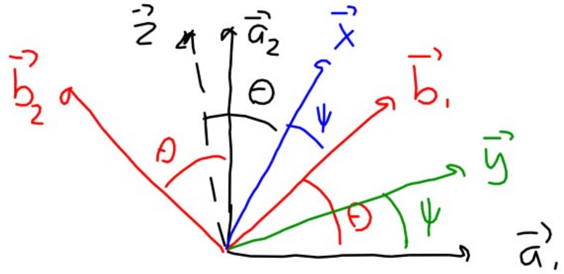
F5' Illustration of the rotation operator



$\mathbb{R}_{ab}$: rotate $a \to b$ an angle θ around axis 3 ($\vec{a}_3$).

Assume $\{\vec{a}_i\}$ is o.n $\Rightarrow$ $\{\vec{b}_i\}$ o.n and $\|\vec{x}\|=1$

$$\underline{x}^b = \begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix} \qquad \underline{x}^a = \begin{bmatrix} \cos(\theta+\psi) \\ \sin(\theta+\psi) \end{bmatrix} \qquad \underline{x}^a = C^a_b \underline{x}^b \overset{\text{o.n.}}{=} R^a_b \underline{x}^b$$

Def: $\underline{y}^a = \underline{x}^b$

$$\underline{x}^a = R^a_b \underline{x}^b = R^a_b \underline{y}^a$$

$$\underline{x}^a = R^a_{ab} \underline{y}^a$$

$$\vec{x} = \mathbb{R}_{ab} \vec{y}$$

Def: $\underline{z}^b = \underline{x}^a$

$$\underline{z}^b = R^a_b \underline{x}^b$$

$$\underline{z}^b = R^b_{ab} \underline{x}^b$$

$$\vec{z} = \mathbb{R}_{ab} \vec{x}$$

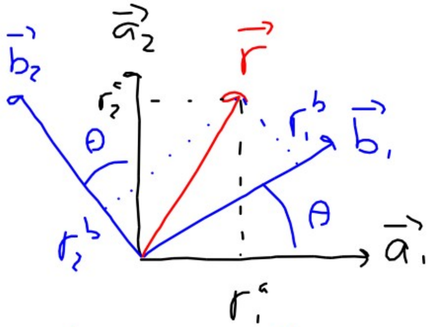
The direction cosine matrix R^a_b works as a rotation operator when used in a single frame. The rotation is the same as we get by rotating from $\{a\}$ to $\{b\}$.

This is called an active interpretation: $R^a_{ab} = R^b_{ab} = R^a_b$

A2.6 Interpretation of the DCM

1. C_b^a is a coordinate transformation matrix (CTM)

$$\underline{r}^a = C_b^a \underline{r}^b$$



This is a passive operation

$$C_b^a \overset{\text{O.n.}}{=} R_b^a = \left[\underline{b}_1^a, \underline{b}_2^a\right] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

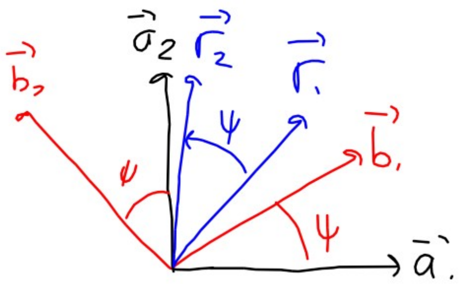
$$\underline{r}^a = C_b^a \underline{r}^b$$

2. C_b^a is a attitude matrix

$$C_b^a = \left[\langle \vec{b}_j, \vec{a}_i^* \rangle\right] = \begin{bmatrix} \langle \vec{b}_1, \vec{a}_1^* \rangle & \langle \vec{b}_2, \vec{a}_1^* \rangle & \langle \vec{b}_3, \vec{a}_1^* \rangle \\ \langle \vec{b}_1, \vec{a}_2^* \rangle & \langle \vec{b}_2, \vec{a}_2^* \rangle & \langle \vec{b}_3, \vec{a}_2^* \rangle \\ \langle \vec{b}_1, \vec{a}_3^* \rangle & \langle \vec{b}_2, \vec{a}_3^* \rangle & \langle \vec{b}_3, \vec{a}_3^* \rangle \end{bmatrix} = \left[\underline{b}_1^a, \underline{b}_2^a, \underline{b}_3^a\right] = C_b^a$$

