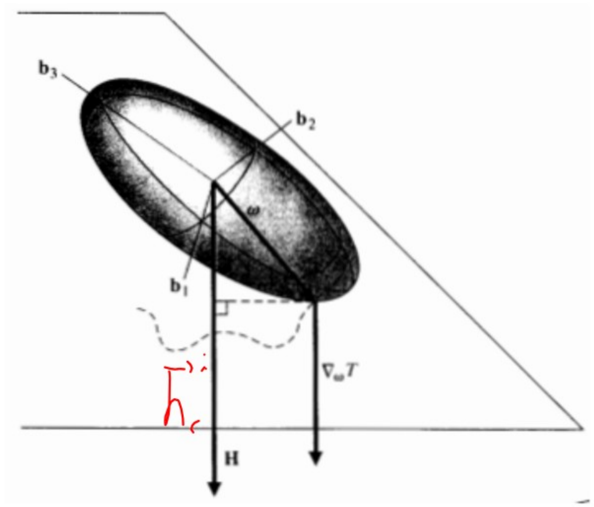
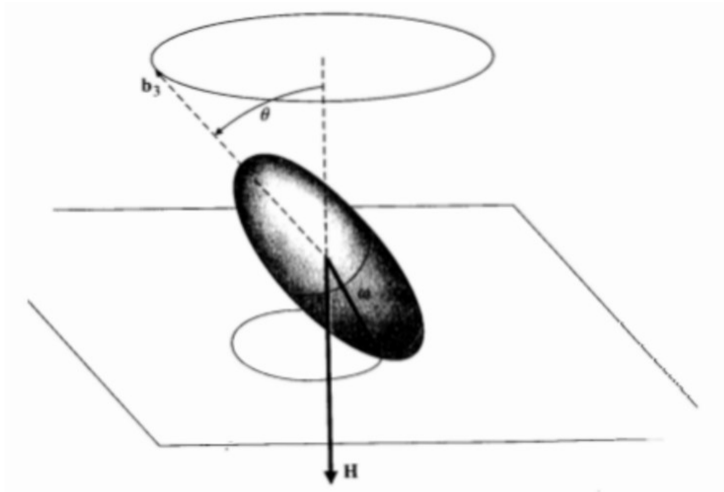
F13

B.3.2 Beskrivelse av bevegelsen sett fra i-systemet

Teorem B.14 ***Bevegelsen av et stivt legeme sett fra treghhetssystemet*** **i**
Bevegelsen av et stivt legeme som ikke er utsatt for et ytre moment er beskrevet, sett fra treghetsrommet, av at den kinetiske energiellipsoida ruller på det invariable plan (plan $\perp$ spinnvektoren $\vec{h}^{\mathbf{i}}$) uten å gli. Rullinga følger polhodet på den kinetiske energiellipsoida (kontaktpunktet mellom det invariable plan og ellipsoida er dermed enden på $\vec{\omega}_b^{\mathbf{i}}$-vektoren).

