Theorems and proofs for Partial differential equations TMA372

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2015

Introduction

This text is written as an aid for those that are taking the course TMA372 Partial differential equations. It contains the recommended theorems and proofs for the year of 2015, taken from the lecture notes of Mohammad Asadzadeh's lectures.

Contents

1 L_∞ error estimates for linear interpolation in an interval

Theorem 1.1. *If $f'' \in L_\infty(a,b)$ and $q = 1$ (two interpolation nodes), then there are interpolation constants c_i independent of f and the length of $[a,b]$ such that*

(i) $||\pi_1 f - f||_{L_\infty(a,b)} \leq c_i(b-a)^2||f''||_{L_\infty(a,b)}$

(ii) $||\pi_1 f - f||_{L_\infty(a,b)} \leq c_i(b-a)||f'||_{L_\infty(a,b)}$

(iii) $||(\pi_1 f)' - f'||_{L_\infty(a,b)} \leq c_i(b-a)||f''||_{L_\infty(a,b)}$.

Proof. Every linear function $p(x)$ on $[a,b]$ can be written as a linear combination of the basis functions

$$\lambda_a(x) = \frac{b-x}{b-a} \quad \text{and} \quad \lambda_b(x) = \frac{x-a}{b-a}.$$

In other words $p(x) = p(a)\lambda_a(x) + p(b)\lambda_b(x)$. Further, we note that

$$\lambda_a(x) + \lambda_b(x) = 1 \quad \text{and} \quad a\lambda_a(x) + b\lambda_b(x) = x.$$

We have that

$$\pi_1 f(x) = f(a)\lambda_a(x) + f(b)\lambda_b(x) \tag{1.1}$$

is the linear function connecting the points $(a,f(a))$ and $(b,f(b))$.

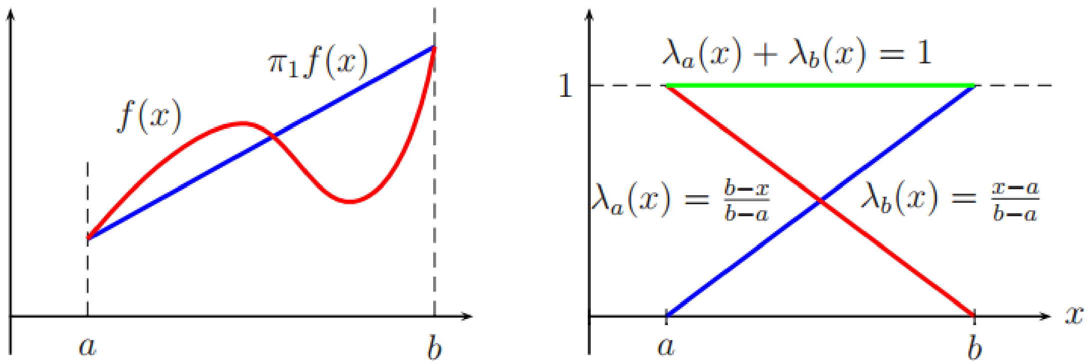

Figure 1: Linear lagrange basis functions.

We now Taylor expand $f(a)$ and $f(b)$ about $x \in (a,b)$:

$$\begin{cases} f(a) = f(x) + (a-x)f'(x) + \frac{1}{2}(a-x)^2 f''(\eta_a), & \eta_a \in [a,x] \\ f(b) = f(x) + (b-x)f'(x) + \frac{1}{2}(b-x)^2 f''(\eta_b), & \eta_b \in [x,b]. \end{cases} \tag{1.2}$$

Now we insert $f(a)$ and $f(b)$ from (1.2) into (1.1), and we get

$$\begin{aligned}\pi_1 f(x) &= f(x)[\lambda_a(x)+\lambda_b(x)] + f'(x)[(a-x)\lambda_a(x)+(b-x)\lambda_b(x)]+\\ &+\frac{1}{2}(a-x)^2 f''(\eta_a)\lambda_a(x)+\frac{1}{2}(b-x)^2 f''(\eta_b)\lambda_b(x) =\\ &= f(x)+\frac{1}{2}(a-x)^2 f''(\eta_a)\lambda_a(x)+\frac{1}{2}(b-x)^2 f''(\eta_b)\lambda_b(x).\end{aligned}$$

Therefore

$$|\pi_1 f(x) - f(x)| = \left|\frac{1}{2}(a-x)^2 f''(\eta_a)\lambda_a(x)+\frac{1}{2}(b-x)^2 f''(\eta_b)\lambda_b(x)\right|. \tag{1.3}$$

Further, we note that in the interval $x \in [a,b]$ it holds that $(a-x)^2 \leq (a-b)^2$ and $(b-x)^2 \leq (a-b)^2$. Also, $\lambda_a(x) \leq 1$ and $\lambda_b(x) \leq 1$, $\forall x \in (a,b)$. Moreover, we have from the definition of the maximum norm that $|f''(\eta_a)| \leq |f''|_{L_\infty(a,b)}$ and $|f''(\eta_b)| \leq |f''|_{L_\infty(a,b)}$. This gives us that (1.3) may be estimated as

$$|\pi_1 f(x) - f(x)| \leq \frac{1}{2}(a-b)^2\cdot 1\cdot |f''|_{L_\infty(a,b)} + \frac{1}{2}(a-b)^2\cdot 1\cdot |f''|_{L_\infty(a,b)}, \tag{1.4}$$

and thus

$$|\pi_1 f(x) - f(x)| \leq (a-b)^2|f''|_{L_\infty(a,b)} \text{ corresponding to } c_i = 1.$$

The other estimates (ii) and (iii) are proved similarly. □

2 BVP ⇔ VF ⇔ MP

$$(\mathrm{BVP})_1 \quad \begin{cases} -\big(a(x)u'(x)\big)' = f(x), & 0 < x < 1,\\ u(0) = u(1) = 0. \end{cases} \tag{2.1}$$

$$(\mathrm{VF})_1 \quad \int_0^1 a(x)u'(x)v'(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x,\ \forall v(x) \in H_0^1 \tag{2.2}$$

