

**Theorem 1.** Given a basis  $\{\vec{p}_i\}$  with dual basis  $\{\vec{p}_i^*\}$ , any vector  $\vec{r}$  can be written as:

$$\vec{r} = \sum_{i=1}^n r_i^p \vec{p}_i, \quad r_i^p = \langle \vec{r}, \vec{p}_i^* \rangle$$

*Proof.* Since  $\{\vec{p}_i\}$  forms a basis, we can write  $\vec{r}$  as:

$$\vec{r} = \sum_{i=1}^n r_i^p \vec{p}_i$$

Now, consider the inner product of an arbitrary vector  $\vec{r}$  and  $\vec{p}_i^*$ :

$$\begin{aligned} \langle \vec{r}, \vec{p}_i^* \rangle &= \left\langle \sum_{j=1}^n r_j^p \vec{p}_j, \vec{p}_i^* \right\rangle \\ &= \sum_{j=1}^n r_j^p \langle \vec{p}_j, \vec{p}_i^* \rangle \\ &= \sum_{j=1}^n r_j^p \delta_{ij} = r_i^p \end{aligned}$$

Thus:

$$r_i^p = \langle \vec{r}, \vec{p}_i^* \rangle$$

□

**Theorem 2.** Given the basis  $\{\vec{p}_i\}$  for the vector space  $\mathcal{V}$ , any linear operator  $\mathbb{A}$  can be represented in  $\mathbb{R}^{n \times n}$  as the matrix  $A^p$ :

$$[\mathbb{A}]^p = A^p = [\langle \mathbb{A} \vec{p}_j, \vec{p}_i^* \rangle]$$

$$\vec{y} = \mathbb{A} \vec{x} \xLeftrightarrow{\text{alg.}} \underline{y}^p = A^p \underline{x}^p$$

*Proof.* Let  $\mathbb{A}$  be a linear operator on  $\mathcal{V}$ , with the basis  $\{\vec{p}_i\}$  and dual basis  $\{\vec{p}_i^*\}$ . Then, let  $\vec{y} = \mathbb{A} \vec{x}$ . Again, we express  $\vec{y}$  and  $\vec{x}$  in the p-frame:

$$\vec{y} = \sum_{i=1}^n y_i^p \vec{p}_i, \quad \vec{x} = \sum_{i=1}^n x_i^p \vec{p}_i$$

From theorem 1, we have:

$$\begin{aligned} y_i^p &= \langle \vec{y}, \vec{p}_i^* \rangle \\ &= \langle \mathbb{A} \vec{x}, \vec{p}_i^* \rangle \end{aligned}$$

Expressing  $\vec{x}$  in the p-frame and using the linearity of  $\mathbb{A}$ , we get

$$\begin{aligned} &= \left\langle \mathbb{A} \sum_{j=1}^n x_j^p \vec{p}_j, \vec{p}_i^* \right\rangle \\ &= \left\langle \sum_{j=1}^n x_j^p \mathbb{A} \vec{p}_j, \vec{p}_i^* \right\rangle \\ y_i^p &= \sum_{j=1}^n x_j^p \langle \mathbb{A} \vec{p}_j, \vec{p}_i^* \rangle \end{aligned}$$

This can be written as the following matrix equation

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \langle \mathbb{A} \vec{p}_1, \vec{p}_1^* \rangle & \dots & \langle \mathbb{A} \vec{p}_n, \vec{p}_1^* \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbb{A} \vec{p}_1, \vec{p}_n^* \rangle & \dots & \langle \mathbb{A} \vec{p}_n, \vec{p}_n^* \rangle \end{bmatrix} \begin{bmatrix} x_1^p \\ \vdots \\ x_n^p \end{bmatrix}$$

or:

$$\underline{y}^p = A^p \underline{x}^p$$

where

$$A^p = [\langle \mathbb{A} \vec{p}_j, \vec{p}_i^* \rangle]$$

□

**Problem 1.** Find the matrix representation of the " $\vec{\omega} \times$ " operator in the orthonormal frame  $\{\vec{p}_i\}$

**Sol.**

To find the matrix representation we use Theorem 2:

$$\begin{aligned} A^p &= [\langle \mathbb{A} \vec{p}_j, \vec{p}_i^* \rangle] \\ &= [\langle \vec{\omega} \times \vec{p}_j, \vec{p}_i^* \rangle] \end{aligned}$$

