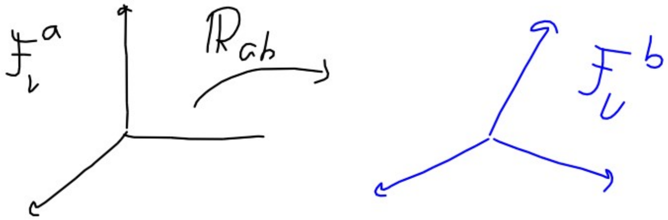
F7 / Relative orientation between two frames can be described by the rotation operator $\mathbb{R}_{ab}$



We have found earlier that:

$$\underbrace{[\mathbb{R}_{ab}]^a}_{R_{ab}^a} = \underbrace{[\mathbb{R}_{ab}]^b}_{R_{ab}^b} = R_b^a$$

Representation of the rotation operator can become time variant:

$$\boxed{[\mathbb{R}_{ab}(t)]^a = R_b^a(t)}$$

For example using Euler angles (3-2-1)

$$R_b^a(t) = R_3(\psi(t))\, R_2(\theta(t))\, R_1(\phi(t))$$

To find the relation between the representation of a point $P(t)$ in the frames $\mathcal{F}_A^a$ and $\mathcal{F}_A^b$ we can use time variant transformation matrices:

$$\boxed{\begin{gathered} T_b^a(t) = \begin{bmatrix} R_b^a(t) & \underline{r}_{ab}(t) \\ \underline{0}^T & 1 \end{bmatrix} \\ \tilde{\underline{r}}_p^a(t) = T_b^a(t)\, \tilde{\underline{r}}_p^b(t) \end{gathered}}$$

In the case of three frames: $\mathcal{F}_A^a, \mathcal{F}_A^b, \mathcal{F}_A^c$

$$\underline{r}^a = T_b^a T_c^b \underline{r}^c \quad \text{(all function of time)}$$

NB! In classical mechanics we use Galileo transformes. We can add relative velocities.

NB! When calculating distances between the positions of points at different times we need to choose only one affine space:

$$P^A(t_2) - P^A(t_1) = \vec{r}_{P^A(t_1)P^A(t_2)}$$

A.5 Derivative in vector spaces and Affine spaces.

The notation has to incorporate two aspects:

1. In which frame $\{a\}$ do we repre(se)nt the vector: $\underline{x}^{a}(t)$
2. In which frame $\{b\}$ do we see the time variation from: $\overset{\circ}{\underline{x}}^{ba}(t)$

In mathematics

Given $f(x,y)$ define the partial derivatives:

$$\frac{\partial f(x,y)}{\partial x} = f_x(x,y) \quad , \quad \frac{\partial f(x,y)}{\partial y} = f_y(x,y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right) = f_{xy}(x,y) \quad , \quad \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right) = f_{yx}(x,y)$$

Here we had two free variables and derivative with respect to these. We will have only one free variable (time t), but we can see the time changes from different frames that moves differently relative to each other.

Introduce notation:

$$\frac{d^a}{dt}\vec{X}(t) = \dot{\vec{X}}^a(t) \qquad : \text{Derivative seen from } \mathcal{F}^a$$

$$\frac{d^b}{dt}\vec{X}(t) = \dot{\vec{X}}^b(t) \qquad : \text{———} \text{ " } \text{———} \ \mathcal{F}^b$$

$$\frac{d^b}{dt}\left(\frac{d^a}{dt}\vec{X}(t)\right) = \ddot{\vec{X}}^{ab}(t)$$

Represent in $\mathcal{F}^c$

$$\left[\dot{\vec{X}}^a(t)\right]^c = \underline{\dot{X}}^{ac}(t), \quad \left[\ddot{\vec{X}}^{ab}(t)\right]^c = \underline{\ddot{X}}^{abc}(t)$$

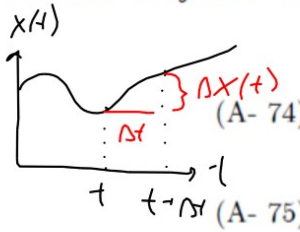
i.e. $\left[\ddot{\vec{X}}^{ab}(t)\right]^c = \ddot{\underline{X}}^{abc}(t)$

Show where the derivatives are seen from, first {a}, then {b}

last superscript shows where we represent the vector {c}

A.5.1 Definisjon av deriverte i vektorrom og affine rom.

