Theorem 1. *Given a basis $\{\vec{p}_i\}$ with dual basis $\{\vec{p}_i^*\}$, any vector $\vec{r}$ can be written as:*

$$\vec{r} = \sum_{i=1}^{n} r_i^p \vec{p}_i, \quad r_i^p = \langle \vec{r}, \vec{p}_i^* \rangle$$

Proof. Since $\{\vec{p}_i\}$ forms a basis, we can write $\vec{r}$ as:

$$\vec{r} = \sum_{i=1}^{n} r_i^p \vec{p}_i$$

Now, consider the inner product of an arbitrary vector $\vec{r}$ and $\vec{p}_i^*$:

$$\begin{aligned}
\langle \vec{r}, \vec{p}_i^* \rangle &= \left\langle \sum_{j=1}^{n} r_j^p \vec{p}_j, \vec{p}_i^* \right\rangle \\
&= \sum_{j=1}^{n} r_j^p \langle \vec{p}_j, \vec{p}_i^* \rangle \\
&= \sum_{j=1}^{n} r_j^p \delta_{ij} = r_i^p
\end{aligned}$$

Thus:

$$r_i^p = \langle \vec{r}, \vec{p}_i^* \rangle$$

□

Theorem 2. *Given the basis $\{\vec{p}_i\}$ for the vector space $\mathcal{V}$, any linear operator $\mathbb{A}$ can be represented in $\mathbb{R}^{n \times n}$ as the matrix A^p:*

$$[\mathbb{A}]^p = A^p = [\langle \mathbb{A}\vec{p}_j, \vec{p}_i{}^* \rangle]$$

$$\vec{y} = \mathbb{A}\vec{x} \overset{\text{alg.}}{\iff} \underline{y}^p = A^p \underline{x}^p$$

Proof. Let $\mathbb{A}$ be a linear operator on $\mathcal{V}$, with the basis $\{\vec{p}_i\}$ and dual basis $\{\vec{p}_i^*\}$. Then, let $\vec{y} = \mathbb{A}\vec{x}$ Again, we express $\vec{y}$ and $\vec{x}$ in the p-frame:

$$\vec{y} = \sum_{i=1}^{n} y_i^p \vec{p}_i, \quad \vec{x} = \sum_{i=1}^{n} x_i^p \vec{p}_i$$

From theorem 1, we have:

$$\begin{aligned}
y_i^p &= \langle \vec{y}, \vec{p}_i^* \rangle \\
&= \langle \mathbb{A}\vec{x}, \vec{p}_i^* \rangle
\end{aligned}$$

Expressing $\vec{x}$ in the p-frame and using the linearity of $\mathbb{A}$, we get

$$\begin{aligned} &= \left\langle \mathbb{A} \sum_{j=1}^{n} x_j^p \vec{p}_j, \vec{p}_i^* \right\rangle \\ &= \left\langle \sum_{j=1}^{n} x_j^p \mathbb{A} \vec{p}_j, \vec{p}_i^* \right\rangle \\ y_i^p &= \sum_{j=1}^{n} x_j^p \left\langle \mathbb{A} \vec{p}_j, \vec{p}_i^* \right\rangle \end{aligned}$$

This can be written as the following matrix equation

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \langle \mathbb{A}\vec{p}_1, \vec{p}_1^* \rangle & \dots & \langle \mathbb{A}\vec{p}_n, \vec{p}_1^* \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbb{A}\vec{p}_n, \vec{p}_1^* \rangle & \dots & \langle \mathbb{A}\vec{p}_n, \vec{p}_n^* \rangle \end{bmatrix} \begin{bmatrix} x_1^p \\ \vdots \\ x_n^p \end{bmatrix}$$

or:

$$\underline{y}^p = A^p \underline{x}^p$$

where

$$A^p = [\langle \mathbb{A}\vec{p}_j, \vec{p}_i{}^* \rangle]$$

$\square$

Problem 1. *Find the matrix representation of the "$\vec{\omega} \times$" operator in the orthonormal frame $\{\vec{p}_i\}$*

Sol.