First we consider  $\vec{\omega} \times \vec{p}_j$  by expressing  $\vec{\omega}$  in the p-frame:

$$\begin{aligned} \vec{\omega} \times \vec{p}_j &= \left( \sum_{i=1}^n \omega_i^p \vec{p}_i \right) \times \vec{p}_j^* \\ &= (\omega_1^p \vec{p}_1 + \omega_2^p \vec{p}_2 + \omega_3^p \vec{p}_3) \times \vec{p}_j^* \end{aligned}$$

Since the p-frame is orthonormal,  $\vec{p}_i^* = \vec{p}_i$ :

$$\vec{\omega} \times \vec{p}_j = (\omega_1^p \vec{p}_1 + \omega_2^p \vec{p}_2 + \omega_3^p \vec{p}_3) \times \vec{p}_j$$

This cross-product can be computed directly for  $j = 1, 2, 3$ :

$$\begin{aligned}\vec{\omega} \times \vec{p}_1 &= -\omega_2^p \vec{p}_3 + \omega_3^p \vec{p}_2 \\ \vec{\omega} \times \vec{p}_2 &= \omega_1^p \vec{p}_3 - \omega_3^p \vec{p}_1 \\ \vec{\omega} \times \vec{p}_3 &= -\omega_1^p \vec{p}_2 + \omega_2^p \vec{p}_1\end{aligned}$$

Then, computing the entries of the matrix  $A^p$ :

$$\begin{aligned}a_{ij} &= \langle \vec{\omega} \times \vec{p}_j, \vec{p}_i \rangle \\ A^p &= \begin{bmatrix} 0 & -\omega_3^p & \omega_2^p \\ \omega_3^p & 0 & -\omega_1^p \\ -\omega_2^p & \omega_1^p & 0 \end{bmatrix}\end{aligned}$$

This matrix is skew-symmetric and is defined by the coordinates of  $\underline{\omega}^p$ , so we denote the matrix representation of " $\vec{\omega} \times$ " as

$$[\vec{\omega} \times]^p = S(\underline{\omega}^p)$$

$$\vec{y} = \vec{\omega} \times \vec{x} \xLeftrightarrow{\text{alg.}} \underline{y}^p = S(\underline{\omega}^p) \underline{x}^p$$

**Theorem 3.** *Given two bases  $\{\vec{p}_i\}$  and  $\{\vec{q}_i\}$  in  $\mathcal{V}$ . Let  $\vec{r}$  and  $\mathbb{A}$  be a vector and a linear operator in  $\mathcal{V}$ , respectively. We then have the following relations between matrix representations in the two bases/frames:*

$$\underline{r}^q = C_p^q \underline{r}^p \quad \text{where} \quad C_p^q = [\langle \vec{p}_j, \vec{q}_i^* \rangle]$$

$$\underline{r}^p = C_q^p \underline{r}^q \quad \text{where} \quad C_q^p = [\langle \vec{q}_j, \vec{p}_i^* \rangle]$$

$$A^q = C_p^q A^p C_q^p \quad \text{and} \quad A^p = C_q^p A^q C_p^q$$

*Proof.* Let  $\underline{r}^q = C_p^q \underline{r}^p$  where  $\underline{r}^q = [r_1^q \ \dots \ r_n^q]^T$  and  $\underline{r}^p = [r_1^p \ \dots \ r_n^p]^T$ . Then,

$$r_i^q = \langle \vec{r}, \vec{q}_i^* \rangle = \sum_{j=1}^n C_{ij} r_j^p$$

Rearranging and expressing  $\vec{r}$  in the p-frame:

$$\sum_{j=1}^n C_{ij} r_j^p = \left\langle \sum_{j=1}^n r_j \vec{p}_j, \vec{q}_i^* \right\rangle$$

By linearity of the inner product, we get

$$\begin{aligned}\sum_{j=1}^n C_{ij} r_j^p &= \sum_{j=1}^n \langle \vec{p}_j, \vec{q}_i^* \rangle r_j^p \\ \implies C_{ij} &= \langle \vec{p}_j, \vec{q}_i^* \rangle\end{aligned}$$

thus

$$C_p^q = [\langle \vec{p}_j, \vec{q}_i^* \rangle]$$

The same procedure can be used to find  $C_q^p$ .