$(\mathrm{MP})_1$ Find $u \in H_0^1 : F(u) \leq F(w)$, $\forall w \in H_0^1$, where $F(w)$ is the total potential energy of the displacement $w(x)$:

$$F(w) = \underbrace{\frac{1}{2}\int_0^1 a(w')^2\,\mathrm{d}x}_{\text{Internal (elastic) energy}} - \underbrace{\int_0^1 fw\,\mathrm{d}x}_{\text{Load potential}} \tag{2.3}$$

Theorem 2.1. *The following two properties are equivalent:*

(i) u satisfies $(\mathrm{BVP})_1$

(ii) u is twice differentiable and satisfies $(\mathrm{VF})_1$

Proof. [$\Rightarrow$] We assume that $a(x)$ is piecewise continuous in (0,1), bounded and strictly positive for $0 \leq x \leq 1$. Let $v(x)$ and $v'(x)$ be square integrable functions, i.e. $v, v' \in L_2(0,1)$. We define the following space:

$$H_0^1 = \left\{ v(x) : \int_0^1 (v(x)^2 + v'(x)^2)\, \mathrm{d}x < \infty, \quad v(0) = v(1) = 0 \right\}.$$

Now multiply $(\mathrm{BVP})_1$ with the test function $v(x) \in H_0^1(0,1)$ and integrate over (0,1). This yields

$$-\int_0^1 \Big(a(x)u'(x)\Big)' v(x)\, \mathrm{d}x = \int_0^1 f(x)v(x)\, \mathrm{d}x.$$

Integration by parts gives us

$$-\Big[a(x)u'(x)v(x)\Big]_0^1 + \int_0^1 a(x)u'(x)v'(x)\, \mathrm{d}x = \int_0^1 f(x)v(x)\, \mathrm{d}x.$$

However, since $v(0) = v(1) = 0$ we get $(\mathrm{VF})_1$:

$$\int_0^1 a(x)u'(x)v(x)\, \mathrm{d}x = \int_0^1 f(x)v(x)\, \mathrm{d}x, \quad \forall v(x) \in H_0^1.$$

[$\Leftarrow$] We start by integrating the left hand side of $(\mathrm{VF})_1$ by parts:

$$-\int_0^1 \Big(a(x)u'(x)\Big)' v(x)\, \mathrm{d}x = \int_0^1 f(x)v(x)\, \mathrm{d}x, \quad \forall v(x) \in H_0^1.$$

This can be rewritten as:

$$\int_0^1 \left\{ -\Big(a(x)u'(x)\Big)' - f(x) \right\} v(x)\, \mathrm{d}x = 0, \quad \forall v(x) \in H_0^1. \tag{2.4}$$

To show that u satisfies $(\mathrm{BVP})_1$ is equivalent to claim that

$$(2.4) \Rightarrow -\Big(a(x)u'(x)\Big)' - f(x) \equiv 0, \quad \forall x \in (0,1).$$

Suppose that the above claim does not hold. Then

$$\exists \xi \in (0,1) : -\Big(a(\xi)u'(\xi)\Big)' - f(\xi) \neq 0,$$

where we may assume WLOG that

$$-\Big(a(\xi)u'(\xi)\Big)' - f(\xi) > 0 \ \ (\text{or } \ < 0).$$

Thus, with the assumptions that $f \in C(0{,}1)$ and $a \in C^1(0{,}1)$, we have by continuity that $\exists \delta > 0$, such that in a δ-neighbourhood of ξ,

$$g(x) := -\Big(a(x)u'(x)\Big)' - f(x) > 0, \quad \forall x \in (\xi - \delta, \xi + \delta).$$

Now choose $v(x)$ from (2.4) as the hat function $v^*(x) > 0$ with $v^*(\xi) = 1$ and the support $I_\delta := (\xi - \delta, \xi + \delta)$. Then $v^*(x) \in H_0^1$ and

$$\int_0^1 \Big\{ -\Big(a(x)u'(x)\Big)' - f(x) \Big\} v^*(x) \, \mathrm{d}x = \int_{I_\delta} \underbrace{g(x)}_{>0} \underbrace{v^*(x)}_{>0} \, \mathrm{d}x > 0.$$

This is a contradiction of (2.4), and thus our claim holds and the proof is complete. $\square$

Theorem 2.2. *The following two properties are equivalent:*

(i) u satisfies $(\mathrm{VF})_1$

(ii) u is the solution for the minimisation problem $(\mathrm{MP})_1$.

In other words

$$\int_0^1 au'v' \, \mathrm{d}x = \int_0^1 fv \, \mathrm{d}x, \forall v \in H_0^1 \iff F(u) \leq F(w), \forall w \in H_0^1$$

Proof. [$\Rightarrow$] For $w \in H_0^1$, let $v = w - u$. Then $v \in H_0^1$ since H_0^1 is a vector space and $u \in H_0^1$. Also

$$\begin{aligned} F(w) = F(u+v) &= \frac{1}{2}\int_0^1 a\Big((u+v)'\Big)^2 \mathrm{d}x - \int_0^1 f(u+v)\,\mathrm{d}x = \\ &= \underbrace{\frac{1}{2}\int_0^1 2au'v'\,\mathrm{d}x}_{(I)} + \underbrace{\frac{1}{2}\int_0^1 a(u')^2\,\mathrm{d}x}_{(II)} + \frac{1}{2}\int_0^1 a(v')^2\,\mathrm{d}x - \\ &\quad - \underbrace{\int_0^1 fu\,\mathrm{d}x}_{(III)} - \underbrace{\int_0^1 fv\,\mathrm{d}x}_{(IV)}. \end{aligned}$$

Using $(\mathrm{VF})_1$ we get that $(I) - (IV) = 0$. Further, by the definition of the functional F, $(II) - (III) = F(u)$. Thus

$$F(w) = F(u) + \frac{1}{2}\int_0^1 a(x)(v'(x))^2 \, \mathrm{d}x,$$

and since $a(x) > 0$ we get that $F(w) \geq F(u)$ and the first implication is proven.

