

# Algebra

Applications and Strategies

# ALGEBRA



## APPLICATIONS AND STRATEGIES

EDWARD D. KIM

PRELIMINARY EDITION

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Preliminary edition

Cover image: AAAA.







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# ACKNOWLEDGEMENTS

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# PREFACE

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Hello to “you” the student. As this course has evolved to support our computer science major, so has the text. The current version of the book is intended to support inquiry-based teaching for understanding that is so crucial for future teachers, while also providing the necessary mathematical foundation and application-based motivation for computer science students. While teaching the course in Spring 2024 using an early version of this edition, I was pleasantly surprised by how many students reported that they, for the first time, saw how useful math could be in the “real world.” I hope that this experience can be replicated in other classes using this text.

This book is intended to be used in a class taught using problem-oriented or inquiry-based methods. Each section begins with a preview of the content that includes an open-ended *Investigate!* motivating question, as well as a structured preview activity. The preview activities are carefully scaffolded to provide an entry-point to the section’s topic and to prime students to engage deeply in the material. Depending on the pace of the class, I have found success assigning only the section preview before class, using the preview activity as in-class group work, or assigning the entire section to be read before class (each section concludes with a small set of reading questions that can be assigned to encourage students to actually read). For those readers using this book for self-study, the organization of the sections will hopefully mimic the style of a rich inquiry-based classroom.

Edward D. Kim

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# How to Use This Book

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In addition to expository text, this book has a few features designed to encourage you to interact with the mathematics.

**Investigate! questions.** Sprinkled throughout the sections (usually at the very beginning of a topic) you will find open-ended questions designed to engage you with the topic soon to be discussed. You really should spend some time thinking about, or even working through, these problems before reading the section. However, don't worry if you cannot find a satisfying solution right away. The goal is to pique your interest, so you will read what is next looking for answers.

**Preview Activities.** Most sections include a structured preview activity. These contain leading questions that you should be able to completely answer before reading the section. The idea is that the questions prime you to engage meaningfully with the new content ahead. If you are using the online version, most of these questions will provide you with immediate feedback so you can be confident moving forward.

**Examples.** I have tried to include the “correct” number of examples. For those examples that include *problems*, full solutions are included. Before reading the solution, try to at least have an understanding of what the problem is asking. Unlike some textbooks, the examples are not meant to be all-inclusive for problems you will see in the exercises. They should not be used as a blueprint for solving other problems. Instead, use the examples to deepen your understanding of the concepts and techniques discussed in each section. Then use this understanding to solve the exercises at the end of each section.

**Exercises.** You get good at math through practice. Each section concludes with practice problems meant to solidify concepts and basic skills presented in that section; the online version provides immediate feedback on these problems. There are then additional exercises that are more challenging and open-ended. These might be assigned as written homework or used in class as group work. Some of the additional exercises have hints or solutions in the back of the book, but use these as little as possible. Struggle is good for you. At the end of each chapter, a larger collection of similar exercises is included (as a sort of “chapter review”) which might bridge the material of different sections in that chapter.

**Interactive Online Version.** For those of you reading this in print or as a PDF, I encourage you to also check out the interactive online version. Many of the preview activities and exercises are interactive and can give you immediate feedback. Some of

these have randomized components, allowing you to practice many similar versions of the same problems until you master the topic.

Hints and solutions to examples are also hidden away behind an extra click to encourage you to think about the problem before reading the solution. There is a good search feature available as well, and the index has expandable links to see the content without jumping to the page immediately. There is also a python scratch pad (the pencil icon) so you can try out some code if you feel so inclined.

Additional interactivity is planned. These “bonus” features will be added on a rolling basis, so keep an eye out!

You can view the interactive version for free at `discrete.openmathbooks.org` or by scanning the QR code below.



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# INTRODUCTION

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Welcome to this algebra class!

## 0.1 WHAT IS DISCRETE MATHEMATICS?

dis·crete / dis'krēt.

