Task 1

Analysis of non-linear and linearized water tank model with and without PID control.

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Model and linearization

Differential equation describing the tank water level:

$$\frac{d}{dt}H = \frac{bV - a\sqrt{H}}{A}$$

This ode is non-linear in $\sqrt{H}$. However, we can approximate this by a first order taylor expansion/linearization in a neighbourhood $H_0 + \hat{H}$ around the stationary point H_0:

$$\sqrt{H_0 + \hat{H}} \approx \sqrt{H_0} + \frac{1}{2\sqrt{H_0}} \cdot (H - H_o)$$

With this linearization we arrive at the linear ODE:

$$\frac{dH}{dt} = \frac{b}{A}V - \frac{a}{2A}\sqrt{H_0}(H - H_0) - \frac{a\sqrt{H_0}}{A}$$

State Space Representation

We can find the state space representation of this system (ignoring the non-homogeneous part):

$$\frac{d}{dt}H = \left[-\tfrac{a}{2A\sqrt{H_0}} \right] H + \left[\tfrac{b}{A} \right] V$$

Alternatively, with our constants and linearization point:

$$\dot{H} = \left[-\tfrac{3\sqrt{10}}{80} \right] H + \left[\tfrac{1}{3} \right] V$$

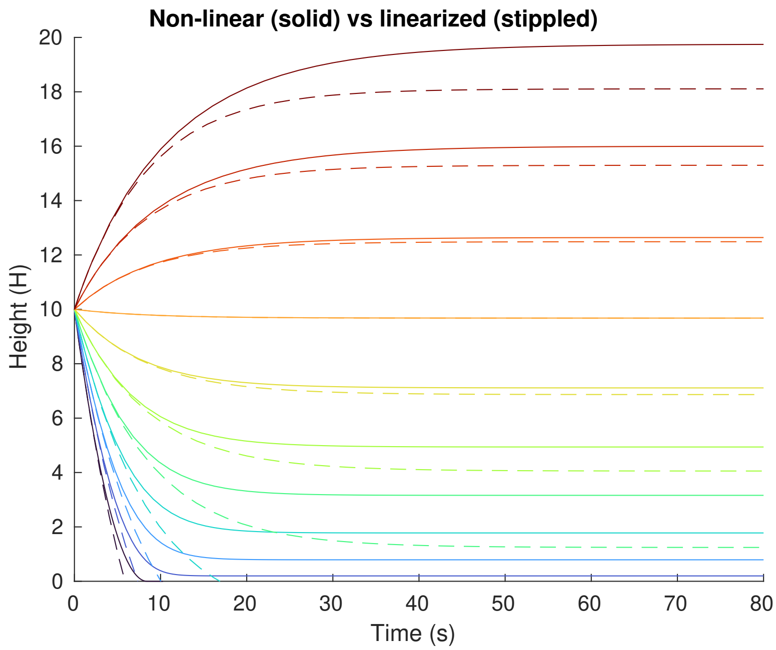
Our state space consists of a one-dimensional state vector H and a one-dimensional control vector V

Simulation of open loop system

For fun, we can simulate the non-linear and linearized differential equations to inspect the accuracy To do this I use matlab's ode45 which can simulate most ordinary differential equations The non-linear and linear ODE's are defined in tank_system_linear.m and tank_system_nonlinear.m, respectively.

```
A = 24; b = 8; a = 18; %water tank parameters

timespan = [0, 80]; %simulaion interval
H0 = 10; %initial level
colormap = turbo(11);

% Solve the and plot the ODE's
figure;
hold on;
for V = 0:10
    [t, H] = ode45(@(t, H) tank_system_nonlinear(t, H, a, b, A, V), timespan, H0);
    plot(t, H, 'Color', colormap(V+1, :));

    [t, H] = ode45(@(t, H) tank_system_linear(t, H, a, b, A, V, H0), timespan, H0);
    plot(t, H,'--','Color', colormap(V+1, :));
end

xlim(timespan);       ylim([0,20]);
xlabel('Time (s)');   ylabel('Height (H)');
title('Non-linear (solid) vs linearized (stippled)');
```

```
Warning: Imaginary parts of complex X and/or Y arguments ignored.
```



Transfer function response

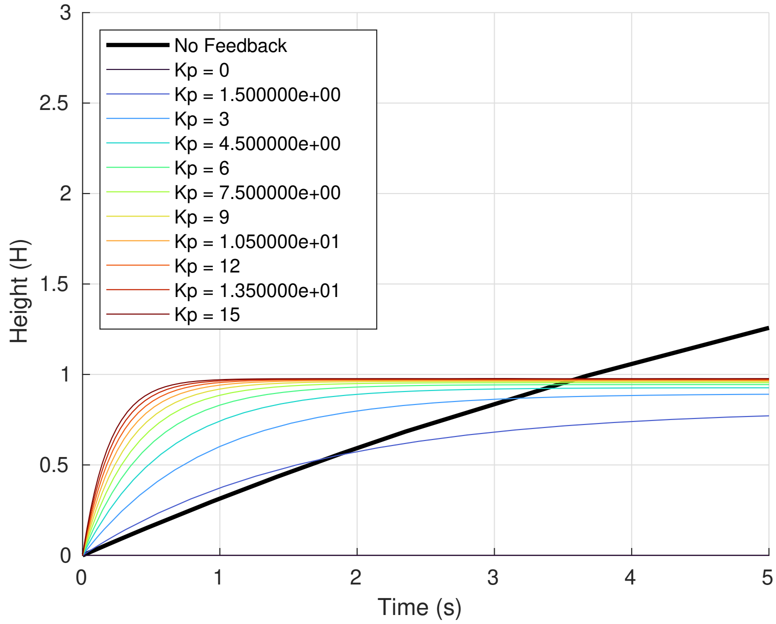
The transfer function is determined by taking the laplace transform of the homogeneous linear ODE:

