F12

Euler equations

Oral Exam

12th Dec – 13th Dec ?

16th Dec – 17th Dec ?

When we put A in the center of mass C the law of angular momentum when $\mathcal{F}^b$ is fixed to the body and $O_b = C = A$:

(1) $$\underline{n}_c^b = J_c^b \dot{\underline{\omega}}_b^{ib\,b} + S(\underline{\omega}_b^{ib}) J_c^b \underline{\omega}_b^{ib}$$

Assume $\mathcal{F}^b$ coinsides with the main axis of the body, i.e. J_c^b is diagonal, and $\underline{n}_c^b = [n_x ; n_y ; n_z]$, $\underline{\omega}_b^{ib} = [\omega_x ; \omega_y ; \omega_z]$, $J_c^b = \mathrm{diag}(J_{xx}^b , J_{yy}^b , J_{zz}^b)$

Then we get the Euler equations (Teorem B.9)

Teorem B.9 ***Eulerlikningene***
Dersom k.s. b velges fast i legemet med origo i A, med akser langs hovedaksene for legemet og A i tillegg tilfredstiller 1 eller 2 :

$A = C = O_b$

1). A ligger i massesenteret.
2). A ligger i ro i treghetsrommet.
kan spinnsatsen skrives på følgende enkle form :

$$\left.\begin{array}{l} n_x = J^b_{xx}\dot{\omega}_x + \omega_y\omega_z(J^b_{zz} - J^b_{yy}) \\ n_y = J^b_{yy}\dot{\omega}_y + \omega_z\omega_x(J^b_{xx} - J^b_{zz}) \\ n_z = J^b_{zz}\dot{\omega}_z + \omega_x\omega_y(J^b_{yy} - J^b_{xx}) \end{array}\right\}, \quad \underline{n}^b_A = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}, \quad \underline{\omega}^{ib}_b = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \qquad \text{(B- 152)}$$

Proof: Insert into 1

$$\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} J^b_{xx}\dot{\omega}_x \\ J^b_{yy}\dot{\omega}_y \\ J^b_{zz}\dot{\omega}_z \end{bmatrix} + \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}\begin{bmatrix} J^b_{xx}\omega_x \\ J^b_{yy}\omega_y \\ J^b_{zz}\omega_z \end{bmatrix} = \begin{bmatrix} J^b_{xx}\dot{\omega}_x + \omega_y\omega_z(J^b_{zz} - J^b_{yy}) \\ J^b_{yy}\dot{\omega}_y + \omega_x\omega_z(J^b_{xx} - J^b_{zz}) \\ J^b_{zz}\dot{\omega}_z + \omega_x\omega_y(J^b_{yy} - J^b_{xx}) \end{bmatrix}$$

Euler equations can be used in two ways:

1) Given the forces ($\underline{n}_c^b$) find the motion ($\underline{\omega}_b^{ib}$). Diff. eq.

2) Given the motion ($\underline{\omega}_b^{ib}$) find the forces ($\underline{n}_c^b$). Alg. eq.

Solution on 2) is the Euler equations

Solution on 1)

2

$$\dot{\omega}_x = \frac{1}{J_{xx}}\left[(J_{yy} - J_{zz})\,\omega_y \omega_z + n_x\right]$$

$$\dot{\omega}_y = \frac{1}{J_{yy}}\left[(J_{zz} - J_{xx})\,\omega_x \omega_z + n_y\right]$$

$$\dot{\omega}_z = \frac{1}{J_{zz}}\left[(J_{xx} - J_{yy})\,\omega_x \omega_y + n_z\right]$$

This eq. is on standard form:

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) \quad , \quad \underline{x}(t_0) \text{ is given}$$

For d.e. 2 : $\underline{x} = \underline{\omega}_b^{ib}$, $\underline{u} = \underline{n}_c^b$

To find the orientation/attitude:

$$\boxed{\begin{aligned} \dot{R}_b^i &= R_b^i\, S(\underline{\omega}_b^{ib}) \quad , \quad R_b^i(t_0) \text{ given} \\ \dot{\underline{\omega}}_b^{ib\,b} &= \underline{f}(\underline{\omega}_b^{ib}, \underline{n}_c^b) \quad , \quad \underline{\omega}_b^{ib}(t_0) \text{ given} \end{aligned}}$$

Insted of a d.e. in DCM R_b^i we can use d.e. for euler angles (A.5)

$$\underline{\dot{\Theta}} = D_b^\theta(\underline{\Theta})\,\underline{w}_b^{ib}$$

$$\underline{\dot{w}}_b^{ibb} = f(\underline{w}_b^{ib}, \underline{n}_c^b)$$

$\underline{\Theta}(t_0)$ and $\underline{w}_b^{ib}(t_0)$ are given

$$\underline{\Theta} = \begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{pmatrix} \quad (3\text{-}2\text{-}1 \text{ euler angles})$$