*Adjective:* Individually separate and distinct.

*Synonyms:* separate - detached - distinct - abstract.

Defining *discrete mathematics* is hard because defining *mathematics* is hard. What is mathematics? The study of numbers? In part yes, but you also study functions and lines and triangles and parallelepipeds and vectors and . . . . Or perhaps you want to say that mathematics is a collection of tools that allow you to solve problems. What sort of problems? Well, those that involve numbers, functions, lines, triangles, . . . . Whatever your conception of what mathematics is, try applying the concept of “discrete” to it, as defined above. Some math fundamentally deals with *stuff* that is individually separate and distinct.

In an algebra or calculus class, you might have found a particular set of numbers (perhaps they constitute the range of a function). You would represent this set as an interval:  $[0, \infty)$  is the range of  $f(x) = x^2$  since the set of outputs of the function are all real numbers 0 and greater. This set of numbers is NOT discrete. The numbers in the set are not separated by much at all. In fact, take any two numbers in the set and there are infinitely many more between them that are also in the set.

Discrete math could still ask about the range of a function, but the set would not be an interval. Consider the function that gives the number of children of each person reading this. What is the range? I’m guessing it is something like  $\{0, 1, 2, 3, 4\}$ . Maybe 5 or 6 is in there too.<sup>1</sup> But certainly nobody reading this has 1.32419 children. This output set *is* discrete because the elements are separate. The inputs to the function also form a discrete set because each input is an individual person.

There are many discrete mathematical objects besides sets of numbers; we will introduce some of these in REMOVED reference. Studying these discrete **structures** is the main focus of discrete mathematics and this book. However, the reason we want to study these structures is because they provide a way to model “real-world” problems.<sup>2</sup>

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<sup>1</sup>Even larger natural numbers for old ladies who live in shoes.

<sup>2</sup>Many of the problems discussed in this book are admittedly contrived and clearly fictional, but hopefully you will see how these toy problems can be generalized to actually represent problems that people would care about in reality.

To get a feel for the subject, let's consider the types of problems you solve in discrete math. Here are a few simple examples:

### ***Investigate!***

*Note: Throughout the book you will see Investigate! activities like this one. Answer the questions in these as best you can to give yourself a feel for what is coming next.*

1. The most popular mathematician in the world is throwing a party for all of his friends. To kick things off, they decide that everyone should shake hands. Assuming all 10 people at the party each shake hands with every other person (but not themselves, obviously) exactly once, how many handshakes take place?
2. At the warm-up event for Oscar's All-Star Hot Dog Eating Contest, Al ate one hot dog. Bob then showed him up by eating three hot dogs. Not to be outdone, Carl ate five. This continued with each contestant eating two more hot dogs than the previous contestant. How many hot dogs did Zeno (the 26th and final contestant) eat? How many hot dogs were eaten in total?
3. After excavating for weeks, you finally arrive at the burial chamber. The room is empty except for two large chests. On each is carved a message (strangely in English):

Exactly one of these chests contains a treasure, while the other is filled with deadly immortal scorpions.

For either chest, if the chest's message is true, then the chest contains treasure.

The problem is, you don't know whether the messages are true or false. What do you do?

4. Back in the days of yore, five small towns decided they wanted to build roads directly connecting each pair of towns. While the towns had plenty of money to build roads as long and as winding as they wished, it was very important that the roads not intersect with each other (as stop signs had not yet been invented). Also, tunnels and bridges were not allowed, for moral reasons. Is it possible for each of these towns to build a road to each of the four other towns without creating any intersections?

As you consider the problems above, don't worry if it is not obvious to you what the solutions are. We are more interested here in what sort of information we need to be able to answer the questions. How can we represent the situation using individually separate and distinct objects? Don't read on until you have thought



about at least this for each of the questions.

