Discrete Mathematics

An Open Introduction, 4th Edition

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4th Edition

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4th Edition

A current version can always be found for free at
http://discrete.openmathbooks.org/

Cover image: *Tiling with Fibonacci and Pascal*.

For Madeline and Teagan

Acknowledgements

This book would not exist if not for "Discrete and Combinatorial Mathematics," by Richard Grassl and Tabitha Mingus. It is the book I learned discrete math out of, and that I taught with the semester before I began writing this text. I wanted to maintain the inquiry-based feel of their book but update, expand, and rearrange some of the material. Some of the best exposition and exercises here were graciously donated from this source.

Thanks go to the graduate students who have co-taught the Discrete Mathematics course with me over the years, including Evan Czysz, Alees Lee, and Sarah Sparks, who helped develop new activities and exercises that have been incorporated into this text. Michelle Morgan provided copy-editing support, and Jennifer Zakotnik-Gutierrez helped code many of the interactive exercises in the online version of the book. Thanks also to Katie Morrison, Nate Eldredge, and Richard Grassl (again) for their suggestions after using parts of this text in their classes.

The online version of the book is written in PreTeXt and hosted on Runestone Academy thanks to the tremendous development work of Rob Beezer, Brad Miller, David Farmer, and Alex Jordan along with the rest of the participants of the pretext-support group (`groups.google.com/g/pretext-support`).

Finally, a thank you to the numerous students who have pointed out typos and made suggestions over the years, and a thanks in advance to those who will do so in the future.

Preface

This text aims to introduce select topics in discrete mathematics at a level appropriate for first- or second-year undergraduate math and computer science majors, especially those who intend to teach middle and high school mathematics. The book began as a set of notes for the Discrete Mathematics course at the University of Northern Colorado. This course serves both as a survey of the topics in discrete math and as the "bridge" course for math majors, as UNC does not offer a separate "introduction to proofs" course. As this course has evolved to support our computer science major, so has the text. The current version of the book is intended to support inquiry-based teaching for understanding that is so crucial for future teachers, while also providing the necessary mathematical foundation and application-based motivation for computer science students. While teaching the course in Spring 2024 using an early version of this edition, I was pleasantly surprised by how many students reported that they, for the first time, saw how useful math could be in the "real world." I hope that this experience can be replicated in other classes using this text.

This book is intended to be used in a class taught using problem-oriented or inquiry-based methods. Each section begins with a preview of the content that includes an open-ended *Investigate!* motivating question, as well as a structured preview activity. The preview activities are carefully scaffolded to provide an entry-point to the section's topic and to prime students to engage deeply in the material. Depending on the pace of the class, I have found success assigning only the section preview before class, using the preview activity as in-class group work, or assigning the entire section to be read before class (each section concludes with a small set of reading questions that can be assigned to encourage students to actually read). For those readers using this book for self-study, the organization of the sections will hopefully mimic the style of a rich inquiry-based classroom.

The topics covered in this text were chosen to match the needs of the students I teach at UNC. The main areas of study are logic and proof, graph theory, combinatorics, and sequences. Induction is covered at the end of the chapter on sequences. Discrete structures are introduced "as needed", but a more thorough treatment of sets and functions is included as a separate chapter, which can be studied independent of the other content. The final chapter covers two additional topics: generating functions and number theory.

While I believe this selection and order of topics is optimal, you should feel free to skip around to what interests you. There are occasionally examples and exercises that rely on earlier material, but I have tried to keep these to a minimum, and they usually can either be skipped or understood without too much additional study. If you are an instructor, you can also create a custom version by editing the PreTeXt source to fit your needs.

Improvements to the 4th Edition. Many of the sections have been rewritten to improve the clarity of the exposition.

- Nearly 300 new exercises, bringing the total to more than 750. These are better divided into preview activity questions, reading questions, practice problems, and additional exercises. Most of the new exercises are interactive for the online version.

- New sections on probability, relations, and discrete structures and their proofs. Some other sections have been split up to make it more likely that a single class period can be devoted to a single topic.

- Improved presentation for the counting chapter with a focus on considering sets of outcomes more than following rules.

- The *Investigate!* activities of the 3rd Edition have been split into two types: *Investigate!* questions and Preview Activities. The former are open-ended questions designed to engage you with the topic soon to be discussed. The latter are structured preview activities that you should be able to completely answer before reading the section.

The previous editions (3rd Edition, released in 2019, 2nd Edition, released in 2016, and the Fall 2015 Edition) will still be available for instructors who wish to use those versions due to familiarity.

I plan to continue improving the book. Some of this will happen in real-time by updating the online versions to include new content (numbering will remain consistent). Thus I encourage you to send along any suggestions and comments as you have them.

Oscar Levin, Ph.D.
University of Northern Colorado, 2024

How to Use This Book

In addition to expository text, this book has a few features designed to encourage you to interact with the mathematics.

***Investigate!* questions.** Sprinkled throughout the sections (usually at the very beginning of a topic) you will find open-ended questions designed to engage you with the topic soon to be discussed. You really should spend some time thinking about, or even working through, these problems before reading the section. However, don't worry if you cannot find a satisfying solution right away. The goal is to pique your interest, so you will read what is next looking for answers.

Preview Activities. Most sections include a structured preview activity. These contain leading questions that you should be able to completely answer before reading the section. The idea is that the questions prime you to engage meaningfully with the new content ahead. If you are using the online version, most of these questions will provide you with immediate feedback so you can be confident moving forward.

Examples. I have tried to include the "correct" number of examples. For those examples that include *problems*, full solutions are included. Before reading the solution, try to at least have an understanding of what the problem is asking. Unlike some textbooks, the examples are not meant to be all-inclusive for problems you will see in the exercises. They should not be used as a blueprint for solving other problems. Instead, use the examples to deepen your understanding of the concepts and techniques discussed in each section. Then use this understanding to solve the exercises at the end of each section.

Exercises. You get good at math through practice. Each section concludes with practice problems meant to solidify concepts and basic skills presented in that section; the online version provides immediate feedback on these problems. There are then additional exercises that are more challenging and open-ended. These might be assigned as written homework or used in class as group work. Some of the additional exercises have hints or solutions in the back of the book, but use these as little as possible. Struggle is good for you. At the end of each chapter, a larger collection of similar exercises is included (as a sort of "chapter review") which might bridge the material of different sections in that chapter.

Interactive Online Version. For those of you reading this in print or as a PDF, I encourage you to also check out the interactive online version. Many of the preview activities and exercises are interactive and can give you immediate feedback. Some of these have randomized components, allowing you to practice many similar versions of the same problems until you master the topic.

Hints and solutions to examples are also hidden away behind an extra click to encourage you to think about the problem before reading the solution. There is a good search feature available as well, and the index has expandable links to see the content without jumping to the page immediately. There is also a python scratch pad (the pencil icon) so you can try out some code if you feel so inclined.

Additional interactivity is planned. These "bonus" features will be added on a rolling basis, so keep an eye out!

You can view the interactive version for free at `discrete.openmathbooks.org` or by scanning the QR code below.

Contents

Chapter 0

Introduction and Preliminaries

Welcome to Discrete Mathematics. If this is your first time encountering the subject, you will probably find discrete mathematics quite different from other math subjects. You might not even know what discrete math is! Hopefully this short introduction will shed some light on what the subject is about and what you can expect as you move forward in your studies.

0.1 What is Discrete Mathematics?

dis·crete / dis'krët.

Adjective: Individually separate and distinct.

Synonyms: separate - detached - distinct - abstract.

Defining *discrete mathematics* is hard because defining *mathematics* is hard. What is mathematics? The study of numbers? In part yes, but you also study functions and lines and triangles and parallelepipeds and vectors and .... Or perhaps you want to say that mathematics is a collection of tools that allow you to solve problems. What sort of problems? Well, those that involve numbers, functions, lines, triangles, ... Whatever your conception of what mathematics is, try applying the concept of "discrete" to it, as defined above. Some math fundamentally deals with *stuff* that is individually separate and distinct.

In an algebra or calculus class, you might have found a particular set of numbers (perhaps they constitute the range of a function). You would represent this set as an interval: $[0, \infty)$ is the range of $f(x) = x^2$ since the set of outputs of the function are all real numbers 0 and greater. This set of numbers is NOT discrete. The numbers in the set are not separated by much at all. In fact, take any two numbers in the set and there are infinitely many more between them that are also in the set.

Discrete math could still ask about the range of a function, but the set would not be an interval. Consider the function that gives the number of children of each person reading this. What is the range? I'm guessing it is something like $\{0, 1, 2, 3, 4\}$. Maybe 5 or 6 is in there too.[1] But certainly nobody reading this has 1.32419 children. This output set *is* discrete because the elements are separate. The inputs to the function also form a discrete set because each input is an individual person.

There are many discrete mathematical objects besides sets of numbers; we will introduce some of these in Section 0.2. Studying these discrete **structures** is the main

[1] Even larger natural numbers for old ladies who live in shoes.

focus of discrete mathematics and this book. However, the reason we want to study these structures is because they provide a way to model "real-world" problems.[2]

To get a feel for the subject, let's consider the types of problems you solve in discrete math. Here are a few simple examples:

Investigate!

Note: Throughout the book you will see Investigate! *activities like this one. Answer the questions in these as best you can to give yourself a feel for what is coming next.*

1. The most popular mathematician in the world is throwing a party for all of his friends. To kick things off, they decide that everyone should shake hands. Assuming all 10 people at the party each shake hands with every other person (but not themselves, obviously) exactly once, how many handshakes take place?

2. At the warm-up event for Oscar's All-Star Hot Dog Eating Contest, Al ate one hot dog. Bob then showed him up by eating three hot dogs. Not to be outdone, Carl ate five. This continued with each contestant eating two more hot dogs than the previous contestant. How many hot dogs did Zeno (the 26th and final contestant) eat? How many hot dogs were eaten in total?

3. After excavating for weeks, you finally arrive at the burial chamber. The room is empty except for two large chests. On each is carved a message (strangely in English):

   Exactly one of these chests contains a treasure, while the other is filled with deadly immortal scorpions.

   For either chest, if the chest's message is true, then the chest contains treasure.

   The problem is, you don't know whether the messages are true or false. What do you do?

4. Back in the days of yore, five small towns decided they wanted to build roads directly connecting each pair of towns. While the towns had plenty of money to build roads as long and as winding as they wished, it was very important that the roads not intersect with each other (as stop signs had not yet been invented). Also, tunnels and bridges were not allowed, for moral reasons. Is it possible for each of these towns to build a road to each of the four other towns without creating any intersections?

[2] Many of the problems discussed in this book are admittedly contrived and clearly fictional, but hopefully you will see how these toy problems can be generalized to actually represent problems that people would care about in reality.

As you consider the problems above, don't worry if it is not obvious to you what the solutions are. We are more interested here in what sort of information we need to be able to answer the questions. How can we represent the situation using individually separate and distinct objects? Don't read on until you have thought about at least this for each of the questions.

Ready? Here are some things you might have thought about:

1. The people at the party are individuals. We can consider the *set* of people. We can also consider sets of pairs of people, since it takes exactly two people to shake hands. So the question is really, how many pairs can you make using elements from a 10-element set?

   For example, if there were three people at the party, conveniently named 1, 2, and 3, then the pairs would be $(1,2)$, $(1,3)$, and $(2,3)$. Or should we include $(2,1)$, $(3,1)$, and $(3,2)$ as well?

2. To count the number of hot dogs eaten, either by an individual or in total, we could use a **sequence** of integers (whole numbers). The nth term in the sequence might represent the number of hot dogs eaten by the nth contestant. We can consider a second sequence, also of integers, that gives the total number of hot dogs eaten by the first n contestants combined.

   The solution to the problem will then be the value of the 26th term in the sequence. To help us find this, we could consider the rate of growth of the sequences, as well as how these two sequences relate to each other.

3. Logic questions also belong under the discrete math umbrella: Each statement can have a *value* of True or False (and there is nothing in-between). To answer questions like that of the chests of scorpions, we must understand the structure of the statements, and how the truth values of the parts of the statements interact to determine the truth value of the whole statement.

4. The last question is about a discrete structure called a **graph**, not to be confused with a graph of a function or set of points. We can use a graph to represent which elements of a set (or towns) are related to each other (or connected by a road). In this case, the question becomes, can we draw a graph with five vertices (towns) and ten edges (roads) such that no two edges intersect?

The four problems above illustrate the four main topics of this book: **combinatorics** (the theory of ways things *combine*; in particular, how to count these ways), **sequences**, **symbolic logic**, and **graph theory**. However, there are other topics that are also considered part of discrete mathematics, including computer science, abstract algebra, number theory, game theory, probability, and geometry (some of these, particularly the last two, have both discrete and non-discrete variants).

Ultimately the best way to learn what discrete math is about is to *do* it. Let's get started! Before we can begin answering more complicated (and fun) problems, we will consider a very brief overview of the types of discrete structures we will be using.

Reading Questions

Each section of the book will end with a small number of *Reading Questions* like the ones below. These are designed to help you reflect on what you have read. In particular, the final reading question asks you to ask a question of your own. Thinking about what you don't yet know is a wonderful way to further your understanding of what you do.

1. Right now, how would you describe what **discrete** mathematics is about, if you were telling your friends about the class you are in? Write one or two sentences.

2. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

0.2 Discrete Structures

0.2.1 Introduction

Investigate!

A double-six domino set consists of tiles containing pairs of numbers, each from 0 to 6. How many tiles are in a double-six domino set? How many dominoes are in a double-nine domino set? How many dominoes are in a double-n domino set?

Try it 0.2.1

Spend a few minutes thinking about the questions above. Then write 2-3 sentences describing your thoughts. You do not need to find a complete solution, but you should describe what you could try and what you think you might need to do to find a solution.

We are taking a problem-solving approach to discrete mathematics: We will consider a large variety of questions that have a discrete feel to them, and consider how to answer those questions (and prove that our answers are correct). This is not the only way to study discrete math. Another approach would be to study the tools used to solve the problems. If we were art students, we could study paintbrushes and easels and the composition of paint, which would be interesting for sure, but I think it is more enjoyable to actually paint those happy little trees.

That said, understanding your tools does help you use them, so in this section, we will consider some basic tools used in discrete mathematics. We will come back to these throughout our studies and understand them better as we need to.

The tools in our subject are called **discrete structures**. They are the mathematical objects that we use to represent parts of the problems we are solving. "Structure" is a good word here, since these "things" have fairly rigid constraints that make them what they are, just like an apartment building is going to have different characteristics than an airplane hangar or a suspension bridge (these are types of physical structures, not mathematical structures, just to be overly clear and destroy the metaphor).

The structures we will use most in discrete math are **sets**, **functions**, **sequences**, **relations**, and **graphs**. We now briefly preview each of these. As we progress through our studies, each will be explored in more detail.

0.2.2 Sets

A **set** is an *unordered* collection of elements. This is fairly vague, but unless we want to spend a whole book trying to understand sets more precisely, it will be good enough for us. It is possible to define all of mathematics using just sets (even numbers can be thought of as sets themselves), but this is also not what we will do. Rather, we want to be able to talk about collections of numbers and other objects, and we will collect them in something we call a **set**.

We can describe sets by saying exactly what elements are members of the set. We could specify this membership in words (e.g., Let A be the set of all natural numbers less than 10), or by explicitly listing all the elements (e.g., $A = \{3, 5, 7\}$), or using something called **set comprehension** (also called **set builder notation**). An example of this is $A = \{x \in \mathbb{N} : x < 10\}$. We would read that as "A is the set of natural numbers that satisfy the property that they are less than 10." More precisely, the $\mathbb{N}$ symbol represents the natural numbers, which is itself a set: $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ (this shows another way to describe a set). The $\in$ symbol means "is an element of". The : is read "such that" and tells us that what comes next is the *condition* that must be true of the set's elements.

> **By the way...** In this book, we define the **natural numbers** to be the whole numbers starting with 0. Not every book includes 0 in this set. It largely depends on what area of mathematics you study.

Since there are multiple ways to describe the same set, we should be careful about what it means for sets to be the same or different. A set is *determined* by its membership, so all four of the following describe exactly the same set:

1. $\{1, 2, 3, 4\}$.
2. $\{1, 2, 1 + 1, 1 + 2, 2 + 2\}$. (How many numbers belong to this set? It's not 5.)
3. $\{2, 4, 1, 3\}$. (All that matters is what elements are in the collection, not what order they were written down in.)
4. $\{x \in \mathbb{N} : x < 5 \text{ and } x \geq 1\}$.

There are lots of things you can *do* with sets, which we will consider in more detail as we need to. We will see that it is often helpful to build new sets from ones we already have (by taking the **union** or **intersection** of sets, for example), to compare sets (asking if one set is a **subset** of another), and to find the number of elements of a set (called its **cardinality**). We might also want to match up elements of one set with another: To do this, we might use a **function**, which we will discuss next. Awesomely, we can also use sets themselves to describe functions. Let's check it out.

0.2.3 Functions

One way to define a **function** is as a rule that assigns each input exactly one output. The output is called the **image** of the input. Functions also come equipped with a **domain**, the set of all inputs, and a **codomain**, the set of all allowable outputs. You might also speak of the **range** of the function, which is the set of all *actual* outputs, or put another way, the set of all *images* of elements from the domain.

We write $f : X \to Y$ to describe a function with name f, domain X and codomain Y. This does not tell us *which* function f is though. To define the function, we must describe the rule. Often this is done with a formula (for example, $f(x) = x^2$ says that each element of the domain is mapped to its square), or in words (like how we just described the squaring function). We could also define a function with a table or a graph.

The key thing that makes a rule a *function* is that there is *exactly one* output for each input. That is, the rule must be a good rule. What output do we assign to the input 7? There can only be one answer for any particular function.

Since a function maps one set (the domain) to another set (the codomain), there is an obvious connection between sets and functions. There is another connection worth considering though: The graph of a function is often described as a ***set*** of points. Here is an example.

Example 0.2.2

Consider the function $f : \{1, 2, 3\} \to \{2, 4, 6\}$ defined by $f(x) = 2x$. If we wanted to plot a graph of this function, we would draw the points $(1, 2)$, $(2, 4)$, and $(3, 6)$ (but we would not connect these points with lines, since we are studying discrete math; the domain only contains three elements, not the infinitely many between them).

So associated with the function is the set of points $\{(1, 2), (2, 4), (3, 6)\}$. In fact, this set of points is associated exactly with this (and only this) function. So we can think of this set of points *as the function itself*.

Of course, we are using a mathematical object here: an **ordered pair**. This is not a set (since sets are unordered). How should we talk about ordered things? We will take this question up in the next section.

There is one more important consideration about how we define a function with a rule. A **closed formula** is one in which each output is given by an explicit rule based solely on its input. This is what most of us think of as a formula. For example, $f(n) = 3n + 1$ is a closed formula, since to find $f(5)$ (say) we take the input 5 and do something to it: Multiply it by 3 and then add 1.

What else could a formula possibly be? A **recursive definition** of a function tells us how to compute the output for a given input *based on other outputs of the function*. For example, we might insist that $f(n) = 2 \cdot f(n-1)$. If we also specify an **initial condition** that $f(0) = 3$, then we can find $f(1) = 2 \cdot 3 = 6$, and then $f(2) = 2 \cdot 6 = 12$, and so on. What is $f(5)$ here? The only way to answer that is to find $f(4)$, which means we need to find $f(3)$, which we could do, since we have computed $f(2)$.

Recursive definitions of functions might be less useful for finding a particular output, but they are often easier to specify for a particular application. We will explore this phenomenon more when we study sequences. Speaking of which...

0.2.4 Sequences

Sometimes we are interested not just in a collection of numbers, but in what *order* those numbers appear. In such cases, we cannot use *sets*, since they do not distinguish between the order of their elements. Instead, we consider **sequences**.

We will consider both **finite** and **infinite** sequences. A finite sequence may be something as simple as $(1, 2, 3)$; that is a sequence with 3 elements, in that particular order. We might also call this an **ordered triple**, the same way that $(7, 3)$ is an ordered pair. In general, we could call this an **n-tuple** if it has n elements (we assume that tuples are ordered).

The key difference between the sequence $(1, 2, 3)$ and the set $\{1, 2, 3\}$ is that we "care" about the order. That is, the sequence $(1, 2, 3)$ is different from the sequence $(2, 3, 1)$, while the set $\{1, 2, 3\}$ is identical to $\{2, 3, 1\}$.

We will often use sequences as a counting tool. For example, a very simple counting question is, "How many wheels do 100 cars have?" Instead of answering just this one question, we could generalize and ask how many wheels n cars have, and get a sequence of answers. This yields the infinite sequence $(4, 8, 12, \ldots)$. The order these multiples of 4 appear in is important, since each number in the sequence corresponds to a specific version of the question.

It is fine to think of a sequence of numbers as an ordered list. We can refer to the **terms** simply as

$$a_0, a_1, a_2, \ldots$$

and might refer to the entire sequence as $(a_n)_{n\geq 0}$ or $(a_n)_{n\in\mathbb{N}}$.

If we want to be a little more precise and more abstract, we can think of a sequence as a *function*. The domain is the natural numbers or some subset of consecutive natural numbers (like $\{1, 2, 3, 4\}$). The codomain is some set. We think of the domain as the positions in the sequence, and the image of those inputs as the terms in the sequence.

For example, we might consider the Fibonacci sequence $(f_n)_{n\geq 1}$, which starts $1, 1, 2, 3, 5, 8, \ldots$. What is the 4th term in the sequence? We might say $f_4 = 3$ (this is assuming the first 1 is the first term and not the 0th term). Note that there can only be one 4th term. There can only be one n-th term for any particular value of n. So for any input (the position of the term), there is only one output (the term). It would be perfectly reasonable to write $f(4) = 3$, and that really does look like a function. But we like to use subscripts.

We can also describe the terms in a sequence using a table. We might write something like the following:

n	1	2	3	4	...
a_n	1	3	6	10	...

This looks exactly like how you would represent a function, even though this table describes the sequence of **triangular numbers** (we will see why they are called this later).

Since sequences are functions, we can use any of the techniques to describe functions to describe sequences. In particular, we might give a **closed formula** for a sequence by explicitly giving the function for the n-th term. For example,

$$a_n = \frac{n(n+1)}{2}.$$

Alternatively, we could define a sequence recursively by saying how to get from one term to the next. This is especially useful for the Fibonacci sequence:

$$f_n = f_{n-1} + f_{n-2}; f_1 = 1, f_2 = 1.$$

Much of our effort in understanding sequences will go into taking a recursive definition and finding a closed formula for the sequence. We will study this, and everything else sequence related in Chapter 4.

0.2.5 Relations

How are the numbers 2 and 6 related? Oh, I know: $2 < 6$. Also, 6 is a multiple of 2. The two numbers are also both even. And here is another fact: They are *not equal*. All four of these are examples of **relations**: less than, multiple of, both even, not equal. And there are many more (infinitely many) relations for pairs of numbers.

The examples above are all **binary relations** in that they relate two elements. You could also consider relations between more than two elements. For example, we could consider the relation "Pythagorean triple" on three numbers that holds precisely if they are the side lengths of a right triangle. So the relation is true of the triple $(3, 4, 5)$, but not of $(4, 5, 6)$ (since $3^2 + 4^2 = 5^2$, but $4^2 + 5^2 \neq 6^2$).

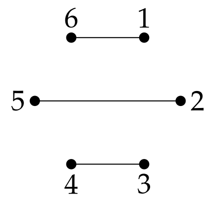
Notice that we can talk about a pair or triple *satisfying* a relation. We might say that a pair *belongs* to the relation. The careful and formal way to define a relation is as a *set* of ordered pairs (or triples, etc.). Consider the (infinite) set of all ordered pairs (a, b) such that $a < b$. Every element of this set contains numbers for which the relation "less than" is true, and every pair of numbers for which the relation is true is a pair in the set. So we can say that this set of pairs ***is*** the relation.

Relations can have some standard properties, and deciding whether a particular relation has a given property can often help us understand the relation better. The less-than relation is, for example, **irreflexive** because there are no elements that are less than themselves. It is also **antisymmetric** since there are no distinct numbers a and b such that $a < b$ and $b < a$. It is also **transitive** since if $a < b$ and $b < c$ then it must follow that $a < c$. These are just a few examples of relation properties though.

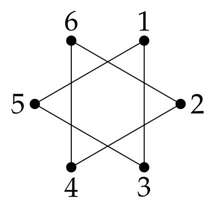
Why would we care about these properties? It turns out that some groups of properties happen together frequently, and for such collections of properties, we can prove general results about the relations that satisfy them. So if we can prove that a given relation is **reflexive**, **symmetric**, and **transitive** (whatever those mean), then we know the relation is an **equivalence relation**, and therefore we know it has a bunch of other properties. A large portion of discrete mathematics is about studying particular types of relations. One of my favorites is a relation that gives us a graph.

0.2.6 Graphs

Consider the set $V = \{1, 2, 3, 4, 5, 6\}$. Which pairs of numbers from that set add up to 7? We could have $\{1, 6\}$ or $\{2, 5\}$ or $\{3, 4\}$. We can picture the set and the interesting (unordered) pairs (i.e., two-element subsets) as a picture called a graph:



On the other hand, we might want to consider pairs of numbers whose sum is even. Then, we would get the following picture.



We call these discrete structures **graphs**. A graph is a type of relation, one that is **symmetric** (if a is related to b, then b is related to a) and **irreflexive** (no element is related to itself).

However, we mostly think of graphs as the drawings of dots and lines, or more precisely as a set of **vertices** together with a set of **edges**, where each edge is a two-element subset of the vertices. Notice that even here, we are using the structure *set* to define the structure *graph*.

Graphs show up in all sorts of real-world applications: In a class, some students are friends with each other, so take the students to be the vertices and the edges to be the friendships. In geography, some countries share a border, so take the countries to be the vertices and connect a pair of vertices with an edge if the countries share a border. Perhaps you are planning a trip and want to fly from Denver to Paris. Is there a direct flight, or must you stop in Newark? That is, does the graph of flights have an edge between Denver and Paris or only between Denver and Newark and between Newark and Paris? When your Amazon driver delivers packages to 10 houses in your neighborhood, how does her app know in which order to deliver the packages? Graph theory!

The study of graphs is a subject in its own right, in which many mathematicians hold doctorate degrees and write hundreds of papers each year. We will scratch the surface of this fascinating subject in Chapter 2

0.2.7 Even More Structures

Our list of structures could go on and on, but we will stop here. We will spend just a little time looking at **multisets**, which are just like sets except that they can have repeated elements. Since this is not a geometry class, we will not consider **finite geometries**, or **designs** (which are somewhere between graphs and geometries). Discrete structures are useful in computer science, but we will stop short of studying **linked lists** or **red-black trees**. Although abstract algebra is a fascinating subject, we will not get to **groups** or **rings** or **metroids** or **POSets** or **Boolean algebras** or .... These are examples of sets on which we define additional operations and study the algebraic structure of how the sets and operations interact.

The point is, discrete mathematics is awesome, and you can spend multiple lifetimes studying it. So what are we waiting for? Let's dive in and solve some problems.

0.2.8 Reading Questions

1. Think back to the domino problem at the beginning of this section. We asked how many dominoes are in a double-six domino *set*. Is this really a set, in our mathematical sense? What discrete structure would you use to represent each domino individually?

2. A double-zero domino set would contain only one domino (both sides showing 0). A double-one set would contain this plus the dominoes $(1,0)$ and $(1,1)$. We can continue in this way, creating a sequence of domino sets. Find the next three terms of the sequence.

$1, 3, \_\,, \_\,, \_\,, \ldots$

3. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

Chapter 1

Logic and Proofs

Logic is the study of consequence. Given a few mathematical statements or facts, we would like to be able to draw some conclusions. For example, if I told you that a particular real-valued function was continuous on the interval $[0, 1]$, and $f(0) = -1$ and $f(1) = 5$, can we conclude that there is some point between $[0, 1]$ where the graph of the function crosses the x-axis? Yes, we can, thanks to the Intermediate Value Theorem from calculus. Can we conclude that there is exactly one point? No. Whenever we find an "answer" in math, we really have a (perhaps hidden) argument.

Mathematics is really about establishing general statements (like the Intermediate Value Theorem). This is done via an argument called a proof. We start with some given conditions, the *premises* of our argument, and from these, we find a consequence of interest, our *conclusion*.

The problem is, as you no doubt know from arguing with friends, not all arguments are *good* arguments. A bad argument is one in which the conclusion does not follow from the premises; i.e., the conclusion is not a consequence of the premises. Logic is the study of what makes an argument good or bad. In other words, logic aims to determine in which cases a conclusion is, or is not, a consequence of a set of premises.

We will start in Section 1.1 by considering statements, the building blocks of arguments. Understanding what counts as a statement and what form statements can take is the first step in understanding arguments. We will take a closer look at how statements can be combined in Section 1.2. Then we will see what mathematical tools we can develop to better analyze these statements and how they interact in Section 1.3. Finally, we will put all of this together in Section 1.4 and Section 1.5 to see how we can use these tools to construct arguments and prove statements.

1.1 Mathematical Statements

Objectives

After completing this section, you should be able to do the following.

1. Identify the logical structure of statements to determine their truth value in terms of the truth values of their parts.
2. Identify the use of quantifiers in a statement, and determine the truth value of the statement based on those quantifiers.
3. Translate between statements in natural language and logical symbols.

1.1.1 Section Preview

Investigate!

While walking through a fictional forest, you encounter three trolls guarding a bridge. Each is either a *knight*, who always tells the truth, or a *knave*, who always lies. The trolls will not let you pass until you correctly identify each as either a knight or a knave. Each troll makes a single statement:

Troll 1: If I am a knave, then there are exactly two knights here.

Troll 2: Troll 1 is lying.

Troll 3: Either we are all knaves, or at least one of us is a knight.

Which troll is which?

Try it 1.1.1

Spend a few minutes thinking about the Investigate problem above. What could you conclude if you knew Troll 1 really was a knave (i.e., their statement was false)? Share your initial thoughts on this.

In order to *do* mathematics, we must be able to *talk* and *write* about mathematics. Perhaps your experience with mathematics so far has mostly involved finding numerical answers to problems. As we embark towards more advanced and abstract mathematics, writing will play a more prominent role in the mathematical process.

In fact, the primary goal of mathematics, as an academic discipline in its own right, is to establish general mathematical truths. How can we know whether these facts, perhaps called *theorems* or *propositions*, are true? We construct valid arguments, called *proofs*, which establish the truth of the statements. Here, an argument is not the sort of thing you have with your Mom when you disagree about what to have for dinner. Rather, we have a technical definition of the term.

Definition 1.1.2 Argument.

An **argument** is a sequence of statements, the last of which is called the **conclusion** and the rest of which are called **premises**.

An argument is said to be **valid** provided the conclusion must be true whenever the premises are all true. An argument is **invalid** if it is not valid; that is, all the premises can be true, and the conclusion could still be false.

An argument is **sound** provided it is valid and all the premises are true. A **proof** of a statement is a sound argument whose conclusion is the statement.

By the way... Our definitions of **argument**, **valid argument**, and **sound argument** are the same ones used in philosophy, the other primary academic discipline concerned with logic and reasoning.

To determine whether we have a proof of a statement, we must decide both whether every premise is true, and whether the argument is valid: whether the conclusion *follows from* the premises. How can we do this?

Example 1.1.3

Consider the following two arguments:

$$\begin{array}{ll} & \text{If Edith eats her vegetables, then she can have a cookie.} \\ & \text{Edith eats her vegetables.} \\ \hline \therefore & \text{Edith gets a cookie.} \end{array}$$

$$\begin{array}{ll} & \text{Florence must eat her vegetables to get a cookie.} \\ & \text{Florence eats her vegetables.} \\ \hline \therefore & \text{Florence gets a cookie.} \end{array}$$

(The symbol "$\therefore$" means "therefore")

Are these arguments valid?

Solution. Do you agree that the first argument is valid but the second argument is not? We will soon develop a better understanding of the logic involved in this analysis, but if your intuition agrees with this assessment, then you are in good shape.

Notice the two arguments look almost identical. Edith and Florence both eat their vegetables. In both cases, there is a connection between the eating of vegetables and cookies. Yet we claim that it is valid to conclude that Edith gets a cookie, but not that Florence does. The difference must be in the connection between eating vegetables and getting cookies. We need to be skilled at reading and comprehending these sentences. Do the two sentences mean the same thing?

Unfortunately, in everyday language we are often sloppy, and you might be tempted to say they are equivalent. But notice that just because Florence *must* eat her vegetables, we have not claimed that doing so would be *enough* (she might also need to clean her room, for example). In everyday (non-mathematical) practice, you might be tempted to say this "other direction" is implied. In mathematics, we never get that luxury.

Remark 1.1.4 The arguments in the example above illustrate another important point: Even if you don't care about the advancement of human knowledge in the field of mathematics, becoming skilled at analyzing arguments is useful. And even if you don't want to give your grandmother a cookie. If you are *using* mathematics to solve problems in some other discipline, it is still necessary to demonstrate that your solution is correct. You better have a good argument that it is!

Since arguments are built up of statements, we must agree on what counts as a statement.

Definition 1.1.5

A **statement** is a declarative sentence that is either true or false.

If the sentences in an argument could not be true or false, there would be no way to determine whether the argument was valid, since validity describes a relationship between the truth values of the premises and conclusions.

The goal of this section is to explore the different "shapes" a statement can take. We will see that more complicated statements can be built up from simpler ones, in ways that entirely determine their truth value based on the truth values of their parts.

Preview Activity

Before reading on to the main content of the section, complete this preview activity to start thinking about the types of questions this section will address.

1. Which of the following sentences should count as statements? That is, for which of the sentences below could you *potentially* claim the sentence was either true or false? Select all that apply.

A. The sum of the first 100 positive integers.

B. What is the sum of the first 100 positive integers?

C. The sum of the first 100 positive integers is 5050.

D. Is the sum of the first 100 positive integers 5050?

E. The sum of the first 100 positive integers is 17.

2. You and your roommate are arguing, and they make the audacious claim that pineapple is good both on pizza and in smoothies. Which of the following are reasonable responses to this claim, from a logical point of view?

A. The statement is false because even though pineapple is good in smoothies, it is NOT good on pizza.

B. The statement is false because while pineapple is good on pizza and pineapple is good in smoothies, a pizza smoothie is never good.

C. The statement is half true because regardless of what you think about pineapple on pizza, we can all agree at least that pineapple is good in smoothies.

D. The statement is false because everyone who likes pineapple on pizza does NOT like pineapple in smoothies.

3. Your roommate now makes an even more outrageous claim: If a superhero movie is part of the Marvel Cinematic Universe, then it is good. Which of the following are reasonable responses to this claim, from a logical point of view?

A. This is false because there are good superhero movies, like Wonder Woman and Dark Knight, that are based on DC Comics, and so not part of the Marvel Cinematic Universe.

B. The statement is false because there is at least one superhero movie that is part of the Marvel Cinematic Universe that is also not good.

C. The statement is false because, for example, Green Lantern is neither Marvel nor good.

D. The statement is true because more than half of the Marvel movies are good.

4. Your roommate just won't let up with their outrageous claims. Now they claim that either every troll is a knave, or there is at least one troll that is a knight. What can you say to this?

A. Yes, this is true because every troll is either a knight or a knave. If it is not the case that *all* trolls are knaves, then there must be *some* troll that is a knight.

B. This is false because some trolls are knights and some other trolls are knaves.

C. The statement is false because there is no way to verify which of the two options is the case.

D. The statement is false because no troll could say that all trolls are knaves, since knaves always lie.

1.1.2 Atomic and Molecular Statements

A **statement** is any declarative sentence which is either true or false. A statement is **atomic** if it cannot be divided into smaller statements, otherwise it is called **molecular**.

Example 1.1.6

These are statements (in fact, *atomic* statements):

- Telephone numbers in the USA have 10 digits.
- The moon is made of cheese.
- 42 is a perfect square.
- Every even number greater than 2 can be expressed as the sum of two primes.
- $3 + 7 = 12$

And these are not statements:

- Would you like some cake?
- The sum of two squares.
- $1 + 3 + 5 + 7 + \cdots + 2n + 1$.
- Go to your room!
- $3 + x = 12$

The reason the sentence "$3 + x = 12$" is not a statement is that it contains a variable. Depending on what x is, the sentence is either true or false, but right now it is neither. One way to make the *sentence* into a *statement* is to specify the value of the variable in some way. This could be done by setting a specific substitution, for example, "$3 + x = 12$ where $x = 9$," which is a true statement. Or you could *capture* the free variable by *quantifying* over it, as in, "For all values of x, $3 + x = 12$," which is false. We will discuss quantifiers in more detail in the subsection Quantifiers and Predicates below.

You can build more complicated (molecular) statements out of simpler (atomic or molecular) ones using **logical connectives**. For example, this is a molecular statement:

Telephone numbers in the USA have 10 digits, and 42 is a perfect square.

Note that we can break this down into two smaller statements. The two shorter statements are *connected* by an "and." We will consider 5 connectives: "and" (Sam is a man, and Chris is a woman), "or" (Sam is a man, or Chris is a woman), "if. . . , then. . . " (if Sam is a man, then Chris is a woman), "if and only if" (Sam is a man if and only if Chris is a woman), and "not" (Sam is not a man). The first four are called **binary connectives** (because they connect two statements) while "not" is an example of a **unary connective** (since it applies to a single statement).

These molecular statements are, of course, still statements, so they must be either true or false. The crucial observation here is that which **truth value** the molecular statement achieves is completely determined by the type of connective and the truth values of the parts. We do not need to know what the parts actually say or whether they have some material connection to each other, only whether those parts are true or false.

To analyze logical connectives, it is enough to consider **propositional variables** (sometimes called *sentential* variables), usually capital letters in the middle of the alphabet: $P, Q, R, S, \ldots$. We think of these as standing in for (usually atomic) statements, but there are only two *values* the variables can achieve: true or false.[1] We also have symbols for the logical connectives: $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\neg$.

Definition 1.1.7 Logical Connectives.

We define the following **logical connectives**.

- $P \wedge Q$ is read "P and Q," and is called a **conjunction**.
- $P \vee Q$ is read "P or Q," and is called a **disjunction**.
- $P \rightarrow Q$ is read "if P then Q," and is called an **implication** or **conditional**.
- $P \leftrightarrow Q$ is read "P if and only if Q," and is called a **biconditional**.
- $\neg P$ is read "not P," and is called a **negation**.

The **truth value** of a statement is determined by the truth value(s) of its part(s), depending on the connectives:

Definition 1.1.8 Truth Conditions for Connectives.

The **truth conditions** for the logical connectives are defined as follows.

- $P \wedge Q$ is true when both P and Q are true.
- $P \vee Q$ is true when P or Q or both are true.
- $P \rightarrow Q$ is true when P is false or Q is true (or both).
- $P \leftrightarrow Q$ is true when P and Q are both true, or both false.
- $\neg P$ is true when P is false.

Each of the above definitions can be represented in a table, called a **truth table**. We simply list what the truth value of the statement is for each possible combination of truth values of the parts.

[1] In computer programming, we should call such variables **Boolean variables**.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

P	$\neg P$
T	F
F	T

Figure 1.1.9 Truth tables for logical connectives.

For example, we can use the truth table for $P \rightarrow Q$ to decide whether the statement, "If 5 is even, then 6 is even," is true or false. Here P is the statement "5 is even," and Q is the statement "6 is even." Since 5 is not even, the statement P is false. Since 6 is even, the statement Q is true. The truth table tells us that the statement $P \rightarrow Q$ is true when P is false and Q is true (the 3rd row). So the statement, "If 5 is even, then 6 is even," is true. (If you don't like that the statement is true, hold on to that thought, and we will hopefully resolve it soon.)

Note that for us, *or* is the **inclusive or** (and not the sometimes used *exclusive or*) meaning that $P \vee Q$ is true when P or Q *or both* P *and* Q are true. As for the other connectives, "and" behaves as you would expect, as does negation. The biconditional (if and only if) might seem a little strange, but you should think of this as saying the two parts of the statements are *equivalent* in that they have the same truth value.

This leaves only the implication $P \rightarrow Q$ which has a slightly different meaning in mathematics than in ordinary usage. However, implications are so common and useful in mathematics that we must develop a level of fluency with their use which warrants a whole section (Section 1.2).

Example 1.1.10

Using the truth conditions for the logical connectives, determine which statements below are true and which are false.

1. 17 is prime, and 17 is odd.
2. 17 is prime, and 18 is prime.
3. 17 is prime, or 18 is prime.
4. 17 is prime, or 19 is prime.
5. If 17 is prime, then 19 is prime.
6. If 18 is prime, then my favorite number is 17.
7. 17 is prime if and only if 19 is prime.
8. 17 is not prime if and only if 19 is not prime.

Solution. First, let's agree on some facts: 17 really is prime and odd, 18 is not either, and 19 is prime.

1. True. Both parts of the conjunction are true, so the entire statement is true.

2. False. The first part is true, but the second part is false, so the entire statement is false.

3. True. The first part is true, so the entire statement is true. As soon as we see one true statement in a disjunction, we can stop checking and declare the entire statement true.

4. True. Since we use the inclusive or, the statement is true when both parts are true.

5. True. Don't be worried that there isn't a good reason that 17 being prime *causes* 19 to be prime. That is not what we mean by a conditional statement. Since the "then" part is true, we know that the statement overall is true.

6. True. The "if" part of the statement is false. That's all we need. I bet you don't even know what my favorite number is, and you don't need to. The statement is true.

7. True. Do both parts have the same truth value? Yes, since they are both true. So the entire statement is true.

8. True as well. Now both parts are false (since both are the negation of a true statement), so the entire statement is true.

The way we define logical connectives and their truth value is very precise and technical. Often, language is not. Part of learning how to communicate mathematics is learning the cultural norms of mathematical language and how to translate statements in ordinary language into these technical statements. This will get easier with practice, so make sure you are talking to lots of people about the math you are studying.

Here are a few examples of how ordinary language might be difficult to translate.

Example 1.1.11

Identify the logical structure of each of the following statements.

1. 4 and 5 are both prime.

2. Only one of 4 or 5 is prime.

3. You must attend every day and do the homework to pass this class.

4. Every number is even or odd.

Solution.

1. Do you agree this is the same statement as "4 is prime, *and* 5 is prime"? Notice that it would not make sense to write this as $P \wedge Q$ where P is "4" and Q is "5 is prime". But if we let P be the statement, "4 is prime," then both parts of the conjunction are statements.

2. Again, we can't just put what is on one side of the "or" as a statement. But if we let P be "4 is prime" and Q be "5 is prime,", then we can write this as $(P \vee Q) \wedge \neg(P \wedge Q)$. That is, either 4 is prime or 5 is prime, and it is not the case that both 4 is prime and 5 is prime.

3. Here is another way you could phrase the same statement: If you pass the class, then it must be the case that you attended every day and that you did the homework. If we agree that this is just a clearer way to state the original statement, then we could illustrate its structure as $P \rightarrow (Q \wedge R)$.

4. Notice that this is not the same as saying, "Every number is even, or every number is odd." Of course, saying, "3 is even or odd," *is* the same as saying, "3 is even, or 3 is odd." Language is confusing!

   We don't yet have the logical technology to translate this statement as anything more than P, where P is the statement, "Every number is even or odd." Luckily, that technology is available, starting... now!

1.1.3 Quantifiers and Predicates

Did you know that all mammals have hair? That every integer is even or odd? That some odd numbers are not prime?

Our goal is to explore how to write statements such as these in mathematical notation to highlight the logical structure of the statements.

This will require considering a new sort of basic sentence called a **predicate**, which is like a statement, but contains a **free variable**. When you replace that variable with a constant of some sort, then the sentence becomes a statement proper. Think of a predicate as making a claim about the values that are substituted for the "placeholder" variable(s).

A predicate can be made into a (true or false) statement by evaluating it at some constant(s), or we can claim that some or all possible constants would make the resulting statement true or false. This is done using **quantifiers**.

Definition 1.1.12 Quantifiers.

The **universal quantifier** is written $\forall$ and is read, "for all." The **existential quantifier** is written $\exists$ and is read, "there exists" or "for some."

We usually write predicates similar to how you write a function, although with capital letters. For example, we might use the predicate $P(x)$ to represent "x is prime". We can then say that $P(7)$ is true (since 7 is prime) and that $P(8)$ is false. Or using quantifiers, we can (falsely) claim that all numbers are prime by writing $\forall x P(x)$ or (truthfully) claim that there is at least one prime number, by writing $\exists x P(x)$.

Example 1.1.13

Translate the statement, "Every number is even or odd," into symbols.

Solution. Before we even start using symbols, it is helpful to rephrase this in a way that captures the logical structure of the statement. What is the claim saying? Given any number, it will either be the case that the number is even, or that the number is odd. In particular, we are not claiming that either all numbers are even or all numbers are odd.

Let's use $E(x)$ to say that x is even, and $O(x)$ to say that x is odd. Then we can write,

$$E(x) \vee O(x)$$

to say that x is even or x is odd. Which x is that true for (according to the claim)? *All* of them. So we write the statement as,

$$\forall x(O(x) \vee E(x)).$$

We added some parentheses to emphasize that the **scope** of the universal quantifier includes both predicates.

Note that if we incorrectly interpreted the statement as claiming that either all numbers are even or all numbers are odd, we could write that as $\forall x O(x) \vee \forall x E(x)$. This is not the same!

Just like we did for propositional logic and the logical connectives, we should decide what it means for a quantified predicate to be true or false. We say $\forall x P(x)$ is true if $P(a)$ is true no matter what constant a we substitute for x. And similarly, $\exists x P(x)$ is true if there is at least one value a for which $P(a)$ is true.

However, we must be careful here. Consider the statement

$$\forall x \exists y(y < x).$$

You would read this, "For every x there is some y such that y is less than x." Note that $<$ is a predicate with two free variables; we have chosen to write it with the symbol between the variables instead of the funky-looking $L(y, x)$ or $<(y, x)$.

Is this statement true? The answer depends on our **domain of discourse**. When we say "for all" x, do we mean all positive integers or all real numbers or all elephants or...? Usually, this information is implied by the context of the statement. In discrete mathematics, we almost always quantify over the *natural numbers*, $0, 1, 2, \ldots$, so let's take that for our domain of discourse here.

For the statement to be true, we need, no matter what natural number we select, for there to be some natural number that is strictly smaller. Perhaps we could let y be $x - 1$? But here is the problem: what if $x = 0$? Then $y = -1$, and that is *not a number!* (in our domain of discourse). Thus we see that the statement is false because there is a number less than or equal to all other numbers. In symbols,

$$\exists x \forall y (y \geq x).$$

We will explore some rules for working with quantifiers and other connectives in Section 1.3. For now, we will focus on translating between informal statements in ordinary language and the more precise language of logic. There is no perfect algorithm for doing this translation, but here are a few useful rules of thumb.

Every blank is blank.

Any statement of the form, "Every P-thing is a Q-thing" can be written as

$$\forall x (P(x) \rightarrow Q(x)).$$

Example: all mammals have hair, becomes $\forall x (M(x) \rightarrow H(x))$, where $M(x)$ means x is a mammal, and $H(x)$ means x has hair.

To make sense of this, think about what we mean by statements like these in terms of sets. We claim that the set of mammals is contained in, or is a subset of, the set of hairy things. What we mean by "A is a subset of B" is precisely that every element of x is an element of y. This can also be expressed by saying that "if x is an element of A, then x is also an element of B."

Some blanks are blank.

Any statement of the form, "Some P-things are Q-things," can be written as

$$\exists x (P(x) \wedge Q(x)).$$

Example: Some cats can swim, becomes $\exists x (C(x) \wedge S(x))$, where $C(x)$ means x is a cat, and $S(x)$ means x can swim.

Again, it is helpful to think of how to express such statements in terms of sets. To say that some cats can swim is to say that there are things that belong both to the set of cats and to the set of swimming things. Such animals belong to the *intersection* of these two sets, which you can describe as belonging to the first set *and* the second set. Existential statements of this form claim that the intersection of the two sets is not empty.

Implicit Quantifiers. It is always a good idea to be precise in mathematics. Sometimes though, we can relax a bit, as long as we all agree on a convention. An example of such a convention is to assume that sentences containing predicates with free variables are intended as statements, where the variables are universally quantified.

For example, do you believe that if a shape is a square, then it is a rectangle? But how can that be true if it is not a statement? To be a little more precise, we have two predicates: $S(x)$ for "x is a square" and $R(x)$ for "x is a rectangle". The *sentence* we are looking at is

$$S(x) \rightarrow R(x).$$

This is neither true nor false, as it is not a statement. But come on! We all know that we meant to consider the statement,

$$\forall x(S(x) \rightarrow R(x)),$$

and this is what our convention tells us to consider. We call the resulting statement the **universal generalization** of the original sentence.

Definition 1.1.14

Given a sentence with free variables, the **universal generalization** of that sentence is the statement obtained by adding enough universal quantifiers to the beginning of the sentence so that all free variables become bound.

Similarly, we will often be a bit sloppy about the distinction between a predicate and a statement. For example, we might write, *let $P(n)$ be the* statement, *"n is prime,"* which is technically incorrect. It is implicit that we mean that we are defining $P(n)$ to be a predicate, which for each n becomes the statement, n is prime.

1.1.4 Reading Questions

1. Match each statement in symbols with its type of statement.

$P \rightarrow Q$	P and Q (conjunction)
$P \vee Q$	If P, then Q, (implication)
$P \wedge Q$	P or Q (disjunction)
$\neg P$	Not P (negation)

2. Consider the sentence, "If $x > 3$, then x is even."

Which of the following statements are true about the sentence? Select all that apply.

A. The sentence is a false statement since it has a free variable.

B. The universal generalization of the sentence is a statement.

C. If you substitute 10 for x, the resulting statement is true.

D. The sentence becomes a true statement no matter what natural number you substitute for x.

3. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

1.1.5 Practice Problems

1. For each sentence below, decide whether it is an atomic statement, a molecular statement, or not a statement at all.

(a) Some say the end is near, and some say we'll see Armageddon soon.

(b) Mom's coming 'round to put it back the way it ought to be.

(c) Learn to swim.

2. Classify each of the sentences below as an atomic statement, a molecular statement, or not a statement at all. If the statement is molecular, say what kind it is (conjuction, disjunction, conditional, biconditional, negation).

(a) Everybody can be fooled sometimes.

(b) Every natural number greater than 1 is either prime or composite.

(c) Go to your room!

(d) The Broncos will win the Super Bowl, or I'll eat my hat.

(e) This shirt is not black.

3. Determine whether each molecular statement below is true or false, or whether it is impossible to determine. Assume you do not know what my favorite number is (but you do know which numbers are prime).

(a) If 4 is my favorite number, then $4 + 1$ is my favorite number.

(b) 8 is my favorite number, and 3 is not prime.

(c) 4 is my favorite number, or 4 is prime.

(d) If 4 is prime, then $2 \cdot 4$ is prime.

(e) If 3 is prime, then 3 is my favorite number.

(f) 8 is my favorite number, and 4 is not prime.

4. Let $P(x, y)$ be the predicate, "person x can be fooled at time y."
Match each statement with its representation in symbols.

It is always true that some people can be fooled.	$\exists x \forall y P(x, y)$
Sometimes everyone can be fooled.	$\forall x \exists y P(x, y)$
Everyone can be fooled sometimes.	$\forall y \exists x P(x, y)$
Some people can be fooled all of the time.	$\exists y \forall x P(x, y)$

5. Your friend believes that you cannot fool everyone at the same time. What is another way of saying this, and how would you write that in symbols (using $P(x, y)$ to say you can fool x at time y).

A. Someone is never fooled. $\exists x \forall y \neg P(x, y)$

B. Everyone is never fooled. $\forall x \forall y \neg P(x, y)$

C. Someone is not fooled sometimes. $\exists x \exists y \neg P(x, y)$

D. Everyone is not fooled sometimes. $\forall x \exists y \neg P(x, y)$

6. Regardless of your beliefs of how many people can be fooled at various times, what could you conclude if we reinterpret $P(x, y)$ to mean $x < y$ and only quantify over the natural numbers (so $\forall x$ means "For all natural numbers," and $\exists x$ means "There exists a natural number")? Select all of the following that apply.

A. $\forall x \exists y P(x, y)$ is true.

B. $\exists x \forall y P(x, y)$ is true.

C. $\forall y \exists x P(x, y)$ is true.

D. $\exists y \forall x P(x, y)$ is true.

E. No matter what $P(x, y)$ means, we can conclude that $\forall x \exists y P(x, y)$ and $\exists y \forall x$ are NOT *logically equivalent*

7. Let $P(x)$ be the predicate, "$17x + 1$ is even."

(a) Is $P(15)$ true or false?

(b) What, if anything, can you conclude about $\exists x P(x)$ from the truth value of $P(15)$?

(c) What, if anything, can you conclude about $\forall x P(x)$ from the truth value of $P(15)$?

8. Let $P(x)$ be the predicate, "$18x + 1$ is even."

(a) Is $P(15)$ true or false?

(b) What, if anything, can you conclude about $\exists x P(x)$ from the truth value of $P(15)$?

(c) What, if anything, can you conclude about $\forall x P(x)$ from the truth value of $P(15)$?

9. Consider the sentence, $\exists x P(x, y) \rightarrow \forall x P(x, y)$. What can we say about this sentence? Select all that apply.

A. The sentence is a statement because it contains quantifiers.

B. The sentence is not a statement because x and z are free variables.

C. The sentence is not a statement because y is a free variable.

D. The universal generalization of the sentence is a statement.

10. Suppose $P(x, y)$ is some binary predicate defined on a very small domain of discourse: just the integers 1, 2, 3, and 4. For each of the 16 pairs of these numbers, $P(x, y)$ is either true or false, according to the following table (x values are rows, y values are columns).

	1	**2**	**3**	**4**
1	T	F	F	F
2	F	T	T	F
3	T	T	T	T
4	F	F	F	F

For example, $P(1, 3)$ is false, as indicated by the F in the first row, third column.
Use the table to decide whether the following statements are true or false.

(a) $\forall y \exists x P(x, y)$.

(b) $\exists x \forall y P(x, y)$.

(c) $\forall x \exists y P(x, y)$.

(d) $\exists y \forall x P(x, y)$.

1.1.6 Additional Exercises

1. Suppose P and Q are the statements: P: Jack passed math. Q: Jill passed math.

(a) Translate "Jack and Jill both passed math" into symbols.

(b) Translate "If Jack passed math, then Jill did not" into symbols.

(c) Translate "$P \vee Q$" into English.

(d) Translate "$\neg(P \wedge Q) \rightarrow Q$" into English.

(e) Suppose you know that if Jack passed math, then so did Jill. What can you conclude if you know that:

i. Jill passed math?
ii. Jill did not pass math?

2. Translate into symbols. Use $E(x)$ for "x is even" and $O(x)$ for "x is odd."

(a) No number is both even and odd.

(b) One more than any even number is an odd number.

(c) There is a prime number that is even.

(d) Between any two numbers there is a third number.

(e) There is no number between a number and one more than that number.

3. For each of the statements below, give a domain of discourse for which the statement is true, and a domain for which the statement is false.

(a) $\forall x \exists y (y^2 = x)$.

(b) $\forall x \forall y (x < y \rightarrow \exists z (x < z < y))$.

(c) $\exists x \forall y \forall z (y < z \rightarrow y \leq x \leq z)$.

1.2 Implications

Objectives

After completing this section, you should be able to do the following.

1. Explain the conditions under which an implication is true.
2. Identify statements as equivalent to a given implication or its converse.
3. Explain the relationship between the truth values of an implication, its converse, and its contrapositive.

1.2.1 Section Preview

Investigate!

Little Timmy's Mom tells him, "If you don't eat all your broccoli, then you will not get any ice cream." Of course, Timmy loves his ice cream, so he quickly eats all his broccoli (which actually tastes pretty good).

After dinner, when Timmy asks for his ice cream, he is told no! Does Timmy have a right to be upset? Why or why not?

By far, the most important type of statement in mathematics is the implication. It is also the least intuitive of our basic molecular statement types. Our goal in this section is to become more familiar with this key concept.

To see why this sort of statement is so prevalent, consider the *Pythagorean Theorem*. Despite what social media might claim, the Pythagorean Theorem is not

$$a^2 + b^2 = c^2.$$

Okay, sure, that has a variable in it, so we must be using the convention to take the universal generalization,

$$\forall a, b, c \in \mathbb{R} \left(a^2 + b^2 = c^2\right).$$

So $1^2 + 5^2 = 2^2$??? Okay, fine. The equation is true as long as a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse. In other words:

> *If* a and b are the lengths of the legs of a right triangle with hypotenuse of length c, *then* $a^2 + b^2 = c^2$.

Math is about making general claims, but a claim is rarely going to be true of absolutely *every* mathematical object. The way we *restrict* our claims to a particular type of object is with an implication: "Take any object you like, *if* it is of the right type, *then* this thing is true about it."

Similarly, as we saw in the Quantifiers and Predicates subsection, when we make claims like “Every square is a rectangle,” we really have an implication: “If something is a square, then it is a rectangle.”

Here is a reminder of what we mean by an implication.

Definition 1.2.1 Implication.

An **implication** (or **conditional**) is a molecular statement of the form

$$P \to Q$$

where P and Q are statements. We say that

- P is the **hypothesis** (or **antecedent**).
- Q is the **conclusion** (or **consequent**).

An implication is *true* provided P is false or Q is true (or both), and *false* otherwise. In particular, the only way for $P \to Q$ to be false is for P to be true *and* Q to be false.

The definition of truth of an implication can also be represented as a truth table:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Figure 1.2.2 The truth table for $P \to Q$.

Does this truth table make sense? Should we believe it? Look in particular at the third row: F, T, T, and consider the implication, “If $5 < 3$ then $5 + 3 = 8$.” Does that statement *feel* true? The truth table says it should be (since $5 < 3$ is false, and $5 + 3 = 8$ is true).

Much of what we will do in the remainder of this section is convince ourselves that this truth table makes sense.

Preview Activity

1. Consider the statement, “If Tommy doesn’t eat his broccoli, then he will not get any ice cream.” Which of the following statements mean the same thing (i.e., will be true in the same situations)? Select all that apply.

A. If Tommy does eat his broccoli, then he will get ice cream.

B. If Tommy gets ice cream, then he ate his broccoli.

C. If Tommy doesn’t get ice cream, then he didn’t eat his broccoli.

D. Tommy ate his broccoli and still didn't get any ice cream.

2. Suppose that your shady uncle offers you the following deal: If you loan him your car, then he will bring you tacos. In which of the following situations would it be fair to say that your uncle is a liar (i.e., that his statement was false)? Select all that apply.

A. You loan him your car. He brings you tacos.

B. You loan him your car. He never buys you tacos.

C. You don't loan him your car. He still brings you tacos.

D. You don't loan him your car. He never brings you tacos.

3. Consider the *sentence*, "If $x \geq 10$, then $x^2 \geq 25$." This sentence becomes a statement when we replace x by a value, or "capture" the x in the scope of a quantifier. Which of the following claims are true (select all that apply)?

A. If we replace x by 15, then the resulting statement is true. (Note, $15^2 = 225$.)

B. If we replace x by 3, then the resulting statement is true.

C. If we replace x by 6, then the resulting statement is true.

D. The universal generalization ("for all x, if $x \geq 10$ the $x^2 \geq 25$") is true.

E. There is a number we could replace x with that makes the statement false.

4. Consider the statement, "If I see a movie, then I eat popcorn" (which happens to be true). Based solely on your intuition of English, which of the following statements mean the same thing? Select all that apply.

A. If I eat popcorn, then I see a movie.

B. If I don't eat popcorn, then I don't see a movie.

C. It is necessary that I eat popcorn when I see a movie.

D. To see a movie, it is sufficient for me to eat popcorn.

E. I only watch a movie if I eat popcorn.

1.2.2 Understanding the Truth Table

The truth value of the implication is determined by the truth values of its two parts. Our definition of the truth conditions for an implication says that there is only one way for an implication to be false: when the hypothesis is true and the conclusion is false.

Example 1.2.3

Consider the statement:

> If Bob gets a 90 on the final, then Bob will pass the class.

This is definitely an implication: P is the statement "Bob gets a 90 on the final," and Q is the statement "Bob will pass the class."

Suppose I made that statement to Bob. In what circumstances would it be fair to call me a liar? What if Bob really did get a 90 on the final, and he did pass the class? Then I have not lied; my statement is true. However, if Bob did get a 90 on the final and did not pass the class, then I lied, making the statement false. The tricky case is this: What if Bob did not get a 90 on the final? Maybe he passes the class, maybe he doesn't. Did I lie in either case? I think not. In these last two cases, P was false, and the statement $P \to Q$ was true. In the first case, Q was true, and so was $P \to Q$. So $P \to Q$ is true when either P is false or Q is true.

Just to be clear, although we sometimes read $P \to Q$ as "P *implies* Q", we are not insisting that there is some *causal* relationship between the statements P and Q (although there might be). "If $x < y$, then $x + 1 < y + 1$," is a true statement (or at least, its universal generalization is). We know it is true because we understand how the two parts interact. If you add 1 to two numbers x and y, then their order does not change. But the statement, "if $1 < 2$, then Euclid studied geometry" is also a true implication.

Example 1.2.4

Decide which of the following statements are true and which are false. Briefly explain.

1. If $1 = 1$, then most horses have 4 legs.

2. If $0 = 1$, then $1 = 1$.

3. If 8 is a prime number, then the 7624th digit of π is an 8.

4. If the 7624th digit of π is an 8, then $2 + 2 = 4$.

Solution. All four of the statements are true. Remember, the only way for an implication to be false is for the *if* part to be true and the *then* part to be false.

1. Here both the hypothesis and the conclusion are true, so the implication is true. It does not matter that there is no meaningful connection between the true mathematical fact and the fact about horses.

