Theorem 1. *Given a basis $\{\vec{p}_i\}$ with dual basis $\{\vec{p}_i^*\}$, any vector $\vec{r}$ can be written as:*

$$\vec{r} = \sum_{i=1}^{n} r_i^p \vec{p}_i, \quad r_i^p = \langle \vec{r}, \vec{p}_i^* \rangle$$

Proof. Since $\{\vec{p}_i\}$ forms a basis, we can write $\vec{r}$ as:

$$\vec{r} = \sum_{i=1}^{n} r_i^p \vec{p}_i$$

Now, consider the inner product of an arbitrary vector $\vec{r}$ and $\vec{p}_i^*$:

$$\langle \vec{r}, \vec{p}_i^* \rangle = \left\langle \sum_{j=1}^{n} r_j^p \vec{p}_j, \vec{p}_i^* \right\rangle$$

$$= \sum_{j=1}^{n} r_j^p \langle \vec{p}_j, \vec{p}_i^* \rangle$$

$$= \sum_{j=1}^{n} r_j^p \delta_{ij} = r_i^p$$

Thus:

$$r_i^p = \langle \vec{r}, \vec{p}_i^* \rangle$$

$\square$

Theorem 2. *Given the basis $\{\vec{p}_i\}$ for the vector space $\mathcal{V}$, any linear operator $\mathbb{A}$ can be represented in $\mathbb{R}^{n \times n}$ as the matrix A^p:*

$$[\mathbb{A}]^p = A^p = [\langle \mathbb{A}\vec{p}_j, \vec{p}_i^* \rangle]$$

$$\vec{y} = \mathbb{A}\vec{x} \overset{\text{alg.}}{\iff} \underline{y}^p = A^p \underline{x}^p$$

Proof. Let $\mathbb{A}$ be a linear operator on $\mathcal{V}$, with the basis $\{\vec{p}_i\}$ and dual basis $\{\vec{p}_i^*\}$. Then, let $\vec{y} = \mathbb{A}\vec{x}$ Again, we express $\vec{y}$ and $\vec{x}$ in the p-frame:

$$\vec{y} = \sum_{i=1}^{n} y_i^p \vec{p}_i, \quad \vec{x} = \sum_{i=1}^{n} x_i^p \vec{p}_i$$

From theorem 1, we have:

$$y_i^p = \langle \vec{y}, \vec{p}_i^* \rangle$$
$$= \langle \mathbb{A}\vec{x}, \vec{p}_i^* \rangle$$

Expressing $\vec{x}$ in the p-frame and using the linearity of $\mathbb{A}$, we get

$$= \left\langle \mathbb{A} \sum_{j=1}^{n} x_j^p \vec{p}_j, \vec{p}_i^* \right\rangle$$

$$= \left\langle \sum_{j=1}^{n} x_j^p \mathbb{A} \vec{p}_j, \vec{p}_i^* \right\rangle$$

$$y_i^p = \sum_{j=1}^{n} x_j^p \langle \mathbb{A} \vec{p}_j, \vec{p}_i^* \rangle$$

This can be written as the following matrix equation

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \langle \mathbb{A}\vec{p}_1, \vec{p}_1^* \rangle & \cdots & \langle \mathbb{A}\vec{p}_n, \vec{p}_1^* \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbb{A}\vec{p}_n, \vec{p}_1^* \rangle & \cdots & \langle \mathbb{A}\vec{p}_n, \vec{p}_n^* \rangle \end{bmatrix} \begin{bmatrix} x_1^p \\ \vdots \\ x_n^p \end{bmatrix}$$

or:

$$\underline{y}^p = A^p \underline{x}^p$$

where

$$A^p = [\langle \mathbb{A}\vec{p}_j, \vec{p}_i^* \rangle]$$

$\square$

Problem 1. *Find the matrix representation of the "$\vec{\omega} \times$" operator in the orthonormal frame $\{\vec{p}_i\}$*

Sol.

To find the matrix representation we use Theorem 2:

$$A^p = [\langle \mathbb{A}\vec{p}_j, \vec{p}_i^* \rangle]$$
$$= [\langle \vec{\omega} \times \vec{p}_j, \vec{p}_i^* \rangle]$$

First we consider $\omega \times \vec{p}_j$ by expressing ω in the p-frame:

$$\vec{\omega} \times \vec{p}_j = \left(\sum_{i=1}^{n} \omega_i^p \vec{p}_i \right) \times \vec{p}_j^*$$
$$= (\omega_1^p \vec{p}_1 + \omega_2^p \vec{p}_2 + \omega_3^p + \vec{p}_3) \times \vec{p}_j^*$$

Since the p-frame is orthonormal, $\vec{p}_i^* = \vec{p}_i$:

$$\vec{\omega} \times \vec{p}_j = (\omega_1^p \vec{p}_1 + \omega_2^p \vec{p}_2 + \omega_3^p \vec{p}_3) \times \vec{p}_j$$

