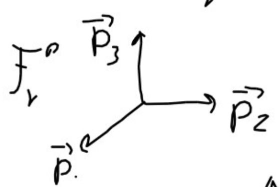
F4 / Example A4 Matrix representation of "$w\times$" operator

Let $\mathcal{F}_r^p$ be orthonormal (o.n.)



$$\vec{w} = w_1\vec{p}_1 + w_2\vec{p}_2 + w_3\vec{p}_3$$

$$\underline{w}^p = [w_1 ; w_2 ; w_3]$$

$$A\vec{a} = \vec{w}\times\vec{a}$$

Definition of cross-product shows that "$w\times$" is a linear operator

$$A^p = \left[\langle A\vec{p}_j , \vec{p}_i^{\,*}\rangle\right] \overset{o.n}{=} \left[\langle \vec{w}\times\vec{p}_j , \vec{p}_i\rangle\right]$$

$$= \left[\langle (w_1\vec{p}_1 + w_2\vec{p}_2 + w_3\vec{p}_3)\times\vec{p}_j , \vec{p}_i\rangle\right]$$

$$\vec{p}_i\times\vec{p}_i = 0 \quad , \quad i = 1,2,3$$

$$\vec{p}_1\times\vec{p}_2 = \vec{p}_3 = -\vec{p}_2\times\vec{p}_1$$

$$\vec{p}_1\times\vec{p}_3 = -\vec{p}_2 = -\vec{p}_3\times\vec{p}_1$$

$$\vec{p}_2\times\vec{p}_3 = \vec{p}_1 = -\vec{p}_3\times\vec{p}_2$$

$$\langle\vec{p}_i , \vec{p}_i\rangle = 1 \quad i = 1,2,3$$

$$\langle\vec{p}_i , \vec{p}_j\rangle = 0 \quad i \neq j$$

$$A^p = \begin{array}{c} \\ i=1 \\ i=2 \\ i=3 \end{array} \begin{array}{c} \begin{array}{ccc} j=1 & j=2 & j=3 \end{array} \\ \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \end{array}$$

$$A^p = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} = S(\underline{w}^p)$$

$$S + S^T = 0$$

Skew symetric form

NB! basisvectors are o.n.

$$\boxed{\vec{b} = \vec{w} \times \vec{a} \quad \overset{\vec{\mathcal{F}}_i^p}{\iff} \quad \underline{b}^p = S(\underline{w}^p)\,\underline{a}^p}$$

We can now write geometrical expressions with geometrical vectors and operators as algebraic equations with matrices.

$$\vec{x} \overset{\mathcal{F}_v^a}{\longleftrightarrow} \underline{x}^a \qquad \mathbb{A} \overset{\mathcal{F}_v^b}{\longleftrightarrow} A^b \qquad \Bigg| \qquad \vec{w}\times \overset{\mathcal{F}_v^a}{\Longleftrightarrow} S(\underline{w}^a) \qquad \text{o.n. basisvectors}$$

NB! $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$

$\vec{a}\to\underline{a}^p \;,\; \vec{b}\to\underline{b}^p \;,\; \vec{c}\to\underline{c}^p$

$$(\underline{a}^p\times\underline{b}^p)\times\underline{c}^p = S\big(S(\underline{a}^p)\underline{b}^p\big)\,\underline{c}^p$$

$$\underline{a}^p\times(\underline{b}^p\times\underline{c}^p) = S(\underline{a}^p)\big(S(\underline{b}^p)\,\underline{c}^p\big) = S(\underline{a}^p)\,S(\underline{b}^p)\,\underline{c}^p$$

A.2.4 Matriserepresentasjon ved bytte av basisvektorer

Problem A.3 Bestem sammenhengen mellom matriserepresentasjonene av vektoren $\vec{r}$ og operatoren $\mathbf{A}$ i hhv q- og p-systemet. Dvs sammenhengen mellom $\underline{r}^q$ og $\underline{r}^p$, A^q og A^p

Teorem A.4 ***Matriserepresentasjon ved bytte av basisvektorer.***
Gitt to basissystemer $\{\vec{q}_i\}$ og $\{\vec{p}_i\}$ i vektorrommet $\mathcal{V}$. La $\vec{r}$ og $\mathbf{A}$ være hhv en vektor og en lineær operator i $\mathcal{V}$. Da har vi følgende sammenhenger mellom matriserepresentasjonene i de to basissystemene :

$$\underline{r}^q = C^q_p \underline{r}^p \quad hvor \quad C^q_p = [\langle \vec{p}_j, \vec{q}_i \rangle] \qquad \text{(A- 10)}$$

$$\underline{r}^p = C^p_q \underline{r}^q \quad hvor \quad C^p_q = [\langle \vec{q}_j, \vec{p}_i \rangle] \qquad \text{(A- 11)}$$