$D_p^\theta(\underline{\Theta})$ is given in A-105

$$D_p^\theta(\underline{\Theta}) = \begin{bmatrix} 1 & \sin\theta_1 \tan\theta_2 & \cos\theta_1 \tan\theta_2 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1/\cos\theta_2 & \cos\theta_1/\cos\theta_2 \end{bmatrix}$$

Here $\underline{X} = \begin{bmatrix} \underline{\Theta} \\ \underline{w}_b^{ib} \end{bmatrix}$

NB! D.e. for $\underline{w}_b^{ib}$ can be solved without solving the d.e. of $\underline{\Theta}$, but not the other way.

B3 Torque free motion of a rigid body

Assume $J_{xx} > J_{yy} > J_{zz}$

We want to determine $\underline{w}_b^{ib}$ and $\underline{\dot{w}}_b^{ib}$

(we will find trajectories, not the time solution)

Ellipsoid of inertia

Matrix of inertia $J_c^b = [J_c^b]^T$

$\Rightarrow$ positive definite matrix and by defining the expression:

$J = \frac{1}{2} \underline{x}^T J_c^b \underline{x}$: ellipsoid when J is constant

If we choose $\underline{F}^b$ along main axis of the body:

$$J = \frac{1}{2} \underline{x}^T \begin{bmatrix} J_{xx} & & \\ & J_{yy} & \\ & & J_{zz} \end{bmatrix} \underline{x} = \frac{1}{2} \sum_{i=1}^{3} J_{ii} x_i^2$$

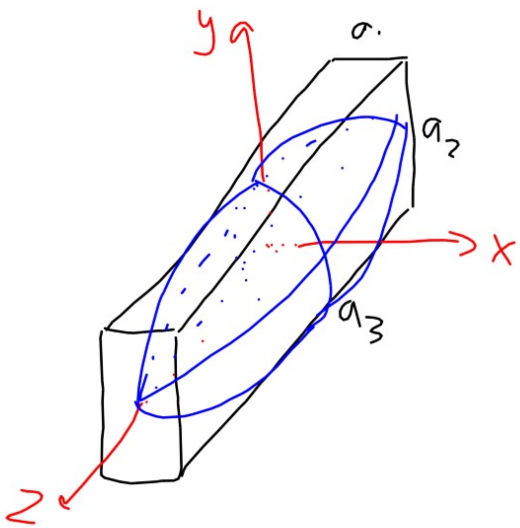
Can be written on standard form of ellipsoids

This e.q on stand. form :

$$\boxed{\frac{x_1^2}{2J/J_{11}} + \frac{x_2^2}{2J/J_{22}} + \frac{x_3^2}{2J/J_{33}} = 1}$$

Half axis : $a_i = \sqrt{2J/J_{ii}}$ for $i = 1, 2, 3$

$a_1 < a_2 < a_3$



axis

$x = 1$

$y = 2$

$z = 3$

The ellipsoid of inertia is a representative of the form of the body.

Kinetic rotation energy ellipsoide

Multiply Euler equations (B.152) (assume $\underline{n}_c^b = \underline{0}$, i.e. torque free motion) withe w_x, w_y and w_z for line 1, 2 and 3 respectively, and summerize.

$$0 = \sum_{i=1}^{3} J_{ii} \dot{w}_i w_i \qquad \begin{pmatrix} 1=x \\ 2=y \\ 3=z \end{pmatrix}$$

Integrate with respect to time $(t_0 \to t)$

$$\int_{t_0}^{t} \sum_{i=1}^{3} J_{ii} w_i \frac{dw_i}{dt} dt = \int_{t_0}^{t} 0 \, dt = 0$$

$$\frac{1}{2}\sum_{i=1}^{3} J_{ii} w_i^2(t) = \frac{1}{2}\sum_{i=1}^{3} J_{ii} w_i^2(t_0) = K_0$$

That means:

$$\boxed{\frac{w_1^2(t)}{2K_0/J_{11}} + \frac{w_2^2(t)}{2K_0/J_{22}} + \frac{w_3^2(t)}{2K_0/J_{33}} = 1}$$

Half axis

$$a_i = \sqrt{\frac{2K_0}{J_{ii}}}$$

$$\int_{t_0}^{t} \sum J_{ii} w_i \, dw_i$$

$$= \sum J_{ii} \left[\frac{1}{2} w_i^2\right]_{t_0}^{t}$$

$$= \frac{1}{2} \sum J_{ii} \left(w_i^2(t) - w_i^2(t_0) \right) = 0$$

We see that the kinetic rotation energy ellipsoid has the same form as the ellipsoid of inertia