$\vec{h}_c^{i}$



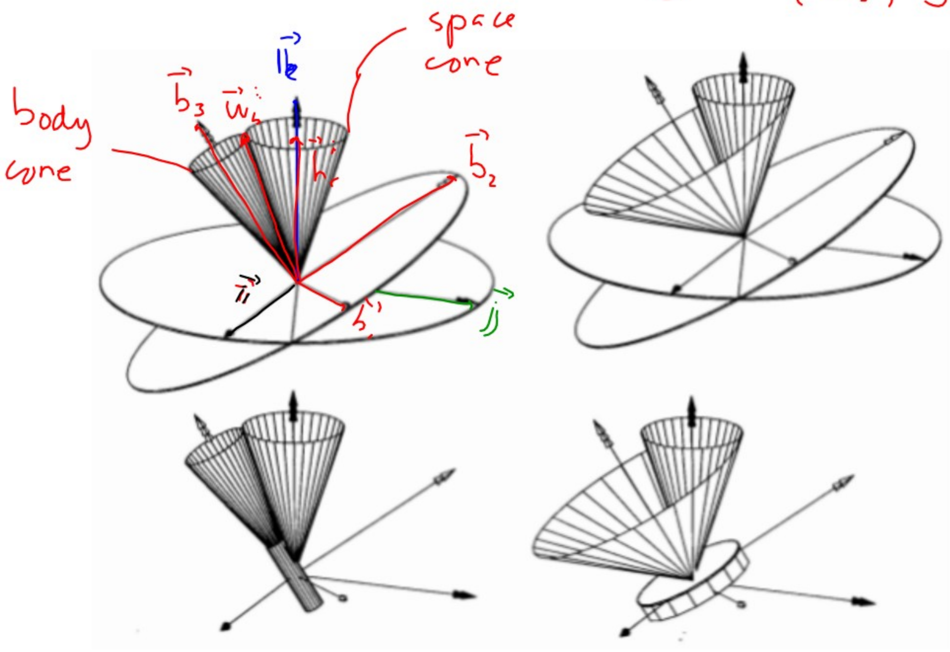


B.4 Torque free motion of axis symmetrical bodies

Assume $J_{xx} = J_{yy} \neq J_{zz}$

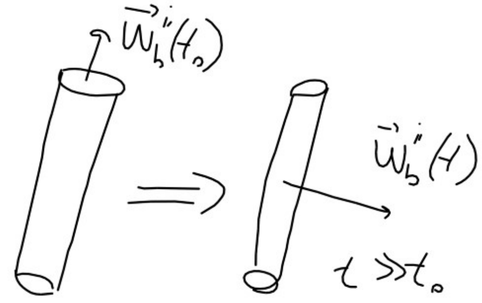
$\mathcal{F}^i : \{ O_i \,;\, \vec{i} \,;\, \vec{j} \,;\, \vec{k} \}$

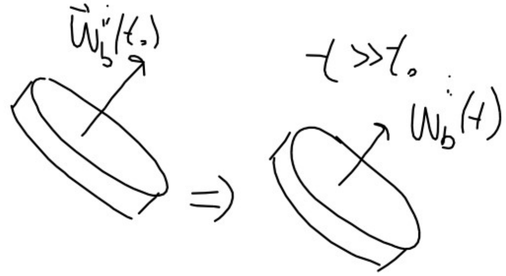
$\mathcal{F}^b : \{ O_b \,;\, \vec{b}_1 \,;\, \vec{b}_2 \,;\, \vec{b}_3 \}$



$J_{xx} = J_{yy} > J_{zz}$ $\quad$ $J_{xx} = J_{yy} < J_{zz}$

What is the long time rotation of a cylinder?

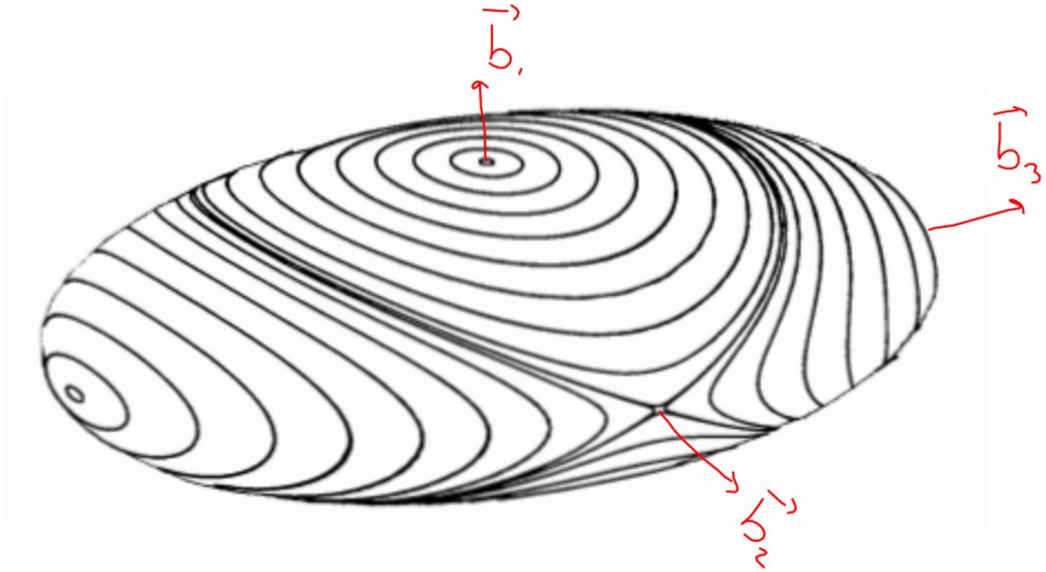




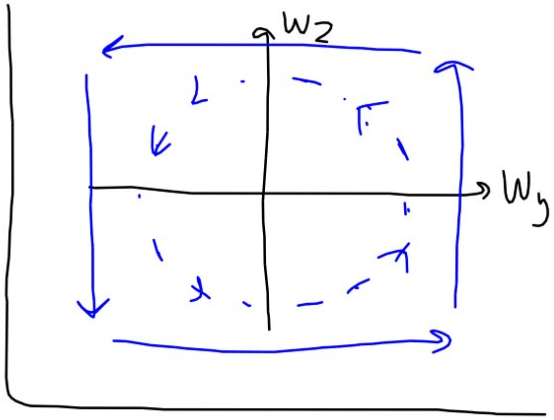
B.3.3 Stabilitet om hovedaksene

(short-time-stability)

