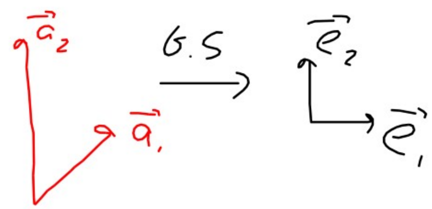
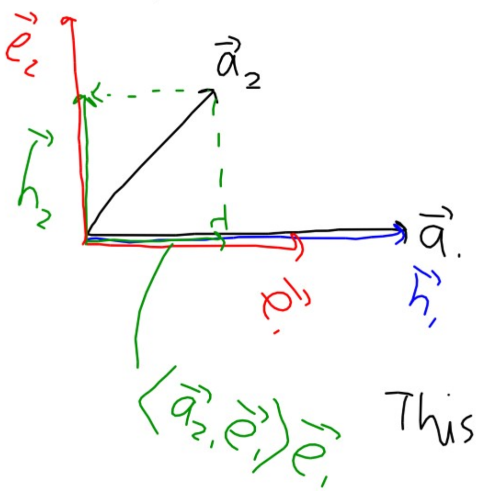
F3

Teorem A.1 ***Grahm-Schmidt ortogonalisering.***
Dersom vi har et sett med basisvektorer $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ kan vi lage et ortonormalt sett av basisvektorer $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ hvor $\langle \vec{e}_i, \vec{e}_j \rangle = \delta_{ij}$ på følgende måte:



$$\left.\begin{aligned} \vec{h}_k &= \vec{a}_k - \sum_{i=1}^{k-1} \langle \vec{a}_k, \vec{e}_i \rangle \, \vec{e}_i \\ \vec{e}_k &= \vec{h}_k \left\| \vec{h}_k \right\|^{-1} \end{aligned}\right\} k = 1, 2, \ldots, n \qquad \text{(A- 3)}$$

Example $n=2$



$$\vec{h}_1 = \vec{a}_1 - \sum_{i=1}^{0} \langle \vec{a}_1, \vec{e}_i \rangle \vec{e}_i = \vec{a}_1$$

$$\vec{e}_1 = \vec{h}_1 / \|\vec{h}_1\| \Rightarrow \|\vec{e}_1\| = 1$$

$$\vec{h}_2 = \vec{a}_2 - \sum_{i=1}^{1} \langle \vec{a}_2, \vec{e}_i \rangle \vec{e}_i = \vec{a}_2 - \langle \vec{a}_2, \vec{e}_1 \rangle \vec{e}_1$$

$$\vec{e}_2 = \vec{h}_2 / \|\vec{h}_2\| \Rightarrow \|\vec{e}_2\| = 1$$

This shows that we can always create orthonormal (o.n) basises

A.2.2 Matrix representation of geometrical vectors

Problem: Given a geometrical vector $\vec{r}$ and a basis (frame) $\{\vec{p}_i\}$, what is the algebraic vector $\underline{r}^p$

Theorem A.2 Column representation (algebraic vector) of $\vec{r} \in V$ is:

$$\vec{r} = r_1^p \vec{p}_1 + r_2^p \vec{p}_2 + \ldots + r_n^p \vec{p}_n = \sum_{i=1}^{n} r_i^p \vec{p}_i$$

where: $r_i^p = \langle \vec{r}, \vec{p}_i^{\,*} \rangle$

$$\underline{r}^p = \begin{bmatrix} r_1^p \\ r_2^p \\ \vdots \\ r_n^p \end{bmatrix} = \left[r_i^p \right]$$

Proof

From linear algebra we know we can write : $\vec{r} = \sum_{i=1}^{n} r_i^p \vec{p}_i$

We first calculate the dual basis $\{\vec{p}_i^*\}$ where $\langle \vec{p}_i, \vec{p}_j^* \rangle = \delta_{ij}$

