F2/ Part A: Mathematical Foundation

In our lectures we go throug the document by O. Hallingstad. Figures and proofs are taken on the white board. Build on linear algebra, matrix theory, ordinary diff. eq.

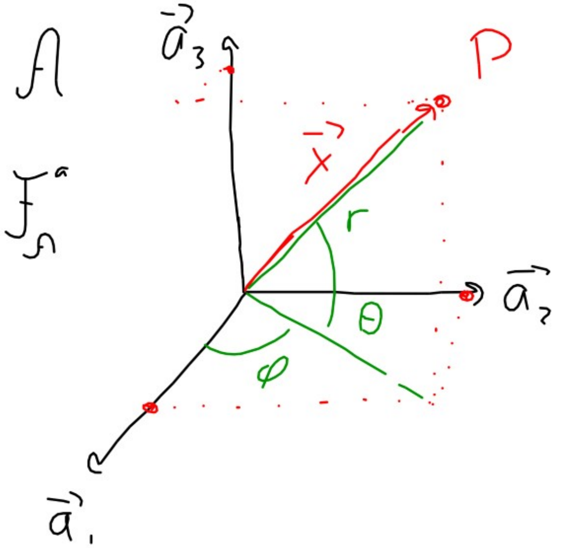
Rom	Rammer	Kommentar
Referanserrom		Fysisk rom bestående av punkter som er i ro i forhold til hverandre. Eng: (observational) frame of reference
Treghetsrom		Et referanserom hvor Newtons 2. lov har sin enkleste form, $\vec{f} = m\vec{a}^{i}$
Vektorrom $\mathcal{V}$	$\mathcal{F}^a_{\mathcal{V}} = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\} = \{\vec{a}_i\} = \{a\}$ ramme a i vektorrom $\mathcal{V}$ med basisvektorer $\vec{a}_i$	Matematisk definert rom med vektorer som objekter.
Affint rom $\mathcal{A}$	$\mathcal{F}^a_{\mathcal{A}} = \{O_a; \vec{a}_1, \vec{a}_2, \vec{a}_3\} = \{a\}$ ramme a i det affine rom $\mathcal{A}$ med origo O_a og basis-vektorer $\vec{a}_i$	Matematisk definert rom med punkter og vektorer som objekter. Brukes som modell for referanse- og treghetsrom.

Definisjon A.1 *Et koordinatsystem $C^a_{\mathcal{A}}$ for et affint rom $\mathcal{A}$ avbilder et punkt P inn i $\mathbb{R}^n$:*

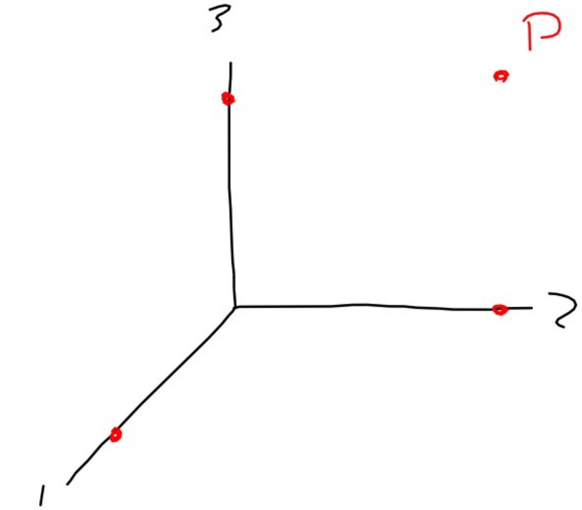
$$C^a_{\mathcal{A}} : P \to \underline{x}^a_P \text{ hvor } P \in \mathcal{A} \text{ og } \underline{x}^a_P \in \mathbb{R}^n$$

$$C^a_{\mathcal{A}}(P) = \underline{x}^a_P$$

Coordinate system



$\mathbb{R}^n$



$$\underline{x}^a_p = \begin{bmatrix} x_1^a \\ x_2^a \\ x_3^a \end{bmatrix}$$ Cartesian coordinates

$$\underline{y}^a_p = \begin{bmatrix} r \\ \phi \\ \Theta \end{bmatrix}$$ spherical coordinate.