[$\Leftarrow$] We assume that $F(u) \leq F(w)\ \forall w \in H_0^1$. Let $g_v(\epsilon) = F(u + \epsilon v)$ for an arbitrary $v \in H_0^1$. Then $(\mathrm{MP})_1$ gives us that g (as a function of ϵ) has a minimum at $\epsilon = 0$. In other words, $\frac{\mathrm{d}}{\mathrm{d}\epsilon} g_v(\epsilon)\Big|_{\epsilon=0} = 0$. We have that

$$g_v(\epsilon) = F(u + \epsilon v) = \frac{1}{2}\int_0^1 a\Big((u + \epsilon v)'\Big)^2 \,\mathrm{d}x - \int_0^1 f(u + \epsilon v)\,\mathrm{d}x =$$
$$= \frac{1}{2}\int_0^1 \{a(u')^2 + a\epsilon^2(v')^2 + 2a\epsilon u'v'\}\,\mathrm{d}x - \int_0^1 fu\,\mathrm{d}x - \epsilon\int_0^1 fv\,\mathrm{d}x,$$

and that the derivative of $g_v(\epsilon)$ is

$$\frac{\mathrm{d}g_v(\epsilon)}{\mathrm{d}\epsilon} = \frac{1}{2}\int_0^1 \{2a\epsilon(v')^2 + 2au'v'\}\,\mathrm{d}x - \int_0^1 fv\,\mathrm{d}x.$$

But, $\frac{\mathrm{d}g_v(\epsilon)}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} = 0$, which gives

$$\int_0^1 au'v'\,\mathrm{d}x - \int_0^1 fv\,\mathrm{d}x = 0,$$

which is our desired variational formulation $(\mathrm{VF})_1$. In conclusion, we can state that $F(u) \leq F(w)\ \forall w \in H_0^1 \Rightarrow (\mathrm{VF})_1$, which completes the proof. $\square$

3 Error estimates for BVP

Definition

A finite element formulation for the Dirichlet BVP (2.1) is given by: find $u_h \in V_h^{(0)}$ such that:

$$(\mathrm{FEM}) \quad \int_0^1 a(x)u_h'(x)v'(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x, \quad \forall v \in V_h^{(0)}, \tag{3.1}$$

where

$$V_h^{(0)} = \{v : v \in C(I, P_1(I_k)), v(0) = v(1) = 0\},$$

$I_k = [x_{k-1}, x_k]$ a subinterval of $I = [0{,}1]$. So, $C(I, P_1(I_k))$ denotes the set of all continuous piecewise linear functions on the partition $\tau_h = \{0 = x_0 < x_1 < \cdots < x_M < x_{M+1} = 1\}$ of I.

Note that $V_h^{(0)}$ is a finite dimensional subspace of H_0^1.

Theorem 3.1. *If $u(x)$ is a solution to the Dirichlet BVP (2.1) and $u_h(x)$ its finite element approximation defined by (3.1), then*

$$||u-u_h||_E \leq ||u-v||_E, \quad \forall v(x) \in V_h^{(0)}. \tag{3.2}$$

In other words, the finite element solution $u_h \in V_h^{(0)}$ is the best approximation in the energy norm of the solution u, by functions in $V_h^{(0)}$.

Proof. Take $v \in V_h^{(0)}$ arbitrarily. We have that

$$\begin{aligned}
||u-u_h||_E^2 &= \int_0^1 a(x)(u'(x)-u_h'(x))^2\,\mathrm{d}x = \\
&= \int_0^1 a(x)\Big(u'(x)-u_h'(x)\Big)\Big(u'(x)-v'(x)+v'(x)-u_h'(x)\Big)\,\mathrm{d}x = \\
&= \int_0^1 a(x)\Big(u'(x)-u_h'(x)\Big)\Big(u'(x)-v'(x)\Big)\,\mathrm{d}x + \\
&+ \underbrace{\int_0^1 a(x)\Big(u'(x)-u_h'(x)\Big)\Big(v'(x)-u_h'(x)\Big)\,\mathrm{d}x}_{=0 \text{ by Galerkin orthogonality since } v-u_h\in V_h^{(0)}\subset H_0^1} = \\
&= \int_0^1 a(x)^{1/2}\Big(u'(x)-u_h'(x)\Big)a(x)^{1/2}\Big(u'(x)-v'(x)\Big)\,\mathrm{d}x \leq \\
&\leq \left(\int_0^1 a(x)\Big(u'(x)-u_h'(x)\Big)^2\mathrm{d}x\right)^{1/2}\left(\int_0^1 a(x)\Big(u'(x)-v'(x)\Big)^2\mathrm{d}x\right)^{1/2} = \\
&= ||u-u_h||_E \cdot ||u-v||_E, \quad \forall v \in V_h^{(0)},
\end{aligned}$$

where we used Cauchy-Schwartz' inequality in the last estimate. The result is thus

$$||u-u_h||_E \leq ||u-v||, \quad \forall v \in V_h^{(0)}.$$

□

Theorem 3.2 (A priori error estimate)**.** *If u and u_h are the solutions to the* (BVP)$_1$ *and and* (FEM) *respectively, then there exists an interpolation constant C_i depending only on $a(x)$ such that*

$$||u-u_h||_E \leq C_i||hu''||_a$$

Proof. Since $\pi_h u(x) \in V_h^{(0)}$, we may choose $v = \pi_h u(x)$ in (3.2) and use an estimate from an interpolation theorem (generalised to the weighted norm $||\cdot||_a$) to get

$$||u-u_h||_E \leq ||u-\pi_h u||_E = ||u'-(\pi_h u)'||_a \leq C_i||hu''||_a = C_i\left(\int_0^1 a(x)h^2(x)u''(x)^2\,\mathrm{d}x\right)^{1/2},$$

which is the desired result and thus the proof is complete. □

Remark 1. *The interpolation theorem is not in the weighted norm. The $a(x)$ dependence of the interpolation constant C_i can be shown as follows:*