Now, consider the equation  $\vec{y} = \mathbb{A}\vec{x}$  where  $\mathbb{A}$  is some linear operator on  $\mathcal{V}$ . We can coordinatize this equation in the p and q-frames:

$$\underline{y}^q = A^q \underline{x}^q \quad \text{and} \quad \underline{y}^p = A^p \underline{x}^p$$

From above, we also have the following relations

$$\underline{y}^q = C_p^q \underline{y}^p \quad \text{and} \quad \underline{y}^p = C_q^p \underline{y}^q$$

By substitution, we derive:

$$\begin{aligned}A^q \underline{x}^q &= \underline{y}^q \\ &= C_q^p \underline{y}^p \\ &= C_q^p A^p \underline{x}^p \\ &= C_q^p A^p C_q^p \underline{x}^q\end{aligned}$$

And thus

$$A^q = C_q^p A^p C_q^p$$

Again, the same procedure can be used to show that  $A^p = C_p^q A^q C_p^q$   $\square$

**Theorem 4.** *The Direction Cosine Matrix (DCM) between two frames whose bases are orthonormal,  $R_p^q$ , is an orthonormal matrix:*

$$(R_p^q)^{-1} = (R_p^q)^T$$

*Proof.* From the definition of the DCM:

$$\begin{aligned}R_p^q &= [\underline{p}_1^q \quad \underline{p}_2^q \quad \underline{p}_3^q] \\ (R_p^q)^T &= \begin{bmatrix} (\underline{p}_1^q)^T \\ (\underline{p}_2^q)^T \\ (\underline{p}_3^q)^T \end{bmatrix}\end{aligned}$$

Then, we compute  $(R_p^q)^T R_p^q$

$$\begin{aligned}(R_p^q)^T R_p^q &= \left[ \left\langle \underline{p}_i^q, \underline{p}_j^q \right\rangle \right] = [\delta_{ij}] \\ &= I\end{aligned}$$

thus,  $(R_p^q)^{-1} = (R_p^q)^T$   $\square$

**Theorem 5.** *The matrix representation of the rotation operator  $\mathbb{R}_{ab}$  in two frames  $\mathcal{F}_{\mathcal{V}}^a$  and  $\mathcal{F}_{\mathcal{V}}^b$  is*

$$[\mathbb{R}_{ab}]^a = [\mathbb{R}_{ab}]^b = C_b^a$$

*Proof.*

$$\begin{aligned} [\mathbb{R}_{ab}]^a &= R_{ab}^a = [\langle \mathbb{R}_{ab} \vec{a}_i, \vec{a}_j^* \rangle] \\ &= [\langle \vec{b}_i, \vec{a}_j^* \rangle] \\ &= C_b^a \end{aligned}$$

Using the "similarity transformation" of the  $R_{ab}^a$ :

$$\begin{aligned} R_{ab}^b &= C_a^b R_{ab}^a C_b^a \\ &= C_a^b C_b^a C_b^a \\ &= C_b^a \end{aligned}$$

And thus  $R_{ab}^b = R_{ab}^a$

□

**Theorem 6.** *The derivative of the rotation matrix  $R_p^q$  is given by*

$$\begin{aligned} \dot{R}_p^q &= S(\underline{w}_p^{qq}) R_p^q \\ &= R_p^q S(\underline{w}_p^{qp}) \end{aligned}$$

*Proof.* Since  $R_p^q$  is a rotation matrix (orthonormal), we have

$$\begin{aligned} (R_p^q)^{-1} &= (R_p^q)^T \\ \implies R_p^q (R_p^q)^T &= I \end{aligned}$$

Taking the derivative on both sides and applying the product rule gives

$$\dot{R}_p^q (R_p^q)^T + R_p^q (\dot{R}_p^q)^T = 0$$

We now define the matrix  $S = \dot{R}_p^q (R_p^q)^T$  such that

$$S + S^T = 0$$

This means S is some skew-symmetric matrix, with form

$$S(\underline{w}) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

From our definition, we have that  $\dot{R}_p^q = S(\underline{\omega})R_p^q$ . Finally, we want to find an interpretation of the vector  $\underline{\omega}$ . Writing this equation out by interpreting  $R_p^q$  as an attitude matrix gives

$$\begin{bmatrix} \dot{p}_1^q & \dot{p}_2^q & \dot{p}_3^q \end{bmatrix} = S(\underline{\omega}) \begin{bmatrix} p_1^q & p_2^q & p_3^q \end{bmatrix}$$

$$\begin{aligned} \dot{p}_i^q &= S(\underline{\omega})p_i^q \\ &= \underline{\omega} \times p_i^q \end{aligned}$$