Teorem B.15 ***Stabiliteten om hovedaksene for et stivt legeme***
Anta b-systemet faller sammen med hovedaksene for det stive legemet og $J^b_{xx} > J^b_{yy} > J^b_{zz}$. *Da gir linearisering om* $\vec{b}_1$*-aksen* (J^b_{xx}) *eller* $\vec{b}_3$*-aksen* (J^b_{zz}) *et lineært system med kompleks konjugerte egenverdier. Linearisering om* $\vec{b}_2$*-aksen* (J^b_{yy}) *gir et lineært system med to egenverdier, den ene ligger i venstre halvplan den andre i høgre.*



B.3.3 Stability around the main axis



1) Rotation around b_1-axis (x-axis)

Assume $|\omega_x| \gg |\omega_y| \approx |\omega_z|$, set $\omega_y \cdot \omega_z = 0$

Euler equations:

$$\dot{\omega}_x = (J_{yy} - J_{zz})\omega_y\omega_z / J_{xx} = 0 \Rightarrow \omega_x(t) = \omega_{x0}$$

$$\dot{\omega}_y = (J_{zz} - J_{xx})\omega_x\omega_z / J_{yy} = \frac{J_{zz} - J_{xx}}{J_{yy}}\omega_{x0}\omega_z = -\alpha_1 \omega_z \Rightarrow \dot{\omega}_y = -\alpha_1 \omega_z$$

$$\dot{\omega}_z = (J_{xx} - J_{yy})\omega_x\omega_y / J_{zz} = \frac{J_{xx} - J_{yy}}{J_{zz}}\omega_{x0}\omega_y = \alpha_2 \omega_y \Rightarrow \dot{\omega}_z = \alpha_2 \omega_y$$

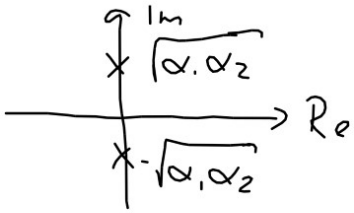
$$\begin{bmatrix} \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} 0 & -\alpha_1 \\ \alpha_2 & 0 \end{bmatrix} \begin{bmatrix} \omega_y \\ \omega_z \end{bmatrix}$$

$$\dot{\underline{x}} = A \underline{x}$$

Eigenvalues:

$$|\lambda \underline{I} - A| = \begin{vmatrix} \lambda & +\alpha_1 \\ \alpha_2 & \lambda \end{vmatrix} = \lambda^2 + \alpha_1\alpha_2 = 0$$

$$\lambda_{1,2} = \pm\sqrt{\alpha_1\alpha_2}\, i$$



We know the solution has form:

$$\boxed{\omega_y(t) = A\sin(\omega t + \phi)} \quad (\omega = 2\pi f)$$

From d.e.

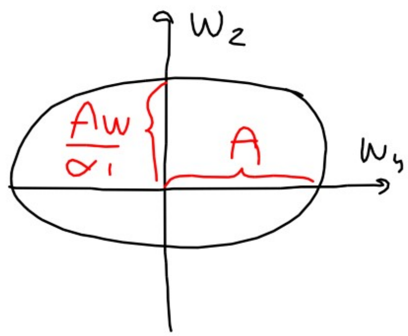
$$\dot{\omega}_y = -\alpha_1 \omega_z$$

$$\boxed{\omega_z(t) = -\frac{1}{\alpha_1} A\omega\cos(\omega t + \phi)}$$

$$\dot{\omega}_z = \frac{1}{\alpha_1} A\omega^2 \sin(\omega t + \phi) = \alpha_2 \omega_y$$

$$\Longrightarrow \quad \omega^2 = \alpha_1 \alpha_2$$

Given $\underline{\omega}_b^{ii}(t_0)$



We can show:

