F10/

Exam is oral

Could all of you that intend to take the exam write me an email at: anders.rodningsby@ffi.no

From F9: Newton's 2. law is valid for all particles (see page 12, Lecture F9)

$$P_i: \quad \vec{f}_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} \vec{f}_{ij} = m_i \vec{a}_i^{\,i} = m_i \ddot{\vec{r}}_i^{\,i} \qquad \text{N.3 } \vec{f}_{ij} = -\vec{f}_{ji}$$

$$P_1: \quad \vec{f}_1 + 0 + \vec{f}_{12} + \vec{f}_{13} + \dots + \vec{f}_{1n} = m_1 \frac{d}{dt}^{i}\left(\frac{d}{dt}^{i} \vec{r}_1\right)$$

$$P_2: \quad \vec{f}_2 + \vec{f}_{21} + 0 + \vec{f}_{23} + \dots + \vec{f}_{2n} = m_2 \frac{d}{dt}^{i}\left(\frac{d}{dt}^{i} \vec{r}_2\right)$$

$$P_3: \quad \vec{f}_3 + \vec{f}_{31} + \vec{f}_{32} + 0 + \dots + \vec{f}_{3n} = m_3 \frac{d}{dt}^{i}\left(\frac{d}{dt}^{i} \vec{r}_3\right)$$

$$\vdots$$

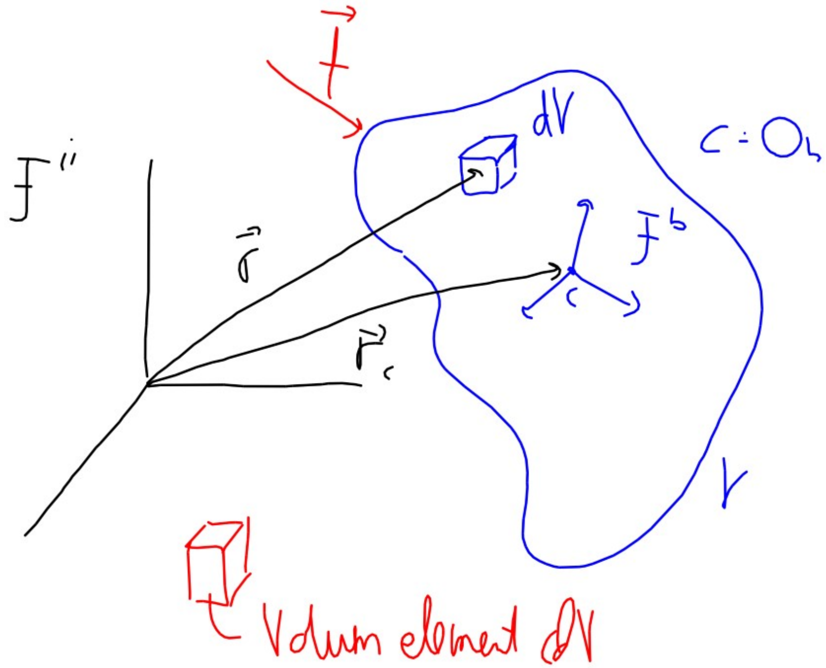
$$P_n: \quad \vec{f}_n + \vec{f}_{n1} + \vec{f}_{n2} + \vec{f}_{n3} + \dots + 0 = m_n \frac{d}{dt}^{i}\left(\frac{d}{dt}^{i} \vec{r}_n\right)$$

$$\sum_{i=1}^{n} P_i: \quad \underbrace{\sum_{i=1}^{n} \vec{f}_i}_{\vec{F}} + \vec{0} \qquad \boxed{\vec{F} = M \vec{a}_c^{\,i}} \qquad \underbrace{\frac{d}{dt}^{i}\left(\frac{d}{dt}^{i} \sum_{i=1}^{n} m_i \vec{r}_i\right)}_{M \ddot{\vec{r}}_c}$$

