**Theorem 1.** *Given a basis* $\{\vec{p}_i\}$ *with dual basis* $\{\vec{p}_i^*\}$, *any vector* $\vec{r}$ *can be written as:*

$$\vec{r} = \sum_{i=1}^{n} r_i^p \vec{p}_i, \quad r_i^p = \langle \vec{r}, \vec{p}_i^* \rangle$$

*Proof.* Since $\{\vec{p}_i\}$ forms a basis, we can write $\vec{r}$ as:

$$\vec{r} = \sum_{i=1}^{n} r_i^p \vec{p}_i$$

Now, consider the inner product of an arbitrary vector $\vec{r}$ and $\vec{p}_i^*$:

$$\begin{aligned}
\langle \vec{r}, \vec{p}_i^* \rangle &= \left\langle \sum_{j=1}^{n} r_j^p \vec{p}_j, \vec{p}_i^* \right\rangle \\
&= \sum_{j=1}^{n} r_j^p \langle \vec{p}_j, \vec{p}_i^* \rangle \\
&= \sum_{j=1}^{n} r_j^p \delta_{ij} = r_i^p
\end{aligned}$$

Thus:

$$r_i^p = \langle \vec{r}, \vec{p}_i^* \rangle$$

□

**Theorem 2.** *Given the basis* $\{\vec{p}_i\}$ *for the vector space* $\mathcal{V}$, *any linear operator* $\mathbb{A}$ *can be represented in* $\mathbb{R}^{n \times n}$ *as the matrix* $A^p$*:*

$$[\mathbb{A}]^p = A^p = [\langle \mathbb{A}\vec{p}_j, \vec{p}_i{}^* \rangle]$$

$$\vec{y} = \mathbb{A}\vec{x} \overset{\text{alg.}}{\Longleftrightarrow} \underline{y}^p = A^p \underline{x}^p$$

*Proof.* Let $\mathbb{A}$ be a linear operator on $\mathcal{V}$, with the basis $\{\vec{p}_i\}$ and dual basis $\{\vec{p}_i^*\}$. Then, let $\vec{y} = \mathbb{A}\vec{x}$ Again, we express $\vec{y}$ and $\vec{x}$ in the p-frame:

$$\vec{y} = \sum_{i=1}^{n} y_i^p \vec{p}_i, \quad \vec{x} = \sum_{i=1}^{n} x_i^p \vec{p}_i$$

From theorem 1, we have:

$$\begin{aligned}
y_i^p &= \langle \vec{y}, \vec{p}_i^* \rangle \\
&= \langle \mathbb{A}\vec{x}, \vec{p}_i^* \rangle
\end{aligned}$$

Expressing $\vec{x}$ in the p-frame and using the linearity of $\mathbb{A}$, we get

$$
\begin{aligned}
&= \left\langle \mathbb{A} \sum_{j=1}^{n} x_j^p \vec{p}_j, \vec{p}_i^* \right\rangle \\
&= \left\langle \sum_{j=1}^{n} x_j^p \mathbb{A} \vec{p}_j, \vec{p}_i^* \right\rangle \\
y_i^p &= \sum_{j=1}^{n} x_j^p \left\langle \mathbb{A} \vec{p}_j, \vec{p}_i^* \right\rangle
\end{aligned}
$$

This can be written as the following matrix equation

$$
\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \langle \mathbb{A}\vec{p}_1, \vec{p}_1^* \rangle & \dots & \langle \mathbb{A}\vec{p}_n, \vec{p}_1^* \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbb{A}\vec{p}_n, \vec{p}_1^* \rangle & \dots & \langle \mathbb{A}\vec{p}_n, \vec{p}_n^* \rangle \end{bmatrix} \begin{bmatrix} x_1^p \\ \vdots \\ x_n^p \end{bmatrix}
$$

or:

$$
\underline{y}^p = A^p \underline{x}^p
$$

where

$$
A^p = [\langle \mathbb{A}\vec{p}_j, \vec{p}_i{}^* \rangle]
$$

□

**Problem 1.** *Find the matrix representation of the "$\vec{\omega} \times$" operator in the orthonormal frame $\{\vec{p}_i\}$*