Når vi skal definere de deriverte av vektorer og punkter må vi starte med det vi kjenner fra matematikken nemlig derivasjon i $\mathbb{R}$ og så generalisere til $\mathbb{R}^n$, $\mathcal{V}$ og $\mathcal{A}$:



1. Derivasjon i $\mathbb{R}$:

$$\dot{x}(t) = \lim_{\Delta t \to 0}\left(\frac{1}{\Delta t}\overbrace{\left(x(t+\Delta t) - x(t)\right)}^{\Delta x(t)}\right) \tag{A- 74}$$

2. Derivasjon i $\mathbb{R}^n$:

$$\dot{\underline{x}}(t) = [\dot{x}_i(t)] \tag{A- 75}$$

fixed

3. Derivasjon i vektorrommet $\mathcal{V}$ sett fra en fast ramme $\mathcal{F}^a_{\mathcal{V}}$:

$$\dot{\vec{x}}^a = \sum_{i=1}^{n} \dot{x}^a_i(t)\,\vec{a}_i \qquad \vec{X}(t) = \sum_{i=1}^{n} X^a_i(t)\,\vec{a}_i \tag{A- 76}$$

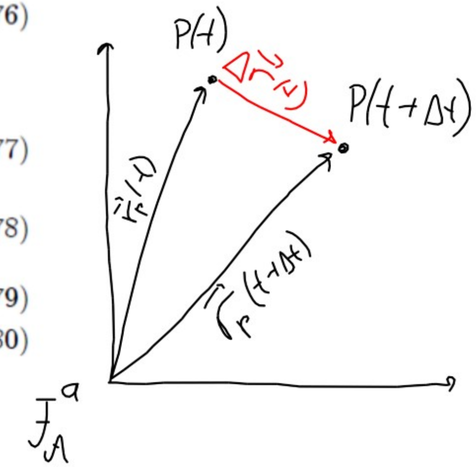
4. Derivasjon i det affine rom $\mathcal{A}$ sett fra en fast ramme $\mathcal{F}^a_{\mathcal{A}}$:

$$\dot{P}^a(t) = \lim_{\Delta t \to 0}\left(\frac{1}{\Delta t}\left(P(t+\Delta t) - P(t)\right)\right) \tag{A- 77}$$

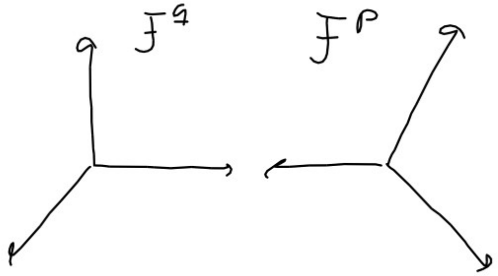
$$= \lim_{\Delta t \to 0}\left(\frac{1}{\Delta t}\underbrace{\left(\vec{r}_{P(t+\Delta t)} - \vec{r}_{P(t)}\right)}_{\Delta\vec{r}(t)}\right) \tag{A- 78}$$

$$= \dot{\vec{r}}^a_P \tag{A- 79}$$

$$= \vec{v}^a_P \tag{A- 80}$$



A5.2 Derivative of DCM



$R_p^q(t)$ gives the relative orientation

Assume R_p^q is o.n. $\Rightarrow (R_p^q)^{-1} = (R_p^q)^T$

Let $R = R_p^q \Rightarrow R \cdot R^T = I$

Take derivative on both sides:

$$\frac{d}{dt}\left(R(t)\, R(t)^T\right) = \frac{d}{dt} I$$

$$\dot{R}(t)\, R^T(t) + R(t) \frac{d}{dt}\left(R^T(t)\right) = 0$$

(1) $$\dot{R}(t)\, R^T(t) + R(t)\left(\dot{R}^T(t)\right) = 0$$

Multiply with $R(t)$ from right:

$$\dot{R}\, \underbrace{R^T R}_{I} + R\,(\dot{R}^T)\, R = 0$$

(2) $$\dot{R} + R\,(\dot{R}^T)\, R = 0$$

Question: Is $(\dot{R})^T = (\dot{R^T})$?

$$\frac{d}{dt}(R^T) = \frac{d}{dt}\left(\left[\underline{p}_1^q, \underline{p}_2^q, \underline{p}_3^q\right]^T\right)$$

$$= \frac{d}{dt}\left(\begin{bmatrix} \underline{p}_1^{q\,T} \\ \underline{p}_2^{q\,T} \\ \underline{p}_3^{q\,T} \end{bmatrix}\right) = \left(\begin{bmatrix} \dot{\underline{p}}_1^{q\,T} \\ \dot{\underline{p}}_2^{q\,T} \\ \dot{\underline{p}}_3^{q\,T} \end{bmatrix}\right) = \left(\frac{d}{dt}R\right)^T$$

