F11 (NB! Remember the figure from F10, page 13)

The body is rigid, i.e. the distance between the molecules (particles) are constant. We have already found the law of angular momentum of a sum of particles (n-particle-system). We want to use integrals insted of summation (Σ).

We let $n \to \infty$: $\sum \to \iiint_M dm$ or $\iiint_V k(\vec{r})\,dV$

M: total mass

V: volume of the body

Particles $\Rightarrow$ massdifferentials

$$\vec{h}_{Ai}^{\,i} = \vec{\rho}_{iA} \times m_i \dot{\vec{\rho}}_{iA}^{\,i}$$

$$\Downarrow$$

$$d\vec{h}_A^{\,i} = \vec{\rho} \times \dot{\vec{\rho}}^{\,i}\, dm$$

$$\vec{h}_A^{\,i} = \iiint_M d\vec{h}_A^{\,i} = \iiint_M \vec{\rho} \times \dot{\vec{\rho}}^{\,i}\, dm$$

$$= \iiint_M \vec{\rho} \times \left(\dot{\vec{\rho}}^{\,b} + \vec{\omega}_b^{\,i} \times \vec{\rho}\right) dm$$

$$\dot{\vec{\rho}}^{\,i} = \frac{d^i}{dt}\vec{\rho} = \frac{d^i}{dt}\left(\sum_{i=1}^{n} \rho_i^b \vec{b}_i\right)$$

$$= \sum \dot{\rho}_i^b \vec{b}_i + \sum \rho_i^b \dot{\vec{b}}_i^{\,i}$$

$$= \dot{\vec{\rho}}^{\,b} + \sum_{i=1}^{n} \rho_i^b \left(\vec{\omega}_b^{\,i} \times \vec{b}_i\right)$$

$$= \dot{\vec{\rho}}^{\,b} + \vec{\omega}_b^{\,i} \times \underbrace{\left(\sum_{i=1}^{n} \rho_i^b \vec{b}_i\right)}_{\vec{\rho}}$$

$$\dot{\vec{\rho}}^{\,i} = \dot{\vec{\rho}}^{\,b} + \vec{\omega}_b^{\,i} \times \vec{\rho}$$

If A is fixed to the body:

$$\vec{h}_A^{\,i} = \iiint_M \vec{\rho} \times (\vec{\omega}_b^{\,i} \times \vec{\rho})\, dm = \underbrace{-\iiint_M dm\, \vec{\rho} \times (\vec{\rho} \times \vec{\omega}_b^{\,i})}_{\mathbb{J}_A} = \mathbb{J}_A\, \vec{\omega}_b^{\,i}$$

Total external torque:

one particle: $\vec{n}_{Ai} = \vec{\rho}_i \times \vec{f}_i$

n particles: $\vec{n}_A = \sum \vec{\rho}_i \times \vec{f}_i \Longrightarrow$ rigid body $\vec{n}_A = \iiint_M \vec{\rho} \times d\vec{f}$

What is the relation between $\vec{n}_A$ and $\vec{h}_A^{\,i}$?

Teorem B.8 ***Spinnsatsen for stive legemer***
Gitt treghetssystemet **i** *og et k.s. b som ligger fast i legemet og har sitt origo i A. Dersom A tilfredstiller 1 eller 2 :*
1). A ligger i massesenteret.
2). A ligger i ro i treghetsrommet. (or constant velocity)
er spinnsatsen på en koordinatuavhengig form :

$$\vec{n}_A = \overset{\cdot}{\vec{h}}{}^{\mathbf{i}}_A$$

Theorem A.19

$$= \overset{\cdot}{\vec{h}}{}^{\mathbf{i}b}_A + \vec{\omega}^{\mathbf{i}}_b \times \vec{h}^{\mathbf{i}}_A \qquad \text{(B- 148)}$$

$$\vec{n}_A = \iiint_M \vec{\rho} \times d\vec{f}$$

$$\overset{\cdot}{\vec{h}}{}^{i}_A = -\iiint_M \vec{\rho} \times (\vec{\rho} \times \overset{\cdot}{\vec{\omega}}{}^{i}_b)\, dm$$

$$= \mathbb{J}_A \overset{\cdot}{\vec{\omega}}{}^{i}_b$$

eller representert i b-systemet :

$$\begin{aligned} \underline{n}^b_A &= \overset{\cdot}{\underline{h}}{}^{\mathbf{i}bb}_A + \underline{\omega}^{\mathbf{i}b}_b \times \underline{h}^{\mathbf{i}b}_A \\ &= J^b \overset{\cdot}{\underline{\omega}}{}^{\mathbf{i}bb}_b + \underline{\omega}^{\mathbf{i}b}_b \times (J^b \underline{\omega}^{\mathbf{i}b}_b) \end{aligned} \qquad \text{(B- 149)}$$

Theorem A.18

$$\overset{\cdot}{\underline{\omega}}{}^{\mathbf{i}bb}_b = R^b_{\mathbf{i}} \overset{\cdot}{\underline{\omega}}{}^{\mathbf{i}}_b$$