We therefore interpret  $\underline{\omega}$  as the angular velocity of the p-frame seen from the q-frame, and represented in the q-frame:

$$\dot{R}_p^q = S(\underline{\omega}_p^{qq})R_p^q$$

Since  $S(\underline{\omega}_p^{qq})$  is a linear operator, we can apply express it using the similarity transform:

$$\begin{aligned} S(\underline{\omega}_p^{qq}) &= [\vec{\omega}_p^q \times]^q \\ &= R_p^q [\vec{\omega}_p^q \times]^p R_q^p \\ &= R_p^q S(\underline{\omega}_p^{qp}) R_q^p \end{aligned}$$

Inserting this into the equation above gives

$$\begin{aligned} \dot{R}_p^q &= S(\underline{\omega}_p^{qq})R_p^q \\ &= R_p^q S(\underline{\omega}_p^{qp}) R_q^p R_q^p \\ &= R_p^q S(\underline{\omega}_p^{qp}) \end{aligned}$$

□

**Theorem 7.** *The derivative of the DCM  $C_p^q$  is given by*

$$\begin{aligned} \dot{C}_p^q &= S(\underline{\omega}_p^{qq})C_p^q \\ &= C_p^q S(\underline{\omega}_p^{qp}) \end{aligned}$$

*Proof.*

$$\begin{aligned} C_p^q &= \begin{bmatrix} p_1^q & p_2^q & p_3^q \end{bmatrix} \\ \dot{C}_p^q &= \begin{bmatrix} \dot{p}_1^q & \dot{p}_2^q & \dot{p}_3^q \end{bmatrix} \end{aligned}$$

Here we use the result from the proof above

$$\begin{aligned} \dot{p}_i^q &= S(\underline{\omega}_p^{qq})p_i^q \\ \dot{C}_p^q &= \begin{bmatrix} S(\underline{\omega}_p^{qq})p_1^q & S(\underline{\omega}_p^{qq})p_2^q & S(\underline{\omega}_p^{qq})p_3^q \end{bmatrix} \\ \dot{C}_p^q &= S(\underline{\omega}_p^{qq})C_p^q \end{aligned}$$

Using the similarity transform we also get  $\dot{C}_p^q = C_p^q S(\underline{\omega}_p^{qp})$

□

**Theorem 8.** *The derivative of a vector  $\vec{r}$  which is fixed in the rotating frame  $\mathcal{F}_V^p(t)$ , seen from the (fixed) frame  $\mathcal{F}_V^q$  is*

$$\dot{\vec{r}}^q = \vec{\omega}_p^q \times \vec{r}$$

*Proof.* The vector  $\vec{r}$  can be expressed in the p-frame as

$$\vec{r} = \sum_{i=1}^3 r_i^p \vec{p}_i$$

Where  $r_i^p$  are constant, as the vector is fixed in the p-frame. Taking the time-derivative seen from the q-frame:

$$\dot{\vec{r}}^q = \sum_{i=1}^3 r_i^p \dot{\vec{p}}_i^q$$

From previously, we have that  $\dot{\vec{p}}_i^q = \vec{\omega}_p^q \times \vec{p}_i^q$ , where  $\vec{\omega}_p^q$  is the angular velocity of the p-frame relative to the q-frame. Inserting this and using the fact that " $\vec{\omega} \times$ " is linear gives

$$\begin{aligned} \dot{\vec{r}}^q &= \sum_{i=1}^3 r_i^p (\vec{\omega}_p^q \times \vec{p}_i^q) \\ &= \vec{\omega}_p^q \times \left( \sum_{i=1}^3 r_i^p \vec{p}_i^q \right) \\ &= \vec{\omega}_p^q \times \vec{r} \end{aligned}$$

□

**Theorem 9.** *The derivative of a vector  $\vec{r}$  which is fixed in the rotating frame  $\mathcal{F}_V^p(t)$ , seen from the (fixed) frame  $\mathcal{F}_V^q$  is*

$$\dot{\vec{r}}^q = \vec{\omega}_p^q \times \vec{r}$$