Since a rigid body can be viewed as a sum of particles (molecules) we have:

$$\boxed{\vec{f} = m\,\vec{a}_c^{\,i}}$$

where $\vec{f}$ is total external force on body, m is the mass and c is center of mass/gravity



$$m = \iiint_V k(\vec{r})\, dV$$ where $k(\vec{r})$ is the mass density

$$\vec{r}_c = \frac{1}{m} \iiint_V \vec{r}\, k(\vec{r})\, dV = \frac{1}{m} \iiint_{z\,y\,x} \vec{r}\, k(\vec{r})\, dx\, dy\, dz$$

B2.2 The law of angular momentum (spinnsatsen)

We want to derive the law of angular momentum of a rigid body that gives the relation between external torque, inertial matrix and angular accelleration.
We do that in 3 steps:

1. The law of angular momentum for one particle
2. ——— " ——— n particles
3. ——— " ——— a rigid body

Teorem B.6 ***Spinnsatsen for en partikkel***
Gitt en partikkel, P, som er utsatt for en kraft, F . La A være et vilkårlig punkt i treghetssystemet **i** *$(\vec{r} = \vec{r}_A + \vec{\rho}_A)$. Da er sammenhengen mellom momentet og spinnet om punktet A:*

(≜ definition)

$$\vec{n}_A = \dot{\vec{h}}^{\mathbf{i}}_A + \vec{\rho}_A \times (m\ddot{\vec{r}}^{\mathbf{i}}_A) \quad \textit{hvor}$$

$$\vec{n}_A \triangleq \vec{\rho}_A \times \vec{F} \tag{B- 141}$$

$$\vec{h}^{\mathbf{i}}_A \triangleq \vec{\rho}_A \times (m\dot{\vec{\rho}}^{\mathbf{i}}_A) \tag{B- 142}$$

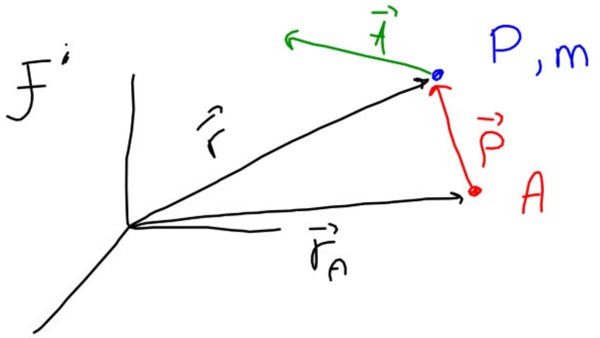
Dersom $\vec{\rho}_A \times (m\ddot{\vec{r}}^{\mathbf{i}}_A) = \vec{0}$, dvs bla når A oppfyller 1 eller 2 :

1). $\ddot{\vec{r}}^{\mathbf{i}}_A = \vec{0}$: A har konstant hastighet i treghetssystemet (ligger f.eks i ro).

2). $\vec{\rho}_A \parallel \ddot{\vec{r}}^{\mathbf{i}}_A$: A akselererer mot/fra partikkel P

så kan spinnsatsen skrives :

$$\boxed{\vec{n}_A = \dot{\vec{h}}^{\mathbf{i}}_A} \tag{B- 143}$$



$$\vec{r} = \vec{r}_A + \vec{\rho}$$

Def: torque around A: $\vec{n}_A = \vec{\rho} \times \vec{f}$

angular mom. around A: $\vec{h}_A^{i} = \vec{\rho} \times (m \dot{\vec{\rho}}^{i})$

NB! Angular momentum and torque may be defined different in other text books!

What is the relation between torque $\vec{n}_A$ and angular momentum $\vec{h}_A^{i}$?