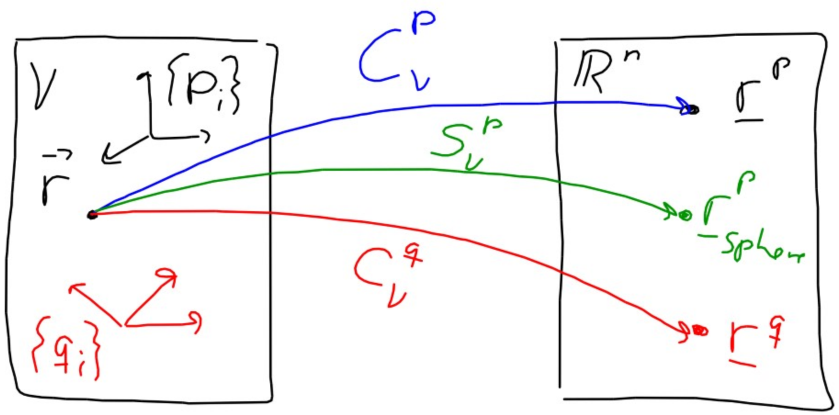
Take innerproduct of $\vec{r}$ with the dual basis $\vec{p}_j^*$

$$\langle \vec{r}, \vec{p}_j^* \rangle = \langle \sum_{i=1}^{n} r_i^p \vec{p}_i, \vec{p}_j^* \rangle = \sum_{i=1}^{n} r_i^p \underbrace{\langle \vec{p}_i, \vec{p}_j^* \rangle}_{\delta_{ij}} = r_j^p$$

i.e. if we switch index:

$$r_i^p = \langle \vec{r}, \vec{p}_i^* \rangle$$

That means: given $\vec{r}$ and $\{\vec{p}_i\}$ then $\underline{r}^p = [\langle \vec{r}, \vec{p}_i^{\,*} \rangle]$



Independent of choise of basis vectors and c.s

Dependent on the choise of basis vectors and type of c.s.

V: n-dimentional vector space

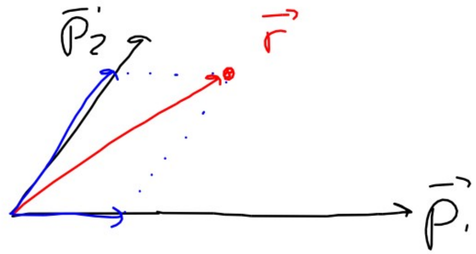
$\mathbb{R}^n$: $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}$ n-din
n-dimentional space of real numbers $\in \mathbb{R}$

C_V^p: coordinate system c.s
(cartesian if o.n axis)

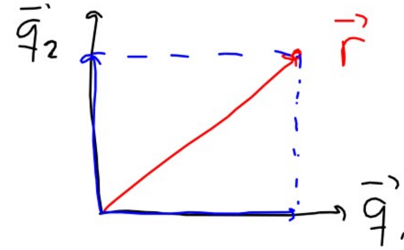
If the basis vectors are orthonormal (o.n.)

$$\langle \vec{p}_i, \vec{p}_j \rangle = \delta_{ij} = \langle \vec{p}_i, \vec{p}_j^{\,*} \rangle$$

$$\Longrightarrow \quad \vec{p}_j = \vec{p}_j^{\,*}$$



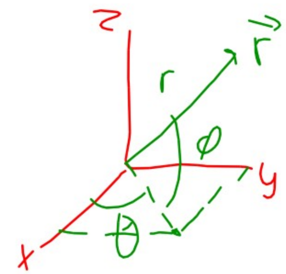
The law of parallellograms
Mathematically we use $\{\vec{p}_i\}$ and $\{\vec{p}_i^{\,*}\}$



Projecting down on the axis
Mathematically use $\{\vec{q}_i\}$

When we use polar- or spherical coordinates we also need to specify a frame (basis)

$$C_v^p(\vec{r}) = \underline{r}^p \quad , \quad S_v^p(\vec{r}) = \begin{bmatrix} r \\ \phi \\ \theta \end{bmatrix}$$



We see that the coordinates a coordinatefunction (C_v^p, S_v^p) gives depend on the frame $\mathcal{F}_v^p$ and how the coordinates are calculated (Cartesian, polar, spherical..). We will just almost use Cartesian coordinates.