2. Here the hypothesis is false and the conclusion is true, so the implication is true.

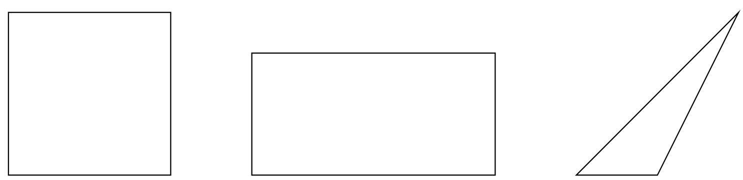
3. I have no idea what the 7624th digit of π is, but this does not matter. Since the hypothesis is false, the implication is automatically true.

4. Regardless of the truth value of the hypothesis, the conclusion is true, making the implication true.

This is a strange example and isn't really how we use implications anyway. This strangeness is not just mathematicians being stubborn though. The truth conditions for implications *must* be like they are for mathematics to make sense. Let's see why.

Example 1.2.5

Consider the statement, "All squares are rectangles," which can also be phrased as, "For all shapes, if the shape is a square, then it is a rectangle." Is this statement true or false? Are we sure? What about the following three shapes?



Solution. Of course the statement is true. A square is a 4-sided plane figure with 4 right angles and 4 equal-length sides, while a rectangle is a 4-sided plane figure with 4 right angles.

However, what we mean when we consider a universal statement like this is that, no matter what we "plug in" for the variable ("the shape" in this case), the resulting statement is true. When the statement is about a particular shape, we have an implication $P \to Q$. This means it must be true that, if the actual shape on the left is a square, then it is a rectangle. Great. The shape is a square (P is true) and is a rectangle (Q is true), so yes, the implication is true.

Is the implication true of the rectangle in the middle? Well, that shape is not a square (P is false), and it is a rectangle (Q is true). But look, we believe that all squares are rectangles, so the statement must be true. Even of a rectangle. The only way this works is if "false implies true" is true!

Similarly, all squares are rectangles is a true statement, even when we look at a triangle. P is false (the triangle is not a square), and Q is false (the triangle is not a rectangle). Thankfully, we defined implications to be true in this case as well.

We have given shapes that illustrate lines 1, 3, and 4 of the truth table for implications (Figure 1.2.2). What shape illustrates line 2? That would need to be a shape that was a square and was not a rectangle.... Of course we can't find one, precisely because the statement is true!

1.2.3 Related Statements

An implication is a way of expressing a relationship between two statements. It is often interesting to ask whether there are other relationships between the statements. Here we introduce some common language to address this question.

Definition 1.2.6 Converse, Contrapositive, and Inverse.

Given an implication $P \to Q$, we say,

- The **converse** is the statement $Q \to P$.
- The **contrapositive** is the statement $\neg Q \to \neg P$.
- The **inverse** is the statement, $\neg P \to \neg Q$.

Example 1.2.7

Consider the implication, "If you clean your room, then you can go to the party." Give the converse, contrapositive, and inverse of this statement

Solution. The converse is, "If you can go to the party, then you clean your room."

The contrapositive is, "If you can't go to the party, then you don't clean your room."

The inverse is, "If you don't clean your room, then you can't go to the party."

Symbolically, both the converse and the contrapositive *switch* the order of the two parts of the statement (or alternatively, think about turning the arrow to point in the other direction). The contrapositive and the inverse take the *negation* of both of the statements. Notice that if you take the converse (switch the order) and then take the contrapositive *of* that converse (switch the order back and negate both parts) you get the inverse. So the inverse is nothing more than the contrapositive of the converse. Or the converse of the contrapositive, which is a fun fact to mention at parties.

When considering statements with quantifiers, we ignore the outside quantifiers when forming the converse, contrapositive, and inverse.

Quantifiers and the Converse, Contrapositive, and Inverse.

A quantified implication $\forall x(P(x) \to Q(x))$ has:

Converse	$\forall x(Q(x) \to P(x))$
Contrapositive	$\forall x(\neg Q(x) \to \neg P(x))$
Inverse	$\forall x(\neg P(x) \to \neg Q(x))$

Note 1.2.8 It is unlikely that we would encounter a statement of the form $\exists x(P(x) \to Q(x))$, since this would be automatically true if there was any x that made $P(x)$ false. But if we did, the same rules would apply to the converse, contrapositive, and inverse as above: Just ignore the quantifier when swapping and/or negating the parts of the implication.

For example, "For all shapes, if the shape is a square, then it is a rectangle" (i.e., all squares are rectangles) has the converse, "For all shapes, if the shape is a rectangle, then it is a square" (so all rectangles are squares).

Well, that's not true! There exist shapes that are rectangles and are NOT squares. Indeed, this is an example of a statement that is true with a false converse. There are lots of examples of this throughout mathematics. There are also examples of true implications that have true converses. You just can't know from the logic.[2]

The contrapositive of "For all shapes, if it is a square, then it is a rectangle" is "For all shapes, if the shape is not a rectangle, then it is not a square." This is true. In fact, ***the contrapositive of a true statement is always true***!

Since the contrapositive of an implication always has the same truth value as its original implication, it can often be helpful to analyze the contrapositive to decide whether an implication is true.

Example 1.2.9

True or false: If you draw any nine playing cards from a regular deck, then you will have at least three cards all of the same suit. Is the converse true?

Solution. True. The original implication is a little hard to analyze because there are so many combinations of nine cards. But consider the contrapositive: if you *don't* have at least three cards all of the same suit, then you don't have nine cards. It is easy to see why this is true. If you don't have at least three cards in a suit, you can have at most two cards of each of the four suits, for a total of at most eight cards.

[2]It turns out the Pythagorean Theorem is one such statement. It is also true that *if* $a^2 + b^2 = c^2$, *then* there is a right triangle with legs of lengths a and b and hypotenuse of length c. So we could have also written the theorem as a biconditional: "a and b are the lengths of the legs of a right triangle with hypotenuse of length c *if and only if* $a^2 + b^2 = c^2$."

The converse: If you have at least three cards of the same suit, then you have nine cards. This is false. You could have three spades and nothing else. Note that to demonstrate that the converse (an implication) is false, we provided an example where the hypothesis is true (you do have three cards of the same suit), but where the conclusion is false (you do not have nine cards). In other words, we find some example that puts us in row 2 of the implication's truth table.

Understanding converses and contrapositives can help understand implications and their truth values:

Example 1.2.10

Suppose I tell Sue that if she gets a 93% on her final, then she will get an A in the class. Assuming that what I said is true, what can you conclude in the following cases:

1. Sue gets a 93% on her final.
2. Sue gets an A in the class.
3. Sue does not get a 93% on her final.
4. Sue does not get an A in the class.

Solution. Note first that whenever $P \to Q$ and P are both true statements, Q must be true as well. For this problem, take P to mean "Sue gets a 93% on her final" and Q to mean "Sue will get an A in the class."

1. We have $P \to Q$ and P, so Q follows. Sue gets an A.
2. You cannot conclude anything. Sue could have gotten the A because she did extra credit, for example. Notice that we do not know that if Sue gets an A, then she gets a 93% on her final. That is the converse of the original implication, so it might or might not be true.
3. The contrapositive of the converse of $P \to Q$ is $\neg P \to \neg Q$, which states that if Sue does not get a 93% on the final, then she will not get an A in the class. But this does not follow from the original implication. Again, we can conclude nothing. Sue could have done extra credit.
4. What would happen if Sue did not get an A but *did* get a 93% on the final? Then P would be true, and Q would be false. This makes the implication $P \to Q$ false! It must be that Sue did not get a 93% on the final. Notice we now have the implication $\neg Q \to \neg P$ which is the contrapositive of $P \to Q$. Since $P \to Q$ is assumed to be true, we know $\neg Q \to \neg P$ is true as well.

As we said above, an implication is not logically equivalent to its converse, but it is possible that both the implication and its converse are true. In this case, when both $P \to Q$ and $Q \to P$ are true, we say that P and Q are equivalent and write $P \leftrightarrow Q$. This is the biconditional we mentioned in Section 1.1.

You can think of "if and only if" statements as having two parts: an implication and its converse. We might say one is the "if" part, and the other is the "only if" part. We also sometimes say that "if and only if" statements have two directions: a forward direction ($P \to Q$) and a backward direction ($P \leftarrow Q$, which is really just sloppy notation for $Q \to P$).

Let's think a little about which part is which. Is $P \to Q$ the "if" part or the "only if" part? Consider an example.

Example 1.2.11

Suppose it is true that I sing if and only if I'm in the shower. We know this means both that if I sing, then I'm in the shower, and also the converse, that if I'm in the shower, then I sing. Let P be the statement, "I sing," and Q be, "I'm in the shower." So $P \to Q$ is the statement "if I sing, then I'm in the shower." Which part of the if and only if statement is this?

What we are really asking for is the meaning of "I sing *if* I'm in the shower" and "I sing *only if* I'm in the shower." When is the first one (the "if" part) *false*? When I am in the shower but not singing. That is the same condition for being false as the statement, "If I'm in the shower, then I sing." So the "if" part is $Q \to P$. On the other hand, to say, "I sing only if I'm in the shower" is equivalent to saying "If I sing, then I'm in the shower," so the "only if" part is $P \to Q$.

It is not especially important to know which part is the "if" or "only if" part, but this does illustrate something very, very important: *There are many ways to state an implication!*

Example 1.2.12

Rephrase the implication, "If I dream, then I am asleep" in as many ways as possible. Then do the same for the converse.

Solution. The following are all equivalent to the original implication:

1. I am asleep if I dream.
2. I dream only if I am asleep.
3. In order to dream, I must be asleep.
4. To dream, it is necessary that I am asleep.
5. To be asleep, it is sufficient to dream.

6. I am not dreaming unless I am asleep.

The following are equivalent to the converse (if I am asleep, then I dream):

1. I dream if I am asleep.
2. I am asleep only if I dream.
3. It is necessary that I dream in order to be asleep.
4. It is sufficient that I be asleep in order to dream.
5. If I don't dream, then I'm not asleep.

Hopefully you agree with the above example. We include the "necessary and sufficient" versions because those are common when discussing mathematics. Let's agree once and for all what they mean.

Definition 1.2.13 Necessary and Sufficient.

- "P is necessary for Q" means $Q \to P$.
- "P is sufficient for Q" means $P \to Q$.
- If P is necessary and sufficient for Q, then $P \leftrightarrow Q$.

To be honest, I have trouble with these if I'm not very careful. I find it helps to keep a standard example for reference.

Example 1.2.14

In a regular deck of cards, the red suits are hearts and diamonds. The black suits are clubs and spades. Thus it is true that, after picking a card, if my card is a spade, then my card is black.

Restate this fact using necessary and sufficient phrasing.

Solution. For my card to be a spade, it is necessary that it is black. However, it is not sufficient for it to be black to say that I am holding a spade (since I could have a club).

I can also say that to have a black card, it is sufficient to have a spade. It is not necessary that I have a spade.

It is helpful to think about the amount of evidence you need. Is knowing that the card is a spade enough evidence to conclude that it is a black card? Yes, that is sufficient! Being a spade is a sufficient condition for the card to be black.

Thinking about the necessity and sufficiency of conditions can also help when writing proofs and justifying conclusions. If you want to establish some mathematical fact, it is helpful to think what other facts would *be enough* (be sufficient) to prove your fact. If you have an assumption, think about what must also be necessary if that hypothesis is true.

1.2.4 Reading Questions

1. It happens to be true that all mammals have hair. Which of the following are also true?

A. Having hair is a necessary condition for being a mammal.

B. Having hair is a sufficient condition for being a mammal.

C. If an animal doesn't have hair, then it is not a mammal.

D. An animal is a mammal only if it has hair.

2. Give an example of a *true* implication (written out in words) that has a *false* converse. Explain why your implication is true and why the converse is false.

3. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

1.2.5 Practice Problems

1. In my safe is a sheet of paper with two shapes drawn on it in colored crayon. One is a circle, and the other is a pentagon. Each shape is drawn in a single color. Suppose you believe me when I tell you that, "If the circle is purple, then the pentagon is orange.

What do you therefore know about the truth value of the following statements?

(a) The circle and the pentagon are both purple.

(b) The circle and the pentagon are both orange.

(c) The circle is not purple, or the pentagon is orange.

(d) If the pentagon is orange, then the circle is purple.

(e) If the pentagon is not orange, then the circle is not purple.

2. Suppose the statement, *"If the square is yellow, then the circle is purple,"* is true. Assume also that the converse is false. Classify each statement below as true or false (if possible).

(a) The circle is purple.

(b) The square is yellow if and only if the circle is not purple.

(c) The square is yellow.

(d) The square is yellow if and only if the circle is purple.

3. Consider the statement, *"If you will give me magic beans, then I will give you a cow."* Decide whether each statement below is the converse, the contrapositive, or neither.

(a) If I will give you a cow, then you will not give me magic beans.

(b) If I will give you a cow, then you will give me magic beans.

(c) If you will not give me magic beans, then I will not give you a cow.

(d) If you will give me magic beans, then I will not give you a cow.

(e) You will give me magic beans, and I will not give you a cow.

(f) If I will not give you a cow, then you will not give me magic beans.

4. You have discovered an old paper on graph theory that discusses the *viscosity* of a graph (which for all you know, is something completely made up by the author). A theorem in the paper claims that "if a graph satisfies *condition (V)*, then the graph is *viscous*." Which of the following are equivalent ways of stating this claim? Which are equivalent to the *converse* of the claim?

(a) Only viscous graphs satisfy condition (V).

(b) For a graph to be viscous, it is necessary that it satisfies condition (V).

(c) A graph is viscous only if it satisfies condition (V).

(d) Satisfying condition (V) is a necessary condition for a graph to be viscous.

(e) A graph is viscous if it satisfies condition (V).

5. Which of the following statements are equivalent to the implication, "*if you win the lottery, then you will be rich,*" and which are equivalent to the converse of the implication?

(a) If you are not rich, then you did not win the lottery.

(b) It is sufficient to win the lottery to be rich.

(c) Either you win the lottery, or else you are not rich.

(d) If you are rich, you must have won the lottery.

(e) You will win the lottery if and only if you are rich.

1.2.6 Additional Exercises

1. Translate into English:

(a) $\forall x(E(x) \rightarrow E(x+2))$.

(b) $\forall x \exists y(\sin(x) = y)$.

(c) $\forall y \exists x(\sin(x) = y)$.

(d) $\forall x \forall y(x^3 = y^3 \rightarrow x = y)$.

2. Consider the statement, "If Oscar eats Chinese food, then he drinks milk."

(a) Write the converse of the statement.

(b) Write the contrapositive of the statement.

(c) Is it possible for the contrapositive to be false? If it was, what would that tell you?

(d) Suppose the original statement is true, and that Oscar drinks milk. Can you conclude anything (about his eating Chinese food)? Explain.

(e) Suppose the original statement is true, and that Oscar does not drink milk. Can you conclude anything (about his eating Chinese food)? Explain.

3. Write each of the following statements in the form, "If …, then …." Careful, some statements may be false (which is fine for the purposes of this question).

(a) To lose weight, you must exercise.

(b) To lose weight, all you need to do is exercise.

(c) Every American is patriotic.

(d) You are patriotic only if you are American.

(e) The set of rational numbers is a subset of the real numbers.

(f) A number is prime if it is not even.

(g) Either the Broncos will win the Super Bowl, or they won't play in the Super Bowl.

4. Consider the implication, "If you clean your room, then you can watch TV." Rephrase the implication in as many ways as possible. Then do the same for the converse.

5. Recall from calculus, if a function is differentiable at a point c, then it is continuous at c, but that the converse of this statement is not true (for example, $f(x) = |x|$ at the point 0). Restate this fact using "necessary and sufficient" language.

6. Consider the statement, "For all natural numbers n, if n is prime, then n is solitary." You do not need to know what *solitary* means for this problem, just that it is a property that some numbers have and others do not.

(a) Write the converse and the contrapositive of the statement, saying which is which. Note: the original statement claims that an implication is true for all n, and it is that implication that we are taking the converse and contrapositive of.

(b) Write the negation of the original statement. What would you need to show to prove that the statement is false?

(c) Even though you don't know whether 10 is solitary (in fact, nobody knows this), is the statement, "If 10 is prime, then 10 is solitary" true or false? Explain.

(d) It turns out that 8 is solitary. Does this tell you anything about the truth or falsity of the original statement, its converse or its contrapositive? Explain.

(e) Assuming that the original statement is true, what can you say about the relationship between the *set* P of prime numbers and the *set* S of solitary numbers. Explain.

1.3 Rules of Logic

Objectives

After completing this section, you should be able to do the following.

1. Use truth tables to determine whether two statements are logically equivalent.
2. Use truth tables to determine whether a deduction rule is valid.
3. Use logical equivalence and deduction rules to simplify statements and make deductions.

1.3.1 Section Preview

Investigate!

Holmes always wears one of the two vests he owns: one tweed and one mint green. He always wears either the green vest or red shoes. Whenever he wears a purple shirt and the green vest, he chooses to not wear a bow tie. He never wears the green vest unless he is also wearing either a purple shirt or red shoes. Whenever he wears red shoes, he also wears a purple shirt. Today, Holmes wore a bow tie. What else did he wear?

Try it 1.3.1

Spend a few minutes thinking about the *Investigate!* question above. Of the six statements in the puzzle, only one is atomic. Use this atomic statement and one other statement to deduce a new statement about what Holmes might (or might not) be wearing. Explain why you think your new statement is true.

Hint. The atomic statement is, "Holmes wore a bow tie." Only one of the molecular statements has this as one of its *atoms*.

Logic studies the ways statements can interact with each other. More precisely, we consider the way the logical form statements can interact. The study of logic does not care about the content of the atomic statements or the meaning of predicates. For example, the claims, "If spiders have six legs, then Sam walks with a limp," and, "If the moon is made of cheese, then cheddar is a type of cheese," are identical from a logical perspective. Logic doesn't care about whether Sam is a spider or the culinary makeup of the moon. Both statements have the same form: They are implications, $P \to Q$.

Of course, in mathematics we often *do* know some relationship between various atomic statements. For example, we know a relationship between being even and being a multiple of 10. That relationship allows us to make claims such as, "If the number I'm thinking of is a multiple of 10, then it is even." Suppose I also told you that I am now thinking of a number that is not even. We can deduce that I am not thinking of a multiple of 10! Crucially, if we accept the truth of the statements here, we can make this deduction without thinking about the nature of numbers. It can feel very liberating and provide much-needed clarity when trying to understand complicated reasoning if we can separate the content from the logical form of arguments.

Our goal in this section is to establish some procedures for analyzing how the truth or falsity of statements interact, based on their logical form. We will see that some molecular statements must be true regardless of whether their atomic parts are true or false, while some statements must always be false. For other statements, it can be that two statements are always true or false together, or that whenever one statement is true, another statement must also be true.

The main method for establishing these relationships will be **truth tables**. There is a very clear procedure for constructing and analyzing truth tables, but for complicated arguments that contain many atomic statements, the truth tables become very large and unwieldy. We will therefore use truth tables to understand some basic equivalences and deductions that can be applied in a sequence of reasoning to construct larger arguments.

Preview Activity

1. Consider the statement, "Whenever Holmes wears a purple shirt and the green vest, he chooses to not wear a bow tie." Let P be the statement, "Holmes wears a purple shirt," G be the statement, "Holmes wears the green vest," and B be the statement, "Holmes wears a bow tie." Which of the following is the best translation of the statement into propositional logic?

A. $(P \wedge G) \rightarrow \neg B$

B. $(P \wedge G) \rightarrow B$

C. $(P \vee G) \rightarrow \neg B$

D. $P \wedge (G \rightarrow B)$

2. Consider the statement, "Holmes never wears the green vest unless he is also wearing either a purple shirt or red shoes." With P and G as in the previous question, and R being the statement, "Holmes wears red shoes," which of the following is the best translation of the statement into propositional logic?

A. $G \rightarrow (P \vee R)$

B. $\neg G \rightarrow (P \vee R)$

C. $(P \vee R) \rightarrow G$

D. $(P \vee R) \to \neg G$

3. Consider the statement, "If you major in math, then you will get a high-paying job," and the statement, "Either you don't major in math, or you will get a high-paying job." In which of the following cases are *both* statements true? Select all that apply.

A. You major in math and get a high-paying job.

B. You major in math and don't get a high-paying job.

C. You don't major in math and do get a high-paying job.

D. You don't major in math and don't get a high-paying job.

1.3.2 Truth Tables

Here's a question about playing Monopoly:

> If you get more doubles than any other player, then you will lose, or if you lose, then you must have bought the most properties.

True or false? We will answer this question and won't need to know anything about Monopoly. Instead, we will look at the logical *form* of the statement.

We need to decide when the statement $(P \to Q) \vee (Q \to R)$ is true. Using the definitions of the connectives in Definition 1.1.8, we see that for this to be true, either $P \to Q$ must be true or $Q \to R$ must be true (or both). Those are true if either P is false or Q is true (in the first case) and Q is false or R is true (in the second case). So—yeah, it gets a bit messy. Luckily, we can make a chart to keep track of all the possibilities with a **truth table**.

The idea is this: On each row, we list a possible combination of T's and F's (Trues and Falses) for each of the propositional variables, and then mark down whether the (molecular) statement in question is true or false in that case. We do this for every possible combination of T's and F's. Then we can clearly see the cases in which the statement is true or false. For complicated statements, we will first fill in values for each part of the statement, as a way of breaking up our task into smaller, more manageable pieces.

Since the truth value of a statement is completely determined by the truth values of its parts and how they are connected, all you need to know is the truth tables for each of the logical connectives, which we have already seen in Figure 1.1.9

The truth tables we consider here all build off the basic ones, applying the basic rules multiple times.

Example 1.3.2

Make a truth table for the statement $\neg P \vee Q$.

Solution. Note that this statement is not $\neg(P \vee Q)$; the negation belongs to P alone. The **main connective** here is the $\vee$, which means we will use that truth table *last*. First, we apply the truth table for $\neg$, and then apply the truth table for $\vee$ using "inputs" from $\neg P$ and Q.

Since there are two variables, there are four possible combinations of T's and F's. Putting this all together gives us the following truth table.

P	Q	$\neg P$	$\neg P \vee Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

We added a column for $\neg P$ to make filling out the last column easier. The entries in the $\neg P$ column were determined by the entries in the P column. Then to fill in the final column, look only at the column for Q and the column for $\neg P$ and use the rule for $\vee$.

Now let's answer our question about Monopoly.

Example 1.3.3

Analyze the statement, "If you get more doubles than any other player, then you will lose, or if you lose, then you must have bought the most properties," using truth tables.

Solution. Represent the statement in symbols as $(P \to Q) \vee (Q \to R)$, where P is the statement, "You get more doubles than any other player," Q is the statement, "You will lose," and R is the statement, "You must have bought the most properties." Now make a truth table.

The truth table must contain 8 rows to account for every possible combination of truth and falsity among the three statements. Here is the full truth table:

P	Q	R	$P \to Q$	$Q \to R$	$(P \to Q) \vee (Q \to R)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

The first three columns are simply a systematic listing of all possible combinations of T and F for the three statements (do you see how you would list the 16 possible combinations for four statements?). The next two columns are determined by the values of P, Q, and R and the definition of implication. Then, the last column is determined by the values in the previous two columns and the definition of $\vee$. It is this final column we care about.

Notice that in each of the eight possible cases, the statement in question is true. So our statement about monopoly is true (regardless of how many properties you own, how many doubles you roll, or whether you win or lose).

The statement about monopoly is an example of a **tautology**, a statement that is necessarily true based on its logical form alone. Tautologies are always true, but they don't tell us much about the world. No knowledge about monopoly was required to determine that the statement was true, and thus knowing that the statement is true tells us nothing about monopoly. It is equally true that "if the moon is made of cheese, then Elvis is still alive, or if Elvis is still alive, then unicorns have 5 legs."

1.3.3 Logical Equivalence

You might have noticed in Example 1.3.2 that the final column in the truth table for $\neg P \vee Q$ is identical to the final column in the truth table for $P \to Q$:

P	Q	$P \to Q$	$\neg P \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

This says that no matter what P and Q are, the statements $\neg P \vee Q$ and $P \to Q$ are either both true or both false. We therefore say these statements are **logically equivalent**.

Definition 1.3.4 Logical Equivalence.

Two (molecular) statements P and Q are **logically equivalent** provided P is true precisely when Q is true. That is, P and Q have the same truth value under any assignment of truth values to their atomic parts.

We write this as $P \equiv Q$.

To verify that two statements are logically equivalent, you can make a truth table for each and check whether the columns for the two statements are identical.

In Section 1.2 we claimed that whenever an implication is true, so is its contrapositive. We can now make this claim as the following theorem.

Theorem 1.3.5

An implication is logically equivalent to its contrapositive. That is,

$$P \to Q \equiv \neg Q \to \neg P.$$

Proof. We simply examine the truth tables.

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$\neg Q$	$\neg P$	$\neg Q \to \neg P$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

(Note that we have the truth value combinations in the same order in both tables, so we can easily see that the final columns are identical.)

Recognizing two statements as logically equivalent can be quite helpful. Rephrasing a mathematical statement can often lend insight into what it is saying, or how to prove or refute it. By using truth tables we can systematically verify that two statements are indeed logically equivalent.

Example 1.3.6

Are the statements, "It will not rain or snow," and, "It will not rain and it will not snow," logically equivalent?

Solution. We want to know whether $\neg(P \vee Q)$ is logically equivalent to $\neg P \wedge \neg Q$. Make a truth table which includes both statements:

P	Q	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

Since the truth values for the two statements are equal in every row, the two statements are logically equivalent.

Notice that this example gives us a way to "distribute" a negation over a disjunction (an "or"). We have a similar rule for distributing over conjunctions ("and"s):

Theorem 1.3.7 De Morgan's Laws.

The negation of a disjunction or conjunction is logically equivalent to a conjunction or disjunction of negations, respectively. That is,

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$

and,

$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q.$$

This suggests there might be a sort of "algebra" you could apply to statements (okay, there is: It is called *Boolean algebra*) to transform one statement into another. We can start collecting useful examples of logical equivalence and apply them in succession to a statement, instead of writing out a complicated truth table.

De Morgan's laws do not directly help us with implications, but as we saw above, every implication can be written as a disjunction:

Implications are Disjunctions.

$$P \to Q \equiv \neg P \vee Q.$$

Example: "If a number is a multiple of 4, then it is even" is equivalent to, "A number is not a multiple of 4, or (else) it is even."

With this and De Morgan's laws, you can take any statement and *simplify* it to the point where negations are only being applied to atomic propositions. Well, except that you could get multiple negations stacked up. But this can be easily dealt with:

Double Negation.

$$\neg\neg P \equiv P.$$

Example: "It is not the case that c is not odd" means "c is odd."

Let's see how we can apply the equivalences we have encountered.

Example 1.3.8

Prove that the statements $\neg(P \to Q)$ and $P \wedge \neg Q$ are logically equivalent without using truth tables.

Solution. We want to start with one of the statements and transform it into the other through a sequence of logically equivalent statements. Start with $\neg(P \to Q)$. We can rewrite the implication as a disjunction, so this is logically equivalent to

$$\neg(\neg P \vee Q).$$

Now apply De Morgan's law to get

$$\neg\neg P \wedge \neg Q.$$

Finally, use double negation to arrive at $P \wedge \neg Q$

Notice that the above example illustrates that the negation of an implication is NOT an implication: It is a conjunction! We saw this before, in Section 1.1, but it is so important and useful, it warrants stating as a theorem.

Theorem 1.3.9 Negation of an Implication.

The negation of an implication is a conjunction:

$$\neg(P \to Q) \equiv P \wedge \neg Q.$$

That is, the only way for an implication to be false is for the hypothesis to be true ***AND*** *the conclusion to be false.*

To verify that two statements are logically equivalent, you can use truth tables or a sequence of logically equivalent replacements. The truth table method, although cumbersome, has the advantage that it can verify that two statements are NOT logically equivalent.

Example 1.3.10

Are the statements $(P \vee Q) \to R$ and $(P \to R) \vee (Q \to R)$ logically equivalent?

Solution. Note that while we could start rewriting these statements with logically equivalent replacements in the hopes of transforming one into another, we will never be sure that our failure is due to their lack of logical equivalence rather than our lack of imagination. So instead, let's make a truth table:

P	Q	R	$(P \vee Q) \to R$	$(P \to R) \vee (Q \to R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

Look at the fourth (or sixth) row. In this case, $(P \to R) \vee (Q \to R)$ is true, but $(P \vee Q) \to R$ is false. Therefore the statements are not logically equivalent.

While we don't have logical equivalence, it is the case that whenever $(P \vee Q) \to R$ is true, so is $(P \to R) \vee (Q \to R)$. This tells us that we can *deduce* $(P \to R) \vee (Q \to R)$ from $(P \vee Q) \to R$, just not the reverse direction.

1.3.4 Equivalence for Quantified Statements

All the examples we have looked at so far have only involved *propositional* logic, where the basic units of logic are statements that are either true or false. It is also possible to say that two statements involving quantifiers and predicates are logically equivalent.

Sometimes the quantifiers have nothing to do with the equivalence. For example,

$$\forall x(P(x) \to Q(x)) \equiv \forall x(\neg P(x) \vee Q(x)).$$

As soon as we replace the x with a constant, we are left with two statements that are logically equivalent based on their propositional form.

Other times, the more interesting times, it is exactly the logic of the quantifiers that makes the statements logically equivalent. What is especially interesting here is that we cannot use truth tables to verify these equivalences!

Instead, we need to reason about the domain of discourse as a set. For example, let's consider how negation interacts with quantifiers.

Consider the claim that "all odd numbers are prime." We might represent this symbolically as $\forall x(O(x) \to P(x))$. The statement clearly is not true, so what *is* true is that "not all odd numbers are prime" (i.e., $\neg\forall x(O(x) \to P(x))$). How do we know? Easy: 9. Yes, 9 is odd but not prime. But is it enough that just one odd number isn't prime?

To dispute a universal claim, you just need *one* single counterexample. You just need to show *there exists* a number for which the claim is false. In our case, we have the equivalence,

$$\neg\forall x(O(x) \to P(x)) \equiv \exists x(O(x) \wedge \neg P(x)).$$

If we ignore the quantifiers for a minute, we are left with

$$\neg(O \to P) \equiv O \wedge \neg P$$

which is exactly an example of Theorem 1.3.9. The new, interesting part is that when we negated the universal quantifier, we got an existential quantifier.

Negating an existential quantifier results in a universal quantifier. This makes sense. If there does not exist something with a property, then everything does not have that property.

Quantifiers and Negation.

$\neg\forall x P(x)$ is equivalent to $\exists x \neg P(x)$.

$\neg\exists x P(x)$ is equivalent to $\forall x \neg P(x)$.

Symbolically, we can pass the negation symbol over a quantifier, but that causes the quantifier to switch type.

Another way to see why this makes sense: Universal quantifiers are like (possibly infinite) conjunctions since they claim that the property is true of this thing, and that thing, and the other thing,... all things. Existential quantifiers are like (possibly infinite) disjunctions: The property is true of at least one thing, maybe this, or that, or the other, or.... De Morgan's laws tell us that when we negate a conjunction, we get a disjunction, and when we negate a disjunction, we get a conjunction. Isn't it great when everything works out as it should?

Example 1.3.11

Suppose we claim that there is no smallest number. We can translate this into symbols as

$$\neg\exists x \forall y(x \leq y).$$

("It is not true that there is a number x such that for all numbers y, x is less than or equal to y.")

However, we know how negation interacts with quantifiers: We can pass a negation over a quantifier by switching the quantifier type (between universal and existential). So the statement above should be *logically equivalent* to

$$\forall x \exists y(y < x).$$

Notice that $y < x$ is the negation of $x \leq y$. This reads, "For every number x there is a number y which is smaller than x." We see that this is another way to make our original claim.

It is important to stress that predicate logic *extends* propositional logic (much like how quantum mechanics extends classical mechanics). Everything that we learned about logical equivalence and deductions still applies. However, predicate logic allows us to analyze statements at a higher resolution, digging down into the individual propositions P, Q, etc.

To do this, we need to understand how quantifiers and connectives interact. We have already seen something about negations and quantifiers. What about the other connectives? Let's look at an example exploring how the universal quantifier and disjunctions can (or cannot) work together.

Example 1.3.12

Consider the two statements,

$$\forall x(P(x) \vee Q(x)) \qquad\qquad \forall x P(x) \vee \forall x Q(x).$$

Are these logically equivalent?

Solution. These statements are NOT logically equivalent. Intuitively, the statement on the left claims that everything is either a P-thing or a Q-thing. The statement on the right claims that either everything is a P-thing or that everything is a Q-thing. These *feel* different.

To be sure, we would like to think of predicates $P(x)$ and $Q(x)$ and some domain of discourse such that one of the statements is true and the other is false. How about we let $P(x)$ be, "x is even" and $Q(x)$ be, "x is odd." Our domain of discourse will be all integers (as that is the set of numbers for which even and odd make sense).

The statement on the left is true! Every number is either even or odd. But is every number even? No. Is every number odd? No. So the statement on the right is false (it is a *false or false*).

Interestingly, the statement on the right implies the statement on the left. That is,

$$(\forall x P(x) \vee \forall x E(x)) \rightarrow \forall x (P(x) \vee Q(x))$$

is always true.

This is similar to a tautology, although we reserve that term for necessary truths in propositional logic. A statement in predicate logic that is necessarily true gets the more prestigious designation of a **law of logic** (or sometimes **logically valid**, but that is less fun).

We can also consider how quantifiers interact with each other.

Example 1.3.13

Can you switch the order of quantifiers? For example, consider the two statements:

$$\forall x \exists y P(x, y) \qquad \text{and} \qquad \exists y \forall x P(x, y).$$

Are these logically equivalent?

Solution. These statements are NOT logically equivalent. To see this, we should provide an interpretation of the predicate $P(x, y)$ which makes one of the statements true and the other false.

Let $P(x, y)$ be the predicate $x < y$. It is true, in the natural numbers, that for all x there is some y greater than that x (since there are infinitely many numbers). However, there is no natural number y which is greater than every number x. Thus it is possible for $\forall x \exists y P(x, y)$ to be true while $\exists y \forall x P(x, y)$ is false.

We cannot do the reverse of this though. If there is some y for which every x satisfies $P(x, y)$, then certainly for every x there is some y which satisfies $P(x, y)$. The first is saying we can find one y that works for every x. The second allows different y's to work for different x's, but nothing is preventing us from using the same y that works for every x. In other words, while we don't have logical equivalence between the two statements, we do have a valid deduction rule:

$$\begin{array}{rl} & \exists y \forall x P(x, y) \\ \hline \therefore & \forall x \exists y P(x, y) \end{array}$$

Put yet another way, this says that the single statement

$$\exists y \forall x P(x, y) \rightarrow \forall x \exists y P(x, y)$$

is always true; it is a law of logic.

1.3.5 Deductions

Earlier, we claimed that the following was a valid argument:

> If Edith eats her vegetables, then she can have a cookie. Edith ate her vegetables. Therefore Edith gets a cookie.

How do we know this is valid? Let's look at the form of the statements. Let P denote, "Edith eats her vegetables" and Q denote, "Edith can have a cookie." The logical form of the argument is then:

$$\begin{array}{rc} & P \to Q \\ & P \\ \hline \therefore & Q \end{array}$$

This is an example of a **deduction rule**, an argument form that is always valid. This one is a particularly famous rule called *modus ponens*. Are you convinced that it is a valid deduction rule? If not, consider the following truth table:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

This is just the truth table for $P \to Q$, but what matters here is that all the lines in the deduction rule have their own column in the truth table. Remember that an argument is valid provided the conclusion must be true given that the premises are true. The premises in this case are $P \to Q$ and P. Which *rows* of the truth table correspond to both of these being true? P is true in the first two rows, and of those, only the first row has $P \to Q$ true as well. And lo-and-behold, in this one case, Q is also true. So if $P \to Q$ and P are both true, we see that Q must be true as well.

Think of deduction rules as a sort of *one-way* form of logical equivalence. Two statements are logically equivalent provided that in every row of the truth table in which the first statement is true, so is the second, and in every row in which the second statement is true, so is the first. A deduction only requires the first of these two parts.

Here are a few more examples.

Example 1.3.14

Show that the following is a valid deduction rule.

$$\begin{array}{rc} & P \to Q \\ & \neg P \to Q \\ \hline \therefore & Q \end{array}$$

Solution. We make a truth table which contains all the lines of the argument form:

P	Q	$P \to Q$	$\neg P$	$\neg P \to Q$
T	T	T	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	F

(we include a column for $\neg P$ just as a helping step to get the column for $\neg P \to Q$).

Now look at all the rows for which both $P \to Q$ and $\neg P \to Q$ are true. This happens only in rows 1 and 3. Hey! In those rows Q is true as well, so the argument form is valid (it is a valid deduction rule).

Example 1.3.15

Decide whether the following is a valid deduction rule.

$$\begin{array}{rc} & P \to R \\ & Q \to R \\ & R \\ \hline \therefore & P \vee Q \end{array}$$

Solution. Let's make a truth table containing all four statements.

P	Q	R	$P \to R$	$Q \to R$	$P \vee Q$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	F
F	F	F	T	T	F

Look at the second-to-last row. Here all three premises of the argument are true, but the conclusion is false. Thus this is not a valid deduction rule.

While we have the truth table in front of us, look at rows 1, 3, and 5. These are the only rows in which all of the statements $P \to R$, $Q \to R$, and $P \vee Q$ are true. It also happens that R is true in these rows as well. Thus we have discovered a new deduction rule we know *is* valid:

$$\begin{array}{rc} & P \to R \\ & Q \to R \\ & P \vee Q \\ \hline \therefore & R \end{array}$$

Quantifier deductions. There are also deduction rules we could write down for quantifiers. For example, such a rule might be:

$$\begin{array}{rc} & \forall x P(x) \\ \hline \therefore & \exists x P(x) \end{array}$$

If everything is a P-thing, then there must be something which is a P-thing.[3] These rules cannot be verified with a truth table, and a full treatment of this sort of predicate logic is beyond the scope of this text.

1.3.6 Reading Questions

1. To check whether two statements are logically equivalent, you can use a truth table. Explain what you would look for in the truth table to conclude that the two statements are logically equivalent. What would tell you they are *not* logically equivalent?

2. To check whether a deduction rule is *valid*, you can use a truth table. Explain what you would look for in the completed truth table to say that the deduction rule is valid, and what would tell you the deduction rule is *not* valid.

3. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

1.3.7 Practice Problems

1. Make a truth table for the statement $(P \wedge Q) \rightarrow (P \vee Q)$.

2. Complete a truth table for the statement $\neg Q \vee (Q \rightarrow P))$. What can you conclude about P and Q if you knew the statement above was false?

3. Construct a truth table for the statement $Q \rightarrow (\neg P \vee R)$.

4. Determine whether the statements $P \rightarrow (Q \vee R)$ and $(P \rightarrow Q) \vee (P \rightarrow R)$ are logically equivalent by completing a truth table for both statements.

5. Determine if the following is a valid deduction rule:

$$\begin{array}{rc} & P \rightarrow Q \\ & \neg Q \\ \hline \therefore & \neg P \end{array}$$

6. Determine if the following is a valid deduction rule:

$$\begin{array}{rc} & P \rightarrow (Q \vee R) \\ & \neg(P \rightarrow Q) \\ \hline \therefore & R \end{array}$$

7. Determine if the following is a valid deduction rule:

[3] Note that this does assume that your domain of discourse is non-empty.

$$\begin{array}{rc} & (P \wedge Q) \to R \\ & \neg P \vee \neg Q \\ \hline \therefore & \neg R \end{array}$$

8. Determine if the following is a valid deduction rule:

$$\begin{array}{rc} & P \to Q \\ & P \wedge \neg Q \\ \hline \therefore & R \end{array}$$

9. Which of the following statements is a *law of logic*? That is, which of the following are true no matter what your domain of discourse is and no matter what you interpret the predicates as meaning? Select all that apply.

A. $\forall x(P(x) \vee \neg P(x))$.

B. $\exists x P(x) \to \forall x P(x)$.

C. $\neg \forall x P(x) \to \exists x P(x)$.

D. $\forall x \exists y P(x,y) \leftrightarrow \exists y \forall x P(x,y)$.

1.3.8 Additional Exercises

1. You stumble upon two trolls playing Stratego®. They tell you:

Troll 1: If we are cousins, then we are both knaves.

Troll 2: We are cousins, or we are both knaves.

Could both trolls be knights? Recall that all trolls are either always-truth-telling knights or always-lying knaves. Explain your answer and how you can use truth tables to find it.

2. Next you come upon three trolls, helpfully wearing name tags. They say:

Pat If either Quinn or I are knights, then so is Ryan.

Quinn Ryan is a knight, and if Pat is a knight, then so am I.

Ryan Quinn is a knave, but Pat and I share the same persuasion.

Create a truth table that includes all three statements. Then use the truth table to determine the persuasion of each troll.

3. Consider the statement about a party, "If it's your birthday or there will be cake, then there will be cake."

(a) Translate the above statement into symbols. Clearly state which statement is P and which is Q.

(b) Make a truth table for the statement.

(c) Assuming the statement is true, what (if anything) can you conclude if you know there will be cake?

(d) Assuming the statement is true, what (if anything) can you conclude if you know there will not be cake?

(e) Suppose you found out that the statement was a lie. What can you conclude?

4. Geoff Poshingten is out at a fancy pizza joint and decides to order a calzone. When the waiter asks what he would like in it, he replies, "I want either pepperoni or sausage. Also, if I have sausage, then I must also include quail. Oh, and if I have pepperoni or quail, then I must also have ricotta cheese."

(a) Translate Geoff's order into logical symbols.

(b) The waiter knows that Geoff is either a liar or a truth-teller (so either everything he says is false, or everything is true). Which is it?

(c) What, if anything, can the waiter conclude about the ingredients in Geoff's desired calzone?

5. Determine whether the following two statements are logically equivalent: $\neg(P \to Q)$ and $P \wedge \neg Q$. Explain how you know you are correct.

6. Simplify the following statements (so that negation only appears right before variables).

(a) $\neg(P \to \neg Q)$.

(b) $(\neg P \vee \neg Q) \to \neg(\neg Q \wedge R)$.

(c) $\neg((P \to \neg Q) \vee \neg(R \wedge \neg R))$.

(d) It is false that if Sam is not a man then Chris is a woman, and that Chris is not a woman.

7. Use De Morgan's Laws and any other logical equivalence facts you know to simplify the following statements. Show all your steps. Your final statements should have negations only appear directly next to the sentence variables or predicates (P, Q, $E(x)$, etc.), and no double negations. It would be a good idea to use only conjunctions, disjunctions, and negations.

(a) $\neg((\neg P \wedge Q) \vee \neg(R \vee \neg S))$.

(b) $\neg((\neg P \to \neg Q) \wedge (\neg Q \to R))$ (careful with the implications).

(c) For both parts above, verify your answers are correct using truth tables. That is, use a truth table to check that the given statement and your proposed simplification are actually logically equivalent.

8. Consider the statement, "If a number is triangular or square, then it is not prime"

(a) Make a truth table for the statement $(T \vee S) \to \neg P$.

(b) If you believed the statement was *false*, what properties would a counterexample need to possess? Explain by referencing your truth table.

(c) If the statement were true, what could you conclude about the number 5657, which is definitely prime? Again, explain using the truth table.

9. Tommy Flanagan was telling you what he ate yesterday afternoon. He tells you, "I had either popcorn or raisins. Also, if I had cucumber sandwiches, then I had soda. But I didn't drink soda or tea." Of course, you know that Tommy is the world's worst liar, and everything he says is false. What did Tommy eat?

Justify your answer by writing all of Tommy's statements using sentence variables (P, Q, R, S, T), taking their negations, and using these to deduce what Tommy actually ate.

10. Can you chain implications together? That is, if $P \to Q$ and $Q \to R$, does that means the $P \to R$? Prove that the following is a valid deduction rule:

$$\begin{array}{rl} & P \to Q \\ & Q \to R \\ \hline \therefore & P \to R \end{array}$$

11. Suppose P and Q are (possibly molecular) propositional statements. Prove that P and Q are logically equivalent if and only if $P \leftrightarrow Q$ is a tautology.

12. Suppose $P_1, P_2, \ldots, P_n$ and Q are (possibly molecular) propositional statements. Suppose further that

$$\begin{array}{rl} & P_1 \\ & P_2 \\ & \vdots \\ & P_n \\ \hline \therefore & Q \end{array}$$

is a valid deduction rule. Prove that the statement

$$(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \to Q$$

is a tautology.

13. Consider the statements below. Translate each into symbols, using the predicate $F(x, y)$ for "person x can be fooled at time y." Decide whether any of the statements are equivalent to each other, or whether any imply any others, in this context or in general.

(a) You can fool some people all of the time.

(b) You can fool everyone some of the time.

(c) You can always fool some people.

(d) Sometimes you can fool everyone.

14. Suppose $P(x)$ is some predicate for which the statement $\forall x P(x)$ is true. Is it also the case that $\exists x P(x)$ is true? In other words, is the statement $\forall x P(x) \rightarrow \exists x P(x)$ always true? Is the converse always true? Assume the domain of discourse is non-empty.

15. Simplifying negations will be especially useful when we try to prove a statement by considering what would happen if it were false. For each statement below, write the *negation* of the statement as simply as possible. Don't just say, "It is false that ..."

(a) Every number is either even or odd.

(b) There is a sequence that is both arithmetic and geometric.

(c) For all numbers n, if n is prime, then $n+3$ is not prime.

16. We can simplify statements in predicate logic using our rules for passing negations over quantifiers before applying logical equivalence to the "inside" propositional part. Simplify the statements below (so negation appears only directly next to predicates).

(a) $\neg\exists x \forall y(\neg O(x) \vee E(y))$.

(b) $\neg\forall x \neg\forall y \neg(x < y \wedge \exists z(x < z \vee y < z))$.

(c) There is a number n for which no other number is less than or equal to n.

(d) It is false that for every number n there are two other numbers which n is between.

17. Simplify the statements below to the point that negation symbols occur only directly next to predicates.

(a) $\neg\forall x \forall y(x < y \vee y < x)$.

(b) $\neg(\exists x P(x) \rightarrow \forall y P(y))$.

1.4 Proofs

Objectives

After completing this section, you should be able to do the following.

- Identify the logical structure of, and distinguish between, *direct proofs*, a *proof by contrapositives*, and a *proof by contradictions*.
- Identify flaws in an incorrect proof and determine whether they are flaws in logic or mathematical concepts.
- Apply definitions to prove statements using basic proof styles.

1.4.1 Section Preview

Investigate!

A **mini sudoku puzzle** is a 4×4 grid of squares, divided into four 2×2 boxes. The goal is to fill each square with a digit from 1 to 4, such that no digit repeats in any row, any column, or any box.

Here is a simple mini sudoku puzzle you can try to solve.

2			1
	4		
		3	

You might notice that the solution to the above puzzle has its four outside corners all different, and its four middle squares all different.

The goal of this *Investigate!* question is to prove that this is not a coincidence: Suppose a mini sudoku puzzle has all different numbers in its four corners (marked with # below). Prove that the center four squares (marked with $*$ below) must also contain different numbers.

#			#
	$*$	$*$	
	$*$	$*$	
#			#

Try it 1.4.1

Try placing numbers into an empty mini sudoku puzzle. See if you can break the statement we were asked to prove in the *Investigate!* activity. What stops you? Briefly explain whether you think the statement is true or false, and why.

Anyone who doesn't believe there is creativity in mathematics clearly has not tried to write proofs. Finding a way to convince the world that a particular statement is necessarily true is a mighty undertaking and can often be quite challenging. There is no guaranteed path to success in the search for proofs. For example, in the summer of 1742, a German mathematician by the name of Christian Goldbach wondered whether every even integer greater than 2 could be written as the sum of two primes. Centuries later, we still don't have a proof of this apparent fact (computers have checked that Goldbach's conjecture holds for all numbers less than 4×10^{18}, but no proof that the statement holds for *all* numbers has been found).

Writing proofs is a bit of an art. Like any art, to be truly great at it, you need some sort of inspiration, as well as some foundational technique. Just as musicians can learn proper fingering, and painters can learn the proper way to hold a brush, we can look at the proper way to construct arguments.

We can view a proof through two distinct but intersecting lenses. First, we can think about the *logical* structure of a proof. Below we will consider three styles of proof that vary precisely in their logical structure. Second, we can look at the *mathematical content* of the proof. How does the proof illustrate understanding of mathematical concepts? Does it use definitions of mathematical objects correctly? How do the definitions interact with each other?

Recall that in Section 1.3 we said that a **tautology** is a necessarily true statement, but that it doesn't tell us anything interesting. Similarly, if a proof relied entirely on the logical form of the statement it was proving, it wouldn't tell us anything interesting about mathematics. Thus all the proofs we consider *must* involve some combining of mathematical concepts in addition to their logical structure.

In this section, we will see examples of how the interaction between logical and mathematical structure plays out. We will think of the logical structure as the *skeleton* or *scaffolding* of the proof and look at the different shapes this skeleton can take. We will then see how the mathematical content of the proof fills in the details of the skeleton, how it adds meat to the bones.

It is often challenging to be careful about proofs when the statements we try to prove seem too obvious or familiar. While we will definitely want to prove simple facts about numbers, like that the sum of two even numbers is even, our familiarity with numbers can make it difficult to take this task seriously. So instead, we will start by proving some facts in what is hopefully a novel setting: mini sudoku puzzles.

Preview Activity

Consider the statement:

If ab is an even number, then a or b is even.

Which of the proofs below appear to be valid proofs of this statement? Note: You can assume all the algebra below is correct (because it is).

1. Suppose a and b are odd. That is, $a = 2k + 1$ and $b = 2m + 1$ for some integers k and m. Then

$$\begin{aligned} ab &= (2k+1)(2m+1) \\ &= 4km + 2k + 2m + 1 \\ &= 2(2km + k + m) + 1. \end{aligned}$$

Therefore ab is odd.

2. Assume that a or b is even -- say it is a (the case where b is even will be identical). That is, $a = 2k$ for some integer k. Then

$$\begin{aligned} ab &= (2k)b \\ &= 2(kb). \end{aligned}$$

Thus ab is even.

3. Suppose that ab is even but a and b are both odd. Namely, $ab = 2n$, $a = 2k + 1$ and $b = 2j + 1$ for some integers n, k, and j. Then

$$\begin{aligned} 2n &= (2k+1)(2j+1) \\ 2n &= 4kj + 2k + 2j + 1 \\ n &= 2kj + k + j + \frac{1}{2}. \end{aligned}$$

But since $2kj + k + j$ is an integer, this says that the integer n is equal to a non-integer, which is impossible.

4. Let ab be an even number, say $ab = 2n$, and a be an odd number, say $a = 2k + 1$.

$$\begin{aligned} ab &= (2k+1)b \\ 2n &= 2kb + b \\ 2n - 2kb &= b \\ 2(n - kb) &= b. \end{aligned}$$

Therefore b must be even.

1.4.2 Direct Proof

The simplest style of proof is **direct proof**. Often all that is required to prove something is a systematic explanation of what everything means. You look at the definitions, carefully explain and *unpack* their meaning, until you see that the conclusion is true.

To illustrate the importance of definitions in a proof, let's give a few careful definitions about mini sudoku puzzles.

Definition 1.4.2 Mini Sudoku Definitions.

A **mini sudoku puzzle** is a partially filled in 4×4 grid of squares, divided into four 2×2 boxes. Each square can be empty or contain a digit from 1 to 4.

We say that a mini sudoku puzzle is **valid** provided no digit from 1 to 4 appears more than once in any row, any column, or any box.

A **solution** to a mini sudoku puzzle is a valid puzzle with no empty squares and every non-empty square of the puzzle unchanged.

We say that a mini sudoku puzzle is **solvable** if there is exactly one solution.

First, let's prove a useful and "obvious" fact about any valid mini sudoku puzzle.

Proposition 1.4.3

Any solution to a mini sudoku puzzle will have each digit from 1 to 4 appear exactly once in each row, in each column, and in each box, appearing a total of four times.

That's obvious, you say! Isn't that exactly what a valid puzzle is? Well, a valid completed puzzle, which is what we mean by a solution, right? Okay, not exactly, since valid means that no digit repeats... isn't that the same thing?

Yes! Exactly! Saying this is a direct proof.

Proof. Suppose you have a solution to a mini sudoku puzzle. That means that you have a 4×4 grid with each square filled in with a digit from 1 to 4 that is valid.[4] Since the puzzle is *valid*, no digit repeats in any row, any column, or any box[5]. Since a row contains four numbers that do not repeat, and there are exactly four possible digits, each of those digits must appear exactly once. This is true for every row, and for every column, and for every box.

For each digit, since it appears exactly once in four different rows, it appears exactly four times. This completes the proof.

Remark 1.4.4 The proof contained one key mathematical idea besides just explaining definitions: If four distinct numbers are chosen from a set of four numbers, then all four numbers are chosen. Perhaps you want to also explain why this is true, or just say it is an example of the pigeonhole principle (we will say what this is soon). How much you explain depends on who you are writing the proof for. A little paranoia when writing proofs is healthy.

Indeed, in most contexts, we wouldn't even need to write out any of the above proof. It would probably be sufficient to say, "Clearly this follows

[4]Definition of **solution**
[5]Definition of **valid**

from the definitions." However, we are currently trying to learn how to write proofs, and it can be useful to be overly pedantic so we can focus on the proof structure and the importance of applying definitions.

Let's prove something about a particular sudoku puzzle.

Example 1.4.5

Prove that for any solution to the mini sudoku puzzle below, if the solution has a 2 in the top-left square (r1c1), then it will contain a 2 in the bottom-right square (r4c4).

	1	3	
			1
			3

Solution. We do not know whether the puzzle is solvable (in fact, it is not), although note that it is valid right now. What we want to prove is that *if* 2 is in the top-left square in any particular solution, *then* that solution contains a 2 in the bottom-right square.

Proof. Let S be a solution to the puzzle, and assume that S contains a 2 in the top-left square. Since S is a valid puzzle, no digit repeats in any row, any column, or any box. Look first at the top row. Since the row already contains 1, 2, and 3, the remaining open square (r1c4) must be a 4.

Now look at column 4 (the right-most column). Since we now know the top-right square is a 4, this column already contains 1, 3, and 4. So the last open square (r4c4) must be a 2.

Thus S contains a 2 in the bottom-right square, which is what we needed to prove. ■

Observe the general form of the argument above. We were trying to prove an implication $P \to Q$: If there was a 2 in r1c1, then there was a 2 in r4c4. We started by assuming P was true. From that, we deduced something, and from that something we deduced Q. This is exactly what a direct proof of an implication $P \to Q$ looks like (we could have had more steps between the P and Q as well).

Assume P. Explain, explain, ..., explain. Therefore Q.

The one additional consideration we must make is that often we are proving a general, universal statement, a statement of the form $\forall x(P(x) \to Q(x))$. To handle the quantifier, we fix an *arbitrary* instance of x. Above, we said, "Let S be a solution to the puzzle." Since we made no additional assumptions about S besides that P

was true about it, we say that S was an *arbitrary* solution.

If we wanted to prove that all squares are rectangles, we first realize that this is the same as saying, "For any shape, if the shape is a square, then it is a rectangle." In symbols, $\forall x(S(x) \rightarrow R(x))$. We will want to assume $P(x)$ is true and deduce $Q(x)$. Which x do we use? An arbitrary one, so our proof can be applied to *all* possible x.

Example 1.4.6

Prove that for any mini sudoku puzzle with three empty squares, if the puzzle has a solution, then the puzzle is solvable.

Solution. Is this obvious? If a puzzle has a solution, then it is solvable, right? Not at all! Look again at the Mini Sudoku Definitions: Just because a puzzle has a solution doesn't mean that it has exactly one solution (i.e., it is solvable). But even if we don't think this needs a proof, let's be paranoid again and use this as an excuse to focus on the logical structure of the proof.

Notice that we are proving that the claim is true no matter what mini sudoku puzzle we start with. We might start with this puzzle:

3		4	2
2	4	3	1
1			4
4	2	1	3

Clearly there is only one way to complete the puzzle: The top row has only one open square, so we can only put one digit in it, and then columns 2 and 3 only have one open square each, so we can fill those uniquely.

Looking at a single example can often be helpful when crafting a proof, but proving a general statement with an example is ***NEVER*** a correct proof.

While there are *only* around 152.58 billion mini sudoku puzzles (and most of those are not valid, fewer have solutions, and even fewer have exactly one empty square), we don't really want to check all possible puzzles. So instead, we fix an arbitrary valid mini sudoku puzzle. We assume that it has a solution and has exactly three open squares. From this, we prove that there is only one possible solution.

Proof. Let P be an arbitrary mini sudoku puzzle. Assume P has exactly three empty squares and that S is a solution.

Since P is arbitrary, we don't know how the three empty squares are arranged. They could all be in different rows, or two could be in the same row, or all three could be in the same row.

If all the empty squares are in different rows, then in each row, there is exactly one empty square. The other three squares are filled with three different digits (since the puzzle is valid), so there is only one choice to fill the empty square. This number must be the number used in the solution S.

Now consider the case where two of the empty squares are in the same row, and the third square is in a different row. The third empty square's row has three different digits, so there is only one choice for the last square, and it must agree with S. Once this is filled in, the other two empty squares must be in two different columns, each of which has three filled-in digits. In each of these columns, the three filled-in digits are different, so there is only one choice for the empty square. So again, any solution must be exactly S.

Finally, if all three empty squares are in the same row, then they are all in different columns. So using the same argument as we did when the empty squares were in different rows, but using columns instead, we see that S is the only solution.

We have considered all possible cases, and in each case, S is the only solution, so P is solvable. ■

Direct proof can, of course, be used to prove statements in mathematics too.

Example 1.4.7

Prove: For all integers n, if n is even, then n^2 is even.

Solution. The format of the proof will be this: Let n be an arbitrary integer. Assume that n is even. Explain explain explain. Therefore n^2 is even.

To fill in the details, we explain what it means for n to be even, and then see what that means for n^2. The *definition* that is relevant here is, "An integer n is **even** if there is an integer k such that $n = 2k$." Here is a complete proof.

Proof. Let n be an arbitrary integer. Suppose n is even. Then $n = 2k$ for some integer k. Now $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Since $2k^2$ is an integer, n^2 is even. ■

Example 1.4.8

Prove: For all integers a, b, and c, if b is a multiple of a, and c is a multiple of b, then c is a multiple of a.

Solution. Even if we don't remember exactly what "is a multiple of" means, we can set up a direct proof for this statement. It will go something like this: Let a, b, and c be arbitrary integers. Assume that b is a multiple of a and that c is a multiple of b. Dot dot dot. Therefore c is a multiple of a.

How do we connect the dots? We say what our hypothesis really means and why this gives us what the conclusion really means. This is where we need the definition of b **is a multiple of** a: This means that $b = ka$ for some integers k. What are we going for? That $c = la$, for some integer l. Here is the complete proof.

Proof. Let a, b, and c be integers. Assume that b is a multiple of a and that c is a multiple of b. So there are integers k and j such that $b = ka$ and $c = jb$. Combining these (through substitution) we get that

$$c = j(ka) = (jk)a.$$

But jk is an integer, so this says that c is a multiple of a. ■

1.4.3 Proof by Contrapositive

Recall that an implication $P \to Q$ is logically equivalent to its contrapositive $\neg Q \to \neg P$. There are plenty of examples of statements that are hard to prove directly, but whose contrapositive can easily be proved using a direct proof. This is all that **proof by contrapositive** does. It gives a direct proof of the contrapositive of the implication. This is enough because the contrapositive is logically equivalent to the original implication.

The skeleton of the proof of $P \to Q$ by contrapositive will always look roughly like this:

Assume $\neg Q$. Explain, explain, ... explain. Therefore $\neg P$.

As before, if there are variables and quantifiers, we set them to be arbitrary elements of our domain.

Example 1.4.9

Prove that if a mini sudoku puzzle is solvable, then it is valid.

Solution. Remember, a puzzle is valid provided no row, column, or box contains a repeated digit. A puzzle is solvable if there is exactly one solution, where a solution is a valid puzzle with no empty squares (that doesn't change any previously filled-in square).

If we try a direct proof, we would start with an arbitrary puzzle and assume it is solvable. We could then "get" the solution, but would need to reason back in time to when the puzzle started. This seems hard. Often, when a proof seems to require breaking things apart, it is easier to try the contrapositive. That's what we will do.

Proof. Let P be an arbitrary mini sudoku puzzle and assume that it is *not* valid. This means that in at least one row, or one column, or one box, some digit appears more than once.

Now suppose we have filled in the empty squares but not changed any previously filled-in squares. The original row, column, or box that contained a duplicate digit will still contain that duplication, so the resulting completed puzzle will not be valid. Thus no solution to P exists, so P is not solvable. ■

We have proved that if a mini sudoku puzzle is not valid, then it is not solvable, which is the contrapositive of what we wanted to prove and so serves as a proof of the original statement.

Here are a couple more mathy examples.

Example 1.4.10

Is the statement, "For all integers n, if n^2 is even, then n is even," true?

Solution. This is the converse of the statement we proved in Example 1.4.7 above using a direct proof. From trying a few examples, this statement appears to be true. So let's prove it.

A direct proof of this statement would require fixing an arbitrary n and assuming that n^2 is even. But it is not at all clear how this would allow us to conclude anything about n. Just because $n^2 = 2k$ does not in itself suggest how we could write n as a multiple of 2.

Try something else: Write the contrapositive of the statement. We get, for all integers n, if n is odd, then n^2 is odd. This looks much more promising.

We need a definition of a number being odd. An integer n is **odd** provided $n = 2k + 1$ for some integer k. Our proof will look something like this:

Let n be an arbitrary integer. Suppose that n is not even. This means that .... In other words .... But this is the same as saying .... Therefore n^2 is not even.

Now we fill in the details.

Proof. We will prove the contrapositive. Let n be an arbitrary integer. Suppose that n is not even, and thus odd. Then $n = 2k + 1$ for some integer k. Now $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Since $2k^2 + 2k$ is an integer, we see that n^2 is odd and therefore not even. ■

Example 1.4.11

Prove that for all integers a and b, if $a + b$ is odd, then a is odd or b is odd.

