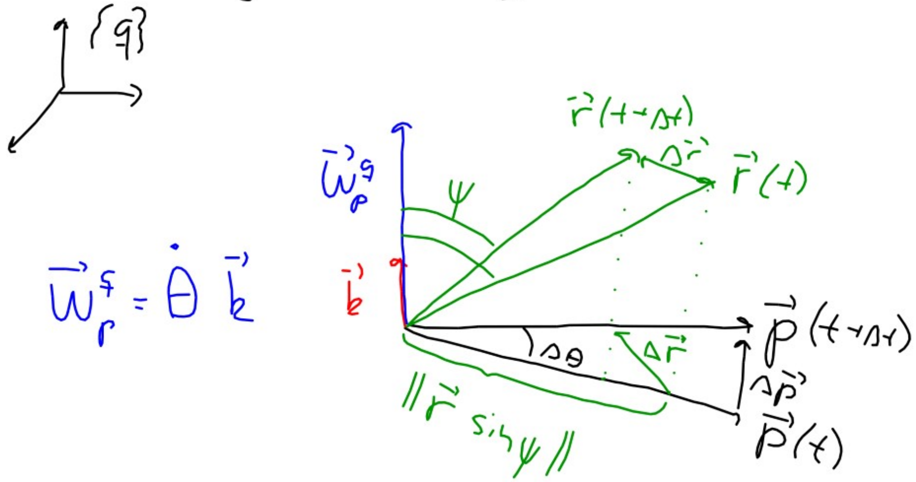
F8/ Angular velocity of rotation for vectors with constant length



* $\vec{r}$ is fixed to $\{\vec{p}\}$
* $\|\vec{r}\|$ is constant
* $\{p\}$ rotates relative to $\{q\}$ with $\vec{\omega}_p^q$
* $\|\vec{k}\| = 1$

$$\|\Delta\vec{r}\| = \|\vec{r}\| \sin\psi\, \Delta\theta$$

The direction of $\Delta\vec{r}$ shall be normal $\perp$ at both $\vec{r}$ and $\vec{k}$, and have a direction given by the r.h.r. for rotation around $\vec{k}$. With unit length this becomes:

$$\frac{\vec{k}\times\vec{r}}{\|\vec{k}\times\vec{r}\|} \text{ where } \|\vec{k}\times\vec{r}\| = \|\vec{k}\|\,\|\vec{r}\| \sin\psi \Rightarrow \text{Unit length: } \frac{\vec{k}\times\vec{r}}{\|\vec{r}\|\sin\psi}$$

$$\Rightarrow \Delta\vec{r} = \Delta\theta \sin\psi\, \|\vec{r}\| \frac{\vec{k}\times\vec{r}}{\|\vec{r}\|\sin\psi} = \Delta\theta\, \vec{k}\times\vec{r} \Rightarrow \boxed{\dot{\vec{r}}^{\,q} = \vec{\omega}_p^q \times \vec{r}}$$

The kinematic problem: Given $\vec{w}_p^q$, what is the differential equation of the attitude matrix or special representations of the attitude matrix?

1. If $R_p^q = [\underline{p}_1^q, \underline{p}_2^q, \underline{p}_3^q]$

$$\boxed{\dot{R}_p^q = S(\underline{w}_p^q)\, R_p^q}$$

2. If $R_p^q = R_3(\theta_3)\, R_2(\theta_2)\, R_1(\theta_1)$: 3-2-1 Euler angles

$$\boxed{\begin{aligned} \dot{\underline{\Theta}} &= D_p^\Theta(\underline{\Theta})\, \underline{w}_p^{qp} \\ &= D_q^\Theta(\underline{\Theta})\, \underline{w}_p^q \end{aligned}}$$

$$\underline{\Theta} = \begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{pmatrix}$$

(see A-105 and A-106 in the note/report)

We solve d.e (differential equations) using numerical methods:

Euler's 1. order method:

SCALAR

a) $\dot{x}(t) = f(x(t))$, $x(t_0)$ given

$$\frac{\Delta x}{\Delta t} = \frac{x(t_{k+1}) - x(t_k)}{\Delta t} = f(x(t_k)) \quad , \quad \Delta t = t_{k+1} - t_k$$

$$x(t_{k+1}) = x(t_k) + f(x(t_k))\,\Delta t \quad , \quad t_k = k \cdot \Delta t \quad , \quad x(t_k) = x_k$$

$$\boxed{x_{k+1} = x_k + \Delta t\, f(x_k) \quad , \quad x_0 \text{ given}}$$

VECTOR

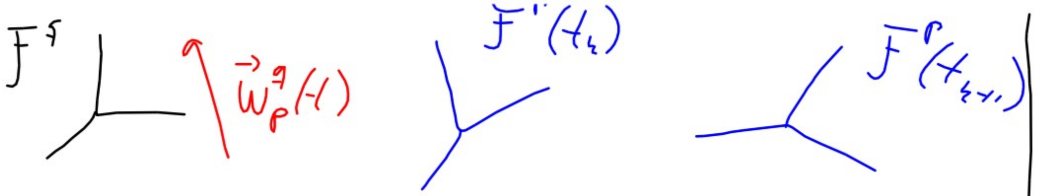
b) $\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t))$, $\underline{x}(t_0)$ given

$$\boxed{\underline{x}_{k+1} = \underline{x}_k + \Delta t\, \underline{f}(\underline{x}_k) \quad , \quad \underline{x}_0 \text{ given}}$$