To find the matrix representation we use Theorem 2:

$$\begin{aligned} A^p &= [\langle \mathbb{A}\vec{p}_j, \vec{p}_i{}^* \rangle] \\ &= [\langle \vec{\omega} \times \vec{p}_j, \vec{p}_i{}^* \rangle] \end{aligned}$$

First we consider $\omega \times \vec{p}_j$ by expressing ω in the p-frame:

$$\begin{aligned} \vec{\omega} \times \vec{p}_j &= \left(\sum_{i=1}^{n} \omega_i^p \vec{p}_i \right) \times \vec{p}_j^* \\ &= (\omega_1^p \vec{p}_1 + \omega_2^p \vec{p}_2 + \omega_3^p + \vec{p}_3) \times \vec{p}_j^* \end{aligned}$$

Since the p-frame is orthonormal, $\vec{p}_i^* = \vec{p}_i$:

$$\vec{\omega} \times \vec{p}_j = (\omega_1^p \vec{p}_1 + \omega_2^p \vec{p}_2 + \omega_3^p \vec{p}_3) \times \vec{p}_j$$

This cross-product can be computed directly for $j = 1, 2, 3$:

$$\vec{\omega} \times \vec{p}_1 = -\omega_2^p \vec{p}_3 + \omega_3^p \vec{p}_2$$
$$\vec{\omega} \times \vec{p}_2 = \omega_1^p \vec{p}_3 - \omega_3^p \vec{p}_1$$
$$\vec{\omega} \times \vec{p}_3 = -\omega_1^p \vec{p}_2 + \omega_2^p \vec{p}_1$$

Then, computing the entries of the matrix A^p:

$$a_{ij} = \langle \vec{\omega} \times \vec{p}_j, \vec{p}_i \rangle$$
$$A^p = \begin{bmatrix} 0 & -\omega_3^p & \omega_2^p \\ \omega_3^p & 0 & -\omega_1^p \\ -\omega_2^p & \omega_1^p & 0 \end{bmatrix}$$

This matrix is skew-symmetric and is defined by the coordinates of $\underline{\omega}^p$, so we denote the matrix representation of "$\vec{\omega}\times$" as

$$[\vec{\omega}\times]^p = S(\underline{\omega}^p)$$

$$\vec{y} = \vec{\omega} \times \vec{x} \overset{\text{alg.}}{\iff} \underline{y}^p = S(\underline{\omega}^p)\underline{x}^p$$

Theorem 3. *Given two bases $\{\vec{p}_i\}$ and $\{\vec{q}_i\}$ in $\mathcal{V}$. Let $\vec{r}$ and $\mathbb{A}$ be a vector and a linear operator in $\mathcal{V}$, respectively. We then have the following relations between matrix representations in the two bases/frames:*

$$\underline{r}^q = C_p^q \underline{r}^p \quad \textit{where} \quad C_p^q = [\langle \vec{p}_j, \vec{q}_i^* \rangle]$$

$$\underline{r}^p = C_q^p \underline{r}^q \quad \textit{where} \quad C_q^p = [\langle \vec{q}_j, \vec{p}_i^* \rangle]$$

$$A^q = C_p^q A^p C_q^p \quad \textit{and} \quad A^p = C_q^p A^q C_p^q$$

Proof. Let $\underline{r}^q = C_p^q \underline{r}^p$ where $\underline{r}^q = \begin{bmatrix} r_1^q & \dots & r_n^q \end{bmatrix}^T$ and $\underline{r}^p = \begin{bmatrix} r_1^p & \dots & r_n^p \end{bmatrix}^T$. Then,

$$r_i^q = \langle \vec{r}, \vec{q}_i^* \rangle = \sum_{j=1}^{n} C_{ij} r_j^p$$

Rearranging and expressing $\vec{r}$ in the p-frame:

$$\sum_{j=1}^{n} C_{ij} r_j^p = \left\langle \sum_{j=1}^{n} r_j \vec{p}_j, \vec{q}_i^* \right\rangle$$

By linearity of the inner product, we get

$$\sum_{j=1}^{n} C_{ij} r_j^p = \sum_{j=1}^{n} \langle \vec{p}_j, \vec{q}_i^* \rangle \, r_j^p$$
$$\Longrightarrow C_{ij} = \langle \vec{p}_j, \vec{q}_i^* \rangle$$

thus

$$C_p^q = [\langle \vec{p}_j, \vec{q}_i^* \rangle]$$

The same procedure can be used to find C_q^p.