Answer: $$\vec{n}_A = \dot{\vec{h}}_A^{i} + \underbrace{\vec{\rho}_A \times (m \ddot{\vec{r}}_A^{i})}_{\text{When is this 0? How to choos A?}}$$

Proof:

$$\vec{n}_A \triangleq \vec{\rho} \times \vec{f} = \vec{\rho} \times (m \ddot{\vec{r}}^{i}) = \vec{\rho} \times \left(m \frac{d^{i\,2}}{dt^2} (\vec{r}_A + \vec{\rho}) \right)$$

$$= \vec{\rho} \times m \ddot{\vec{r}}_A^{i} + \underbrace{\vec{\rho} \times (m \ddot{\vec{\rho}}^{i})}_{= \dot{\vec{h}}_A^{i} \; ?}$$

$$\vec{h}_A^{i} \triangleq \vec{\rho} \times (m \dot{\vec{\rho}}^{i})$$

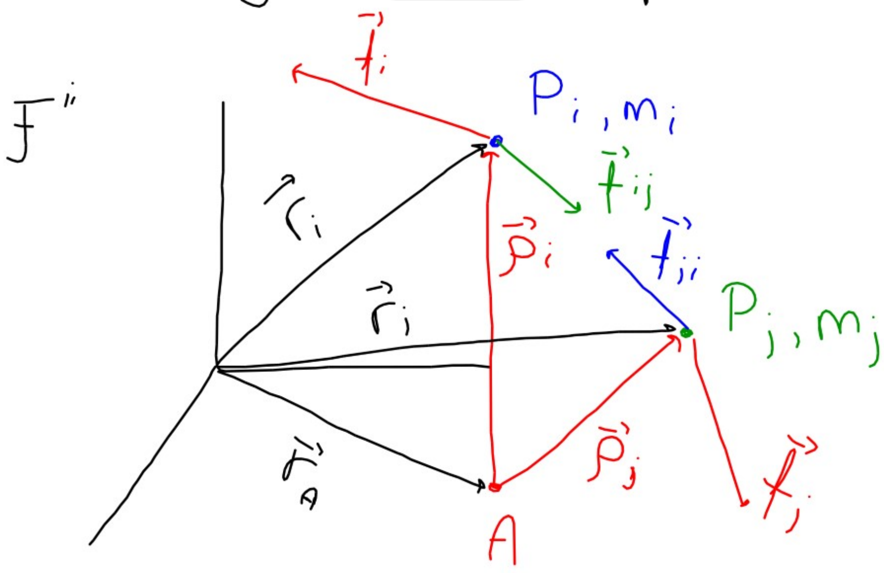
$$\dot{\vec{h}}_A^{i} = \underbrace{\dot{\vec{\rho}}^{i} \times (m \dot{\vec{\rho}}^{i})}_{= \vec{0}} + \vec{\rho} \times (m \ddot{\vec{\rho}}^{i})$$

Yes, they are equal!

$$\Longrightarrow \boxed{\vec{n}_A = \dot{\vec{h}}_A^{i} + \vec{\rho} \times m \ddot{\vec{r}}_A^{i}}$$

q.e.d.

Law of ang. mom. for n-particles



A: arbitrary point

$\mathcal{F}^i$: inertial frame

P_i : particle i

m_i : mass of particle i

$\vec{f}_i$: external force on P_i

$\vec{f}_{ij}$: force on P_i from P_j

Assume : $\vec{f}_{ij} = -\vec{f}_{ji}$ and

$\vec{f}_{ij} \parallel \vec{r}_i - \vec{r}_j = \vec{r}_{ij}$, i.e

Central forces

For P_i we have:

$$\vec{n}_{Ai} = \vec{\rho} \times \left(\vec{f}_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} \vec{f}_{ij} \right)$$

$$\dot{\vec{h}}_{Ai} = \vec{\rho}_i \times \left(m_i \ddot{\vec{\rho}}_i \right)$$

For n-particles:

$$\vec{n}_A = \sum_{i=1}^{n} \vec{n}_{Ai}$$: Total torque around A (= total external torque around A)

$$\dot{\vec{h}}_A = \sum_{i=1}^{n} \dot{\vec{h}}_{Ai}$$: Total angular momentum around A

What is the relation between $\vec{n}_A$ and $\dot{\vec{h}}_A$?