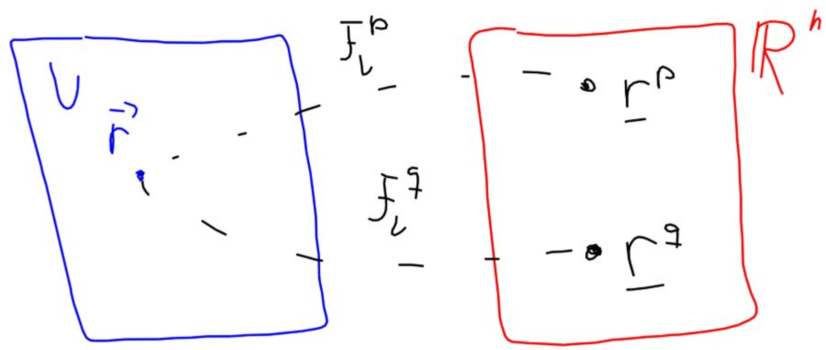
$$C^q_p = [C^p_q]$$

$$A^q = C^q_p A^p C^p_q \text{ og } A^p = C^p_q A^q C^q_p \qquad \text{(A- 12)}$$

C^q_p og C^p_q kalles retningskosinmatriser (RKM). (Vi skal senere se at den kan brukes i mange sammenhenger og har navn deretter. Ovenfor brukes den som en koordinattransformasjonsmatrise, KTM.)

$$\underline{r}^p = C^p_q \underline{r}^q$$

$$\underline{r}^q = \underbrace{C^q_p C^p_q}_{I} \underline{r}^q$$



C^q_p :

Direction Cosine Matrix

(DCM)

Proof: $\underline{r}^q = C_p^q \underline{r}^p$

$$\vec{r} = \sum_{j=1}^{n} r_j^p \vec{p}_j = \sum_{j=1}^{n} r_j^q \vec{q}_j$$

$$\langle \vec{r}, \vec{q}_i^{\,*} \rangle = \left\langle \sum_{j=1}^{n} r_j^q \vec{q}_j, \vec{q}_i^{\,*} \right\rangle = \left\langle \sum_{j=1}^{n} r_j^p \vec{p}_j, \vec{q}_i^{\,*} \right\rangle$$

$$= \sum_{j=1}^{n} r_j^q \overbrace{\langle \vec{q}_j, \vec{q}_i^{\,*} \rangle}^{\partial_{ij}} = \sum_{j=1}^{n} r_j^p \langle \vec{p}_j, \vec{q}_i^{\,*} \rangle$$

$$r_i^q = \sum_{j=1}^{n} \langle \vec{p}_j, \vec{q}_i^{\,*} \rangle r_j^p$$

$$C_p^q = \begin{bmatrix} \langle \vec{p}_1, \vec{q}_1^{\,*} \rangle & \langle \vec{p}_2, \vec{q}_1^{\,*} \rangle & \cdots \\ \langle \vec{p}_1, \vec{q}_2^{\,*} \rangle & & \\ \vdots & & \\ \langle \vec{p}_1, \vec{q}_n^{\,*} \rangle & \cdots & \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \\ r_i^q \\ \\ \end{bmatrix}}_{\underline{r}^q} = \underbrace{\begin{bmatrix} \\ \langle \vec{p}_1, \vec{q}_i^{\,*} \rangle \langle \vec{p}_2, \vec{q}_i^{\,*} \rangle \cdots \langle \vec{p}_n, \vec{q}_i^{\,*} \rangle \\ \\ \end{bmatrix}}_{C_p^q} \underbrace{\begin{bmatrix} r_1^p \\ r_2^p \\ \vdots \\ r_n^p \end{bmatrix}}_{\underline{r}^p}$$

$$\Rightarrow \boxed{\underline{r}^q = C_p^q \underline{r}^p, \quad C_p^q = \left[\langle \vec{p}_j, \vec{q}_i^{\,*} \rangle \right]}$$