Solution. The problem with trying a direct proof is that it will be hard to separate a and b from knowing something about $a + b$. On the other hand, if we know something about a and b separately, then combining them might give us information about $a + b$. The contrapositive of the statement we are trying to prove is: for all integers a and b, if a and b are even, then $a + b$ is even. Thus our proof will have the following format:

Let a and b be integers. Assume that a and b are both even. la la la. Therefore $a + b$ is even.

Here is a complete proof.

Proof. Let a and b be integers. Assume that a and b are even. Then $a = 2k$ and $b = 2l$ for some integers k and l. Now $a + b = 2k + 2l = 2(k + l)$. Since $k + l$ is an integer, we see that $a + b$ is even, completing the proof. ■

Note that our assumption that a and b are even is the negation of a or b is odd. We used De Morgan's law here.

Direct proofs and proofs by contrapositive can be used when proving *implications*. Remember that some statements that are not explicitly written as an implication can be rephrased as one.

Example 1.4.12

Consider the statement, "For every prime number p, either $p = 2$, or p is odd." We can rephrase this as, "For every prime number p, if $p \neq 2$, then p is odd." Now try to prove it.

Use the following as the definition of a prime number: An integer $p > 1$ is **prime** provided it has exactly two factors, namely 1 and p.

Solution.

Proof. Let p be an arbitrary prime number. Assume p is not odd. So p is divisible by 2. Since p is prime, it must have exactly two divisors, and it has 2 as a divisor, so p must be divisible by only 1 and 2. Therefore $p = 2$. This completes the proof (by contrapositive). ∎

1.4.4 Proof by Contradiction

Take a step back and consider what it would mean if the conclusion of an argument were false. There are two reasons this could happen. Either the logic in the argument is faulty, or at least one assumption must be false. A **proof by contradiction** exploits this second possibility. We start with a single assumption and construct a *valid* proof that leads to a false conclusion; the only possibility is that the single assumption was false. The false conclusion is the "contradiction," which just means a necessarily false statement (technically a statement of the form $P \wedge \neg P$).

The general form of a proof by contradiction to prove a statement P is,

> Assume $\neg P$ (that P is not true). This means that..., which tells us..., so we can say... But that is a contradiction, so P must in fact be true.

Note that if we think of this style of argument as a direct proof of something, it is a direct proof of

$$\neg P \to \text{contradiction}.$$

So we have a valid proof of this implication. How can a true implication have a false conclusion? Recall the truth table for an implication (Figure 1.2.2). The only row in which the implication is true but the conclusion is false is row 4, and here the hypothesis is also false. So $\neg P$ is false, which is to say P is true.

Once you start writing proofs by contradiction, it becomes very natural. Let's see how with a sudoku proof.

Example 1.4.13

Prove that any solution to the mini sudoku puzzle below must contain a 3 in the top-right corner (r1c4).

	2		
	3		
3			
			1

Solution. There are multiple ways we could prove this, which is often the case for proofs by contradiction. While we would always start with the same initial assumption (the opposite of what we want to prove), where the contradiction is found can vary.

Proof. Let S be a solution to the puzzle, but assume that S does *not* contain a 3 in the top-right corner. Then the top row must contain a 3 somewhere else. It cannot be in column 1, since it already has a 3. It cannot be in column 2, since that square is already filled in (with the 2). It also cannot be in column 3, since if it was, there would be no place to put a three in the bottom row. So there would be no 3 in the top row, contradicting that S is a solution.

Therefore S must contain a 3 in the top-right corner. ■

Example 1.4.14

Prove that there is no solution to the sudoku puzzle below.

4			
	1		
			4

Solution. Look at the logical format of this statement:

$$\neg \exists S(S \text{ is a solution }).$$

Using the rules for negation of quantifiers, this is the same as

$$\forall S(S \text{ is not a solution}).$$

If we were to prove this directly, we would need to consider all possible solutions and show that they are not valid. That seems quite challenging.

On the other hand, if we try a proof by contradiction, we get to assume the negation of the statement we are asked to prove. That is, we can assume that there *is* a solution. That is a single solution we can reason about. Much more manageable.

Proof. Suppose, for the sake of contradiction, that there *does* exist a solution to the puzzle. This solution must have a 4 in the bottom-left box. But the only squares that can hold a 4 are either in row 4 or column 1 (or both). In either case, this contradicts that there is a 4 already in column 1 and row 4. ■

Here are three examples of proofs by contradiction about numbers:

Example 1.4.15

Prove that $\sqrt{2}$ is irrational.

Solution.

Proof. Suppose not. Then $\sqrt{2}$ is equal to a fraction $\frac{a}{b}$. Without loss of generality, assume $\frac{a}{b}$ is in lowest terms (otherwise reduce the fraction). So,

$$2 = \frac{a^2}{b^2}$$

$$2b^2 = a^2.$$

Thus a^2 is even, and as such a is even. So $a = 2k$ for some integer k, and $a^2 = 4k^2$. We then have,

$$2b^2 = 4k^2$$

$$b^2 = 2k^2.$$

Thus b^2 is even, and as such b is even. Since a is also even, we see that $\frac{a}{b}$ is not in lowest terms, a contradiction. Thus $\sqrt{2}$ is irrational. ■

Example 1.4.16

Prove: There are no integers x and y such that $x^2 = 4y + 2$.

Solution.

Proof. We proceed by contradiction. So suppose there *are* integers x and y such that $x^2 = 4y + 2 = 2(2y + 1)$. So x^2 is even. We have seen that this implies that x is even. So $x = 2k$ for some integer k. Then $x^2 = 4k^2$. This in turn gives $2k^2 = (2y + 1)$. But $2k^2$ is even, and $2y + 1$ is odd, so these cannot be equal. Thus we have a contradiction, so there must not be any integers x and y such that $x^2 = 4y + 2$. ■

Example 1.4.17

The **Pigeonhole Principle**: If more than n pigeons fly into n pigeonholes, then at least one pigeonhole will contain at least two pigeons. Prove this!

Solution.

Proof. Suppose, contrary to stipulation, that each of the pigeonholes contains at most one pigeon. Then at most, there will be n pigeons. But we assumed that there are more than n pigeons, so this is impossible. Thus there must be a pigeonhole with more than one pigeon. ■

While we phrased this proof as a proof by contradiction, we could have also used a proof by contrapositive since our contradiction was simply the negation of the hypothesis. Sometimes this will happen, in which case you can use either style of proof. There are examples, however, where the contradiction occurs "far away" from the original statement.

1.4.5 Summary of Proof Styles

We have considered three styles of proof: direct proof, proof by contrapositive, and proof by contradiction. It can be challenging to decide which style of proof to use on a given problem, and no rule will always tell us what to do. Often, there are multiple ways you can proceed in a proof, which is one reason math is so exciting.

A good starting point when writing proofs is to consider what the initial assumption would be with each style, and what the conclusion you would be looking for is. A proof is a little like a kids-menu maze. There is a *START* and an *EXIT*, and your goal is to find your way from one to the other. Sometimes it helps to work your way in from both sides and hopefully meet in the middle.

Starts and Ends Proofs.

To prove an implication $P \to Q$:

Direct	Start: Assume P. End: Therefore Q.
Contrapositive	Start: Assume $\neg Q$. End: Therefore $\neg P$.
Contradiction	Start: Assume $\neg(P \to Q)$. End: ...which is a contradiction.

You can use a proof by contradiction even if you are not trying to prove an implication, but if the statement is an implication, then assuming $\neg(P \to Q)$ is really powerful. Remember, the only way for an implication to be false is for P to be true and Q to be false. So we are actually assuming $P \wedge \neg Q$. Aha! P is what we assume in a direct proof. $\neg Q$ is what we assume in a proof by contrapositive. So a *proof by* *contradiction is like doing the other two proofs at the same time, and meeting in the middle!*

To illustrate this, let's prove the fact from the *Investigate!* activity. We will prove that if a mini sudoku puzzle has all different numbers in its corners (marked with a through d below), then the center four squares (marked with $*$ below) must also contain different numbers (in any solution).

a			b
	$*$	$*$	
	$*$	$*$	
c			d

We will give three proofs, first a direct proof, then a proof by contrapositive, and finally a proof by contradiction.

Proof. Let P be a sudoku puzzle with all different numbers in its corners: a in the top-left, b in the top-right, c in the bottom-left, and d in the bottom-right. Let S be any solution to the puzzle.

In S, whatever digit a is must appear in row 2. Since a is already in the top-left box, it cannot appear in columns 1 or 2, so it must appear in columns 3 or 4. If it is in column 3, then a appears in one of the center squares. If not, then it is in column 4. In this latter case, we ask where a appears in row 3: It cannot be in column 1 or 4, so it must be in column 2 or 3, and thus in one of the center squares.

The same argument can now be applied to each of the other three outer squares. Thus, in any solution to the puzzle, the center four squares must contain the digits a through d, all different.

Now we will prove the same statement by contrapositive.

Proof. Let P be a mini sudoku puzzle with numbers in its corners. Let S be any solution to the puzzle. Assume that the center four squares do not contain all different numbers in S. Then there must be some common number, say n, in two of the center squares.

Since n cannot appear twice in a row, it must be that n is in two center squares diagonally from each other: either in r2c2 and r3c3, or in r2c3 and r3c2. In either case, where could n be in rows 1 and 4? It cannot be in columns 2 or 3, so it must be in columns 1 and 4. This means that n will be in opposite outer corners, meaning that the digits in the four corners are not all different.

Finally, we will prove the same statement by contradiction.

Proof. Let P be a mini sudoku puzzle and assume that the four outer corners are all different, but in some solution S, the center four squares are not all different.

Suppose a, the digit in the top-left corner, appears twice in the center four squares. The only way this can happen is if a appears in r3c2 and r2c3, as shown below.

a			b
	*	a	
	a	*	
c			d

But now, where can the fourth a in the go in the solution? It must be in r4c4, contradicting that the four corners are all different. An analogous argument leads to a contradiction for each of the other possible outer corner digits being repeated in the center. Thus, the center four squares must contain all different numbers.

1.4.6 Reading Questions

1. Which of the following would be the best first line of a *direct proof* if you wanted to prove the statement, "For all sets A of single-digit numbers, if $|A| = 6$, then A contains an even number."

A. Suppose there exists a set A of single-digit numbers with $|A| = 6$ but that contains only odd numbers.

B. Fix an arbitrary set A of single-digit numbers, and assume $|A| = 6$.

C. Suppose A is a set of single-digit numbers with $|A| \neq 6$.

D. Let A be a set of single-digit numbers that contains an even number.

E. Let A be a set of single-digit numbers, and assume that A does not contain any even numbers.

2. Which of the following would be the best first line of a *proof by contrapositive* if you wanted to prove the statement, "For all sets A of single-digit numbers, if $|A| = 6$, then A contains an even number."

A. Suppose there exists a set A of single-digit numbers with $|A| = 6$ but that contains only odd numbers.

B. Fix an arbitrary set A of single-digit numbers, and assume $|A| = 6$.

C. Suppose A is a set of single-digit numbers with $|A| \neq 6$.

D. Let A be a set of single-digit numbers that contains an even number.

E. Let A be a set of single-digit numbers, and assume that A does not contain any even numbers.

3. Which of the following would be the best first line of a *proof by contradiction* if you wanted to prove the statement, "For all sets A of single-digit numbers, if

$|A| = 6$, then A contains an even number."

A. Suppose there exists a set A of single-digit numbers with $|A| = 6$ but that contains only odd numbers.

B. Fix an arbitrary set A of single-digit numbers, and assume $|A| = 6$.

C. Suppose A is a set of single-digit numbers with $|A| \neq 6$.

D. Let A be a set of single-digit numbers that contains an even number.

E. Let A be a set of single-digit numbers, and assume that A does not contain any even numbers.

4. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

1.4.7 Practice Problems

1. Arrange some of the statements below to form a correct proof of the following statement: "For any integer n, if n is even, then $7n$ is even."

- Let n be an arbitrary integer, and assume $7n$ is even.
- Let n be an arbitrary integer, and assume $7n$ is odd.
- Since 7 is odd and the product of an odd number and an odd number is odd,
- Since an even number divided by 7 must be odd,
- n must be even.
- $7n$ must be odd.
- Let n be an arbitrary integer, and assume n is even.
- Since the product of any number with an even number is even,
- $7n$ must be even.

2. Arrange some of the statements below to form a correct proof of the following statement: "For any integer n, if $7n$ is even, then n is even."

- Let n be an arbitrary integer, and assume n is even.
- Since the 7 is odd and the product of an odd number with an even number is even,
- $7n$ must be even.
- Let n be an arbitrary integer, and assume $7n$ is even.

- Since an even number divided by 7 must be even,
- n must be even.
- Let n be an arbitrary integer, and assume n is odd.
- Since 7 is odd and the product of an odd number and an odd number is odd,
- $7n$ must be odd.

3. Consider the statement, "For any numbers a and b, if $a + b$ is odd, then either a or b is odd".

Give a valid proof of the statement using a *proof by contrapositive*. Arrange some statements below to complete the proof.

- Let a and b be integers, and assume that $a + b$ is odd.
- Let a and b be integers, and assume that if $a + b$ is odd, then either a or b is odd.
- Let a and b be integers, and assume both are even.
- The sum of two even integers must also be even.
- Therefore $a + b$ is even.
- Let a and b be integers and assume that $a + b$ is odd but a and b are both even.
- The sum of two odd integers must be even.
- But then $a + b$ is both even and odd, a contradiction.

4. Consider the same statement, "For any numbers a and b, if $a + b$ is odd, then either a or b is odd."

Give a valid proof of the statement, this time using a *proof by contradiction* using some of the statements below.

- Let a and b be integers, and assume that $a + b$ is odd.
- Let a and b be integers, and assume that if $a + b$ is odd, then either a or b is odd.
- Let a and b be integers, and assume both are even.
- Let a and b be integers, and assume that $a + b$ is odd but a and b are both even.
- Therefore $a + b$ is even.
- The sum of two even integers must also be even.

- The sum of two odd integers must be even.
- But then $a + b$ is both even and odd, a contradiction.

5. Below are three statements together with a possible first line of a proof of that statement. In each case, say whether the first line is the start of a direct proof, a proof by contrapositive, or a proof by contradiction.

(a) ***Statement:*** For every integer n, the number $7n - 1$ is divisible by 6.

First line: Suppose there were some integer n for which $7n - 1$ was not divisible by 6.

(b) ***Statement:*** For any integer n, if n is prime, then n is solitary

First line: Let n be an integer, and assume n is not solitary.

(c) ***Statement:*** If a shape is a pentagon, then its interior angles add up to 480 degrees.

First line: Consider an arbitrary shape, and assume it is a pentagon.

6. What would the first line be for a proof in each style, of the following statement: "If a function $f : A \to B$ is a bijection, then $|A| = |B|$."

Assume $f : A \to B$ is a bijection	Direct proof
Assume $f : A \to B$ is a bijection and $\lvert A\rvert \neq \lvert B\rvert$	Proof by contrapositive
Assume $\lvert A\rvert \neq \lvert B\rvert$	Proof by contradiction

1.4.8 Additional Exercises

1. For a given predicate $P(x)$, you might believe that the statements $\forall x P(x)$ or $\exists x P(x)$ are either true or false. How would you decide if you were correct in each case? You have four choices: You could give an example of an element n in the domain for which $P(n)$ is true or for which $P(n)$ is false, or you could argue that no matter what n is, $P(n)$ is true or is false.

(a) What would you need to do to prove $\forall x P(x)$ is true?

(b) What would you need to do to prove $\forall x P(x)$ is false?

(c) What would you need to do to prove $\exists x P(x)$ is true?

(d) What would you need to do to prove $\exists x P(x)$ is false?

2. Consider the statement, "For all integers a and b, if $a + b$ is even, then a and b are even."

(a) Write the contrapositive of the statement.

(b) Write the converse of the statement.

(c) Write the negation of the statement.

(d) Is the original statement true or false? Prove your answer.

(e) Is the contrapositive of the original statement true or false? Prove your answer.

(f) Is the converse of the original statement true or false? Prove your answer.

(g) Is the negation of the original statement true or false? Prove your answer.

3. For each of the statements below, say what method of proof you should use to prove them. Then say how the proof starts and how it ends. Bonus points for filling in the middle.

(a) There are no integers x and y such that x is a prime greater than 5 and $x = 6y + 3$.

(b) For all integers n, if n is a multiple of 3, then n can be written as the sum of consecutive integers.

(c) For all integers a and b, if $a^2 + b^2$ is odd, then a or b is odd.

4. Consider the statement, "For all integers n, if n is even then $8n$ is even."

(a) Prove the statement. What sort of proof are you using?

(b) Is the converse true? Prove or disprove.

5. The game TENZI comes with 40 six-sided dice (each numbered 1 to 6). Suppose you roll all 40 dice.

(a) Prove that there will be at least seven dice that land on the same number.

(b) How many dice would you have to roll before you were guaranteed that some four of them would all match or all be different? Prove your answer.

6. Prove that for all integers n, it is the case that n is even if and only if $3n$ is even. That is, prove both implications: If n is even, then $3n$ is even, and if $3n$ is even, then n is even.

7. Prove that $\sqrt{3}$ is irrational.

8. Consider the statement, "For all integers a and b, if a is even and b is a multiple of 3, then ab is a multiple of 6."

(a) Prove the statement. What sort of proof are you using?

(b) State the converse. Is it true? Prove or disprove.

9. Prove the statement, "For all integers n, if $5n$ is odd, then n is odd." Clearly state the style of proof you are using.

10. Prove the statement, "For all integers a, b, and c, if $a^2 + b^2 = c^2$, then a or b is even."

11. Suppose that you would like to prove the following implication:

For all numbers n, if n is prime then n is solitary.

Write out the beginning and end of the argument if you were to prove the statement,

(a) Directly

(b) By contrapositive

(c) By contradiction

You do not need to provide the middle parts of the proofs (since you do not know what solitary means). However, make sure that you give the first few and last few lines of the proofs so that we can see the logical structure you would follow.

12. Suppose you have a collection of rare 5-cent stamps and 8-cent stamps. You desperately need to mail a letter and, having no other stamps available, decide to dip into your collection. The question is, what amounts of postage can you make?

(a) Prove that if you only use an even number of both types of stamps, the amount of postage you make must be even.

(b) Suppose you made an even amount of postage. Prove that you used an even number of at least one of the types of stamps.

(c) Suppose you made exactly 72 cents of postage. Prove that you used at least 6 of at least one type of stamp.

13. Prove: $x = y$ if and only if $xy = \dfrac{(x+y)^2}{4}$. Note, you will need to prove two "directions" here, the "if" and the "only if" part.

14. Prove that log(7) is irrational.

15. Prove that there are no integer solutions to the equation $x^2 = 4y + 3$.

16. Prove that every prime number greater than 3 is either one more or one less than a multiple of 6.

17. Your "friend" has shown you a "proof" he wrote to show that $1 = 3$. Here is the proof:

Proof. I claim that $1 = 3$. Of course we can do anything to one side of an equation as long as we also do it to the other side. So subtract 2 from both sides. This gives $-1 = 1$. Now square both sides, to get $1 = 1$. And we all agree this is true.

What is going on here? Is your friend's argument valid? Is the argument a proof of the claim $1 = 3$? Carefully explain using what we know about logic.

18. A standard deck of 52 cards consists of 4 suits (hearts, diamonds, spades, and clubs) each containing 13 different values (Ace, 2, 3, . . . , 10, J, Q, K). If you draw some number of cards at random, you might or might not have a pair (two cards with the same value) or three cards all of the same suit. However, if you draw enough cards, you will be guaranteed to have these. For each of the following, find the smallest number of cards you would need to draw to be guaranteed having the specified cards. Prove your answers.

(a) Three of a kind (for example, three 7's).

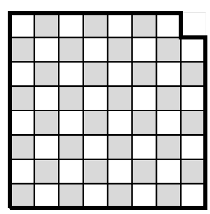
(b) A flush of five cards (for example, five hearts).

(c) Three cards that are either all the same suit or all different suits.

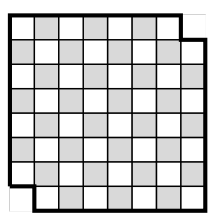
19. Suppose you are at a party with 19 of your closest friends (so including you, there are 20 people there). Explain why there must be at least two people at the party who are friends with the same number of people at the party. Assume friendship is always reciprocated.

20. Your friend has given you his list of 115 best Doctor Who episodes (in order of greatness). It turns out that you have seen 60 of them. Prove that there are at least two episodes you have seen that are exactly four episodes apart on your friend's list.

21. Suppose you have an $n \times n$ chessboard, but your dog has eaten one of the corner squares. You have dominoes that each cover exactly two squares of the board. Can you cover the remaining squares on the board with non-overlapping dominoes? What needs to be true about n? Give necessary and sufficient conditions (that is, say exactly which values of n work and which do not work). Prove your answers.



22. What if your $n \times n$ chessboard is missing two opposite corners? Prove that no matter what n is, you will not be able to cover the remaining squares with non-overlapping dominoes.



1.5 Proofs about Discrete Structures

Objectives

After completing this section, you should be able to do the following.

- Read and comprehend definitions related to discrete structures, so you can apply the definitions correctly.
- Write proofs about discrete structures.

1.5.1 Section Preview

Investigate!

Suppose there are 15 people at a party. Most people know each other already, but there are still some people who decide to shake hands. Is it possible for everyone at the party to shake hands with exactly three other people?

So far we have seen how the logical form of a statement can inform how to build the scaffolding of a proof. This can only get us so far though: To flesh out the proof skeleton requires an understanding of the mathematical objects and structures the proofs are about. Some of this can come from carefully reading definitions. Yet there is also some less concrete understanding and intuition that comes from working with the objects and structures that can lead to that "ah-ha!" moment of inspiration that suggests how to proceed with a proof.

By the way... Why are we writing proofs? Besides practice in becoming better reasoners, diving into careful proofs about discrete structures is a way to learn more about the structures themselves. They are a playground for exploring mathematics, to help us build intuition for mathematical structures. So we study structures to help us write proofs about them, and we write proofs about them to help understand the structures. Bootstrapping!

Another reason to shift our focus toward proofs about discrete structures is that doing so illustrates an important feature of mathematics: abstraction. We have been proving particular facts about particular problems. We might even start to notice similarities between the proofs for some statements. This might be due to the underlying mathematical structures that the problems are (secretly) about. If we prove the general facts about these structures, then we can apply these "theorems" to many different problems.

Some discrete structures lend themselves to particular styles of proof and some "standard" proof techniques can apply to particular structures. We will see some of this here, but mostly we take this opportunity to remind ourselves of some of the basic definitions and properties for discrete structures, and use the proofs about them to help understand these better.

PREVIEW ACTIVITY

In this preview activity, we will explore some basic properties of sets and functions. Later in this section, we will write proofs about these ideas.

1. Remember that a set is just a collection of elements. Here are two definitions about sets:

a. A set A is a subset of a set B, written $A \subseteq B$, provided every element in A is also an element of B.

b. Given sets A and B, the union of A and B, written $A \cup B$, is the set containing every element that is in A or B or both.

Let's build some examples.

(a) Let $B = \{1, 3, 5, 7, 9\}$. Give an example of a set A containing 3 elements that is a subset of B.

What is $A \cup B$ for the set A you gave as an example?

(b) Give an example of two distinct sets A and B such that $A \cup B = B$.

For the example you gave, is $A \subseteq B$?

(c) Find examples, if they exist, of sets A and B such that $A \cup B \neq B$.

For the example you gave, is $A \subseteq B$?

2. Which of the following are always true?

A. For any sets A and B, $A \cup B \subseteq B$.

B. For any sets A and B, $B \subseteq A \cup B$.

C. For any sets A and B, if $A \subseteq B$, then $A \cup B \subseteq B$.

D. For any sets A and B, if $A \cup B = B$, then $A \subseteq B$.

3. For any function $f : \mathbb{N} \to \mathbb{N}$ and any set $A \subseteq \mathbb{N}$, we can define the image of A under f to be the set of all outputs of f when the input is an element of A. We write this as $f(A) = \{f(x) : x \in A\}$.

For the following tasks, let's explore the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x^2 - 3x + 8$.

(a) Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$. Find $f(A)$ and $f(B)$. Then find $f(A) \cup f(B)$.

(b) Now find $A \cup B$ and $f(A \cup B)$.

(c) Give an example, if one exists, of two distinct sets A and B such that $A \subseteq B$ and $f(A) \subseteq f(B)$.

Give an example, if one exists, of two distinct sets A and B such that $A \subseteq B$ but $f(A) \nsubseteq f(B)$.

1.5.2 Proofs about Sets

Recall that a set is an unordered collection of elements. We can describe a set by listing these elements, or by specifying a property that all elements in the set satisfy. For example,

$$A = \{1, 2, 3, 4, 5\},$$

or

$$B = \{x \in \mathbb{N} : x < 10\}.$$

The second set here is the set of natural numbers $(0, 1, 2, \ldots)$ less than 10. Notice that every element in A is also an element of B. Here is a definition that captures that idea.

Definition 1.5.1

A set A is a **subset** of a set B, written $A \subseteq B$, provided every element of A is also an element of B.

The set B is sometimes called a **superset** of A.

We say A is a **proper subset** of B, written $A \subset B$, provided $A \subseteq B$ and $A \neq B$. In other words, if every element in A is an element in B, and there is at least one element in B that is *not* in A.

Example 1.5.2

Let $A = \{x \in \mathbb{N} : x < 5\}$ and $B = \{x \in \mathbb{N} : x^2 < 10\}$. Is $B \subseteq A$? Is B a *proper* subset of A?

Solution. We are asking whether every natural number less than 5 is also a natural number whose square is less than 10. Okay, we could just write out the elements of the sets: $A = \{0, 1, 2, 3, 4\}$ and $B = \{0, 1, 2, 3\}$ (since $3^2 = 9$ and $4^2 = 16$). So $B \subseteq A$. But $B \neq A$, so in fact $B \subset A$.

The sets in the example above were small, and it is easy enough to write down the elements of the sets. However, we can also prove subset relationships between sets if this isn't practical or even possible (perhaps the sets are infinite). Let's look carefully at how we could have reasoned about the example above.

We claimed that every element of B was also an element of A. Another way to say this: For all numbers n, *if* n is an element of B, *then* n is also an element of A. Recognizing this as a conditional statement, we can proceed to give a direct, contrapositive, or contradiction proof of the fact. Here a direct proof would be perfectly acceptable. Let's try it:

Proof. Let n be an element of the set B. Then $n^2 < 10$, by the definition of B. Since $4^2 = 16$, we must have that $n < 4$. By the definition of A, and the fact that $4 < 5$, we see that $n \in A$.

To be clear, this proof is *way* more than we would normally do for this example, but its format should be illuminating. Proving that one set is a subset of another is really the same as proving an implication!

To give an example of how we can apply the definition of "subset" in a more general setting, let's prove a basic fact about subsets.

Proposition 1.5.3

For any sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof. We will give a direct proof. Let A, B, and C be sets, and assume that $A \subseteq B$ and $B \subseteq C$. We will prove that $A \subseteq C$.

Let x be an element of A. Since $A \subseteq B$, we know that $x \in B$. Since $B \subseteq C$, we know that $x \in C$. Therefore, $A \subseteq C$.

Now let's prove a fact about numbers: Every multiple of 9100 is also a multiple of 13. We could factor 9100, but here is an easier way. The set of multiples of 9100 is a subset of the set of multiples of 91. And the set of multiples of 91 is a subset of the set of multiples of 13. Now apply the proposition above.

The proof of Proposition 1.5.3 is what is sometimes called an **element chasing** proof. By the definition of subset, $A \subseteq B$ means every element of A is an element of B, or equivalently, for all x, if x is an element of A, then x is an element of B. One way to prove this is to "chase" the element x from A to B.

Example 1.5.4

Prove that if $A \subseteq B$, then $A \cup B \subseteq B$. Recall that $A \cup B$ is the **union** of sets A and B, and contains all elements that are in A or B or both.

Solution. We will write a direct proof. So we will assume that $A \subseteq B$ and prove that $A \cup B \subseteq B$. Our desired conclusion is a statement about subsets, so let's do an element chasing proof for it.

Proof. Let A and B be sets and assume $A \subseteq B$. Now let x be an element in $A \cup B$. This means that x is an element of A, or x is an element of B, or both.[6]

Consider the cases. If x is an element of A, then since $A \subseteq B$, we know that x is an element of B. On the other hand, if x is not an element of A, then x must be an element of B (since x is in $A \cup B$). In either case, x is an element of B. Therefore, $A \cup B \subseteq B$. ■

We can actually prove a strong statement: $A \subseteq B$ if and only if $A \cup B = B$. You are asked to do this in the exercises.

1.5.3 Proofs about Functions

A function $f : A \to B$ is a rule that assigns each element of the set A (the domain) to exactly one element of the set B (the codomain). It is any rule: There doesn't have to be a formula or rationale for it; we just need to match up elements from A to elements in B. For example, we could let A be the set of students enrolled in a particular Discrete Math course and let B be the set of months of the year. Now define the function $f : A \to B$ to be the rule that assigns to each student the month in which their birthday falls. Since every student has an assigned month, and no more than one month, this is a function.

Here is a definition of a particular type of function.

Definition 1.5.5

A function $f : A \to B$ is **injective** (or **one-to-one**) provided every element in B is the image of at most one element in A. In other words, no element in B is the *output* for more than one *input* from A.

In the example below, we use *two-line notation* to describe a function. The top row contains the inputs, and the bottom row lists the corresponding outputs. So $f : \{1, 2, 3, 4\} \to \{a, b, c, d\}$ might be defined as,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \end{pmatrix},$$

which means that $f(1) = a$, $f(2) = b$, $f(3) = c$, and $f(4) = d$.

Example 1.5.6

Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6, 8\}$. Consider the functions $f : A \to B$ and $g : A \to B$ defined by,

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 8 & 6 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 4 \end{pmatrix}.$$

Which of these functions is injective?

Solution. The function f is injective: Each element of B is the image of at most one element of A. The function g is not injective: The element 4 in B is the image of both 1 and 3 in A.

Consider the student-to-birth-month function again. Could this possibly be injective? Or put another way, must there be two students in the course with the same birth month (which would say the function is *not* injective)? The answer seems to depend on how many students are in the class.

[6] From the definition of **union**.

But let's pause and think about the more general fact about functions we have here. Let's prove the following fact. Recall that $|A|$ denotes the **cardinality** (size) of the set A: the number of elements in A.

Proposition 1.5.7

Suppose $f : A \to B$ is a function with A and B both finite sets. If $|A| > |B|$, then f is not *injective.*

Proof. We will give a proof of the contrapositive: If f is injective, then $|A| \leq |B|$. Let $|B| = n$. Since f is injective, each element of B must be the output for *at most* one element of A. Thus there are at most n elements in A that get mapped to B by f. But the definition of a function requires that every element of the domain is mapped to exactly one element of the codomain, so there must be at most n elements in A.

Does this proof remind you of our pigeonhole-like proofs? It should, since this is precisely (one of) the careful formulations of the pigeonhole principle. We could have proved the fact about students sharing a birth month, but now we can just apply the proposition above and be done. When you apply a theorem or proposition to directly prove another result, we call the latter result a **corollary**.

Corollary 1.5.8

Suppose a class has 25 students. Then at least two students share the same birth month.

Proof. Consider the function that maps each student to their birth month. Since the domain has 25 elements and the codomain has 12 elements, the function is not injective, by Proposition 1.5.7. Therefore, at least two students share the same birth month.

Functions always have inputs from a *set* (called the **domain**) and outputs in a set as well (called the **codomain**). This naturally leads to facts to consider about the interaction between sets and functions.

Definition 1.5.9

Given a function $f : X \to Y$ and a set $A \subseteq X$, we define the **image of A under f** to be the set $f(A) = \{f(a) \in Y : a \in A\}$. That is, $f(A)$ is the set of all outputs of the function for inputs in A.

Example 1.5.10

Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = 2n$. Let $A = \{1, 2, 3\}$. Find $f(A)$.

Solution. Evaluate each element of A by f.

$$f(1) = 2; \qquad f(2) = 4; \qquad f(3) = 6.$$

We want the set of these outputs. So $f(A) = \{2, 4, 6\}$.

Now let's prove something.

Proposition 1.5.11

Let $f : X \to Y$ be a function, and let A and B be subsets of X. If $A \subseteq B$, then $f(A) \subseteq f(B)$.

Proof. Let f, A, and B be as in the proposition. Assume that $A \subseteq B$. Now consider an element $y \in f(A)$. By definition, this means that there is some $a \in A$ such that $f(a) = y$. Since $a \in A$ and $A \subseteq B$, we have that $a \in B$. Then by definition,

$$y = f(a) \in f(B).$$

Since y was an arbitrary element of $f(A)$, we have proved that $f(A) \subseteq f(B)$.

Notice that the proof above is an element chasing proof again. This makes sense as soon as you remember that $f(A)$ and $f(B)$ are just the names of sets. To prove one set is a subset of another, we chase elements from the subset to the superset.

1.5.4 Proofs about Relations

A **relation** on a set A is a set of ordered pairs of elements from A. We can think of a relation as a way to describe a type of relationship between elements of A. For example, we might have a relation on the set of people at a party that describes who is friends with whom. We might have a relation on the set of natural numbers that describes which pairs of numbers are related by the relation $x < y$.

Relations permeate all of mathematics, often without us even thinking of them. Whenever we make a statement about two elements of a set, we are implicitly defining a relation. The statement is true when the pair is in the relation. For example, the statement, "3 is less than 5," is true because the pair $(3, 5)$ is in the relation $<$. In fact, using the language we developed in the subsection Quantifiers and Predicates, we can say that a relation is just a predicate, where the variables come from the same set.

Often relations have special symbols like "$=$" or "$\leq$" or "$\perp$". When we talk about a general relation, we will either use $\sim$ and write $x \sim y$, or use a capital letter like R, and write $R(x, y)$ or xRy or even $(x, y) \in R$ (these all mean the same thing).

When we study relations, we try to identify properties that relations might have. Here is an example of a very common property.

Definition 1.5.12

A relation R on a set A is **transitive** provided for all $x, y, z \in A$, if xRy and yRz, then xRz.

Example 1.5.13

Consider the relation $\sim$ on the set of students in your Discrete Math course that holds of two students, provided they have some other class together. Is this relation transitive?

Solution. No, not necessarily (although for some sets of students it could be). For example, suppose Alice has another class with Bruce, say Introduction to Programming. Carlos is not in Intro to Programming, but he and Bruce are both in Organic Chemistry. So then Alice~Bruce and Bruce~Carlos, but it might not be the case that Alice~Carlos (since Alice need not be in Organic Chemistry with Carlos).

Proving that a relation is *not* transitive takes nothing more than finding a counterexample, which means finding three elements a, b, and c such that $a \sim b$, $b \sim c$, but $a \nsim c$ (remember, the only way for an implication to be false is for the hypothesis to be true and the conclusion to be false).

Perhaps slightly more interesting would be proof that a relation is transitive.

Example 1.5.14

Consider the set of all students in your Discrete Mathematics class, and define the relation $\sim$ that holds of students a and b (so $a \sim b$ is true), provided a is taller than b. Prove that this relation is transitive.

Solution. The definition of transitive is an implication, so we can try a direct proof.

Proof. Let a, b, and c be arbitrary students in your Discrete Math course. Assume, $a \sim b$ and $b \sim c$. That means that a is taller than b, and that b is taller than c. But then surely a must be even taller than c than they are taller than b, so we have that $a \sim c$ is true as well. Thus $\sim$ is transitive on this set. ■

1.5.5 Proofs about Graphs

We will spend all of Chapter 2 studying proofs about graphs since this is such a rich area of mathematics. As a preview, here is an example of how graph proofs can go.

A **graph** is a set V of **vertices** and a set E of **edges**. The edges are two-element subsets of the vertices, and we can think of them as representing relationships between the vertices. Note, this is an abstract definition of a graph using sets, but we often draw graphs using dots for the vertices connected by lines for the edges, as this gives us a nice picture of what is going on.

Since graphs represent a type of relationship between elements (vertices), we can use graphs to represent many real-world problems. For example, the vertices of a graph might represent people at a party. Each edge can represent a handshake between two people. So if we wondered whether it is possible for the 15 people at a party to each shake hands with exactly 3 people there, we are really asking whether there is a graph with 15 vertices where each vertex belongs to 3 edges. ("Belongs to"?? Yes, because an edge is a two-element subset of the vertices, so if an edge "touches" or "comes out of" a vertex, that means the vertex belongs to that particular two-element subset.)

Here is a definition related to this idea.

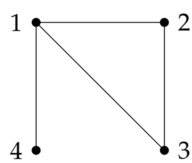
Definition 1.5.15

Let v be a vertex in a graph G. The **degree** of v, written $d(v)$, is the number of edges that contain v, i.e., the number of edges **incident** to v.

Example 1.5.16

Consider the graph G with vertices $V = \{1, 2, 3, 4\}$ and edges $E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}\}$. What is the degree of each vertex in G?

Solution. It might be helpful to picture the graph:



We have $d(1) = 3$, $d(2) = 2$, $d(3) = 2$, and $d(4) = 1$. You can see this by counting how many edges are incident to each vertex, or by counting how many edges (subsets) each vertex belongs to.

So is it possible for 15 people to each shake hands with exactly three people in their group? Well, is there a graph with 15 vertices, all of degree 3? The answer is no!

One way you can see this is if you ask how many edges such a graph would have. Each vertex is incident to three edges, so counting incidences, we get $15 \cdot 3 = 45$. But every edge is incident to two vertices, so we have counted each edge twice. So the number of edges in such a graph would be $45/2 = 22.5$. But the number of edges in a graph must be a whole number, so there is no such graph.

This suggests that we can say something more in general. The following proposition is a simple consequence of the Handshake Lemma 2.1.8, which we will prove in Section 2.1. Here we give a complete proof of this particular formulation of it.

Proposition 1.5.17

In any graph, the number of vertices with odd degree must be even.

Proof. We will prove this by contradiction. Suppose there was a graph with an odd number of vertices with odd degree. Consider the sum of the degrees of all the vertices. The sum for the odd-degree vertices would be odd (since the sum of an odd number of odd numbers is odd). The sum of the even-degree vertices will be even (any sum of even numbers must be even). The sum of an odd number and an even number is odd. Thus the sum of all the degrees will be odd.

However, the number of edges in a graph is half the sum of the degrees (by Lemma 2.1.8, or simply because each edge contributes one to the count of the degree of two vertices). Since the number of edges is a whole number, we see that the sum of the degrees must be even. This contradicts what we found in the previous paragraph.

Therefore, in any graph, the number of vertices with odd degree must be even.

1.5.6 Reading Questions

1. Which of the following is the definition of a function $f : A \to B$ being injective?

A. Every element of B is the image of at most one element of A.

B. The domain A is a larger set than the codomain B.

C. Every element of A is sent to at most one element of B.

D. The codomain B is no smaller than the domain A.

2. When would you most likely use element chasing as part of a proof?

A. When proving that one set is a subset of another.

B. When proving that a function is injective.

C. When proving that a relation is transitive.

D. When proving that a graph has an odd number of edges.

3. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

1.5.7 Practice Problems

1. Given sets A and B, the **intersection** of A and B, written $A \cap B$, is the set of all elements that are in both A and B.

Suppose you wanted to prove that if $A \cap B = B$ then $B \subseteq A$.

Which would be a good start to this proof if you used a direct proof?

A. Let a be an element of $A \cap B$.

B. Let b be an element of B.

C. Let a be an element of A.

D. Suppose there is an element b in B that is not in A.

2. Suppose you wanted to prove that for all sets A and B that $A \cap B \subseteq A$. Which of the following would be a good start to a proof by contradiction?

A. Suppose there is an element a in A that is not in $A \cap B$.

B. Suppose there is an element a in $A \cap B$ that is not in A.

C. Let a be an element of A.

D. Let a be an element of $A \cap B$.

3. Arrange some of the statements below to form a correct proof of the following statement: "For any sets A and B, if $B \subseteq A \cap B$ then $B \subseteq A$."

- Therefore $B \subseteq A \cap B$
- Since $A \cap B$ contains all the elements that are in both A and B, b is an element of A.
- Then b is an element of $A \cap B$ since $B \subseteq A \cap B$.
- Let b be an element of $A \cap B$.
- Suppose $B \subseteq A$.
- Suppose $B \subseteq A \cap B$, and let b be an element of B.
- Then b is an element of B since $B \subseteq A \cap B$.
- Therefore $B \subseteq A$.
- Suppose $A \subseteq B$.

4. Prove that for any sets A and B, $(A \cap B) \cup A = A$

Arrange the statements below to form a correct proof.

- So in particular, x is an element of A.
- Second, we will prove that $A \subseteq (A \cap B) \cup A$.
- Let x be an element of $(A \cap B) \cup A$.
- Therefore $A \subseteq (A \cap B) \cup A$.
- First we will prove that $(A \cap B) \cup A \subseteq A$.

- Therefore $(A \cap B) \cup A \subseteq A$.
- Since $(A \cap B) \cup A \subseteq A$ and $A \subseteq (A \cap B) \cup A$, we have $(A \cap B) \cup A = A$.
- Then x is an element of $(A \cap B) \cup A$, since x is in A or in the other set.
- Then x is an element of $A \cap B$, or x is an element of A.
- Let x be an element of A.

5. Let $f : X \to Y$ be a function and let $B \subseteq Y$ be a subset of the codomain. Define the **inverse image** of B under f to be the set $f^{-1}(B) = \{x \in X : f(x) \in B\}$. That is, it is all the elements in the domain that are mapped to elements in B.

Prove that if $B_1 \subseteq B_2$ are subsets of the codomain, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$. Arrange some of the statements below to form a correct proof.

- Thus $B_1 \subseteq B_2$.
- This means that $f(a)$ is an element of B_1.
- Therefore b is an element of B_2.
- Therefore $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.
- Suppose $B_1 \subseteq B_2$.
- Since $B_1 \subseteq B_2$, $f(a)$ is an element of B_2.
- This then means that a is an element of $f^{-1}(B_2)$.
- Let b be an element of B_1.
- Let a be an element of $f^{-1}(B_1)$.

1.5.8 Additional Exercises

1. Prove that for any two sets A and B, $A \subseteq B$ if and only if $A \cup B = B$.

2. The **intersection** of sets A and B, denoted $A \cap B$, is the set of all elements that are in both A and B.

Prove that for any two sets A and B, $A \subseteq B$ if and only if $A \cap B = A$.

3. Prove that for any sets A, B, and C, if $A \cup B \subseteq C$, then $A \subseteq C$ and $B \subseteq C$.

4. Prove that for any sets A, B, and C, if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

5. The **difference** of sets A and B, written $A \setminus B$, is the set of all elements that are in A but not in B.

The **empty set**, written $\emptyset$, is the set that contains no elements.

Prove that if $A \setminus B = A$ then $A \cap B = \emptyset$.

6. Prove that if $A \setminus B = B \setminus A$ then $A = B$.

7. Let $f : X \to Y$ be a function, and let A and B be subsets of X.

(a) Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$.

(b) Find an example of a function and two sets A and B such that $f(A \cap B) \neq f(A) \cap f(B)$.

8. Let $f : X \to Y$ be a function, and let A and B be subsets of X.

(a) Prove that $f(A \cup B) \subseteq f(A) \cup f(B)$.

(b) Prove that $f(A) \cup f(B) \subseteq f(A \cup B)$

(c) What can you conclude from the two proofs above?

9. Given a function $f : X \to Y$ and a set $B \subseteq Y$, we define the **inverse image** of B under f as the set $f^{-1}(B) = \{x \in X : f(x) \in B$. That is, it is all the elements in the domain that are mapped to elements in B.

(a) For $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n^2$, what are each of the following sets?

(a) $f^{-1}(\{1, 4, 9\})$
(b) $f^{-1}(\{2, 3, 5, 7\})$
(c) $f^{-1}(\{1, 2, \ldots, 10\})$

(b) Prove that for any set $C \subseteq X$, $C \subseteq f^{-1}(f(C))$.

(c) Give an example of a function f and a set C such that $C \neq f^{-1}(f(C))$.

(d) Prove that for any set $D \subseteq Y$, $f(f^{-1}(D)) \subseteq D$.

(e) Give an example of a function f and a set D such that $f(f^{-1}(D)) \neq D$.

10. Let $f : X \to Y$ be a function, and let A and B be subsets of Y. Prove that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

11. Let $f : X \to Y$ be a function, and let A and B be subsets of Y. Prove that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

12. For each relation below, determine whether it is transitive. If it is, prove it. If it is not, give a counterexample.

(a) The relation "|" (divides) on $\mathbb{Z}$ defined by $a|b$ provided b is a multiple of a.

(b) The relation "$\leq$" (less than or equal to) on $\mathbb{R}$.

(c) The relation "$\perp$" (is perpendicular to) on the set of lines in the plane.

(d) The relation "$\sim$" (is similar to) on the set of triangles in the plane (two triangles are similar if they have the same angles, but are not necessarily the same size).

1.6 Chapter Summary

We have considered logic both as its own sub-discipline of mathematics and as a means to help us better understand and write proofs. We noticed that mathematical statements have a particular logical form, and analyzing that form can help make sense of the statement.

At the most basic level, a statement might combine simpler statements using *logical connectives*. We often make use of variables and *quantify* over those variables. How to resolve the truth or falsity of a statement based on these connectives and quantifiers is what logic is all about. From this, we can decide whether two statements are logically equivalent or if one or more statements (logically) imply another.

When writing proofs (in any area of mathematics) our goal is to explain why a mathematical statement is true. Thus it is vital that our argument implies the truth of the statement. To be sure of this, we first must know what it means for the statement to be true, as well as ensure that the statements that make up the proof correctly imply the conclusion. A firm understanding of logic is required to check whether a proof is correct.

There is, however, another reason that understanding logic can be helpful. Understanding the logical structure of a statement often gives clues for how to write a proof of the statement.

This is not to say that writing proofs is always straight-forward. Consider again the *Goldbach conjecture*:

> Every even number greater than 2 can be written as the sum of two primes.

We are not going to try to prove the statement here, but we can at least say what a proof might look like, based on the logical form of the statement. Perhaps we should write the statement to highlight the quantifiers and connectives:

> For all integers n, if n is even and greater than 2, then there exist integers p and q such that p and q are prime, and $n = p + q$.

What would a direct proof look like? Since the statement starts with a universal quantifier, we would start, ``Let n be an arbitrary integer." The rest of the statement is an implication. In a direct proof we assume the "if" part, so the next line would be, "Assume n is greater than 2 and is even." I have no idea what comes next, but eventually, we would need to find two prime numbers p and q (depending on n) and explain how we know that $n = p + q$.

Or maybe we try a proof by contradiction. To do this, we first assume the negation of the statement we want to prove. What is the negation? From what we have studied we should be able to see that it is,

> There is an integer n such that n is even and greater than 2, but for all integers p and q, either p or q is not prime, or $n \neq p + q$.

Could this statement be true? A proof by contradiction would start by assuming it was and eventually conclude with a contradiction, proving that our assumption of truth was incorrect. And if you can find such a contradiction, you will have proved one of the most famous open problems in mathematics. Good luck.

Chapter Review

1. Complete a truth table for the statement $\neg P \to (Q \wedge R)$.

2. Suppose you know that the statement "if Peter is not tall, then Quincy is fat and Robert is skinny" is false. What, if anything, can you conclude about Peter and Robert if you know that Quincy is indeed fat? Explain (you may reference Question 1.6.1).

3. Are the statements $P \to (Q \vee R)$ and $(P \to Q) \vee (P \to R)$ logically equivalent? Explain your answer.

4. Is the following a valid deduction rule? Explain.

$$\begin{array}{rc} & P \to Q \\ & P \to R \\ \hline \therefore & P \to (Q \wedge R). \end{array}$$

5. Write the negation, converse and contrapositive for each of the statements below.

(a) If the power goes off, then the food will spoil.

(b) If the door is closed, then the light is off.

(c) $\forall x (x < 1 \to x^2 < 1)$.

(d) For all natural numbers n, if n is prime, then n is solitary.

(e) For all functions f, if f is differentiable, then f is continuous.

(f) For all integers a and b, if $a \cdot b$ is even, then a and b are even.

(g) For every integer x and every integer y, there is an integer n such that if $x > 0$ then $nx > y$.

(h) For all real numbers x and y, if $xy = 0$ then $x = 0$ or $y = 0$.

(i) For every student in Math 228, if they do not understand implications, then they will fail the exam.

6. Consider the statement, "For all integers n, if n is even and $n \leq 7$, then n is negative or $n \in \{0, 2, 4, 6\}$."

(a) Is the statement true? Explain why.

(b) Write the negation of the statement. Is it true? Explain.

(c) State the contrapositive of the statement. Is it true? Explain.

(d) State the converse of the statement. Is it true? Explain.

7. Consider the statement: $\forall x(\forall y(x + y = y) \rightarrow \forall z(x \cdot z = 0))$.

(a) Explain what the statement says in words. Is this statement true? Be sure to state what you are taking the universe of discourse to be.

(b) Write the converse of the statement, both in words and in symbols. Is the converse true?

(c) Write the contrapositive of the statement, both in words and in symbols. Is the contrapositive true?

(d) Write the negation of the statement, both in words and in symbols. Is the negation true?

8. Simplify the following.

(a) $\neg(\neg(P \wedge \neg Q) \rightarrow \neg(\neg R \vee \neg(P \rightarrow R)))$.

(b) $\neg\exists x \neg\forall y \neg\exists z(z = x + y \rightarrow \exists w(x - y = w))$.

9. Consider the statement, "For all integers n, if n is odd, then $7n$ is odd."

(a) Prove the statement. What sort of proof are you using?

(b) Prove the converse. What sort of proof are you using?

10. Suppose you break your piggy bank and scoop up a handful of 22 coins (pennies, nickels, dimes, and quarters).

(a) Prove that you must have at least 6 coins of a single denomination.

(b) Suppose you have an odd number of pennies. Prove that you must have an odd number of at least one of the other types of coins.

(c) How many coins would you need to scoop up to be sure that you either had 4 coins that were all the same or 4 coins that were all different? Prove your answer.

11. You come across four trolls playing bridge. They declare:

Troll 1: All trolls here see at least one knave.

Troll 2: I see at least one troll that sees only knaves.

Troll 3: Some trolls are scared of goats.

Troll 4: All trolls are scared of goats.

Are there any trolls that are not scared of goats? Recall, of course, that all trolls are either knights (who always tell the truth) or knaves (who always lie).

Chapter 2

Graph Theory

Graph theory has existed as a branch of mathematics for only a short time; the first book on graph theory was published less than 100 years ago. While the first problem related to what we now call graph theory dates back to 1735, it has been the advent of computers that has shown the subject's true utility. It is a subject with simple beauty and surprising depth. Many of the main areas of graph theory can be understood with almost no mathematical prerequisites, yet new research in the subject generates hundreds of peer-reviewed research papers each year.

In this chapter, we will explore just a few of the ways you can use graphs and their properties to solve problems that show up in computer science, mathematics, and almost every other applied science.

2.1 Problems and Definitions

Objectives

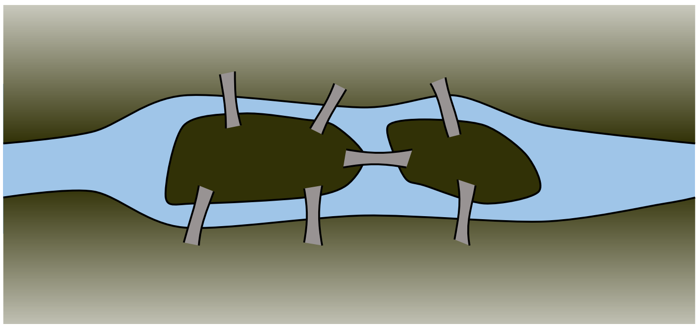
After completing this section, you should be able to do the following.

1. Use the language of graph theory to describe properties of graphs.
2. Utilize multiple representations of graphs.
3. Apply the Handshake Lemma to answer questions about graphs and problems they represent.

2.1.1 Section Preview

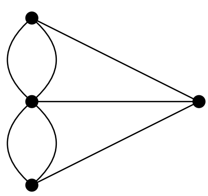
Investigate!

In the time of Euler, in the town of Königsberg in Prussia, there was a river containing two islands. The islands were connected to the banks of the river by seven bridges (as seen below). The bridges were very beautiful, and on their days off, townspeople would spend time walking over the bridges. As time passed, a question arose: Was it possible to plan a walk so that you cross each bridge once and only once? Euler was able to answer this question. Are you?



Graph Theory is a relatively new area of mathematics, first studied by the super famous mathematician Leonhard Euler in 1735. Since then it has blossomed into a powerful tool used in nearly every branch of science and is currently an active area of mathematics research.

The problem above, known as the *Seven Bridges of Königsberg*, is the problem that originally inspired graph theory. Consider a "different" problem: Below is a drawing of four dots connected by some lines. Is it possible to trace over each line once and only once (without lifting your pencil, starting and ending on a dot)?



There is an obvious connection between these two problems. Any path in the dot and line drawing corresponds exactly to a path over the bridges of Königsberg.

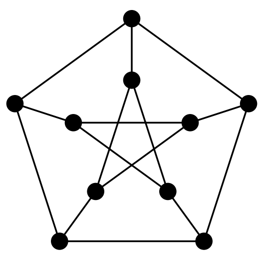
Pictures like this dot and line drawing are called **graphs** (although technically, the picture above is a **multigraph**). Graphs are made up of a collection of dots called **vertices** and lines connecting those dots called **edges**. When two vertices are connected by an edge, we say they are **adjacent**. The nice thing about looking at graphs instead of pictures of rivers, islands, and bridges is that we now have a mathematical object to study. We have distilled the "important" parts of the bridge picture for the problem. It does not matter how big the islands are, what the bridges are made out of, if the river contains alligators, etc. All that matters is which land masses are connected to which other land masses, and how many times. This was the great insight Euler had.

We will return to the question of finding paths through graphs in Section 2.4. In this section, we will explore various ways that graphs can be used to represent, or *model*, real-world problems. Along the way, we will introduce some basic definitions, terminology, and notation that will be used in the rest of the chapter.

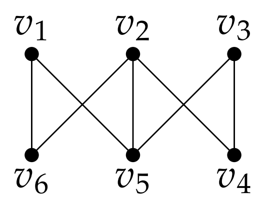
Preview Activity

To get a feel for graphs and the types of questions we want to ask about them, let's explore four examples of graphs.

G_1:



G_2:



G_3:

vertex	**adjacent to**
a	b, c
b	a, f
c	a, d, e
d	c, e, f
e	c, d
f	b, d

G_4:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The graph G_3 is presented as an **adjacency list** where each vertex gets a list of which other vertices it is adjacent to. The graph G_4 is presented as an **adjacency matrix** where the rows and columns correspond to the vertices and the entries are 1 if the vertices are adjacent and 0 otherwise. Before answering the questions below, it might be helpful to draw a more traditional representation of these graphs.

1. First, let's count the number of vertices and edges in each graph.

2. We call the number of edges incident to a particular vertex (i.e., the number of edges "coming out of" the vertex) the degree of the vertex. A list of the degrees of all the vertices in non-increasing order is called a degree sequence for the graph. Find the degree sequence for each graph.

3. We often care about paths between vertices in a graph. A graph is connected if there is a path between every pair of vertices. Sometimes there is a path that starts at a vertex and eventually comes back to itself, which is called a cycle.

(a) Which of the graphs are connected?

(b) Which of the graphs contain cycles?

(c) Graphs that are connected and contain no cycles are called trees. For each graph, how many edges must you remove to turn it into a tree? (If it is already a tree, the answer would be 0.)

4. For which of the graphs is it possible to draw the graph in such a way that no edges cross?

5. Suppose we color each vertex of a graph so that adjacent vertices always have different colors. The smallest number of colors needed to do this is called the chromatic number of the graph. Find the chromatic number for each graph.

Note: If the graphs represented friendships between people, then the chromatic number would tell us the minimum number of groups we would need if we wanted to divide up everyone into groups of people who were not yet friends.

2.1.2 What is a Graph?

Before we start studying graphs, we need to agree upon what a graph is. While we almost always think of graphs as pictures (dots connected by lines), this is fairly ambiguous. Do the lines need to be straight? Does it matter how long the lines are or how large the dots are? Can there be two lines connecting the same pair of dots? Can one line connect three dots?

The way we avoid ambiguities in mathematics is to provide concrete and rigorous *definitions*. Crafting good definitions is not easy, but it is incredibly important. The definition is the agreed-upon starting point from which all truths in mathematics proceed. Is there a graph with no edges? We have to look at the definition to see if this is possible.

We want our definition to be precise and unambiguous, but it also must agree with our intuition for the objects we are studying. It needs to be useful: We *could* define a graph to be a six-legged mammal, but that would not let us solve any problems about bridges. Instead, here is the (now) standard definition of a graph.

Definition 2.1.1 Graph.

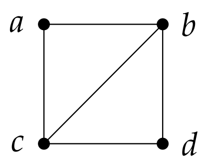
A **graph** is an ordered pair $G = (V, E)$ consisting of a nonempty set V (called the **vertices**) and a set E (called the **edges**) of two-element subsets of V.

Strange. Nowhere in the definition is there talk of dots or lines. From the definition, a graph could be

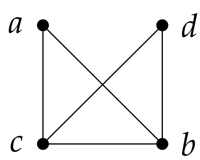
$$(\{a,b,c,d\},\{\{a,b\},\{a,c\},\{b,c\},\{b,d\},\{c,d\}\}).$$

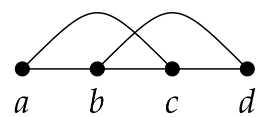
Here we have a graph with four vertices (the letters a,b,c,d) and five edges (the pairs $\{a,b\},\{a,c\},\{b,c\},\{b,d\},\{c,d\}$)).

Looking at sets and sets of 2-element sets is difficult to process. That is why we often draw a representation of these sets. We put a dot down for each vertex, and connect two dots with a line precisely when those two vertices are one of the 2-element subsets in our set of edges. Thus one way to draw the graph described above is this:



However we could also have drawn the graph differently. For example either of these:





We should be careful about what it means for two graphs to be "the same." Actually, given our definition, this is easy: Are the vertex sets equal? Are the edge sets equal? We know what it means for sets to be equal, and graphs are nothing but a pair of two special sorts of sets.

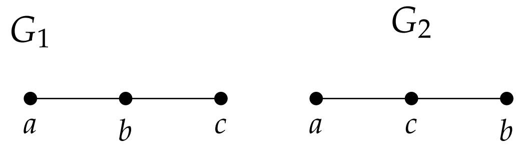
Example 2.1.2

Are the graphs below equal?

$$G_1 = (\{a,b,c\},\{\{a,b\},\{b,c\}\}); \qquad G_2 = (\{a,b,c\},\{\{a,c\},\{c,b\}\}).$$

Solution. No. Here the vertex sets of each graph are equal, which is a good start. Also, both graphs have two edges. In the first graph, we have edges $\{a,b\}$ and $\{b,c\}$, while in the second graph we have edges $\{a,c\}$ and $\{c,b\}$. Of course, $\{b,c\} = \{c,b\}$, so that is not the problem. The issue is that $\{a,b\} \neq \{a,c\}$. Since the edge sets of the two graphs are not equal (as sets), the graphs are not equal (as graphs).

Even if two graphs are not *equal*, they might be *basically* the same. The graphs in the previous example could be drawn like this:



Graphs that are basically the same (but perhaps not equal) are called **isomorphic**. We will give a precise definition of this term after a quick example:

Example 2.1.3

Consider the graphs:

$$G_1 = (V_1, E_1) \text{ where } V_1 = \{a, b, c\} \text{ and } E_1 = \{\{a, b\}, \{a, c\}, \{b, c\}\};$$

$$G_2 = (V_2, E_2) \text{ where } V_2 = \{u, v, w\} \text{ and } E_2 = \{\{u, v\}, \{u, w\}, \{v, w\}\}.$$

Are these graphs the same?

Solution. The two graphs are NOT equal. It is enough to notice that $V_1 \neq V_2$ since $a \in V_1$ but $a \notin V_2$. However, both of these graphs consist of three vertices with edges connecting every pair of vertices. We can draw them as follows:

Clearly we want to say these graphs are basically the same, so while they are not equal, they will be *isomorphic*. We can rename the vertices of one graph and get the second graph as the result.

Intuitively, graphs are **isomorphic** if they are basically the same, or better yet, if they are the same except for the names of the vertices. To make the concept of renaming vertices precise, we give the following definitions:

Definition 2.1.4

An **isomorphism** between two graphs G_1 and G_2 is a bijection $f : V_1 \to V_2$ between the vertices of the graphs such that $\{a, b\}$ is an edge in G_1 if and only if $\{f(a), f(b)\}$ is an edge in G_2.

Two graphs are **isomorphic** if there is an isomorphism between them. In this case we write $G_1 \cong G_2$.

An isomorphism is simply a function which renames the vertices. It must be a bijection so every vertex gets a new name. These newly named vertices must be connected by edges precisely when they were connected by edges with their old names.

Example 2.1.5

Decide whether the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are equal or isomorphic.

$V_1 = \{a, b, c, d\}$, $E_1 = \{\{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}\}$
$V_2 = \{a, b, c, d\}$, $E_2 = \{\{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}\}$

Solution. The graphs are NOT equal, since $\{a, d\} \in E_1$ but $\{a, d\} \notin E_2$. However, since both graphs contain the same number of vertices and the same number of edges, they *might* be isomorphic (this is not enough in most cases, but it is a good start).

We can try to build an isomorphism. How about we say $f(a) = b$, $f(b) = c$, $f(c) = d$ and $f(d) = a$. This is definitely a bijection, but to make sure that the function is an isomorphism, we must make sure it *respects the edge relation*. In G_1, vertices a and b are connected by an edge. In G_2, $f(a) = b$ and $f(b) = c$ are connected by an edge. So far, so good, but we must check the other three edges. The edge $\{a, c\}$ in G_1 corresponds to $\{f(a), f(c)\} = \{b, d\}$, but here we have a problem. There is no edge between b and d in G_2. Thus f is NOT an isomorphism.