Now, consider the equation $\vec{y} = \mathbb{A}\vec{x}$ where $\mathbb{A}$ is some linear operator on $\mathcal{V}$. We can coordinatize this equation in the p and q-frames:

$$\underline{y}^q = A^q \underline{x}^q \quad \text{and} \quad \underline{y}^p = A^p \underline{x}^p$$

From above, we also have the following relations

$$\underline{y}^q = C_p^q \underline{y}^p \quad \text{and} \quad \underline{y}^p = C_q^p \underline{y}^q$$

By substitution, we derive:

$$\begin{aligned} A^q \underline{x}^q &= \underline{y}^q \\ &= C_q^p \underline{y}^p \\ &= C_q^p A^p \underline{x}^p \\ &= C_q^p A^p C_q^p \underline{x}^q \end{aligned}$$

And thus

$$A^q = C_q^p A^p C_q^p$$

Again, the same procedure can be used to show that $A^p = C_p^q A^q C_p^q$ $\square$

Theorem 4. *The Direction Cosine Matrix (DCM) between two frames whose bases are orthonormal, R_p^q, is an orthonormal matrix:*

$$(R_p^q)^{-1} = (R_p^q)^T$$

Proof. From the definition of the DCM:

$$R_p^q = \begin{bmatrix} \underline{p}_1{}^q & \underline{p}_2^q & \underline{p}_3^q \end{bmatrix}$$
$$(R_p^q)^T = \begin{bmatrix} (\underline{p}_1^q)^T \\ (\underline{p}_2^q)^T \\ (\underline{p}_3^q)^T \end{bmatrix}$$

Then, we compute $(R_p^q)^T R_p^q$

$$\begin{aligned} (R_p^q)^T R_p^q &= \left[\left\langle \underline{p}_i^q, \underline{p}_j^q \right\rangle \right] = [\delta_{ij}] \\ &= I \end{aligned}$$

thus, $(R_p^q)^{-1} = (R_p^q)^T$ $\square$

Theorem 5. *The matrix representation of the rotation operator $\mathbb{R}_{ab}$ in two frames $\mathcal{F}_{\mathcal{V}}^{a}$ and $\mathcal{F}_{\mathcal{V}}^{b}$ is*

$$[\mathbb{R}_{ab}]^{a} = [\mathbb{R}_{ab}]^{b} = C_{b}^{a}$$

Proof.

$$\begin{aligned}
[\mathbb{R}_{ab}]^{a} = R_{ab}^{a} &= \left[\left\langle \mathbb{R}_{ab}\vec{a}_i, \vec{a}_j^{*}\right\rangle\right] \\
&= \left[\left\langle \vec{b}_i, \vec{a}_j^{*}\right\rangle\right] \\
&= C_{b}^{a}
\end{aligned}$$

Using the "similarity transformation" of the R_{ab}^{a}:

$$\begin{aligned}
R_{ab}^{b} &= C_{a}^{b} R_{ab}^{a} C_{b}^{a} \\
&= C_{a}^{b} C_{b}^{a} C_{b}^{a} \\
&= C_{b}^{a}
\end{aligned}$$

And thus $R_{ab}^{b} = R_{ab}^{a}$

□

Theorem 6. *The derivative of the rotation matrix R_{p}^{q} is given by*

$$\begin{aligned}
\dot{R}_{p}^{q} &= S(\underline{w}_{p}^{qq}) R_{p}^{q} \\
&= R_{p}^{q} S(\underline{w}_{p}^{qp})
\end{aligned}$$

Proof. Since R_{p}^{q} is a rotation matrix (orthonormal), we have

$$\begin{aligned}
&(R_{p}^{q})^{-1} = (R_{p}^{q})^{T} \\
\implies &R_{p}^{q}(R_{p}^{q})^{T} = I
\end{aligned}$$

Taking the derivative on both sides and applying the product rule gives

$$\dot{R}_{p}^{q}(R_{p}^{q})^{T} + R_{p}^{q}(\dot{R}_{p}^{q})^{T} = 0$$

We now define the matrix $S = \dot{R}_{p}^{q}(R_{p}^{q})^{T}$ such that

$$S + S^{T} = 0$$

This means S is some skew-symmetric matrix, with form

$$S(\underline{w}) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

From our definition, we have that $\dot{R}_p^q = S(\underline{\omega})R_p^q$. Finally, we want to find an interpretation of the vector $\underline{\omega}$. Writing this equation out by interpreting R_p^q as an attitude matrix gives

$$\begin{bmatrix}\dot{\underline{p}}_1^q & \dot{\underline{p}}_2^q & \dot{\underline{p}}_3^q\end{bmatrix} = S(\underline{\omega})\begin{bmatrix}\underline{p}_1^q & \underline{p}_2^q & \underline{p}_3^q\end{bmatrix}$$

$$\begin{aligned}\dot{\underline{p}}_i^q &= S(\underline{\omega})\underline{p}_i^q \\ &= \underline{\omega} \times \underline{p}_i^q\end{aligned}$$