I.e.

$$\frac{d}{dt}\left(R^T\right) = \frac{d}{dt}\left(R^{-1}\right) = \left(\frac{d}{dt}R\right)^T$$

In general: $\frac{d}{dt}A^{-1} \neq \left(\frac{d}{dt}A\right)^{-1}$

Eq. 1 can be written:

1 $$\dot{R}R^T + \left(\dot{R}R^T\right)^T = 0$$

$$S := \dot{R}R^T \implies S + S^T = 0$$

I.e. S: scew symmetric matrix

$$S = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} = S(\underline{w})$$

$$\vec{w} \times \vec{a} \implies S(\underline{w}^q)\,\underline{a}^q$$

Eq. 2 can be written:

$$\dot{R} + \underbrace{R\dot{R}^T}_{S^T} R = 0$$

$$\dot{R} = -S^T R = S R$$

$$\boxed{\dot{R}_p^q = S(\underline{w})\, R_p^q} \quad , \quad R_p^q(t_0) \text{ given}$$

R_p^q is an attitude matrix

$$R_p^q = [\underline{p}_1^q , \underline{p}_2^q , \underline{p}_3^q]$$

$$\dot{R}_p^q = [\dot{\underline{p}}_1^q , \dot{\underline{p}}_2^q , \dot{\underline{p}}_3^q]$$

$$= S(\underline{w})[\underline{p}_1^q , \underline{p}_2^q , \underline{p}_3^q]$$

$$= [S(\underline{w})\underline{p}_1^q , S(\underline{w})\underline{p}_2^q , S(\underline{w})\underline{p}_3^q]$$

$$\dot{\underline{p}}_i^q = S(\underline{w})\,\underline{p}_i^q$$
$$= \underline{w} \times \underline{p}_i^q$$

We see that $\underline{w}$ affects the calculation of the derivative of the rotating basis vectors seen from the q-frame. We interpret therefore $\underline{w}$ as the angular velocity to the p-frame seen from the q-frame, and we use the notation $\underline{w}_p^q = \underline{w}_p^{qq}$ because we see the derivative from the q-frame and we represent in the q-frame.

We therefore write : $\boxed{\dot{R}_p^q = S(\underline{w}_p^q)\,R_p^q}$

$S(\underline{w}_p^q) = S(\underline{w}_p^{qq})$ is the representation of the operator " $\vec{w}_p^q \times$ " in the q-frame, but linear operators can also be represented in other frames using the similarity transformation.

$$S(\underline{w}_p^{qq}) = R_p^q \, S(\underline{w}_p^{qp}) \, R_q^p$$

$$\dot{R}_p^q = S(\underline{w}_p^{qq}) R_p^q = R_p^q \, S(\underline{w}_p^{qp}) \underbrace{R_q^p \, R_p^q}_{I}$$

$$\boxed{\dot{R}_p^q = S(\underline{w}_p^q) R_p^q = R_p^q \, S(\underline{w}_p^{qp})}$$

$$\underline{w}_p^{qq} = R_p^q \, \underline{w}_p^{qp}$$

Proof of A.17

$$C_p^q = [\underline{p}_1^q, \underline{p}_2^q, \underline{p}_3^q]$$

$$\dot{C}_p^q = [\dot{\underline{p}}_1^{qq}, \dot{\underline{p}}_2^{qq}, \dot{\underline{p}}_3^{qq}] = [\underline{w}_p^{qq} \times \underline{p}_1^q, \underline{w}_p^{qq} \times \underline{p}_2^q, \underline{w}_p^{qq} \times \underline{p}_3^q]$$

$$= [S(\underline{w}_p^{qq})\underline{p}_1^q, S(\underline{w}_p^{qq})\underline{p}_2^q, S(\underline{w}_p^{qq})\underline{p}_3^q]$$

$$= S(\underline{w}_p^{qq})[\underline{p}_1^q, \underline{p}_2^q, \underline{p}_3^q] = S(\underline{w}_p^{qq}) C_p^q$$

Similarity transformation: $S(\underline{w}_p^{qq}) C_p^q = C_p^q S(\underline{w}_p^{qp}) C_q^p C_p^q = C_p^q S(\underline{w}_p^{qp})$

$$\boxed{\dot{C}_p^q = S(\underline{w}_p^q) C_p^q = C_p^q S(\underline{w}_p^{qp})} \quad , \quad \underline{w}_p^{qq} = C_p^q \underline{w}_p^{qp}$$