Law of ang. momentum

hvor spinnet er definert ved :

$$\underline{h}^{\mathbf{i}b}_A = J^b \underline{\omega}^{\mathbf{i}b}_b$$

$$J^b_A = \left[\mathbb{J}_A\right]^b$$

Treghetsmatrisa J^b *beregnes via treghetsmomenta,* J^b_{ii}*, og treghetsprodukta,* J^b_{ij} *:*

$$J^b = \begin{bmatrix} J^b_{xx} & -J^b_{xy} & -J^b_{xz} \\ -J^b_{yx} & J^b_{yy} & -J^b_{yz} \\ -J^b_{zx} & -J^b_{zy} & J^b_{zz} \end{bmatrix} \tag{B- 150}$$

$$\begin{bmatrix} J^b_{xx} & -J^b_{xy} & -J^b_{xz} \\ -J^b_{yx} & J^b_{yy} & -J^b_{yz} \\ -J^b_{zx} & -J^b_{zy} & J^b_{zz} \end{bmatrix} = \begin{bmatrix} \int_M \left(y^2+z^2\right) dm & -\int_M xydm & -\int_M xzdm \\ -\int_M xydm & \int_M \left(x^2+z^2\right) dm & -\int_M yzdm \\ -\int_M xzdm & -\int_M yzdm & \int_M \left(x^2+y^2\right) dm \end{bmatrix} \tag{B- 151}$$

Dvs treghetsmatrisa er symmetrisk.

NB! Here is $\rho^b = \begin{pmatrix} \rho_1^b \\ \rho_2^b \\ \rho_3^b \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Summary: Kinetic eq. for the centre of mass ($c = A = O_b$)

Linear Momentum $\vec{p}_c^{\,i} = m \vec{v}_c^{\,i}$ Angular Momentum $\vec{h}_c^{\,i} = -\iiint_M \vec{\rho} \times (\vec{\rho} \times \vec{\omega}_b^{\,i})\, dm = \underbrace{\mathbb{J}_c}\, \vec{\omega}_b^{\,i}$

Operator (mass operator)

$\Rightarrow$ Matrix of inertia when rep. in the b-frame

N.2 $\vec{f} = \dot{\vec{p}}_c^{\,i} = \frac{d^i}{dt}(m \cdot \vec{v}_c^{\,i}) = \dot{\vec{p}}_c^{\,ib} + \vec{\omega}_b^{\,i} \times \vec{p}_c^{\,i}$ | $\underline{f}^i = m \dot{\underline{v}}_c^{\,i}$, $\underline{f}^b = \dot{\underline{p}}_c^{\,ibb} + \underline{\omega}_b^{\,ib} \times \underline{p}_c^{\,ib}$

Law of ang. momentum $\vec{n}_c = \dot{\vec{h}}_c^{\,i} = \frac{d^i}{dt}(\mathbb{J}_c \vec{\omega}_b^{\,i}) = \dot{\vec{h}}_c^{\,ib} + \vec{\omega}_b^{\,i} \times \vec{h}_c^{\,i}$ | $\underline{n}_c^{\,i} = \dot{\underline{h}}_c^{\,i}$, $\underline{n}_c^{\,b} = \dot{\underline{h}}_c^{\,ibb} + \underbrace{\underline{\omega}_b^{\,ib} \times}_{S(\underline{\omega}_b^{\,ib})} \underline{h}_c^{\,ib}$

How to calculate the ang. momentum : $\vec{h}_c^{\,i} = \mathbb{J}_c \vec{\omega}_b^{\,i}$

If we represent $\vec{\rho}$ in $\mathcal{F}^i$ the matrix representation of $\mathbb{J}_c$ will be time varying, but represented in the $\mathcal{F}^b$ it will be tim invariant.
We therefore choose to calculate $\vec{h}_c^{\,i}$ in the body frame $(\mathcal{F}^b)$

$$\vec{h}_c^{\,i} = -\iiint_M \vec{\rho} \times (\vec{\rho} \times \vec{\omega}_b^{\,i})\, dm = \mathbb{J}_c \vec{\omega}_b^{\,i}, \quad \text{where } O_b = A = C$$

$$\underline{h}_c^{ib} = -\iiint_M \underline{\rho}^b \times (\underline{\rho}^b \times \underline{\omega}_b^{ib})\, dm = -\iiint_M S(\underline{\rho}^b)\, S(\underline{\rho}^b)\, \underline{\omega}_b^{ib}\, dm$$

$$= \underbrace{\left(-\iiint_M S(\underline{\rho}^b)\, S(\underline{\rho}^b)\, dm\right)}_{J_c^b = [\mathbb{J}_c]^b} \underline{\omega}_b^{ib} = J_c^b\, \underline{\omega}_b^{ib}$$

$$\boxed{\begin{aligned} \underline{h}_c^{ib} &= J_c^b \, \underline{w}_b^{ib} \\ \underline{n}_c^b &= \dot{\underline{h}}_c^{ibb} + \underline{w}_b^{ib} \times \underline{h}_c^{ib} = J_c^b \, \dot{\underline{w}}_b^{ibb} + S(\underline{w}_b^{ib}) \, J_c^b \, \underline{w}_b^{ib} \end{aligned}}$$