We therefore interpret $\underline{\omega}$ as the angular velocity of of the p-frame seen from the q-frame, and represented in the q-frame:

$$\dot{R}_p^q = S(\underline{\omega}_p^{qq})R_p^q$$

Since $S(\underline{\omega}_p^{qq})$ is a linear operator, we can apply express it using the similarity transform:

$$\begin{aligned}S(\underline{\omega}_p^{qq}) &= \left[\vec{\omega}_p^q \times\right]^q \\ &= R_p^q \left[\vec{\omega}_p^q \times\right]^p R_q^p \\ &= R_p^q S(\underline{\omega}_p^{qp}) R_q^p\end{aligned}$$

Inserting this into the equation above gives

$$\begin{aligned}\dot{R}_p^q &= S(\underline{\omega}_p^{qq})R_p^q \\ &= R_p^q S(\underline{\omega}_p^{qp}) \cancel{R_q^p R_p^q} \\ &= R_p^q S(\underline{\omega}_p^{qp})\end{aligned}$$

□

Theorem 7. *The derivative of the DCM C_p^q is given by*

$$\begin{aligned}\dot{C}_p^q &= S(\underline{w}_p^{qq})C_p^q \\ &= C_p^q S(\underline{w}_p^{qp})\end{aligned}$$

Proof.

$$\begin{aligned}C_p^q &= \begin{bmatrix}\underline{p}_1^q & \underline{p}_2^q & \underline{p}_3^q\end{bmatrix} \\ \dot{C}_p^q &= \begin{bmatrix}\dot{\underline{p}}_1^q & \dot{\underline{p}}_2^q & \dot{\underline{p}}_3^q\end{bmatrix}\end{aligned}$$

Here we use the result from the proof above

$$\dot{\underline{p}}_i^q = S(\underline{\omega}_p^{qq})\underline{p}_i$$

$$\dot{C}_p^q = \begin{bmatrix}S(\underline{\omega}_p^{qq})\underline{p}_1^q & S(\underline{\omega}_p^{qq})\underline{p}_2^q & S(\underline{\omega}_p^{qq})\underline{p}_3^q\end{bmatrix}$$

$$\dot{C}_p^q = S(\underline{\omega}_p^{qq})C_p^q$$

Using the similarity transform we also get $\dot{C}_p^q = C_p^q S(\underline{\omega}_p^{qp})$ □

Theorem 8. *The derivative of a vector $\vec{r}$ which is fixed in the rotating frame $\mathcal{F}^p_{\mathcal{V}}(t)$, seen from the (fixed) frame $\mathcal{F}^q_{\mathcal{V}}$ is*

$$\dot{\vec{r}}^q = \vec{\omega}^q_p \times \vec{r}$$

Proof. The vector $\vec{r}$ can be expressed in the p-frame as

$$\vec{r} = \sum_{i=1}^{3} r^p_i \vec{p}_i$$

Where r^p_i are constant, as the vector is fixed in the p-frame. Taking the time-derivative seen from the q-frame:

$$\dot{\vec{r}}^q = \sum_{i=3}^{3} r^p_i \dot{\vec{p}}^q_i$$

From previously, we have that $\dot{\vec{p}}_i = \vec{\omega}^q_p \times \vec{p}_i$, where $\vec{\omega}^q_p$ is the angular velocity of the the p-frame relative to the q-frame. Inserting this and using the fact that "$\vec{\omega}\times$" is linear gives

$$\begin{aligned}
\dot{\vec{r}}^q &= \sum_{i=1}^{3} r^p_i (\vec{\omega}^p_q \times \dot{\vec{p}}^q_i) \\
&= \vec{\omega}^q_p \times \left(\sum_{i=1}^{n} r^p_i \vec{p}^q_i \right) \\
&= \vec{\omega}^q_p \times \vec{r}
\end{aligned}$$

□

Theorem 9. *The derivative of a vector $\vec{r}(t)$ which is time-varying in a rotating frame $\mathcal{F}^p_{\mathcal{V}}(t)$, seen from the (fixed) frame $\mathcal{F}^q_{\mathcal{V}}$ is*

$$\dot{\vec{r}}^q = \vec{\omega}^q_p \times \vec{r}$$

Proof. We again coordinatize $\vec{r}(t)$, now with time-varying coordinates $r^p_i(t)$:

$$\vec{r}(t) = \sum_{i=1}^{n} r^p_i(t) \vec{p}_i(t)$$

Taking the derivative seen from the q-frame and applying the product rule gives

$$\begin{aligned}
\dot{\vec{r}}^q &= \sum_{i=1}^{3} \dot{r_i}^{pp} \vec{p}_i + \sum_{i=1}^{3} r^p_i \dot{\vec{p}}^q_i \\
\dot{\vec{r}}^q &= \dot{\vec{r}}^p + \vec{\omega}^q_p \times \vec{r}
\end{aligned}$$