(can choose $J = K_0$)

$\underline{w}_b^{ib}(t)$ has to stay on the kin. rot. energy ellipsoid

Angular momentum ellipsoide

When external torque is 0 $(\vec{n}_c = \vec{0})$ we get $\overset{\circ}{\vec{h}}{}_c^{i,i} = \vec{0}$

and $\|\vec{h}_c^{i}\| = h_0$ (constant length) and the same direction as seen from $\mathcal{F}^i$

The angular momentum seen from $\mathcal{F}^b$

$$\underline{h}_c^{ib} = J_c^b \underline{w}_b^{ib} = \begin{bmatrix} J_{xx} w_x \\ J_{yy} w_y \\ J_{zz} w_z \end{bmatrix}$$

length is constant:

$$\left(\underline{h}_c^{ib}\right)^T \underline{h}_c^{ib} = h_0^2$$

$$\sum_{i=1}^{3} J_{ii}^2 w_i^2 = h_0^2$$

Writen on stand. form:

$$\boxed{\frac{w_1^2(t)}{h_0^2/J_{11}^2} + \frac{w_2^2(t)}{h_0^2/J_{22}^2} + \frac{w_3^2(t)}{h_0^2/J_{33}^2} = 1}$$

Half axis: $a_i = \frac{h_0}{J_{ii}}$, $i = 1,2,3$

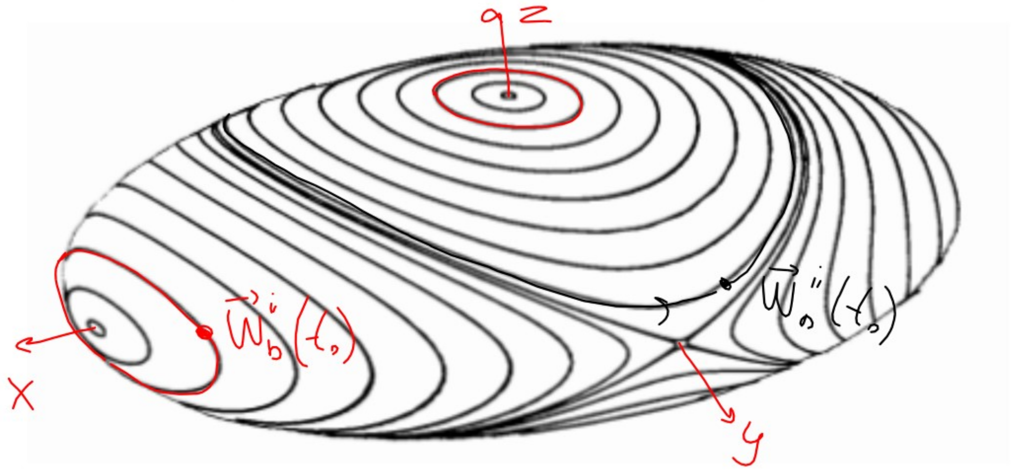
We see that the kinetic rotation energy ellipsoide and the angular momentum ellipsoide have not equal half axises.

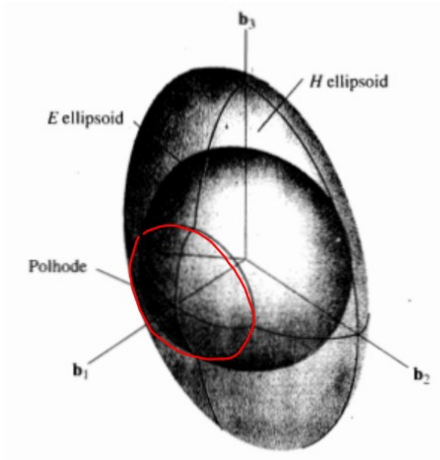
$\Rightarrow$ different forms on the ellipsoides but they need to intersect

$\underline{w}_b^{ib}(t)$ has to stay on the intersection (pole head) of the two ellipsoides

B.3.1 Beskrivelse av bevegelsen sett fra b-systemet

Teorem B.13 ***Bevegelsen av et stivt legeme sett fra det roterende b-systemet***
Anta b-systemet faller sammen med hovedaksene for det stive legemet. For et stivt legeme som ikke er utsatt for ytre moment beveger vinkelhastighetsvektoren ($\underline{\omega}_b^{ib}$) seg da, sett fra b-systemet, på skjeringa (polhode) mellom spinnellipsoida og den kinetiske rotasjonsenergiellipsoida. Bevegelsen til det stive legemet er i hvert øyeblikk en ren rotasjon om vinkelhastighetsvektoren.





Benoit ellipsoide