$$\begin{aligned} ||u' - (\pi_h u)'||_a &= \left(\int_0^1 a(x)\Big(u'(x) - (\pi_h u)'(x)\Big)^2 \,\mathrm{d}x\right)^{1/2} \leq \\ &\leq \Big(\max_{x\in[0,1]} a(x)^{1/2}\Big) \cdot ||u' - (\pi_h u)'||_{L_2} \leq c_i \Big(\max_{x\in[0,1]} a(x)^{1/2}\Big) ||hu''||_{L_2} = \\ &= c_i \Big(\max_{x\in[0,1]} a(x)^{1/2}\Big) \left(\int_0^1 h(x)^2 u''(x)^2 \,\mathrm{d}x\right)^{1/2} \leq \\ &\leq c_i \frac{\max\limits_{x\in[0,1]} a(x)^{1/2}}{\min\limits_{x\in[0,1]} a(x)^{1/2}} \left(\int_0^1 a(x) h(x)^2 u''(x)^2 \,\mathrm{d}x\right)^{1/2}. \end{aligned}$$

So, we have that

$$C_i = c_i \frac{\max\limits_{x\in[0,1]} a(x)^{1/2}}{\min\limits_{x\in[0,1]} a(x)^{1/2}},$$

where c_i is the interpolation constant from a theorem (second estimate in theorem 3.3 from the lecture notes).

Theorem 3.3 (A posteriori error estimate)**.** *There is an interpolation constant c_i depending only on $a(x)$ such that the error in the finite element approximation of the* (BVP) *satisfies*

$$||e(x)||_E \leq c_i \left(\int_0^1 \frac{1}{a(x)} h(x)^2 R(u_h(x))^2 \,\mathrm{d}x\right)^{1/2},$$

where $R(u_h(x)) = f + (a(x)u_h'(x))'$ is the residual and $e(x) := u(x) - u_h(x) \in H_0^1$.

Proof. We start with the definition of the energy norm:

$$\begin{aligned} ||e(x)||_E^2 &= \int_0^1 a(x)(e'(x))^2 \,\mathrm{d}x = \int_0^1 a(x)\Big(u'(x) - u_h'(x)\Big)e'(x) \,\mathrm{d}x = \\ &= \int_0^1 a(x)u'(x)e'(x) \,\mathrm{d}x - \int_0^1 a(x)u_h'(x)e'(x) \,\mathrm{d}x. \end{aligned}$$

Further, we know that $e \in H_0^1$, so the variational formulation $(VF)_1$ gives

$$\int_0^1 a(x)u'(x)e'(x) \,\mathrm{d}x = \int_0^1 f(x)e(x) \,\mathrm{d}x.$$

This gives

$$||e(x)||_E^2 = \int_0^1 f(x)e(x) \,\mathrm{d}x - \int_0^1 a(x)u_h'(x)e'(x) \,\mathrm{d}x.$$

We now add and subtract the interpolant $\pi_h e(x)$ and its derivative $(\pi_h e)'(x)$ to e and e' in the integrands, which yields

$$\begin{aligned}||e(x)||_E^2 &= \int_0^1 f(x)\Big(e(x) - \pi_h e(x)\Big)\,\mathrm{d}x + \underbrace{\int_0^1 f(x)\pi_h e(x)\,\mathrm{d}x}_{(i)} - \\ &- \int_0^1 a(x)u_h'(x)\Big(e'(x) - (\pi_h e)'(x)\Big)\,\mathrm{d}x - \underbrace{\int_0^1 a(x)u_h'(x)(\pi_h e)'(x)\,\mathrm{d}x}_{(ii)}.\end{aligned}$$

We now have that $(i)-(ii) = 0$, since $u_h(x)$ is a solution of (FEM) and $\pi_h e(x) \in V_h^{(0)}$. Thus we have

$$\begin{aligned}||e(x)||_E^2 &= \int_0^1 f(x)\Big(e(x) - \pi_h e(x)\Big)\,\mathrm{d}x - \int_0^1 a(x)u_h'(x)\Big(e'(x) - (\pi_h e)'(x)\Big)\,\mathrm{d}x = \\ &= \int_0^1 f(x)\Big(e(x) - \pi_h e(x)\Big)\,\mathrm{d}x - \sum_{k=1}^{M+1}\int_{x_{k-1}}^{x_k} a(x)u_h'(x)\Big(e'(x) - (\pi_h e)'(x)\Big)\,\mathrm{d}x.\end{aligned}$$

Further, we integrate by parts in the integrals in the sum:

$$\begin{aligned}&-\int_{x_{k-1}}^{x_k} a(x)u_h'(x)\Big(e'(x) - (\pi_h e)'(x)\Big)\,\mathrm{d}x = \\ &= -\Big[a(x)u_h'(x)\Big(e(x) - \pi_h e(x)\Big)\Big]_{x_{k_1}}^{x_k} + \int_{x_{k-1}}^{x_k}\Big(a(x)u_h'(x)\Big)'\Big(e(x) - \pi_h e(x)\Big)\,\mathrm{d}x.\end{aligned}$$

Now we use the fact that $e(x_k) = \pi_h e(x_k)$ for $k = 1,2,\ldots,M+1$, where x_k are the interpolation nodes. This makes the boundary terms vanish and we have

$$-\int_{x_{k-1}}^{x_k} a(x)u_h'(x)\Big(e'(x) - (\pi_h e)'(x)\Big)\,\mathrm{d}x = \int_{x_{k-1}}^{x_k}\Big(a(x)u_h'(x)\Big)'\Big(e(x) - \pi_h e(x)\Big)\,\mathrm{d}x.$$