Not all hope is lost, however. Just because f is not an isomorphism does not mean that there is no isomorphism at all. We can try again. At this point it might be helpful to draw the graphs to see how they should match up.

Alternatively, notice that in G_1, the vertex a is adjacent to every other vertex. In G_2, there is also a vertex with this property: c. So build the bijection $g : V_1 \to V_2$ by defining $g(a) = c$ to start with. Next, where should we send b? In G_1, the vertex b is only adjacent to vertex a. There is exactly one vertex like this in G_2, namely d. So let $g(b) = d$. As for the last two, in this example, we have a free choice: let $g(c) = b$ and $g(d) = a$ (switching these would be fine as well).

We should check that this really is an isomorphism. It is definitely a bijection. We must make sure that the edges are respected. The four edges in G_1 are

$$\{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}.$$

Under the proposed isomorphism these become

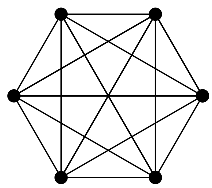
$$\{g(a), g(b)\}, \{g(a), g(c)\}, \{g(a), g(d)\}, \{g(c), g(d)\}$$

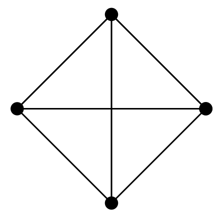
$$\{c, d\}, \{c, b\}, \{c, a\}, \{b, a\},$$

which are precisely the edges in G_2. Thus g is an isomorphism, so $G_1 \cong G_2$

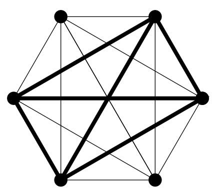
Sometimes we will talk about a graph with a special name (like K_n or the *Petersen graph*) or perhaps draw a graph without any labels. In this case, we are really referring to *all* graphs isomorphic to any copy of that particular graph. A collection of isomorphic graphs is often called an **isomorphism class**.[1]

There are other relationships between graphs that we care about, other than equality and being isomorphic. For example, compare the following pair of graphs:





These are definitely not isomorphic, but notice that the graph on the right looks like it might be part of the graph on the left, especially if we draw it like this:



We would like to say that the smaller graph is a *subgraph* of the larger.

We should give a careful definition of this. In fact, there are two reasonable notions for what a subgraph should mean.

Definition 2.1.6 Subgraphs.

We say that $G' = (V', E')$ is a **subgraph** of $G = (V, E)$, and write $G' \subseteq G$, provided $V' \subseteq V$ and $E' \subseteq E$.

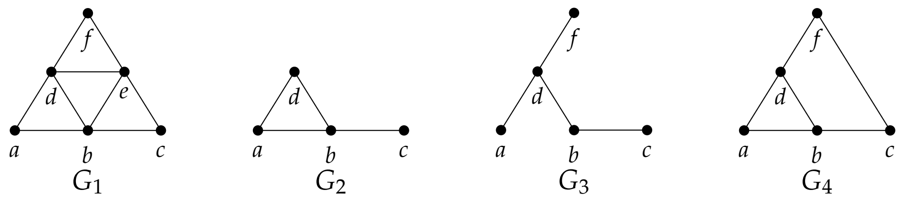
We say that $G' = (V', E')$ is an **induced subgraph** of $G = (V, E)$ provided $V' \subseteq V$ and every edge in E whose vertices are still in V' is also an edge in E'.

Notice that every induced subgraph is also an ordinary subgraph, but not conversely. Think of a subgraph as the result of deleting some vertices and edges from the larger graph. For the subgraph to be an induced subgraph, we can still delete vertices, but now we only delete those edges that included the deleted vertices.

[1]This is not unlike geometry, where we might have more than one copy of a particular triangle. There instead of *isomorphic* we say *congruent*.

Example 2.1.7

Consider the graphs:



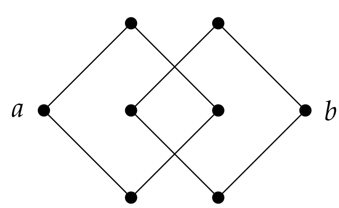
Here both G_2 and G_3 are subgraphs of G_1. But only G_2 is an *induced* subgraph. Every edge in G_1 that connects vertices in G_2 is also an edge in G_2. In G_3, the edge $\{a, b\}$ is in E_1 but not E_3, even though vertices a and b are in V_3.

The graph G_4 is NOT a subgraph of G_1, even though it looks like all we did is remove vertex e. The reason is that in E_4 we have the edge $\{c, f\}$, but this is not an element of E_1, so we don't have the required $E_4 \subseteq E_1$.

Back to some basic graph theory definitions. Notice that all the graphs we have drawn above have the property that no pair of vertices is connected more than once, and no vertex is connected to itself. Graphs like these are sometimes called **simple**, although we will just call them *graphs*. This is because our definition of a graph says that the edges form a set of 2-element subsets of the vertices. Remember that it doesn't make sense to say a set contains an element more than once. So no pair of vertices can be connected by an edge more than once. Also, since each edge must be a set containing two vertices, we cannot have a single vertex connected to itself by an edge.

That said, there are times we want to consider double (or more) edges and single-edge loops. For example, the "graph" we drew for the Bridges of Königsberg problem had double edges because there really are two bridges connecting a particular island to the near shore. We will call these objects **multigraphs**. This is a good name: A *multiset* is a set in which we are allowed to include a single element multiple times.

The graphs above are also **connected**: you can get from any vertex to any other vertex by following some path of edges. A graph that is not connected can be thought of as two separate graphs drawn close together. For example, the following graph is NOT connected because there is no path from a to b:



Vertices in a graph do not always have edges between them. If we add all possible edges, then the resulting graph is called **complete**. That is, a graph is complete if every pair of vertices is connected by an edge. Since a graph is determined completely by which vertices are adjacent to which other vertices, there is only one complete graph with a given number of vertices. We give these a special name: K_n is the complete graph on n vertices.

Each vertex in K_n is adjacent to $n-1$ other vertices. We call the number of edges emanating from a given vertex the **degree** of that vertex. So every vertex in K_n has degree $n-1$. How many edges does K_n have? One might think the answer should be $n(n-1)$, since we count $n-1$ edges n times (once for each vertex). However, each edge is incident to 2 vertices, so we counted every edge exactly twice. Thus there are $n(n-1)/2$ edges in K_n. Alternatively, we can say there are $\binom{n}{2}$ edges, since to draw an edge we must choose 2 of the n vertices.

In general, if we know the degrees of all the vertices in a graph, we can find the number of edges. The sum of the degrees of all vertices will always be *twice* the number of edges, since each edge adds to the degree of two vertices. Notice this means that the sum of the degrees of all vertices in any graph must be even!

This is our first example of a general result about all graphs. It seems innocent enough, but we will use it to prove all sorts of other statements. So let's give it a name and state it formally.

Lemma 2.1.8 Handshake Lemma.

In any graph, the sum of the degrees of vertices in the graph is always twice the number of edges.

The handshake lemma[2] is sometimes called the *degree sum formula*, and can be written symbolically as

$$\sum_{v \in V} d(v) = 2e.$$

Here we are using the notation $d(v)$ for the degree of the vertex v.

One use for the lemma is to actually find the number of edges in a graph. To do this, you must be given the **degree sequence** for the graph (or be able to find it from other information). This is a list of every degree of every vertex in the graph, generally written in non-increasing order.

Example 2.1.9

How many vertices and edges must a graph have if its degree sequence is

$$(4, 4, 3, 3, 3, 2, 1)?$$

Solution. The number of vertices is easy to find. It is the number of degrees in the sequence: 7. To find the number of edges, we compute the degree sum

$$4 + 4 + 3 + 3 + 3 + 2 + 1 = 20,$$

[2]A *lemma* is a mathematical statement that is primarily of importance in that it is used to establish other results.

so the number of edges is half this: 10.

The handshake lemma also tells us what is not possible.

Example 2.1.10

At a recent math seminar, 9 mathematicians greeted each other by shaking hands. Is it possible that each mathematician shook hands with exactly 7 people at the seminar?

Solution. It seems like this should be possible. Each mathematician chooses one person to not shake hands with. But this cannot happen. We are asking whether a graph with 9 vertices can have each vertex have degree 7. If such a graph existed, the sum of the degrees of the vertices would be $9 \cdot 7 = 63$. This would be twice the number of edges (handshakes) resulting in a graph with 31.5 edges. That is impossible. Thus at least one (in fact an odd number) of the mathematicians must have shaken hands with an *even* number of people at the seminar.

We can generalize the previous example to get the following proposition.[3]

Proposition 2.1.11

In any graph, the number of vertices with odd degree must be even.

Proof. Suppose there were a graph with an odd number of vertices with odd degree. Then the sum of the degrees in the graph would be odd, which is impossible, by the handshake lemma.

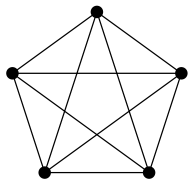
We will consider further applications of the handshake lemma in the exercises.

One final definition: We say a graph is **bipartite** if the vertices can be divided into two sets, A and B, with no two vertices in A adjacent and no two vertices in B adjacent. The vertices in A can be adjacent to some or all of the vertices in B. If each vertex in A is adjacent to all the vertices in B, then the graph is a **complete bipartite graph**, and gets a special name: $K_{m,n}$, where $|A| = m$ and $|B| = n$.

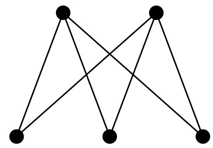
Named Graphs. Some graphs are used more than others and get special names.

K_n	The complete graph on n vertices.
$K_{m,n}$	The complete bipartite graph with sets of m and n vertices.
C_n	The cycle on n vertices, just one big loop.
P_n	The path on $n+1$ vertices (so n edges), just one long path.

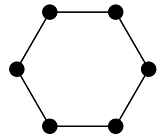
[3] A **proposition** is a general statement in mathematics, similar to a theorem, although generally of lesser importance.



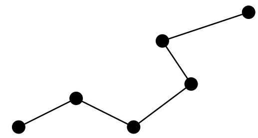
K_5



$K_{2,3}$



C_6



P_5

Graph Theory Definitions. There are a lot of definitions to keep track of in graph theory. Here is a glossary of the terms we have already used and will soon encounter.

Graph	A collection of **vertices**, some of which are connected by **edges**. More precisely, a pair of sets V and E, where V is a set of vertices and E is a set of 2-element subsets of V.
Adjacent	Two vertices are **adjacent** if they are connected by an edge. Two edges are **adjacent** if they share a vertex.
Bipartite graph	A graph for which it is possible to divide the vertices into two disjoint sets such that there are no edges between any two vertices in the same set.
Complete bipartite graph	A bipartite graph for which every vertex in the first set is adjacent to every vertex in the second set.
Complete graph	A graph in which every pair of vertices is adjacent.
Connected	A graph is **connected** if there is a path from any vertex to any other vertex.
Chromatic number	The minimum number of colors required in a proper vertex coloring of the graph.
Cycle	A path (see below) that starts and stops at the same vertex, but contains no other repeated vertices.
Degree of a vertex	The number of edges incident to a vertex.
Euler trail	A walk which uses each edge exactly once.
Euler circuit	An Euler trail which starts and stops at the same vertex.
Multigraph	A **multigraph** is just like a graph but can contain multiple edges between two vertices as well as single edge loops (that is an edge from a vertex to itself).
Path	A **path** is a walk that doesn't repeat any vertices (or edges) except perhaps the first and last. If a path starts and ends at the same vertex, it is called a **cycle**.

Planar A graph which can be drawn (in the plane) without any edges crossing.

Subgraph We say that H is a **subgraph** of G if every vertex and edge of H is also a vertex or edge of G. We say H is an **induced** subgraph of G if every vertex of H is a vertex of G and each pair of vertices in H are adjacent in H if and only if they are adjacent in G .

Tree A connected graph with no cycles. (If we remove the requirement that the graph is connected, the graph is called a **forest**.) The vertices in a tree with degree 1 are called **leaves**.

Vertex coloring An assignment of colors to each of the vertices of a graph. A vertex coloring is **proper** if adjacent vertices are always colored differently.

Walk A sequence of vertices such that consecutive vertices (in the sequence) are adjacent (in the graph). A walk in which no edge is repeated is called a **trail**, and a trail in which no vertex is repeated (except possibly the first and last) is called a **path**.

2.1.3 Reading Questions

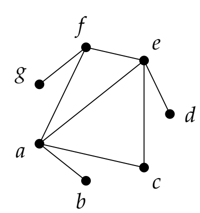
1. Is there more than one graph with five vertices and six edges? Explain what this question even means and how you would answer it.

2. If a graph has 10 vertices, each with degree 4, how many edges does it have?

3. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

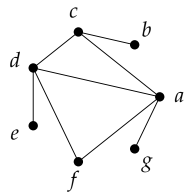
2.1.4 Practice Problems

1. Consider the graph G below.

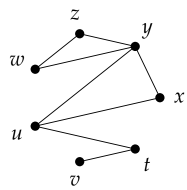


Which of the following graphs are isomorphic to G? Select all that apply.

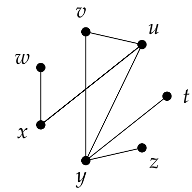
(a)



(b)

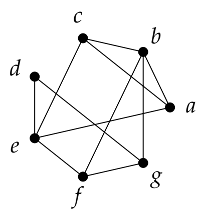


(c)

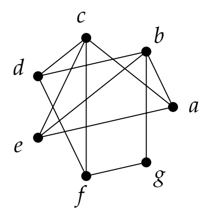


2. Which of the following graphs are isomorphic to each other?

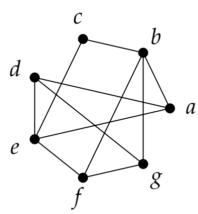
(a)



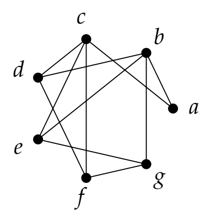
(c)



(b)



(d)



3. The graph G_1 has 8 vertices all of degree 6. How many edges does G_1 have?

The graph G_2 has 7 vertices all of degree k, and 7 edges. What is k?

The graph G_3 has all vertices of degree 4, and 16 edges. How many vertices does G_3 have?

4. Suppose a graph has degree sequence $(8,7,7,6,6,6,5,4,4,3)$. How many edges must the graph have?

5.

(a) What is the largest n such that P_n is a subgraph of K_5?

(b) What is the largest n such that C_n is a subgraph of K_5?

(c) What is the largest n such that P_n is an *induced* subgraph of K_5?

(d) What is the largest n such that C_n is an *induced* subgraph of K_5?

2.1.5 Additional Exercises

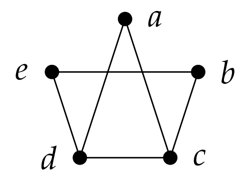
1. If 10 people each shake hands with each other, how many handshakes took place? What does this question have to do with graph theory?

2. Among a group of 5 people, is it possible for everyone to be friends with exactly 2 of the people in the group? What about 3 of the people in the group?

3. Is it possible for two *different* (non-isomorphic) graphs to have the same number of vertices and the same number of edges? What if the degrees of the vertices in the two graphs are the same (so both graphs have vertices with degrees 1, 2, 2, 3, and 4, for example)? Draw two such graphs or explain why not.

4. Are the two graphs below equal? Are they isomorphic? If they are isomorphic, give the isomorphism. If not, explain.

Graph 1: $V = \{a, b, c, d, e\}$, $E = \{\{a, b\}, \{a, c\}, \{a, e\}, \{b, d\}, \{b, e\}, \{c, d\}\}$.
Graph 2:



5. Consider the following two graphs:

$$G_1 \quad V_1 = \{a, b, c, d, e, f, g\}$$
$$E_1 = \{\{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}, \{b, e\}, \{b, f\}, \{c, g\}, \{d, e\}, \{e, f\}, \{f, g\}\}.$$
$$G_2 \quad V_2 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\},$$
$$E_2 = \{\{v_1, v_4\}, \{v_1, v_5\}, \{v_1, v_7\}, \{v_2, v_3\}, \{v_2, v_6\}, \{v_3, v_5\}, \{v_3, v_7\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_5, v_7\}\}.$$

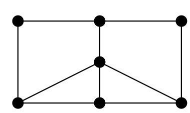
(a) Let $f : G_1 \to G_2$ be a function that takes the vertices of Graph 1 to vertices of Graph 2. The function is given by the following table:

x	a	b	c	d	e	f	g
$f(x)$	v_4	v_5	v_1	v_6	v_2	v_3	v_7

Does f define an isomorphism between Graph 1 and Graph 2?

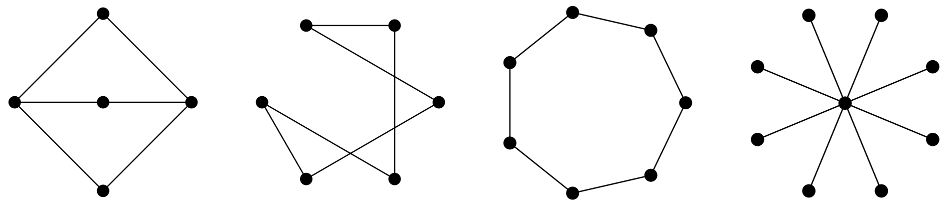
(b) Define a new function g (with $g \neq f$) that defines an isomorphism between Graph 1 and Graph 2.

(c) Is the graph pictured below isomorphic to Graph 1 and Graph 2? Explain.



6. What is the largest number of edges possible in a graph with 10 vertices? What is the largest number of edges possible in a *bipartite* graph with 10 vertices? What is the largest number of edges possible in a *tree* with 10 vertices?

7. Which of the graphs below are bipartite? Justify your answers.



8. For which $n \geq 3$ is the graph C_n bipartite?

9. For each of the following, try to give two *different* unlabeled graphs with the given properties, or explain why doing so is impossible.

(a) Two different trees with the same number of vertices and the same number of edges. A tree is a connected graph with no cycles.

(b) Two different graphs with 8 vertices all of degree 2.

(c) Two different graphs with 5 vertices all of degree 4.

(d) Two different graphs with 5 vertices all of degree 3.

10. Decide whether the statements below about subgraphs are true or false. For those that are true, briefly explain why (1 or 2 sentences). For any that are false, give a counterexample.

(a) Any subgraph of a complete graph is also complete.

(b) Any *induced* subgraph of a complete graph is also complete.

(c) Any subgraph of a bipartite graph is bipartite.

(d) Any subgraph of a tree is a tree.

11. We often define graph theory concepts using set theory. For example, given a graph $G = (V, E)$ and a vertex $v \in V$, we define

$$N(v) = \{u \in V : \{v, u\} \in E\}.$$

We define $N[v] = N(v) \cup \{v\}$. The goal of this problem is to figure out what all this means.

(a) Let G be the graph with vertices V and edges E given by

$$V = \{a, b, c, d, e, f\}, \; E = \{\{a, b\}, \{a, e\}, \{b, c\}, \{b, e\}, \{c, d\}, \{c, f\}, \{d, f\}, \{e, f\}\}.$$

Find $N(a)$, $N[a]$, $N(c)$, and $N[c]$.

(b) What are the largest and smallest possible values for $|N(v)|$ and $|N[v]|$ (the sizes of these sets) for the graph in part (a)? Explain.

(c) Give an example of a graph $G = (V, E)$ (probably different from the one above) for which $N[v] = V$ for some vertex $v \in V$. Is there a graph for which $N[v] = V$ for *all* $v \in V$? Explain.

(d) Give an example of a graph $G = (V, E)$ for which $N(v) = \emptyset$ for some $v \in V$. Is there an example of such a graph for which $N[u] = V$ for some other $u \in V$ as well? Explain.

(e) Describe in words what $N(v)$ and $N[v]$ mean in general.

12. A graph is a way of representing the relationships between elements in a set: An edge between the vertices x and y tells us that x is related to y (which we can write as $x \sim y$). Not all sorts of relationships can be represented by a graph, though. For each relationship described below, either draw the graph or explain why the relationship cannot be represented by a graph.

(a) The set $V = \{1, 2, \ldots, 9\}$ and the relationship $x \sim y$ when $x - y$ is a non-zero multiple of 3.

(b) The set $V = \{1, 2, \ldots, 9\}$ and the relationship $x \sim y$ when y is a multiple of x.

(c) The set $V = \{1, 2, \ldots, 9\}$ and the relationship $x \sim y$ when $0 < |x - y| < 3$.

13. Consider graphs with n vertices. Remember, graphs do not need to be *connected*.

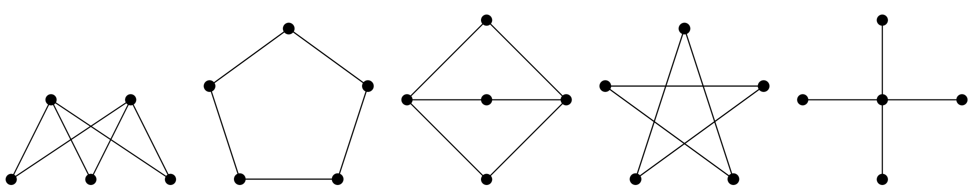
(a) How many edges must the graph have to guarantee at least one vertex has degree two or more? Prove your answer.

(b) How many edges must the graph have to guarantee all vertices have degree two or more? Prove your answer.

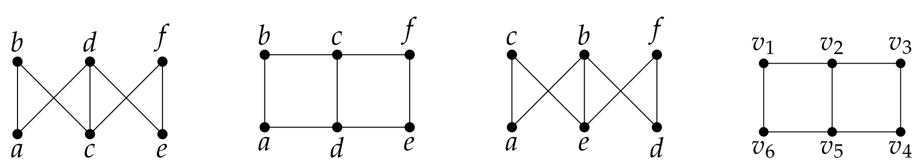
14. Prove that any graph with at least two vertices must have two vertices of the same degree.

15. Suppose G is a connected graph with $n > 1$ vertices and $n - 1$ edges. Prove that G has a vertex of degree 1.

16. Which (if any) of the graphs below are the same?



The graphs above are unlabeled. Usually we think of a graph as having a specific set of vertices. Which (if any) of the graphs below are the same?



Actually, all the graphs above are just *drawings* of graphs. A graph is really an abstract mathematical object consisting of two sets V and E, where E is a set of 2-element subsets of V. Are the graphs below the same or different?

Graph 1: $V = \{a, b, c, d, e\}$,
$E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{b, c\}, \{d, e\}\}$.

Graph 2: $V = \{v_1, v_2, v_3, v_4, v_5\}$,
$E = \{\{v_1, v_3\}, \{v_1, v_5\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_5\}, \{v_4, v_5\}\}$.

2.2 Trees

Objectives

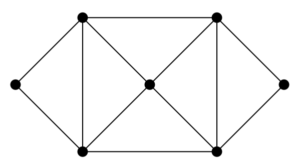
After completing this section, you should be able to do the following.

1. Prove basic facts about trees.
2. Use theorems about trees to solve problems.
3. Identify and construct spanning trees.

2.2.1 Section Preview

Investigate!

Consider the graph drawn below.



1. Find a subgraph with the smallest number of edges that is still connected and contains all the vertices.
2. Find a subgraph with the largest number of edges that doesn't contain any cycles.
3. What do you notice about the number of edges in your examples above? Is this a coincidence?

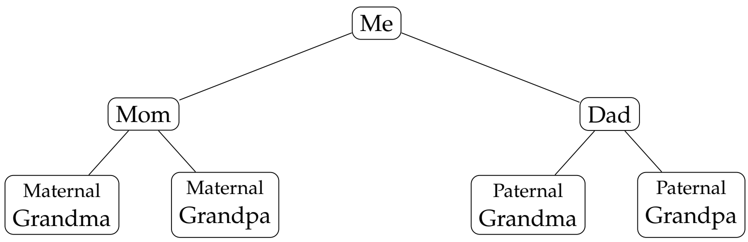
One very useful and common approach to studying graph theory is to restrict your focus to graphs of a particular kind. For example, you could try to really understand just complete graphs or just bipartite graphs, instead of trying to understand all graphs in general. That is what we are going to do now, looking at *trees*. Hopefully by the end of this section we will have a better understanding of this class of graph, and also understand why it is important enough to warrant its own section.

Definition 2.2.1 Trees and Forests.

A **tree** is a connected graph containing no cycles.[4]

A **forest** is a graph containing no cycles. Note that this means that a connected forest is a tree.

Does the definition above agree with your intuition for what graphs we should call trees? Try thinking of examples of trees, and make sure they satisfy the definition. One thing to keep in mind is that while the trees we study in graph theory are related to trees you might see in other subjects, the correspondence is not exact. For instance, in anthropology, you might study family trees, like the one below,



So far so good, but while your grandparents are (probably) not blood relatives, if we go back far enough, it is likely that they did have *some* common ancestor. If you trace the tree back from you to that common ancestor, then down through your other grandparent, you would have a cycle, and thus the graph would not be a tree.

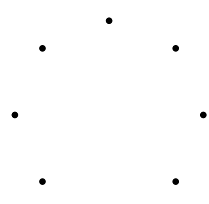
You might also have seen something called a *decision tree* (such as the algorithm for deciding whether a series converges or diverges). Sometimes these too contain cycles, as the decision for one node might lead you back to a previous step.

Both the examples of trees above also have another feature worth mentioning: There is a clear order to the vertices in the tree. The definition of a tree does not include this added structure, although we can impose such a structure by considering **rooted trees**, where we simply designate one vertex as the *root*. We will consider such trees in more detail later in this section.

In this section, we will explore some basic properties of trees, which will serve as an excellent introduction to writing proofs about graphs. We will also consider a special kind of tree, called a **spanning tree**, which is a tree that includes all the vertices of a connected graph. Finally, we will briefly consider rooted trees.

Preview Activity

1. Take out a piece of paper and draw 8 vertices in a circle.



[4]Sometimes this is stated as "a tree is an acyclic connected graph;" "acyclic" is just a fancy word for "containing no cycles."

We are going to add edges to this graph following some requirements.

(a) First, add the fewest number of edges possible so that the resulting graph is connected. That is, there must be a path between any pair of vertices (a path can use more than one edge, of course).

How many edges are in the graph you drew?

(b) Was the resulting graph you found a tree?

(c) Now start over with an empty graph again. This time, add the largest number of edges possible so that the resulting graph contains no cycles.

How many edges are in the graph you drew?

(d) Was the resulting graph you found a tree?

2.2.2 Properties of Trees

We wish to really understand trees. This means we should discover properties of trees: what makes them special and what is special about them.

A tree is a connected graph with no cycles. Is there anything else we can say? It would be nice to have other equivalent conditions for a graph to be a tree. That is, we would like to know whether there are any graph theoretic properties that all trees have, and perhaps even that *only* trees have.

To get a feel for the sorts of things we can say, we will consider three *propositions* about trees. These will also illustrate important proof techniques that apply to graphs in general, and happen to be a little easier for trees.

Our first proposition gives an alternate definition for a tree. That is, it gives necessary and sufficient conditions for a graph to be a tree.

Proposition 2.2.2

A graph T is a tree if and only if between every pair of distinct vertices of T there is a unique path.

Proof. This is an "if and only if" statement, so we must prove two implications. We start by proving that if T is a tree, then between every pair of distinct vertices there is a unique path.

Assume T is a tree, and let u and v be distinct vertices (if T only has one vertex, then the conclusion is satisfied automatically). We must show two things to show that there is a unique path between u and v: that there is a path, and that there is not more than one path. The first of these is automatic; since T is a tree, it is connected, so there is a path between any pair of vertices.

To show the path is unique, we suppose there are two paths between u and v, and get a contradiction. The two paths might start out the same, but since they are different, there is some first vertex u' after which the two paths diverge. However, since the two paths both end at v, there is some first vertex after u' that they have in common, call it v'. Now consider the two paths from u' to v'. Taken together, these form a cycle, which contradicts our assumption that T is a tree.

Now we consider the converse: If between every pair of distinct vertices of T there is a unique path, then T is a tree. So assume the hypothesis: Between every pair of distinct vertices of T there is a unique path. To prove that T is a tree, we must show it is connected and contains no cycles.

The first half of this is easy: T is connected, because there is a path between every pair of vertices. To show that T has no cycles, we assume it does, for the sake of contradiction. Let u and v be two distinct vertices in a cycle of T. Since we can get from u to v by going clockwise or counterclockwise around the cycle, there are two paths from u and v, contradicting our assumption.

We have established both directions so we have completed the proof.

Read the proof above very carefully. Notice that both directions had two parts: the existence of paths, and the uniqueness of paths (which related to the fact that there were no cycles). In this case, these two parts were really separate. In fact, if we just considered graphs with no cycles (a forest), then we could still do the parts of the proof that explore the uniqueness of paths between vertices, even if there might not *exist* paths between vertices.

This observation allows us to state the following *corollary*.[5]

Corollary 2.2.3

A graph F is a forest if and only if between any pair of vertices in F there is at most one path.

We do not give a proof of the corollary (it is, after all, supposed to follow directly from the proposition), but for practice, you are asked to give a careful proof in the exercises. When you do so, try to use proof by contrapositive instead of proof by contradiction.

Our second proposition tells us that all trees have **leaves**: vertices of degree one.

Proposition 2.2.4

Any tree with at least two vertices has at least two vertices of degree one.

Proof. We give a proof by contradiction. Let T be a tree with at least two vertices, and suppose, contrary to stipulation, that there are not two vertices of degree one.

Let P be a path in T of longest possible length. Let u and v be the endpoints of the path. Since T does not have two vertices of degree one, at least one of these must have degree two or higher. Say that it is u. We know that u is adjacent to a vertex in the path P, but now it must also be adjacent to another vertex, call it u'.

Where is u'? It cannot be a vertex of P, because if it was, there would be two distinct paths from u to u': the edge between them, and the first part of P (up to

[5]A corollary is another sort of provable statement, like a proposition or theorem, but one that follows direction from another already established statement, or its proof.

u'). But u' also cannot be outside of P, for if it was, there would be a path from u' to v that was longer than P, which has the longest possible length.

This contradiction proves that there must be at least two vertices of degree one. In fact, we can say a little more: u and v must *both* have degree one.

The proposition is quite useful when proving statements about trees, because we often prove statements about trees by **induction**. This is a proof technique that we investigate fully in Section 4.5, but for now, all we need to understand is that it is useful to compare a given tree to smaller trees. To show something is true of a tree with v vertices, we will assume the thing is true of all trees with $v - 1$ vertices. By removing a vertex of degree 1, we get this smaller tree, and then we just need to show that putting the vertex back doesn't mess us up.

Is there a tree with exactly 7 vertices and 7 edges? Try to draw one. Could a tree with 7 vertices have only 5 edges? There is a good reason that these seem impossible to draw.

Proposition 2.2.5

Let T be a tree with v vertices and e edges. Then $e = v - 1$.

We will prove that this proposition is true for all possible values of $v \geq 1$. Note that if $v = 1$, then the tree must have 0 edges, so yes, $e = v - 1$. We could then look at trees with $v = 2$ vertices (there is only one, and it has $e = 1 = v - 1$ edges) and then all trees with $v = 3$ vertices, and then $v = 4$, and so on, but that would take forever. Literally!

Instead, we will do a version of *proof by contradiction* called the **minimal criminal**. We will assume the proposition is not true (just like we would start any proof by contradiction). This means that there is some *smallest* tree for which it isn't true. If this smallest counterexample (i.e., minimal criminal) has v vertices, then we are guaranteed that *all* trees with $v - 1$ vertices are *not* counterexamples. Let's see how this works out.

Proof. Suppose, for the sake of contradiction, that the proposition is not true for all trees. Let T be a tree for which the proposition is not true, with the smallest number of vertices among all the counterexamples. Let v be the number of vertices of T.

Since the only tree with one vertex has zero edges, that cannot be our tree T, so we can assume $v \geq 2$. In particular, we know by Proposition 2.2.4 that T must contain at least one vertex of degree 1, call it v_0.

Let T' be the tree resulting from removing v_0 from T (together with its incident edge). Since we removed a leaf, T' is still a tree (the unique paths between pairs of vertices in T' are the same as the unique paths between them in T).

Now T' has $v-1$ vertices. Since T was the smallest tree for which the proposition isn't true, we know that the proposition is true for T'. So T' must have one fewer edges than vertices; that is, T' has $v - 2$ edges. But T has one more edge than T', so it has $v - 1$ edges. This contradicts our assumption that T does not satisfy the proposition, which completes our proof.

2.2.3 Spanning Trees

One of the advantages of trees is that they give us a few simple ways to travel through the vertices. If a connected graph is not a tree, then we can still use these traversal algorithms if we identify a subgraph that *is* a tree.

First we should consider if this even makes sense. Given any connected graph G, will there always be a subgraph that is a tree? Well, that is actually too easy: You could just take a single vertex of G. If we want to use this subgraph to tell us how to visit all vertices, then we want our subgraph to include all of the vertices. We call such a tree a **spanning tree**.

Definition 2.2.6

Given a connected graph G, a **spanning tree** of G is a subgraph of G which is a tree and includes all the vertices of G.

It turns out that every connected graph has one (and usually many).

Theorem 2.2.7

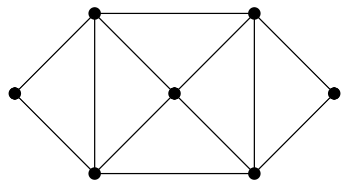
Every connected graph has a spanning tree.

How do we know? We can give an algorithm for *finding* a spanning tree! Start with a connected graph G. If there is no cycle, then G is already a tree and we are done. If there is a cycle, let e be any edge in that cycle and consider the new graph $G_1 = G - e$ (i.e., the graph you get by deleting e). This tree is still connected: Since e belonged to a cycle, there were at least two paths between its incident vertices. Now repeat: If G_1 has no cycles, we are done; otherwise define G_2 to be $G_1 - e_1$, where e_1 is an edge in a cycle in G_1. Keep going. This process must eventually stop, since there are only a finite number of edges to remove. The result will be a tree, and since we never removed any vertex, a *spanning* tree.

This is by no means the only algorithm for finding a spanning tree. You could have started with the empty graph and added edges that belong to G, as long as adding them would not create a cycle. You have some choices as to which edges you add first: You could always add an edge adjacent to edges you have already added (after the first one, of course), or add them using some other order. Which spanning tree you end up with depends on these choices.

Example 2.2.8

Find two different spanning trees of the graph,



Solution. Here are two spanning trees.

Although we will not consider this in detail, these algorithms are usually applied to *weighted* graphs. Here every edge has some weight or cost assigned to it. The goal is to find a spanning tree that has the smallest possible combined weight. Such a tree is called a **minimum spanning tree**. Finding the minimum spanning tree uses basically the same algorithms as we described above, but when picking an edge to add, you always pick the smallest (or when removing an edge, you always remove the largest).[6]

2.2.4 Rooted Trees

So far, we have thought of trees only as a particular kind of graph. However, it is often useful to add additional structure to trees to help solve problems. Data is often structured like a tree. This book, for example, has a tree structure: Draw a vertex for the book itself. Then draw vertices for each chapter, connected to the book vertex. Under each chapter, draw a vertex for each section, connecting it to the chapter it belongs to. The graph will not have any cycles; it will be a tree, but a tree with a clear hierarchy which is not present if we don't identify the "book vertex" as the "top".

As soon as one vertex of a tree is designated as the **root**, then every other vertex on the tree can be characterized by its position relative to the root. This works because there is a unique path between any two vertices in a tree. So from any vertex, we can travel back to the root in exactly one way. This also allows us to describe how distinct vertices in a rooted tree are related.

If two vertices are adjacent, then we say one of them is the **parent** of the other, which is called the **child** of the parent. Of the two, the parent is the vertex that is closer to the root. Thus the root of a tree is a parent, but is not the child of any vertex (and is unique in this respect: All non-root vertices have *exactly one* parent).

Not surprisingly, the child of a child of a vertex is called the **grandchild** of the vertex (and it is the **grandparent**). More generally, we say that a vertex v is a

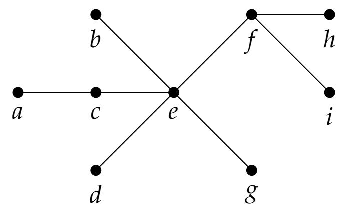
[6] If you add the smallest edge adjacent to edges you have already added, you are doing *Prim's algorithm*. If you add the smallest edge in the entire graph, you are following *Kruskal's algorithm*.

descendent of a vertex u provided u is a vertex on the path from v to the root. Then we would call u an **ancestor** of v.

For most trees (in fact, all except paths with one end the root), there will be pairs of vertices neither of which is a descendant of the other. We might call these cousins or siblings. In fact, vertices u and v are called **siblings** provided they have the same parent. Note that siblings are never adjacent (do you see why?).

Example 2.2.9

Consider the tree below.



If we designate vertex f as the root, then e, h, and i are the children of f, and are siblings of each other. Among the other things we cay say are that a is a child of c, and a descendant of f. The vertex g is a descendant of f, in fact, is a grandchild of f. Vertices g and d are siblings, since they have the common parent e.

Notice how this changes if we pick a different vertex for the root. If a is the root, then its lone child is c, which also has only one child, namely e. We would then have f the child of e (instead of the other way around), and f is the descendant of a, instead of the ancestor. f and g are now siblings.

All of this flowery language helps us describe how to *navigate* through a tree. Traversing a tree, visiting each vertex in some order, is a key step in many algorithms. Even if the tree is not rooted, we can always form a rooted tree by picking any vertex as the root. Here is an example of why doing so can be helpful.

Example 2.2.10

Explain why every tree is a bipartite graph.

Solution. To show that a graph is bipartite, we must divide the vertices into two sets, A and B, so that no two vertices in the same set are adjacent. Here is an algorithm that does just this.

Designate any vertex as the root. Put this vertex in set A. Now put all of the children of the root in set B. None of these children are adjacent (they are siblings), so we are good so far. Now put into A every child of every vertex in B (i.e., every grandchild of the root). Keep going until all vertices have been assigned one of the sets, alternating between A and B every "generation." That is, a vertex is in set B if and only if it is the child of a vertex in set A.

The key to how we partitioned the tree in the example was to know which vertex to assign to a set next. We chose to visit all vertices in the same generation before any vertices of the next generation. This is usually called a **breadth-first search** (we say "search" because you often traverse a tree looking for vertices with certain properties).

In contrast, we could also have partitioned the tree in a different order. Start with the root; put it in A. Then look for one child of the root to put in B. Then find a child of that vertex, into A, and then find its child, into B, and so on. When you get to a vertex with no children, retreat to its parent, and see if the parent has any other children. So we travel as far from the root as fast as possible, then backtrack until we can move forward again. This is called **depth-first search**.

These algorithmic explanations can serve as a proof that every tree is bipartite, although care needs to be spent to prove that the algorithms are *correct*. Another approach to prove that all trees are bipartite, using induction, is requested in the exercises.

2.2.5 Reading Questions

1. Suppose T is a tree with 10 vertices. Which of the following statements must be true about T? Select all that apply.

A. T has a unique path between every pair of vertices.

B. If you removed any edge from T, the resulting graph would be disconnected.

C. If you added any edge between two vertices in T (that were not already adjacent), the resulting graph would have a cycle.

D. T has exactly two vertices of degree one.

2. If a tree has 20 vertices, how many edges does it have?

3. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

2.2.6 Practice Problems

1. Are the following statements true or false?

(a) Every tree is bipartite.

(b) Every bipartite graph is a tree.

(c) There is a forest with more edges than vertices.

(d) There is a tree with 6 vertices and 6 edges.

(e) Every tree with 7 vertices has the same number of edges.

2. A forest contains 30 vertices and 21 edges. How many connected components does the graph have?

3. A connected graph with 22 vertices contains 29 edges. Without knowing which particular graph this is, what is the smallest and largest possible number of edges you can remove to get a spanning tree?

Smallest number of edges to remove: _________

Largest number of edges to remove: _________

4. The average degree of a tree is 1.992 (that is, if you sum the degrees of vertices and divide by the number of vertices, you get 1.992).

How many vertices does the tree have?

5. A tree contains some number of leaves (degree 1 vertices) and four non-leaf vertices. The degrees of the non-leaf vertices are 8, 6, 5, and 3. How many leaves does the tree have?

Smallest number of leaves possible: _________

Largest number of leaves possible: _________

2.2.7 Additional Exercises

1. Which of the following graphs are trees?

(a) $G = (V,E)$ with $V = \{a,b,c,d,e\}$ and $E = \{\{a,b\},\{a,e\},\{b,c\},\{c,d\},\{d,e\}\}$

(b) $G = (V,E)$ with $V = \{a,b,c,d,e\}$ and $E = \{\{a,b\},\{b,c\},\{c,d\},\{d,e\}\}$

(c) $G = (V,E)$ with $V = \{a,b,c,d,e\}$ and $E = \{\{a,b\},\{a,c\},\{a,d\},\{a,e\}\}$

(d) $G = (V,E)$ with $V = \{a,b,c,d,e\}$ and $E = \{\{a,b\},\{a,c\},\{d,e\}\}$

2. For each degree sequence below, decide whether it must always, must never, or could possibly be a degree sequence for a tree. Remember, a degree sequence lists out the degrees (number of edges incident to the vertex) of all the vertices in a graph in non-increasing order.

(a) $(4,1,1,1,1)$

(b) $(3,3,2,1,1)$

(c) $(2,2,2,1,1)$

(d) $(4,4,3,3,3,2,2,1,1,1,1,1,1,1)$

3. For each degree sequence below, decide whether it must always, must never, or could possibly be a degree sequence for a tree. Justify your answers.

(a) $(3,3,2,2,2)$

(b) $(3,2,2,1,1,1)$

(c) $(3,3,3,1,1,1)$

(d) $(4, 4, 1, 1, 1, 1, 1, 1)$

4. Suppose you have a graph with v vertices and e edges that satisfies $v = e + 1$. Must the graph be a tree? Prove your answer.

5. Prove that any graph (not necessarily a tree) with v vertices and e edges that satisfies $v > e + 1$ will NOT be connected.

6. If a graph G with v vertices and e edges is connected and has $v < e + 1$, must it contain a cycle? Prove your answer.

7. We define a **forest** to be a graph with no cycles.

(a) Explain why this is a good name. That is, explain why a forest is a union of trees.

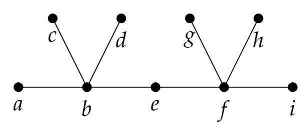
(b) Suppose F is a forest consisting of m trees and v vertices. How many edges does F have? Explain.

(c) Prove that any graph G with v vertices and e edges that satisfies $v < e + 1$ must contain a cycle (i.e., not be a forest).

8. Give a careful proof of Corollary 2.2.3: A graph is a forest if and only if there is at most one path between any pair of vertices. Use proof by contrapositive (and not a proof by contradiction) for both directions.

9. Give a careful *minimal criminal* proof that every tree is bipartite.

10. Consider the tree drawn below.



(a) Suppose we designate vertex e as the root. List the children, parents, and siblings of each vertex. Does any vertex other than e have grandchildren?

(b) Suppose e is *not* chosen as the root. Does our choice of root vertex change the *number* of children e has? The number of grandchildren? How many are there of each?

(c) In fact, pick any vertex in the tree and suppose it is not the root. Explain why the number of children of that vertex does not depend on which other vertex is the root.

(d) Does the previous part work for other trees? Give an example of a different tree for which it holds. Then either prove that it always holds or give an example of a tree for which it doesn't.

11. Let T be a rooted tree that contains vertices u, v, and w (among others, possibly). Prove that if w is a descendant of both u and v, then u is a descendant of v or v is a descendant of u.

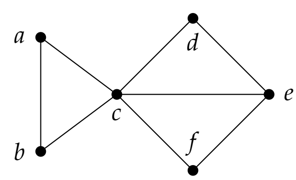
12. Unless it is already a tree, a given graph G will have multiple spanning trees. How similar or different must these be?

(a) Must all spanning trees of a given graph be isomorphic to each other? Explain why or give a counterexample.

(b) Must all spanning trees of a given graph have the same number of edges? Explain why or give a counterexample.

(c) Must all spanning trees of a graph have the same number of leaves (vertices of degree 1)? Explain why or give a counterexample.

13. Find all spanning trees of the graph below. How many different spanning trees are there? How many different spanning trees are there *up to isomorphism* (that is, if you grouped all the spanning trees by which are isomorphic, how many groups would you have)?



14. Give an example of a graph that has exactly 7 different spanning trees. Note, it is acceptable for some or all of these spanning trees to be isomorphic.

15. Prove that every connected graph which is not itself a tree must have at least three different (although possibly isomorphic) spanning trees.

16. Consider edges that must be in every spanning tree of a graph. Must every graph have such an edge? Give an example of a graph that has exactly one such edge.

2.3 Planar Graphs

Objectives

After completing this section, you should be able to do the following.

1. Distinguish between planar and non-planar graphs.
2. Use Euler's formula to prove that certain graphs are non-planar.
3. Apply Euler's formula to polyhedra.

2.3.1 Section Preview

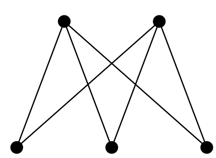
Investigate!

When a connected graph can be drawn without any edges crossing, it is called **planar**. When a planar graph is drawn in this way, it divides the plane into regions called **faces**.

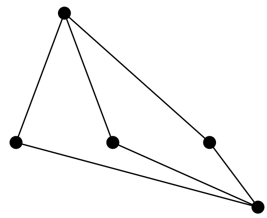
1. Draw, if possible, two different planar graphs with the same number of vertices, edges, and faces.
2. Draw, if possible, two different planar graphs with the same number of vertices and edges, but a different number of faces.

When is it possible to draw a graph so that none of the edges cross? If this *is* possible, we say the graph is **planar** (since you can draw it on the *plane*).

Notice that the definition of planar includes the phrase "it is possible to." This means that even if a graph does not look like it is planar, it still might be. Perhaps you can redraw it in a way in which no edges cross. For example, this is a planar graph:



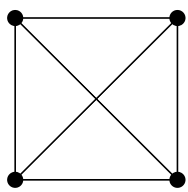
That is because we can redraw it like this:

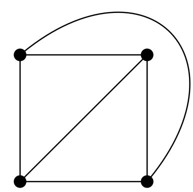


The graphs are the same, so if one is planar, the other must be, too. However, the original drawing of the graph was not a **planar representation** of the graph.

When a planar graph is drawn without edges crossing, the edges and vertices of the graph divide the plane into regions. We will call each region a **face**. The graph above has 3 faces (yes, we *do* include the "outside" region as a face). The number of faces does not change no matter how you draw the graph (as long as you do so without the edges crossing), so it makes sense to ascribe the number of faces as a property of the planar graph.

WARNING: you can only count faces when the graph is drawn in a planar way. For example, consider these two representations of the same graph:





If you try to count faces using the graph on the left, you might say there are 5 faces (including the outside). But drawing the graph with a planar representation shows that in fact there are only 4 faces.

PREVIEW ACTIVITY

1.

(a) Draw a connected planar graph with 5 vertices and 5 edges. How many faces (including the "outside" face) does your graph have?

(b) Now add a single edge to your graph, between two vertices that are not already adjacent. Assuming the resulting graph is still planar, list the number of vertices, edges, and faces it now has.

(c) Now add another edge to the graph, this time to a new vertex. Assuming the resulting graph is still planar, list the number of vertices, edges, and faces it now has.

2. Now draw at least three more connected, planar graphs, each with at least six vertices. Count the number of vertices v, edges e, and faces f for each graph and record your data in the table below.

v	e	f

3. Do you notice any patterns? What happens to the numbers if you add an edge between two non-adjacent vertices? What happens if you add a new vertex and connect it to an existing vertex?

Conjecture an expression that involves the number of vertices v, the number of edges e, and the number of faces f that remains constant for all connected planar graphs. What is that constant?

Hint. You might conjecture an expression like $\frac{v+e}{f}$. But this is not right, because there is a planar graph for which this would be $\frac{5+5}{2} = 5$ and another planar graph for which the expression would be $\frac{6+7}{3} \neq 5$.

What sort of expression will stay constant if v and e both increase by 1? And also stay constant if e and f both increase by 1?

4. A cube is made of six squares, each of which shares an edge with each of its neighbors. Vertices of the cube join three of the squares.

(a) How many vertices, edges, and faces does a cube have?

(b) Does this match the relationship you conjectured above?

2.3.2 Euler's Formula for Planar Graphs

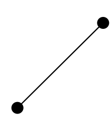
There is a connection between the number of vertices (v), the number of edges (e), and the number of faces (f) in any connected planar graph. This relationship is called Euler's formula.

Euler's Formula for Planar Graphs.

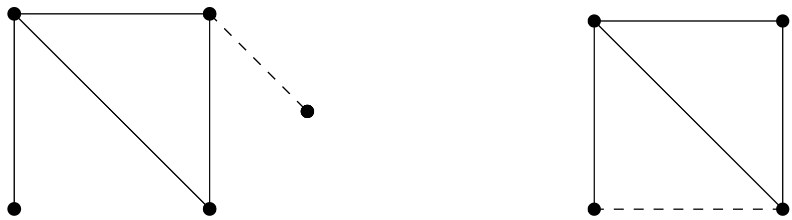
For any connected planar graph with v vertices, e edges, and f faces, we have

$$v - e + f = 2.$$

Why is Euler's formula true? One way to convince ourselves of its validity is to draw a planar graph step by step. Start with the graph P_2:



Any connected graph (besides just a single isolated vertex) must contain this subgraph. Now we build up to our graph by adding edges and vertices. Each step will consist of either adding a new vertex connected by a new edge to part of your graph (so creating a new "spike") or by connecting two vertices already in the graph with a new edge (completing a circuit).



What do these "moves" do? When adding the spike, the number of edges increases by 1, the number of vertices increases by 1, and the number of faces remains the same. But this means that $v - e + f$ does not change. Completing a circuit adds one edge, adds one face, and keeps the number of vertices the same. So again, $v - e + f$ does not change.

Since we can build any graph using a combination of these two moves, and doing so never changes the quantity $v - e + f$, that quantity will be the same for all graphs. But notice that our starting graph P_2 has $v = 2$, $e = 1$, and $f = 1$, so $v - e + f = 2$.

The argument we have outlined above is not quite correct, since we made the unjustified assumption that all graphs can be built up from P_2 using only the two moves we described. To avoid this issue, we can use a minimal criminal argument. You are asked to do this in the exercises, but the idea is essentially the same as we have here, except that we start with a minimal connected planar graph that does not satisfy the formula, then *remove* either an edge or a vertex (and its edge) to get a smaller connected planar graph that does satisfy the formula. But just like the adding moves we have described above, removing an edge or a vertex does not change the quantity $v - e + f$.

2.3.3 Non-planar Graphs

Investigate!

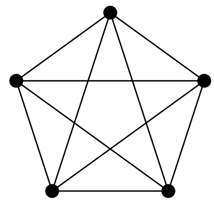
For the complete graphs K_n, we would like to be able to say something about the number of vertices, edges, and (if the graph is planar) faces. Let's first consider K_3:

1. How many vertices does K_3 have? How many edges?
2. If K_3 is planar, how many faces should it have?

Repeat parts (1) and (2) for K_4, K_5, and K_{23}.

What about complete bipartite graphs? How many vertices, edges, and faces (if it were planar) does $K_{7,4}$ have? For which values of m and n are K_n and $K_{m,n}$ planar?

Not all graphs are planar. If there are too many edges and too few vertices, then some of the edges will need to intersect. The smallest graph where this happens is K_5.



If you try to redraw this without edges crossing, you quickly get into trouble. There seems to be one edge too many. In fact, we can prove that no matter how you draw it, K_5 will always have edges crossing.

Theorem 2.3.1

K_5 is not planar.

Proof. The proof is by contradiction. So assume that K_5 is planar. Then the graph must satisfy Euler's formula for planar graphs. K_5 has 5 vertices and 10 edges, so we get

$$5 - 10 + f = 2,$$

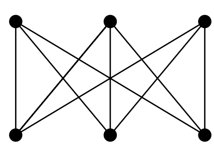
which says that if the graph is drawn without any edges crossing, there would be $f = 7$ faces.

Now consider how many edges surround each face. Each face must be surrounded by at least 3 edges. Let B be the total number of *boundaries* around all the faces in the graph. Thus we have that $3f \leq B$. But also $B = 2e$, since each edge is used as a boundary exactly twice. Putting this together we get

$$3f \leq 2e.$$

But this is impossible, since we have already determined that $f = 7$ and $e = 10$, and $21 \not\leq 20$. This is a contradiction, so in fact K_5 is not planar.

The other simplest graph which is not planar is $K_{3,3}$



Proving that $K_{3,3}$ is not planar answers the classic houses and utilities puzzle: it is not possible to connect each of three houses to each of three utilities without the lines crossing.

Theorem 2.3.2

$K_{3,3}$ is not planar.

Proof. Again, we proceed by contradiction. Suppose $K_{3,3}$ were planar. Then by Euler's formula, there will be 5 faces, since $v = 6$, $e = 9$, and $6 - 9 + f = 2$.

How many boundaries surround these 5 faces? Let B be this number. Since each edge is used as a boundary twice, we have $B = 2e$. Also, $B \geq 4f$ since each face is surrounded by 4 or more boundaries. We know this is true because $K_{3,3}$ is bipartite, so does not contain any 3-edge cycles. Thus

$$4f \leq 2e.$$

But this would say that $20 \leq 18$, which is clearly false. Thus $K_{3,3}$ is not planar.

Note the similarities and differences in these proofs. Both are proofs by contradiction, and both start with using Euler's formula to derive the (supposed) number of faces in the graph. Then we find a relationship between the number of faces and the number of edges based on how many edges surround each face. This is the only difference. In the proof for K_5, we got $3f \leq 2e$ and for $K_{3,3}$ we had $4f \leq 2e$. The coefficient of f is the key. It is the smallest number of edges that could surround any face. If some number of edges surround a face, then these edges form a cycle. So that number is the size of the smallest cycle in the graph.

In general, if we let g be the size of the smallest cycle in a graph (g stands for *girth*, which is the technical term for this) then for any planar graph we have $gf \leq 2e$. When this disagrees with Euler's formula, we know for sure that the graph cannot be planar.[7]

2.3.4 Polyhedra

Investigate!

A cube is an example of a convex polyhedron. It contains 6 identical squares for its faces, 8 vertices, and 12 edges. The cube is a **regular polyhedron** (also known as a **Platonic solid**) because each face is an identical regular polygon and each vertex joins an equal number of faces.

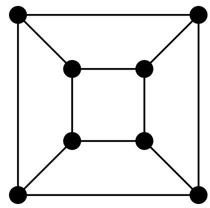
There are exactly four other regular polyhedra: the tetrahedron, octahedron, dodecahedron, and icosahedron, with 4, 8, 12, and 20 faces respectively. How many vertices and edges do each of these have?

Another area of mathematics where you might have heard the terms "vertex," "edge," and "face" is geometry. A **polyhedron** is a geometric solid made up of flat polygonal faces joined at edges and vertices. We are especially interested in **convex**

[7]Note that for technical reasons, the girth of a graph without any cycles (a forest) is defined to be infinity, and in this case, we definitely don't have $gf \leq 2e$, even though all trees are planar.

polyhedra, which means that any line segment connecting two points on the interior of the polyhedron must be entirely contained inside the polyhedron.[8]

Notice that since $8 - 12 + 6 = 2$, the vertices, edges, and faces of a cube satisfy Euler's formula for planar graphs. This is not a coincidence. We can represent a cube as a planar graph by projecting the vertices and edges onto the plane. One such projection looks like this:



In fact, *every* convex polyhedron can be projected onto the plane without edges crossing. Think of placing the polyhedron inside a sphere, with a light at the center of the sphere. The edges and vertices of the polyhedron cast a shadow onto the interior of the sphere. You can then cut a hole in the sphere in the middle of one of the projected faces and "stretch" the sphere to lie down flat on the plane. The face that was punctured becomes the "outside" face of the planar graph.

The point is, we can apply what we know about graphs (in particular planar graphs) to convex polyhedra. Since every convex polyhedron can be represented as a planar graph, we see that Euler's formula for planar graphs holds for all convex polyhedra as well. We also can apply the same sort of reasoning we use for graphs in other contexts to convex polyhedra. For example, we know that there is no convex polyhedron with 11 vertices all of degree 3, as this would make $33/2$ edges.

Example 2.3.3

Is there a convex polyhedron consisting of three triangles and six pentagons? What about three triangles, six pentagons, and five heptagons (7-sided polygons)?

Solution. How many edges would such polyhedra have? For the first proposed polyhedron, the triangles would contribute a total of 9 edges, and the pentagons would contribute 30. However, this counts each edge twice (as each edge borders exactly two faces), giving $39/2$ edges, an impossibility. There is no such polyhedron.

The second polyhedron does not have this obstacle. The extra 35 edges contributed by the heptagons give a total of $74/2 = 37$ edges. So far so good. Now how many vertices does this supposed polyhedron have? We can use Euler's formula. There are 14 faces, so we have $v - 37 + 14 = 2$ or equivalently $v = 25$. But now use the vertices to count the edges again. Each vertex must

[8] An alternative definition for convex is that the internal angle formed by any two faces must be less than 180 deg.

have degree *at least* three (that is, each vertex joins at least three faces since the interior angle of all the polygons must be less that 180°), so the sum of the degrees of vertices is at least 75. Since the sum of the degrees must be exactly twice the number of edges, this says that there are strictly more than 37 edges. Again, there is no such polyhedron.

To conclude this application of planar graphs, consider the regular polyhedra. We claimed there are only five. How do we know this is true? We can prove it using graph theory.

Theorem 2.3.4

There are exactly five regular polyhedra.

Proof. Recall that all the faces of a regular polyhedron are identical regular polygons and that each vertex has the same degree. Consider four cases, depending on the type of regular polygon.

Case 1: Each face is a triangle. Let f be the number of faces. There are then $3f/2$ edges. Using Euler's formula, we have $v - 3f/2 + f = 2$ so $v = 2 + f/2$. Now each vertex has the same degree, say k. So the number of edges is also $kv/2$. Putting this together gives

$$e = \frac{3f}{2} = \frac{k(2 + f/2)}{2},$$

which says

$$k = \frac{6f}{4 + f}.$$

Both k and f must be positive integers. Note that $\frac{6f}{4+f}$ is an increasing function for positive f, bounded above by a horizontal asymptote at $k = 6$. Thus the only possible values for k are 3, 4, and 5. Each of these is possible. To get $k = 3$, we need $f = 4$ (this is the tetrahedron). For $k = 4$ we take $f = 8$ (the octahedron). For $k = 5$ take $f = 20$ (the icosahedron). Thus there are exactly three regular polyhedra with triangles for faces.

Case 2: Each face is a square. Now we have $e = 4f/2 = 2f$. Using Euler's formula, we get $v = 2 + f$, and counting edges using the degree k of each vertex gives us

$$e = 2f = \frac{k(2 + f)}{2}.$$

Solving for k gives

$$k = \frac{4f}{2 + f} = \frac{8f}{4 + 2f}.$$

This is again an increasing function, but this time the horizontal asymptote is at $k = 4$, so the only possible value that k could take is 3. This produces 6 faces, and we have a cube. There is only one regular polyhedron with square faces.

Case 3: Each face is a pentagon. We perform the same calculation as above, this time getting $e = 5f/2$ so $v = 2 + 3f/2$. Then

$$e = \frac{5f}{2} = \frac{k(2 + 3f/2)}{2},$$

so

$$k = \frac{10f}{4 + 3f}.$$

Now the horizontal asymptote is at $\frac{10}{3}$. This is less than 4, so we can only hope to have $k = 3$. We can do so by using 12 pentagons, getting the dodecahedron. This is the only regular polyhedron with pentagons as faces.

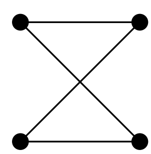
Case 4: Each face is an n-gon with $n \geq 6$. Following the same procedure as above, we deduce that

$$k = \frac{2nf}{4 + (n-2)f},$$

which will be increasing to a horizontal asymptote of $\frac{2n}{n-2}$. When $n = 6$, this asymptote is at $k = 3$. Any larger value of n will give an even smaller asymptote. Therefore no regular polyhedra exist with faces larger than pentagons.[9]

2.3.5 Reading Questions

1. Is the graph shown below planar? Explain your answer.



2. Suppose you draw a graph with 10 vertices and 14 edges in such a way that no edges cross. How many faces could your graph have? Explain your answer(s).

3. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

2.3.6 Practice Problems

1. Are the following statements true or false?

(a) K_7 is planar

(b) K_6 is not planar

(c) K_5 is planar

(d) $K_{3,3}$ is planar

[9]Notice that you can tile the plane with hexagons. This is an infinite planar graph; each vertex has degree 3. These infinitely many hexagons correspond to the limit as $f \to \infty$ to make $k = 3$.

(e) $K_{4,4}$ is not planar

(f) $K_{4,5}$ is planar

(g) $K_{3,10}$ is not planar

(h) $K_{2,5}$ is not planar

2. Suppose G is a planar connected graph. It has 15 edges, and 6 faces. How many vertices does G have?

3. Suppose a connected graph has 9 vertices, and every vertex has degree 3.

a. How many edges does the graph have?

b. If the graph were planar, how many faces would it have?

4. Let's prove that K_{11} is not planar:

First, how many vertices and how many edges does K_{11} have?

If we assume that K_{11} were planar, then how many faces *would* it have?

However, since every face is bounded by at least _____ edges, and every edge borders exactly _____ faces, we can get a bound on the number of faces. What is the largest number of faces possible based on this line of reasoning?

$f \leq$ __________.

This is a contradiction, so K_{11} is not planar. QED.

5. Suppose the graph is planar but not connected, and has 7 components. Draw enough examples to derive a variant of Euler's formula for this case.

$v - e + f =$ __________.

2.3.7 Additional Exercises

1. Is it possible for a planar graph to have 6 vertices, 10 edges, and 5 faces? Explain.

2. The graph G has 6 vertices with degrees $2, 2, 3, 4, 4, 5$. How many edges does G have? Could G be planar? If so, how many faces would it have? If not, explain.

3. Is it possible for a connected graph with 7 vertices and 10 edges to be drawn so that no edges cross and create 4 faces? Explain.