**Sol.**

To find the matrix representation we use Theorem 2:

$$
\begin{aligned}
A^p &= [\langle \mathbb{A}\vec{p}_j, \vec{p}_i{}^* \rangle] \\
&= [\langle \vec{\omega} \times \vec{p}_j, \vec{p}_i{}^* \rangle]
\end{aligned}
$$

First we consider $\omega \times \vec{p}_j$ by expressing $\omega$ in the p-frame:

$$
\begin{aligned}
\vec{\omega} \times \vec{p}_j &= \left( \sum_{i=1}^{n} \omega_i^p \vec{p}_i \right) \times \vec{p}_j^* \\
&= (\omega_1^p \vec{p}_1 + \omega_2^p \vec{p}_2 + \omega_3^p + \vec{p}_3) \times \vec{p}_j^*
\end{aligned}
$$

Since the p-frame is orthonormal, $\vec{p}_i^* = \vec{p}_i$:

$$
\vec{\omega} \times \vec{p}_j = (\omega_1^p \vec{p}_1 + \omega_2^p \vec{p}_2 + \omega_3^p \vec{p}_3) \times \vec{p}_j
$$

This cross-product can be computed directly for $j = 1, 2, 3$:

$$\begin{aligned}
\vec{\omega} \times \vec{p}_1 &= -\omega_2^p \vec{p}_3 + \omega_3^p \vec{p}_2 \\
\vec{\omega} \times \vec{p}_2 &= \omega_1^p \vec{p}_3 - \omega_3^p \vec{p}_1 \\
\vec{\omega} \times \vec{p}_3 &= -\omega_1^p \vec{p}_2 + \omega_2^p \vec{p}_1
\end{aligned}$$

Then, computing the entries of the matrix $A^p$:

$$\begin{aligned}
a_{ij} &= \langle \vec{\omega} \times \vec{p}_j, \vec{p}_i \rangle \\
A^p &= \begin{bmatrix} 0 & -\omega_3^p & \omega_2^p \\ \omega_3^p & 0 & -\omega_1^p \\ -\omega_2^p & \omega_1^p & 0 \end{bmatrix}
\end{aligned}$$

This matrix is skew-symmetric and is defined by the coordinates of $\underline{\omega}^p$, so we denote the matrix representation of "$\vec{\omega}\times$" as

$$[\vec{\omega}\times]^p = S(\underline{\omega}^p)$$

$$\vec{y} = \vec{\omega} \times \vec{x} \overset{\text{alg.}}{\iff} \underline{y}^p = S(\underline{\omega}^p)\underline{x}^p$$

**Theorem 3.** *Given two bases $\{\vec{p}_i\}$ and $\{\vec{q}_i\}$ in $\mathcal{V}$. Let $\vec{r}$ and $\mathbb{A}$ be a vector and a linear operator in $\mathcal{V}$, respectively. We then have the following relations between matrix representations in the two bases/frames:*

$$\underline{r}^q = C_p^q \underline{r}^p \quad \textit{where} \quad C_p^q = [\langle \vec{p}_j, \vec{q}_i^{\,*} \rangle]$$

$$\underline{r}^p = C_q^p \underline{r}^q \quad \textit{where} \quad C_q^p = [\langle \vec{q}_j, \vec{p}_i^{\,*} \rangle]$$

$$A^q = C_p^q A^p C_q^p \quad \textit{and} \quad A^p = C_q^p A^q C_p^q$$

*Proof.* Let $\underline{r}^q = C_p^q \underline{r}^p$ where $\underline{r}^q = \begin{bmatrix} r_1^q & \dots & r_n^q \end{bmatrix}^T$ and $\underline{r}^p = \begin{bmatrix} r_1^p & \dots & r_n^p \end{bmatrix}^T$. Then,

$$r_i^q = \langle \vec{r}, \vec{q}_i^{\,*} \rangle = \sum_{j=1}^{n} C_{ij} r_j^p$$

Rearranging and expressing $\vec{r}$ in the p-frame:

$$\sum_{j=1}^{n} C_{ij} r_j^p = \left\langle \sum_{j=1}^{n} r_j \vec{p}_j, \vec{q}_i^{\,*} \right\rangle$$

By linearity of the inner product, we get

$$\sum_{j=1}^{n} C_{ij} r_j^p = \sum_{j=1}^{n} \langle \vec{p}_j, \vec{q}_i^* \rangle \, r_j^p$$
$$\implies C_{ij} = \langle \vec{p}_j, \vec{q}_i^* \rangle$$

thus

$$C_p^q = [\langle \vec{p}_j, \vec{q}_i^* \rangle]$$

The same procedure can be used to find $C_q^p$.