3. C_b^a is a rotation matrix (RM)

$$C_b^a = \left[\mathbb{R}_{ab}\right]^a = \left[\mathbb{R}_{ab}\right]^b$$



$$\vec{r}_2 = \mathbb{R}_{ab}\vec{r}_1 \iff \underline{r}_2^a = \left[\mathbb{R}_{ab}\right]^a \underline{r}_1^a = R_{ab}^a \underline{r}_1^a = C_b^a \underline{r}_1^a$$

$$\underline{r}_2^b = \left[\mathbb{R}_{ab}\right]^b \underline{r}_1^b = R_{ab}^b \underline{r}_1^b = C_b^a \underline{r}_1^b$$

This is an active operation. Vectors are rotated.

The coordinate transformation matrix (CTM) C_b^a that transform a vector in the b-frame to the a-frame ($\underline{r}_1^a = C_b^a \underline{r}_1^b$) work as a rotationmatrix when used in a single frame, ($\underline{r}_2^a = C_b^a \underline{r}_1^a$), and rotates the vector in the same way we have to rotate the a-frame to get to the b-frame

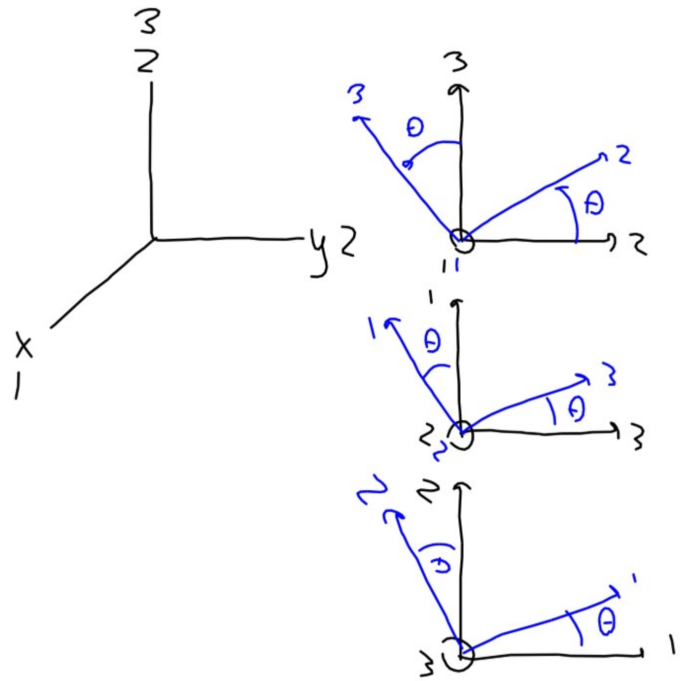
A.2.7 Representasjon av ortogonale RKM

Vi skal i dette avsnittet se på ulike måter å representere ortogonale RKM.

Eulervinkelrepresentasjon av RKM

Elementære RKM. Gitt rammene q og p. Dersom en tenker seg at rammene opprinnelig var sammenfallende fås den endelige p-ramma ved å dreie den en vinkel θ om q_i-aksen. Vi har følgende elementære RKMer:

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{bmatrix}, \; R_2 = \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix}, \; R_3 = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(A- 30)}$$



$$R_1(\theta) = \left(R^{\text{black}}_{\text{blue}}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_2(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Calculation rules for elementary DCM

$$R_i(\theta_1+\theta_2) = R_i(\theta_1)\,R_i(\theta_2) = R_i(\theta_2)\,R_i(\theta_1)$$