$$\omega = \omega_{x0}\sqrt{\frac{(J_{xx}-J_{zz})(J_{xx}-J_{yy})}{J_{yy}J_{zz}}}$$

$$A = \frac{\omega_y(0)}{\sin\phi}$$

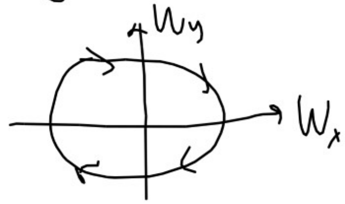
$$\phi = \arctan\left(\frac{-\omega\,\omega_y(0)}{\alpha_1\,\omega_z(0)}\right)$$

ii) Rotation around b_3-axis (z-axis)

Assume $|\omega_z| \gg |\omega_x| \approx |\omega_y|$, $\omega_x\omega_y = 0$

$\Rightarrow$ Complex conjugated eigen values

and $\omega_z(t) = \omega_{z0}$



iii) Rotation around b_2-axis (y-axis)

Assume $|\omega_y| \gg |\omega_x| \approx |\omega_z|$, $\omega_x \omega_z = 0$

$$\dot{\omega}_x = \frac{J_{yy} - J_{zz}}{J_{xx}} \omega_{y0} \omega_z = \beta_1 \omega_z \quad , \beta_1 > 0$$

$$\dot{\omega}_y = \frac{J_{xx} - J_{zz}}{J_{yy}} \omega_x \omega_z = 0 \Rightarrow \omega_y(t) = \omega_{y0}$$

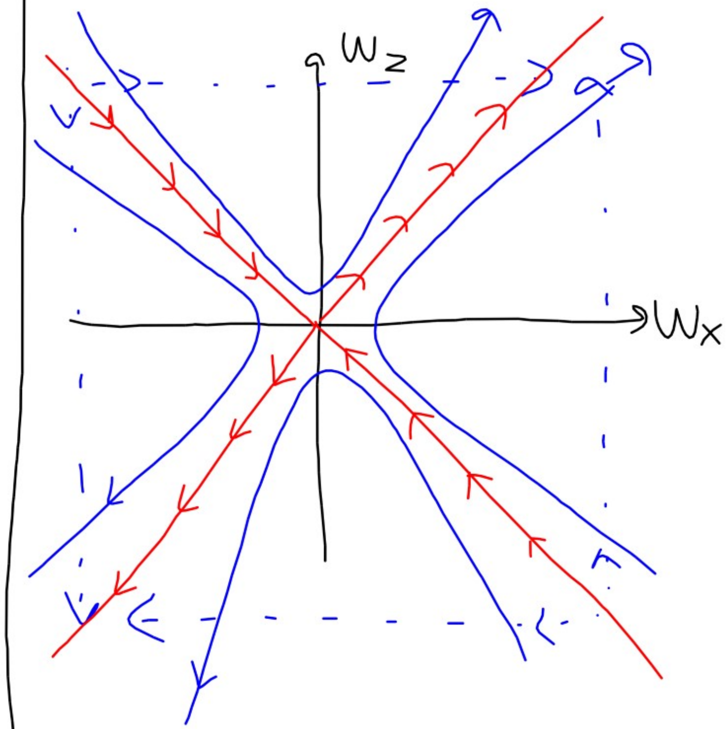
$$\dot{\omega}_z = \frac{J_{xx} - J_{yy}}{J_{zz}} \omega_{y0} \omega_x = \beta_2 \omega_x \quad , \beta_2 > 0$$

$$\begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} 0 & \beta_1 \\ \beta_2 & 0 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_z \end{bmatrix}$$

$$\dot{\underline{x}} = A \underline{x}$$

$$\dot{\omega}_x = \beta_1 \omega_z$$

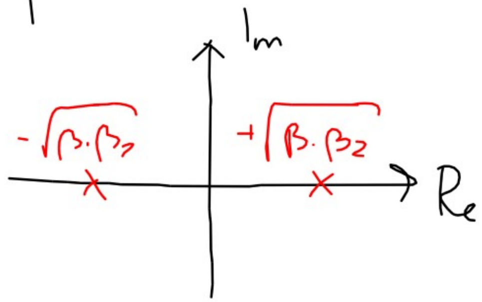
$$\dot{\omega}_z = \beta_2 \omega_x$$



Eigenvalues:

$$|\lambda I - A| = \begin{vmatrix} \lambda & -\beta_1 \\ -\beta_2 & \lambda \end{vmatrix} = \lambda^2 - \beta_1\beta_2 = 0$$

$$\lambda_{1,2} = \pm\sqrt{\beta_1\beta_2}$$



$$\begin{bmatrix} w_x(t) \\ w_z(t) \end{bmatrix} = M \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{-\lambda_1 t} \end{bmatrix} M^{-1} \begin{bmatrix} w_x(0) \\ w_z(0) \end{bmatrix}$$