Now, consider the equation $\vec{y} = \mathbb{A}\vec{x}$ where $\mathbb{A}$ is some linear operator on $\mathcal{V}$. We can coordinatize this equation in the p and q-frames:

$$\underline{y}^q = A^q \underline{x}^q \quad \text{and} \quad \underline{y}^p = A^p \underline{x}^p$$

From above, we also have the following relations

$$\underline{y}^q = C_p^q \underline{y}^p \quad \text{and} \quad \underline{y}^p = C_q^p \underline{y}^q$$

By substitution, we derive:

$$\begin{aligned} A^q \underline{x}^q &= \underline{y}^q \\ &= C_q^p \underline{y}^p \\ &= C_q^p A^p \underline{x}^p \\ &= C_q^p A^p C_q^p \underline{x}^q \end{aligned}$$

And thus

$$A^q = C_q^p A^p C_q^p$$

Again, the same procedure can be used to show that $A^p = C_p^q A^q C_p^q$ □

**Theorem 4.** *The Direction Cosine Matrix (DCM) between two frames whose bases are orthonormal, $R_p^q$, is an orthonormal matrix:*

$$(R_p^q)^{-1} = (R_p^q)^T$$

*Proof.* From the definition of the DCM:

$$R_p^q = \begin{bmatrix} \underline{p}_1{}^q & \underline{p}_2^q & \underline{p}_3^q \end{bmatrix}$$
$$(R_p^q)^T = \begin{bmatrix} (\underline{p}_1^q)^T \\ (\underline{p}_2^q)^T \\ (\underline{p}_3^q)^T \end{bmatrix}$$

Then, we compute $(R_p^q)^T R_p^q$

$$\begin{aligned} (R_p^q)^T R_p^q &= \left[ \left\langle \underline{p}_i^q, \underline{p}_j^q \right\rangle \right] = [\delta_{ij}] \\ &= I \end{aligned}$$

thus, $(R_p^q)^{-1} = (R_p^q)^T$ □

**Theorem 5.** *The matrix representation of the rotation operator $\mathbb{R}_{ab}$ in two frames $\mathcal{F}_{\mathcal{V}}^a$ and $\mathcal{F}_{\mathcal{V}}^b$ is*

$$[\mathbb{R}_{ab}]^a = [\mathbb{R}_{ab}]^b = C_b^a$$

*Proof.*

$$\begin{aligned}
[\mathbb{R}_{ab}]^a = R_{ab}^a &= \left[\langle \mathbb{R}_{ab}\vec{a}_i, \vec{a}_j^* \rangle\right] \\
&= \left[\left\langle \vec{b}_i, \vec{a}_j^* \right\rangle\right] \\
&= C_b^a
\end{aligned}$$

Using the "similarity transformation" of the $R_{ab}^a$:

$$\begin{aligned}
R_{ab}^b &= C_a^b R_{ab}^a C_b^a \\
&= C_a^b C_b^a C_b^a \\
&= C_b^a
\end{aligned}$$

And thus $R_{ab}^b = R_{ab}^a$

$\square$

**Theorem 6.** *The derivative of the rotation matrix $R_p^q$ is given by*

$$\begin{aligned}
\dot{R}_p^q &= S(\underline{w}_p^{qq}) R_p^q \\
&= R_p^q S(\underline{w}_p^{qp})
\end{aligned}$$

*Proof.* Since $R_p^q$ is a rotation matrix (orthonormal), we have

$$\begin{aligned}
(R_p^q)^{-1} &= (R_p^q)^T \\
\implies R_p^q (R_p^q)^T &= I
\end{aligned}$$