A coordinate system is a function from a affine space into $\mathbb{R}^n$. The coordinates depends on both the frame and wheter we choose spherical, cartesian or other coordinates.

A.2 Vektorrom

Jeg vil bruke følgende notasjon og forkortelser i tidsinvariante vektorrom:

$\vec{x}$ $\in \mathcal{A}$ or $\mathcal{V}$	Geometrisk vektor
$\underline{x}$ $\in \mathbb{R}^n$ $\vec{y} = A\vec{x}$	Algebraisk vektor (kolonnematrise)
$\mathbf{A}$	Operator
$A^q = [\mathbf{A}]^q$ $\underline{y}^q = A^q \underline{x}^q$	Operatoren $\mathbf{A}$ representert i q-systemet
$\{\vec{q}_i\}$	Basissystemet q
$\{\vec{q}_i^*\}$	Det duale basissystem for q
$S(\underline{\omega}^q) \equiv [\vec{\omega}\times]^q$	Matriserepresentasjon av "$\vec{\omega}\times$"-operatoren i q-systemet.
$< \vec{a}, \vec{b} >$	Indreproduktet av $\vec{a}$ og $\vec{b}$.
$\underline{x}^T = [x_1, x_2, \ldots, x_n]$	Transponert vektor (Matlab skrivemåte)
$\underline{x} = [x_i]\,, \quad x_i = [\underline{x}]_i$	Kolonnematrise med generelt element x_i
$D = [d_{ij}]\,, \quad d_{ij} = [D]_{ij}$	Matrise med generelt element d_{ij}
c_φ	$\cos(\varphi)$
s_φ	$\sin(\varphi)$
C_a^b	Retningskosinmatrise (Direction cosine matrix)
R_a^b	Ortogonal retningskosinmatrise
k.s.	Koordinatsystem
$C_\mathcal{V}^a$	k.s. a i vektorrommet $\mathcal{V}$
$C_\mathcal{A}^a$	k.s. a i det affine rom $\mathcal{A}$
RKM (DCM)	Retningskosinmatrise
$\mathbb{K}, \mathbb{R}, \mathbb{C}$	$\underbrace{\text{Skalarkropp}}_{\mathbb{K}}$, $\underbrace{\text{mengden av reelle tall}}_{\mathbb{R}}$, $\underbrace{\text{mengden av komplekse tall}}_{\mathbb{C}}$

$\mathcal{A}$: affine space

$\mathcal{V}$: vector space

$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \underline{x}^T = [x_1\ x_2\ \ldots\ x_n]$

A2.1 Definitions

$\mathbb{K}$: Scalar body: Defined rules of calculation (real and complex numb.)

$\mathbb{N}$: Natural numbers: $1, 2, 3, \ldots$ do not create a scalar body

$\mathbb{Z}$: Whole numbers: $\ldots -2, -1, 0, 1, 2 \ldots$

$$2 \cdot \left(\frac{1}{2}\right) = 1$$

$\forall$: for all

$\exists$: it exist

$\in$: element of

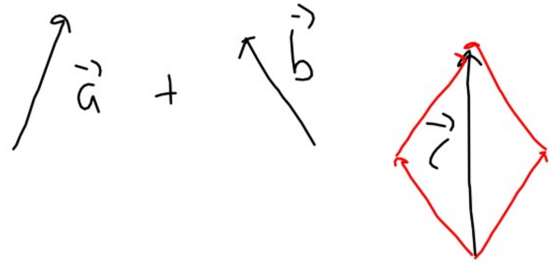
Linear vector spaces

A vector space is defined over a scalar body (uses numbers from the scalar body)

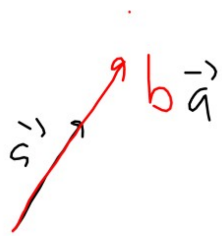
- Vector addition (+)
- Scalar multiplication $(\cdot)$

Examples of vectors

Arrows in the 2D-plane or 3D-space

(+) 