Ready? Here are some things you might have thought about:

1. The people at the party are individuals. We can consider the *set* of people. We can also consider sets of pairs of people, since it takes exactly two people to shake hands. So the question is really, how many pairs can you make using elements from a 10-element set?

For example, if there were three people at the party, conveniently named 1, 2, and 3, then the pairs would be (1, 2), (1, 3), and (2, 3). Or should we include (2, 1), (3, 1), and (3, 2) as well?

2. To count the number of hot dogs eaten, either by an individual or in total, we could use a **sequence** of integers (whole numbers). The  $n$ th term in the sequence might represent the number of hot dogs eaten by the  $n$ th contestant. We can consider a second sequence, also of integers, that gives the total number of hot dogs eaten by the first  $n$  contestants combined.

The solution to the problem will then be the value of the 26th term in the sequence. To help us find this, we could consider the rate of growth of the sequences, as well as how these two sequences relate to each other.

3. Logic questions also belong under the discrete math umbrella: Each statement can have a *value* of True or False (and there is nothing in-between). To answer questions like that of the chests of scorpions, we must understand the structure of the statements, and how the truth values of the parts of the statements interact to determine the truth value of the whole statement.
4. The last question is about a discrete structure called a **graph**, not to be confused with a graph of a function or set of points. We can use a graph to represent which elements of a set (or towns) are related to each other (or connected by a road). In this case, the question becomes, can we draw a graph with five vertices (towns) and ten edges (roads) such that no two edges intersect?

The four problems above illustrate the four main topics of this book: **combinatorics** (the theory of ways things *combine*; in particular, how to count these ways), **sequences**, **symbolic logic**, and **graph theory**. However, there are other topics that are also considered part of discrete mathematics, including computer science, abstract algebra, number theory, game theory, probability, and geometry (some of these, particularly the last two, have both discrete and non-discrete variants).

Ultimately the best way to learn what discrete math is about is to *do* it. Let's get started! Before we can begin answering more complicated (and fun) problems, we will consider a very brief overview of the types of discrete structures we will be using.

## READING QUESTIONS

Each section of the book will end with a small number of *Reading Questions* like the ones below. These are designed to help you reflect on what you have read. In

particular, the final reading question asks you to ask a question of your own. Thinking about what you don't yet know is a wonderful way to further your understanding of what you do.

1. Right now, how would you describe what **discrete** mathematics is about, if you were telling your friends about the class you are in? Write one or two sentences.
2. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

# EXPRESSIONS AND EQUATIONS

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Text before the first section.

## 1.1 OPERATIONS

Algebra provides a powerful way of solving many kinds of questions Algebra achieves this by enhancing arithmetic. Arithmetic focuses on the result of operations like addition and multiplication when using numbers that are constant. The contribution of algebra is to enhance arithmetic with variables and with geometry, with a focus on how the numbers appearing a problem *relate* to each other. With these extra ways of thinking, algebra allows us to answer many questions that are really hard to think about using arithmetic alone.

I want your experience in math to be as smooth and as enjoyable as possible. I want this for you, even if you can recall being frustrated with mathematics. To achieve that goal, I am deliberately writing this book for you the student, and not for your teacher. I hope to take each part of algebra that has the potential to be challenging and *really* break it down step-by-step. This does mean that I may ask you to try something different from the way you have done it in the past. I might also ask you to think about things that you haven't really thought about much before. I hope you'll give it a shot: what do you have to lose by trying this subject in a new way? In fact, I encourage you to really think about the language used in mathematics.

**Note 1.1.1** Pay attention to the language of mathematics.

As an example of this, when writing  $2^x$  be sure to say one of the following:

- “2 to the  $x$ ”
- “2 to the  $x$ th power”
- “2 raised to the  $x$ ”
- “2 raised to the  $x$ th power”

If we only say “2” then slightly pause to say “ $x$ ” this focuses on the specific individual symbols. The bigger problem is that saying “2” followed by “ $x$ ” is taken to mean  $2x$ . Why is this a problem? Because  $2^x$  and  $2x$  are not equal. In fact, when  $x = 3$  then  $2^x$  simplifies to 8, but  $2x$  simplifies to 6.