Taking the derivative on both sides and applying the product rule gives

$$\dot{R}_p^q (R_p^q)^T + R_p^q (\dot{R}_p^q)^T = 0$$

We now define the matrix $S = \dot{R}_p^q (R_p^q)^T$ such that

$$S + S^T = 0$$

This means S is some skew-symmetric matrix, with form

$$S(\underline{w}) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

From our definition, we have that $\dot{R}_p^q = S(\underline{\omega})R_p^q$. Finally, we want to find an interpretation of the vector $\underline{\omega}$. Writing this equation out by interpreting $R_p^q$ as an attitude matrix gives

$$\begin{bmatrix}\underline{\dot{p}}_1^q & \underline{\dot{p}}_2^q & \underline{\dot{p}}_3^q\end{bmatrix} = S(\underline{\omega})\begin{bmatrix}\underline{p}_1^q & \underline{p}_2^q & \underline{p}_3^q\end{bmatrix}$$

$$\begin{aligned}\underline{\dot{p}}_i^q &= S(\underline{\omega})\underline{p}_i^q \\ &= \underline{\omega} \times \underline{p}_i^q\end{aligned}$$

We therefore interpret $\underline{\omega}$ as the angular velocity of of the p-frame seen from the q-frame, and represented in the q-frame:

$$\dot{R}_p^q = S(\underline{\omega}_p^{qq})R_p^q$$

Since $S(\underline{\omega}_p^{qq})$ is a linear operator, we can apply express it using the similarity transform:

$$\begin{aligned}S(\underline{\omega}_p^{qq}) &= \left[\vec{\omega}_p^q \times\right]^q \\ &= R_p^q \left[\vec{\omega}_p^q \times\right]^p R_q^p \\ &= R_p^q S(\underline{\omega}_p^{qp}) R_q^p\end{aligned}$$

Inserting this into the equation above gives

$$\begin{aligned}\dot{R}_p^q &= S(\underline{\omega}_p^{qq})R_p^q \\ &= R_p^q S(\underline{\omega}_p^{qp})\cancel{R_q^p R_p^q} \\ &= R_p^q S(\underline{\omega}_p^{qp})\end{aligned}$$

□

**Theorem 7.** *The derivative of the DCM $C_p^q$ is given by*

$$\begin{aligned}\dot{C}_p^q &= S(\underline{w}_p^{qq})C_p^q \\ &= C_p^q S(\underline{w}_p^{qp})\end{aligned}$$

*Proof.*

$$\begin{aligned}C_p^q &= \begin{bmatrix}\underline{p}_1^q & \underline{p}_2^q & \underline{p}_3^q\end{bmatrix} \\ \dot{C}_p^q &= \begin{bmatrix}\underline{\dot{p}}_1^q & \underline{\dot{p}}_2^q & \underline{\dot{p}}_3^q\end{bmatrix}\end{aligned}$$

Here we use the result from the proof above

$$\underline{\dot{p}}_i^q = S(\underline{\omega}_p^{qq})\underline{p}_i$$

$$\dot{C}_p^q = \begin{bmatrix}S(\underline{\omega}_p^{qq})\underline{p}_1^q & S(\underline{\omega}_p^{qq})\underline{p}_2^q & S(\underline{\omega}_p^{qq})\underline{p}_3^q\end{bmatrix}$$

$$\dot{C}_p^q = S(\underline{\omega}_p^{qq})C_p^q$$

Using the similarity transform we also get $\dot{C}_p^q = C_p^q S(\underline{\omega}_p^{qp})$ □

**Theorem 8.** *The derivative of a vector $\vec{r}$ which is fixed in the rotating frame $\mathcal{F}^p_\mathcal{V}(t)$, seen from the (fixed) frame $\mathcal{F}^q_\mathcal{V}$ is*

$$\dot{\vec{r}}^q = \vec{\omega}^q_p \times \vec{r}$$

*Proof.* The vector $\vec{r}$ can be expressed in the p-frame as

$$\vec{r} = \sum_{i=1}^{3} r^p_i \vec{p}_i$$

Where $r^p_i$ are constant, as the vector is fixed in the p-frame. Taking the time-derivative seen from the q-frame:

$$\dot{\vec{r}}^q = \sum_{i=3}^{3} r^p_i \dot{\vec{p}}^q_i$$

From previously, we have that $\dot{\vec{p}}_i = \vec{\omega}^q_p \times \vec{p}_i$, where $\vec{\omega}^q_p$ is the angular velocity of the the p-frame relative to the q-frame. Inserting this and using the fact that "$\vec{\omega}\times$" is linear gives

$$\begin{aligned}
\dot{\vec{r}}^q &= \sum_{i=1}^{3} r^p_i (\vec{\omega}^p_q \times \dot{\vec{p}}^q_i) \\
&= \vec{\omega}^q_p \times \left( \sum_{i=1}^{n} r^p_i \vec{p}^q_i \right) \\
&= \vec{\omega}^q_p \times \vec{r}
\end{aligned}$$