(+) $\vec{c} = \vec{a} + \vec{b} = \vec{b} + \vec{a}$

$(\cdot)$ 

$\|\vec{a}\| = a$

$\|\vec{b}\| = b$

Colummatrices with dimension n (n-tuplets of numbers)

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad (+)\ \underline{x} + \underline{y} = [x_i] + [y_i] = [x_i + y_i] \quad (\circ)\ a\underline{x} = a[x_i] = [a x_i]$$

n'th order polynomials

$$\vec{a} = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

$$\vec{b} = b_n x^n + b_{n-1} x^{n-1} + \dots + b_1 x + b_0$$

$$(+)\quad \vec{a} + \vec{b} = (a_n + b_n) x^n + (a_{n-1} + b_{n-1}) x^{n-1} + \dots + (a_1 + b_1) x + (a_0 + b_0)$$

$$(\cdot)\quad c\vec{a} = c a_n x^n + c a_{n-1} x^{n-1} + \dots + c a_1 x + c a_0$$

Basis

Linear independent vectors $\{\vec{q}_i\}$

$$a_1\vec{q}_1 + a_2\vec{q}_2 + \ldots + a_n\vec{q}_n = 0 \quad \forall\ \vec{q}_i \neq 0 \iff \text{all } a_i = 0$$

Given the basis $\{\vec{q}_i\} \in V$ all vectors $\vec{v} \in V$ can be written as: $\vec{v} = v_1\vec{q}_1 + v_2\vec{q}_2 + \ldots + v_n\vec{q}_n$

Inner product

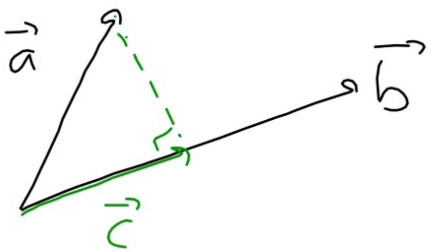
Used to i.e. calculate the length of a vector, orthogonality and to project one vektor down to another.

Examples

Algebraic vector

$\underline{x}, \underline{y} \in \mathbb{R}^n$ $\quad \langle \underline{x}, \underline{y} \rangle = \underline{x}^T \underline{y} = [x_1\ x_2 \ldots x_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$

Geometrical vectors



$\langle \vec{a}, \vec{b} \rangle = \|\vec{a}\| \, \|\vec{b}\| \cos \angle \vec{a}\vec{b}$

$\|\vec{c}\| = \|\vec{a}\| \cos \angle \vec{a}\vec{b}$

If $\|\vec{b}\| = b = 1$ then $\vec{b} \langle \vec{a}, \vec{b} \rangle = \vec{c}$ Project $\vec{a}$ down on $\vec{b}$

$\|\vec{a}\| = \langle \vec{a}, \vec{a} \rangle^{1/2} = a$ norm/length of $\vec{a}$, $|a|$ = number of a

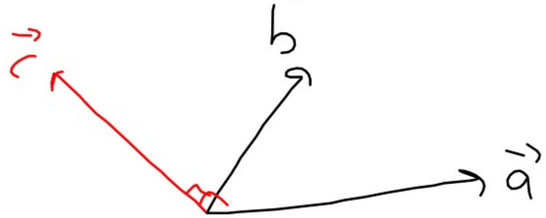
$|a| = |-a|$

Cross product

NB! $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$, $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

Examples

- Geometrical vector



$\vec{c} = \vec{a} \times \vec{b}$ $\quad \| \vec{c} \| = \left| \| \vec{a} \| \| \vec{b} \| \sin \angle \vec{a}\vec{b} \right|$

- Algebraic vectors

$$\underline{c}^q = \underline{a}^q \times \underline{b}^q = S(\underline{a}^q)\, \underline{b}^q$$

where $\{\vec{q}_i\}$ is orthogonal with unit length

$$S(\underline{d}) = \begin{bmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{bmatrix}$$