This cross-product can be computed directly for $j = 1, 2, 3$:

$$\vec{\omega} \times \vec{p}_1 = -\omega_2^p \vec{p}_3 + \omega_3^p \vec{p}_2$$
$$\vec{\omega} \times \vec{p}_2 = \omega_1^p \vec{p}_3 - \omega_3^p \vec{p}_1$$
$$\vec{\omega} \times \vec{p}_3 = -\omega_1^p \vec{p}_2 + \omega_2^p \vec{p}_1$$

Then, computing the entries of the matrix A^p:

$$a_{ij} = \langle \vec{\omega} \times \vec{p}_j, \vec{p}_i \rangle$$

$$A^p = \begin{bmatrix} 0 & -\omega_3^p & \omega_2^p \\ \omega_3^p & 0 & -\omega_1^p \\ -\omega_2^p & \omega_1^p & 0 \end{bmatrix}$$

This matrix is skew-symmetric and is defined by the coordinates of $\underline{\omega}^p$, so we denote the matrix representation of "$\vec{\omega} \times$" as

$$[\vec{\omega} \times]^p = S(\underline{\omega}^p)$$

$$\vec{y} = \vec{\omega} \times \vec{x} \overset{\text{alg.}}{\iff} \underline{y}^p = S(\underline{\omega}^p) \underline{x}^p$$

Theorem 3. *Given two bases $\{\vec{p}_i\}$ and $\{\vec{q}_i\}$ in $\mathcal{V}$. Let $\vec{r}$ and $\mathbb{A}$ be a vector and a linear operator in $\mathcal{V}$, respectively. We then have the following relations between matrix representations in the two bases/frames:*

$$\underline{r}^q = C_p^q \underline{r}^p \quad \text{where} \quad C_p^q = [\langle \vec{p}_j, \vec{q}_i^* \rangle]$$

$$\underline{r}^p = C_q^p \underline{r}^q \quad \text{where} \quad C_q^p = [\langle \vec{q}_j, \vec{p}_i^* \rangle]$$

$$A^q = C_p^q A^p C_q^p \quad \text{and} \quad A^p = C_q^p A^q C_p^q$$

Proof. Let $\underline{r}^q = C_p^q \underline{r}^p$ where $\underline{r}^q = \begin{bmatrix} r_1^q & \cdots & r_n^q \end{bmatrix}^T$ and $\underline{r}^p = \begin{bmatrix} r_1^p & \cdots & r_n^p \end{bmatrix}^T$. Then,

$$r_i^q = \langle \vec{r}, \vec{q}_i^* \rangle = \sum_{j=1}^n C_{ij} r_j^p$$

Rearranging and expressing $\vec{r}$ in the p-frame:

$$\sum_{j=1}^n C_{ij} r_j^p = \left\langle \sum_{j=1}^n r_j \vec{p}_j, \vec{q}_i^* \right\rangle$$

By linearity of the inner product, we get

$$\sum_{j=1}^{n} C_{ij} r_j^p = \sum_{j=1}^{n} \langle \vec{p}_j, \vec{q}_i^* \rangle\, r_j^p$$

$$\implies C_{ij} = \langle \vec{p}_j, \vec{q}_i^* \rangle$$

thus

$$C_p^q = [\langle \vec{p}_j, \vec{q}_i^* \rangle]$$

The same procedure can be used to find C_q^p.

Now, consider the equation $\vec{y} = \mathbb{A}\vec{x}$ where $\mathbb{A}$ is some linear operator on $\mathcal{V}$. We can coordinatize this equation in the p and q-frames:

$$\underline{y}^q = A^q \underline{x}^q \quad \text{and} \quad \underline{y}^p = A^p \underline{x}^p$$

From above, we also have the following relations

$$\underline{y}^q = C_p^q \underline{y}^p \quad \text{and} \quad \underline{y}^p = C_q^p \underline{y}^q$$

By substitution, we derive:

$$A^q \underline{x}^q = \underline{y}^q$$
$$= C_q^p \underline{y}^p$$
$$= C_q^p A^p \underline{x}^p$$
$$= C_q^p A^p C_q^p \underline{x}^q$$

And thus

$$A^q = C_q^p A^p C_q^p$$

Again, the same procedure can be used to show that $A^p = C_p^q A^q C_p^q$ □

Theorem 4. *The Direction Cosine Matrix (DCM) between two frames whose bases are orthonormal, R_p^q, is an orthonormal matrix:*

$$(R_p^q)^{-1} = (R_p^q)^T$$

Proof. From the definition of the DCM:

$$R_p^q = \begin{bmatrix} \underline{p}_1{}^q & \underline{p}_2^q & \underline{p}_3^q \end{bmatrix}$$

$$(R_p^q)^T = \begin{bmatrix} (\underline{p}_1^q)^T \\ (\underline{p}_2^q)^T \\ (\underline{p}_3^q)^T \end{bmatrix}$$