□

**Theorem 9.** *The derivative of a vector $\vec{r}(t)$ which is time-varying in a rotating frame $\mathcal{F}^p_\mathcal{V}(t)$, seen from the (fixed) frame $\mathcal{F}^q_\mathcal{V}$ is*

$$\dot{\vec{r}}^q = \vec{\omega}^q_p \times \vec{r}$$

*Proof.* We again coordinatize $\vec{r}(t)$, now with time-varying coordinates $r^p_i(t)$:

$$\vec{r}(t) = \sum_{i=1}^{n} r^p_i(t) \vec{p}_i(t)$$

Taking the derivative seen from the q-frame and applying the product rule gives

$$\begin{aligned}
\dot{\vec{r}}^q &= \sum_{i=1}^{3} \dot{r_i}^{pp} \vec{p}_i + \sum_{i=1}^{3} r^p_i \dot{\vec{p}}^q_i \\
\dot{\vec{r}}^q &= \dot{\vec{r}}^p + \vec{\omega}^q_p \times \vec{r}
\end{aligned}$$

□

**Theorem 10.** *Given two frames $\mathcal{F}^q_\mathcal{V} = \{O_q; \vec{q}_1, \vec{q}_2, \vec{q}_3\}$ and $\mathcal{F}^q_\mathcal{V} = \{O_p; \vec{p}_1, \vec{p}_2, \vec{p}_3\}$ and a point $P$. Let*

$$\begin{aligned} \vec{r} &= P - O_q \\ \vec{\rho} &= P - O_p \\ \vec{r}_{qp} &= O_p - O_q \end{aligned}$$

*Then, we have the following relations for the velocity and acceleration seen from the q- and p-systems:*

$$\begin{aligned} \vec{r} &= \vec{r}_{qp} + \vec{\rho} \\ \dot{\vec{r}}^q &= \dot{\vec{r}}^q_{qp} + \dot{\vec{\rho}}^p + \vec{\omega}^q_p \times \vec{\rho} \\ \ddot{\vec{r}}^{qq} &= \ddot{\vec{r}}^{qq}_{qp} + \ddot{\vec{\rho}}^{pp} + \dot{\vec{\omega}}^{qq}_p \times \vec{\rho} + \vec{\omega}^q_p \times \left(\vec{\omega}^q_p \times \vec{\rho}\right) + 2\vec{\omega}^q_p \times \dot{\vec{\rho}}^p \end{aligned}$$

*Proof.* The first relation of the position vectors can be computed directly

$$\begin{aligned} &\vec{r}_{qp} + \vec{\rho} = (O_p - O_q) + (P - O_p) \\ &\vec{r}_{qp} + \vec{\rho} = \vec{r} \end{aligned}$$

In this construction, the position vector $\vec{\rho}$ and its derivatives can be considered as in the proof for theorem 9, so we have

$$\dot{\vec{\rho}}^q = \dot{\vec{\rho}}^p + \vec{\omega}^q_p \times \vec{\rho}$$

We take the q-frame derivative of both sides of the position vector equation, and apply this theorem recursively for $\vec{\rho}$ and its derivatives, as well as the product rule for the cross-product derivatives

$$\begin{aligned} \vec{r} &= \vec{r}_{qp} + \vec{\rho} \\ \dot{\vec{r}}^q &= \dot{\vec{r}}^q_{qp} + \dot{\vec{\rho}}^q \\ &= \underline{\dot{\vec{r}}^q_{qp} + \dot{\vec{\rho}}^p + \vec{\omega}^q_p \times \vec{\rho}} \end{aligned}$$