In general

$$R_a^c = R_b^c R_a^b \neq R_a^b R_b^c$$

$$\left(R_i(\theta)\right)^T = R_i(-\theta) = \left(R_i(\theta)\right)^{-1} \qquad \left(R_b^a\right)^T = R_a^b$$

Rotation sequences

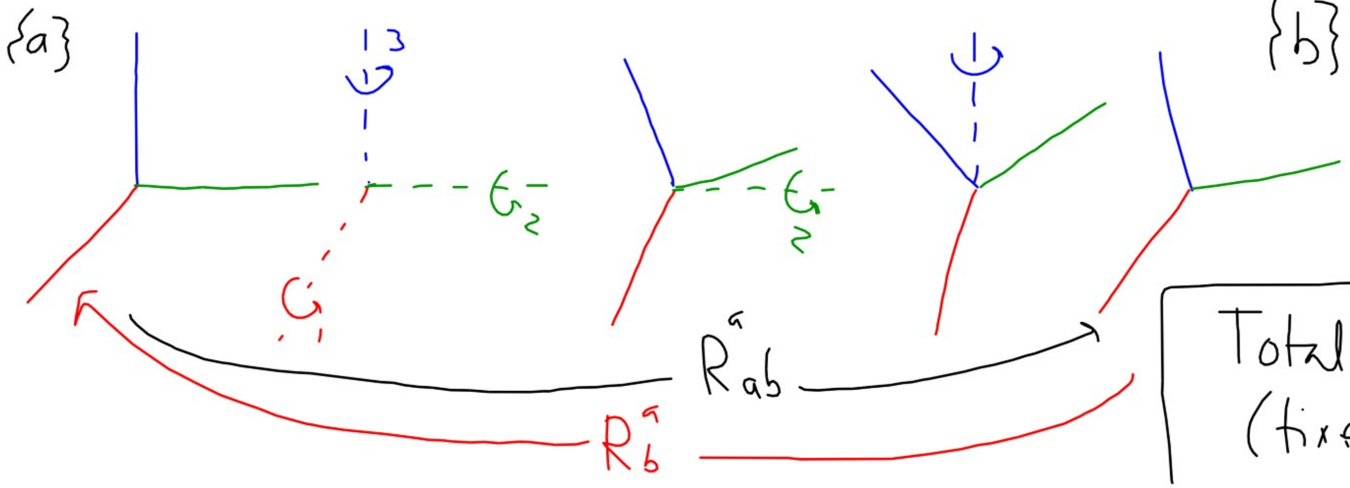
1. Rotation around new axis (Euler angles)

Rotation 1 :	1	1	2	2	3	3	1	1	2	2	3	3
Rotation 2 :	2	3	1	3	1	2	2	3	1	3	1	2
Rotation 3 :	3	2	3	1	2	1	1	1	2	2	3	3