$$\frac{H(s)}{V(s)} = \frac{\frac{b}{A}}{s + 0.16\frac{a}{A}}$$

Here, I simulate and plot the closd loop step responses for varying proportional gains:

```
close;

H0 = 0; %initial tank level
V = 1; %step response no feedback
H_desired = 1; %setpoint
colormap = turbo(11);

step_timespan = [0, 5];
figure;
grid on;
hold on;
```

```
%without feedback:
Kp = 0;
[t, H] = ode45(@(t, H) tank_system_step(t, H, a, b, A, H_desired, H0, V), timespan, H0);
plot(t, H, 'Color', 'k', 'LineWidth', 2, 'DisplayName', 'No Feedback');

for i = 0:10
    Kp = i*1.5;
    [t, H] = ode45(@(t, H) tank_system_step_feedback(t, H, a, b, A, H_desired, H0, Kp), time
    plot(t, H, 'Color', colormap(i+1, :), 'DisplayName', sprintf('Kp = %d', Kp));
end

xlim(step_timespan);       ylim([0,3]);
xlabel('Time (s)');   ylabel('Height (H)');
legend('Location', 'northwest');
```



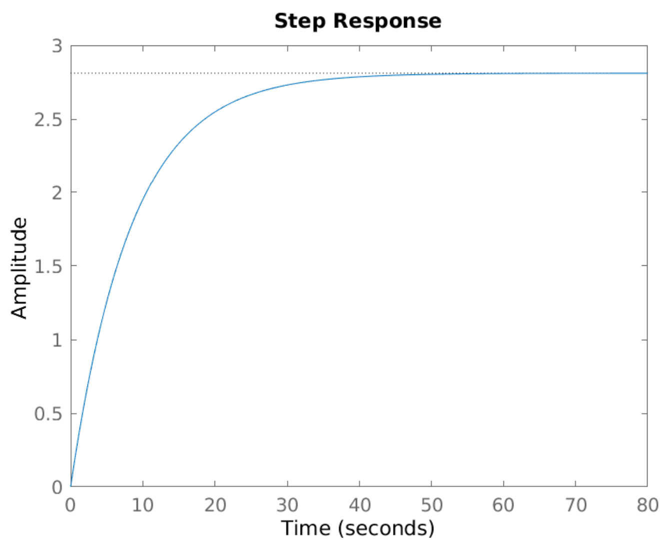
Since the system is a first-order linear system without disturbances, it behaves nicely and doesn't reach any oscillations.

Step response with control

Now we develop a complete PID controller for our plant using the transfer function of the linearized model.

```
close;
s = tf('s');
```

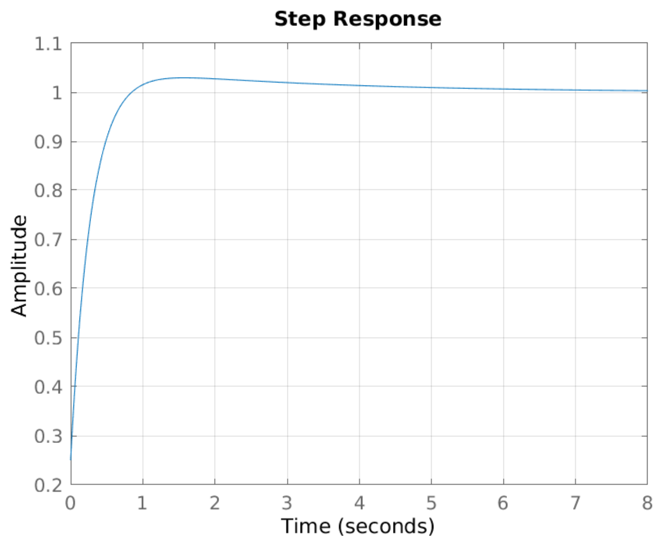
```
A = 24; b = 8; a = 18; %water tank parameters
tf_linear = (b/A)/(s+1/(2*sqrt(10))*(a/A)); %defining transfer function as given in the task
step(tf_linear);
```



Now we tune all the gains to achieve the desired response

```
Kp = 15;
Ki = 5;
Kd = 1;

H = tf_linear; %plant transfer function
C = Kp + Ki/s + Kd*s; %PID transfer function

CH = C*H; %closed loop transfer function
closed_tf = CH/(CH+1);
step(closed_tf);

info = stepinfo(closed_tf);
overshoot     = info.Overshoot;
rise_time     = info.RiseTime;
settling_time = info.SettlingTime;

% Display results
fprintf('Overshoot: %.2f%%\n', overshoot);
fprintf('Rise Time: %.2f s\n', rise_time);
fprintf('Settling Time: %.2f s\n', settling_time);
grid on;
```

```
Overshoot: 2.95%
Rise Time: 0.49 s
Settling Time: 2.95 s
```



Applying the control to the non-linear and linearized plants in simulink

With our conservative gains [Kp, Ki, Kd] = [15, 5, 1] We achieved a nice response.

Now, I plot the response for a setpoint of 10 an an initial tank level of 9, for both the non-linear and linear systems.

See simulink scope plots!