Now we sum over k to get

$$-\int_0^1 a(x)u_h'(x)\Big(e'(x) - (\pi_h e)'(x)\Big)\,\mathrm{d}x = \int_0^1\Big(a(x)u_h'(x)\Big)'\Big(e(x) - \pi_h e(x)\Big)\,\mathrm{d}x,$$

where we interpret $\Big(a(x)u_h'(x)\Big)$ on each local subinterval $[x_{k-1}, x_k]$, since u_h' in general is discontinuous which implies that u_h'' does not exist globally on [0,1]. Thus

$$\begin{aligned}||e(x)||_E^2 &= \int_0^1 f(x)\Big(e(x) - \pi_h e(x)\Big)\,\mathrm{d}x + \int_0^1\Big(a(x)u_h'(x)\Big)'\Big(e(x) - \pi_h e(x)\Big)\,\mathrm{d}x = \\ &= \int_0^1\Big[f(x) + \Big(a(x)u_h'(x)\Big)'\Big]\Big[e(x) - \pi_h e(x)\Big]\,\mathrm{d}x.\end{aligned}$$

Now we let the residual error be $R(u_h(x)) = f(x) + \Big(a(x)u_h'(x)\Big)'$. This is a well-defined function except in in the set $\{x_k\}, k = 1, \ldots, M-1$ since $\Big(a(x_k)u_h'(x_k)\Big)'$ is not defined there. By now using Cauchy-Schwartz' inequality we get

$$\begin{aligned}||e(x)||_E^2 &= \int_0^1 R(u_h(x))\Big(e(x) - \pi_h e(x)\Big)\,\mathrm{d}x = \\ &= \int_0^1 \frac{1}{\sqrt{a(x)}} h(x) R(u_h(x)) \sqrt{a(x)} \left(\frac{e(x) - \pi_h e(x)}{h(x)}\right) \mathrm{d}x \leq \\ &\leq \left(\int_0^1 \frac{1}{a(x)} h^2(x) R^2(u_h(x))\,\mathrm{d}x\right)^{1/2} \left(\int_0^1 a(x) \left(\frac{e(x) - \pi_h e(x)}{h(x)}\right)^2 \mathrm{d}x\right)^{1/2}.\end{aligned}$$

Now, by the definition of the L_2-norm we get that

$$\left|\left|\frac{e(x) - \pi_h e(x)}{h(x)}\right|\right|_a = \left(\int_0^1 a(x) \left(\frac{e(x) - \pi_h e(x)}{h(x)}\right)^2 \mathrm{d}x\right)^{1/2}. \tag{3.3}$$

To estimate equation (3.3) we can use an interpolation estimate for $e(x)$ in each subinterval which gives us

$$\left|\left|\frac{e(x) - \pi_h e(x)}{h(x)}\right|\right|_a \leq C_i ||e'(x)||_a = C_i ||e(x)||_E,$$

for a constant C_i dependent on $a(x)$. Thus,

$$||e(x)||_E^2 \leq \left(\int_0^1 \frac{1}{a(x)} h^2(x) R^2(u_h(x))\,\mathrm{d}x\right)^{1/2} C_i ||e(x)||_E,$$

which completes the proof. □

4 Stability estimates for IVP

Here we shall consider the following problem.

$$\begin{cases} \dot{u}(t) + a(t)u(t) = f(t), & 0 < t \leq T \\ u(0) = u_0, \end{cases} \tag{4.1}$$

where $f(t)$ is the source term and $a(t)$ a bounded function. If $a(t) \geq 0$, the problem is called parabolic, while $a(t) \geq \alpha > 0$ yields a dissipative problem.

Theorem 4.1 (Stability estimates). *Using the solution formula*

$$u(t) = u_0 e^{-A(t)} + \int_0^t e^{-(A(t)-A(s))} f(s)\,\mathrm{d}s, \tag{4.2}$$

where $A(s) = \int_0^t a(s)\,\mathrm{d}s$ *and* $e^{A(t)}$ *is the integrating factor, we can derive the following stability estimates:*

(i) If $a(t) \geq \alpha > 0$, *then*

$$|u(t)| \leq e^{-\alpha t}|u_0| + \frac{1}{\alpha}(1 - e^{-\alpha t}) \max_{0 \leq s \leq t} |f(s)|.$$

(ii) If $a(t) \geq 0$, *i.e.* $\alpha = 0$ *(the parabolic case), then*

$$|u(t)| \leq |u_0| + \int_0^t |f(s)|\,\mathrm{d}s \quad \textit{or} \quad |u(t)| \leq |u_0| + ||f||_{L_1(0,t)}.$$

Proof. We begin with *(i)*. $A(t) = \int_0^t a(s)\,\mathrm{d}s$ will be an increasing function of t, when $a(t) \geq \alpha > 0$, $A(t) \geq \alpha t$. This gives us that

$$A(t) - A(s) = \int_0^t a(r)\,\mathrm{d}r - \int_0^s a(r)\,\mathrm{d}r = \int_t^s a(r)\,\mathrm{d}r \geq \alpha(t-s).$$

Thus we get $e^{-A(t)} \leq e^{-\alpha t}$ and $e^{-(A(t)-A(s))} \leq e^{-\alpha(t-s)}$. Now we use equation (4.2) to get

$$|u(t)| \leq |u_0|e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)}|f(s)|\,\mathrm{d}s. \tag{4.3}$$

By integrating we will get the following

$$|u(t)| \leq e^{-\alpha t}|u_0| + \max_{0 \leq s \leq t} |f(s)| \left[\frac{1}{\alpha} e^{-\alpha(t-s)}\right]_{s=0}^{s=t}, \text{ i.e.}$$

$$|u(t)| \leq e^{-\alpha t}|u_0| + \frac{1}{\alpha}(1 - e^{-\alpha t}) \max_{0 \leq s \leq t} |f(s)|.$$

This proves *(i)*.

We go further to *(ii)*. We let $\alpha = 0$ in equation (4.3), so we get

$$|u(t)| \leq |u_0| + \int_0^t |f(s)|\,\mathrm{d}s,$$

which completes the proof. □

5 Lax-Milgram

Firstly, we recall:

$$(u,v) = \int_0^1 u'(x)v'(x)\,\mathrm{d}x \quad \text{and} \quad \ell(v) = \int_0^1 f(x)v(x)\,\mathrm{d}x.$$

Secondly, in this section we will use the following expressions.