$$\begin{aligned} \ddot{\vec{r}}^{qq} &= \frac{{}^q d}{dt}\left[\dot{\vec{r}}^q_{qp} + \dot{\vec{\rho}}^p + \vec{\omega}^q_p \times \vec{\rho}\right] \\ &= \ddot{\vec{r}}^{qq}_{qp} + \ddot{\vec{\rho}}^{pq} + \left\{\dot{\vec{\omega}}^{qq}_p \times \vec{\rho} + \vec{\omega}^q_p \times \dot{\vec{\rho}}^q\right\} \\ &= \ddot{\vec{r}}^{qq}_{qp} + \left(\ddot{\vec{\rho}}^{pp} + \vec{\omega}^q_p \times \dot{\vec{\rho}}^p\right) + \left\{\dot{\vec{\omega}}^{qq}_p \times \vec{\rho} + \vec{\omega}^q_p \times \left(\dot{\vec{\rho}}^p + \vec{\omega}^q_p \times \vec{\rho}\right)\right\} \\ &= \ddot{\vec{r}}^{qq}_{qp} + \ddot{\vec{\rho}}^{pp} + \vec{\omega}^q_p \times \dot{\vec{\rho}}^p + \dot{\vec{\omega}}^{qq}_p \times \vec{\rho} + \vec{\omega}^q_p \times \dot{\vec{\rho}}^p + \vec{\omega}^q_p \times (\vec{\omega}^q_p \times \vec{\rho}) \\ \ddot{\vec{r}}^{qq} &= \underline{\ddot{\vec{r}}^{qq}_{qp} + \ddot{\vec{\rho}}^{pp} + \dot{\vec{\omega}}^{qq}_p \times \vec{\rho} + \vec{\omega}^q_p \times \left(\vec{\omega}^q_p \times \vec{\rho}\right) + 2\vec{\omega}^q_p \times \dot{\vec{\rho}}^p} \end{aligned}$$

$\square$

**Theorem 11.** *Assume Newton's 3rd law holds for the force between particles, i.e* $\vec{f}_{ij} = -\vec{f}_{ji}$. *Then, the total external force* $\vec{F}$ *is equal to the total mass* $M$ *times the acceleration of the center of mass* $\vec{a}_c^{\mathbf{i}}$ *seen from the inertial frame:*

$$\vec{F} = M\frac{{}^{\mathbf{i}}d^{\mathbf{i}}d}{dt^2}\vec{r}_c = M\vec{a}_c^{\mathbf{i}}$$

*Proof.* The center of mass is defined as

$$\vec{r}_c = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{r}_i$$

Where $m_i$ is the mass of particle $i$ with position vector $\vec{r}_i$. The acceleration of the center of mass in the inertial frame $\vec{a}_c^{\mathbf{i}}$ is found by differentiating twice

$$\begin{aligned}
\vec{a}_c^{\mathbf{i}} &= \ddot{\vec{r}}_c^{\mathbf{ii}} \\
&= \frac{1}{M}\sum_{i=1}^{n} m_i\ddot{\vec{r}}_i^{\mathbf{ii}} \\
\implies M\vec{a}_c^{\mathbf{i}} &= \sum_{i=1}^{n} m_i\vec{a}_i^{\mathbf{i}}
\end{aligned}$$

Newton's law for a single particle states

$$m_i\vec{a}_i^{\mathbf{i}} = \vec{F}_i$$

Where $\vec{F}_i$ is the total force on the particle, i.e is the sum of internal and external forces on the particle

$$\vec{F}_i = \vec{f}_i + \sum_{j=1}^{n}\vec{f}_{ji} = m_i\vec{a}_i^{\mathbf{i}}$$

Substituting this into the n-particle equation gives

$$M\vec{a}_c^{\mathbf{i}} = \sum_{i=1}^{n}\left(\vec{f}_i + \sum_{j=1}^{n} f_{ji}\right)$$

Since $f_{ij} + f_{ji} = 0$, the contribution from internal forces is zero, and the sum reduces to

$$\begin{aligned}
M\vec{a}_c^{\mathbf{i}} &= \sum_{i=1}^{n}\vec{f}_i \\
M\vec{a}_c^{\mathbf{i}} &= \vec{F}
\end{aligned}$$

□