### 1.1.1 ORDER OF OPERATIONS

We will take a little time to make sure that everyone is on the same page regarding the Order of Operations. Before we go any further, we should describe what we

mean by an expression.

### Definition 1.1.2

An **expression** is mathematical notation representing a number.

**By the way...** An expression may consist of just a single constant such as 3 or  $\frac{4}{7}$ , or can be a variable representing a number such as  $x$  or  $y$ , or can be a combination of constants and variables connected by operations such as addition, subtraction, multiplication, division, and exponentiation.

### Example 1.1.3

Both  $-40$  and  $\sqrt{23}$  are examples of expressions. Both of these expressions are *constants*.

### Example 1.1.4

Both  $9x - 8$  and  $x^2 + 3x + 31$  are examples of expressions which mention the variable  $x$ . In the first expression, the variable  $x$  is written once. In the second expression, the variable  $x$  is written twice.

### Example 1.1.5

The expression  $x^2 + 2x + y^2 - 6y$  mentions two variables.

Please note that an expression does not contain an equal sign. For example,  $x^2 + 2x + y^2 - 6y = 22$  is not an expression. The notation that we just wrote instead states that one expression is equal to another expression.

We need the Order of Operations because this it is easy to misinterpret an expression if we do not all agree on how to read expressions. The Order of Operations is a set of rules that tells us the order in which to evaluate (and more generally read) an expression.

### Definition 1.1.6

An expression must always be simplified and read by following the **Order of Operations**:

1. Parentheses
2. Exponents
3. Multiplication and Division (from left to right)

#### 4. Addition and Subtraction (from left to right)

**Remark 1.1.7** By the time we get to the third part of the Order of Operations, all exponents would have been evaluated. At this point, we look for any multiplication or division. If there is both multiplication and division, we evaluate them from left to right. It is not true that multiplication must be done before division. All divisions and multiplications that we see have the same level of precedence, and we evaluate them scanning from left to right in that order.

**Remark 1.1.8** Similarly, by the time we get to the last part of Order of Operations, all multiplications and divisions would have been handled. That means that what remains of our expressions should only have addition and subtraction operations remaining. These should be handled from left to right.

**Remark 1.1.9** The Order of Operations is sometimes remembered by the acronym PEMDAS, which stands for Parentheses, Exponents, Multiplication, Division, Addition, and Subtraction.

Because of the way that PEMDAS is often taught, many people mistakenly believe that multiplication must be done before division, and addition must be done before subtraction. This is not true. For this reason, some people prefer the acronym GEMA, which stands for Grouping symbols, Exponents, Multiplication and Division, Addition and Subtraction. In the acronym GEMA, the “G” is used to indicate that there are many kinds of grouping symbols, not just parentheses. Also, the multiplication and division are addressed together in the “M”, and addition and subtraction are addressed together in the “A”.

#### Example 1.1.10

Simplify the expression  $5^2 - 30 \div 3 + 2 \cdot 6$ .

**Solution.** We will simplify the expression by following the Order of Operations. First, notice that there are no parentheses, so we move on to the next part of the Order of Operations. There is a place where the expression has exponents, so we zoom in on  $5^2$  and simplify this portion of the expression to 25. So, the expression given to us becomes  $25 - 30 \div 3 + 2 \cdot 6$ .

Now there is no more exponents. Next, we look for any multiplication or division. We see both multiplication and division, so we evaluate them from left to right. The left-most multiplication or division we see is the division

$30 \div 3$ , which simplifies to 10. This gives us the expression  $25 - 10 + 2 \cdot 6$ . Continuing to scan from left to right for any multiplications or divisions, we see the multiplication  $2 \cdot 6$ , which simplifies to 12. This gives us the expression  $25 - 10 + 12$ .