□

Theorem 10. *Given two frames* $\mathcal{F}_{\mathcal{V}}^{q} = \{O_q; \vec{q}_1, \vec{q}_2, \vec{q}_3\}$ *and* $\mathcal{F}_{\mathcal{V}}^{q} = \{O_p; \vec{p}_1, \vec{p}_2, \vec{p}_3\}$ *and a point P. Let*

$$\begin{aligned}\vec{r} &= P - O_q \\ \vec{\rho} &= P - O_p \\ \vec{r}_{qp} &= O_p - O_q\end{aligned}$$

Then, we have the following relations for the velocity and acceleration seen from the q- and p-systems:

$$\begin{aligned}\vec{r} &= \vec{r}_{qp} + \vec{\rho} \\ \dot{\vec{r}}^{q} &= \dot{\vec{r}}_{qp}^{q} + \dot{\vec{\rho}}^{p} + \vec{\omega}_p^q \times \vec{\rho} \\ \ddot{\vec{r}}^{qq} &= \ddot{\vec{r}}_{qp}^{qq} + \ddot{\vec{\rho}}^{pp} + \dot{\vec{\omega}}_p^{qq} \times \vec{\rho} + \vec{\omega}_p^q \times \left(\vec{\omega}_p^q \times \vec{\rho}\right) + 2\vec{\omega}_p^q \times \dot{\vec{\rho}}^{p}\end{aligned}$$

Proof. The first relation of the position vectors can be computed directly

$$\begin{aligned}\vec{r}_{qp} + \vec{\rho} &= (O_p - O_q) + (P - O_p) \\ \vec{r}_{qp} + \vec{\rho} &= \vec{r}\end{aligned}$$

In this construction, the position vector $\vec{\rho}$ and its derivatives can be considered as in the proof for theorem 9, so we have

$$\dot{\vec{\rho}}^{q} = \dot{\vec{\rho}}^{p} + \vec{\omega}_p^q \times \vec{\rho}$$

We take the q-frame derivative of both sides of the position vector equation, and apply this theorem recursively for $\vec{\rho}$ and its derivatives, as well as the product rule for the cross-product derivatives

$$\begin{aligned}\vec{r} &= \vec{r}_{qp} + \vec{\rho} \\ \dot{\vec{r}}^{q} &= \dot{\vec{r}}_{qp}^{q} + \dot{\vec{\rho}}^{q} \\ &= \underline{\dot{\vec{r}}_{qp}^{q} + \dot{\vec{\rho}}^{p} + \vec{\omega}_p^q \times \vec{\rho}}\end{aligned}$$

$$\begin{aligned}\ddot{\vec{r}}^{qq} &= \frac{{}^{q}d}{dt}\left[\dot{\vec{r}}_{qp}^{q} + \dot{\vec{\rho}}^{p} + \vec{\omega}_p^q \times \vec{\rho}\right] \\ &= \ddot{\vec{r}}_{qp}^{qq} + \ddot{\vec{\rho}}^{pq} + \left\{\dot{\vec{\omega}}_p^{qq} \times \vec{\rho} + \vec{\omega}_p^q \times \dot{\vec{\rho}}^{q}\right\} \\ &= \ddot{\vec{r}}_{qp}^{qq} + \left(\ddot{\vec{\rho}}^{pp} + \vec{\omega}_p^q \times \dot{\vec{\rho}}^{p}\right) + \left\{\dot{\vec{\omega}}_p^{qq} \times \vec{\rho} + \vec{\omega}_p^q \times \left(\dot{\vec{\rho}}^{p} + \vec{\omega}_p^q \times \vec{\rho}\right)\right\} \\ &= \ddot{\vec{r}}_{qp}^{qq} + \ddot{\vec{\rho}}^{pp} + \vec{\omega}_p^q \times \dot{\vec{\rho}}^{p} + \dot{\vec{\omega}}_p^{qq} \times \vec{\rho} + \vec{\omega}_p^q \times \dot{\vec{\rho}}^{p} + \vec{\omega}_p^q \times (\vec{\omega}_p^q \times \vec{\rho}) \\ \ddot{\vec{r}}^{qq} &= \underline{\ddot{\vec{r}}_{qp}^{qq} + \ddot{\vec{\rho}}^{pp} + \dot{\vec{\omega}}_p^{qq} \times \vec{\rho} + \vec{\omega}_p^q \times \left(\vec{\omega}_p^q \times \vec{\rho}\right) + 2\vec{\omega}_p^q \times \dot{\vec{\rho}}^{p}}\end{aligned}$$

$\square$