1. Bilinear: $a(u,v)$ satisfies the same properties as scalar products, however it does not need to be symmetric.
2. Bounded: $|a(u,v)| \leq \beta||u||\,||v||$, $\beta > 0$ constant.
3. Elliptic: $a(u,v) \geq \alpha||v||^2$, $\alpha > 0$ constant.

Theorem 5.1 (Lax-Milgram theorem)**.** *Assuming $\ell(v)$ is a bounded, linear functional on V (a Hilbert space with scalar product (u,v) and norm $||u|| = \sqrt{(u,u)}$) and $a(u,v)$ is bilinear bounded and elliptic in V, then there is a unique $u \in V$ such that*

$$a(u,v) = \ell(v), \quad \forall v \in V.$$

Proof. We begin by consider the variational formulation (VF) and the minimisation problem (MP) in abstract forms:

(V) Find $u \in \mathcal{H}_0^1$ such that $(u,v) = \ell(v)\ \forall v \in \mathcal{H}_0^1$

(M) Find $u \in \mathcal{H}_0^1$ such that $F(u) = \min_{v \in \mathcal{H}_0^1} F(v)$, with $F(v) = \frac{1}{2}||v||^2 - \ell(v)$.

We have before proven that (V) and (M) are equivalent. We will now show that (M) has a unique solution and thus we have proven the same for (V) since they are equivalent, which in turn results in that we've proven the Lax-Milgram theorem.

We begin by noting that $\exists$ a real number σ such that $F(v) > \sigma$, $\forall v \in \mathcal{H}_0^1$ (otherwise it would not be possible to minimise F). In other words, we can write

$$F(v) = \frac{1}{2}||v||^2 - \ell(v) \geq \frac{1}{2}||v||^2 - \gamma||v||,$$

where γ is the constant bounding ℓ, i.e. $|\ell(v)| \leq \gamma||v||$. However,

$$0 \leq \frac{1}{2}(||v|| - \gamma)^2 = \frac{1}{2}||v||^2 - \gamma||v|| + \frac{1}{2}\gamma^2,$$

thus we have

$$F(v) \geq \frac{1}{2}||v||^2 - \gamma||v|| \geq -\frac{1}{2}\gamma^2.$$

Let now σ^* be the largest real number σ such that

$$F(v) > \sigma, \ \forall v \in \mathcal{H}_0^1. \tag{5.1}$$

Take now a sequence of functions $\{u_k\}_{k=0}^{\infty}$ such that

$$F(u_k) \to \sigma^*. \tag{5.2}$$

To show that *there exists a unique* solution for (V) and (M) we will use the following:

(i) It is always possible to find a sequence $\{u_k\}_{k=0}^{\infty}$ such that $F(u_k) \to \sigma^*$ (because $\mathbb{R}$ is complete).

(ii) The parallelogram law:

$$||a+b||^2 + ||a-b||^2 = 2||a||^2 + 2||b||^2.$$

Using (ii) and the linearity of ℓ we get

$$\begin{aligned} ||u_k - u_j||^2 &= 2||u_k||^2 + 2||u_j||^2 - ||u_k + u_j||^2 - 4\ell(u_k) - 4\ell(u_j) + 4\ell(u_k + u_j) = \\ &= 2||u_k||^2 - 4\ell(u_k) + 2||u_j||^2 - 4\ell(u_j) - ||u_k + u_j||^2 + 4\ell(u_k + u_j) = \\ &= 4F(u_k) + 4F(u_j) - 8F\left(\frac{u_k + u_j}{2}\right), \end{aligned}$$

where we used the definition $F(v) = \frac{1}{2}||v||^2 - \ell(v)$, with $v = u_k, v = u_j$ and $v = (u_k + u_j)/2$. Also, by the linearity of ℓ we got:

$$-||u_k + u_j||^2 + 4\ell(u_k + u_j) = -4\left|\left|\frac{u_k + u_j}{2}\right|\right|^2 + 8\ell\left(\frac{u_k + u_j}{2}\right) = -8F\left(\frac{u_k + u_j}{2}\right).$$

Now, since $F(u_k) \to \sigma^*$ and $F(u_j) \to \sigma^*$ we have that

$$||u_k + u_j||^2 \leq 4F(u_k) + 4F(u_j) - 8\sigma^* \xrightarrow[k,j\to\infty]{} 0.$$

We have thus shown that $\{u_k\}_{k=0}^{\infty}$ is a Cauchy sequence. Since $\{u_k\} \subset \mathcal{H}_0^1$ and $\mathcal{H}_0^1$ is complete, $\{u_k\}$ is a convergent sequence. In other words, $\exists u \in \mathcal{H}_0^1 : u = \lim_{k\to\infty} u_k$. So by the continuity of F we have

$$\lim_{k\to\infty} F(u_k) = F(u). \tag{5.3}$$

Now (5.2) and (5.3) gives us that $F(u) = \sigma^*$ and by (5.1) and the definition of σ^* we have

$$F(u) < F(v), \quad \forall v \in \mathcal{H}_0^1.$$

Thus we have shown that there is a unique solution for (M), which means there is a unique solution for (V) since (M) $\Leftrightarrow$(V). Also, we didn't use symmetry of the scalar products, which means that we've proven the Lax-Milgram theorem with the bilinear, bounded and elliptic $a(u,v)$. $\square$

6 Error estimates for the Poisson equation

In this section we will study the Poisson equation in higher dimensions:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \subset \mathbb{R}^d,\, d = 2{,}3 \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

Here Ω is a bounded domain in $\mathbb{R}^d$, with polygonal boundary $\partial\Omega$.