4. Is it possible for a graph with 10 vertices and edges to be a connected planar graph? Explain.

5. Is there a connected planar graph with an odd number of faces where every vertex has degree 6? Prove your answer.

6. I'm thinking of a polyhedron containing 12 faces. Seven are triangles and four are quadrilaterals. The polyhedron has 11 vertices including those around the mystery face. How many sides does the last face have?

7. Consider some classic polyhedrons.

(a) An *octahedron* is a regular polyhedron made up of 8 equilateral triangles (it sort of looks like two pyramids with their bases glued together). Draw a planar graph representation of an octahedron. How many vertices, edges, and faces does an octahedron (and your graph) have?

(b) The traditional design of a soccer ball is a (spherical projection of a) truncated icosahedron. This consists of 12 regular pentagons and 20 regular hexagons. No two pentagons are adjacent (so the edges of each pentagon are shared only by hexagons). How many vertices, edges, and faces does a truncated icosahedron have? Explain how you arrived at your answers. Bonus: draw the planar graph representation of the truncated icosahedron.

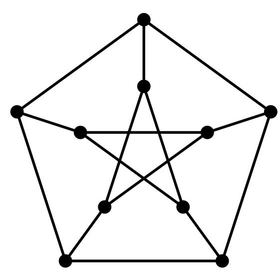
(c) Your "friend" claims that he has constructed a convex polyhedron out of 2 triangles, 2 squares, 6 pentagons, and 5 octagons. Prove that your friend is lying. Hint: each vertex of a convex polyhedron must border at least three faces.

8. Prove Euler's formula using a minimal criminal argument, where minimum means smallest number of edges

9. Prove Euler's formula using a minimal criminal argument, where minimum means smallest number of *vertices*.

10. Euler's formula ($v - e + f = 2$) holds for all *connected* planar graphs. What if a graph is not connected? Suppose a planar graph has two components. What is the value of $v - e + f$ now? What if it has k components?

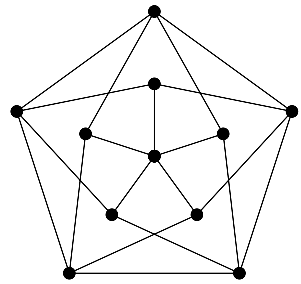
11. Prove that the **Petersen graph** (below) is not planar.



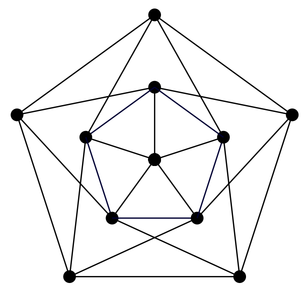
12. Prove that any planar graph with v vertices and e edges satisfies $e \leq 3v - 6$.

13. Prove that any planar graph must have a vertex of degree 5 or less.

14. Give a careful proof that the graph below is not planar.



15. Explain why we cannot use the same sort of proof we did in Exercise 2.3.7.14 to prove that the graph below is not planar. Then explain how you know the graph is not planar anyway.



2.4 Euler Trails and Circuits

Objectives

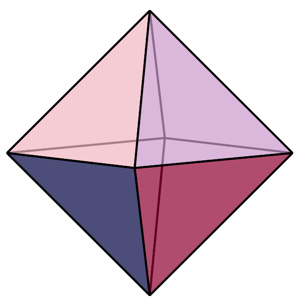
After completing this section, you should be able to do the following.

1. Identify whether a graph or multigraph has an Euler trail or circuit.
2. Justify why the necessary condition for a graph having an Euler trail is necessary.
3. Distinguish between Euler trails and Hamilton paths, and decide which is more appropriate to use for a given problem.

2.4.1 Section Preview

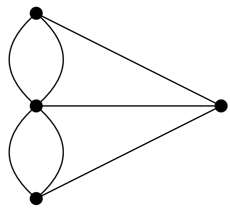
Investigate!

A spider is standing on one face of an octahedron (a polyhedron with eight triangular faces). She wants to crawl along the solid from face to face so that she crosses each edge exactly once. Is this possible? If so, how?



If we start at a vertex and trace along edges to get to other vertices, we create a *walk* through the graph. More precisely, a **walk** in a graph is a sequence of vertices such that every vertex in the sequence is adjacent to the vertices before and after it in the sequence. If the walk travels along every edge exactly once, then the walk is called an **Euler trail** (or **Euler walk** or **Euler path**). If, in addition, the starting and ending vertices are the same (so you trace along every edge exactly once and end up where you started), then the walk is called an **Euler circuit** (or **Euler tour**). Of course if a graph is not connected, there is no hope of finding such a trail or circuit. For the rest of this section, assume all the graphs discussed are connected.

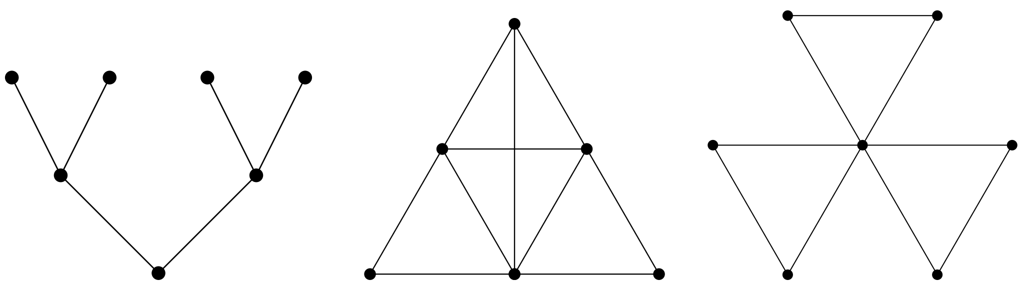
The bridges of Königsberg problem is really a question about the existence of Euler trails. There will be a route that crosses every bridge exactly once if and only if the multigraph below has an Euler trail:



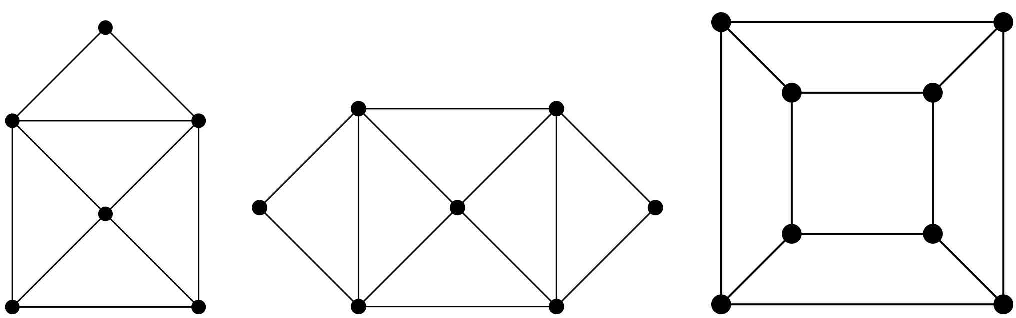
This graph is small enough that we could actually check every possible walk that does not reuse edges, and in doing so convince ourselves that there is no Euler trail (let alone an Euler circuit). On small graphs that do have an Euler trail, it is usually not difficult to find one. Our goal is to find a quick way to check whether a graph has an Euler trail or circuit, even if the graph is quite large.

Preview Activity

Which of the graphs below have an Euler trail? Which have an Euler circuit?



G_1 G_2 G_3

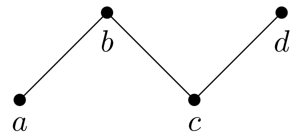


G_4 G_5 G_6

1. Write down the degree sequence of the graphs above.

What might the connection be between the degree sequence and the existence of an Euler trail or circuit?

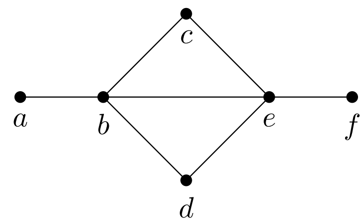
2. One way to write down an Euler trail or circuit is to list the *edges* in order. Each edge will be a pair of vertices, and to indicate what direction we travel over that edge, we can write it as an ordered pair rather than a set. For example, consider this graph:



There are two Euler trails we could write:

$$(a, b), (b, c), (c, d) \qquad \text{or} \qquad (d, c), (c, b), (b, a).$$

(a) Write down an Euler trail for the graph below.



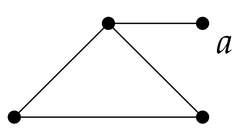
For each vertex, write down its degree and the number of times it appears in your list of edges.

(b) Suppose you have a graph with degree sequence $(4, 2, 2, 2, 2)$ that has an Euler trail. How many times will the name of the degree 4 vertex appear in your list of edges?

(c) Suppose you have a graph with an Euler trail written as a list of edges. What can you conclude about a vertex that appears exactly 3 times in the list? Select all the choices that could be true.

2.4.2 Conditions for Euler Trails

One way to guarantee that a graph does *not* have an Euler circuit is to include a "spike," a vertex of degree 1.



The vertex a has degree 1, and if you try to make an Euler circuit, you see that you will get stuck at the vertex. It is a dead end. That is, unless you start there. But then there is no way to return, so there is no hope of finding an Euler circuit. There is however an Euler trail. It starts at the vertex a, then loops around the triangle. You will end at the vertex of degree 3.

You run into a similar problem whenever you have a vertex of any odd degree.

If you start at such a vertex, you will not be able to end there (after traversing every edge exactly once). After using one edge to leave the starting vertex, you will be left with an even number of edges emanating from the vertex. Half of these could be used for returning to the vertex, the other half for leaving. So you return, then leave. Return, then leave. The only way to use up all the edges is to use the last one by leaving the vertex. On the other hand, if you have a vertex with odd degree at which you do not start a trail, then you will eventually get stuck at that vertex. The trail will use pairs of edges incident to the vertex to arrive and leave again. Eventually all but one of these edges will be used up, leaving only an edge to arrive by, and none to leave again.

What all this says is that if a graph has an Euler trail and two vertices with odd degree, then the Euler trail must start at one of the odd-degree vertices and end at the other. In such a situation, every other vertex *must* have an even degree since we need an equal number of edges to get to those vertices as to leave them. How could we have an Euler circuit? The graph could not have any odd-degree vertex as an Euler trail would have to start there or end there, but not both. Thus for a graph to have an Euler circuit, all vertices must have even degree.

The converse is also true: if all the vertices of a graph have even degree, then the graph has an Euler circuit, and if there are exactly two vertices with odd degree, the graph has an Euler trail. To prove this is a little tricky, but the basic idea is that you will never get stuck because there is an "outbound" edge for every "inbound" edge at every vertex. If you try to make an Euler trail and miss some edges, you will always be able to "splice in" a circuit using the edges you previously missed.

Euler Trails and Circuits.

- A graph has an Euler circuit if and only if the degree of every vertex is even.
- A graph has an Euler trail if and only if there are at most two vertices with odd degree.

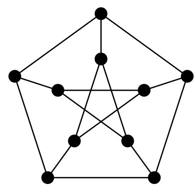
Since the bridges of Königsberg graph has all four vertices with odd degree, there is no Euler trail through the graph. Thus there is no way for the townspeople to cross every bridge exactly once.

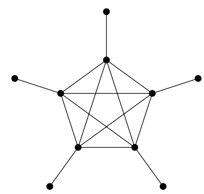
2.4.3 Hamilton Paths

Suppose you wanted to tour Königsberg in such a way that you visit each land mass (the two islands and both banks) exactly once. This can be done. In graph theory terms, we are asking whether there is a path that visits every vertex exactly once. Such a path is called a **Hamilton path** (or **Hamiltonian path**). We could also consider **Hamilton cycles**, which are Hamilton paths that start and stop at the same vertex.

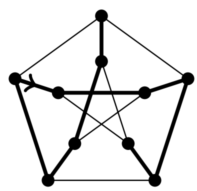
Example 2.4.1

Determine whether the graphs below have a Hamilton path.





Solution. The graph on the left has a Hamilton path (many different ones, actually), as shown here:



The graph on the right does not have a Hamilton path. You would need to visit each of the "outside" vertices, but as soon as you visit one, you get stuck. Note that this graph does not have an Euler trail, although there are graphs with Euler trails but no Hamilton paths.

It appears that finding Hamilton paths would be easier because graphs often have more edges than vertices, so there are fewer requirements to be met. However, nobody knows whether this is true. There is no known simple test for whether a graph has a Hamilton path. For small graphs this is not a problem, but as the size of the graph grows, it gets harder and harder to check whether there is a Hamilton path. In fact, this is an example of a question which as far as we know is too difficult for computers to solve in general, as it is an example of a problem that is NP-complete.

2.4.4 Reading Questions

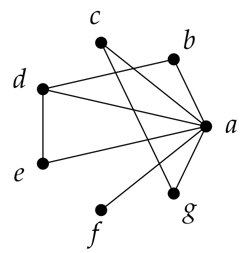
1. Is there a graph that has an Euler circuit but not an Euler trail? Explain your answer.

2. Can a tree have an Euler tour? Can a tree have an Euler circuit? Explain your answers.

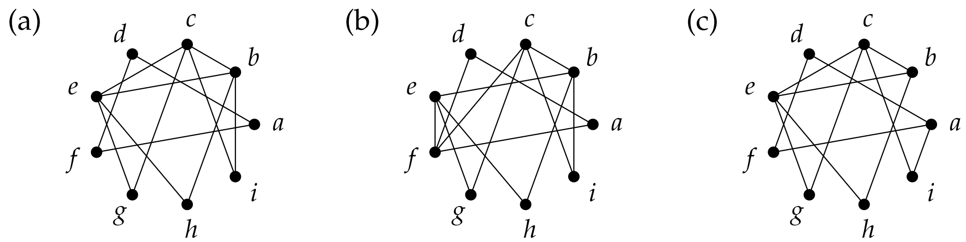
3. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

2.4.5 Practice Problems

1. Find an Euler trail in the following graph.



2. Which of the graphs below have an Euler circuit? If a graph has an Euler circuit, find it.



3. Determine which of the following graphs have an Euler circuit or an Euler trail.

(a)

$$
\begin{aligned}
V &= \{a, b, c, d, e, f\} \\
E &= \{ab, af, bc, cd, ce, cf, de, ef\}
\end{aligned}
$$

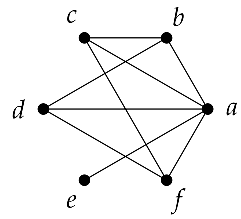
(b) The graph given by the adjacency list:

u	v,w,y,z
v	u,w,x,y
w	u,v,x,z
x	v,w,y,z
y	u,v,x,z
z	u,w,x,y

(c) The graph given by the adjacency matrix:

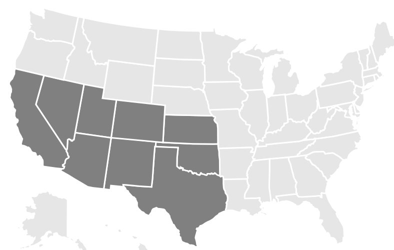
$$
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}
$$

4. Add an edge to the following graph (between a pair of vertices that are not already adjacent) to make it have an Euler trail. Then find the Euler trail.



2.4.6 Additional Exercises

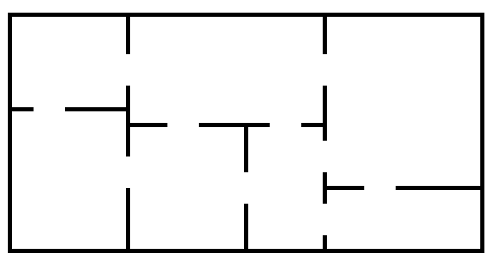
1. You and your friends want to tour the southwest by car. You will visit the nine states below, with the following rather odd rule: You must cross each border between neighboring states exactly once (so, for example, you must cross the Colorado-Utah border exactly once). Can you do it? If so, does it matter where you start your road trip? What fact about graph theory solves this problem?



2. Which of the following graphs contain an Euler trail? Which contain an Euler circuit?

(a) K_4 (b) K_5 (c) $K_{5,7}$ (d) $K_{2,7}$ (e) C_7 (f) P_7

3. Edward A. Mouse has just finished his brand new house. The floor plan is shown below:



(a) Edward wants to give a tour of his new pad to a lady-mouse friend. Is it possible for them to walk through every doorway exactly once? If so, in which rooms must they begin and end the tour? Explain.

(b) Is it possible to tour the house visiting each room exactly once (not necessarily using every doorway)? Explain.

(c) After a few mouse-years, Edward decides to remodel. He would like to add some new doors between the rooms he has. Of course, he cannot add any doors to the exterior of the house. Is it possible for each room to have an odd number of doors? Explain.

4. For which n does the graph K_n contain an Euler circuit? Explain.

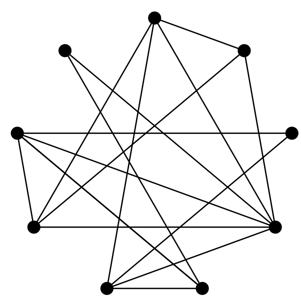
5. For which m and n does the graph $K_{m,n}$ contain an Euler trail? An Euler circuit? Explain.

6. For which n does K_n contain a Hamilton path? A Hamilton cycle? Explain.

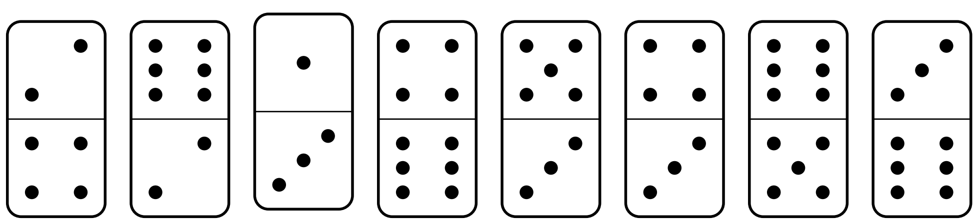
7. For which m and n does the graph $K_{m,n}$ contain a Hamilton path? A Hamilton cycle? Explain.

8. A bridge builder has come to Königsberg and would like to add bridges so that it *is* possible to travel over every bridge exactly once. How many bridges must be built?

9. Below is a graph representing friendships between a group of students (each vertex is a student and each edge is a friendship). Is it possible for the students to sit around a round table in such a way that every student sits between two friends? What does this question have to do with trails?



10. On the table rest 8 dominoes, as shown below. If you were to line them up in a single row, so that any two sides touching had matching numbers, what would the sum of the two end numbers be?

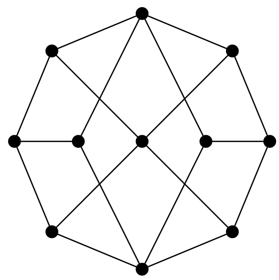


11. Is there anything we can say about whether a graph has a Hamilton path based on the degrees of its vertices?

(a) Suppose a graph has a Hamilton path. What is the maximum number of vertices of degree one the graph can have? Explain why your answer is correct.

(b) Find a graph that does not have a Hamilton path even though no vertex has degree one. Explain why your example works.

12. Consider the following graph:



(a) Find a Hamilton path. Can your path be extended to a Hamilton cycle?

(b) Is the graph bipartite? If so, how many vertices are in each "part"?

(c) Use your answer to part (b) to prove that the graph has no Hamilton cycle.

(d) Suppose you have a bipartite graph G in which one part has at least two more vertices than the other. Prove that G does not have a Hamilton path.

2.5 Coloring

Objectives

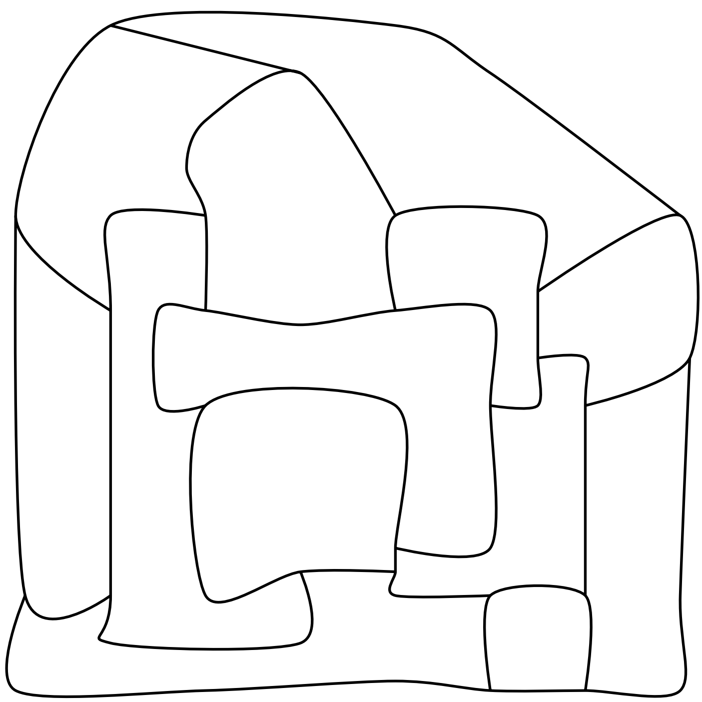
After completing this section, you should be able to do the following.

1. Determine the chromatic number of a graph.
2. Determine the chromatic index of a graph
3. Decide whether using the chromatic number or chromatic index is more appropriate to solve particular problems.

2.5.1 Section Preview

Investigate!

Mapmakers in the fictional land of Euleria have drawn the borders of the various dukedoms of the land. To make the map pretty, they wish to color each region. Adjacent regions must be colored differently, but it is perfectly fine to color two distant regions with the same color. What is the fewest colors the mapmakers can use and still accomplish this task?



Perhaps the most famous graph theory problem is how to color maps.

> Given any map of countries, states, counties, etc., how many colors are needed to color each region on the map so that neighboring regions are colored differently?

Actual map makers usually use around seven colors. For one thing, they require watery regions to be a specific color, and with a lot of colors it is easier to find a permissible coloring. We want to know whether there is a smaller palette that will work for any map.

How is this related to graph theory? Well, if we place a vertex in the center of each region (say in the capital of each state) and then connect two vertices if their states share a border, we get a graph. Coloring regions on the map corresponds to coloring the vertices of the graph. Since neighboring regions cannot be colored the same, our graph cannot have vertices colored the same when those vertices are adjacent.

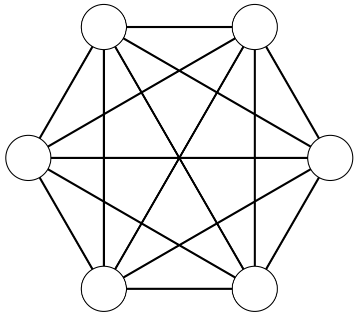
In general, given any graph G, a coloring of the vertices is called (not surprisingly) a **vertex coloring**. If the vertex coloring has the property that adjacent vertices are colored differently, then the coloring is called **proper**. Every graph has a proper vertex coloring. For example, you could color every vertex with a different color. But often you can do better. The smallest number of colors needed to get a proper vertex coloring is called the **chromatic number** of the graph, written $\chi(G)$.

Our goal in this section is to see how graph coloring can be used to solve some problems and to understand some basic properties of graph coloring.

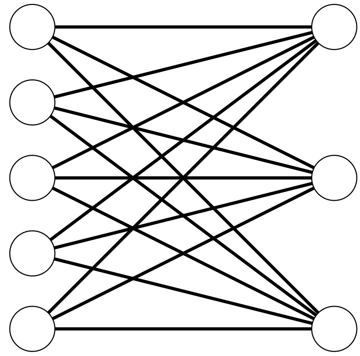
Preview Activity

For each graph below:

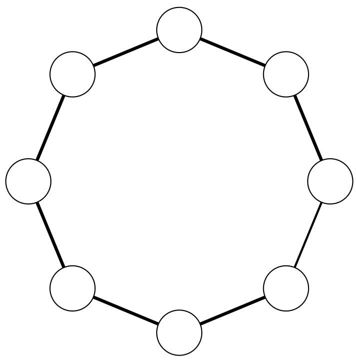
- Find a proper vertex coloring using some number of colors. That is, color vertices using any number of colors but in such a way that no pair of adjacent vertices have the same color.
- Find the *fewest* number of colors you need to properly color the vertices of the graph. This is called the **chromatic number** of the graph. Think about how you know your answer is correct.
- Can you generalize? Can you conclude anything about the chromatic number for particular sorts of graphs?



1.



2.



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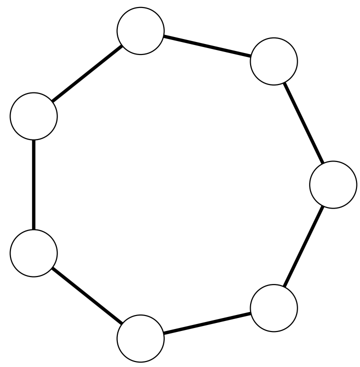

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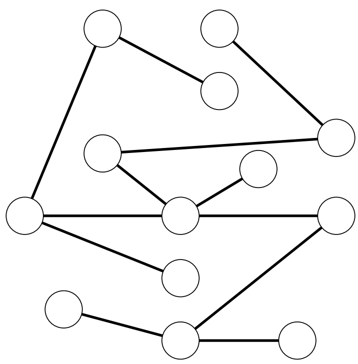
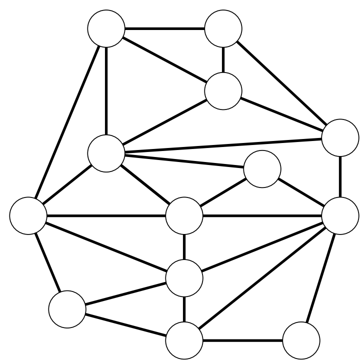


5.



6.

2.5.2 Coloring Vertices

Investigate!

The math department plans to offer 10 classes next semester. Some classes cannot run at the same time (perhaps they are taught by the same professor, or are required for seniors).

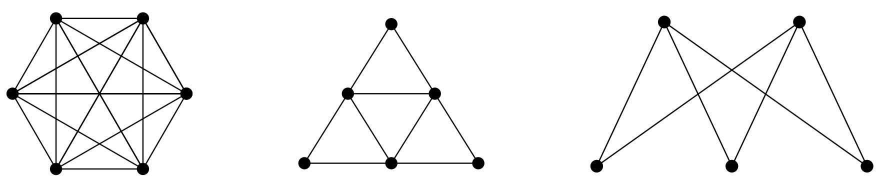
Class:	Conflicts with:
A	D I
B	D I J
C	E F I
D	A B F
E	C H I
F	C D I
G	J
H	E I J
I	A B C E F H
J	B G H

How many different time slots are needed to teach these classes (and which should be taught at the same time)? More importantly, how could we use graph coloring to answer this question?

The best way to get a feel for the chromatic number is to actually try to color some graphs.

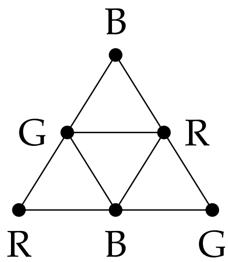
Example 2.5.1

Find the chromatic number of the graphs below.



Solution. The graph on the left is K_6. The only way to properly color the graph is to give every vertex a different color (since every vertex is adjacent to every other vertex). Thus the chromatic number is 6.

The middle graph can be properly colored with just 3 colors (Red, Blue, and Green). For example:



There is no way to color it with just two colors, since there are three vertices mutually adjacent (i.e., a triangle). Thus the chromatic number is 3.

The graph on the right is just $K_{2,3}$. As with all bipartite graphs, this graph has chromatic number 2: color the vertices on the top row red and the vertices on the bottom row blue.

It appears that there is no limit to how large chromatic numbers can get. It should not come as a surprise that K_n has chromatic number n. So how could there possibly be an answer to the original map coloring question? If the chromatic number of a graph can be arbitrarily large, then it seems like there would be no upper bound to the number of colors needed for any map. But there is.

The key observation is that while it is true that for any number n there is a graph with chromatic number n, only some graphs arrive as representations of maps. If you convert a map to a graph, the edges between vertices correspond to borders between the countries. So you should be able to connect vertices in such a way that the edges do not cross. In other words, the graphs representing maps are all *planar*!

So the question is, what is the largest chromatic number of any planar graph? The answer is the best-known theorem of graph theory:

Theorem 2.5.2 The Four Color Theorem.

If G is a planar graph, then the chromatic number of G is less than or equal to 4. Thus any map can be properly colored with 4 or fewer colors.

We will not prove this theorem. Really. Even though the theorem is easy to state and understand, the proof is not. In fact, there is currently no “easy” known proof of the theorem. The current best proof still requires powerful computers to check an *unavoidable set* of 633 *reducible configurations*. The idea is that every graph must contain one of these reducible configurations (this fact also needs to be checked by a computer) and that reducible configurations can, in fact, be colored in 4 or fewer colors.

Cartography is certainly not the only application of graph coloring. There are plenty of situations in which you might wish to partition the objects in question so that related objects are not in the same set. For example, you might wish to store chemicals safely. To avoid explosions, certain pairs of chemicals should not be stored in the same room. By coloring a graph (with vertices representing chemicals and edges representing potential negative interactions), you can determine the smallest number of rooms needed to store the chemicals.

Here is a further example:

Example 2.5.3

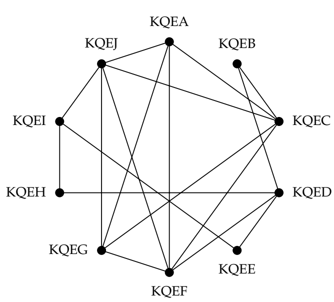
Radio stations broadcast their signal at certain frequencies. However, there are a limited number of frequencies to choose from, so nationwide, many stations use the same frequency. This works because the stations are far enough apart that their signals will not interfere; no one radio could pick them up at the same time.

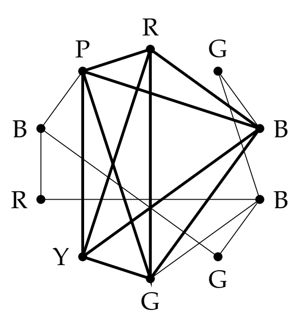
Suppose 10 new radio stations are to be set up in a currently unpopulated (by radio stations) region. The radio stations that are close enough to each other to cause interference are recorded in the table below. What is the fewest number of frequencies the stations could use?

	KQEA	KQEB	KQEC	KQED	KQEE	KQEF	KQEG	KQEH	KQEI	KQEJ
KQEA			x			x	x			x
KQEB			x	x						
KQEC	x	x				x	x			x
KQED		x			x	x		x		
KQEE				x					x	
KQEF	x		x	x			x			x
KQEG	x		x			x				x
KQEH				x					x	
KQEI					x			x		x
KQEJ	x		x			x	x		x	

Solution. Represent the problem as a graph with vertices as the stations and edges when two stations are close enough to cause interference. We are looking for the chromatic number of the graph. Vertices that are colored identically represent stations that can have the same frequency.

This graph has chromatic number 5. A proper 5-coloring is shown on the right. Notice that the graph contains a copy of the complete graph K_5, so no fewer than 5 colors can be used.





In the example above, the chromatic number was 5, but this is not a counterexample to the Four Color Theorem 2.5.2, since the graph representing the radio stations is not planar. It would be nice to have some quick way to find the chromatic number of a (possibly non-planar) graph. It turns out nobody knows whether an efficient algorithm for computing chromatic numbers exists.

While we might not be able to find the exact chromatic number of a graph easily, we can often give a reasonable range for the chromatic number. In other words, we can give upper and lower bounds for the chromatic number.

This is not very difficult: for every graph G, the chromatic number of G is at least 1 and at most the number of vertices of G.

What? You want *better* bounds on the chromatic number? Well, you are in luck.

A **clique** in a graph is a set of vertices all of which are pairwise adjacent. In other words, a clique of size n is just a copy of the complete graph K_n. We define the **clique number** of a graph to be the largest n for which the graph contains a clique of size n. Any clique of size n cannot be colored with fewer than n colors, so we have a nice lower bound:

Theorem 2.5.4

The chromatic number of a graph G is at least the clique number of G.

There are times when the chromatic number of G is *equal* to the clique number. These graphs have a special name; they are called **perfect**. If you know that a graph is perfect, then finding the chromatic number is simply a matter of searching for the largest clique.[10] However, not all graphs are perfect.

For an upper bound, we can improve on "the number of vertices" by looking at the degrees of vertices. Let $\Delta(G)$ be the largest degree of any vertex in the graph G. One reasonable guess for an upper bound on the chromatic number is $\chi(G) \leq \Delta(G) + 1$. Why is this reasonable? Starting with any vertex, it together with all of its neighbors can always be colored in $\Delta(G) + 1$ colors, since at most we are talking about $\Delta(G) + 1$ vertices in this set. Now fan out! At any point, if you consider an already colored vertex, some of its neighbors might be colored, some might not. But no matter what, that vertex and its neighbors could all be colored distinctly, since there are at most $\Delta(G)$ neighbors, plus the one vertex being considered.

In fact, there are examples of graphs for which $\chi(G) = \Delta(G) + 1$. For any n, the complete graph K_n has chromatic number n, but $\Delta(K_n) = n - 1$ (since every vertex is adjacent to every *other* vertex). Additionally, any *odd* cycle will have chromatic number 3, but the degree of every vertex in a cycle is 2. It turns out that these are the only two types of examples where we get equality, a result known as Brooks' Theorem.

[10]There are special classes of graphs that can be proved to be perfect. One such class is the set of **chordal** graphs, which have the property that every cycle in the graph contains a **chord**—an edge between two vertices in the cycle which are not adjacent in the cycle.

Theorem 2.5.5 Brooks' Theorem.

Any graph G satisfies $\chi(G) \leq \Delta(G)$, unless G is a complete graph or an odd cycle, in which case $\chi(G) = \Delta(G) + 1$.

The proof of this theorem is *just* complicated enough that we will not present it here (although you are asked to prove a special case in the exercises). The adventurous reader is encouraged to find a book on graph theory to find suggestions for how to prove the theorem.

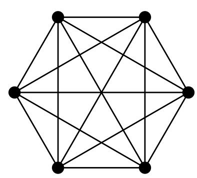
2.5.3 Coloring Edges

The chromatic number of a graph tells us about coloring vertices, but we could also ask about coloring edges. Just like with vertex coloring, we might insist that adjacent edges must be colored differently. Here, we are thinking of two edges as being adjacent if they are incident to the same vertex. The least number of colors required to properly color the edges of a graph G is called the **chromatic index** of G, written $\chi'(G)$.

Example 2.5.6

Six friends decide to spend the afternoon playing chess. Everyone will play everyone else once. They have plenty of chess sets, but nobody wants to play more than one game at a time. Games will last an hour (thanks to their handy chess clocks). How many hours will the tournament last?

Solution. Represent each player with a vertex and put an edge between two players if they play each other. In this case, we get the graph K_6:



We must color the edges; each color represents a different hour. Since different edges incident to the same vertex will be colored differently, no player will be playing two different games (edges) at the same time. Thus we need to know the chromatic index of K_6.

Notice that for sure $\chi'(K_6) \geq 5$, since there is a vertex of degree 5. It turns out, 5 colors is enough (go find such a coloring). Therefore the friends will play for 5 hours.

Interestingly, if one of the friends in the above example left, the remaining 5 chessletes would still need 5 hours: the chromatic index of K_5 is also 5.

In general, what can we say about the chromatic index? Certainly $\chi'(G) \geq \Delta(G)$. But how much higher could it be? Only a little higher.

Theorem 2.5.7 Vizing's Theorem.

For any graph G, the chromatic index $\chi'(G)$ is either $\Delta(G)$ or $\Delta(G)+1$.

At first, this theorem makes it seem like the chromatic index might not be very interesting. However, deciding which case a graph is in is not always easy. Graphs for which $\chi'(G) = \Delta(G)$ are called *class 1*, while the others are called *class 2*. Bipartite graphs always satisfy $\chi'(G) = \Delta(G)$, so are class 1 (this was proved by König in 1916, decades before Vizing proved his theorem in 1964). In 1965 Vizing proved that all planar graphs with $\Delta(G) \geq 8$ are of class 1, but this does not hold for all planar graphs with $2 \leq \Delta(G) \leq 5$. Vizing conjectured that all planar graphs with $\Delta(G) = 6$ or $\Delta(G) = 7$ are class 1; the $\Delta(G) = 7$ case was proved in 2001 by Sanders and Zhao; the $\Delta(G) = 6$ case is still open.

Ramsey Theory. There is another interesting way we might consider coloring edges, quite different from what we have discussed so far. What if we colored every edge of a graph either red or blue? Can we do so without, say, creating a *monochromatic* triangle (i.e., an all red or all blue triangle)? Certainly, for some graphs the answer is yes. Try doing so for K_4. What about K_5? K_6? How far can we go?

The problem above is not too difficult and is a fun exercise. We could extend the question in a variety of ways. What if we had three colors? What if we were trying to avoid other graphs? Surprisingly, very little is known about these questions. For example, we know that you need to go up to K_{17} in order to force a monochromatic triangle using three colors, but nobody knows how big you need to go with more colors. Similarly, we know that using two colors, K_{18} is the smallest graph that forces a monochromatic copy of K_4, but the best we have to force a monochromatic K_5 is a range, somewhere from K_{43} to K_{49}. If you are interested in these sorts of questions, this area of graph theory is called Ramsey theory. Check it out.

2.5.4 Reading Questions

1. True or false: if a graph contains a vertex of degree 5, then the chromatic number of the graph is at least 5. Explain.

2. In your own words, explain the difference between chromatic number and chromatic index.

3. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

2.5.5 Practice Problems

1. Each of the following problems can be solved by finding either the chromatic number of a graph or the chromatic index of a graph.

For each problem, say whether you should find a proper coloring of the vertices or of the edges of the graph to solve the problem. Note: you likely don't have enough information to actually solve the problem, but this is okay. Just say in principle whether this is an edge-coloring or vertex-coloring application.

(a) Professor Snape stores potion ingredients in as few cabinets as possible, but some ingredients can't be stored in the same cabinet because they could interact dangerously. How many cabinets are required?

(b) At a speed dating event, everyone must spend 5 minutes talking to another person before moving on to the next. How long must the event be?

(c) Five students will interview with four companies for internships at a job fair. How many time slots are needed for interviews?

(d) The math department wants to schedule common, day-long midterm exams over Spring Break, but obviously a student in two classes has to have their exams for those classes on different days (and this is the only problem with this plan). How many days are needed for exams?

2. What is the chromatic number of each graph?

P_{13} ____________
C_4 ____________
C_{13} ____________
$K_{6,8}$ ____________
K_7 ____________

3. What is the chromatic ***index*** of each graph?

P_{14} ____________
C_{12} ____________
C_5 ____________
$K_{4,9}$ ____________
K_8 ____________

4. The following statements are about the chromatic number $\chi(G)$ and the chromatic index $\chi'(G)$ of graphs. We use $\Delta(G)$ for the maximum degree of G.
Are the following statements true or false?

(a) If a graph contains a vertex of degree 6, then the chromatic index of the graph is at least 6.

(b) $\chi'(G) \geq \Delta(G)$.

(c) The chromatic index of any planar graph is at most 4.

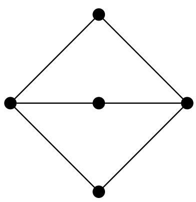
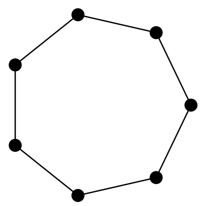
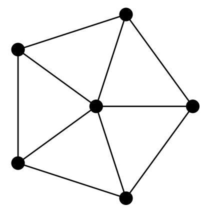
(d) For any cycle, the chromatic index is equal to the chromatic number.

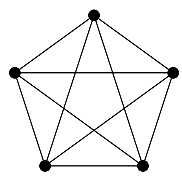
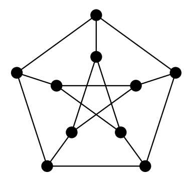
2.5.6 Additional Exercises

1. What is the smallest number of colors you need to properly color the vertices of $K_{4,5}$? That is, find the chromatic number of the graph.

2. Draw a graph with chromatic number 6 (i.e., which requires 6 colors to properly color the vertices). Could your graph be planar? Explain.

3. Find the chromatic number of each of the following graphs.

4. A group of 10 friends decides to head up to a cabin in the woods (where nothing could possibly go wrong). Unfortunately, a number of these friends have dated each other in the past, and things are still a little awkward. To get to the cabin, they need to divide up into some number of cars, and no two people who dated should be in the same car.

(a) What is the smallest number of cars you need if all the relationships were strictly heterosexual? Represent an example of such a situation with a graph. What kind of graph do you get?

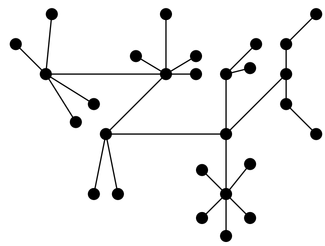
(b) Because a number of these friends dated there are also conflicts between friends of the same gender, listed below. Now what is the smallest number of conflict-free cars they could take to the cabin?

Friend	A	B	C	D	E	F	G	H	I	J
Conflicts	CFJ	J	AEF	H	CFG	ACEGI	EFI	D	AFG	B

5. What is the smallest number of colors that can be used to color the vertices of a cube so that no two adjacent vertices are colored identically?

6. Prove the chromatic number of any tree is two. Recall, a tree is a connected graph with no cycles.

(a) Consider the tree below. If you color the left-most vertex red, what should its neighbor be colored? What should the neighbors of the neighbor be colored? Describe a procedure to color the tree below using two colors.



(b) If you used the same procedure to color a cycle, will you always be able to color it with two colors?

(c) Prove that your procedure from part (a) always works for any tree.

(d) Now, give a different proof, this time using induction, that every tree has chromatic number 2.

7. The two problems below can be solved using graph coloring. For each problem, represent the situation with a graph, say whether you should be coloring vertices or edges and why, and use the coloring to solve the problem.

(a) Your Quidditch league has 5 teams. You will play a tournament next week in which every team will play every other team once. Each team can play at most one match each day, but there is plenty of time in the day for multiple matches. What is the fewest number of days over which the tournament can take place?

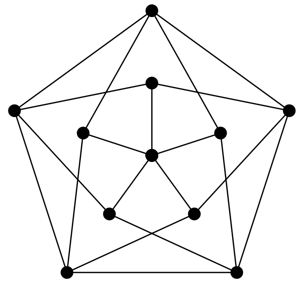
(b) Ten members of Math Club are driving to a math conference in a neighboring state. However, some of these students have dated in the past, and things are still a little awkward. Each student lists which other students they refuse to share a car with; these conflicts are recorded in the table below. What is the fewest number of cars the club needs to make the trip? Do not worry about running out of seats, just avoid the conflicts.

Student:	A	B	C	D	E	F	G	H	I	J
Conflicts:	BEJ	ADG	HJ	BF	AI	DJ	B	CI	EHJ	ACFI

8. Prove the 6-color theorem: every planar graph has chromatic number 6 or less. Do not assume the 4-color theorem (whose proof is MUCH harder), but you may assume the fact that every planar graph contains a vertex of degree at most 5.

9. Not all graphs are perfect. Give an example of a graph with chromatic number 4 that does not contain a copy of K_4. That is, there should be no 4 vertices all pairwise adjacent.

10. Find the chromatic number of the graph below and prove you are correct.



11. Prove that any connected graph G which contains at least one vertex of degree less than $\Delta(G)$ (the maximal degree of all vertices in G) has chromatic number at most $\Delta(G)$.

12. You have a set of magnetic alphabet letters (one of each of the 26 letters in the alphabet) that you need to put into boxes. For obvious reasons, you don't want to put two consecutive letters in the same box. What is the fewest number of boxes you need (assuming the boxes are able to hold as many letters as they need to)?

13. Suppose you colored the edges of a graph either red or blue (not requiring that adjacent edges be colored differently). What must be true of the graph to guarantee some vertex is incident to three edges of the same color? Prove your answer.

14. Prove that if you color every edge of K_6 either red or blue, you are guaranteed a monochromatic triangle (that is, an all-red or an all-blue triangle).

2.6 Relations and Graphs

Objectives

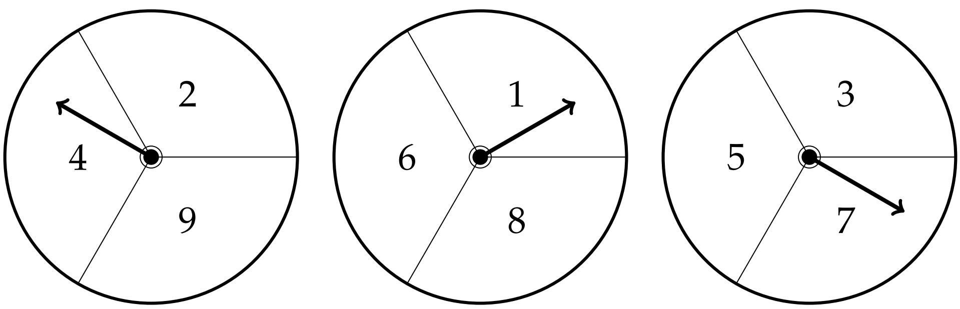
After completing this section, you should be able to do the following.

1. Explain the relationship between a graph and a relation.
2. Determine whether a relation is reflexive, symmetric, or transitive.
3. Use an equivalence relation to partition a set and use a partition to define an equivalence relation.

2.6.1 Section Preview

Investigate!

Consider the three spinners below.



A B C

If you and a friend each pick a different spinner and spin them, we can consider the nine possible outcomes. For example, between spinners A and B, the outcomes are

$$(2,1),(2,6),(2,8),(4,1),(4,6),(4,8),(9,1),(9,6),(9,8).$$

This suggests that spinner A will win five out of nine times.

Compare the other combinations of spinners. Which spinner is best?

In this section, we will explore a generalization of a graph, called a **relation**. We will see how a relation can be represented by a graph and how a graph can be used to represent a relation. We will also consider some properties that a relation might have, and how these properties can be used to classify relations into different types.

Preview Activity

In a given month, some days are more similar than others. For example, the 3rd of the month is more like the 24th than it is like the 15th. What does this possibly mean? We will explore two ways in which this is true.

1. We will say that two numbers between 1 and 31 are related, written $a \sim b$ if their difference is a multiple of 7. So for example, $3 \sim 24$, since $24 - 3 = 3 \cdot 7$, but $3 \nsim 15$ since $15 - 3 = 12$ which is not a multiple of 7.

(a) Which of the following are true? That is, which of the following pairs of numbers are related as we have defined above?

- ☐ $4 \sim 14$
- ☐ $7 \sim 14$
- ☐ $10 \sim 17$
- ☐ $17 \sim 24$
- ☐ $10 \sim 24$
- ☐ $20 \sim 10$
- ☐ $31 \sim 3$
- ☐ $25 \sim 25$

(b) Which of the following statements are true about the $\sim$ relation in this case?

- ☐ $a \sim a$ for every number a
- ☐ $a \nsim a$ for any number a
- ☐ For any numbers a and b, if $a \sim b$, then $b \sim a$
- ☐ For any numbers a and b, if $a \sim b$ and $b \sim a$, then $a = b$
- ☐ For any numbers a and b, if $a \sim b$ and $b \sim c$, then $a \sim c$

(c) We will write $[a]$ for the set of all numbers related to a. For example, $[7] = \{7, 14, 21, 28\}$. Find each of the following:

- $[1] =$ ______________________;
- $[2] =$ ______________________;
- $[3] =$ ______________________;
- $[4] =$ ______________________;
- $[5] =$ ______________________;
- $[6] =$ ______________________.

Are there any numbers that are in more than one of the sets $[a]$ above?

2. When you divide a multiple of 7 by 7, you get a whole number. If you divide another number by 7, you can either write the result as a decimal or as a quotient and a remainder. For example, $19 \div 7$ is 2 with a remainder of 5, since we can write $19 = 2 \cdot 7 + 5$. The remainder is also called the modulus. When programming in python (and many other languages), the modulus operator is written as `%`. For example, `19 % 7` is `5`. Try this out for a few numbers.

(a) Find all the numbers a between 1 and 31 that are $5 \mod 7$. That is, find all a such that `a % 7 = 5`.

(b) Since the modulus is a function, each number has exactly one modulus when divided by 7. This means that the moduli partition the numbers from 1 to 31: every number belongs to exactly one of the sets of numbers with a particular modulus. We have already found the set for modulus 5. Find the other sets.

- `a % 7 = 0`: ________________;
- `a % 7 = 1`: ________________;
- `a % 7 = 2`: ________________;
- `a % 7 = 3`: ________________;
- `a % 7 = 4`: ________________;
- `a % 7 = 6`: ________________.

(c) We can use the moduli to define a relation on the numbers from 1 to 31. We will say that $a \sim b$ if `a % 7 = b % 7`. In other words, two numbers are related if they belong to the same set of the partition we found above.

Which of the following are true? That is, which of the following pairs of numbers are related by this modulus relation?

- ☐ $4 \sim 14$
- ☐ $7 \sim 14$
- ☐ $10 \sim 17$
- ☐ $17 \sim 24$
- ☐ $10 \sim 24$
- ☐ $20 \sim 10$
- ☐ $31 \sim 3$
- ☐ $25 \sim 25$

2.6.2 Relations Generally

A graph is a way to represent some ways that different objects are related. We have seen how to use graphs to represent which people are friends, or which classes have time conflicts, or which radio stations are too close to have the same frequency. Not all ways in which things can be related can be represented by a graph, however. In this section, we will consider the more general concept of a relation and see how those might be related to graphs.

Consider the example of the relation between students and classes that holds when a student is in that class (in a particular semester). This is a relation between two different sets (the students and the classes). If we used a graph to illustrate this relation, the graph would be *bipartite*, since two students are never related to each other, and two classes are never related to each other.

A graph is really a set of vertices and a set of edges: $G = (V, E)$; each element of the set E is a two-element subset of V. If we want to draw attention to the bipartiteness of the graph, we can split up V into its two sets and write $G = ((A, B), E)$. In this notation, for the graph to be bipartite, we want each edge to be a pair (a, b) where a is an element of A and b is an element of B. In other words, each edge is an element of the **Cartesian product** of A and B, written,

$$A \times B = \{(a, b) : a \in A,\ b \in B\}.$$

(Another way to say this is that $A \times B$ is "the set of all ordered pairs of elements from A and B.")

Note 2.6.1 There is one subtlety here we should point out: the bipartite graph we have described really has *directed edges* from A to B since we are considering *ordered* pairs. Our definition of a graph has edges as two-element *subsets* of vertices, and subsets are not ordered. As long as A and B are disjoint sets, there is no confusion here, but we see relations in which A and B share some elements, but we still care about the order. More on that soon.

This example exactly illustrates what a general binary relation is. Here is the careful definition.

Definition 2.6.2

A **binary relation** is a set of ordered pairs. We say the binary relation is a **relation on sets** A **and** B provided the ordered pairs are a subset of $A \times B$. We say a binary relation is a **relation on a set** A provided the ordered pairs are a subset of $A \times A$.

Note that $A \times A$ is just the set of all ordered pairs where both coordinates are elements from A.

Example 2.6.3

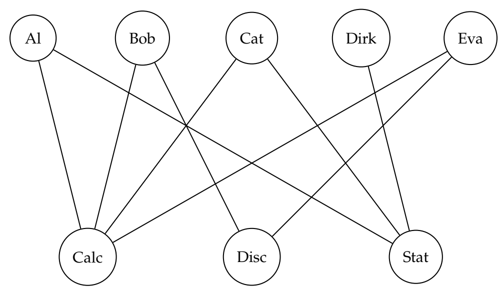
Consider a set A of students and a set B of classes. Say $A = \{\text{Al, Bob, Cat, Dirk, Eva}\}$ and $B = \{\text{Calculus, Discrete, Statistics}\}$. Everyone except Dirk is in Calculus, Bob and Eva are in Discrete, and Al, Cat, and Dirk are in Statistics.

We can define a relation T of "is taking" A and B that holds of a student and class precisely if that student is taking that class. Write this relation as a subset of $A \times B$ and draw its bipartite graph.

Solution. To write the relation precisely, we just give the set of ordered pairs:

$$\begin{aligned} T = \{&(\text{Al}, \text{Calculus}), (\text{Al}, \text{Statistics}), \\ &(\text{Bob}, \text{Calculus}), (\text{Bob}, \text{Discrete}) \\ &(\text{Cat}, \text{Calculus}), (\text{Cat}, \text{Statistics}) \\ &(\text{Dirk}, \text{Statistics}) \\ &(\text{Eva}, \text{Calculus}), (\text{Eva}, \text{Discrete})\} \end{aligned}$$

We can draw this relation as a bipartite graph:



Example 2.6.4

Consider the relation M for "is a multiple of" on the set $A = \{1, 2, 3, 4, 5, 6\}$. Write this as a set of ordered pairs. Does this relation create a graph?

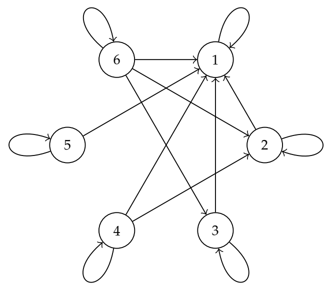
Solution. First, let's think about which elements should be related and which should not. We know that 6 is a multiple of 2, so the relation is true of the pair $(6, 2)$, but 6 is not a multiple of 4, so the pair $(6, 4)$ does not satisfy the relation. More precisely, we say that $(6, 2) \in M$ but $(6, 4) \notin M$.

Let's list all the elements of M:

$$\begin{aligned} M = \{&(1, 1), (2, 1), (2, 2), (3, 1), (3, 3), (4, 1), (4, 2), (4, 4), \\ &(5, 1), (5, 5), (6, 1), (6, 2)(6, 3), (6, 6)\}. \end{aligned}$$

This relation is not a graph for two reasons: first, elements are related to themselves, and second, the order of elements in the relation is not symmetric (6 is related to 2, but 2 is not related to 6, for example).

We can, however, still draw something like a graph to illustrate this relation: Since the direction of the relation matters, we will have *directed* edges. This alone would create a **directed graph**. Since vertices can have edges going to themselves, we would call the structure a **multigraph**, so the relation can be thought of as a **directed multigraph**.



A binary relation R on sets A and B can always be "turned around" to give a relation on sets B and A. That is, R says how things in A are related to things in B; those things in B are related to things in A, just in an *inverse* (backward) way. We will call this new relation the **inverse** of R.

Definition 2.6.5

Given a binary relation R, define R^{-1} to be the **inverse** of R as the set

$$R^{-1} = \{(b, a) : (a, b) \in R\}.$$

The relation T from Example 2.6.3 said that a given student was in a particular class. The inverse relation T^{-1} says that a given class has a particular student in it. For example, (Calculus, Al) is an element of T^{-1}. Graphically, there won't be any difference in the picture, although we could put the set of classes on top and students on bottom.

The relation M from Example 2.6.4 gave us that, for example, 6 is a multiple of 2, since $(6, 2) \in M$. For the inverse, we have $(2, 6) \in M^{-1}$, which means that 2 is a factor of 6. For this example, we have represented a relation as a directed multigraph. The graph of the inverse will look exactly the same, but all the arrows will point in the opposite direction.

Another way to create a new relation is to combine two relations. Suppose in addition to the "is taking" relation from Example 2.6.3, we define a relation "is taught by" that matches up each course with its instructor. Perhaps Professor X teaches Calculus. In that case, since Al is taking Calculus, and Calculus is taught by Professor X, we can conclude that Al is taking a class with Professor X.

Definition 2.6.6

Let R be a relation from set A to B and S be a relation from B to C. The **composition** of R and S is

$$S \circ R = \{(a, c) \in A \times C : (a, b) \in R \text{ and } (b, c) \in S \text{ for some } b \in B\}$$

Note the order in which we wrote the two relations: it's $S \circ R$, not $R \circ S$. The reason we do this (which might seem backward) is that it agrees with the usual notation for composition of *functions*. In fact, functions are nothing but a specific type of relation!

Example 2.6.7

Write out the relation $P \circ T$, composing the relations of Example 2.6.3 and $P = \{(\text{Calculus}, \text{Prof X}), (\text{Discrete}, \text{Prof L}), (\text{Statistics}, \text{Prof X}), (\text{Statistics}, \text{Prof S})\}$. Note that here we are saying the statistic course is co-taught by professors X and S.

Solution. The relation we are looking for is a relation between students and professors. We can start with each element of T and *extend* it by following the course to the professor(s) via P. So for example, start with Cat, and notice that $(\text{Cat}, \text{Calculus}) \in T$. Now look at what Calculus is related to via P: $(\text{Calculus}, \text{Prof X}) \in P$. Thus we can push these together to conclude that $(\text{Cat}, \text{Prof X}) \in P \circ T$.

An alternative approach would be to simply consider which pairs of students and professors are linked by a common class. Should the pair $(\text{Al}, \text{Prof L})$ be an element of the composition? No, because there is no class that is both taken by Al and taught by Professor L. On the other hand, we can conclude the $(\text{Al}, \text{Prof X}) \in P \circ T$ since there is a common course. In fact, the common course could be Calculus or Statistics (it doesn't matter how many common middle steps there are, as long as there is at least one).

Here is the complete relation:

$$\begin{aligned} P \circ T = \{&(\text{Al}, \text{Prof X}), (\text{Al}, \text{Prof S}), (\text{Bob}, \text{Prof X}), (\text{Bob}, \text{Prof L}),\\ &(\text{Cat}, \text{Prof X}), (\text{Cat}, \text{Prof S}), (\text{Dirk}, \text{Prof X}), (\text{Dirk}, \text{Prof S}),\\ &(\text{Eva}, \text{Prof X}), (\text{Eva}, \text{Prof L})\} \end{aligned}$$

Something interesting often happens when you compose a relation with its inverse. You might have seen something like this for *functions* in calculus or algebra: $f(x) = e^x$ has an inverse function $f^{-1}(x) = \ln(x)$, and we know that $f(f^{-1}(x)) = e^{\ln(x)} = x$. But functions are relations in which every "input" has exactly one "output", so perhaps it is not surprising that composing with an inverse just gives you the identity function. When inputs can have multiple outputs, and outputs can have multiple inputs, then we get something more.

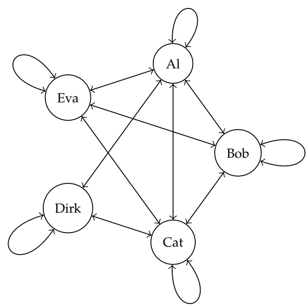
Example 2.6.8

Describe the relation $T^{-1} \circ T$ (using our familiar relation T from Example 2.6.3). What does this tell us about students and classes?

Solution. By following the relation, we could start by saying Al is in Calculus, and Calculus is taken by Al, so Al is related to Al. But also , Calculus is taken by Bob, so now Al is also related to Bob. Is Bob related to Al as well? Yes, since Bob is taking Calculus and Calculus is taken by Al.

The composition is telling us which pairs of students have at least one class in common. That's almost all pairs of students, except that Dirk is not related to Bob or Eva, since they don't take Statistics, and that is the only class he takes.

Instead of writing out the relation (which will almost be all of $A \times A$), here is the graph representation.



While we drew a directed multigraph here, the arrows going both ways mean that we really could have drawn just a multigraph.

2.6.3 Properties of Relations

From this point on, we will just consider relations on a single set (so from a set to itself). To help understand these relations, let's consider some basic properties a relation might or might not have.

Definition 2.6.9 Reflexive, Symmetric, and Transitive.

Ler R be a relation on the set A. We say,

- R is **reflexive** provided $(a, a) \in A$ for all $a \in A$.
- R is **symmetric** provided, for all $a, b \in A$, if $(a, b) \in R$ then $(b, a) \in R$.
- R is **transitive** provided, for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

Let's examine each of these properties carefully.

It will be helpful to consider a few standard examples of relations on sets as we go. Most relations we consider here will be written using *infix* notation, just meaning that we put the relation symbol between the two things it is relating. For example, the *less than* relation is almost always written as $2 < 6$ rather than writing $(2, 6) \in <$.

Example 2.6.10 Reflexive and non-reflexive relations.

A relation is reflexive when every element is related to itself. The following are reflexive relations:

- The "less-than-or-equal-to" relation on any set of numbers. Is it the case that $3 \leq 3$? More importantly, is every number no greater than itself? Since the answer is yes, this relation is reflexive.
- The "within 3" relation, that holds of two numbers a and b provided $|a - b| \leq 3$. To prove that this is reflexive, we simply note that $|a - a| = 0 \leq 3$.
- The "is a multiple of" relation from Example 2.6.4. Note that the directed multigraph for this relation had loops at every vertex.

However, these relations are not reflexive:

- The "sums to zero" relation, that holds on numbers a and b if $a + b = 0$. Note that while $0 + 0 = 0$, so $(0, 0)$ is an element of the relation, every other number is not related to itself.
- Any relation that is described by a graph. Remember, graphs cannot have edges looping back to a single vertex, so the edge relation on a graph is not reflexive. (A multigraph could be reflexive or not).

If no element is related to itself (such as in the edge relation for a graph), then we call the relation **irreflexive**. Of course, some relations are neither reflexive nor irreflexive.

Checking that a relation is reflexive is relatively easy. The other two properties are phrased as implications, which makes them a little more complex.

Example 2.6.11 Symmetric and non-symmetric relations.

Essentially, symmetric relations are the ones that work "both ways." More precisely, if a is related to b, then b is also related to a.

The following relations are symmetric.

- The "within 3" relation: if $|a - b| \leq 3$ then $|b - a| \leq 3$.
- The "sums to zero" relation: if $a + b = 0$, then certainly $b + a = 0$.
- For any graph, the edge relation is symmetric. Of course, for *directed* graphs this is usually not true.

On the other hand, these relations are not symmetric.

- $\leq$ is not symmetric. All we need to do to prove that a relation is not symmetric is to find some a and b such that $a \leq b$ but $b \nleq a$. Well, $3 \leq 4$, but $4 \nleq 3$. QED.

- The "is a multiple of" relation is not symmetric. 6 is a multiple of 2 but 2 is not a multiple of 6.

Relations that are not symmetric could in fact be **antisymmetric**, meaning the *only* elements for which both (a, b) and (b, a) are in the relation is when $a = b$. Note that there are relations that are neither symmetric nor antisymmetric.