J_c^b : Matrix of inertia

$$J_c^b = -\iiint_M S(\underline{\rho}^b) S(\underline{\rho}^b) \, dm \quad , \quad \underline{\rho}^b = [\rho_1 ; \rho_2 ; \rho_3]$$

$$= -\iiint_M \begin{bmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & -\rho_1 \\ -\rho_2 & \rho_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & -\rho_1 \\ -\rho_2 & \rho_1 & 0 \end{bmatrix} dm$$

$$= + \iiint_M \begin{bmatrix} \rho_2^2 + \rho_3^2 & -\rho_1\rho_2 & -\rho_1\rho_3 \\ -\rho_2\rho_1 & \rho_1^2 + \rho_3^2 & -\rho_2\rho_3 \\ -\rho_3\rho_1 & -\rho_3\rho_2 & \rho_1^2 + \rho_2^2 \end{bmatrix} dm$$

$$J_c^b = [\mathbb{J}_c]^b$$

Momentum of inertia

$$[J_c^b]_{ii} = \iiint_M (\rho_j^2 + \rho_k^2) \, dm , \quad \begin{matrix} j \neq k \\ \neq \quad \neq \\ i \quad\; i \end{matrix}$$

$$[J_c^b]_{ij} = -\iiint_M \rho_i \rho_j \, dm , \quad i \neq j$$

Product of inertia

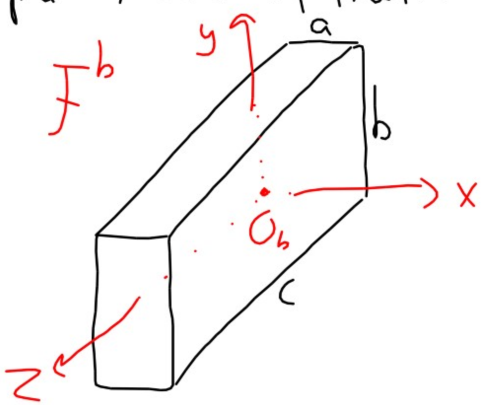
J_c^b is a real, symmetrical, pos. def. matrix ($\mathrm{Det}(J_c^b) > 0$)

$\Rightarrow$ matrix has real eigenvalues and orthogonal eigenvectors.

Eigenvectors are called the main axis of the body. I.e. if $\mathcal{F}^b$ has its basis vectors along the body's main axis we get:

$$J_c^b = \mathrm{diag}\left(J_{xx}^b, J_{yy}^b, J_{zz}^b\right)$$

Example: Matrix of inertia of a brick:



Assume: $a < b < c$

$\mathcal{F}^b$ has origo in the centre of mass O_b

$$\underline{r}^b = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Cross-terms:

$$\left[J_c^b\right]_{xy} = -\iiint_M xy\,dm = -\iiint xy\,k\,dx\,dy\,dz \quad , \quad k : \text{mass density (constant)}$$

$$= -k \int_{-c/2}^{c/2}\int_{-b/2}^{b/2} y \left(\int_{-a/2}^{a/2} x\,dx\right) dy\,dz = 0 \quad \text{due to symmetry}$$

Diag elements:

$$\left[J_c^b\right]_{xx} = k \iiint (y^2+z^2)\,dx\,dy\,dz = k\iint (y^2+z^2)\left(\int_{-a/2}^{a/2} 1\,dx\right) dy\,dz$$

$$= k\,a \iint (y^2+z^2)\,dy\,dz = ak\iint y^2\,dy\,dz + ak\iint z^2\,dy\,dz$$

$$= akc\int_{-b/2}^{b/2} y^2\,dy + akb\int_{-c/2}^{c/2} z^2\,dz$$

$$= akc\left[\frac{1}{3}y^3\right]_{-b/2}^{b/2} + akb\left[\frac{1}{3}z^3\right]_{-c/2}^{c/2} = akc\,\frac{2}{3}\cdot\frac{1}{8}b^3 + akb\,\frac{2}{3}\cdot\frac{1}{8}c^3$$

$$= \underbrace{k\,abc}_{M}\,\frac{b^2}{12} + \underbrace{k\,abc}_{M}\,\frac{c^2}{12} = \frac{M}{12}\left(b^2+c^2\right)$$

$$J_c^b = \mathrm{diag}\left(\frac{M}{12}(b^2 + c^2) , \frac{M}{12}(a^2 + c^2) , \frac{M}{12}(a^2 + b^2) \right)$$

We had $a < b < c \Rightarrow J_{xx}^b > J_{yy}^b > J_{zz}^b$