Teorem B.7 ***Spinnsatsen for et n-partikkel system***
Gitt et system av n partikler hvor partikkel i er utsatt for den ytre kraft $\vec{F}_i$ og krafta $\vec{f}_{ij}$ fra partikkel j ($j = 1, \ldots, n$) antas å oppfylle Newtons 3.lov ($\vec{f}_{ij} = -\vec{f}_{ji}$) og i tillegg være en sentralkraft (ligger langs $\vec{r}_i - \vec{r}_j$). Da vil for et vilkårlig punkt A i treghetssystemet $\mathbf{i}$*:*

Arbitrary A

$$\boxed{\vec{n}_A = \dot{\vec{h}}_A^{\mathbf{i}} + \sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \times \ddot{\vec{r}}_A^{\mathbf{i}} \quad \textit{hvor}} \tag{B- 144}$$

$$\vec{n}_A \stackrel{\triangle}{=} \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \underset{\vec{f}_i}{\vec{F}_i} \quad \textit{totalt ytre moment om A} \tag{B- 145}$$

$$\vec{h}_A \stackrel{\triangle}{=} \sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \times \dot{\vec{\rho}}_{Ai}^{\mathbf{i}} \quad \textit{totalt spinn om A} \tag{B- 146}$$

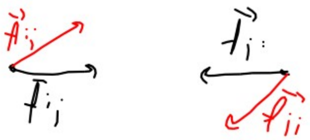
Dersom $\sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \times \ddot{\vec{r}}_A^{\mathbf{i}} = 0$*, dvs bl.a. når A oppfyller 1, 2 eller 3 :*
1). $\sum_{i=1}^{n} m_i \vec{\rho}_{Ai} = \vec{0}$ *: A er i massesenteret.* normal to do.
2). $\ddot{\vec{r}}_A^{\mathbf{i}} = \vec{0}$ *: A har konstant hastighet i treghetsrommet (f.eks. i ro).*
3). $\sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \parallel \ddot{\vec{r}}_A^{\mathbf{i}}$*: A akselererer mot massesenteret.*
så kan spinnsatset for et n-partikkel system skrives :

N.2

$$\boxed{\vec{n}_A = \dot{\vec{h}}_A^{\mathbf{i}}} \tag{B- 147}$$

$\vec{f} = m\vec{a}_c^i = \dot{\vec{p}}_c^i$

$\vec{f}_{ij} = -\vec{f}_{ji}$ are central forces



$\vec{f}_{ij} = -\vec{f}_{ji}$ are not central forces

<u>Proof</u>

Summariz $\dot{\vec{h}}_{Ai}$ over all i and do the same for $\vec{n}_{Ai}$.

Use Newton's 2. law and see that all cross-terms are removed because $\vec{f}_{ij} + \vec{f}_{ji} = \vec{0}$ and they are central forces.

$$\vec{n}_A = \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \left(\vec{f}_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} \vec{f}_{ij} \right)$$

$$\begin{aligned} = \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \vec{f}_i \quad & \underbrace{\begin{aligned} &+ \vec{\rho}_{A1} \times \left(0 + \vec{f}_{12} + \dots + \vec{f}_{1n} \right) \\ &+ \vec{\rho}_{A2} \times \left(\vec{f}_{21} + 0 + \dots + \vec{f}_{2n} \right) \\ &\ \ \vdots \\ &+ \vec{\rho}_{An} \times \left(\vec{f}_{n1} + \vec{f}_{n2} + \dots + 0 \right) \end{aligned}}_{= \vec{0}} \end{aligned}$$

$$\boxed{\vec{n}_A = \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \vec{f}_i}$$