Now there are no more multiplications or divisions, we look for any additions or subtractions, starting from the left. The left-most addition or subtraction we see is the subtraction  $25 - 10$ , which simplifies to 15. This gives us the expression  $15 + 12$ . Continuing to scan from left to right for any additions or subtractions, we see the addition  $15 + 12$ , which simplifies to 27.

Because this was our first example of applying the Order of Operations, we wanted to be very thorough to explain each step. To present our work, we start from the original expression and after writing an equal sign (to indicate that what we will write next is equal) write a simplified version of the expression. We continue this process until we reach the final simplified expression. For this example, we have  $5^2 - 30 \div 3 + 2 \cdot 6 = 25 - 30 \div 3 + 2 \cdot 6 = 25 - 10 + 2 \cdot 6 = 25 - 10 + 12 = 15 + 12 = 27$ .

It is also acceptable to write each expressions on their own lines, as follows:

$$\begin{aligned} 5^2 - 30 \div 3 + 2 \cdot 6 &= 25 - 30 \div 3 + 2 \cdot 6 \\ &= 25 - 10 + 2 \cdot 6 \\ &= 25 - 10 + 12 \\ &= 15 + 12 \\ &= 27 \end{aligned}$$

Note that when presenting our work *vertically* we still include the equal signs to indicate that each expression is equal to the previous expression.

**Remark 1.1.11** When simplifying any expression, it is important to include equal signs to indicate that each expression is equal to the previous expression.

**Remark 1.1.12** While simplifying expressions, ensure that the next expression you write is truly equal to the previous expression, instead of just writing the portion of the expression that is changing.

### Example 1.1.13

Simplify the expression  $3 \cdot (6 - 4)^2 \div 2$ .

**Solution 1.** We can present our work horizontally, continuing to always write to the right of an equal sign like this  $3 \cdot (6 - 4)^2 \div 2 = 3 \cdot 2^2 \div 2 = 3 \cdot 4 \div 2 = 12 \div 2 = 6$ .

**Solution 2.** We can instead present our work vertically

$$\begin{aligned} 3 \cdot (6 - 4)^2 \div 2 &= 3 \cdot 2^2 \div 2 \\ &= 3 \cdot 4 \div 2 \\ &= 12 \div 2 \\ &= 6 \end{aligned}$$

### Example 1.1.14

Simplify the expression  $200 - 4^2 \div 8 \times 5 + 6$ .

**Solution 1.** We can present our work horizontally, always writing to the right of the equal sign like this:  $200 - 4^2 \div 8 \times 5 + 6 = 200 - 16 \div 8 \times 5 + 6 = 200 - 2 \times 5 + 6 = 200 - 10 + 6 = 190 + 6 = 196$ .

**Solution 2.** We can instead present our work vertically:

$$\begin{aligned} 200 - 4^2 \div 8 \times 5 + 6 &= 200 - 16 \div 8 \times 5 + 6 \\ &= 200 - 2 \times 5 + 6 \\ &= 200 - 10 + 6 \\ &= 190 + 6 \\ &= 196 \end{aligned}$$

It is important for us to apply the Order of Operations not *only* to simplify expressions that contain only constants, but to also apply the Order of Operations when to interpret expressions that contain variables.

Sequencing the operations in a given expression

Using only the OOO, the expressions are equal or it cannot be determined based on

## 1.1.2 PICTURES AND LANGUAGE

sticks, rectangles, terms, factors Text of section.

Text of section.

Text of section.

Text of section.

Text of section.

Text of section.

## 1.2 EXPRESSIONS WITH FRACTIONS

Text of section.

The top and bottom of a fraction are in (hidden) parentheses. Affects calculator input.

Mixed fractions, improper fractions, decimal answers



## 1.3 SOLVING EQUATIONS

Intro to equations.

Technique with one copy of the variable.

Multiple terms.