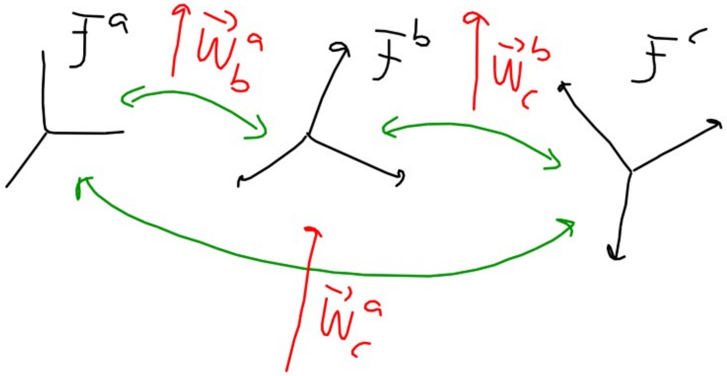
MATRIX

c) $\dot{R}(t) = S(\underline{w}(t))R(t)$, $R(t_0)$ given

$$\boxed{R_{k+1} = R_k + \Delta t\, S(\underline{w}_k)R_k \quad , \quad R_0 \text{ given}}$$



Angular velocity and ang. accelleration in different basis systems



$$\vec{w}_c^{a} = \vec{w}_b^{a} + \vec{w}_c^{b}$$

Derivative seen from F^a

$$\dot{\vec{w}}_c^{aa} = \dot{\vec{w}}_b^{aa} + \dot{\vec{w}}_c^{ba}$$

What is $\dot{\vec{w}}_c^{ba}$ expressed by $\dot{\vec{w}}_c^{bb}$

$$\vec{w}_c^{b} = \sum w_{ci}^{bb} \vec{b}_i$$

$$\frac{d^a}{dt}\vec{w}_c^{b} = \frac{d^a}{dt}\left(\sum w_{ci}^{bb} \vec{b}_i\right)$$

$$= \sum \dot{w}_{ci}^{bbb} \vec{b}_i + \underbrace{\sum w_{ci}^{bb} \vec{w}_b^{a} \times \vec{b}_i}_{\vec{w}_b^{a} \times \sum w_{ci}^{bb} \vec{b}_i}$$

$$\dot{\vec{w}}_c^{ba} = \dot{\vec{w}}_c^{bb} + \vec{w}_b^{a} \times \vec{w}_c^{b}$$

$$\boxed{\dot{\vec{\omega}}_c^{aa} = \dot{\vec{\omega}}_b^{aa} + \dot{\vec{\omega}}_c^{bb} + \vec{\omega}_b^{a} \times \vec{\omega}_c^{b}}$$

Natural to represent the derivative (actually 2. derivative) in the same frame as the 1. derivative.

Angular velocities and their derivatives in the case we have algebraic vectors can either be found by representing the geometrical equations in desirable frames or by differensiating directly:

$$\underline{\omega}_c^{aa} = \underline{\omega}_b^{aa} + \underline{\omega}_c^{ba}$$

$$= \underline{\omega}_b^{aa} + R_b^a \, \underline{\omega}_c^{bb}$$

: It is better to both represent and find derivative in the same frame.

We had:

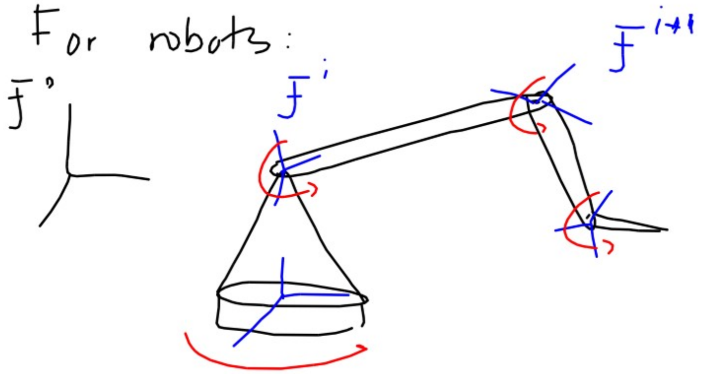
$$\underline{\omega}_c^{aa} = \underline{\omega}_b^{aa} + R_b^a \underline{\omega}_c^{bb}$$