Using the language of inverse relations, a relation is symmetric if and only if the relation is equal to its inverse.

Example 2.6.12 Transitive and non-transitive relations.

If a is related to b, and b is related to c, does that mean a is related to c? If this is true no matter what a, b, and c are, then we say the relation is transitive.

Here are some transitive relations.

- $\leq$. Suppose $a \leq b$ and $b \leq c$. Then clearly $a \leq c$.
- The "is a multiple of" relation. This is a good one to write a proof for: Suppose a is a multiple of b and that b is a multiple of c. Then $a = bk$ for some integer k and $b = cj$ for some integer j. By substitution, $a = cjk$, so a is a multiple of c.

However, the following relations are not transitive.

- The "within 3" relation is not transitive. All we need to do is find three numbers that fail to meet the condition. How about 1, 3, and 5? Here 1 is within 3 of 3, and 3 is within 3 of 5, but $|1 - 5| = 4$ so the relation does not hold of $(1, 5)$.
- The "sums to zero" relation is not transitive. Notice that we never claimed that a, b, and c need to be different numbers. Let $a = 5$, $b = -5$, and $c = 5$. Then $a + b = 0$ and $b + c = 0$, so the relation holds of (a, b) and (b, c). But $a + c = 10$ so the relation does not hold of (a, c).

The edge relation for a graph might or might not be transitive. What would a graph look like if its edge relation was transitive?

2.6.4 Equivalence Relations

Now we will do something very typical for mathematics: We will look at our most common types of relations, consider what properties these have, and then classify other relations that also have these properties as a specific class of relations.

The relation we are all most familiar with is **equality**. Which properties of relations does the equality relation possess? Certainly everything is equal to itself, so equality is *reflexive*. If $a = b$, then $b = a$, so equality is *symmetric*. If $a = b$ and $b = c$, then $a = c$, so equality is *transitive*.

What other relations are *reflexive*, *symmetric*, and *transitive*? Exactly those relations that behave like equality. We call such relations **equivalence relations**.

Definition 2.6.13 Equivalence Relation.

A relation that is reflexive, symmetric, and transitive is called an **equivalence relation**.

Remark 2.6.14 Another example of a type of relation that is modeled after a classic relation is a **partial order**. This is a relation that is reflexive, *antisymmetric*, and transitive, just like less-than-or-equal-to. Perhaps you noticed already that the subset relation, written $\subseteq$, feels a lot like $\leq$. This is because $\subseteq$ is also a partial order. So is the "is a multiple of" relation we saw above.

There are lots of interesting things we can say about partial orders and the sets they partially order, called **partially ordered sets** or **PoSets**. Another time.

None of the examples we have considered so far in this section have been equivalence relations, but they are ubiquitous in mathematics. They are so common that it is easy to overlook them as anything worth saying something about at all. Let's see some examples.

Example 2.6.15

Prove that the relation $\equiv_2$, which holds of two integers if their difference is even, is an equivalence relation. That is, $a \equiv_2 b$ if and only if $b - a = 2k$ for some integer k.

Solution. We simply check the three required properties.

1. $\equiv_2$ is reflexive: for any integer a, we have $a - a = 0$ and $0 = 2k$ for $k = 0$, so $a \equiv_2 a$.

2. $\equiv_2$ is symmetric: Fix arbitrary integers a and b, and assume $a \equiv_2 b$. That means that $b - a = 2k$ for some integer k. What about $a - b$? Well, we will have $a - b = 2(-k)$, and $-k$ is an integer, so we have $b \equiv_2 a$.

3. $\equiv_2$ is transitive: Fix arbitrary integers a, b, and c and assume $a \equiv_2 b$ and $b \equiv_2 c$. This means that $b - a = 2k$ and $c - b = 2j$ for some integers k and j. What about $c - a$? Well,

$$c - a = (c - b) + (b - a) = 2k + 2j = 2(k + j).$$

Since $k + j$ is an integer, we see that $a \equiv_2 c$ as required.

Example 2.6.16

Let's call two graphs "degree-sequence-equivalent" if they have the same degree sequence. Is this an equivalence relation?

Solution. Yes it is. Clearly every graph has the same degree sequence as itself, so the relation is reflexive. If G_1 has the same degree sequence as G_2, then G_2 has the same degree sequence as G_1, so the relation is transitive. Finally, if G_1 has the same degree sequence as G_2, which has the same degree sequence as G_3, then they all have the same degree sequence, so G_1 has the same degree sequence as G_3 (i.e., the relation is transitive).

This example is almost too obvious. That's because we said that two things are related if a well-defined property of those things was *equal*, and equality satisfies the properties of an equivalence relation. Our next goal is to try to make better sense of this and see that it is exactly what gives us an equivalence relation.

2.6.5 Equivalence Classes and Partitions

Given *any* relation R, we can look at the *set* of elements that are related to a particular element. For the "is taking" relation, we can ask what classes Al is taking (i.e., the classes related to Al). For the "is a multiple of", we can ask which numbers 6 a multiple of. One way to study the relation is to study the sets of things related to each element.

Definition 2.6.17

Let R be a relation on the set A, and let a be an element of A. The **relation class** of a, written $[a]$ is the set of all elements b such that $(a, b) \in R$ (the set of b that are related to a by R). That is,

$$[a] = \{b \in A : (a, b) \in R\}.$$

When R is an equivalence relation, we call relation classes **equivalence classes**.

Example 2.6.18

Find the relation classes for the "is a multiple of" relation on the set $A = \{1, 2, 3, 4, 5, 6\}$.

Solution. There will be six relation classes since each element has a relation class. They are:

$$\begin{aligned}
[1] &= \{1\} \\
[2] &= \{1, 2\} \\
[3] &= \{1, 3\}
\end{aligned}$$

$$[4] = \{1, 2, 4\}$$
$$[5] = \{1, 5\}$$
$$[6] = \{1, 2, 3, 6\}.$$

For example, we found [4] by considering all pairs $(4, b)$ that satisfied the relation: 4 is a multiple of 1, 2, and 4, so those are the possible values of b that we find.

Look back at the directed multigraph for this relation shown in the solution to Example 2.6.4. What are the relation classes? They are nothing but the neighbors of each vertex (where neighbor means you follow the arrows in the correct direction).

Example 2.6.19

Find the equivalence classes for the $\equiv_2$ relation on the integers.

Solution. On no! Our set is infinite, so we will have infinitely many relation classes? Well, we better get started...

What numbers are related to 1? Remember, we want integers whose difference with 1 is a multiple of 2. So 3 for sure. Also 5, and 7, and -1, and -3, and... all the odds? Yes, because the difference of any two odd numbers is even. We can also say that the difference between any two even numbers is even, so the equivalence class of 2 will contain all the even numbers. So far we have:

$$[1] = \{\ldots, -3, -1, 1, 3, 5, \ldots\}$$
$$[2] = \{\ldots, -4, -2, 2, 4, 6, \ldots\}$$

Actually, I think we are done. While [3], [4], [5], and so on are all completely valid equivalence classes, the elements that are related to 3 will be exactly the elements related to 1, since $1 \equiv_2 3$. This is because the relation is transitive! If $3 \equiv_2 b$, then we know $1 \equiv_2 b$.

So we have exactly two equivalence classes. Every integer is in exactly one of these two equivalence classes, and the equivalence class of any integer is exactly the class it belongs to.

Examine the two examples above carefully. For the "is a multiple of" relation, which is NOT an equivalence relation, some elements belong to more than one (different) relation class. But for $\equiv_2$, which is an equivalence relation, every element is in exactly one equivalence class. This is no accident. To make sense of this, we will define a new term.

Definition 2.6.20

Given a non-empty set A, a **partition** of A is a set P of non-empty subsets of A such that every element of A is in exactly one element of P.

That definition has a lot of symbols and sets involved. It's really not complicated though: A partition is a way to break up a set into *disjoint* subsets that *cover* the whole set. That the subsets are disjoint means no element is in *more than one* subset. That the subsets cover the set means every element is in *at least one* subset.

Example 2.6.21

Give a few different partitions of the set $A = \{1, 2, 3, 4, 5\}$.

Solution. There are so many choices here! One choice is:

$$P_1 = \{\{1, 2, 3\}, \{4, 5\}\}.$$

That's a partition of A into two subsets. Another partition:

$$P_2 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}.$$

Another:

$$P_3 = \{\{1, 3, 5\}, \{2, 4\}\},$$

which happens to be a partition into even and odd numbers. We also have the trivial partition:

$$P_4 = \{\{1, 2, 3, 4, 5\}\}.$$

Note: that is a single set inside the set P_4.

It is sometimes a little confusing to think of the elements of a partition as subsets since the partition is a set, and a set of sets can be difficult to talk about. We sometimes call the partition a *collection* and each of the elements of the partition *parts* or *blocks*.

Now the big idea: For any equivalence relation, the equivalence classes form a partition, and for any partition, we can define a relation "are in the same subset" which will be an equivalence relation. We have already seen that the equivalence classes of $\equiv_2$ form a partition. Let's go the other direction.

Example 2.6.22

Define an equivalence relation $\sim$ on the set $A = \{1, 2, 3, 4, 5\}$ from the partition $P = \{\{1, 4\}, \{2, 3, 5\}\}$.

Solution. Say two elements of A are equivalent provided they belong to the same element of P. That is, $1 \sim 4$ and $4 \sim 1$, and $2 \sim 3$, $2 \sim 5$, $3 \sim 5$, $3 \sim 2$,

$5 \sim 2$, and $5 \sim 3$. Wait. We also have $1 \sim 1$, $2 \sim 2$, and so on, since every number is in the same subset as itself.

It is clear from looking that this relation is reflexive, symmetric, and transitive, so is an equivalence relation.

Theorem 2.6.23

Given any equivalence relation R on a set A, the equivalence classes form a partition of A.

Given any partition $P = \{B_1, B_2, \ldots,\}$ of a set A, the relation $\equiv_P$ defined by $a \equiv_P b$ if and only if a and b belong to the same block of P, is an equivalence relation.

Further, the equivalence classes for the equivalence relation formed by a partition are exactly the original partition, and the equivalence relation built by the partition of its equivalence classes is exactly the original equivalence relation.

2.6.6 Reading Questions

1. Consider the relation R defined on the integers that holds of a and b precisely if $b - a \geq 4$. So for example, $(2,7) \in R$ but $(8,6) \notin R$. Which of the following properties of relations does R have?

A. R is reflexive.

B. R is irreflexive.

C. R is symmetric.

D. R is antisymmetric.

E. R is transitive.

2. Not all graphs have a transitive edge relation. But do some of them? If so, give an example and explain why it is transitive. If not, explain why.

3. After reading this section, what questions do you have? Ask at least one question about this section that you are curious about.

2.6.7 Practice Problems

1. Which of the following relations on the integers are reflexive?

(a) $x \sim y$ if and only if $x + y$ is odd.

(b) $x \sim y$ if and only if $x + y$ is positive.

(c) $x \sim y$ if and only if $xy \geq 0$.

(d) $x \sim y$ if and only if xy is positive.

(e) $x \sim y$ if and only if $y - x$ is a multiple of 10.

2. Which of the following relations on the integers are symmetric?

(a) $x \sim y$ if and only if $x + y$ is odd.

(b) $x \sim y$ if and only if $x + y$ is positive.

(c) $x \sim y$ if and only if $xy \geq 0$.

(d) $x \sim y$ if and only if xy is positive.

(e) $x \sim y$ if and only if $y - x$ is a multiple of 10.

3. Which of the following relations on the integers are transitive?

(a) $x \sim y$ if and only if $x + y$ is odd.

(b) $x \sim y$ if and only if $x + y$ is positive.

(c) $x \sim y$ if and only if $xy \geq 0$.

(d) $x \sim y$ if and only if xy is positive.

(e) $x \sim y$ if and only if $y - x$ is a multiple of 10.

4. Define relations $R_1, \ldots, R_6$ on $1, 2, 3, 4$ by

(a) $R_1 = \{(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$,

(b) $R_2 = \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)\}$.

(c) $R_3 = \{(2,4),(4,2)\}$.

(d) $R_4 = \{(1,2),(2,3),(3,4)\}$,

(e) $R_5 = \{(1,1),(2,2),(3,3),(4,4)\}$,

(f) $R_6 = \{(1,3),(1,4),(2,3),(2,4),(3,1),(3,4)\}$,

For each of the relations, determine whether it is any of reflexive, irreflexive, symmetric, antisymmetric, and transitive.

5. For each of the following relations on the set of all people, determine whether the relation is any of reflexive, irreflexive, symmetric, anitsymmetric, and transitive.

(a) $a \sim b$ if and only if a is older than b.

(b) $a \sim b$ if and only if a and b have a common grandparent.

(c) $a \sim b$ if and only if a has the same first name as b.

(d) $a \sim b$ if and only if a and b were born on the same day.

6. For the following relations on the set of all integers, determine whether the relation is any of reflexive, irreflexive, symmetric, antisymmetric, and transitive:

(a) $x \sim y$ if and only if $x + y = 0$.

(b) $x \sim y$ if and only if $x - y$ is an integer.

(c) $x \sim y$ if and only if $x = 2y$.

(d) $x \sim y$ if and only if $xy > 1$.

2.6.8 Additional Exercises

1. Consider the relation $>$ on the set $\{1, 2, \ldots, 8\}$.

(a) Draw the directed graph of the relation $>$.

(b) Draw the directed graph for the inverse relation $>^{-1}$.

(c) Is the inverse relation of $>$ the same as the relation $\leq$? Explain.

2. True or false: for any relation R on a set A, the relation R^{-1} is symmetric if and only if R is symmetric. Justify your answer.

3. True or false: for any relation R on a set A, the composition of R with its inverse, $R \circ R^{-1}$, is always reflexive. Justify your answer.

4. Find, if possible, an example of a relation on the set $\{1, 2, 3, 4\}$ that is reflexive and symmetric, but not transitive. If such a relation exists, draw the directed multigraph of the relation and list the ordered pairs that define it. Explain your answers.

5. Find, if possible, an example of a relation on the set $\{1, 2, 3, 4\}$ that is reflexive and transitive, but not symmetric. If such a relation exists, draw the directed multigraph of the relation and list the ordered pairs that define it. Explain your answers.

6. Find, if possible, an example of a relation on the set $\{1, 2, 3, 4\}$ that is symmetric and transitive, but not reflexive. If such a relation exists, draw the directed multigraph of the relation and list the ordered pairs that define it. Explain your answers.

7. What is wrong with the following argument that any relation that is symmetric and transitive must be reflexive?

> Suppose R is a relation on a set A that is symmetric and transitive. Since R is symmetric, if aRb, then bRa holds. Since R is transitive, if aRb and bRa, then aRb holds. Since this is true for all elements a, we have that aRa is true for all a in A, so R is reflexive.

8. Suppose R is an equivalence relation on the set $A = \{1, 2, \ldots, 6\}$. What could the directed multigraph for R look like? Give at least two different examples of such R and their graphs to illustrate your answer.

9. Consider the relation R on the set $A = \{1, 2, 3, 4, 5\}$ defined by $R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1), (2, 1), (3, 1), (4, 1), (5, 1)\}$. Is R an equivalence relation? Justify your answer.

Regardless of your answer, what do the relation classes $[a]$ for $a \in R$ look like? Can you tell whether R is an equivalence relation from this information?

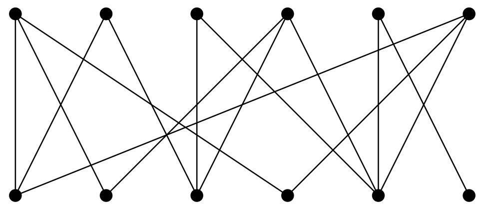
10. Consider the "loner" relation on a set of students that describes friendships, and holds *only* between a student and themselves (i.e., nobody is friends with anyone other than themselves). Is this an equivalence relation? Justify your answer. If it is an equivalence relation, what do the equivalence classes look like?

2.7 Matching in Bipartite Graphs

Investigate!

Given a bipartite graph, a **matching** is a subset of the edges for which every vertex belongs to exactly one of the edges. Our goal in this activity is to discover some criteria for when a bipartite graph has a matching.

Does the graph below contain a matching? If so, find one.



Not all bipartite graphs have matchings. Draw as many fundamentally different examples of bipartite graphs that do NOT have matchings. Your goal is to find all the possible obstructions to a graph having a perfect matching. Write down the *necessary* conditions for a graph to have a matching (that is, fill in the blank: If a graph has a matching, then ________). Then ask yourself whether these conditions are sufficient (is it true that if ________, then the graph has a matching?).

We conclude with one more example of a graph theory problem to illustrate the variety and vastness of the subject.

Suppose you have a bipartite graph G. This will consist of two sets of vertices A and B with some edges connecting some vertices of A to some vertices in B (but of course, no edges between two vertices both in A or both in B). A **matching of** A is a subset of the edges for which each vertex of A belongs to exactly one edge of the subset, and no vertex in B belongs to more than one edge in the subset. In practice, we will assume that $|A| = |B|$ (the two sets have the same number of vertices), so this says that every vertex in the graph belongs to exactly one edge in the matching.[11]

Some context might make this easier to understand. Think of the vertices in A as representing students in a class, and the vertices in B as representing presentation topics. We put an edge from a vertex $a \in A$ to a vertex $b \in B$ if student a would like to present on topic b. Of course, some students would want to present on more than one topic, so their vertex would have degree greater than 1. As the teacher, you want to assign each student their own unique topic. Thus you want to find a matching of A: you pick some subset of the edges so that each student gets matched up with exactly one topic, and no topic gets matched to two students.[12]

[11]What we are calling a *matching* is sometimes called a *perfect matching* or *complete matching*. This is because it is interesting to look at non-perfect matchings as well. We will call those *partial* matchings.

[12]The standard example for matchings used to be the *marriage problem* in which A consisted of the

The question is: when does a bipartite graph contain a matching of A? To begin to answer this question, consider what could prevent the graph from containing a matching. This will not necessarily tell us a condition when the graph *does* have a matching, but at least it is a start.

One way G could not have a matching is if there is a vertex in A not adjacent to any vertex in B (so having degree 0). What else? What if two students both like the same topic, and no others? Then after assigning that one topic to the first student, there is nothing left for the second student to like, so it is very much as if the second student has degree 0. Or what if three students like only two topics between them? Again, after assigning one student a topic, we reduce this to the previous case of two students liking only one topic. We can continue this way with more and more students.

It should be clear at this point that if there is a group of n students who as a group like $n-1$ or fewer topics, then no matching is possible. This is true for any value of n, and any group of n students.

To make this more graph-theoretic, say you have a set $S \subseteq A$ of vertices. Define $N(S)$ to be the set of all the **neighbors** of vertices in S. That is, $N(S)$ contains all the vertices (in B) that are adjacent to at least one of the vertices in S. (In the student/topic graph, $N(S)$ is the set of topics liked by the students of S.) Our discussion above can be summarized as follows:

Matching Condition.

If a bipartite graph $G = \{A, B\}$ has a matching of A, then

$$|N(S)| \geq |S|$$

for all $S \subseteq A$.

Is the converse true? Suppose G satisfies the matching condition $|N(S)| \geq |S|$ for all $S \subseteq A$ (every set of vertices has at least as many neighbors as vertices in the set). Does that mean that there is a matching? Surprisingly, yes. The obvious necessary condition is also sufficient.[13] This is a theorem first proved by Philip Hall in 1935.[14]

Theorem 2.7.1 Hall's Marriage Theorem.

Let G be a bipartite graph with sets A and B. Then G has a matching of A if and only if

$$|N(S)| \geq |S|$$

men in the town, B the women, and an edge represented a marriage that was agreeable to both parties. A matching then represented a way for the town elders to marry off everyone in the town, no polygamy allowed. We have chosen a more progressive context for the sake of political correctness.

[13]This happens often in graph theory. If you can avoid the obvious counterexamples, you often get what you want.

[14]There is also an infinite version of the theorem, which was proved by Marshal Hall, Jr. The name is a coincidence though as the two Halls are not related.

for all $S \subseteq A$.

There are quite a few different proofs of this theorem – a quick internet search will get you started.

In addition to its application to marriage and student presentation topics, matchings have applications all over the place. We conclude with one such example.

Example 2.7.2

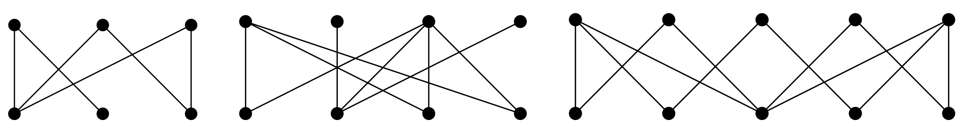
Suppose you deal 52 regular playing cards into 13 piles of 4 cards each. Prove that you can always select one card from each pile to get one of each of the 13 card values Ace, 2, 3, . . . , 10, Jack, Queen, and King.

Solution. Doing this directly would be difficult, but we can use the matching condition to help. Construct a graph G with 13 vertices in the set A, each representing one of the 13 card values, and 13 vertices in the set B, each representing one of the 13 piles. Draw an edge between a vertex $a \in A$ to a vertex $b \in B$ if a card with value a is in the pile b. Notice that we are just looking for a matching of A; each value needs to be found in the piles exactly once.

We will have a matching if the matching condition holds. Given any set of card values (a set $S \subseteq A$), we must show that $|N(S)| \geq |S|$. That is, the number of piles that contain those values is at least the number of different values. But what if it wasn't? Say $|S| = k$. If $|N(S)| < k$, then we would have fewer than $4k$ different cards in those piles (since each pile contains 4 cards). But there are $4k$ cards with the k different values, so at least one of these cards must be in another pile, a contradiction. Thus the matching condition holds, so there is a matching, as required.

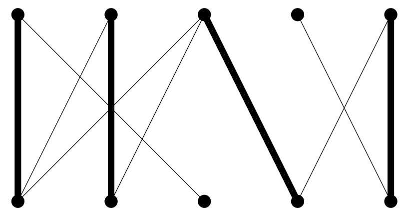
Exercises

1. Find a matching of the bipartite graphs below or explain why no matching exists.



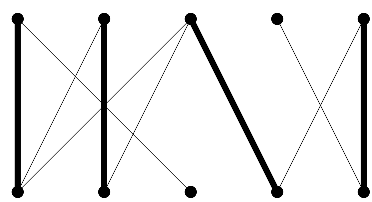
2. A bipartite graph that doesn't have a matching might still have a **partial matching**. By this we mean a set of *edges* for which no vertex belongs to more than one edge (but possibly belongs to none). Every bipartite graph (with at least one edge) has a partial matching, so we can look for the largest partial matching in a graph.

Your "friend" claims that she has found the largest partial matching for the graph below (her matching is in bold). She explains that no other edge can be added, because all the edges not used in her partial matching are connected to matched vertices. Is she correct?

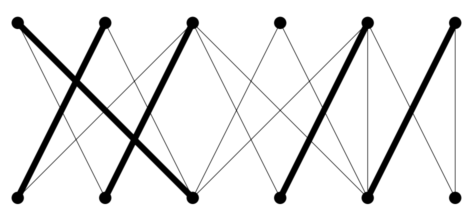


3. One way you might check to see whether a partial matching is maximal is to construct an **alternating path**. This is a sequence of adjacent edges, which alternate between edges in the matching and edges not in the matching (no edge can be used more than once). If an alternating path starts and stops with an edge *not* in the matching, then it is called an **augmenting path**.

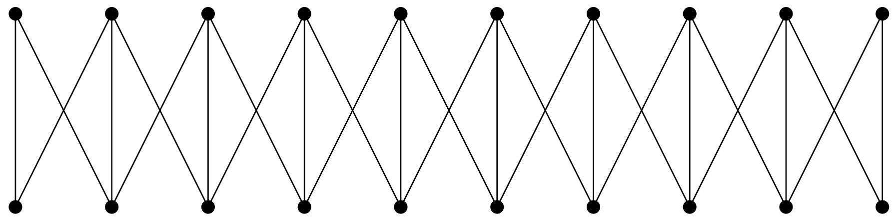
(a) Find the largest possible alternating path for the partial matching of your friend's graph. Is it an augmenting path? How would this help you find a larger matching?



(b) Find the largest possible alternating path for the partial matching below. Are there any augmenting paths? Is the partial matching the largest one that exists in the graph?



4. The two richest families in Westeros have decided to enter into an alliance by marriage. The first family has 10 sons, the second has 10 girls. The ages of the kids in the two families match up. To avoid impropriety, the families insist that each child must marry someone either their own age, or someone one position younger or older. In fact, the graph representing agreeable marriages looks like this:



The question: how many different acceptable marriage arrangements which marry off all 20 children are possible?

(a) How many marriage arrangements are possible if we insist that there are exactly 6 boys who marry girls not their own age?

(b) Could you generalize the previous answer to arrive at the total number of marriage arrangements?

(c) How do you know you are correct? Try counting in a different way. Look at smaller family sizes and get a sequence.

(d) Can you give a recurrence relation that fits the problem?

5. We say that a set of vertices $A \subseteq V$ is a **vertex cover** if every edge of the graph is incident to a vertex in the cover (so a vertex cover covers the *edges*). Since V itself is a vertex cover, every graph has a vertex cover. The interesting question is about finding a **minimal** vertex cover, one that uses the fewest possible number of vertices.

(a) Suppose you had a matching of a graph. How can you use that to get a minimal vertex cover? Will your method always work?

(b) Suppose you had a minimal vertex cover for a graph. How can you use that to get a partial matching? Will your method always work?

(c) What is the relationship between the size of the minimal vertex cover and the size of the maximal partial matching in a graph?

6. For many applications of matchings, it makes sense to use bipartite graphs. You might wonder, however, whether there is a way to find matchings in graphs in general.

(a) For which n does the complete graph K_n have a matching?

(b) Prove that if a graph has a matching, then $|V|$ is even.

(c) Is the converse true? That is, do all graphs with $|V|$ even have a matching?

(d) What if we also require the matching condition? Prove or disprove: If a graph with an even number of vertices satisfies $|N(S)| \geq |S|$ for all $S \subseteq V$, then the graph has a matching.

2.8 Chapter Summary

Hopefully this chapter has given you some sense of the wide variety of graph theory topics as well as why these studies are interesting. There are many more interesting areas to consider, and the list is increasing all the time; graph theory is an active area of mathematical research.

One reason graph theory is such a rich area of study is that it deals with such a fundamental concept: Any pair of objects can either be related or not related. What the objects are and what "related" means varies depending on context, and this leads to many applications of graph theory to science and other areas of math. The objects can be countries, and two countries can be related if they share a border. The objects could be land masses that are related if there is a bridge between them. The objects could be websites that are related if there is a link from one to the other. Or we can be completely abstract: The objects are vertices that are related if there is an edge between them.

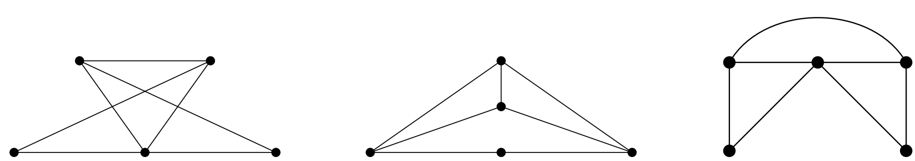
What question we ask about the graph depends on the application, but often leads to deeper, general and abstract questions worth studying in their own right. Here is a short summary of the types of questions we have considered:

- Can the graph be drawn in the plane without edges crossing? If so, how many regions does this drawing divide the plane into?
- Is it possible to color the vertices of the graph so that related vertices have different colors using a small number of colors? How many colors are needed?
- Is it possible to trace over every edge of a graph exactly once without lifting your pencil? What other sorts of "paths" might a graph possess?
- Can you find subgraphs with certain properties? For example, when does a (bipartite) graph contain a subgraph in which all vertices are only related to one other vertex?

Not surprisingly, these questions are often related to each other. For example, the chromatic number of a graph cannot be greater than 4 when the graph is planar. Whether the graph has an Euler trail depends on how many vertices each vertex is adjacent to (and whether those numbers are always even or not). Even the existence of matchings in bipartite graphs can be proved using paths.

Chapter Review

1. Which (if any) of the graphs below are the same? Which are different? Explain.



2. Which of the graphs in the previous question contain Euler trails or circuits? Which of the graphs are planar?

3. Draw a graph that has an Euler circuit but is not planar.

4. Draw a graph that does not have an Euler trail and is also not planar.

5. Consider the graph $G = (V, E)$ with $V = \{a, b, c, d, e, f, g\}$ and $E = \{ab, ac, af, bg, cd, ce\}$ (here we are using the shorthand for edges: ab really means $\{a, b\}$, for example).

(a) Is the graph G isomorphic to $G' = (V', E')$ with $V' = \{t, u, v, w, x, y, z\}$ and $E' = \{tz, uv, uy, uz, vw, vx\}$? If so, give the isomorphism. If not, explain how you know.

(b) Find a graph G'' with 7 vertices and 6 edges which is NOT isomorphic to G, or explain why this is not possible.

(c) Write down the *degree sequence* for G. That is, write down the degrees of all the vertices, in decreasing order.

(d) Find a connected graph G''' with the same degree sequence of G which is NOT isomorphic to G, or explain why this is not possible.

(e) What kind of graph is G? Is G complete? Bipartite? A tree? A cycle? A path? A wheel?

(f) Is G planar?

(g) What is the chromatic number of G? Explain.

(h) Does G have an Euler trail or circuit? Explain.

6. If a graph has 10 vertices and 10 edges and contains an Euler circuit, must it be planar? How many faces would it have?

7. Suppose G is a graph with n vertices, each having degree 5.

(a) For which values of n does this make sense?

(b) For which values of n does the graph have an Euler trail?

(c) What is the smallest value of n for which the graph might be planar? (tricky)

8. At a school dance, 6 girls and 4 boys take turns dancing (as couples) with each other.

(a) How many couples dance if every girl dances with every boy?

(b) How many couples dance if everyone dances with everyone else (regardless of gender)?

(c) Explain what graphs can be used to represent these situations.

9. Among a group of n people, is it possible for everyone to be friends with an odd number of people in the group? If so, what can you say about n?

10. Your friend has challenged you to create a convex polyhedron containing 9 triangles and 6 pentagons.

(a) Is it possible to build such a polyhedron using *only* these shapes? Explain.

(b) You decide to also include one heptagon (seven-sided polygon). How many vertices does your new convex polyhedron contain?

(c) Assuming you are successful in building your new 16-faced polyhedron, could every vertex be the joining of the same number of faces? Could each vertex join either 3 or 4 faces? If so, how many of each type of vertex would there be?

11. Is there a convex polyhedron that requires 5 colors to properly color the vertices of the polyhedron? Explain.

12. How many edges does the graph $K_{n,n}$ have? For which values of n does the graph contain an Euler circuit? For which values of n is the graph planar?

13. The graph G has 6 vertices with degrees $1, 2, 2, 3, 3, 5$. How many edges does G have? If G was planar, how many faces would it have? Does G have an Euler trail?

14. What is the smallest number of colors you need to properly color the vertices of K_7. Can you say whether K_7 is planar based on your answer?

15. What is the smallest number of colors you need to properly color the vertices of $K_{3,4}$? Can you say whether $K_{3,4}$ is planar based on your answer?

16. Prove that $K_{3,4}$ is not planar. Do this using Euler's formula, not just by appealing to the fact that $K_{3,3}$ is not planar.

17. A dodecahedron is a regular convex polyhedron made up of 12 regular pentagons.

(a) Suppose you color each pentagon with one of three colors. Prove that there must be two adjacent pentagons colored identically.

(b) What if you use four colors?

(c) What if instead of a dodecahedron you colored the faces of a cube?

18. Decide whether the following statements are true or false. Prove your answers.

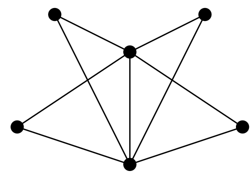
(a) If two graphs G_1 and G_2 have the same chromatic number, then they are isomorphic.

(b) If two graphs G_1 and G_2 have the same number of vertices and edges and have the same chromatic number, then they are isomorphic.

(c) If two graphs are isomorphic, then they have the same chromatic number.

19. If a planar graph G with 7 vertices divides the plane into 8 regions, how many edges must G have?

20. Consider the graph below:



(a) Does the graph have an Euler trail or circuit? Explain.

(b) Is the graph planar? Explain.

(c) Is the graph bipartite? Complete? Complete bipartite?

(d) What is the chromatic number of the graph?

21. For each part below, say whether the statement is true or false. Explain why the true statements are true, and give counterexamples for the false statements.

(a) Every bipartite graph is planar.

(b) Every bipartite graph has chromatic number 2.

(c) Every bipartite graph has an Euler trail.

(d) Every vertex of a bipartite graph has even degree.

(e) A graph is bipartite if and only if the sum of the degrees of all the vertices is even.

22. Consider the statement, "If a graph is planar, then it has an Euler trail."

(a) Write the converse of the statement.

(b) Write the contrapositive of the statement.

(c) Write the negation of the statement.

(d) Is it possible for the contrapositive to be false? If it was, what would that tell you?

(e) Is the original statement true or false? Prove your answer.

(f) Is the converse of the statement true or false? Prove your answer.

23. Let G be a connected graph with v vertices and e edges. Use mathematical induction to prove that if G contains exactly one cycle (among other edges and vertices), then $v = e$.

Note: This is asking you to prove a special case of Euler's formula for planar graphs, so do not use that formula in your proof.

Chapter 3

Counting

One of the first things you learn in mathematics is how to count. Now we want to count large collections of things quickly and precisely. For example:

- In a group of 10 people, if everyone shakes hands with everyone else exactly once, how many handshakes take place?
- How many ways can you distribute 10 Girl Scout cookies to 7 Boy Scouts?
- How many anagrams are there of "anagram"?

Before tackling questions like these, let's look at the basics of counting.

3.1 Pascal's Arithmetical Triangle

Objectives

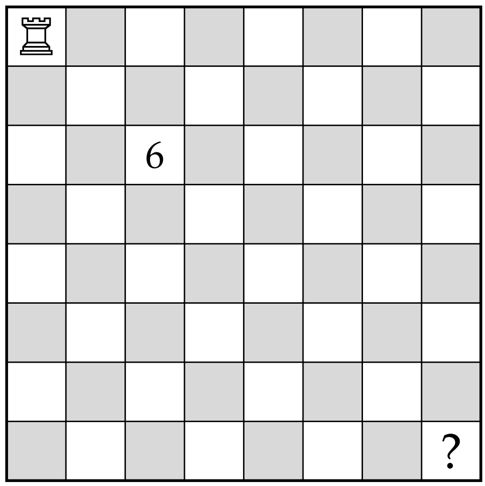
After completing this section, you should be able to do the following.

1. Use Pascal's triangle to answer counting questions about lattice paths, bit strings, and subsets.
2. Explain how Pascal's triangle is generated and how it relates to counting questions.
3. Explain why Pascal's triangle is related to so many different types of counting problems.

3.1.1 SECTION PREVIEW

Investigate!

In chess, a rook can move only in straight lines (not diagonally). How many ways can the rook in the top-left corner travel to the bottom-right corner of the board, moving only down and to the right?



Also, what does this have to do with counting how many pizzas you can order if you use half of the 14 available toppings?

PREVIEW ACTIVITY

Let's find some of the numbers of paths that the rook can take to get to various squares in the chessboard.

1. The 6 in the square in the 3rd row and column represents that there are 6 different paths to that square, even though there are only four squares the rook must move through to get there. One path is DDRR (down down right right). List all 6 paths.

2. How many paths are there to the square in row 4, column 2 (diagonally down and to the left of the 6)? List out all the paths as D/R strings.

How many paths is this? That is, what number goes in that square of the chessboard?

3. Now let's find the paths to the square in row 4, column 3 (directly below the 6).

First, list all the paths that end with an R.

Next, list all the paths that end with a D.

Are there any other paths? In total, how many paths are there to this square?

4. Continue filling in the chessboard, either counting D/R strings directly or using your observation from the previous task. What is the number in the lower right corner of the chessboard?

In 1653, Blaise Pascal, concerned with questions that would lay the foundation of probability theory, collected several facts about a triangular array of numbers in his *Treatise on Arithmetical Triangle*. This arrangement of numbers appeared as early as the 10th century in China, India, and Persia. The Chinese and Persian treatment of the triangle was in service of what we would now consider algebra: finding nth roots, essentially solving polynomial equations. The numbers in the triangle appear as solutions to counting problems in Indian texts: from six *tastes*, how many combinations of one, or two, or three,... can you make? European mathematicians in the 14th century presented the triangle as a table of **figurate numbers** (numbers that can be arranged in a geometric shape), which were themselves the centerpiece of the work of Pythagoras and his followers.

So what is this remarkable triangle that holds the secrets of so many different mathematical problems? Behold, Pascal's triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1
1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1
1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1
1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1
1 16 120 560 1820 4368 8008 11440 12870 11440 8008 4368 1820 560 120 16 1

Figure 3.1.1 Pascal's triangle.

Spend some time gazing at the beauty of this triangle. What do you notice? What do you wonder? Look specifically at the 5th row (we call the 1 on the top row 0, so row 5 is 1, 5, 10, 10, 5, 1). How do the numbers in this row relate to the numbers above them? Notice that $5 = 1 + 4$ and $10 = 4 + 6$. Does this occur anywhere else in the triangle?

Indeed, every number in the triangle is the *sum of the two numbers above it*. Let's take this as our *definition* of Pascal's triangle. We can then generate as many rows of the triangle as we like. It is this additive definition that was used in China and Persia to find nth roots, and we will briefly mention this use at the end of this section. However, we are interested in counting questions, so our main goal now is to observe how the numbers of Pascal's triangle are answers to a variety of counting questions.

Here are some apparently different discrete objects we can count: lattice paths, bit strings, subsets, and pizzas. We will give an example of each type of counting problem (and say what these things even are). As we will see, the numbers in Pascal's triangle are the answers to all of these questions.

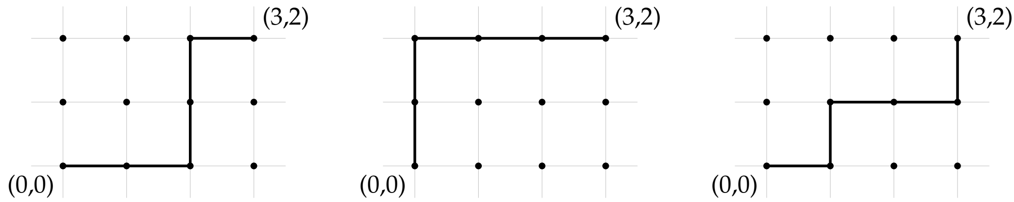
Before we jump in, a little bit of notation. Let's give each number in Pascal's triangle a name, based on its position. Think of each number as being in a row and a column: rows are counted down, starting at 0, and columns are counted in from the left, also starting at 0. The entry in row n and column k will be denoted $\binom{n}{k}$. For example, the $\binom{6}{3} = 20$, since that is the value in row 6, column 3. For reasons that will become clear soon, we pronounce $\binom{n}{k}$ as "n **choose** k." We can rewrite the triangle with these names:

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

3.1.2 Lattice Paths

The **integer lattice** is the set of all points in the Cartesian plane for which both the x and y coordinates are integers. If you like to draw graphs on graph paper, the lattice is the set of all the intersections of the grid lines.

A **lattice path** is one of the shortest possible paths connecting two points on the lattice, moving only horizontally and vertically. For example, here are three possible lattice paths from the point $(0, 0)$ to $(3, 2)$:



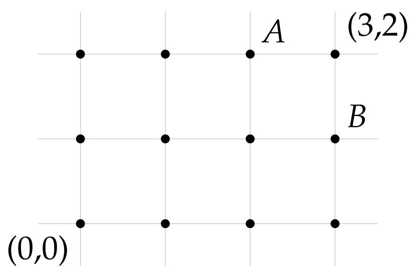
Notice that to ensure the path is the *shortest* possible, each move must be either to the right or up. Additionally, in this case, no matter what path we take, we must make three steps right and two steps up. No matter in what order we make these steps, there will always be five steps. Thus each path has **length** five.

The counting question we will ask is this: *how many* lattice paths are there between $(0,0)$ and $(3,2)$? In this case, drawing all the paths wouldn't take too long. Or we could list each path as a string of "directions" such as $xxyyx$, $yyxxx$, or $xyxxy$, which correspond to the three paths drawn above, where an x means travel one unit in the x direction, and similarly for y. We would get the following ten paths:

$$\begin{array}{cccccc} & xxxyy & xxyxy & xyxxy & yxxxy & \\ xxyyx & xyxyx & yxxyx & xyyxx & yxyxx & yyxxx. \end{array}$$

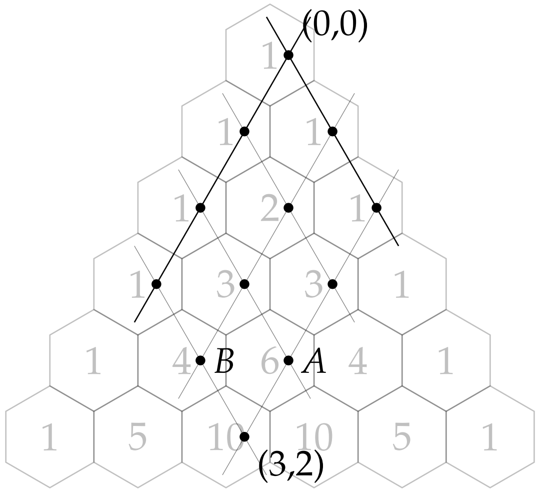
When the distance between starting and stopping points is larger, we will want to find a more efficient way to count the paths.

Let's take what we learned from the rook paths (which are, gasp, actually lattice paths). Consider the lattice shown below:



Any lattice path from (0,0) to (3,2) must pass through exactly one of A and B. The point A is 4 steps away from (0,0) and two of them are in the x direction. The last step is also in the x direction, so the paths from (0,0) to (3,2) that pass through A are exactly the six strings we listed above that end in an x. For the paths that pass through point B, the last step will be in the y direction, so the paths from (0,0) to (3,2) that pass through B are exactly the four strings we listed above that end in a y. So the total number of paths to (3,2) is just 6 + 4.

The general observation here is that to find the number of paths that start at $(0,0)$ and end at (m,n), we can find the number of paths to the point directly to the left of the endpoint, $(m-1,n)$ and add the number of paths to the point directly below the endpoint, $(m,n-1)$. This is exactly the same way that Pascal's triangle is generated! Indeed, if we rotate the lattice appropriately, so the point $(0,0)$ is at the top of the triangle and the axes along the sides of the triangle, we see that the numbers in Pascal's triangle give us exactly the number of paths to each lattice point.



To make this observation helpful for actually finding the number of paths from the origin to a given point, we note that it is the *length* of the path that determines the *row* of Pascal's triangle, and the number of steps in the y direction that says how far into the triangle we are -- the *column* of Pascal's triangle.

Example 3.1.2

How many lattice paths are there from $(0, 0)$ to $(4, 7)$?

Solution. The length of these paths is $4 + 7 = 11$. Look at the 11th row of Pascal's triangle:

$$1, 11, 55, 165, 330, 462, 462, 330, 165, 55, 11, 1.$$

Count to the 7th position (remembering that the 1 is in position 0) which gives us $\binom{11}{7} = 330$ different paths.

3.1.3 Bit Strings

"Bit" is short for "binary digit," so a **bit string** is a string of binary digits. The **binary digits** are simply the numbers 0 and 1. All of the following are bit strings:

$$1001 \quad 0 \quad 1111 \quad 1010101010.$$

The number of bits (0's or 1's) in the string is the **length** of the string; the strings above have lengths 4, 1, 4, and 10 respectively. We also can ask how many of the bits are 1's. The number of 1's in a bit string is the **weight** of the string; the weights of the above strings are 2, 0, 4, and 5 respectively.

Definition 3.1.3 Bit Strings.

- An n**-bit string** is a bit string of length n. That is, it is a string containing n symbols, each of which is a bit, either 0 or 1.
- The **weight** of a bit string is the number of 1's in it.
- $\mathbf{B}_k^n$ is the set of all n-bit strings of weight k.

For example, the elements of the set $\mathbf{B}_2^3$ are the bit strings 011, 101, and 110. Those are the only strings containing three bits, exactly two of which are 1's.

The counting questions: How many 5-bit strings have weight 3? In other words, we are asking for the cardinality $|\mathbf{B}_3^5|$.

Let's just list them and see how many there are.

$$\begin{array}{cccccc} & 11100 & 11010 & 10110 & 01110 & \\ 11001 & 10101 & 01101 & 10011 & 01011 & 00111. \end{array}$$

Great. Ten of them. Actually, I have a confession: I didn't type all of these from scratch. Instead I just modified the list of 10 lattice paths from (0,0) to (3,2) that we found earlier. Each x became a 1 and each y became a 0. After all, any lattice path with length n that requires k steps in the x direction can be represented by a string of n symbols of two types, with k of those symbols being of one type. Whether we call the two symbols x and y or we call them 1 and 0 will not change *how many* strings we get.

It is not surprising then that the same relationship between Pascal's triangle and lattice paths holds for bit strings. Look at the 10 strings above. The first row contains all the bit strings of $\mathbf{B}_3^5$ that end in a 0. Before that ending 0, we have a string in $\mathbf{B}_3^4$, since it must have length 4 and weight 3 (the ending 0 increases the length, but not the weight). The second row contains all the bit strings of $\mathbf{B}_3^5$ that end in a 1. Before that ending 1, we have a string in $\mathbf{B}_2^4$, since it must have length 4 and weight 2 (the ending 1 increases the length and the weight). So the number of 5-bit strings of weight 3 is the sum of the number of 4-bit strings of weight 3 and the number of 4-bit strings of weight 2. In symbols:

$$|\mathbf{B}_3^5| = |\mathbf{B}_3^4| + |\mathbf{B}_2^4|.$$

Now we have two good reasons to believe that Pascal's triangle tells us the number of bit strings of a given weight: There is a one-to-one correspondence between lattice paths and bit strings, and the same recursive relationship holds for bit strings as it does for generating Pascal's triangle. So we can now use the triangle to count bit strings.

Example 3.1.4

How many 11-bit strings have weight 5?

Solution. There will be $\binom{11}{5}$ such strings. From Pascal's triangle, we see that $\binom{11}{5} = 462$

3.1.4 Subsets and Pizzas

A **subset** of a set A is any set all of whose elements are also in A. Think of starting with the set A and removing some (or none or all) of its elements: the resulting set is a subset of A. (More information about sets can be found in Section 0.2 and Section 5.1.)

Suppose we look at the set $A = \{1, 2, 3, 4, 5\}$. How many subsets of A contain exactly 3 elements? Let's list them all:

$$\begin{array}{cccccc} & \{1,2,3\} & \{1,2,4\} & \{1,3,4\} & \{2,3,4\} & \\ \{1,2,5\} & \{1,3,5\} & \{2,3,5\} & \{1,4,5\} & \{2,4,5\} & \{3,4,5\}. \end{array}$$

Again, we see there are ten. In fact, we have listed them in the same order as we listed the ten 5-bit strings of weight 3 and the ten lattice paths from (0,0) to (3,2). Wait, does this even make sense? In what way is a subset the same as a bit-string?

Think of each bit in a bit string as representing one of the elements in a set. The set A has five elements, so we need five bits to represent a subset of A. If the bit in position n is a 0, that means we do *not* include n in our subset, while a 1 in that position tells us that n is in the subset. Three 1's means we have said, "yes" to three elements.

Example 3.1.5

Which subsets of $\{1, 2, 3, 4, 5, 6\}$ correspond to the bit strings below?

101011 001000 111111 000000

Solution. Here we are not fixing the weight of the strings, so our subsets will not all have the same size. Here is the correspondence:

$$\begin{array}{c|l} 101011 & \{1,3,5,6\} \\ 001000 & \{3\} \\ 111111 & \{1,2,3,4,5,6\} \\ 000000 & \emptyset \end{array}$$

The last subset is the **empty set**: the set that contains no elements (we could have also written $\{\}$). This is a subset of *every* set!

Remark 3.1.6 What we have done here is give a *bijection* between the set of 5-bit strings of weight 3 and the set of 3-element subsets of A. A **bijection** is a function $f : X \to Y$ such that each element of Y is the image of exactly one element from X. You can prove that if there is a bijection between two sets, then they have the same number of elements. This is a common counting technique we will use in the upcoming sections.

This example illustrates that, once again, Pascal's triangle can give us the answer to a counting question. The number of k-element subsets of a set with n elements is the same as the number of n-bit strings of weight k, and that is the number in row n, column k of the triangle: $\binom{n}{k}$.

Example 3.1.7

How many subsets of the set $\{a, b, c, d, e, f, g\}$ have exactly 4 elements?

Solution. The set contains 7 elements, so the number of 4-element subsets is the same as the number of 7-bit strings of weight 4, namely $\binom{7}{4} = 35$.

At this point I'm sure we are all getting pretty hungry, so let's get some pizza. But which pizza shall we order? Let's not overdo it and just choose three toppings from the ten available. How many different pizzas can we order?

Aha! So that's why we care about counting subsets!! Each pizza choice is nothing more than a 3-element subset of the set of 10 toppings. We now know how to count this: $\binom{10}{3} = 120$ different pizzas.

What if we want an Italian soda with dinner? Let's say we want to add two different flavored syrups from the 13 available. How many different sodas are possible? This is just the number of 2-element subsets of a set with 13 elements: $\binom{13}{2} = 78$.

This is why we pronounce $\binom{n}{k}$ as "n choose k". It is the number of ways to choose k items from a collection of n items, since choosing k elements is exactly how you build a k-element subset.

We can view counting lattice paths as choosing k out of n things: Of the n steps on the path, we choose k of them to be in the x direction. Bit strings can also be thought of in this way: Of the n bits in the string, we choose k of them to be 1's.

We can now answer all sorts of real-world counting problems, as long as they are really nothing more than asking for the number of subsets of a set. Pascal's triangle contains all these answers.

Counting Subsets.

The number of k-element subsets of a set with n elements is the number in row n, column k of Pascal's triangle: $\binom{n}{k}$, which we read as "n choose k." This is the number of ways to choose k items from a collection of n items.

3.1.5 Algebra?

Earlier we said that one of the original uses for Pascal's triangle was to solve problems in *algebra*. What does counting subsets (or bit strings or lattice paths) have to do with algebra?

Suppose you expand the binomial expression $(x+1)^6$ (i.e., multiply the binomial $x+1$ by itself six times). This can be tedious to do by hand, but a computer algebra system such as SageMath can do this easily.

```
expand((x+1)^6)
```

```
x^6 + 6*x^5 + 15*x^4 + 20*x^3 + 15*x^2 + 6*x + 1
```

Do the coefficients look familiar? Consider the 6th row of Pascal's triangle:

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1.$$

Why are these the coefficients?

Try it 3.1.8

Modify the SageMath code above to expand $(x+1)^{10}$. What is the coefficient of x^6?

To see why this is more than just a coincidence, let's look at the expansion of $(x+y)^3$ and do it very carefully. We are really multiplying out

$$(x+y)(x+y)(x+y).$$

This means we must distribute the binomials, which looks like the following. (We will use a different typeface for each version of the x and y to keep track of where everything comes from.)

$$\begin{aligned}(x+y)^3 =& (x+y)(\mathbf{x}+\mathbf{y})(\mathcal{X}+\mathcal{Y})\\ =& [(x+y)(\mathbf{x}+\mathbf{y})]\mathcal{X} + [(x+y)(\mathbf{x}+\mathbf{y})]\mathcal{Y}\\ =& [(x+y)\mathbf{x} + (x+y)\mathbf{y}]\mathcal{X} + [(x+y)\mathbf{x} + (x+y)\mathbf{y}]\mathcal{Y}\\ =& [x\mathbf{x} + y\mathbf{x} + x\mathbf{y} + y\mathbf{y}]\mathcal{X} + [x\mathbf{x} + y\mathbf{x} + x\mathbf{y} + y\mathbf{y}]\mathcal{Y}\\ =& x\mathbf{x}\mathcal{X} + y\mathbf{x}\mathcal{X} + x\mathbf{y}\mathcal{X} + y\mathbf{y}\mathcal{X} + x\mathbf{x}\mathcal{Y} + y\mathbf{x}\mathcal{Y} + x\mathbf{y}\mathcal{Y} + y\mathbf{y}\mathcal{Y}.\end{aligned}$$

This repeated distribution results in a sum of terms, each the product of three variables. We see that each term is the result of *choosing* either the x or the y from each of the binomials. For example, the term $x\mathbf{y}\mathcal{X}$ is the result of choosing the x from the first binomial, the $\mathbf{y}$ from the second, and the $\mathcal{X}$ from the third.

Say we want to find the coefficient of the x^2y therm. We collect like terms, collecting all the terms in which we have chosen x two times (and y the other one time). Alternatively, the x^2y term comes from all the strings with two x and one y, just like a bit string or lattice path. No matter how you think of it, the result is that we have $\binom{3}{2} = 3$ terms with the form x^2y.

Hopefully it is clear that this generalizes to the expansion of $(x+y)^n$ for any positive integer n. This is known as the *binomial theorem.*

Theorem 3.1.9 Binomial Theorem.

The nth row of Pascal's triangle gives the coefficients of the expansion of $(x+y)^n$. That is, for any positive integer n,

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.,$$

so the coefficient of $x^k y^{n-k}$ is $\binom{n}{k}$.

For this reason, the numbers in Pascal's triangle are often called **binomial coefficients**.

Example 3.1.10

Without multiplying out the binomial, give the expansion of $(x+y)^5$.

Solution. We take the 5th row of Pascal's triangle: $1, 5, 10, 10, 5, 1$. The expansion is then,

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

3.1.6 Reading Questions

1. Why is the number of lattice paths from $(0,0)$ to $(3,5)$ the same as the number of 8-bit strings with weight 5?

2. Which of the following counting questions have the answer $\binom{11}{5}$? Select all that apply.

A. How many lattice paths are there from $(0,0)$ to $(11,5)$?

B. How many subsets of $\{1, 2, \ldots, 11\}$ contain exactly 5 elements?

C. How many 11-bit strings have weight 5?

D. How many ways can you select 5 flavors of ice cream for a giant sundae from 11 available flavors?

3. The number of subsets of $\{1, 2, \ldots, 8\}$ of size 3 is the same as the number of subsets of $\{1, 2, \ldots, 7\}$ of size either 2 or 3. Explain why this makes sense.

4. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

3.1.7 Practice Problems

1. Use Pascal's triangle to find the numeric values of the following.

(a) $\binom{7}{2}$ (c) $\binom{8}{3}$ (e) $\binom{8}{7}$

(b) $\binom{7}{3}$ (d) $\binom{10}{5}$ (f) $\binom{13}{2}$

2. Compute the following sums of the rows of Pascal's triangle.

$$\binom{2}{0} + \binom{2}{1} + \binom{2}{2}.$$

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}.$$

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}.$$

Now based on your work in the above calculations, give a guess for the sum of the 13th row of Pascal's triangle.

3. Consider the lattice paths from (3,4) to (8,10).
How long is each such path?
How many steps are in the x-direction?
How many different paths are there?

4. How many lattice paths are there from (a, b) to (c, d)? Give your answer as a binomial coefficient, so $\binom{n}{k}$, but say what n and k are.

5. Consider the bit string 100110010.
What is the length of the bit string?
What is the weight of the bit string?
How many bit strings (including this one) have the same length of weight as it?

6. Consider the set $\mathbf{B}_3^8$ of 8-bit strings of weight 3.
How many 1s are there in each bit-string? How many 0s are there in each bit-string?
How many bit strings are there in the set?
How many 8-bit strings of weight 5 are there?

7. Suppose you are ordering a calzone from *D.P. Dough*. You want 5 distinct toppings, chosen from their list of 10 vegetarian toppings.

a. How many choices do you have for your calzone?

b. How many choices do you have for your calzone if you refuse to have green pepper as one of your toppings?

c. How many choices do you have for your calzone if you *insist* on having green pepper as one of your toppings?

How do the three questions above relate to each other? Do you see why this makes sense?

8. How many subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ have six elements?

Of those, how many contain the number 1?

Of the total number of 6-element subsets, how many do NOT contain the number 1?

9. What is the coefficient of x^{14} in $(x + 3)^{16}$?

10. What is the coefficient of x^9 in the expansion of $(x + 2)^{17} + x^4(x + 3)^{21}$?

3.1.8 Additional Exercises

1. How many lattice paths are there from $(0, 0)$ to $(8, 3)$. How many lattice paths are there from $(0, 0)$ to $(3, 8)$? Why does it make sense that these two numbers are the same? Explain your reasoning.

2. Suppose you are counting lattice paths from $(0, 0)$ to $(4, 3)$. We know that the number of such paths is $\binom{4+3}{4} = \binom{7}{4} = 35$. Here is another way to count these paths: Consider the cases for where your first step in the y-direction is. There are five different options here; compute the number of paths for each case.

(a) How many paths are there from $(0, 0)$ to $(4, 3)$ where the first step is in the y-direction (i.e., paths that pass through the point $(0, 1)$)?

(b) How many paths are there from $(0, 0)$ to $(4, 3)$ that first move one step in the x-direction, then one step in the y-direction (so they pass through the points $(1, 0)$ and then $(1, 1)$)?

(c) List the remaining three cases and how many paths there are for each case.

(d) Verify that the sum of the paths in each case is 35. Then describe where all these numbers are in Pascal's triangle. Find another instance of this pattern and verify that it works.

3. Explain why the coefficient of x^5y^3 is the same as the coefficient of x^3y^5 in the expansion of $(x + y)^8$?

4. Consider the expansion of $(x + y)^5$.

(a) Use the binomial theorem to write out the expansion of $(x + y)^5$.

(b) Using the expanded version of $(x + y)^5$, multiply (using the distributive property) $(x + y)^5 \cdot (x + y)$. Simplify your answer, but show all your steps.

(c) How does your previous answer relate to what you get when you apply the binomial theorem to $(x + y)^6$?

3.2 Combining Outcomes

Objectives

After completing this section, you should be able to do the following.

1. Apply the sum and product principles to count the number of outcomes of an event.

2. Solve counting problems using a combination of the sum and product principles.

3. Justify the product principle in terms of the sum principle.

3.2.1 Section Preview

Investigate!

A standard deck of playing cards contains 52 cards. There are four suits: clubs, diamonds, hearts, and spades. Each suit contains cards of 13 values: Ace, 2, 3,..., 10, Jack, Queen, and King.

We call the hearts and diamonds **red cards** (with clubs and spades the **black cards**). In each suit, the Jack, Queen, and King are called **face cards** (the others are call **number cards**).

Suppose you pick two cards from a deck. How many different combinations will have the first card be a red card and the second card be a face card?

Combinatorics is about counting, but looking at the word, we see it really deals with how things *combine*. In this section, we will consider two very different ways that we can combine the outcomes we count in combinatorics problems.

Preview Activity

1. A restaurant offers 8 appetizers and 14 entrées. How many choices do you have if:

   a. you will eat one dish, either an appetizer or an entrée?

   b. you are extra hungry and want to eat both an appetizer and an entrée?

2. Think about the methods you used to solve the questions about appetizers and entrées. Can you frame these as rules for how to combine two numbers of things when counting? Write down the rules for these methods.

3. Do your rules work? Let's apply them to another counting problem to check. Given that a standard deck of playing cards has 12 face cards and 26 red cards, answer the following.

(a) How many ways can you select a card which is either a face card *or* a red card?

(b) How many ways can you select a card which is both a face card *and* a red card?

3.2.2 What are *Outcomes*?

Before we start counting, a quick note about how we ask counting problems. Often counting problems are asked terms of how many ways something can happen. That is, there is some **event** that can result in different **outcomes**, and it is the different outcomes that are counted. Alternatively, we can take a less "active" view by asking for the number of elements in a set, such as the set of bit strings of a particular length and weight. However, this too could be rephrased as a question about an event: if you randomly select one of the bit strings (from the set of bit strings), how many things can happen?

So counting the elements in a set or counting the number of outcomes of an event is not really different. Let's think of every counting problem as asking for the number of elements in a *set* of *outcomes*.

At the most basic level, counting is not hard at all: Just write all the outcomes in a numbered list and see how many numbers you use. Let me tell you about my bow tie collection. I have,

1. A purple bow tie.
2. A green bow tie.
3. A striped bow tie.
4. A paisley bow tie.

How many bow ties do I have (how many ways can I select a bow tie)? Well, the largest number in my numbered list is 4, so I have four bow ties.[1] Of course, as we saw in the previous section, we can use shortcuts to avoid listing out all of the outcomes, such as using Pascal's triangle to find the number of items in our list.

Where counting begins to get interesting is when we want to make a new list that combines two or more lists we have already counted. For example, suppose that in addition to my list of four bow ties, I had a list of seven pairs of novelty socks. I could ask how many choices I have if I wanted to wear either a cool bow tie or a pair of novelty socks. Okay, combine my two lists to make a new list with 11 items: Perhaps first list the four bow ties and then keep counting (starting with 5 for the first pair of socks) until you have listed the seven additional items, ending at 11.

[1] Obviously this example is quite outdated; only four bow ties? What sort of mathematician do you take me for?

In this example, we have combined our two sets of outcomes (picking a bow tie and picking a pair of socks) to get a new set of outcomes (picking a bow tie or pair of socks). That resulting set of outcomes contained 11 elements.

Now consider the question: How many choices do I have if I want to wear both a bow tie and a pair of novelty socks? Again, I need to combine my two sets of outcomes to get a new set of outcomes. But this new set of outcomes will contain 28 elements. How did I know that I should multiply the number of elements in each of my original sets, instead of adding them? We are once again combining outcomes: The resulting set of outcomes that contains 28 elements that are all outcomes consisting of a bow tie and a pair of socks. Now read the previous paragraph! They sound very close to each other!

The difference between the two scenarios is subtle. There are two ways we can think of combining outcomes:

1. We can combine the *sets* of outcomes.

2. We can combine the *outcomes* in the sets.

Being able to distinguish between these operations is a key skill in combinatorics.