A2.3 Matrix representation of linear operator

A operator is a function that given a vector calculates a new vector:

$$\mathbb{O}(\vec{r}) = \vec{a} \quad : V \rightarrow V$$

We will look at linear operators

Def A.10 Linear operator

Operator $\mathbb{A}$ is linear $\overset{\text{if and only if}}{\iff}$ $\forall\, \vec{x}, \vec{y} \in V$ and $\forall\, a, b \in \mathbb{R}$ then:

$$\boxed{\mathbb{A}(a\vec{x} + b\vec{y}) = a\mathbb{A}\vec{x} + b\mathbb{A}\vec{y}}$$

Example: $\vec{w} \times \vec{r}$

$$\mathbb{A} = \vec{w}\times$$

$$\begin{aligned}
\mathbb{A}(a\vec{x} + b\vec{y}) &= \vec{w} \times (a\vec{x} + b\vec{y}) \\
&= \vec{w} \times a\vec{x} + \vec{w} \times b\vec{y} \\
&= a\underbrace{\vec{w}\times}_{\mathbb{A}}\vec{x} + b\underbrace{\vec{w}\times}_{\mathbb{A}}\vec{y} \\
&= a\mathbb{A}\vec{x} + b\mathbb{A}\vec{y}
\end{aligned}$$

That means the cross product "$\vec{w}\times$" is a linear operator!

Theorem A.3 Matrix representation of a linear operator

Given $\{\vec{p}_i\}$ in the vector space V. Then any linear operator $\mathbb{A}$ can be represented in $\mathbb{R}^{n\times n}$ as the matrix A^p. We have:

$$\boxed{\begin{gathered} A^p = [a^p_{ij}] = [\langle \mathbb{A}\vec{p}_j, \vec{p}_i^{\,*}\rangle] \\ \vec{y} = \mathbb{A}\vec{x} \underset{\mathcal{F}_V^p}{\Longleftrightarrow} \underline{y}^p = A^p \underline{x}^p \end{gathered}}$$

Proof of theorem:

Given $\mathcal{F}_V^p = \{\vec{p}_i\}$, $\mathbb{A}: \vec{y} = \mathbb{A}\vec{x}$

From before: $\vec{y} = \sum_{i=1}^{n} y_i^p \vec{p}_i$, $\vec{x} = \sum_{j=1}^{n} x_j^p \vec{p}_j$, $\vec{y} = \mathbb{A}\vec{x}$

$$y_i^p = \langle \vec{y}, \vec{p}_i^{\,*} \rangle = \langle \mathbb{A}\vec{x}, \vec{p}_i^{\,*} \rangle$$

$$= \langle \mathbb{A}\Big(\sum_{j=1}^{n} x_j^p \vec{p}_j\Big), \vec{p}_i^{\,*} \rangle$$

lin. opr.

$$= \langle \sum_{j=1}^{n} x_j^p \mathbb{A}\vec{p}_j , \vec{p}_i^{\,*} \rangle$$

$$= \sum_{j=1}^{n} x_j^p \underbrace{\langle \mathbb{A}\vec{p}_j, \vec{p}_i^{\,*} \rangle}_{a_{ij}^p}$$

$$y_i^p = \sum_{j=1}^{n} a_{ij}^p x_j^p$$

$$\begin{bmatrix} y_1^p \\ y_2^p \\ y_3^p \end{bmatrix} = \begin{bmatrix} a_{11}^p & a_{12}^p & a_{13}^p \\ a_{21}^p & a_{22}^p & a_{23}^p \\ a_{31}^p & a_{32}^p & a_{33}^p \end{bmatrix} \begin{bmatrix} x_1^p \\ x_2^p \\ x_3^p \end{bmatrix}$$

$$\underline{y}^p = A^p \quad \underline{x}^p$$

$$\underline{y}^p = A^p \underline{x}^p$$

where $A^p = [a_{ij}^p] = [\langle \mathbb{A}\vec{p}_j, \vec{p}_i^{\,*} \rangle]$

Q.e.D