Take derivative:

$$\boxed{\dot{\underline{\omega}}_c^{aaa} = \dot{\underline{\omega}}_b^{aaa} + S(\underline{\omega}_b^{aa}) R_b^a \underline{\omega}_c^{bb} + R_b^a \dot{\underline{\omega}}_c^{bbb}}$$

$$= \dot{\underline{\omega}}_b^{aaa} + R_b^a S(\underbrace{\underline{\omega}_b^{ab}}_{R_a^b \underline{\omega}_b^{aa}}) \underline{\omega}_c^{bb} + R_b^a \dot{\underline{\omega}}_c^{bbb}$$

For robots:



$\underline{\omega}_i^{0,i}$: angular velocity of link i (frame i) seen from link 0 (frame 0) (inertial space), represented in link i (frame i)

$$\underline{\omega}_{i+1}^{0,i+1} = \underline{\omega}_i^{0,i+1} + \underline{\omega}_{i+1}^{i,i+1}$$

$$\boxed{\underline{\omega}_{i+1}^{0,i+1} = R_i^{i+1}\,\underline{\omega}_i^{0,i} + \underline{\omega}_{i+1}^{i,i+1}}$$

$$\boxed{\dot{\underline{\omega}}_{i+1}^{0,i+1,i+1} = S(\underline{\omega}_i^{i+1,i+1})\,R_i^{i+1}\,\underline{\omega}_i^{0,i} + R_i^{i+1}\,\dot{\underline{\omega}}_i^{0,i,i} + \dot{\underline{\omega}}_{i+1}^{i,i+1,i+1}}$$

Theorem A.18 Derivative of angular velocity

$$\boxed{\overset{\circ}{\underline{\omega}}{}_b^{abb} = R_a^b \, \dot{\underline{\omega}}{}_b^{aaa}}$$

$\Rightarrow$ Angular accelleration transform in the same way as angular velocities

Proof:

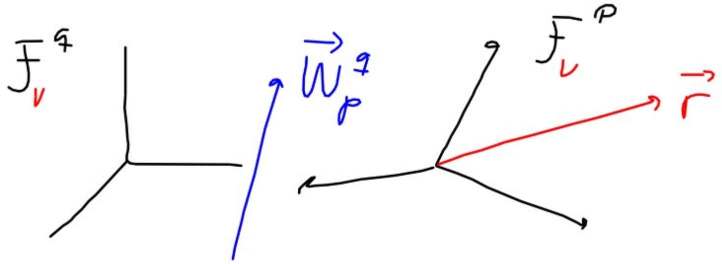
$$\underline{\omega}_b^{ab} = R_a^b \underline{\omega}_b^{aa}$$

$$\overset{\circ}{\underline{\omega}}{}_b^{abb} = \underbrace{\dot{R}_a^b \underline{\omega}_b^{aa}}_{0?} + R_a^b \dot{\underline{\omega}}{}_b^{aaa}$$

$$\dot{R}_a^b \underline{\omega}_b^{aa} = S(\underline{\omega}_a^{bb}) R_a^b \underline{\omega}_b^{aa} = S(\underline{\omega}_a^{bb}) \underline{\omega}_b^{ab} = -\underline{\omega}_a^{bb} \times \underline{\omega}_a^{bb} = 0$$

$$\vec{\omega}_a^b = -\vec{\omega}_b^a$$

A 5.3 Derivative of vectors



Assume $\vec{r}$ has constant length and is fixed relative to $\mathcal{F}_v^p$.

We have showed that:

$$\boxed{\dot{\vec{r}}^q = \vec{\omega}_p^q \times \vec{r}}$$

Assume $\vec{r}$ is changed seen from $\mathcal{F}_v^p$

$$\vec{r} = \sum r_i^p \vec{p}_i$$

$$\dot{\vec{r}}^q = \sum \dot{r}_i^p \vec{p}_i + \sum r_i^p \dot{\vec{p}}_i^q$$

$$= \sum \dot{r}_i^p \vec{p}_i + \sum r_i^p \vec{\omega}_p^q \times \vec{p}_i$$

$\vec{p}_i$ has constant length

$$\boxed{\dot{\vec{r}}^q = \dot{\vec{r}}^p + \vec{\omega}_p^q \times \vec{r}}$$

For algebraic vectors the corresponding equations are:

Constant vector seen from $\mathcal{F}^p$:

$$\underline{r}^q = R^q_p \underline{r}^p$$

$$\boxed{\dot{\underline{r}}^{qq} = S(\underline{\omega}^q_p) R^q_p \underline{r}^p = S(\underline{\omega}^q_p) \underline{r}^q} \quad = \underline{\omega}^q_p \times \underline{r}^q$$

$$(\dot{\underline{r}}^p = \underline{0})$$

Time varying vector seen from $\mathcal{F}^p$

$$\boxed{\dot{\underline{r}}^{qq} = S(\underline{\omega}^q_p) R^q_p \underline{r}^p + R^q_p \dot{\underline{r}}^{pp}}$$