Example 3.2.1

Suppose your refrigerator door has some magnetic letters and numerals on it: There are five letters (A, B, C, D, and E) and three numerals (1, 2, and 3). Your brother says that if you combine these sets of magnets, you will get a collection of 8 magnets. Your sister says that you can combine these magnets in 15 ways.

What sets of outcomes are your siblings thinking of?

Solution. Your brother is thinking of combining the sets of outcomes. The set of outcomes for the letters is $\{A, B, C, D, E\}$ and the set of outcomes for the numerals is $\{1, 2, 3\}$. The set of outcomes for the combined set of magnets is $\{A, B, C, D, E, 1, 2, 3\}$, which contains 8 elements.

Your sister is thinking of combining the outcomes themselves. The set of outcomes is $\{A1, A2, A3, B1, B2, B3, C1, C2, C3, D1, D2, D3, E1, E2, E3\}$, which contains 15 elements.

3.2.3 The Sum and Product Principles

Let's carefully write down the rules we can use to combine sets of outcomes and outcomes in sets.

Principle 3.2.2 Sum Principle.

If an event A results in m outcomes, and event B results in n disjoint *outcomes, then the event "A or B" results in $m + n$ outcomes.*

It is important that the events be **disjoint**: i.e., that there is no way for A and B to both happen at the same time. For example, a standard deck of 52 cards contains 26 red cards and 12 face cards. However, the number of ways to select a card which is either red or a face card is not $26 + 12 = 38$. This is because there are 6 cards which are both red and face cards.

Example 3.2.3

How many two-letter "words" start with either A or B? (A **word** is just a sequence of letters; it doesn't have to be English, or even pronounceable.)

Solution. First, how many two-letter words start with A? We just need to select the second letter, which can be accomplished in 26 ways. So there are 26 words starting with A. There are also 26 words that start with B. To select a word that starts with either A or B, we can pick the word from the first 26 or the second 26, for a total of 52 words.

We have two sets of outcomes: the words starting with A and the words starting with B. We create a new set of outcomes by combining the sets to get the set of words that start with either A or B, and by the sum principle, this new set contains 52 outcomes.

We often use the principle when the sets of outcomes need to be counted using other counting techniques.

Example 3.2.4

How many 10-bit strings of weight 6 start with either 11 or 00?

Solution. We learned how to count bit strings with a fixed length and weight in Section 3.1. The total number of 10-bit strings of weight 6 is $\binom{10}{6} = 210$ (where that value was found using Pascal's triangle). However, we want to count just those that start with 11 or 00 (and not those that start with 10 or 01).

Since there are no bit strings that start with both 11 and 00, we can apply the sum principle. We will compute the size of each set of outcomes and then add the results.

How many 10-bit strings of weight 6 start with 11? If the string starts with 11, then of the remaining eight bits, there must be four 1's, so that the total weight is 6. This is just like asking for the number of 8-bit strings with weight 4, so there are $\binom{8}{4} = 70$ such strings. Similarly, there are $\binom{8}{6} = 28$ strings that start with 00 (all six 1's must be in the remaining 8 bits). By the sum principle, the total number of strings we want to count is $70 + 28 = 98$.

We can also use the sum principle *indirectly* in a way that might be called the *subtraction principle* (but isn't).

Example 3.2.5

How many 10-bit strings of weight 6 have at least one 1 in their first three bits?

Solution. There are lots of different ways a string can have at least one 1 in its first three bits: It could start with 100, 010, 001, 101, 110, 011, or 111. We could count the number of strings that start with each of them and then apply the sum principle a bunch of times. But let's not.

Suppose the number of strings we want to count is x. Let y be the number of 10-bit strings of weight 6 that *do not* have a 1 in any of their first three bits. What is $x + y$? This is the number of 10-bit strings of weight 6 in total, which is $\binom{10}{6} = 210$. It is also easy to find y: Bit strings with length 10 and weight 6 that start with 000 must end in a bit string with length 7 and weight 6. There are $\binom{7}{6} = 7$ of these.

Thus, we have

$$x + 7 = 210,$$

so $x = 210 - 7 = 203$.

While we didn't want to use the sum principle multiple times in the previous example, we could have. The sum principle works with more than two events. Say, in addition to your four bow ties and seven pairs of socks, you could also pick one of five belt buckles. How many choices do you have now? You would have $4 + 7 + 5 = 16$ options.

Example 3.2.6

How many two-letter words start with one of the 5 vowels?

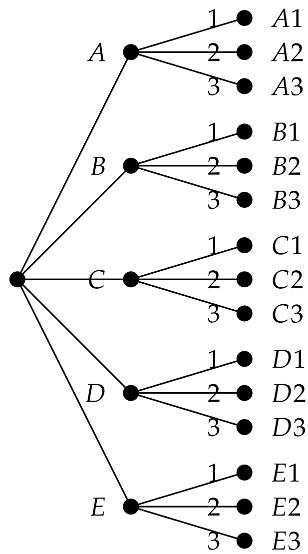
Solution. There are 26 two-letter words starting with A, another 26 starting with E, and so on. By the sum principle, the total number of words will then be $26 + 26 + 26 + 26 + 26 = 130$ Of course it would be easier to just multiply $5 \cdot 26$.

Note that in the previous example, when using the sum principle on a bunch of outcome sets of the same size, it is quicker to multiply. Let's take advantage of this observation and state it as a new principle.

Principle 3.2.7 Product Principle.

If event A can occur in m ways, and each possibility for A allows for exactly n ways for event B, then the event "A and B" can occur in $m \cdot n$ ways.

A nice way to picture the product principle is to think of the outcomes as "levels" in a tree. To illustrate this, consider the counting question of Example 3.2.1: How many ways can you select a letter from $\{A, B, C, D, E\}$ followed by a numeral from $\{1, 2, 3\}$? Here is a tree picture that shows what is going on.



Each leaf of the tree corresponds to an outcome. The tree shows that the three outcomes of the second level are repeated five times, once for each outcome of the first level.

Example 3.2.8

How many two-letter words start with one of the 5 vowels? This time use the product principle to answer the question.

Solution. How can we think of selecting a two-letter word as a sequence of two events that both happen? Event A is "selecting the vowel that starts the word." Event B is "selecting the second letter for the word." By the product principle, there are $5 \cdot 26 = 130$ ways for the event "A and B" to occur.

Of course the solutions to Example 3.2.6 and Example 3.2.8 are the same, but the outcomes we are combining and the ways we are combining them are different.

When applying the sum principle, we combined five disjoint sets of outcomes. One such set of outcomes was,

$$\{AA, AB, AC, AD, \ldots, AZ\}$$

and another was,

$$\{EA, EB, EC, ED, \ldots, EZ\}.$$

That is, each set of outcomes was a set of 2-letter words, starting with a different vowel. The sum principle combined these five sets to create a new set that contained exactly the same elements in the original sets, just all together in one set.

On the other hand, when we applied the product principle, we combined the outcomes in two sets. The first set was,

$$\{A, E, I, O, U\}$$

and the second set was,

$$\{A, B, C, D, \ldots, Z\}.$$

The product principle combines the elements in those sets to make a set of new outcomes (outcomes not present in any of the sets being combined).

As with the sum principle, we often use the product principle to define sets of outcomes that first need to be counted carefully.

Example 3.2.9

How many lattice paths from $(0,0)$ to $(10,10)$ pass through the point $(4,7)$?

Solution. We can count lattice paths with Pascal's triangle, but how do we ensure that the path passes through a particular point? Well, any path that passes through $(4,7)$ must first go from $(0,0)$ to $(4,7)$, and then complete its journey with a path from $(4,7)$ to $(10,10)$.

The number of paths from $(0,0)$ to $(4,7)$ is $\binom{11}{4} = 330$.

The number of paths from $(4,7)$ to $(10,10)$ is $\binom{9}{3} = 84$. (The paths have length 9 and with 6 steps in the x direction.)

Now combine these with the... wait, what principle? To get a path from $(0,0)$ to $(10,10)$ that passes through $(4,7)$, we need to first select a path from $(0,0)$ to $(4,7)$ and then *add on* a path from $(4,7)$ to $(10,10)$. So is this the sum principle???

No! Remember that the sum principle combines the two sets of outcomes. If we applied the sum principle, we would get a larger set containing the $330 + 84$ paths that are contained in each individual set. None of these are paths we want to count. We want to combine the different elements in our sets of outcomes in every possible combination. That is why the product principle is correct here.

Thus, the total number of paths we want to count is $330 \cdot 84 = 27720$.

The product principle generalizes to more than two events as well.

Example 3.2.10

How many license plates can you make out of three letters followed by three numerical digits?

Solution. Here we have six events: the first letter, the second letter, the third letter, the first digit, the second digit, and the third digit. The first three events can each happen in 26 ways; the last three can each happen in 10 ways. So the total number of license plates will be $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10$, using the product principle.

Does this make sense? Think about how we would pick a license plate. How many choices would we have? First, we need to pick the first letter. There are 26 choices. Now for each of those, there are 26 choices for the second letter: 26 second letters with first letter A, 26 second letters with first letter B, and so on. We add 26 to itself 26 times. Or quicker: there are $26 \cdot 26$ choices for the first two letters.

Now for each choice of the first two letters, we have 26 choices for the third letter. That is, 26 third letters for the first two letters AA, 26 choices for the third letter after starting AB, and so on. There are $26 \cdot 26$ of these 26 third letter choices, for a total of $(26 \cdot 26) \cdot 26$ choices for the first three letters. And for each of these $26 \cdot 26 \cdot 26$ choices of letters, we have a bunch of choices for the remaining digits.

In fact, there are going to be exactly 1000 choices for the numbers. We can see this because there are 1000 three-digit numbers (000 through 999). This is 10 choices for the first digit, 10 for the second, and 10 for the third. The product principle says we multiply: $10 \cdot 10 \cdot 10 = 1000$.

All together, there were 26^3 choices for the three letters and 10^3 choices for the numbers, so we have a total of $26^3 \cdot 10^3$ choices of license plates.

The previous example had an answer that used exponents. So is there an **exponential principle** we should learn? We don't separate one out as there really isn't anything different in the way we think of constructing our outcome sets. But we will often see the product principle applied to the same event repeated multiple times. Here are two examples.

Example 3.2.11

Suppose you roll a 12-sided die seven times, recording the number that appears after each roll. How many sequences of seven rolls are possible?

Solution. Each roll can result in 12 different outcomes, so by the product principle, the total number of sequences is 12^7.

Example 3.2.12

How many different pizzas can you make if you can choose from 10 toppings and you can have any number of toppings on your pizza?

Solution. If we want to apply the product principle, we must think of the compound event of picking a pizza as the result of combining some number of individual events in every possible combination. What are those events? There are multiple answers to this question, but the following is the most useful.

First, decide whether you want to include anchovies on your pizza or not. This event could result in two outcomes (yes or no to the anchovies). Second, decide whether you want black olives. Then Canadian bacon, then diced tomatoes, etc. Each event has two outcomes, and we must combine all ten of these outcomes to create a single outcome corresponding to a single pizza. Thus, by the product principle, there are 2^{10} possible pizzas.

By the way, another approach to this problem is to "code" each pizza as a bit-string, where a 1 means the topping is included and a 0 means it is not. Then the number of pizzas is the number of bit-strings of length 10. Each bit in the string has two possible values, so there are 2^{10} 10-bit strings.

Careful: "and" doesn't mean "times." For example, how many playing cards are both red and a face card? Not $26 \cdot 12$. The answer is 6, and we needed to know something about cards to answer that question.

Example 3.2.13 Counting functions.

How many functions $f : \{1, 2, 3, 4, 5\} \to \{a, b, c, d\}$ are there?

Solution. Remember that a function sends each element of the domain to exactly one element of the codomain. To determine a function, we just need to specify the image of each element in the domain. Where can we send 1? There are 4 choices. Where can we send 2? Again, 4 choices. What we have here is 5 "events" (picking the image of an element in the domain) each of which can happen in 4 ways (the choices for that image). Thus there are $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^5$ functions.

This is more than just an example of how we can use the product principle in a particular counting question. What we have here is a general interpretation of certain applications of the product principle using rigorously defined mathematical objects: functions. Whenever we have a counting question that asks for the number of outcomes of a repeated event, we can interpret that as asking for the number of functions from $\{1, 2, \ldots, n\}$ (where n is the number of times the event is repeated) to $\{1, 2, \ldots, k\}$ (where k is the number of ways that event can occur).

3.2.4 Combining Principles

Let's return to this section's *Investigate!* question: How many ways can you select two cards from a standard deck of 52, so that the first one is a red card and the second one is a face card?

This looks a little like the product principle since the outcomes consist of pairs of cards. There are 26 red cards that are outcomes for the first event, and 12 face cards that are outcomes for the second event. However, the answer is not $26 \cdot 12$. The problem is that while there are 26 ways for the first card to be selected, it is not the case that *for each* of those, there are 12 ways to select the second card. If the first card was both red and a face card, then there would be only 11 choices for the second card.

In Section 3.3 we will explore some ways to adjust for counting problems when events are not disjoint, but we can solve our card problem now if we are careful.

Think about the entire set of two-card outcomes. Can we split these outcomes into two disjoint sets (as to apply the sum principle) such that each set can be formed using the product principle?

Of all the outcomes, we first count those that start with a red, non-face card. Of the 26 red cards, there are 6 that are face cards, so there are 20 red, non-face cards. For each of these, there are 12 face cards that can be selected as the second card. By the product principle, the number of two-card outcomes starting with a red, non-face card is $20 \cdot 12 = 240$.

Now, what outcomes have we not yet counted? It is exactly the two-card outcomes that start with a red face card. There are 6 cards we could start with, and for each there are 11 choices for the second card. So the number of two-card outcomes starting with a red face card is $6 \cdot 11 = 66$.

Finally, apply the sum principle to these two disjoint sets of outcomes to see that the total number of two-card outcomes is $240 + 66 = 306$.

We conclude this section with two more examples of how you can use both the sum and product principles in a single counting problem.

Example 3.2.14

How many lattice paths from $(0, 0)$ to $(9, 9)$ pass through either $(2, 6)$ or $(6, 2)$?

Solution. We are quite fortunate that no paths from $(0, 0)$ to $(9, 9)$ pass through both $(2, 6)$ and $(6, 2)$. This means we can apply the sum principle to the two sets of outcomes: the paths that pass through $(2, 6)$ and the paths that pass through $(6, 2)$. We will apply the product principle to determine the number of paths in each of these disjoint sets of outcomes.

The paths from $(0, 0)$ to $(9, 9)$ that pass through $(2, 6)$ are each the concatenation of paths from $(0, 0)$ to $(2, 6)$ and paths from $(2, 6)$ to $(9, 9)$. There are $\binom{8}{2} = 28$ paths for the first part and $\binom{10}{7} = 120$ paths for the second part, so applying the product principle gives us $28 \cdot 120 = 3360$ paths that pass through $(2, 6)$.

The paths from $(0, 0)$ to $(9, 9)$ that pass through $(6, 2)$ can similarly be calculated as

$$\binom{8}{6} \cdot \binom{10}{3} = 28 \cdot 120 = 3360$$

(and upon reflection, it is not surprising that the two numbers are the same, since each path that passes through $(2, 6)$ can be reflected to create a path that passes through $(6, 2)$).

Thus, the total number of paths that pass through either $(2, 6)$ or $(6, 2)$ is $3360 + 3360 = 6720$.

Example 3.2.15

How many two-digit numbers, using only the digits $\{1, 2, 3, 4, 5\}$ have the *sum* of their digits even?

Solution. First think about the set of outcomes. Some two-digit numbers that have the sum of their digits even are $11, 13, 15, 22, 24, 31, \ldots$. Okay, maybe it isn't all that difficult to just list them all (and this would be a nice way to check our work). Let's try to use the sum and product principles anyway.

What do we notice about all these numbers? It looks like both digits must be even or both digits must be odd, to make the sum even. We can consider these two cases, counting each case with the product principle.

Counting "odd-odd" numbers: three choices for the first digit and three choices for the second digit. So there are $3 \cdot 3 = 9$ odd-odd numbers.

Counting "even-even" numbers: two choices for the first digit and two choices for the second digit. So there are $2 \cdot 2 = 4$ even-even numbers.

Combined, there are $9 + 4 = 13$ two-digit numbers with an even sum of digits.

3.2.5 Reading Questions

1. How will you decide if you should add or multiply when combining numbers in a counting problem? Explain how you are currently thinking about this.

2. Your cousin is trying to solve a counting problem about how many different routes he can take between his dorm and his classroom. He has 9 routes that make sense, and similarly, there are 9 routes to go back from his classroom to his dorm.

How many round trips are possible? Your cousin says 18, because after going from his dorm to the classroom, he has to add on a route back to the dorm. Is he right? Why or why not?

3. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

3.2.6 Practice Problems

1. Your wardrobe consists of 6 shirts, 3 pairs of pants, and 15 bow ties. How many different outfits can you make?

2. For your college interview, you must wear a tie. You own 8 regular (boring) ties and 3 (cool) bow ties.

a. How many choices do you have for your neck-wear?

b. You realize that the interview is for clown college, so you should probably wear both a regular tie and a bow tie. How many choices do you have now?

c. For the rest of your outfit, you have 2 shirts, 7 skirts, 3 pants, and 8 dresses. You want to select either a shirt to wear with a skirt or pants, or just a dress. How many outfits do you have to choose from?

3. We usually write numbers in decimal form (or base 10), meaning numbers are composed using 10 different "digits" $\{0, 1, \ldots, 9\}$. Sometimes though it is useful to write numbers hexadecimal or base 16. Now there are 16 distinct digits that can be used to form numbers: $\{0, 1, \ldots, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}\}$. So for example, a 3 digit hexadecimal number might be 2B8.

a. How many 3-digit hexadecimals are there in which the first digit is E or F?

b. How many 6-digit hexadecimals start with a letter (A-F) and end with a numeral (0-9)?

c. How many 4-digit hexadecimals start with a letter (A-F) or end with a numeral (0-9) (or both)?

4. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$

a. How many subsets are there total?

b. How many subsets have $\{2, 3, 5\}$ as a subset?

c. How many subsets contain at least one odd number?

d. How many subsets contain exactly one even number?

5. Let $S = \{1, 2, 3, 4, 5, 6\}$.

a. How many subsets are there of cardinality 4?

b. How many subsets of cardinality 4 have $> \{2, 3, 5\}$ as a subset?

c. How many subsets of cardinality 4 contain at least one odd number?

d. How many subsets of cardinality 4 contain exactly one even number?

6. Let $A = \{1, 2, 3, 4, 5, 6\}$.

(a) How many subsets of A are there?

(b) How many subsets of A contain exactly 5 elements?

(c) How many subsets of A contain only even numbers?

(d) How many subsets of A contain an even number of elements?

7. You break your piggy-bank to discover lots of pennies and nickels. You start arranging these in rows of 4 coins.

a. You find yourself making rows containing an equal number of pennies and nickels. For fun, you decide to lay out every possible such row. How many coins will you need?

b. How many coins would you need to make all possible rows of 4 coins (not necessarily with equal number of pennies and nickels)?

8. How many 11-bit strings contain 4 or more 1's?

9. How many subsets of $\{0, 1, \ldots, 9\}$ have cardinality 7 or more?

10. How many shortest lattice paths start at $(5, 5)$ and

a. end at $(12, 12)$?

b. end at $(12, 12)$ and pass through $(8, 9)$?

c. end at $(12, 12)$ and avoid $(8, 9)$?

3.2.7 Additional Exercises

1. Your Blu-ray collection consists of 9 comedies and 7 horror movies. Give an example of a question for which the answer is:

(a) 16.

(b) 63.

2. The number 735000 factors as $2^3 \cdot 3 \cdot 5^4 \cdot 7^2$. How many divisors does it have? Explain your answer using the multiplicative principle.

3.3 Non-Disjoint Outcomes

Objectives

After completing this section, you should be able to do the following.

1. Use Venn diagrams to count the number of outcomes in the union of non-disjoint sets.
2. Apply the principle of inclusion/exclusion for two and three sets.
3. Explain why the principle of inclusion/exclusion works.

3.3.1 Section Preview

Investigate!

A recent buzz marketing campaign for *The Pie Hole* surveyed patrons on their pie preferences. People were asked whether they enjoyed (A) Apple, (B) Blueberry, or (C) Cherry pie (respondents answered yes or no to each type of pie, and could say yes to more than one type). The following table shows the results of the survey.

Pies enjoyed:	A	B	C	AB	AC	BC	ABC
Number of people:	20	13	26	9	15	7	5

How many of those surveyed enjoy at least one of the types of pie? Also, explain why the answer is not 95.

In Section 3.2 we explored the sum principle as a way of combining two (or more) sets of *disjoint* outcomes. What happens if the outcomes are not disjoint?

Something different needs to be done. For example, when counting the number of playing cards that are either face cards or red cards, we cannot apply the sum principle to claim there are $12 + 26 = 38$ cards. The problem is exactly that six cards are both face cards and red cards. There are a few different ways we can handle this that we will explore in this section.

Preview Activity

1. Consider the eight bit strings of length 3. Let's find the number of strings that start with 1 or have weight 2 (i.e., contain exactly two 1s).

(a) List all the 3-bit strings that start with 1.

(b) List all the 3-bit strings that have weight 2.

(c) Now list all the 3-bit strings that start with 1 or have weight 2. But be lazy: Don't list them from scratch. Use your lists from the two tasks above.

(d) How many strings are there in the lists?

- Strings that start with 1: _____
- Strings of weight 2: _____
- Strings that start with 1 or have weight 2: _____
- Strings that both start with a 1 *and* have weight 2: _____

2. How did you combine your two lists above? Explain how you did it. Then think of another method you could have used, and explain how that would be different.

3. Xiang and Omari are discussing their favorite mathematicians throughout history. Xiang has a list of 7 favorites, while Omari has a list of 5. They discover that they have 3 mathematicians in common. How many mathematicians are on their combined list?

3.3.2 Counting with Venn Diagrams

To understand how to deal with combining sets of outcomes that are not disjoint, it will be helpful to use some notation from set theory, which we will briefly review here.

Let A be the set of outcomes for the first event and B be the set of outcomes for the second event. When we ask for the number of ways either event can happen, the new set of outcomes consists of all outcomes that are in A, or B, or both. This is nothing more than the **union** of the set A and B, written $A \cup B$.

The sum principle only applies when no outcome belongs to both events. The elements that are common to sets A and B are the sets in their **intersection**, written $A \cap B$. The intersection of two sets is itself a set. If sets A and B are **disjoint**, then there are *no* elements in the intersection, so the intersection is the **empty set**, written $\emptyset$.

We are counting the *number* of outcomes, so what we are really interested in is the **size** (or **cardinality**) of the sets. We write the size of a set X as $|X|$.

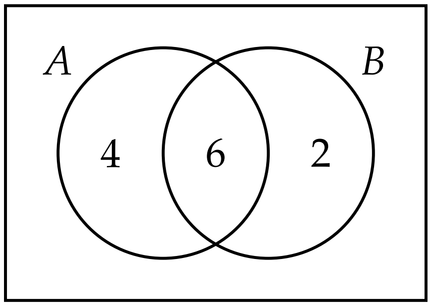
With this notation in hand, we can restate the sum principle as follows.

Principle 3.3.1 Sum Principle (with sets).

Given two sets A and B, if $A \cap B = \emptyset$, then

$$|A \cup B| = |A| + |B|\,.$$

Now consider what happens to the sum principle when the sets are NOT disjoint. Suppose we want to find $|A \cup B|$ and know that $|A| = 10$ and $|B| = 8$. If we knew that the sets were disjoint, then $|A \cup B|$ would be 18. But if we don't have that the sets are disjoint, we need more information We must know how many of the 8 elements in B are also elements of A. Suppose we also know that $|A \cap B| = 6$. Now we can say exactly how many elements are in A, and, of those, how many are in B and how many are not (6 of the 10 elements are in B, so 4 are in A but not in B). We could fill in a Venn diagram as follows:



This says there are 6 elements in $A \cap B$, 4 elements in A but not B (which we can write as $|A \setminus B| = 4$), and 2 elements in B but not A (written $|B \setminus A| = 2$). Now *these* three sets *are* disjoint, so we can use the sum principle to find the number of elements in $A \cup B$. It is $6 + 4 + 2 = 12$.

Example 3.3.2

How many 7-bit strings of weight 4 start with 11 or end with 00, or both?

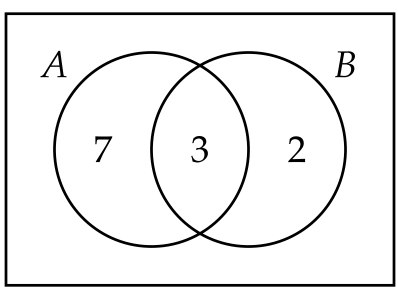
Solution. Let A be the set of 7-bit strings of weight 4 that start with 11. We have $|A| = \binom{5}{2} = 10$. Let B be the set of 7-bit strings of weight 4 that end with 00. We have $|B| = \binom{5}{4} = 5$.

But it is not true that $|A \cup B| = 10 + 5 = 15$, since strings like 1101100 both start with 11 and end with 00. That is included in the 10 strings starting with 11 and the five strings ending with 00. We must find the number of strings that belong to both A and B. We have

$$|A \cap B| = \binom{3}{2} = 3$$

(and indeed, there are three strings in the intersection: 1111000, 1110100, and 1101100).

We can now fill in a Venn diagram:



The circle for A contains a total of 10 elements, the 3 that also belong to B

and 7 more that belong to just A. The circle for B contains a total of 5 elements, the 3 that also belong to A and 2 more that belong to just B. The number of elements in $A \cup B$ is the sum of the elements in each region

$$||A \cup B| = 7 + 3 + 2 = 12.$$

We can do something similar with three sets.

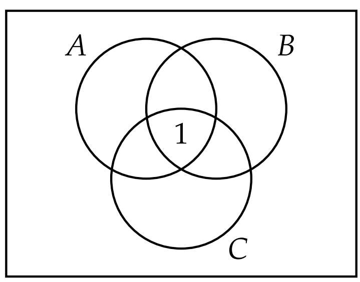
Example 3.3.3

An examination in three subjects, Algebra, Biology, and Chemistry, was taken by 41 students. The following table shows how many students failed in each single subject and in their various combinations:

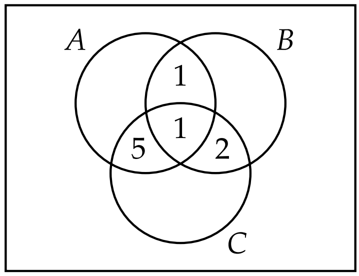
Subject:	A	B	C	AB	AC	BC	ABC
Failed:	12	5	8	2	6	3	1

How many students failed at least one subject?

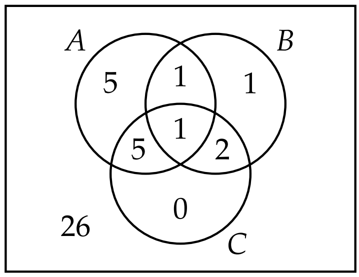
Solution. The answer is not 37, even though the sum of the numbers above is 37. For example, while 12 students failed Algebra, 2 of those students also failed Biology, 6 also failed Chemistry, and 1 of those failed all three subjects. In fact, that 1 student who failed all three subjects is counted a total of 7 times in the total 37. To clarify things, let us think of the students who failed Algebra as the elements of the set A, and similarly for sets B and C. The one student who failed all three subjects is the lone element of the set $A \cap B \cap C$. Thus, in Venn diagrams:



Now let's fill in the other intersections. We know $A \cap B$ contains 2 elements, but 1 element has already been counted. So we should put a 1 in the region where A and B intersect (but C does not). Similarly, we calculate the cardinality of $(A \cap C) \setminus B$, and $(B \cap C) \setminus A$:



Next, we determine the numbers that should go in the remaining regions, including outside of all three circles. This last number is the number of students who did not fail any subject:



We found 5 goes in the "A only" region because the entire circle for A needed to have a total of 12, and 7 were already accounted for. Similarly, we calculate the "B only" region to contain only 1 student and the "C only" region to contain no students.

Thus the number of students who failed at least one class is 15 (the sum of the numbers in each of the eight disjoint regions). The number of students who passed all three classes is 26: the total number of students, 41, less the 15 who failed at least one class.

Note that we can also answer other questions. For example, how many students failed just Chemistry? None. How many passed Algebra but failed both Biology and Chemistry? This corresponds to the region inside both B and C but outside of A, containing 2 students.

Here is an interesting math fact.

$$12 + 5 + 8 - 2 - 6 - 3 + 1 = 15.$$

Is it a coincidence that this way of combining the numbers of students who failed each subject in the example above gives the same number of students who failed at least one subject? This suggests that there might be a simpler way to find the size of a union of non-disjoint sets.

3.3.3 The Principle of Inclusion/Exclusion

It would be nice to write an algebraic formula that captures what we have done in the previous examples. Even better would be for this algebraic formula to be a generalization of the case where the sets *are* disjoint.

Consider again the example where $|A| = 10$, $|B| = 8$ and $|A \cap B| = 6$. We said that $|A \cup B| = 12$, found by adding $4 + 6 + 2$.

If A and B had been disjoint, then $|A \cup B|$ would have been $|A| + |B| = 10 + 8 = 18$. We see that is off by exactly 6, which just so happens to be $|A \cap B|$. So perhaps we guess,

$$|A \cup B| = |A| + |B| - |A \cap B| .$$

This makes sense! When we add the number of elements in A to the number of elements in B, we have counted the six elements that belong to both sets exactly twice. So if we subtract them out, we have counted them exactly once.

In other words, we have:

Theorem 3.3.4 Cardinality of a Union (2 Sets).

For any finite sets A and B,

$$|A \cup B| = |A| + |B| - |A \cap B| .$$

Example 3.3.5

How many 7-bit strings of weight 4 start with 11 or end with 00, or both? Use Theorem 3.3.4 and compare to Example 3.3.2.

Solution. We have $|A| = \binom{5}{2} = 10$, $|B| = \binom{5}{4} = 5$, and $|A \cap B| = \binom{3}{2} = 3$. Therefore:

$$|A \cup B| = 10 + 5 - 3 = 12.$$

This makes sense since the three bit strings that start with 11 and end with 00 are counted in the 10 strings starting with 11 and again among the 5 strings ending with 00, so we have over-counted these exactly one time before we subtract them out.

For three sets, we can also count the elements in the union by carefully removing elements we have counted multiple times. Since there are more ways for the sets to overlap, the formula is more complicated.

Theorem 3.3.6 Cardinality of a Union (3 Sets).

For any finite sets A, B, and C,

$$\begin{aligned}|A \cup B \cup C| = & |A| + |B| + |C| \\ & - |A \cap B| - |A \cap C| - |B \cap C| \\ & + |A \cap B \cap C| .\end{aligned}$$

To determine how many elements are in at least one of A, B, or C we add up all the elements in each of those sets. However, when we do that, any element in both

A and B is counted twice. Also, each element in both A and C is counted twice, as are elements in B and C, so we take each of those out of our sum once. But now what about the elements which are in $A \cap B \cap C$ (in all three sets)? We added them in three times, but also removed them three times. They have not yet been counted. Thus we add those elements back in at the end.

Example 3.3.7

How many of the numbers in $\{1, 2, \ldots, 50\}$ are multiples of 2, 3, or 5?

Solution. Of the numbers in $\{1, 2, \ldots, 50\}$, let A be the set of multiples of 2, B be the set of multiples of 3, and C be the set of multiples of 5. We have $|A| = 25$, $|B| = 16$, and $|C| = 10$. (These numbers can be found by division; for example, since a third of the numbers are multiples of three, we can compute $50/3 = 16.66$, so there are 16 multiples of 3 in the set.) Some of the numbers in the set are multiples of both 2 and 3, such as $6, 12, 18, \ldots$ all the multiples of 6. There are $|A \cap B| = 8$ such numbers. Similarly, there are $|A \cap C| = 5$ multiples of both 2 and 5 (multiples of 10), and $|B \cap C| = 3$ multiples of both 3 and 5 (multiples of 15). There is $|A \cap B \cap C| = 1$ multiples of all three (just the number 30).

Using Theorem 3.3.6, we have:

$$|A \cup B \cup C| = 25 + 16 + 10 - 8 - 5 - 3 + 1 = 36.$$

Let's use this example to understand the theorem better. Consider the number 21. In which sets is it counted? It is in B, but not A or C, so it is counting among the 16 multiples of 3 and not counted in any of the other sets, so it is included exactly once in our complete count. Similarly for all other numbers that are multiples of just one of 2, 3, or 5.

The number 18 is in sets A and B, and so also in $A \cap B$. It is counted in the 25 multiples of 2, the 16 multiples of 3, and the 8 multiples of 6. So it is added twice in our total and subtracted once, meaning it is counted exactly one time. The same will be true of all numbers that belong to exactly two of the individual sets.

What about 30? It belongs to all the sets. It is therefore added to our total three times, when we add the multiples of 2, 3, and 5. But it is also subtracted three times, once for each of the three pairs of sets. So it is not counted at all in our total... until we add it back in in the last step, after which it is counted exactly once.

This process of adding in, then subtracting out, then adding back in, and so on is called the **Principle of Inclusion/Exclusion**, or simply **PIE**. This principle can be applied to any number of sets, but the formula becomes more and more complicated the more sets you have. For example, this is what PIE looks like for four sets.

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D|$$

$$-|A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|$$
$$+|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|$$
$$-|A \cap B \cap C \cap D|\,.$$

Phew! And for five sets it is... just kidding. But even if we don't write it down, hopefully we all see what you *would* have to write down. In practice, PIE for more than three sets is only used when we can simplify the computation by recognizing that, for example, all the intersections of two sets will have the same size. Some examples of this are explored in Section 3.8.

3.3.4 Overlaps and the Product Principle

Everything we have considered so far in this section has been about how the sum principle applies when the sets of outcomes are not disjoint. Do we need to be similarly worried about overlaps when applying the product principle?

Not really.

One use of the product principle is to compute probabilities, and in that context, we do make a distinction between events being **independent** or **dependent**. We will explore this more in Section 3.7. Essentially, to find the probability of two events occuring, we can multiply their respective probabilities if and only if the two events are *independent*: if the outcome of one event does not affect the outcome of the other.

While finding probability is closely related to counting outcomes, the product principle itself does not require that sets of outcomes be disjoint or make a distinction between dependent or independent events. What *does* matter is that the number of outcomes for the second event does not change based on which outcome was the result of the first event.

A distinction we *do* often make when applying the product principle is whether the outcomes of events can be "repeated." Consider the following example.

Example 3.3.8

How many 3-letter words (sequences of three letters) are there when,

1. repeats are allowed?
2. repeats are not allowed?

Solution. Being careful with the product principle, we are combining three events: A is the event of selecting the first letter, B is the event of selecting the second letter, and C is the event of selecting the third letter. If we want to count the number of words when letters in the word can be repeated, then each event contains 26 outcomes. So the number of 3-letter words is $26^3 = 17,576$ when repeats are allowed.

When repeats are not allowed, then the number of outcomes for event A is still 26, but the number of outcomes for B is only 25, since no matter what the outcome of event A is, that letter cannot be selected for the second position in the word. Similarly, no matter what the outcomes of event A and B are, the number of outcomes for event C is always 24: any letter not selected in the first two events. Thus the number of 3-letter words containing no repeated letters is $26 \cdot 25 \cdot 24 = 15,600$.

It's a little strange that the actual set of outcomes for event B is different for each possible outcome of event A. This doesn't matter though, as long as the *size* of the outcome set doesn't change.

We will explore problems such as the previous example in much more depth next in Section 3.4. Before that though, here is a final example of where the product principle does NOT easily work.

Example 3.3.9

Explain why the product principle does NOT apply to the following counting question: How many 3-letter words have their letters in alphabetical order?

Solution. The first thing we might try is to make event A be selecting the first letter, B be selecting the second letter, and C be selecting the third letter. There are 26 outcomes to event A. How many outcomes are there for event B? If we selected w as the first letter, then we can select a letter in the set $\{w, x, y, z\}$ as our second letter, so it appears there are 4 outcomes for event B. However, if we selected a as our first letter, then any letter could be used for the second letter, so event B has 26 outcomes. Oh no!

The problem is that the number of outcomes for the second (and third) letter in the word changes depending on the outcome of the previous event. The events are not independent, but more than that, their *size* is not independent, so the product principle does not apply.

We will see how to answer the counting problem above in the case that letters can be repeated in Section 3.5. If we restricted the question further and required that the letters be distinct (and in alphabetical order), we already know how to answer the question. Since for any *set* of three letters, there is exactly one 3-letter word that has those letters in alphabetical order, we can simply count the number of sets of three letters. This is $\binom{26}{3}$. Then we just look at Pascal's triangle to find the value... oh shoot. Our triangle doesn't go down that far. I guess we should think of a way to compute the value without the triangle. Read on!

3.3.5 Reading Questions

1. Which of the following best describes what the Principle of Inclusion/Exclusion is used for?

   A. To count the size of a union of two or more not-necessarily disjoint sets.

   B. To count the size of the union of two or more sets, as long as they are disjoint.

C. To count the size of the intersection of two or more independent sets of outcomes.

D. To find the size of the intersection of dependent sets of outcomes.

2. Why does the Principle of Inclusion/Exclusion *add* the size of the intersection of three sets, rather than subtract? Explain in your own words.

3. What questions do you have after reading this section? Write at least one question about the section you are curious about.

3.3.6 Practice Problems

1. Suppose you have sets A and B with $|A| = 9$ and $|B| = 19$.

(a) What is the largest possible value for $|A \cap B|$?

(b) What is the smallest possible value for $|A \cap B|$?

(c) What are the possible values for $|A \cup B|$?

2. If $|A| = 5$ and $|B| = 2$, what is $|A \cup B| + |A \cap B|$?

3. A group of college students was asked about their TV watching habits. Of those surveyed, 30 students watch *The Walking Dead*, 24 watch *The Blacklist*, and 28 watch *Game of Thrones*. Additionally, 18 watch *The Walking Dead* and *The Blacklist*, 10 watch *The Walking Dead* and *Game of Thrones*, and 12 watch *The Blacklist* and *Game of Thrones*. There are 7 students who watch all three shows. How many students surveyed watched at least one of the shows?

4. In a recent survey, 100 students reported whether they liked their potatoes mashed, French-fried, or twice-baked. 54 liked them mashed, 39 liked French fries, and 51 liked twice baked potatoes. Additionally, 18 students liked both mashed and French-fried potatoes, 28 liked French fries and twice baked potatoes, 29 liked mashed and baked, and 11 liked all three styles. How many students *hate* potatoes? Explain why your answer is correct.

5. How many 13-bit strings (that is, bit strings of length 13) are there which:

a. Start with the sub-string 011?

b. Have weight 8 (i.e., contain exactly 8 1's) and start with the sub-string 011?

c. Either start with 011 or end with 01 (or both)?

d. Have weight 8 and either start with 011 or end with 01 (or both)?

6. For how many $n \in \{1, 2, \ldots, 715\}$ is n a multiple of one or more of 4, 3, or 5?

7. How many positive integers less than 1400 are multiples of 4, 7, or 9? Use the Principle of Inclusion/Exclusion.

8. We want to build 13 letter "words" using only the first $n = 12$ letters of the alphabet. For example, if $n = 5$ we can use the first 5 letters, $\{a, b, c, d, e\}$ (Recall, words are just strings of letters, not necessarily actual English words.)

a. How many of these words are there total?

b. How many of these words contain no repeated letters?

c. How many of these words start with the sub-word "ade"?

d. How many of these words either start with "ade" or end with "be" or both?

e. How many of the words containing no repeats also do not contain the sub-word "bed"?

9. Gridtown USA, besides having excellent donut shops, is known for its precisely laid out grid of streets and avenues. Streets run east-west, and avenues north-south, for the entire stretch of the town, never curving and never interrupted by parks or schools or the like.

Suppose you live on the corner of 4th and 4th and work on the corner of 19th and 19th. Thus you must travel 30 blocks to get to work as quickly as possible.

a. How many different routes can you take to work, assuming you want to get there as quickly as possible?

a. Now suppose you want to stop and get a donut on the way to work, from your favorite donut shop on the corner of 18th Ave. and 16th St. How many routes to work, stopping at the donut shop, can you take (again, ensuring the shortest possible route)?

a. Disaster Strikes Gridtown: there is a pothole on 7th Ave. between 6th St. and 7th St. How many routes to work can you take avoiding that unsightly (and dangerous) stretch of road?

a. The pothole has been repaired (phew!) and a new donut shop has opened on the corner of 7th Ave. and 6th St. How many routes to work drive by one or the other (or both) donut shops? Hint: the donut shops serve PIE.

3.3.7 Additional Exercises

1. Let A, B, and C be sets.

(a) Find $|(A \cup C) \setminus B|$ provided $|A| = 50$, $|B| = 45$, $|C| = 40$, $|A \cap B| = 20$, $|A \cap C| = 15$, $|B \cap C| = 23$, and $|A \cap B \cap C| = 12$.

(b) Describe a set in terms of A, B, and C with cardinality 26.

2. For how many three-digit numbers (100 to 999) is the *sum of the digits* even? (For example, 343 has an even sum of digits: $3 + 4 + 3 = 10$ which is even.) Find the answer and explain why it is correct in at least two *different* ways.

3.4 Combinations and Permutations

Objectives

After completing this section, you should be able to do the following.

1. Correctly decide between using a combination or a permutation when solving a counting problem.
2. Apply the correct combination or permutation to solve a counting problem.
3. Explain the relationship between combinations and permutations and why their formulas are correct.

3.4.1 Section Preview

Investigate!

You have decided to decorate your magic wand with bands of different colored tape. You have 10 different colors to choose from, and you will use five of them to create five different stripes of color. How many different wand designs are possible?

The product principle gives us a way to count the number of outcomes when each outcome is made by combining smaller pieces. A typical example of this is to count the number of three-letter "words"; each outcome (word) we count is made up of a combination of three smaller pieces (letters). Since there are 26 choices for each smaller piece, there are $26 \cdot 26 \cdot 26 = 26^3$ possible outcomes.

The product principle does not require that each piece that is being combined be chosen from a set of the same size. We will use this observation to create a standard way to count outcomes when the pieces are chosen from a fixed set, but without allowing for any piece to be used more than once. For example, we can ask how many 3-letter words there are that contain distinct letters.

These arrangements are called **permutations**. We will also consider another counting technique where we count **combinations**, which is related but counts something different. We will explore how these two counting techniques are related and how they can be used to solve a wide range of counting problems.

Preview Activity

You have a bunch of poker chips that come in five different colors: red, blue, green, purple, and yellow.

1. How many two-chip stacks are possible where the bottom chip must be red or blue?

(a) List all possible two-chip stacks. For example, the stack with a red chip on bottom and a green chip on top can be listed as "RG".

(b) Using the additive principle, we notice that there are _____ stacks that have blue on the bottom, another _____ stacks that have red on the bottom, so there are a total of __________ possible stacks.

(c) If we use the multiplicative principle, then there are _____ choices for the bottom chip and _____ choices for the top chip, so there are __________ possible stacks.

2. How many different three-chip stacks are possible if the bottom chip must be red or blue and the top chip must be green, purple, or yellow?

Hint. How does this question relate to the previous question? Is there something we can do to the 10 two-chip stacks to make them into three-chip stacks?

3. How many different three-chip stacks are there in which no color is repeated?

(a) First, how many three-chip stacks with no repeated color have blue on the bottom and green in the middle?

And how many three-chip stacks with no repeated color have blue on the bottom and yellow in the middle?

In fact, for any stack with blue on the bottom and some other color in the middle, there are _____ possible stacks.

(b) If we insist that blue is on the bottom, how many choices do we have for the color of the middle chip?

Combining this with the answer from the previous question, how many three-chip stacks with no repeated color have blue on the bottom?

(c) Of course, we didn't need to start with blue on the bottom. How many choices do we have for the color of the bottom chip?

So how many three-chip stacks with no repeated color are there?

(d) How many four-chip sticks are there with no repeated color?

4. Suppose you wanted to take three chips with different colors and put them in your pocket.

(a) One outcome is taking the blue, green, and purple chips. How many of the three-chip stacks of different color chips correspond to this single pocketful?

Hint. With these three colors, how many choices do you have for which chip is on the bottom? In the middle? On top?

(b) How many different stacks of chips would result in picking up the red, yellow, and green chips?

(c) So of the _____ possible three-chip stacks, we can group the chips into groups of size _____, where each group corresponds to the same pocketful of chips. How many different pocketfuls of chips are there?

(d) How many different pocketfuls of chips are there if you take four chips?

3.4.2 Counting Sequences

A **permutation** is a (possible) rearrangement of objects. For example, there are 6 permutations of the letters a, b, c:

$$abc, \quad acb, \quad bac, \quad bca, \quad cab, \quad cba.$$

In terms of our discrete structures, each permutation is really a *sequence* or *tuple* of a fixed length.

We know that we have them all listed above —there are 3 choices for which letter we put first, then 2 choices for which letter comes next, which leaves only 1 choice for the last letter. The multiplicative principle says we multiply $3 \cdot 2 \cdot 1$.

Example 3.4.1

How many sequences (permutations) are there of the letters a, b, c, d, e, f?

Solution. We do NOT want to try to list all of the length 6 sequences of these letters. However, if we did, we would need to pick a letter to write down first. There are 6 choices for that letter. For each choice of the first letter, there are 5 choices for the second letter (we cannot repeat the first letter; we are rearranging letters and only have one of each), and for each of those, there are 4 choices for the third, 3 choices for the fourth, 2 choices for the fifth, and finally only 1 choice for the last letter. So there are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ permutations of the 6 letters.

A piece of notation is helpful here: $n!$, read "n factorial", is the product of all positive integers less than or equal to n (for reasons of convenience, we also define $0!$ to be 1). So the number of permutations of 6 letters, as seen in the previous example is $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. This generalizes:

Theorem 3.4.2 Permutations of n Elements.

There are $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$ permutations of n (distinct) elements.

Example 3.4.3 Counting Bijective Functions.

How many functions $f : \{1, 2, \ldots, 8\} \to \{1, 2, \ldots, 8\}$ are *bijective*?

Solution. Remember what it means for a function to be bijective: Each element in the codomain must be the image of exactly one element of the domain. Using two-line notation, we could write one of these bijections as

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 5 & 8 & 7 & 6 & 2 & 4 \end{pmatrix}.$$

What we are really doing is just rearranging the elements of the codomain, so we are creating a permutation of 8 elements. In fact, "permutation" is another term used to describe bijective functions from a finite set to itself.

If you believe this, then you see the answer must be $8! = 8 \cdot 7 \cdot \cdots \cdot 1 = 40320$. You can see this directly as well: For each element of the domain, we must pick a distinct element of the codomain to map to. There are 8 choices for where to send 1, then 7 choices for where to send 2, and so on. We multiply using the multiplicative principle.

Sometimes we do not want to permute all of the letters/numbers/elements we are given.

Example 3.4.4

How many four-letter "words" can you make from the letters a through g, with no repeated letters?

Solution. This is just like the problem of permuting four letters, only now we have more choices for each letter. For the first letter, there are 7 choices. For each of those, there are 6 choices for the second letter. Then there are 5 choices for the third letter and 4 choices for the last letter. The total number of words is $7 \cdot 6 \cdot 5 \cdot 4 = 840$.

This is not 7! because we never multiplied by 3, 2, or 1. We could write it using 7! though, if we cancel the 3, 2, and 1. Thus we could write the answer as

$$\frac{7!}{3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 7 \cdot 6 \cdot 5 \cdot 4.$$

In general, we can ask how many permutations exist of k objects choosing those objects from a larger collection of n objects. (In the example above, $k = 4$, and $n = 7$.) We write this number $P(n, k)$ and sometimes call it a **k-permutation of n elements**.

From the example above, we see that to compute $P(n, k)$ we must apply the multiplicative principle to k numbers, starting with n and counting backwards. For example

$$P(10, 4) = 10 \cdot 9 \cdot 8 \cdot 7.$$

Notice that $P(10,4)$ starts out looking like 10!, but we stop after 7. We can formally account for this "stopping" by dividing away the part of the factorial we do not want:

$$P(10,4) = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{10!}{6!}.$$

Careful: The factorial in the denominator is not 4! but rather $(10-4)!$.

Definition 3.4.5 k-permutations of n elements.

$P(n,k)$ is the number of **k-permutations of n elements**, the number of ways to *arrange* k objects chosen from n distinct objects.

Theorem 3.4.6

The number of k-permutations of n elements is

$$P(n,k) = \frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-(k-1)).$$

Note that when $n = k$, we have $P(n,n) = \frac{n!}{(n-n)!} = n!$ (since we defined 0! to be 1). This makes sense —we already know $n!$ gives the number of permutations of all n objects.

Example 3.4.7 Counting injective functions.

How many functions $f : \{1,2,3\} \to \{1,2,3,4,5,6,7,8\}$ are *injective*?

Solution. Note that it doesn't make sense to ask for the number of *bijections* here, as there are none (because the codomain is larger than the domain, there are no surjections). But for a function to be injective, we just can't use an element of the codomain more than once.

We need to pick an element from the codomain to be the image of 1. There are 8 choices. Then we need to pick one of the remaining 7 elements to be the image of 2. Finally, one of the remaining 6 elements must be the image of 3. So the total number of functions is $8 \cdot 7 \cdot 6 = P(8,3)$.

What this demonstrates in general is that the number of injections $f : A \to B$, where $|A| = k$ and $|B| = n$, is $P(n,k)$.

3.4.3 Counting Sets

Let's consider another way to count sequences: First count sets, then arrange them.

Example 3.4.8

Your basketball team has 12 players. Assuming everyone can play every position, how many ways can you choose 5 players to be on the court at the same time?

Solution. This question is actually too vague. Do we mean how many ways can we select five players? Or do we mean how many ways can we pick five players to fill the five positions?[2] Let's answer both of these questions.

First, if we just want to select five out of the 12 players, that is just like picking five out of 12 pizza toppings (although less delicious). We know that there are $\binom{12}{5}$ ways to do this, and from Pascal's triangle we know that this is 792.

On the other hand, if we wanted to pick five players for the five different positions,... well, we could start by picking one of the 792 different *sets* of five players, and then permute them into the five positions. Of the five players on the court, we pick one of the five to be the point guard, then one of the remaining four to be the shooting guard, and so on. This gives us $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$ ways to arrange the players. So the total number of ways to pick five players for the five positions is

$$\binom{12}{5} \cdot 5! = 792 \cdot 120 = 95,040.$$

Wait. We could have found that number directly. Without choosing the five players first, we have 12 choices for the point guard, then 11 choices for the shooting guard, and so on. So the total number of ways to pick five players for the five positions is simply the permutation

$$P(12,5) = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = 95,040.$$

Thank goodness that is the same answer!

The example above illustrates a second way to compute the number of k-permutations of n elements: First select which k elements will be in the permutation, then count how many ways there are to arrange them. Once you have selected the set of k objects, we know there are $k!$ ways to arrange (permute) them. But how do you select k objects from the n? You have n objects, and you need to *choose* k of them. You can do that in $\binom{n}{k}$ ways.

Using the multiplicative principle to combine the two steps, we get another formula for $P(n,k)$:

$$P(n,k) = \binom{n}{k} \cdot k!.$$

[2] I'm told the five positions are called point guard, shooting guard, small forward, power forward, and center. Who knew?

This is HUGE!
We have a closed formula for $P(n, k)$ already. We can substitute that in:

$$\frac{n!}{(n-k)!} = \binom{n}{k} \cdot k!.$$

If we then divide both sides by $k!$, we get a closed formula for $\binom{n}{k}$.

Theorem 3.4.9 Closed Formula for $\binom{n}{k}$.

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)(n-2)\cdots(n-(k-1))}{k(k-1)(k-2)\cdots 1}.$$

Another name for the collections of things that $\binom{n}{k}$ counts is **combinations**. We say that $\binom{n}{k}$ counts the number of k-element combinations of n elements. Sometimes we even use the notation $C(n, k)$ instead of $\binom{n}{k}$.

Clearly combinations and permutations are very closely related. The formulas to count each are very similar; there is just an extra $k!$ in the denominator of $\binom{n}{k}$. That extra $k!$ accounts for the fact that $\binom{n}{k}$ does not distinguish between the different orders that the k objects can appear in. We are just selecting (or choosing) the k objects, not arranging them.

Example 3.4.10

You decide to have a dinner party. Even though you are incredibly popular and have 14 different friends, you only have enough chairs to invite 6 of them.

1. How many choices do you have for which 6 friends to invite?
2. What if you need to decide not only which friends to invite but also where to seat them along your long table? How many choices do you have then?

Solution.

1. You must simply choose 6 friends from a group of 14. This can be done in $\binom{14}{6}$ ways. We can find this number either by using Pascal's triangle or the closed formula: $\frac{14!}{8!\cdot 6!} = 3003$.
2. Here you must count all the ways you can permute 6 friends chosen from a group of 14. So the answer is $P(14, 6)$, which can be calculated as $\frac{14!}{8!} = 2162160$.

   Notice that we can think of this counting problem as a question about counting functions: How many injective functions are there from your set of 6 chairs to your set of 14 friends (the functions are injective because you can't have a single chair go to two of your friends).

How are these numbers related? Notice that $P(14,6)$ is *much* larger than $\binom{14}{6}$. This makes sense. $\binom{14}{6}$ picks 6 friends, but $P(14,6)$ arranges the 6 friends as well as picks them. In fact, we can say exactly how much larger $P(14,6)$ is. In both counting problems we choose 6 out of 14 friends. For the first one, we stop there, at 3003 ways. But for the second counting problem, each of those 3003 choices of 6 friends can be arranged in exactly 6! ways. So now we have $3003 \cdot 6!$ choices and that is exactly 2162160.

Alternatively, look at the first problem another way. We want to select 6 out of 14 friends, but we do not care about the order they are selected in. To select 6 out of 14 friends, we might try this:

$$14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9.$$

This is a reasonable guess, since we have 14 choices for the first guest, then 13 for the second, and so on. But the guess is wrong (in fact, that product is exactly $2162160 = P(14,6)$). It distinguishes between the different orders in which we could invite the guests. To correct for this, we could divide by the number of different arrangements of the 6 guests (so that all of these would count as just one outcome). There are precisely 6! ways to arrange 6 guests, so the correct answer to the first question is

$$\frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{6!}.$$

Note that another way to write this is

$$\frac{14!}{8! \cdot 6!}.$$

which is what we had originally.

Perhaps "combination" is a misleading label. We don't mean it like a combination lock (where the order would definitely matter). Perhaps a better metaphor is a combination of flavors — you just need to decide which flavors to combine, not the order in which to combine them.

So how do you know when to use a combination and when to use a permutation?

Does order matter? Almost every source you look at about combinations and permutations will make some variation of the following claim.

> Use a permutation when the order matters, and use a combination when the order does not matter.

What does this even mean? And why is it an absolutely awful way to distinguish between the two counting techniques? Let's explore with an example.

Suppose you are proposing a new lottery game. In the game, five numbered balls will be randomly shot out of the machine that holds balls numbered 1 to 50. If a player correctly picks the five numbers that come out, they win the jackpot.

Of course, we want to know the odds of winning this lottery game, which requires us to know how many different outcomes there are. The balls fly out of the machine randomly, so one result might be to get the balls

$$26, 5, 42, 17, 33.$$

If you were holding the ticket that had numbers $5, 17, 26, 33, 47$, do you expect to have won? Most lottery games would say you have, since you have the same numbers, just in a different order.

In this sense, the *order* of the numbers does not matter. A better way to say this is that *any* order of the same numbers is considered the same outcome, and is counted only once. So really, we are counting the *sets* of five numbers, not the *sequences* of five numbers. Each outcome is a combination, so the number of outcomes should be $C(52, 5) = \binom{52}{5}$.

On the other hand, if we must pick the numbers in the same order they come out of the machine, then we are really matching a sequence of numbers to a sequence of numbers. The number of sequences is $P(52, 5)$. In this case, we would say that the order matters.

Okay, so far so good. But what about the following example?

Example 3.4.11

How many three letter "words" are there in which

1. the letters appear in alphabetical order?
2. the letters in the word can come in any order?

Solution. Does "order matter" for the first question? In some sense, absolutely! We only want to count words where the letters are in the correct order. But from the standard combination/permutation sense of order mattering, it does not.

Much better would be to ask ourselves whether we should represent each outcome as a set of letters or a sequence of letters. It is true that words are a sequence of letters, but are we counting all possible sequences? Nope, just one sequence for each set of letters. Ah, yes, set of letters. Each outcome corresponds exactly to one set of three letters. We are counting sets, so we count combinations. The number of outcomes is $C(26, 3) = \binom{26}{3}$.

For the second question, it might look like the order shouldn't matter. Well, it doesn't matter to the machine spitting out the letters, but it would matter if we were trying to match the letters. No, let's think about it like this: Are we counting every possible sequence of letters? Yes! So these are permutations, and the number of outcomes is $P(26, 3)$.

Another deceptive use of order appears when counting bit strings. Recall that an n-bit string of weight k is a string of n bits (0s and 1s) in which k of the bits are 1s.

For example, the 4-bit strings of weight 2 are

$$1100,\ 1010,\ 1001,\ 0110,\ 0101,\ 0011.$$

Does order matter for bit strings? Of course it does! It is the only thing that matters. All of the strings above contain exactly the same number of 0s and 1s; they are only distinguished by the order of the bits.

And yet, the number of n-bit strings of weight k is $\binom{n}{k}$ and NOT a permutation. What is going on here?

Nothing is broken; we are just not thinking about the right objects about which to consider the order. When we *choose* k out of n things for a bit string, it is the *positions* we are choosing (to fill with 1s, say). It does not matter in what order we choose those positions, just what set of positions we choose. That is, the bit string 1001 is the result of choosing positions 1 and 4 to put 1s into. If we chose positions 4 and then 1, we would get the same bit string.

To summarize, the question of whether order matters can lead us astray when deciding between a combination and a permutation. Instead, we should decide whether the outcomes we are counting are sets or sequences. If we are counting sets, think combination. If we are counting sequences, think permutation.

3.4.4 The Quotient Principle

We have two ways to write the numerical relationship between the numbers of combinations and permutations. Using the product principle, we have,

$$P(n,k) = \binom{n}{k} \cdot k!$$

which can be rewritten as,

$$\binom{n}{k} = \frac{P(n,k)}{k!}.$$

This second formula suggests that there might be a *quotient principle* that we could use to justify it.

Let's think about what division means. One way to think of the division problem $24 \div 6 = 4$, for example, is saying that if you have 24 things that you divide into groups of size 6, the number of groups will be 4.[3]

Now let's look at all the 3-permutations of a set of size 4: For example, all the ways to make 3-letter words using the letters a, b, c, d. We know the number of such words is $P(4,3) = 4 \cdot 3 \cdot 2 = 24$. It is helpful to actually list all these out.

$$\begin{array}{cccccc} abc & acb & bac & bca & cab & cba \\ abd & adb & bad & bda & dab & dba \\ acd & adc & cad & cda & dac & dca \\ bcd & bdc & cbd & cdb & dbc & dcb \end{array}$$

[3]The other way to interpret this statement is that if you divide 24 things into six groups, then each group will have size 4, but this is less useful for what we will do.

Look at the first row. What do all these permutations have in common? These are exactly the permutations that use the letters a, b, c in different orders. It is not surprising that there are 6 of these, since the number of ways to arrange three elements is $3! = 6$. Similarly, the second row has 6 permutations that use the letters a, b, d in different orders, and so on.

The point is, if we wanted to just count how many *sets* of three of the four letters we have, each set corresponds exactly to one of the rows in the table. There are 24 elements in the table, and 6 elements in each row, so there must be $24 \div 6 = 4$ rows. And of course, we are not surprised because $\binom{4}{3} = 4$.

A more mathematically rigorous explanation for this phenomenon is to use the language of equivalence relations and partitions from Section 2.6. We start with permutations, and then we define an *equivalence relation* on the permutations by saying two permutations are equivalent provided they contain exactly the same elements. From the equivalence relation we get a *partition* of our set of permutations into *equivalence classes*. The combinations we want to count are precisely the equivalence classes. Since each equivalence class has the same size, we can find the number of classes by dividing the number of permutations by the size of each class.

This sort of quotient principle is also useful for solving questions where the answer isn't obviously either a permutation or combination.

Example 3.4.12

You have decided to decorate your magic wand with bands of different colored tape. You have 10 different colors to choose from, and you will use five of them to create five different stripes of color. How many different wand designs are possible?

Solution. Our first attempt to solve this problem might be to think of each outcome we are trying to count as a 5-permutation of 10 colors. That would give us an answer of $P(10, 5) = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 30,240$. So why is this not correct? Let's start making a list of the outcomes that are possible.

1. Red, blue, green, yellow, orange.
2. Red, blue, green, yellow, purple.
3. Blue, green, red, yellow, orange.
4. Orange, yellow, green, blue, red.
5. etc.

While starting this list, perhaps we asked ourselves whether red, red, blue, blue, red should be on the list, and recognized that it shouldn't. Another thing we might have considered is whether items 1 and 3 are really different. They are, since one wand has red as an outside color while the other one does not.

But what about items 1 and 4? They have the same five colors, but in a different order. However, if you were to spin the magic wand around, like magicians are apt to do, you could easily end up with wand 4 from wand 1. Aha! We see that each wand is counted exactly twice in our list of permutations: two permutations are equivalent if they are just the *reverse* of each other.

Since we can group the permutations into groups of size 2, and each group corresponds to a single wand, we see that the correct answer is,

$$\frac{P(10,5)}{2} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{2} = 15,120.$$

Tada!

3.4.5 Reading Questions

1. True or false: The number of sequences of two distinct digits from 0-9 is *twice* the number of sets of two distinct digits from 0-9. Briefly explain.

2. True or false: The number of sequences of three distinct digits from 0-9 is *3 times* the number of sets of three distinct digits from 0-9. Briefly explain.

3. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

3.4.6 Practice Problems

1. A pizza parlor offers 8 toppings.

a. How many 2-topping pizzas could they put on their menu? Assume double toppings are not allowed.

b. How many total pizzas are possible, with between zero and 8 toppings (but not double toppings) allowed?

c. The pizza parlor will list the 8 toppings in two equal-sized columns on their menu. How many ways can they arrange the toppings in the left column?