Further notation we are going to use is

$$\tau_h = \{K : \cup K = \Omega\},$$

where τ_h is a triangulation of the domain Ω by the element K with the maximum diagonal $h = \max \operatorname{diag}(K)$. Also, we shall consider continous, piecewise linear approximations for the solution $u \in H_0^1(\Omega)$ in a finite dimensional subspace defined as:

$$V_h^{(0)} = \{v(x) : v \text{ continous, piecewise linear on } \tau_h, v = 0 \text{ on } \partial\Omega\}.$$

Theorem 6.1 (A priori error estimate for the Poisson equation)**.** *Let $e = u - U$ be the error in the continous, piecewise linear approximation U of the solution u of the Poisson equation (6.1). Then*

$$||\nabla e|| = ||\nabla(u - U)|| \leq C||hD^2u||,$$

for some constant C, and where

$$D^2u = (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)^{1/2}$$

Proof. For $e = u - U$ we have $\nabla e = \nabla u - \nabla U = \nabla(u - U)$. We consider the variational formulation of the problem:

$$\text{(VF)} \qquad \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x, \quad \forall v \in H_0^1(\Omega), \tag{6.2}$$

and the same for the approximation U:

$$(V_h^{(0)}) \qquad \int_\Omega \nabla U \cdot \nabla v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x, \quad \forall v \in V_h^{(0)}. \tag{6.3}$$

We now subtract equation (6.3) from (6.2) where we restrict v to $V_h^{(0)}$, and we get the Galerkin orthogonality:

$$\int_\Omega (\nabla u - \nabla U) \cdot \nabla v \, \mathrm{d}x = \int_\Omega \nabla e \cdot \nabla v \, \mathrm{d}x = 0, \quad \forall v \in V_h^{(0)}. \tag{6.4}$$

Further, we get that

$$||\nabla e||^2 = \int_\Omega \nabla e \cdot \nabla e \, \mathrm{d}x = \int_\Omega \nabla e \cdot \nabla(u-U) \, \mathrm{d}x = \int_\Omega \nabla e \cdot \nabla u \, \mathrm{d}x - \int_\Omega \nabla e \cdot \nabla U \, \mathrm{d}x.$$

By using the Galerkin orthogonality (6.4), which we can do since $U \in V_h^{(0)}$, we get that

$$\int_\Omega \nabla e \cdot \nabla U \, \mathrm{d}x = 0.$$

Thus, we can remove the ∇U term and insert $\int_\Omega \nabla e \cdot \nabla v \, \mathrm{d}x = 0$, $\forall v \in V_h^{(0)}$:

$$||\nabla e||^2 = \int_\Omega \nabla e \cdot \nabla u \, \mathrm{d}x - \int_\Omega \nabla e \cdot \nabla v \, \mathrm{d}x = \int_\Omega \nabla e \cdot \nabla(u-v) \, \mathrm{d}x \leq ||\nabla e|| \, ||\nabla(u-v)||.$$

This means that

$$||\nabla(u-U)|| \leq ||\nabla(u-v)||, \quad \forall v \in V_h^{(0)}. \tag{6.5}$$

Thus, ∇U is, in the L_2-norm, closer to the exact solution ∇u than any other ∇v, $v \in V_h^{(0)}$. In other words, measuring in H_0^1-norm, the error $u - U$ is orthogonal to $V_h^{(0)}$. It is possible to show that there is an interpolant $v \in V_h^{(0)}$ such that

$$||\nabla(u-v)|| \leq C||hD^2u||, \tag{6.6}$$

where $h = h(x)$ =diag(K), $x \in K$ and C is a constant independent of h. If we now combine equations (6.5) and (6.6) we get

$$||\nabla e|| = ||\nabla(u-U)|| \leq C||hD^2u||,$$

which indicates that the error is small if $h(x)$ is sufficiently small depending on D^2u. And thus the proof is done. □

Theorem 6.2 (A posteriori error estimate for the Poisson equation)**.** *Let u be the solution of the Poisson equation (6.1) and U its continous, piecewise linear finite element approximation. Then there is a constant C independent of u and h, such that*

$$||u-U|| \leq C||h^2 r||,$$

where $r = f + \Delta_h U$ is the residual with Δ_h being the discrete Laplacian defined by

$$(\Delta_h U, v) = \sum_{K \in \tau_h} (\nabla U, \nabla v)_K.$$

Proof. We begin by considering the dual problem

$$\begin{cases} -\Delta\varphi(x) = & e(x), & x \in \Omega \\ \varphi(x) = 0, & x \in \partial\Omega, & e(x) = u(x) - U(x). \end{cases}$$

Then we have $e(x) = 0$, $\forall x \in \partial\Omega$, and by using Green's formula we get that

$$||e||^2 = \int_\Omega ee \,\mathrm{d}x = \int_\Omega e(-\Delta\varphi) \,\mathrm{d}x = \int_\Omega \nabla e \cdot \nabla\varphi \,\mathrm{d}x.$$

By the Galerkin orthogonality and the boundary condition $\varphi(x) = 0$, $\forall x \in \partial\Omega$, we get

$$\begin{aligned}||e||^2 &= \int_\Omega \nabla e \cdot \nabla\varphi \,\mathrm{d}x - \int_\Omega \nabla e \cdot \nabla v \,\mathrm{d}x = \int_\Omega \nabla e \cdot \nabla(\varphi - v) \,\mathrm{d}x = \\ &= \int_\Omega (-\Delta e)(\varphi - v) \,\mathrm{d}x \leq ||h^2 r||\, ||h^{-2}(\varphi - v)|| \leq \\ &\leq C||h^2 r||\, ||\Delta\varphi|| \leq C||h^2 r||\, ||e||,\end{aligned}$$

where we use that $-\Delta e = -\Delta u + \Delta_h U = f + \Delta_h U = r$ is the residual and choose v as an interpolant of φ. Hence, the final result becomes

$$||u - U|| \leq C||h^2 r||.$$

□

7 Stability and energy estimates for the heat equation in $\mathbb{R}^n$

We will consider the initial boundary value problem for the heat conduction in R^n:

$$\begin{cases} \dot{u}(x,t) - \Delta u(x,t) = f(x,t), & \Omega \subset \mathbb{R}^d,\ d = 1,2,3 \quad \text{(DE)} \\ u(x,t) = 0, & 0 < t < T,\ \text{(BC)} \\ u(x,0) = u_0(x), & x \in \Omega\ \text{(IC)} \end{cases} \tag{7.1}$$