Rewriting as multiple terms.

Strategy comparison.

## 1.4 EQUATIONS WITH FRACTIONS

Text of section.

## 1.5 EXPONENTS

Text of section.

## 1.6 RADICALS

Text of section.

## **1.7 FACTORING AND EXPANSION**

Text of section.

## 1.8 RATIONAL EXPRESSIONS

Text of section.

## 1.9 SOLVING EQUATIONS REVISITED

Text of section.

## 1.10 SAMPLE CODE

### Objectives

After completing this section, you should be able to do the following.

1. Explain the conditions under which an implication is true.
2. Identify statements as equivalent to a given implication or its converse.
3. Explain the relationship between the truth values of an implication, its converse, and its contrapositive.

### 1.10.1 SECTION PREVIEW

#### Investigate!

While walking through a fictional forest, you encounter three trolls guarding a bridge. Each is either a *knight*, who always tells the truth, or a *knave*, who always lies. The trolls will not let you pass until you correctly identify each as either a knight or a knave. Each troll makes a single statement:

Troll 1: If I am a knave, then there are exactly two knights here.

Troll 2: Troll 1 is lying.

Troll 3: Either we are all knaves, or at least one of us is a knight.

Which troll is which?

#### Try it 1.10.1

Spend a few minutes thinking about the Investigate problem above. What could you conclude if you knew Troll 1 really was a knave (i.e., their statement was false)? Share your initial thoughts on this.

#### Definition 1.10.2 Argument.

An **argument** is a sequence of statements, the last of which is called the **conclusion** and the rest of which are called **premises**.

An argument is said to be **valid** provided the conclusion must be true whenever the premises are all true. An argument is **invalid** if it is not valid; that is, all the premises can be true, and the conclusion could still be false.

An argument is **sound** provided it is valid and all the premises are true. A



**proof** of a statement is a sound argument whose conclusion is the statement.

**By the way...** Our definitions of **argument**, **valid argument**, and **sound argument** are the same ones used in philosophy, the other primary academic discipline concerned with logic and reasoning.

### Example 1.10.3

Consider the following two arguments:

If Edith eats her vegetables, then she can have a cookie.
Edith eats her vegetables.
$\therefore$ Edith gets a cookie.

Florence must eat her vegetables to get a cookie.
Florence eats her vegetables.
$\therefore$ Florence gets a cookie.

(The symbol “ $\therefore$ ” means “therefore”)

Are these arguments valid?

**Solution.** Do you agree that the first argument is valid but the second argument is not? We will soon develop a better understanding of the logic involved in this analysis, but if your intuition agrees with this assessment, then you are in good shape.

Notice the two arguments look almost identical. Edith and Florence both eat their vegetables. In both cases, there is a connection between the eating of vegetables and cookies. Yet we claim that it is valid to conclude that Edith gets a cookie, but not that Florence does. The difference must be in the connection between eating vegetables and getting cookies. We need to be skilled at reading and comprehending these sentences. Do the two sentences mean the same thing?

Unfortunately, in everyday language we are often sloppy, and you might be tempted to say they are equivalent. But notice that just because Florence *must* eat her vegetables, we have not claimed that doing so would be *enough* (she might also need to clean her room, for example). In everyday (non-mathematical) practice, you might be tempted to say this “other direction” is implied. In mathematics, we never get that luxury.

**Remark 1.10.4** The arguments in the example above illustrate another important point: Even if you don’t care about the advancement of human knowledge

in the field of mathematics, becoming skilled at analyzing arguments is useful. And even if you don't want to give your grandmother a cookie. If you are *using* mathematics to solve problems in some other discipline, it is still necessary to demonstrate that your solution is correct. You better have a good argument that it is!

## SELECTED HINTS

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## SELECTED SOLUTIONS

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## LIST OF SYMBOLS

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$\therefore$	“therefore”	19





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## **Colophon**

This book was authored in PreTeXt.