12 sequences

3-2-1 Euler angles

2. Rotation around fixed axis (12 sequences)



R^a_{ab} : rotate $a \to b$

R^a_b : transform $b \to a$

$R^a_b = R^a_{ab} = R^b_{ab}$

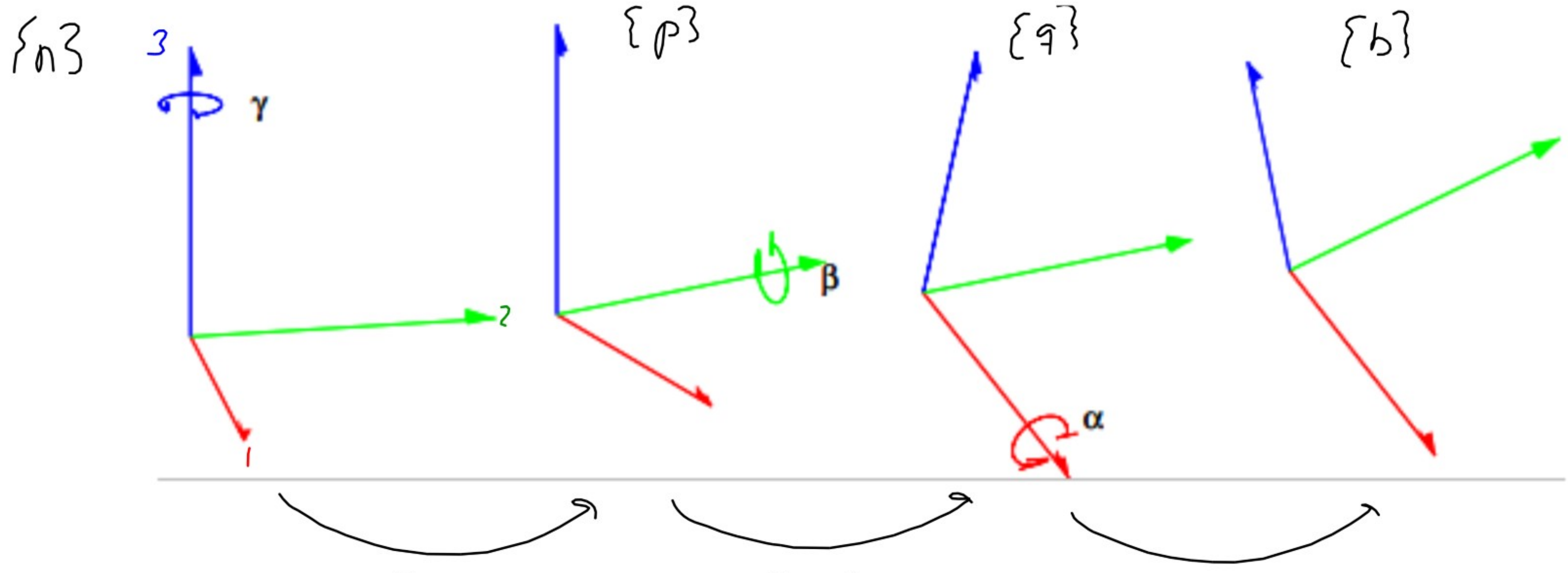
Total of 24 sequences
(fixed + new axis)

Example

What is R_b^a when we rotate γ around axis 3, β around axis 2, and α around axis 1?

Fixed axis: $R_b^a = R_1(\alpha) R_2(\beta) R_3(\gamma)$

New axis : $R_b^a = R_3(\gamma) R_2(\beta) R_1(\alpha)$



$$R_3(\gamma) \qquad R_2(\beta) \qquad R_1(\alpha)$$

$$R_{np} = R^n_p \qquad R_{pq} = R^p_q \qquad R_{qb} = R^q_b$$

$$R_{nb} = R^n_b = R^n_p R^p_q R^q_b = R_3(\gamma)\, R_2(\beta)\, R_1(\alpha)$$

Teorem A.8 ***Sammenheng mellom rotasjon om nye og faste akser***
Tre rotasjoner om nye akser (eulervinkler) gir den samme endelige stilling som de samme rotasjoner tatt i omvendt rekkefølge om faste akser, dvs :

$$ {}^{E}R_p^q(\alpha_i, \beta_j, \gamma_k) = {}^{F}R_p^q(\gamma_k, \beta_j, \alpha_i) \qquad \text{(A- 35)} $$

$$ {}^{E}R_p^q(\alpha_i, \beta_j, \gamma_k) = R_i(\alpha)\, R_j(\beta)\, R_k(\gamma) $$

$$ {}^{F}R_p^q(\gamma_k, \beta_j, \alpha_i) = R_i(\alpha)\, R_j(\beta)\, R_k(\gamma) $$

Direct problem

Given sequence of rotation and the angles (fixed or new axis) find the DCM.