Proof A.12 $\quad A^q = C^q_p A^p C^p_q$

$\mathcal{F}_r^q$: ① $\underline{y}^q = A^q \underline{x}^q$ ② $\underline{y}^q = C^q_p \underline{y}^p$

$\mathcal{F}_r^p$: ③ $\underline{y}^p = A^p \underline{x}^p$ ④ $\underline{x}^p = C^p_q \underline{x}^q$

$$\underline{y}^q \overset{①}{=} A^q \underline{x}^q \overset{②}{=} C^q_p \underline{y}^p \overset{③}{=} C^q_p A^p \underline{x}^p \overset{④}{=} \underbrace{C^q_p A^p C^p_q}_{A^q} \underline{x}^q$$

$$\boxed{A^q = C^q_p A^p C^p_q}$$

Similarity transformation

$$\underline{y}^q = A^q \underline{x}^q = C^q_p A^p \underbrace{C^p_q \underline{x}^q}_{\underline{x}^p}$$

$$C^q_p \underbrace{A^p \underline{x}^p}_{\underline{y}^p}$$

$$\underbrace{C^q_p \underline{y}^p}_{\underline{y}^q}$$

Eksempel A.5 Teorem A.5 ***RKM for to ortonormale basissystem***

Dersom vi har to ortonormale basisvektorsettet $\{\vec{q}_i\}$ *og* $\{\vec{p}_i\}$, *dvs*

$$\begin{aligned} \langle \vec{q}_i, \vec{q}_j \rangle &= \delta_{ij} \\ \langle \vec{p}_i, \vec{p}_j \rangle &= \delta_{ij} \end{aligned} \tag{A- 13}$$

så vil de duale basissystema være lik basissystema

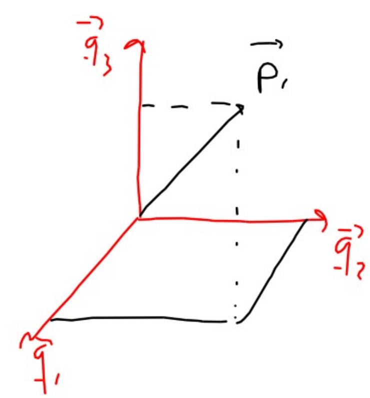
$$\begin{aligned} \vec{q}_i &= \vec{q}_i^*, \quad i = 1, 2, \ldots, n \\ \vec{p}_i &= \vec{p}_i^*, \quad i = 1, 2, \ldots, n \end{aligned} \tag{A- 14}$$

Dette gir

$$C_p^q = [\langle \vec{p}_j, \vec{q}_i \rangle] = [\cos(\angle \vec{p}_j \vec{q}_i)] = R_p^q \tag{A- 15}$$

Dette viser hvorfor C_p^q *kalles en* **retningskosinmatrise**. *Vi vil innføre en spesiell notasjon i dette tilfellet med ortonormale basissystemer og betegner en ortonormal RKM med* R_p^q.

$$R_p^q = \begin{bmatrix} \cos\angle \vec{p}_1 \vec{q}_1 & \cos\angle \vec{p}_2 \vec{q}_1 & \cos\angle \vec{p}_3 \vec{q}_1 \\ \cos\angle \vec{p}_1 \vec{q}_2 & \cos\angle \vec{p}_2 \vec{q}_2 & \cos\angle \vec{p}_3 \vec{q}_2 \\ \cos\angle \vec{p}_1 \vec{q}_3 & \cos\angle \vec{p}_2 \vec{q}_3 & \cos\angle \vec{p}_3 \vec{q}_3 \end{bmatrix} = \begin{bmatrix} \underline{p}_1^q & \underline{p}_2^q & \underline{p}_3^q \end{bmatrix}$$



See that the i'th column in R_p^n represents the basisvector $\vec{p}_i$ represented in the q-basis system

Example (Oppgave A.1) Innerproduct in $\mathbb{R}^n$ o.n. basisvectors

$$\langle \vec{a}, \vec{b} \rangle = \left\langle \sum_{i=1}^{n} a_i \vec{p}_i \,,\, \sum_{j=1}^{n} b_j \vec{p}_j \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j \underbrace{\langle \vec{p}_i, \vec{p}_j \rangle}_{\delta_{ij}} = \sum_{i=1}^{n} a_i b_i = (\underline{a}^p)^T \underline{b}^p$$

$$n=3 \quad \langle \vec{a}, \vec{b} \rangle = (\underline{a}^p)^T \underline{b}^p = \|\vec{a}\| \, \|\vec{b}\| \cos \angle \vec{a}\vec{b}$$