Then, we compute $(R_p^q)^T R_p^q$

$$(R_p^q)^T R_p^q = \left[\left\langle \underline{p}_i^q, \underline{p}_j^q \right\rangle\right] = [\delta_{ij}]$$
$$= I$$

thus, $(R_p^q)^{-1} = (R_p^q)^T$ □

Theorem 5. *The matrix representation of the rotation operator* $\mathbb{R}_{ab}$ *in two frames* $\mathcal{F}_\mathcal{V}^a$ *and* $\mathcal{F}_\mathcal{V}^b$ *is*

$$[\mathbb{R}_{ab}]^a = [\mathbb{R}_{ab}]^b = C_b^a$$

Proof.

$$[\mathbb{R}_{ab}]^a = R_{ab}^a = \left[\langle \mathbb{R}_{ab}\vec{a}_i, \vec{a}_j^* \rangle\right]$$
$$= \left[\langle \vec{b}_i, \vec{a}_j^* \rangle\right]$$
$$= C_b^a$$

Using the "similarity transformation" of the R_{ab}^a:

$$R_{ab}^b = C_a^b R_{ab}^a C_b^a$$
$$= C_a^b C_b^a C_b^a$$
$$= C_b^a$$

And thus $R_{ab}^b = R_{ab}^a$

$\square$

Theorem 6. *The derivative of the rotation matrix* R_p^q *is given by*

$$\dot{R}_p^q = S(\underline{w}_p^{qq})R_p^q$$
$$= R_p^q S(\underline{w}_p^{qp})$$

Proof. Since R_p^q is a rotation matrix (orthonormal), we have

$$(R_p^q)^{-1} = (R_p^q)^T$$
$$\implies R_p^q (R_p^q)^T = I$$

Taking the derivative on both sides and applying the product rule gives

$$\dot{R}_p^q (R_p^q)^T + R_p^q (\dot{R}_p^q)^T = 0$$

We now define the matrix $S = \dot{R}_p^q (R_p^q)^T$ such that

$$S + S^T = 0$$

This means S is some skew-symmetric matrix, with form

$$S(\underline{w}) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

From our definition, we have that $\dot{R}^q_p = S(\underline{\omega})R^q_p$. Finally, we want to find an interpretation of the vector $\underline{\omega}$. Writing this equation out by interpreting R^q_p as an attitude matrix gives

$$\begin{bmatrix} \dot{\underline{p}}^q_1 & \dot{\underline{p}}^q_2 & \dot{\underline{p}}^q_3 \end{bmatrix} = S(\underline{\omega}) \begin{bmatrix} \underline{p}^q_1 & \underline{p}^q_2 & \underline{p}^q_3 \end{bmatrix}$$

$$\dot{\underline{p}}^q_i = S(\underline{\omega})\underline{p}^q_i$$
$$= \underline{\omega} \times \underline{p}^q_i$$

We therefore interpret $\underline{\omega}$ as the angular velocity of of the p-frame seen from the q-frame, and represented in the q-frame:

$$\dot{R}^q_p = S(\underline{\omega}^{qq}_p)R^q_p$$

Since $S(\underline{\omega}^{qq}_p)$ is a linear operator, we can apply express it using the similarity transform:

$$S(\underline{\omega}^{qq}_p) = \left[\vec{\omega}^q_p \times \right]^q$$
$$= R^q_p \left[\vec{\omega}^q_p \times \right]^p R^p_q$$
$$= R^q_p S(\underline{\omega}^{qp}_p) R^p_q$$

Inserting this into the equation above gives

$$\dot{R}^q_p = S(\underline{\omega}^{qq}_p)R^q_p$$
$$= R^q_p S(\underline{\omega}^{qp}_p) \cancel{R^p_q} \cancel{R^q_p}$$
$$= R^q_p S(\underline{\omega}^{qp}_p)$$

$\square$

Theorem 7. *The derivative of the DCM C^q_p is given by*

$$\dot{C}^q_p = S(\underline{w}^{qq}_p)C^q_p$$
$$= C^q_p S(\underline{w}^{qp}_p)$$

Proof.

$$C^q_p = \begin{bmatrix} \underline{p}^q_1 & \underline{p}^q_2 & \underline{p}^q_3 \end{bmatrix}$$
$$\dot{C}^q_p = \begin{bmatrix} \dot{\underline{p}}^q_1 & \dot{\underline{p}}^q_2 & \dot{\underline{p}}^q_3 \end{bmatrix}$$

Here we use the result from the proof above

$$\dot{\underline{p}}^q_i = S(\underline{\omega}^{qq}_p)\underline{p}_i$$

$$\dot{C}^q_p = \begin{bmatrix} S(\underline{\omega}^{qq}_p)\underline{p}^q_1 & S(\underline{\omega}^{qq}_p)\underline{p}^q_2 & S(\underline{\omega}^{qq}_p)\underline{p}^q_3 \end{bmatrix}$$
$$\dot{C}^q_p = S(\underline{\omega}^{qq}_p)C^q_p$$

Using the similarity transform we also get $\dot{C}^q_p = C^q_p S(\underline{\omega}^{qp}_p)$ $\square$