Theorem 7.1 (Energy estimates)**.** *Let $f \equiv 0$. Then the solution u of the heat equation (7.1) satisfies the following stability estimates:*

$$\max\left(||u||(t), 2\int_0^t ||\nabla u||^2(s) \,\mathrm{d}s\right) \leq ||u_0||^2 \tag{7.2}$$

$$||\nabla u||(t) \leq \frac{1}{\sqrt{2t}}||u_0|| \tag{7.3}$$

$$\left(\int_0^t s||\Delta u||^2(s) \,\mathrm{d}s\right)^{1/2} \leq \frac{1}{2}||u_0|| \tag{7.4}$$

$$||\Delta u||(t) \leq \frac{1}{\sqrt{2}t}||u_0|| \tag{7.5}$$

$$\int_\epsilon^t ||\dot{u}||(s) \,\mathrm{d}s \leq \frac{1}{2}\sqrt{\ln\frac{t}{\epsilon}}||u_0|| \tag{7.6}$$

Proof. Let $f \equiv 0$. To derive the first two estimates, we multiply (7.1) by u and integrate over Ω:

$$\int_\Omega \dot{u}u \,\mathrm{d}x - \int_\Omega (\Delta u)u \,\mathrm{d}x = 0. \tag{7.7}$$

We note that $\dot{u}u = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}u^2$, so by using Green's formula with the Dirichlet boundary condition $u = 0$ on $\partial\Omega$ we get

$$-\int_\Omega (\Delta u)u \,\mathrm{d}x = -\int_{\partial\Omega} (\nabla u \cdot \mathbf{n})u \,\mathrm{d}s + \int_\Omega \nabla u \cdot \nabla u \,\mathrm{d}x = \int_\Omega |\nabla u|^2 \,\mathrm{d}x.$$

We can thus write equation (7.7) as:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2 \,\mathrm{d}x + \int_\Omega |\nabla u|^2 \,\mathrm{d}x = 0 \Leftrightarrow \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||u||^2 + ||\nabla u||^2 = 0, \tag{7.8}$$

where $||\cdot||$ denotes the $L_2(\Omega)$-norm. Now we substitute t with s and integrate equation (7.8) over $s \in (0,t)$ to get

$$\frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}||u||^2(s)\,\mathrm{d}s + \int_0^t ||\nabla u||^2(s)\,\mathrm{d}s = \frac{1}{2}||u||^2(t) - \frac{1}{2}||u||^2(0) + \int_0^t ||\nabla u||^2 \,\mathrm{d}s = 0.$$

Now by inserting $u(0) = u_0$ we have

$$||u||^2(t) + 2\int_0^t ||\nabla u||(s)\,\mathrm{d}s = ||u_0||^2.$$

Thus we have the first two estimates:

$$||u||(t) \leq ||u_0||, \quad \text{and} \quad \int_0^t ||\nabla u||^2(s)\,\mathrm{d}s \leq \frac{1}{2}||u_0||^2.$$

To derive estimates (7.3) and (7.4) we now instead multiply $\dot{u} - \Delta u = 0$ with $-t\Delta u$ and integrate over Ω to get

$$-t\int_\Omega \dot{u}\Delta u \,\mathrm{d}x + t\int_\Omega (\Delta u)^2 \,\mathrm{d}x = 0. \tag{7.9}$$

Green's formula ($u = 0$ on $\partial\Omega \Rightarrow \dot{u} = 0$ on $\partial\Omega$) gives us that

$$\int_\Omega \dot{u}\Delta u \,\mathrm{d}x = -\int_\Omega \nabla\dot{u} \cdot \nabla u \,\mathrm{d}x = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\nabla u||^2,$$

so equation (7.9) can be rewritten as

$$t\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\nabla u||^2 + t||\Delta u||^2 = 0.$$

Further, the above equation can be rewritten using the relation $t\frac{\mathrm{d}}{\mathrm{d}t}||\nabla u||^2 = \frac{\mathrm{d}}{\mathrm{d}t}(t||\nabla u||^2) - ||\nabla u||^2$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t||\nabla u||^2\right) + 2t||\Delta u||^2 = ||\nabla u||^2.$$

Again, we substitute t with s and integrate over $(0,t)$:

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\left(s||\nabla u||^2\right)\,\mathrm{d}s + \int_0^t 2s||\Delta u||^2(s)\,\mathrm{d}s = \int_0^t ||\nabla u||^2(s)\,\mathrm{d}s.$$

Now, from equation (7.2) we have that the last equality is estimated as

$$t||\nabla u||^2(t) + 2\int_0^t s||\Delta u||^2(s)\,\mathrm{d}s \leq \frac{1}{2}||u_0||^2.$$

From this we have

$$||\nabla u||(t) \leq \frac{1}{\sqrt{2t}}||u_0|| \quad \text{and} \quad \left(\int_0^t s||\Delta u||^2(s)\,\mathrm{d}s\right)^{1/2} \leq \frac{1}{2}||u_0||,$$

which are (7.3) and (7.4). The estimate (7.5) is proved analogously.

To prove the last estimate we use $\dot{u} = \Delta u$ and (7.5) to write

$$\begin{aligned}\int_\epsilon^t ||\dot{u}||(s)\,\mathrm{d}s &= \int_\epsilon^t ||\Delta u||(s)\,\mathrm{d}s = \int_\epsilon^t \frac{1}{\sqrt{s}}\sqrt{s}||\Delta u||(s)\,\mathrm{d}s \leq \\ &\leq \left(\int_\epsilon^t \frac{1}{s}\,\mathrm{d}s\right)^{1/2}\left(\int_\epsilon^t s||\Delta u||^2(s)\,\mathrm{d}s\right)^{1/2} \leq \frac{1}{2}\sqrt{\ln\frac{t}{\epsilon}}||u_0||.\end{aligned}$$

In the last two inequalities we used Cauchy-Schwartz inequality and (7.4), respectively. Thus all estimates are proven, and the proof is done. □