Teorem A.6 ***RKM R_p^q er en ortogonal matrise***
Retningskosinmatrisa mellom to rammer som begge har ortonormale basisvektorer, R_p^q, er en ortonormal matrise. Dvs

$$(R_p^q)^{-1} = (R_p^q)^T \tag{A- 16}$$

Proof:

$$R_p^q = [\underline{p}_1^q, \underline{p}_2^q, \underline{p}_3^q], \quad (R_p^q)^T = \begin{bmatrix} (\underline{p}_1^q)^T \\ (\underline{p}_2^q)^T \\ (\underline{p}_3^q)^T \end{bmatrix} \Rightarrow (R_p^q)^T R_p^q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{I} = (R_p^q)^{-1} R_p^q$$

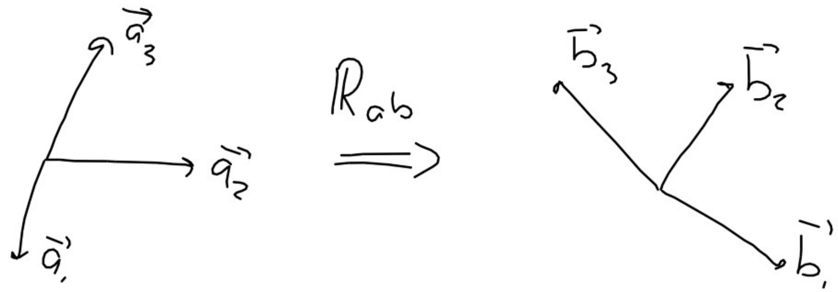
$$(\underline{p}_i^q)^T \underline{p}_j^q = \langle \underline{p}_i^q, \underline{p}_j^q \rangle = \delta_{ij}$$

$$\Rightarrow (R_p^q)^T = (R_p^q)^{-1}$$

A2.5 Matrix representation of rotation operator

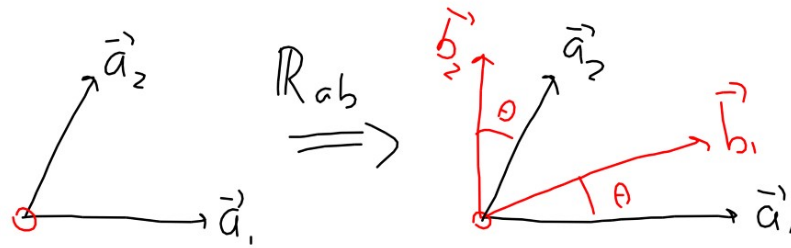
Def A.11: A rotation operator is a linear operator $R_{ab} : V \to V$ defined by $\vec{b}_i = R_{ab}\vec{a}_i$, $i = 1, 2, \ldots, n$

3D)



All basisvectors are rotated the same angle around the same axis.

2D)



Question: What is the representation of $\mathbb{R}_{ab}$ in $\mathcal{F}_v^a$ and $\mathcal{F}_v^b$. That means what is: $[\mathbb{R}_{ab}]^a$ and $[\mathbb{R}_{ab}]^b$

Theorem A.7 Matrix representation of the rotation operator $\mathbb{R}_{ab}$ in $\mathcal{F}_v^a$ and $\mathcal{F}_v^b$ is:

$$\boxed{[\mathbb{R}_{ab}]^a = [\mathbb{R}_{ab}]^b = C_b^a}$$

NB! $\mathbb{R}_{ab} \Longleftrightarrow C_b^a$

Define:

$$R_{ab}^a = [\mathbb{R}_{ab}]^a = C_b^a$$

$$R_{ab}^b = [\mathbb{R}_{ab}]^b = C_b^a$$

Differentiate between physically to rotate a vector ($\mathbb{R}_{ab} \rightarrow C_b^a$) and to transform a vector between two different frames C_b^a

Proof:

$$[\mathbb{R}_{ab}]^a = R^a_{ab} = \left[\langle \mathbb{R}_{ab}\vec{a}_j, \vec{a}_i^{\,*}\rangle\right] = \left[\langle \vec{b}_j, \vec{a}_i^{\,*}\rangle\right] = C^a_b$$

$$[\mathbb{R}_{ab}]^b = R^b_{ab} = C^b_a R^a_{ab} C^a_b = \underbrace{C^b_a C^a_b}_{I} C^a_b = C^a_b$$