$= \vec{0}$ because:

$$\vec{\rho}_{Ai} \times \vec{f}_{ij} + \vec{\rho}_{Aj} \times \vec{f}_{ji}$$

$$= \vec{\rho}_{Ai} \times \vec{f}_{ij} - \vec{\rho}_{Aj} \times \vec{f}_{ij}$$

$$= \left(\vec{\rho}_{Ai} - \vec{\rho}_{Aj} \right) \times \vec{f}_{ij}$$

$$= \vec{0}$$

since $\vec{\rho}_{Ai} - \vec{\rho}_{Aj} \parallel \vec{f}_{ij}$

(central forces)

$$\vec{n}_A = \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \vec{f}_i = \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \left(m_i \ddot{\vec{r}}_i^{\,i}\right) = \sum_{i=1}^{n} \vec{\rho}_{Ai} \times m_i \left(\frac{d^{i\,2}}{dt^2}\left(\vec{r}_i\right)\right)$$

$$= \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \left(m_i \frac{d^{i\,2}}{dt^2}\left(\vec{r}_A + \vec{\rho}_{Ai}\right)\right)$$

$$= \sum_{i=1}^{n} \vec{\rho}_{Ai} \times m_i \ddot{\vec{r}}_A^{\,i} + \underbrace{\sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \times \ddot{\vec{\rho}}_{Ai}^{\,i}}_{\dot{\vec{h}}_A^{\,i}\ ?}$$

Yes, they are!

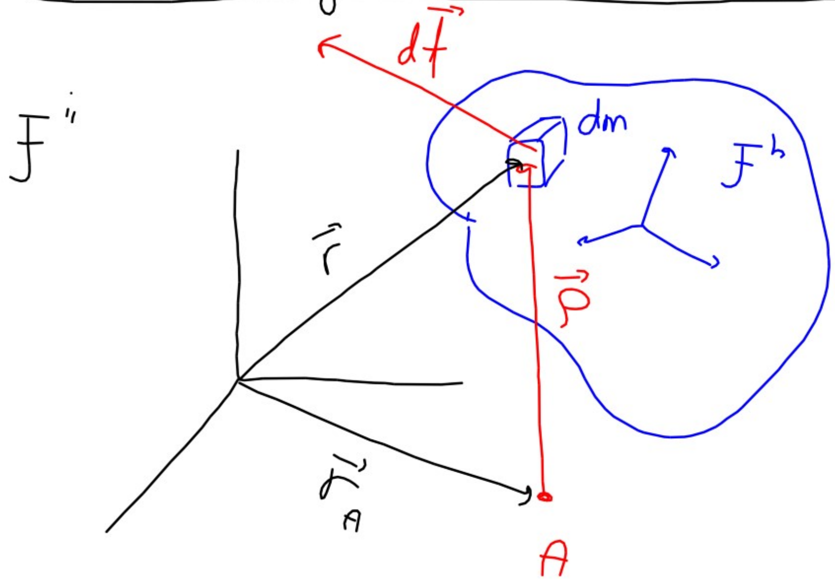
$$\dot{\vec{h}}_A^{\,i} = \frac{d^{i}}{dt}\left(\sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \times \dot{\vec{\rho}}_{Ai}^{\,i}\right)$$

$$= \underbrace{\sum_{i=1}^{n} m_i \dot{\vec{\rho}}_{Ai}^{\,i} \times \dot{\vec{\rho}}_{Ai}^{\,i}}_{=\vec{0}\ \text{since parallell vectors}} + \sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \times \ddot{\vec{\rho}}_{Ai}^{\,i}$$

$$\Rightarrow \boxed{\vec{n}_A = \dot{\vec{h}}_A^{\,i} + \sum_{i=1}^{n} \vec{\rho}_{Ai} \times m_i \ddot{\vec{r}}_A^{\,i}}$$

q.e.d.

Law of angular momentum of a rigid body



$\mathcal{F}^i$: inertial frame

$\mathcal{F}^b$: body frame (fixed to the body)

$d\vec{f}$: external force acting on the mass element $dm = \rho(\vec{r})\, dV$ where $\rho(\vec{r})$ is the mass density

b - body