Scew-symmetrical form of $\underline{d}$

Dyad product

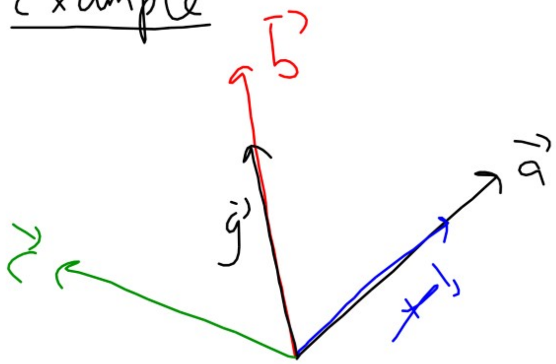
Used to represent operators

Def. $\vec{a}\vec{b}$: dyad product

$$\vec{a}\vec{b}\cdot\vec{c} = \vec{a}\langle\vec{b},\vec{c}\rangle$$

$$\vec{d}\cdot\vec{a}\vec{b} = \langle\vec{d},\vec{a}\rangle\vec{b}$$

Example



$$\vec{f} = \vec{a}\vec{b}\cdot\vec{c} = \vec{a}\langle\vec{b},\vec{c}\rangle$$

$$\vec{g} = \vec{c}\cdot\vec{a}\vec{b} = \langle\vec{c},\vec{a}\rangle\vec{b}$$

We can use the dyad to represent rotation- and projection operators.

Projection operator down on direction $\vec{a}$, $\|\vec{a}\| = 1$

$$\mathbb{P}(\vec{a})\vec{b} = \vec{a}\,\|\vec{b}\|\cos\angle\vec{a}\vec{b}$$

$$= \vec{a}\langle\vec{a},\vec{b}\rangle$$

$$= \vec{a}\vec{a}\cdot\vec{b}$$

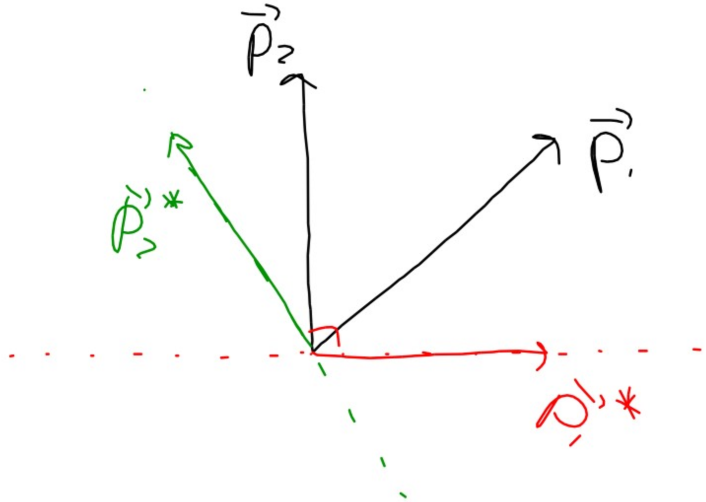
Def. A.9 Dual basis

Used to decompose vectors using a given basis (frame)

NB! Dual basis do not require orthogonal basisvectors

Def. The dual basis $\{\vec{p}_i^{\,*}\}$ to the basis $\{\vec{p}_i\}$ is defined as:

$$\langle \vec{p}_i , \vec{p}_i^{\,*} \rangle = \delta_{ij} \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases} \qquad \delta_{ij} : \text{Kronecker delta}$$



$$\langle \vec{p}_1 , \vec{p}_1^{\,*} \rangle = 1 \; , \; \langle \vec{p}_2 , \vec{p}_1^{\,*} \rangle = 0 \; . \; \vec{p}_1^{\,*} \perp \vec{p}_2$$

$$\langle \vec{p}_1 , \vec{p}_2^{\,*} \rangle = 0 \; , \; \langle \vec{p}_2 , \vec{p}_2^{\,*} \rangle = 1$$

Dual basis:

We will show later that $\vec{v} = \sum_{i=1}^{n} v_i^p \cdot \vec{p}_i$ where $v_i^p = \langle \vec{v}, \vec{p}_i^{\,*} \rangle$

$$\Rightarrow \underline{v}^p = \begin{bmatrix} v_1^p \\ v_2^p \\ v_3^p \end{bmatrix}$$