$$\alpha_i \quad \beta_j \quad \gamma_k \rightarrow R_p^q$$

axis

angles

Gives always a unique (entydig) matrix R_p^q

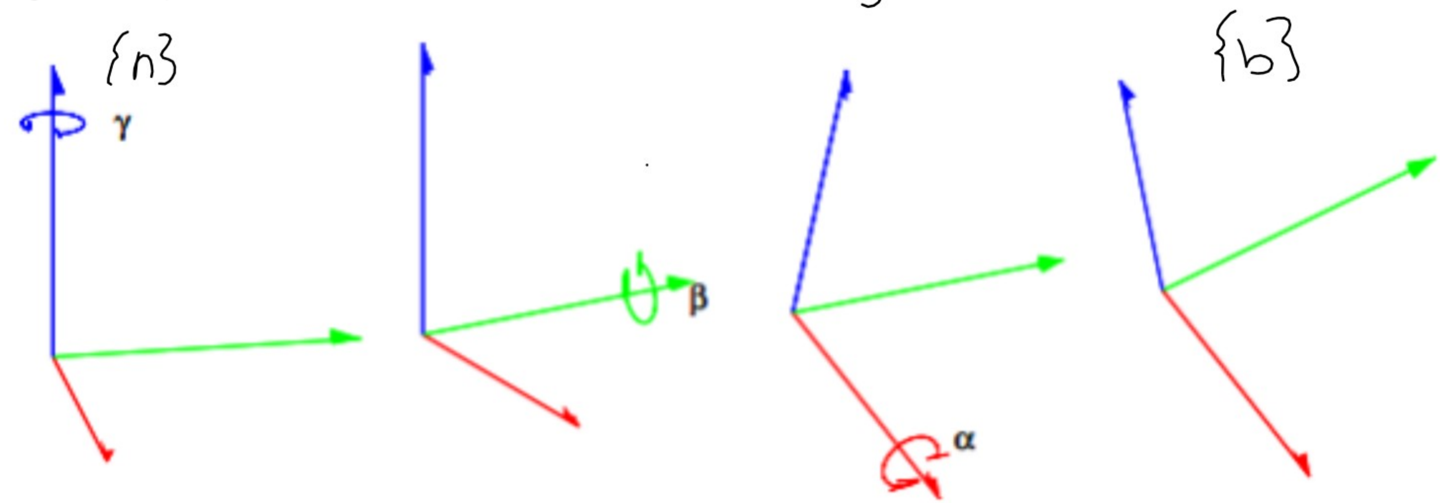
Inverse problem

Given R_p^q, find the angles α, β, γ for a given rotation sequence.

Can not always find a unige (entydig) solution $\Rightarrow$ we have singularities.

(for 3-2-1 sequence (Euler) if rotating 90° around axis2)

Example A.6 3-2-1 Euler angles



$R_b^n = R_3(\gamma) R_2(\beta) R_1(\alpha)$

b: body frame

n: navigation frame

Eksempel A.6 3-2-1 Eulervinkler.
Ved simulering av fly og båter bruker en ofte følgende stillingsmatrise:

$$R_b^n = {}^E R_b^n(\theta_3, \theta_2, \theta_1) = R_3(\theta_3) R_2(\theta_2) R_1(\theta_1) \qquad r_{31} = -\sin(\theta_2) \qquad \text{(A- 36)}$$

Multipliseres de elementære RKM sammen får vi :

$$ {}^E R_b^n(\theta_3, \theta_2, \theta_1) = \begin{bmatrix} c_{\theta_3} c_{\theta_2} & c_{\theta_3} s_{\theta_2} s_{\theta_1} - s_{\theta_3} c_{\theta_1} & c_{\theta_3} s_{\theta_2} c_{\theta_1} + s_{\theta_3} s_{\theta_1} \\ s_{\theta_3} c_{\theta_2} & s_{\theta_3} s_{\theta_2} s_{\theta_1} + c_{\theta_3} c_{\theta_1} & s_{\theta_3} s_{\theta_2} c_{\theta_1} - c_{\theta_3} s_{\theta_1} \\ -s_{\theta_2} & c_{\theta_2} s_{\theta_1} & c_{\theta_2} c_{\theta_1} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \qquad \text{(A- 37)}$$