \mathbf{e}^{\tilde{X}_t + \tilde{Y}_t}$:

$$T \mathbf{e}^{\tilde{T}_t^*} = \varpi \tau W N \mathbf{e}^{\tilde{\tau}_t + \tilde{W}_t + \tilde{N}_t} + (1 - \varpi) \tau W^* N^* \mathbf{e}^{\tilde{\tau}_t + \tilde{W}_t^* + \tilde{N}_T^*}$$

Using $\mathbf{e}^{\tilde{X}_t} \approx 1 + \tilde{X}_t$:

$$T^*(1 + \tilde{T}_t^*) = \varpi \tau W N (1 + \tilde{\tau}_t + \tilde{W}_t + \tilde{N}_t) + (1 - \varpi) \tau W^* N^* (1 + \tilde{\tau}_t + \tilde{W}_t^* + \tilde{N}_T^*)$$

Subtract (??):

$$T^*(1 + \tilde{T}_t^*) - T^* = \varpi \tau W N (1 + \tilde{\tau}_t + \tilde{W}_t + \tilde{N}_t) - \varpi \tau W N + (1 - \varpi) \tau W^* N^* (1 + \tilde{\tau}_t + \tilde{W}_t^* + \tilde{N}_T^*) - (1 - \varpi) \tau W^* N^*$$

$$T^*[(1 + \tilde{T}_t^*) - 1] = \varpi \tau W N [(1 + \tilde{\tau}_t + \tilde{W}_t + \tilde{N}_t) - 1] + (1 - \varpi) \tau W^* N^* [(1 + \tilde{\tau}_t + \tilde{W}_t^* + \tilde{N}_T^*) - 1]$$

$$T^* \tilde{T}_t^* = \varpi \tau W N (\tilde{\tau}_t + \tilde{W}_t + \tilde{N}_t) + (1 - \varpi) \tau W^* N^* (\tilde{\tau}_t + \tilde{W}_t^* + \tilde{N}_T^*)$$

Divide by T^* :

$$\boxed{\tilde{T}_t^* = \tau \frac{W N}{T^*} \times \varpi (\tilde{\tau}_t + \tilde{W}_t + \tilde{N}_t) + \tau \frac{W^* N^*}{T^*} \times (1 - \varpi) (\tilde{\tau}_t + \tilde{W}_t^* + \tilde{N}_T^*)} \quad (\text{A.279})$$