<u>Unstable</u>

A.6 Matrix calculation in cybernetics

Standard equation : $\dot{\underline{x}} = A\underline{x}$, $\underline{x}(0) = \underline{x}_0$

Eigenvalues to matrix A: $|\lambda \underline{I} - A| = 0 \iff n^{th}$ order equation of λ

$$\lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0 = 0 \Rightarrow \lambda_i \ , \ i = 1,2,\dots,n$$

MatLab:

$[M, \Lambda] = eig(A)$

Eigenvectors: $\underbrace{(\lambda_i \underline{I} - A)}_{\det = 0} \underline{m}_i = \underline{0}$ } I.e. we do not have a unique solution. We can choose $\|\underline{m}_i\| = 1$

If $\lambda_i \neq \lambda_j$ for all $i \neq j$ we can show that $\{\underline{m}_i\}$ is linear independent, and can form a basis.

$M = [\underline{m}_1, \underline{m}_2, \ldots, \underline{m}_n]$ Eigenvector matrix

We had d.e. $\dot{\underline{x}} = A\underline{x}$, $\underline{x}(0) = \underline{x}_0$, since we want to use M as a DCM we need to introduce a unique notation for the two frames the transformation is between $\{m\}$ and $\{q\}$

I.e $\dot{\underline{x}}^q = A^q \underline{x}^q$, $\underline{x}^q(0) = \underline{x}_0^q$

$$M = M_m^q = [\underline{m}_1^q, \underline{m}_2^q, \ldots, \underline{m}_n^q]$$

$$\underline{x}^q = M_m^q \underline{x}^m$$

We had: $\dot{\underline{x}}^q = A^q \underline{x}^q = M_m^q \dot{\underline{x}}^m = A^q M_m^q \underline{x}^m$

$$\Rightarrow \dot{\underline{x}}^m = \underbrace{(M_m^q)^{-1} A^q M_m^q}_{A^m = \underline{\Lambda}^m = [\lambda_i \delta_{ij}] = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)} \underline{x}^m$$

General $(M^q_m)^{-1} \neq (M^q_m)^T$

We have: $\dot{\underline{x}}^m = \underline{\Lambda}^m \underline{x}^m$, $\underline{x}^n(0) = (M^q_m)^{-1} \underline{x}^q(0)$

$$\Updownarrow$$

$\dot{x}_i^m = \lambda_i x_i^m$, $x_i^m(0)$ given

$x_i^m(t) = e^{\lambda_i(t-t_0)} x_i^m(t_0) = e^{\lambda_i t} x_i^m(0)$, since $t_0 = 0$

$$\Updownarrow$$

$$\boxed{\underline{x}^m(t) = \underbrace{e^{\underline{\Lambda}^m t}}_{=\left[e^{\lambda_i t} \delta_{ij}\right]} \underline{x}^m(0)}$$

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

MatLab $\Rightarrow$ expm(X)

exp(X) $\rightarrow$ element by element

i.e. $\dot{\underline{x}}^q = A^q \underline{x}^q$, $\underline{x}^q(0)$ given

has solution:

$$\underline{x}^q(t) = M^q_m \, \underline{x}^m(t)$$

$$\underline{x}^q(t) = M^q_m \, e^{\Lambda^m t} \left(M^q_m\right)^{-1} \underline{x}^q(0)$$

$$\underline{x}^q(t) = e^{M^q_m \Lambda^m (M^q_m)^{-1} t} \, \underline{x}^q(0)$$

$$\underline{x}^q(t) = e^{A^q t} \, \underline{x}^q(0)$$

If M is invertible:

$$e^{MXM^{-1}} = M e^{X} M^{-1}$$

where $e^{A^q t} = \underbrace{M^q_m e^{\Lambda^m t} \left(M^q_m\right)^{-1}}_{\phi(t,0)} = \underbrace{\left(\underline{I} + \frac{1}{1!} A^q t + \frac{1}{2!} A^q A^q t^2 + \ldots\right)}_{\text{Used as numerical calculation}}$