2. A combination lock consists of a dial with 41 numbers on it. To open the lock, you turn the dial to the right until you reach the first number, then to the left until you get to the second number, then to the right again to the third number. The numbers must be distinct. How many different combinations are possible?

3. Using the digits 2 through 9, find the number of different 7-digit numbers such that:

a. Digits can be used more than once.

b. Digits cannot be repeated, but can come in any order.

c. Digits cannot be repeated and must be written in increasing order.

4. In an attempt to clean up your room, you have purchased a new floating shelf to put some of your 30 books you have stacked in a corner. These books are all by different authors. The new book shelf is large enough to hold 19 of the books. Careful: Before answering the next two questions, ask yourself which answer should be larger.

a. How many ways can you select and arrange 19 of the 30 books on the shelf? Notice that here we will allow the books to end up in any order.

b. How many ways can you arrange 19 of the 30 books on the shelf if you insist they must be arranged alphabetically by author?

5. An *anagram* of a word is just a rearrangement of its letters. How many different anagrams of "ambidextrously" are there?

6. How many anagrams are there of the word "seeded" that start with the letter "s"?

7. How many anagrams are there of "goggles"?

8. On a business retreat, your company of 40 businessmen and businesswomen goes golfing.

a. You need to divide up into foursomes (groups of 4 people): a first foursome, a second foursome, and so on. How many ways can you do this?

b. After all your hard work, you realize that in fact, you want each foursome to include one of the 10 Board members (who are among the 40 golfers already). How many ways can you do this?

9. How many different seating arrangements are possible for King Arthur and his 13 knights around their round table?

10. Consider sets A and B with $|A| = 13$ and $|B| = 22$.

a. How many functions $f : A \to B$ are there?

b. How many functions $f : A \to B$ are injective?

11. Consider functions $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5, 6\}$.

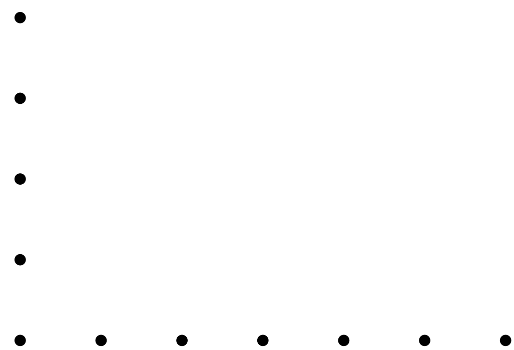
a. How many functions are there total?

b. How many functions are injective?

c. How many of the injective functions are *increasing*? To be increasing means that if $a < b$ then $f(a) < f(b)$, or in other words, the outputs get larger as the inputs get larger.

3.4.7 Additional Exercises

1. How many triangles are there with vertices from the points shown below? Note that we are not allowing degenerate triangles – ones with all three vertices on the same line – but we do allow non-right triangles. Explain why your answer is correct.



2. We have seen that the formula for $P(n, k)$ is $\dfrac{n!}{(n-k)!}$. Your task here is to explain *why* this is the right formula.

(a) Suppose you have 12 chips, each a different color. How many different stacks of 5 chips can you make? Explain your answer and why it is the same as using the formula for $P(12, 5)$.

(b) Using the scenario of the 12 chips again, what does 12! count? What does 7! count? Explain.

(c) Explain why it makes sense to divide 12! by 7! when computing $P(12, 5)$ (in terms of the chips).

(d) Does your explanation work for numbers other than 12 and 5? Explain the formula $P(n, k) = \frac{n!}{(n-k)!}$ using the variables n and k.

3.5 Counting Multisets

Objectives

After completing this section, you should be able to do the following.

1. Identify counting problems whose outcomes can be represented by multisets.
2. Represent outcomes of counting problems using multisets and sticks and stones diagrams.
3. Solve counting problems using sticks and stones.

3.5.1 Section Preview

Investigate!

Skittles come in five "flavors". How many different handfuls of 8 skittles are possible?

Suppose you have baked 8 identical cupcakes to give to your top five favorite discrete math teachers. How many ways can you distribute the cupcakes?

Why are the answers to the two counting questions above the same?

We know how to solve lots of types of counting problems now. If each outcome in a set of outcomes we are counting can be represented by a sequence, we count it as a permutation (if the terms in the sequence don't repeat) or use the product principle (if they do). If we are counting outcomes for which we don't distinguish between different arrangements of the terms (if order doesn't matter) then we think of the outcome as a set and count it as a combination. However, sets never allow an element to be repeated; each element is either in a set or not.

So we have a glaring hole in our counting repertoire. What if we want to count outcomes that are collections of terms for which we do not distinguish between the order the terms appear in, but can contain terms more than once? In other words, how can we complete the following table?

	Distinguished Arrangements?	
	Yes	No
Repeats OK	Sequences Prod. Principle n^k	
No repeats	Sequences Permutations $P(n,k)$	Sets Combinations $\binom{n}{k}$

What we want to count are some sort of set-like structure but one that *does* permit elements in the set to appear more than once. Such structures are called **multisets**.

Definition 3.5.1

A **multiset** is an unordered collection of elements, each of which can appear any number of times. The number of times an element appears is called its **multiplicity**.

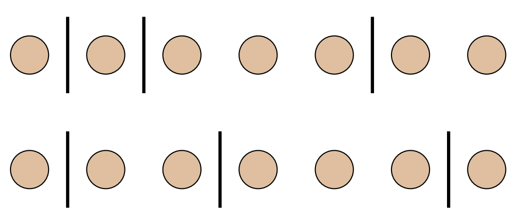
Multisets are written using the same notation as sets: a comma-separated list in braces, such as $\{1,2,2,5\}$.

In this section, we will explore how multisets can be used to represent a wide range of counting problems. We will then develop a way to translate multisets into a special type of bit-string so that we can use the numbers in Pascal's triangle to count the number of multisets.

Preview Activity

1. Find a collection of identical objects, perhaps pennies or sugar cubes. Also grab a few dividers, which could be toothpicks, matches, or pens.

Line up the pennies in a single row. We will divide the row into some number of groups by placing the toothpicks in the spaces between the pennies. We will distinguish between which order the groups come in. For example, two different ways to divide 7 pennies into 4 groups would look like this:



We will allow for one or more groups to contain no pennies (by having two toothpicks next to each other or before or after all of the pennies).

By lining up the correct number of pennies and sticks, count the number of ways you can divide a row of pennies into the given number of groups.

(a) If you want to divide a row of pennies into 4 groups, how many toothpicks will you need?

(b) How many ways are there to separate a row of three pennies into two groups?

(c) How many ways are there to separate a row of four pennies into two groups?

(d) How many ways are there to separate a row of five pennies into two groups?

(e) How many ways are there to separate a row of three pennies into three groups?

(f) How many ways are there to separate a row of four pennies into three groups?

(g) How many ways are there to separate a row of five pennies into three groups?

(h) Based on your answers above, make a conjecture about how many ways you could separate a row of seven pennies into four groups.

Hint. Look for the numbers you found in the previous questions in Pascal's triangle.

3.5.2 Have Some Cookies

Consider the following counting problem:

> You have 7 cookies to give to 4 kids. How many ways can you do this?

Take a moment to think about how you might solve this problem. You may assume that it is acceptable to give a kid no cookies. Also, the cookies are all identical and the order in which you give out the cookies does not matter (so giving cookies to the second kid before the first kid does not count as a separate outcome as the other way around).

Before solving the problem, here is a wrong answer: You might guess that the answer should be 4^7 because, for each of the 7 cookies, there are 4 choices of kids to which you can give the cookie. This is reasonable, but wrong. To see why, consider a few possible outcomes: We could assign the first six cookies to kid A, and the seventh cookie to kid B. Another outcome would assign the first cookie to kid B and the six remaining cookies to kid A. Both outcomes are included in the 4^7 answer. But for our counting problem, both outcomes are really the same – kid A gets six cookies and kid B gets one cookie. This would have been the correct answer if the cookies were all different, but they are not.

What do outcomes actually look like? How can we represent them? One approach would be to write an outcome as a string of four numbers like this:

$$3112,$$

which represents the outcome in which the first kid gets 3 cookies, the second and third kids each get 1 cookie, and the fourth kid gets 2 cookies. Represented this way, the order in which the numbers occur matters. 1312 is a different outcome, because the first kid gets one cookie instead of 3. Each number in the string can be any integer between 0 and 7. But the answer is not 7^4. We need the *sum* of the numbers to be 7.

Another way we might represent outcomes is to write a string of seven letters:

ABAADCD,

which represents that the first cookie goes to kid A, the second cookie goes to kid B, the third and fourth cookies go to kid A, and so on. In fact, this outcome is identical to the previous one—A gets 3 cookies, B and C get 1 each, and D gets 2. Each of the seven letters in the string can be any of the 4 possible letters (one for each kid), but the number of such strings is not 4^7, because here order does *not* matter. In fact, another way to write the same outcome is

AAABCDD.

This will be the preferred representation of the outcome. Since we can write the letters in any order, we might as well write them in *alphabetical* order for the purposes of counting. So we will write all the A's first, then all the B's, and so on. In fact, since we do not distinguish between the different arrangements, it is like we are writing a set, just that this **multiset** can contain an element more than one time.

So we now have two useful ways to think of the outcomes.

1. As a multiset containing 7 elements, each of which is one of the four kids (or letter representing their names). For example,

   $$\{A, A, A, B, C, D, D\}.$$

   Each time a kid is in the multiset, it means that kid gets a cookie, so the **multiplicity** of the kid in the multiset is the number of cookies the kid gets.

2. As a string of 4 non-negative numbers with a sum of 7. For example,

   3112.

   The position of the number in the string represents the kid, and the number itself represents the number of cookies that kid gets.

Before we think about how to count the number of outcomes represented this way, here are a couple more examples of counting problems we can represent in these ways.

Example 3.5.2

You grab a handful of ten jelly beans from a bag that contains six flavors. Write down three possible outcomes using both multisets and strings of numbers.

Solution. Multisets are the more natural way to think of this problem since each handful is a multiset of flavors. So for example we could have,

$$\{R, R, G, G, G, B, B, B, B, Y\}$$

meaning you got two red, three green, four blue, and one yellow jelly bean (and no purple or orange), or

$$\{R, R, R, R, R, R, R, R, R, R\}$$

which means you got ten red jelly beans. You could also have

$$\{R, G, B, Y, P, O, O, O, O, O\}$$

meaning you have one each of red, green, blue, and yellow, and five orange jelly beans.

The corresponding sequences of six numbers summing to 10 are

$$2, 3, 4, 1, 0, 0; \qquad 10, 0, 0, 0, 0, 0; \qquad 1, 1, 1, 1, 1, 5.$$

(We added commas between numbers since not every number is a single digit.) Notice that for this representation we need to agree on a fixed order of the flavors, which was not alphabetical in this case, but the same order we used when listing the multisets.

Example 3.5.3

You have 12 identical copies of your favorite Discrete Math book that you want to put on 5 bookshelves. Write down three possible outcomes using both multisets and strings of numbers.

Solution. Here the sequences of numbers are more natural, since we can just say how many books go on each shelf. We will need 5 numbers that add up to 12. For example, we could have

1. $3, 3, 3, 2, 1$, meaning 3 books on the first shelf, 3 on the second, 3 on the third, 2 on the fourth, and 1 on the fifth.
2. $0, 0, 12, 0, 0$, meaning all 12 books are on the middle shelf.
3. $1, 1, 1, 1, 8$, meaning one book on each of the first four shelves and 8 on the last.

The corresponding multisets are

1. $\{1,1,1,2,2,2,3,3,3,4,4,5\}$, meaning shelf 1 is assigned a book three times, shelf 2 is assigned a book three times, etc. Notice that there are 12 elements in the multiset, one for each book. The numbers here represent the shelves; the multiplicity of the number is the number of books on that shelf.
2. $\{3,3,3,3,3,3,3,3,3,3,3,3\}$. The only shelf in the multiset is shelf 3, with multiplicity 12, meaning that shelf 3 gets 12 books.
3. $\{1,2,3,4,5,5,5,5,5,5,5,5\}$. Here shelf 1 gets one book, shelf 2 gets one book, shelf 3 gets one book, shelf 4 gets one book, and shelf 5 gets the rest.

This last example could also have been asked as a question about solving equations with non-negative integer solutions. In particular, we could have asked for solutions to the equation $a + b + c + d + e = 12$. Think of each variable as saying how many books go on each shelf. We will ask questions directly like this, but realizing how to represent other questions as such an equation can be useful.

3.5.3 Representing Multisets with Bit Strings

Now let's return to the original problem of distributing 7 cookies to 4 kids and actually count the number of outcomes.

When we were counting plain old sets, we saw that the numbers in Pascal's triangle gave us the counts. In fact, we saw this was true because we could represent each set as a bit string. Whenever an element was in the set, we would denote that with a 1, and if the element was not in the set, we would mark its absence with a 0.

This is essentially what we did with multisets but had to use strings of non-negative numbers beyond just 0 and 1, since an element in a multiset can appear more than 0 or 1 times. But if we could translate those strings of numbers into some other sort of bit string, then we could use Pascal's triangle to count the number of multisets.

Here is how we can do this. Given a multiset such as

$$\{A,A,A,B,C,D,D\}$$

we have a number sequence representation as

$$3,1,1,2.$$

Well, instead of individual numbers, write each as a sequence of that many 1s. So this example becomes

$$111,1,1,11.$$

This is slightly awkward since we are using commas to separate the numbers. To make this clearer, let's switch to two different symbols. We will call them **sticks and**

stones, where the stone represents a 1 and the stick represents a comma. So the sequence

$$111, 1, 1, 11$$

becomes

$$\circ\circ\circ|\circ|\circ|\circ\circ.$$

This is a string of ten symbols, 7 of which are stones and 3 of which are sticks.

This is fantastic! Whatever two symbols we use (you might also see these called **stars and bars** or **balls and bins**), we can use Pascal's triangle to count the number of ways to arrange them. The number of 10-bit strings of weight 3 (or weight 7) is $\binom{10}{3} = 120$.

In terms of cookies, we can view this sticks and stones diagram as saying after how many cookies we stop giving cookies to the first kid and start giving cookies to the second kid. And then after how many do we switch to the third kid? And after how many do we switch to the fourth? So

$$\circ\circ\circ|\circ|\circ|\circ\circ$$

means three cookies go to the first kid; then we switch and give one cookie to the second kid, then switch, one to the third kid, switch, two to the fourth kid. Notice that we need 7 stones and 3 sticks – one stone for each cookie, and one stick for each switch between kids, so one fewer sticks than there are kids (we don't need to switch after the last kid – we are done).

While we are at it, we can also answer a related question: How many ways are there to distribute 7 cookies to 4 kids so that each kid gets at least one cookie? What can you say about the corresponding sticks and stones charts? The charts must start and end with at least one stone (so that kids A and D) get cookies, and also no two sticks can be adjacent (so that kids B and C are not skipped). One way to ensure this is to place sticks only in the spaces *between* the stones. With 7 stones, there are 6 spots between the stones, so we must choose 3 of those 6 spots to fill with bars. Thus there are $\binom{6}{3}$ ways to distribute 7 cookies to 4 kids giving at least one cookie to each kid.

Another (and more general) way to approach this modified problem is to first give each kid one cookie. Now the remaining 3 cookies can be distributed to the 4 kids without restrictions. So we have 3 stones and 3 sticks for a total of 6 symbols, 3 of which must be bars. So again we see that there are $\binom{6}{3}$ ways to distribute the cookies.

Sticks and stones can be used in counting problems other than kids and cookies. Here are a few examples:

Example 3.5.4

Your favorite mathematical ice cream parlor offers 10 flavors. How many milkshakes could you create using exactly 6, not necessarily distinct scoops? The order you add the flavors does not matter (they will be blended up anyway), but you are allowed repeats. So one possible shake is triple chocolate, double cherry, and mint chocolate chip.

Solution. We get six scoops, each of which could be one of ten possible flavors. Represent each scoop as a star. Think of going down the counter one flavor at a time: You see vanilla first, and skip to the next, chocolate. You say yes to chocolate three times (use three stones), then switch to the next flavor. You keep skipping until you get to cherry, which you say yes to twice. Another switch and you are at mint chocolate chip. You say yes once. Then you keep switching until you get past the last flavor, never saying yes again (since you already have said yes six times). There are ten flavors to choose from, so we must switch from considering one flavor to the next nine times. These are the nine bars.

Now that we are confident that we have the right number of sticks and stones, we answer the question simply: There are 6 stones and 9 bars, so 15 symbols. We need to pick 9 of them to be bars, so the number of milkshakes possible is

$$\binom{15}{9}.$$

Example 3.5.5

How many 7 digit phone numbers are there in which the digits are non-increasing? That is, every digit is less than or equal to the previous one.

Solution. We need to decide on 7 digits, so we will use 7 stones. The sticks will represent a switch from each possible single-digit number down to the next smaller one. So the phone number 866-5221 is represented by the sticks and stones chart

$$|\circ||\circ\circ|\circ|||\circ\circ|\circ|.$$

There are 10 choices for each digit (0-9), so we must switch between choices 9 times. We have 7 stones and 9 bars, so the total number of phone numbers is

$$\binom{16}{9}.$$

Example 3.5.6

How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13.$$

(An **integer solution** to an equation is a solution in which the unknown must have an integer value.)

1. where $x_i \geq 0$ for each x_i?
2. where $x_i > 0$ for each x_i?
3. where $x_i \geq 2$ for each x_i?

Solution. This problem is just like giving 13 cookies to 5 kids. We need to say how many of the 13 units go to each of the 5 variables. In other words, we have 13 stones and 4 bars (the sticks are like the "+" signs in the equation).

1. If x_i can be 0 or greater, we are in the standard case with no restrictions. So 13 stones and 4 sticks can be arranged in $\binom{17}{4}$ ways.
2. Now each variable must be at least 1. So give one unit to each variable to satisfy that restriction. Now there are 8 stones left, and still 4 bars, so the number of solutions is $\binom{12}{4}$.
3. Now each variable must be 2 or greater. So before any counting, give each variable 2 units. We now have 3 remaining stones and 4 bars, so there are $\binom{7}{4}$ solutions.

Counting with Functions. Many of the counting problems in this section might at first appear to be examples of counting *functions*. After all, when we try to count the number of ways to distribute cookies to kids, we are assigning each cookie to a kid, just like you assign elements of the domain of a function to elements in the codomain. However, the number of ways to assign 7 cookies to 4 kids is $\binom{10}{7} = 120$, while the number of functions $f : \{1, 2, 3, 4, 5, 6, 7\} \to \{a, b, c, d\}$ is $4^7 = 16384$. What is going on here?

When we count functions, we consider the following two functions, for example, to be different:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ a & b & c & c & c & c & c \end{pmatrix} \qquad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ b & a & c & c & c & c & c \end{pmatrix}.$$

But these two functions would correspond to the *same* cookie distribution: Kids a and b each get one cookie, and kid c gets the rest (and none for kid d).

The point: Elements of the domain are distinguished, but cookies are indistinguishable. This is analogous to the distinction between permutations (like counting functions) and combinations (not).

3.5.4 Reading Questions

1. Which of the following counting questions are NOT an example of a question you would use sticks and stones to solve?

A. How many ways can you distribute six unique gifts to three friends?

B. How many ways can you distribute six identical gifts to three friends?

C. How many different combinations of numbers can you get if you roll three identical 6-side dice?

D. How many different three-scoop milkshakes can you make when each scoop of ice cream can be one of six different flavors?

2. When you count outcomes using sticks and stones, does order matter? Do you allow repeats? What do you mean by your answers (the order of what, the repeat of what)?

3. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

3.5.5 Practice Problems

1. A multiset is a collection of objects, just like a set, but can contain an object more than once (the order of the elements still doesn't matter). For example, $\{1, 1, 2, 5, 5, 7\}$ is a multiset of size 6.

a. How many *sets* of size 9 can be made using the 10 numeric digits 0 through 9?

b. How many *multi*sets of size 9 can be made using the 10 numeric digits 0 through 9?

2. Using the digits 2 through 7, find the number of different 5-digit numbers such that:

a. Digits cannot be repeated and must be written in increasing order. (*Increasing* means *strictly* increasing. For example, the digits of 134 are increasing, but the digits of 133 are not.)

b. Digits *can* be repeated and must be written in *non-decreasing* order. (Now the digits don't need to be strictly increasing; 133 has digits non-decreasing.)

3. After gym class you are tasked with putting the 16 identical dodgeballs away into 10 bins.

a. How many ways can you do this if there are no restrictions?

b. How many ways can you do this if each bin must contain at least one dodgeball?

4. How many integer solutions are there to the equation $x + y + z = 12$ for which

a. x, y, and z are all positive?

b. x, y, and z are all non-negative?

c. x, y, and z are all greater than or equal to -3.

5. When playing Yahtzee, you roll five regular 6-sided dice. How many different outcomes are possible from a single roll? The order of the dice does not matter.

When playing Super-Yahtzee, you roll 4 regular 10-sided dice. Now how any different outcomes are possible from a single roll?

6. Your friend tells you she has 9 coins in her hand (just pennies, nickels, dimes, and quarters). If you guess how many of each kind of coin she has, she will give them to you. If you guess randomly, what is the probability that you will be correct?

7. How many integer solutions to $x_1 + x_2 + x_3 + x_4 = 35$ are there for which $x_1 \geq 4$, $x_2 \geq 4$, $x_3 \geq 1$, and $x_4 \geq 4$?

8. Consider functions $f : \{1, 2, 3, 4, 5, 6\} \to \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

a. How many of these functions are strictly increasing? Explain. (A function is strictly increasing provided if $a < b$, then $f(a) < f(b)$.)

b. How many of the functions are non-decreasing? Explain. (A function is non-decreasing provided if $a < b$, then $f(a) \leq f(b)$.)

9. *Conic*, your favorite math themed fast food drive-in offers 28 flavors which can be added to your soda. You have enough money to buy a large soda with 7 added flavors. How many different soda concoctions can you order if:

a. You refuse to use any of the flavors more than once?

b. You refuse repeats but care about the order in which the flavors are added?

c. You allow yourself multiple shots of the same flavor?

d. You allow yourself multiple shots, and care about the order in which the flavors are added?

3.5.6 Additional Exercises

1. Each of the counting problems below can be solved with sticks and stones. For each, say what outcome the diagram

$$\circ\circ\circ|\circ||\circ\circ|$$

represents, if there are the correct number of sticks and stones for the problem. Otherwise, say why the diagram does not represent any outcome, and what a correct diagram would look like.

(a) How many ways are there to select a handful of 6 jellybeans from a jar that contains 5 different flavors?

(b) How many ways can you distribute 5 identical lollipops to 6 kids?

(c) How many 6-letter words can you make using the 5 vowels in alphabetical order?

(d) How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 6$.

2. Solve the three counting problems below. Then say why it makes sense that they all have the same answer. That is, say how you can interpret them as each other.

(a) How many ways are there to distribute 8 cookies to 3 kids?

(b) How many solutions in non-negative integers are there to $x + y + z = 8$?

(c) How many different packs of 8 crayons can you make using crayons that come in red, blue, and yellow?

3. We have represented multisets with sticks and stones diagrams, then counted the number of sticks and stones diagrams to tell us the number of multisets. This is only a valid process if every multiset corresponds to exactly one sticks and stones diagram, and vice versa.

(a) Clearly write down the rule for how to convert a multiset into a sticks and stones diagram. That is, describe the function f that takes a multiset as input and outputs a sticks and stones diagram.

(b) Prove that the function f is a bijection. That is, prove that every sticks and stones diagram corresponds to exactly one multiset, and vice versa.

3.6 Combinatorial Proofs

Objectives

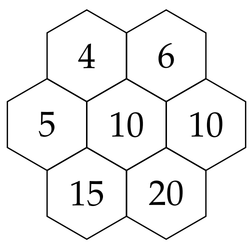
After completing this section, you should be able to do the following.

1. Write counting problems that have a given answer.
2. Write two different solutions to a counting problem.
3. Prove binomial identities using combinatorial proofs.

3.6.1 Section Preview

Investigate!

Look at any cell in the interior Pascal's triangle and the six numbers that surround it. For example, you might look at this cell:



Of the six numbers surrounding our selected cell, we will divide them into two groups of three, alternating between the groups. So for this example, we have a group with 4, 10, and 15, and a second group with 5, 6, and 20. But notice:

$$4 \cdot 10 \cdot 15 = 600 = 5 \cdot 6 \cdot 20.$$

Does this work no matter what center cell you pick? Why??

One of the coolest things about combinatorics is that you can often answer the same counting question in dramatically different ways. When we recognize this about a particular problem, we can often generalize the question to reveal two different expressions that must represent the same quantity. The counting problem itself becomes a proof of the equality of the two expressions. This style of proof is called a **combinatorial proof**.

Preview Activity

It is often possible to find the answer to a counting question in two different ways. Doing so results in two formulas that give the answer, which might look different, but must be equal (since they are both the correct answer to the same question). This provides a proof of the equality of the two formulas, called a **combinatorial proof**.

Let's explore a couple of examples.

1. Consider the set of 7-bit strings of weight 3. That is, strings of 0s and 1s that are seven characters long and have exactly three 1s. How many such strings are there?

   (a) Give the answer as a single binomial coefficient.

   (b) Now count just those 7-bit strings of weight 3 that start with a 1. How many are there?

   (c) Now count just those 7-bit strings of weight 3 that start with 01. How many are there?

   (d) Now count just those 7-bit strings of weight 3 that start with 001. How many are there?

   (e) Continue this process until you have counted all 7-bit strings of weight 3, as a *sum* of binomial coefficients. What is this sum?

2. Consider the counting question "How many ways can you permute the letters of the word *STATISTICS*?" Note that this is not just a permutation, since there are repeated letters.

   (a) How many ways can you select three of the ten positions in the anagram to be occupied by the letter *S*?

   (b) How many ways can you select three of the remaining seven positions in the anagram to be occupied by the letter *T*?

   (c) Continue with this approach until you have found an expression for the number of ways to permute the letters of *STATISTICS* as a product of binomial coefficients. Write out this product.

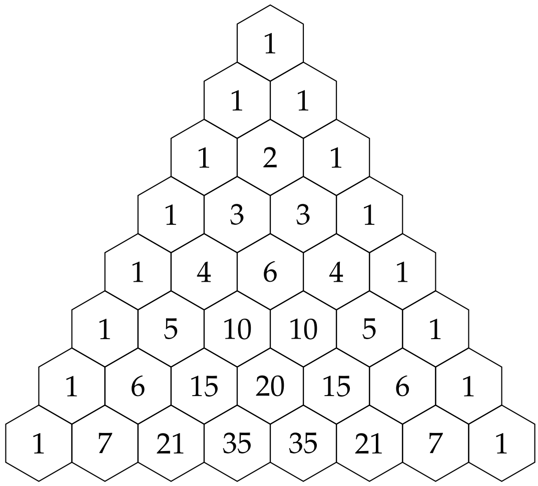
   (d) Now answer the counting question again, this time starting by asking how many ways you can select positions for the letter *A* first. Continue in any way you like until you have found a different expression for the number of ways to permute the letters of *STATISTICS* as a product of binomial coefficients. Write out this product.

   (e) Try yet another approach. What is wrong with saying the answer is 10!? This is too large, but we can correct it by dividing to account for outcomes that are equivalent. What should you divide by?

   Write your answer as a quotient of factorials. Do you get the same answer as before?

3.6.2 PATTERNS IN PASCAL'S TRIANGLE

Have a look again at Pascal's triangle. Forget for a moment where it comes from. Just look at it as a mathematical object. What do you notice?



There are lots of patterns hidden away in the triangle, enough to fill a reasonably sized book. Here are just a few of the most obvious ones:

1. The entries on the border of the triangle are all 1.

2. Any entry not on the border is the sum of the two entries above it.

3. The triangle is symmetric. In any row, entries on the left side are mirrored on the right side.

4. The sum of all entries on a given row is a power of 2. (You should check this!)

We would like to state these observations in a more precise way, and then prove that they are correct. Now each entry in Pascal's triangle is in fact a binomial coefficient. The 1 on the very top of the triangle is $\binom{0}{0}$. The next row (which we will call row 1, even though it is not the top-most row) consists of $\binom{1}{0}$ and $\binom{1}{1}$. Row 4 (the row 1, 4, 6, 4, 1) consists of the binomial coefficients

$$\binom{4}{0} \binom{4}{1} \binom{4}{2} \binom{4}{3} \binom{4}{4}.$$

Given this description of the elements in Pascal's triangle, we can rewrite the above observations as follows:

1. $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$.

2. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

3. $\binom{n}{k} = \binom{n}{n-k}$.

4. $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

Each of these is an example of a **binomial identity** : an identity (i.e., equation) involving binomial coefficients.

Our goal is to establish these identities. We wish to prove that they hold for all values of n and k. These proofs can be done in many ways. One option would be to give algebraic proofs, using the formula for $\binom{n}{k}$:

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}.$$

Here's how you might do that for the second identity above.

Example 3.6.1

Give an algebraic proof for the binomial identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Solution.

Proof. By the definition of $\binom{n}{k}$, we have

$$\binom{n-1}{k-1} = \frac{(n-1)!}{(n-1-(k-1))!(k-1)!} = \frac{(n-1)!}{(n-k)!(k-1)!}$$

and

$$\binom{n-1}{k} = \frac{(n-1)!}{(n-1-k)!k!}.$$

Thus, starting with the right-hand side of the equation:

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(n-k)!(k-1)!} + \frac{(n-1)!}{(n-1-k)!\,k!} \\
&= \frac{(n-1)!k}{(n-k)!\,k!} + \frac{(n-1)!(n-k)}{(n-k)!\,k!} \\
&= \frac{(n-1)!(k+n-k)}{(n-k)!\,k!} \\
&= \frac{n!}{(n-k)!\,k!} \\
&= \binom{n}{k}.
\end{aligned}$$

The second line (where the common denominator is found) works because $k(k-1)! = k!$ and $(n-k)(n-k-1)! = (n-k)!$. ■

This is certainly a valid proof but also is entirely useless. Even if you understand the proof perfectly, it does not tell you *why* the identity is true. A better approach would be to explain what $\binom{n}{k}$ *means* and then say why that is also what $\binom{n-1}{k-1} + \binom{n-1}{k}$ means. Let's see how this works for the four identities we observed above.

Example 3.6.2

Explain why $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$.

Solution. What do these binomial coefficients tell us? Well, $\binom{n}{0}$ gives the number of ways to select 0 objects from a collection of n objects. There is only one way to do this, namely to not select any of the objects. Thus $\binom{n}{0} = 1$. Similarly, $\binom{n}{n}$ gives the number of ways to select n objects from a collection of n objects. There is only one way to do this: Select all n objects. Thus $\binom{n}{n} = 1$.

Alternatively, we know that $\binom{n}{0}$ is the number of n-bit strings with weight 0. There is only one such string, the string of all 0's. So $\binom{n}{0} = 1$. Similarly $\binom{n}{n}$ is the number of n-bit strings with weight n. There is only one string with this property, the string of all 1's.

Another way: $\binom{n}{0}$ gives the number of subsets of a set of size n containing 0 elements. There is only one such subset, the empty set. $\binom{n}{n}$ gives the number of subsets containing n elements. The only such subset is the original set (of all elements).

Example 3.6.3

Explain why $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Solution. The easiest way to see this is to consider bit strings. $\binom{n}{k}$ is the number of bit strings of length n containing k 1's. Of all of these strings, some start with a 1 and the rest start with a 0. First consider all the bit strings which start with a 1. After the 1, there must be $n-1$ more bits (to get the total length up to n) and exactly $k-1$ of them must be 1's (as we already have one, and we need k total). How many strings are there like that? There are exactly $\binom{n-1}{k-1}$ such bit strings, so of all the length n bit strings containing k 1's, $\binom{n-1}{k-1}$ of them start with a 1. Similarly, there are $\binom{n-1}{k}$ which start with a 0 (we still need $n-1$ bits and now k of them must be 1's). Since there are $\binom{n-1}{k}$ bit strings containing $n-1$ bits with k 1's, that is the number of length n bit strings with k 1's which start with a 0. Therefore $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Another way: Consider the question, how many ways can you select k pizza toppings from a menu containing n choices? One way to do this is just $\binom{n}{k}$. Another way to answer the same question is to first decide whether or not you want anchovies. If you do want anchovies, you still need to pick $k-1$ toppings, now from just $n-1$ choices. That can be done in $\binom{n-1}{k-1}$ ways. If you do not want anchovies, then you still need to select k toppings from $n-1$ choices (the anchovies are out). You can do that in $\binom{n-1}{k}$ ways. Since the choices with anchovies are disjoint from the choices without anchovies, the total choices are $\binom{n-1}{k-1} + \binom{n-1}{k}$. But wait. We answered the same question in two different ways, so the two answers must be the same. Thus $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

You can also explain (prove) this identity by counting subsets, or even lattice paths.

Example 3.6.4

Prove the binomial identity $\binom{n}{k} = \binom{n}{n-k}$.

Solution. Why is this true? $\binom{n}{k}$ counts the number of ways to select k things from n choices. On the other hand, $\binom{n}{n-k}$ counts the number of ways to select $n-k$ things from n choices. Are these really the same? Well, what if instead of selecting the $n-k$ things you choose to exclude them. How many ways are there to choose $n-k$ things to exclude from n choices? Clearly this is $\binom{n}{n-k}$ as well (it doesn't matter whether you include or exclude the things once you have chosen them). And if you exclude $n-k$ things, then you are including the other k things. So the set of outcomes should be the same.

Let's try the pizza counting example like we did above. How many ways are there to pick k toppings from a list of n choices? On the one hand, the answer is simply $\binom{n}{k}$. Alternatively, you could make a list of all the toppings you don't want. To end up with a pizza containing exactly k toppings, you need to pick $n-k$ toppings to not put on the pizza. You have $\binom{n}{n-k}$ choices for the toppings you don't want. Both of these ways give you a pizza with k toppings, in fact all the ways to get a pizza with k toppings. Thus these two answers must be the same: $\binom{n}{k} = \binom{n}{n-k}$.

You can also prove (explain) this identity using bit strings, subsets, or lattice paths. The bit string argument is nice: $\binom{n}{k}$ counts the number of bit strings of length n with k 1's. This is also the number of bit strings of length n with k 0's (just replace each 1 with a 0 and each 0 with a 1). But if a string of length n has k 0's, it must have $n-k$ 1's. And there are exactly $\binom{n}{n-k}$ strings of length n with $n-k$ 1's.

Example 3.6.5

Prove the binomial identity $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

Solution. Let's do a "pizza proof" again. We need to find a question about pizza toppings which has 2^n as the answer. How about this: If a pizza joint offers n toppings, how many pizzas can you build using any number of toppings from no toppings to all toppings, using each topping at most once?

On one hand, the answer is 2^n. For each topping, you can say "yes" or "no," so you have two choices for each topping.

On the other hand, divide the possible pizzas into disjoint groups: the pizzas with no toppings, the pizzas with one topping, the pizzas with two toppings, etc. If we want no toppings, there is only one pizza like that (the empty pizza, if you will), but it would be better to think of that number as $\binom{n}{0}$ since we choose 0 of the n toppings. How many pizzas have 1 topping? We need to choose 1 of the n toppings, so $\binom{n}{1}$. We have:

- Pizzas with 0 toppings: $\binom{n}{0}$
- Pizzas with 1 topping: $\binom{n}{1}$
- Pizzas with 2 toppings: $\binom{n}{2}$
- $\vdots$
- Pizzas with n toppings: $\binom{n}{n}$.

The total number of possible pizzas will be the sum of these, which is exactly the left-hand side of the identity we are trying to prove.

Again, we could have proved the identity using subsets, bit strings, or lattice paths (although the lattice path argument is a little tricky).

Hopefully this gives some idea of how explanatory proofs of binomial identities can go. It is worth pointing out that more traditional proofs can also be beautiful.[4] For example, consider the following rather slick proof of the last identity.

Expand the binomial $(x+y)^n$:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x \cdot y^{n-1} + \binom{n}{n}y^n.$$

Let $x = 1$ and $y = 1$. We get:

$$(1+1)^n = \binom{n}{0}1^n + \binom{n}{1}1^{n-1}1 + \binom{n}{2}1^{n-2}1^2 + \cdots + \binom{n}{n-1}1 \cdot 1^{n-1} + \binom{n}{n}1^n.$$

Of course this simplifies to:

$$(2)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}.$$

Something fun to try: Let $x = 1$ and $y = 2$. Neat huh?

3.6.3 More Proofs

The explanatory proofs given in the above examples are typically called **combinatorial proofs**. In general, to give a combinatorial proof for a binomial identity, say $A = B$, you do the following:

[4] Most every binomial identity can be proved using mathematical induction, using the recursive definition for $\binom{n}{k}$. We will discuss induction in Section 4.5.

1. Find a counting problem you will be able to answer in two ways.
2. Explain why one answer to the counting problem is A.
3. Explain why the other answer to the counting problem is B.

Since both A and B are the answers to the same question, we must have $A = B$.

The tricky thing is coming up with the question. This is not always obvious, but it gets easier the more counting problems you solve. You will start to recognize types of answers as the answers to types of questions. More often what will happen is that you will be solving a counting problem and happen to think up two different ways of finding the answer. Now you have a binomial identity, and the proof is right there. The proof *is* the problem you just solved together with your two solutions.

For example, consider this counting question:

> How many 10-letter words use exactly four A's, three B's, two C's, and one D?

Let's try to solve this problem. We have 10 spots for letters to go. Four of those need to be A's. We can pick the four A-spots in $\binom{10}{4}$ ways. Now where can we put the B's? Well there are only 6 spots left; we need to pick 3 of them. This can be done in $\binom{6}{3}$ ways. The two C's need to go in two of the 3 remaining spots, so we have $\binom{3}{2}$ ways of doing that. That leaves just one spot of the D, but we could write that 1 choice as $\binom{1}{1}$. Thus the answer is:

$$\binom{10}{4}\binom{6}{3}\binom{3}{2}\binom{1}{1}.$$

But why stop there? We can find the answer another way too. First let's decide where to put the one D: we have 10 spots and we need to choose 1 of them, so this can be done in $\binom{10}{1}$ ways. Next, choose one of the $\binom{9}{2}$ ways to place the two C's. We now have 7 spots left, and three of them need to be filled with B's. There are $\binom{7}{3}$ ways to do this. Finally the A's can be placed in $\binom{4}{4}$ (that is, only one) ways. So another answer to the question is

$$\binom{10}{1}\binom{9}{2}\binom{7}{3}\binom{4}{4}.$$

Interesting. This gives us the binomial identity:

$$\binom{10}{4}\binom{6}{3}\binom{3}{2}\binom{1}{1} = \binom{10}{1}\binom{9}{2}\binom{7}{3}\binom{4}{4}.$$

Here are a couple more binomial identities with combinatorial proofs.

Example 3.6.6

Prove the identity

$$1n + 2(n-1) + 3(n-2) + \cdots + (n-1)2 + n1 = \binom{n+2}{3}.$$

Solution. To give a combinatorial proof we need to think up a question we can answer in two ways: one way needs to give the left-hand side of the identity, and the other way needs to be the right-hand side of the identity. Our clue to what question to ask comes from the right-hand side: $\binom{n+2}{3}$ counts the number of ways to select 3 things from a group of $n+2$ things. Let's name those things $1, 2, 3, \ldots, n+2$. In other words, we want to find 3-element subsets of those numbers (since order should not matter, subsets are exactly the right thing to think about). We will have to be a bit clever to explain why the left-hand side also gives the number of these subsets. Here's the proof.

Proof. Consider the question "How many 3-element subsets are there of the set $\{1, 2, 3, \ldots, n+2\}$?" We answer this in two ways:

Answer 1: We must select 3 elements from the collection of $n+2$ elements. This can be done in $\binom{n+2}{3}$ ways.

Answer 2: Break this problem up into cases by what the middle number in the subset is. Say each subset is $\{a, b, c\}$ written in increasing order. We count the number of subsets for each distinct value of b. The smallest possible value of b is 2, and the largest is $n+1$.

When $b = 2$, there are $1 \cdot n$ subsets: 1 choice for a and n choices (3 through $n+2$) for c.

When $b = 3$, there are $2 \cdot (n-1)$ subsets: 2 choices for a and $n-1$ choices for c.

When $b = 4$, there are $3 \cdot (n-2)$ subsets: 3 choices for a and $n-2$ choices for c.

And so on. When $b = n+1$, there are n choices for a and only 1 choice for c, so $n \cdot 1$ subsets.

Therefore the total number of subsets is

$$1n + 2(n-1) + 3(n-2) + \cdots + (n-1)2 + n1.$$

Since Answer 1 and Answer 2 are answers to the same question, they must be equal. Therefore

$$1n + 2(n-1) + 3(n-2) + \cdots + (n-1)2 + n1 = \binom{n+2}{3}.$$

■

Example 3.6.7

Prove the binomial identity

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Solution. We will give two different proofs of this fact. The first will be very similar to the previous example (counting subsets). The second proof is a little slicker, using lattice paths.

Proof. Consider the question: "How many pizzas can you make using n toppings when there are $2n$ toppings to choose from?"

Answer 1: There are $2n$ toppings, from which you must choose n. This can be done in $\binom{2n}{n}$ ways.

Answer 2: Divide the toppings into two groups of n toppings (perhaps n meats and n veggies). Any choice of n toppings must include some number from the first group and some number from the second group. Consider each possible number of meat toppings separately:

0 meats: $\binom{n}{0}\binom{n}{n}$, since you need to choose 0 of the n meats and n of the n veggies.

1 meat: $\binom{n}{1}\binom{n}{n-1}$, since you need 1 of n meats so $n-1$ of n veggies.

2 meats: $\binom{n}{2}\binom{n}{n-2}$. Choose 2 meats and the remaining $n-2$ toppings from the n veggies.

And so on. The last case is n meats, which can be done in $\binom{n}{n}\binom{n}{0}$ ways.

Thus the total number of pizzas possible is

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0}.$$

This is not quite the left-hand side ... yet. Notice that $\binom{n}{n} = \binom{n}{0}$ and $\binom{n}{n-1} = \binom{n}{1}$ and so on, by the identity in Example 3.6.4. Thus we do indeed get

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2.$$

Since these two answers are answers to the same question, they must be equal, and thus

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

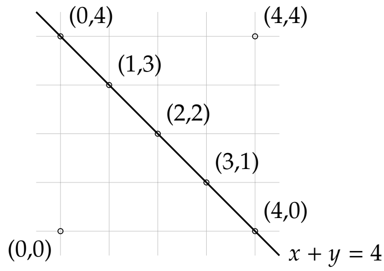
■

For an alternative proof, we use lattice paths. This is reasonable to consider because the right-hand side of the identity reminds us of the number of paths from $(0,0)$ to (n,n).

Proof. Consider the question: How many lattice paths are there from $(0,0)$ to (n,n)?

Answer 1: We must travel $2n$ steps, and n of them must be in the up direction. Thus there are $\binom{2n}{n}$ paths.

Answer 2: Note that any path from $(0,0)$ to (n,n) must cross the line $x+y=n$. That is, any path must pass through exactly one of the points: $(0,n),(1,n-1),(2,n-2),\ldots,(n,0)$. For example, this is what happens in the case $n=4$:



How many paths pass through $(0,n)$? To get to that point, you must travel n units, and 0 of them are to the right, so there are $\binom{n}{0}$ ways to get to $(0,n)$. From $(0,n)$ to (n,n) takes n steps, and 0 of them are up. So there are $\binom{n}{0}$ ways to get from $(0,n)$ to (n,n). Therefore there are $\binom{n}{0}\binom{n}{0}$ paths from $(0,0)$ to (n,n) through the point $(0,n)$.

What about through $(1,n-1)$? There are $\binom{n}{1}$ paths to get there (n steps, 1 to the right) and $\binom{n}{1}$ paths to complete the journey to (n,n) (n steps, 1 up). So there are $\binom{n}{1}\binom{n}{1}$ paths from $(0,0)$ to (n,n) through $(1,n-1)$.

In general, to get to (n,n) through the point $(k,n-k)$ we have $\binom{n}{k}$ paths to the midpoint and then $\binom{n}{k}$ paths from the midpoint to (n,n). So there are $\binom{n}{k}\binom{n}{k}$ paths from $(0,0)$ to (n,n) through $(k,n-k)$.

All together then, the total paths from $(0,0)$ to (n,n) passing through exactly one of these midpoints is

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2.$$

Since these two answers are answers to the same question, they must be equal, and thus

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2=\binom{2n}{n}.$$

∎

3.6.4 Reading Questions

1. Which of the following describes the overall strategy for a combinatorial proof?

A. Ask a counting question that can be answered in two ways (and answer it).

B. Ask two counting questions that can both be answered in the same way (and answer the questions).

C. Simplify the algebraic expressions for each side of the combinatorial identity.

D. Assume the identity fails to hold for some smallest value, and get a contradiction by looking at a smaller value.

2. Write a counting question that you could use to establish the identity:

$$\binom{x+y}{x} = \binom{x+y}{y}.$$

3. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

3.6.5 Practice Problems

1. Create a combinatorial proof of the identity $10 + 10 = 2 \cdot 10$.

- Either:

  Consider the question, "How many two-digit numbers start with a 3 or 4?"

  Or:

  Consider the question, "How many 2-topping pizzas can you make choosing from 10 toppings?"

- The first way to answer this is $10 + 10$.

- Either:

  This is because there are 10 numbers that start with 3, and another 10 that start with 4.

  Or:

  This is because there are 10 choices for the first topping, and 10 choices for the second topping.

- A second answer to the question is $2 \cdot 10$.

- Either:

  This is because you have 2 choices for the first digit, and 10 choices for the second digit.

Or:

This is because you must either choose 2 identical toppings or two different toppings.

- Since both expressions answer the same question, they must be equal. Therefore $10 + 10 = 2 \cdot 10$.

2. If you were asked to give a combinatorial proof of the identity $\binom{n}{2}\binom{n-2}{k} = \binom{n}{k}\binom{n-k}{2}$, which of the following would be reasonable questions to use?

A. How many ways can you select 2 flavors of ice cream from n choices and k toppings for your sundae?

B. How many ways can you select k bow ties to pack in your checked bag and 2 more bow ties to pack in your carry-on, from a collection of n bow ties?

C. There are n students in Math Club. How many ways can you pick a subset of size 2 or k that will run your fall fundraiser?

D. From a class of n students, how many ways can you select 2 to be prefects and another k to be on the party planning committee?

3.6.6 Additional Exercises

1. Give a combinatorial proof of the identity $2 + 2 + 2 = 3 \cdot 2$.

2. Suppose you own x fezzes and y bow ties. Of course, x and y are both greater than 1.

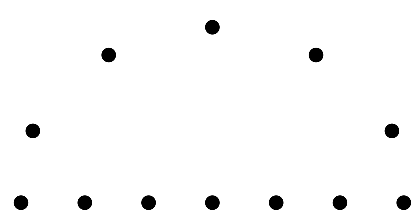
(a) How many combinations of fez and bow tie can you make? You can wear only one fez and one bow tie at a time. Explain.

(b) Explain why the answer is *also* $\binom{x+y}{2} - \binom{x}{2} - \binom{y}{2}$. (If this is what you claimed the answer was in part (a), try it again.)

(c) Use your answers to parts (a) and (b) to give a combinatorial proof of the identity

$$\binom{x+y}{2} - \binom{x}{2} - \binom{y}{2} = xy.$$

3. How many triangles can you draw using the dots below as vertices?

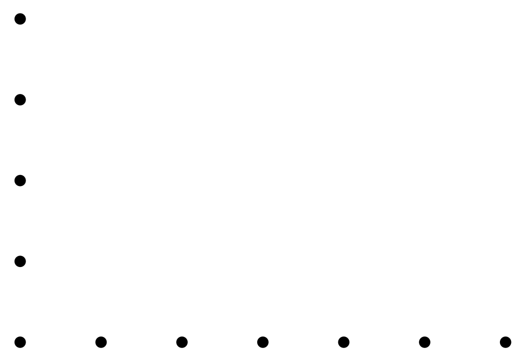


(a) Find an expression for the answer which is the sum of three terms involving binomial coefficients.

(b) Find an expression for the answer which is the difference of two binomial coefficients.

(c) Generalize the above to state and prove a binomial identity using a combinatorial proof. Say you have x points on the horizontal axis and y points in the semi-circle.

4. Consider all the triangles you can create using the points shown below as vertices. Note that we are not allowing degenerate triangles (ones with all three vertices on the same line), but we do allow non-right triangles.



(a) Find the number of triangles, and explain why your answer is correct.

(b) Find the number of triangles again, using a different method. Explain why your new method works.

(c) State a binomial identity that your two answers above establish (that is, give the binomial identity that your two answers are a proof for). Then generalize this using m's and n's.

5. A woman is getting married. She has 15 best friends but can only select 6 of them to be her bridesmaids, one of which needs to be her maid of honor. How many ways can she do this?

(a) What if she first selects the 6 bridesmaids, and then selects one of them to be the maid of honor?

(b) What if she first selects her maid of honor, and then 5 other bridesmaids?

(c) Explain why $6\binom{15}{6} = 15\binom{14}{5}$.

6. Consider the identity:

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

(a) Is this true? Try it for a few values of n and k.

(b) Use the formula for $\binom{n}{k}$ to give an algebraic proof of the identity.

(c) Give a combinatorial proof of the identity.

7. Give a combinatorial proof of the identity $\binom{n}{2}\binom{n-2}{k-2} = \binom{n}{k}\binom{k}{2}$.

8. Consider the binomial identity

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}.$$

(a) Give a combinatorial proof of this identity. Hint: What if some number of a group of n people wanted to go to an escape room, and among those going, one needed to be the team captain?

(b) Give an alternate proof by multiplying out $(1+x)^n$ and taking derivatives of both sides.

9. Give a combinatorial proof for the identity $1 + 2 + 3 + \cdots + n = \binom{n+1}{2}$.

10. Consider the bit strings in $\mathbf{B}_2^6$ (bit strings of length 6 and weight 2).

(a) How many of those bit strings start with 1?

(b) How many of those bit strings start with 01?

(c) How many of those bit strings start with 001?

(d) Are there any other strings we have not counted yet? Which ones, and how many are there?

(e) How many bit strings are there total in $\mathbf{B}_2^6$?

(f) What binomial identity have you just given a combinatorial proof for?

11. Let's count **ternary** digit strings, that is, strings in which each digit can be 0, 1, or 2.

(a) How many ternary digit strings contain exactly n digits?

(b) How many ternary digit strings contain exactly n digits and n 2's.

(c) How many ternary digit strings contain exactly n digits and $n-1$ 2's. (Hint: Where can you put the non-2 digit, and then what could it be?)

(d) How many ternary digit strings contain exactly n digits and $n-2$ 2's. (Hint: See previous hint.)

(e) How many ternary digit strings contain exactly n digits and $n-k$ 2's.

(f) How many ternary digit strings contain exactly n digits and no 2's. (Hint: What kind of a string is this?)

(g) Use the above parts to give a combinatorial proof for the identity

$$\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + 2^3\binom{n}{3} + \cdots + 2^n\binom{n}{n} = 3^n.$$

12. How many ways are there to rearrange the letters in the word "rearrange"? Answer this question in at least two different ways to establish a binomial identity.

13. Establish the identity below using a combinatorial proof.

$$\binom{2}{2}\binom{n}{2} + \binom{3}{2}\binom{n-1}{2} + \binom{4}{2}\binom{n-2}{2} + \cdots + \binom{n}{2}\binom{2}{2} = \binom{n+3}{5}.$$

14. In Example 3.6.5 we established that the sum of any row in Pascal's triangle is a power of two. Specifically,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

The argument given there used the counting question, "How many pizzas can you build using any number of n different toppings?" To practice, give new proofs of this identity using different questions.

(a) Use a question about counting subsets.

(b) Use a question about counting bit strings.

(c) Use a question about counting lattice paths.

15.

(a) The Stanley Cup is decided in a best of 7 tournament between two teams. In how many ways can your team win? Let's answer this question two ways:

i. How many of the 7 games does your team need to win? How many ways can this happen?

ii. What if the tournament goes all 7 games? So you win the last game. How many ways can the first 6 games go down?

iii. What if the tournament goes just 6 games? How many ways can this happen? What about 5 games? 4 games?

iv. What are the two different ways to compute the number of ways your team can win? Write down an equation involving binomial coefficients (that is, $\binom{n}{k}$'s). What pattern in Pascal's triangle is this an example of?

(b) Generalize. What if the rules changed, and you played a best of 9 tournament (5 wins required)? What if you played an n game tournament with k wins required to be named champion?

16. Let $k_1, k_2, \ldots, k_j$ be a list of positive integers that sum to n (i.e., $\sum_{i=1}^{j} k_i = n$). Use two graphs containing n vertices to explain why

$$\sum_{i=1}^{j} \binom{k_i}{2} \le \binom{n}{2}.$$

3.7 Applications to Probability

Objectives

After completing this section, you should be able to do the following.

1. Use counting techniques to compute probabilities of events.
2. Understand the basic rules of probability.
3. Compute probabilities of compound events, both independent and dependent.

3.7.1 Section Preview

Investigate!

Suppose you, like the 17th-century French nobleman Chevalier de Mere, liked gambling on the outcome of rolling fair 6-sided dice (each numbered 1 to 6). Would you bet him that he couldn't roll at least one 6 in four rolls of a single die? What about betting that he couldn't roll at least one double-6 in 24 rolls of both dice?

To make these decisions, we should decide

1. How likely is it that in four rolls of a single die, there will be at least one 6?
2. How likely is it that in 24 rolls of two dice, there will be at least one double-6?
3. Since the ratio 4 : 6 is equal to the ratio 24 : 36, should the probability of these events be the same? That's what the Chevalier de Mere thought. Do you?

Here is a python script that can help you get a feel for the questions above. You can switch between the two questions by commenting and uncommenting out the appropriate lines (lines that start with a # are comments). See how lucky you are!

```
import random

for i in range(4):
    die = random.randint(1,6)
    print(f"You rolled a {die}")

#for i in range(24):
```

```
#    die1 = random.randint(1,6)
#    die2 = random.randint(1,6)
#    print(f"You rolled a {die1} and a {die2}")
```

If you know some python, you might want to modify the script to run the experiment 1000 times and see how many of those are “wins”.

We can get a feel for probability *empirically* by observing how frequently events occur when an experiment is repeated many times. It often happens, as it did with the Chevalier de Mere, that our intuition about probability is not quite right. Using the counting techniques we have studied, we can explain why our intuition is off and what the true probabilities are.

Most of the questions about counting we have considered in this chapter can also be asked as a question about probability. For example: How many passwords of length 8 can you make using just lower-case letters? What is the probability that randomly selecting 8 lower-case letters will give you your password?

While the subject of probability is vast and complex, the basics of discrete probability are little more than counting. So here we will take a brief look at how our study of counting can help us understand probability.

Preview Activity

Suppose you were in a class of 30 students. How likely is it that at least two of the students were born on the same day of the year?

Assume that all days are equally likely and that nobody was born on February 29th. Would you believe the answer is more than 25%? More than 50%? More than 70%??? Let’s find the answer.

1. First, what should we mean by probability? If you roll a fair six-sided die, what is the probability of rolling a 6?

What is the probability of rolling an even number?

2. We will define the probability of an event as the number of ways the event can happen divided by the total number of things that can happen.

(a) Suppose you roll two dice (one red and one green). How many total outcomes are there?

(b) Of those outcomes, how many have *different* numbers on the two dice?

Hint. How many sequences of two different numbers can you make using the numbers 1 to 6?

(c) Combining the two numbers your found above, what is the probability that two dice will show different numbers?

(d) What is the probability that you will get three different numbers when rolling three dice? (Assume the dice are different colors).

3. Now to birthdays. There are 365 days in a year.

(a) How many possible sequences of 30 birthdays are there?

(b) How many possible sequences of 30 birthdays contain no repeats?

(c) What is the probability that 30 people have no repeated birthdays?

(d) Among the 30 people, either they all have different birthdays or at least two share a birthday. Since this is certain, its probability is 1. So what is the probability that at least two people (out of the 30) share a birthday?

(e) What is the smallest number of people you would need to have a greater than 90% chance that at least two share a birthday?

3.7.2 Computing Probabilities

Think about how we use the language of probability in our everyday lives. We might say that tossing a coin has a 50% chance of coming up heads. Or that when rolling two dice, having the sum of the dice result in a 7 is more likely than having the sum be a 2. Casinos certainly rely on certain pairs of cards being consistently more likely than others when setting payouts for Blackjack. All of this assumes that there is some *randomness* to events, and that even in this randomness, there is some *consistency* to what can happen. We will assume this model of reality.

The things we can assign probabilities to are called **random experiments**. These can have different possible **outcomes**. We will call the (finite) *set* of possible outcomes to a random experiment the **sample space** (we will usually denote this set as S). By definition, performing a random experiment will always result in exactly one outcome from the sample space.

Throughout this section, we will always assume the **uniform probability distribution**, which means that we insist that each outcome in the sample space is equally likely. Then the probability of any particular outcome in the sample space S is exactly $\frac{1}{|S|}$.

Note 3.7.1 The uniform probability distribution is a common and reasonable assumption to make, but it does preclude us from asking some questions. For example, throwing a dart at a dartboard is not uniformly distributed, and similarly, rolling weighted dice would not be. What is the probability that a thumbtack lands point up? But how would we even start to answer these questions? We would have to make some assumptions about what the probabilities of the outcomes actually are (perhaps via some repeated experiments).

There are other reasons to study different probability distributions, and this is a major topic of study in a course in probability theory.

Example 3.7.2

Suppose you flip two fair coins (a penny and a nickel). What is the sample space of possible outcomes? What is the probability of getting two heads?

Solution. The same space is the set of all possible outcomes of the experiment, which in this case is the set $\{HH, HT, TH, TT\}$. The probability of getting two heads is then $\frac{1}{4}$. In fact, every outcome has probability $\frac{1}{4}$ since there are 4 outcomes in the sample space.

Finding probabilities of *outcomes* really is this easy. Where things get more fun is if we look for the probability of an **event**: a subset of the sample space. For a particular random experiment, there might be lots of different events we ask about, and they do not need to be mutually exclusive. An event can also be a set containing just a single outcome or might contain no outcomes.

For example, suppose you roll a fair 6-side die. The sample space contains six outcomes $\{1, 2, 3, 4, 5, 6\}$. Some events we might care about include rolling an even number (the subset $\{2, 4, 6\}$), rolling a number less than 3 (the set $\{1, 2\}$), or rolling a number less than 10 (the subset $\{1, 2, 3, 4, 5, 6\}$). In fact, we now know that there are exactly $2^6 = 64$ different events we could ask about, since there are 64 subsets of the sample space.

What does our intuition suggest about the example events described above? Rolling an even number should be just as likely as rolling an odd number, so we hope that the probability of rolling an even number is $\frac{1}{2}$. Similarly, the probability of rolling a number less than 3 should be $\frac{1}{3}$ since a third of the possible outcomes are less than 3. What about rolling a number less than 10? Well, this *must* happen, so it would be 100%, which as a fraction is just 1.

Consistent with our intuition, we define the probability of an event as follows.

Definition 3.7.3

Suppose a random experiment has sample space S. The **probability** of an event E is the number of outcomes in E divided by the number of outcomes in S. We write this as $P(E) = \frac{|E|}{|S|}$.

Example 3.7.4

Suppose you roll a regular 6-sided die (each side contains a number from 1 to 6). What is the probability that you will roll an even number?

Solution. The sample space is the set $\{1, 2, 3, 4, 5, 6\}$ of possible rolls. The event, call it E for even, is the set of outcomes $\{2, 4, 6\}$. Thus the probability of E occurring is

$$P(E) = \frac{3}{6} = \frac{1}{2}.$$

We have spent a lot of effort learning how to count the size of sets. We can then use this to compute probabilities by counting the size of the sample space (set) and the size of the event (set).

Example 3.7.5

If you draw 5 cards from a regular deck of 52 cards, what is the probability that you will draw 4-of-a-kind?

Solution. First, let's count the sample space, which will consist of all 5-card hands. The order of the cards in a hand is not important, so we will just count 5-element subsets of the 52 cards. The sample space therefore contains $\binom{52}{5}$ elements. (This number is just under 2.6 million: $2,598,960$ to be exact.)

Now, how many of those will be 4-of-a-kind? One way we could count this would be to first select which of the 13 values will be the 4-of-a-kind, which can be done in $\binom{13}{1} = 13$ ways. What about the other card in the hand? Well, there are 48 other cards it could be, so the number of 4-of-a-kind hands is $13 \cdot 48 = 624$.

This makes the probability of getting 4-of-a-kind,

$$P(\text{4-of-a-kind}) = \frac{13 \cdot 48}{\binom{52}{5}} \approx 0.00024.$$

An important subtlety: Whenever counting the size of the sample space and the event, we must make sure that we are really counting the number of elements *of* the sample space that are in the event. In particular, if we count *subsets* of cards in the sample space (using a combination instead of using a permutation to count sequences of cards) then we must count the number of *subsets* of cards in the event.

Interestingly, we can find the probability of getting 4-of-a-kind using permutations too: The number of 5-card sequences is $P(52,5) = 311,875,200$. Finding the number of 4-of-a-kind sequences is a little more complicated. There are 13 possible values for the 4-of-a-kind, and 48 remaining cards for the fifth card. But those five cards can be arranged in 5! different ways. So the number of 4-of-a-kind sequences is $13 \cdot 48 \cdot 5!$. This gives,

$$P(\text{4-of-a-kind}) = \frac{13 \cdot 48 \cdot 5!}{P(52,5)} \approx 0.00024.$$

Is this close to the same answer we had before? It is *exactly* the same (we can verify this by noticing the extra 5! in both the numerator and denominator).

While picking between combinations and permutations (as long as you pick the same for both the sample space and the event) will give you the same probability, this is not always true, as you are asked to explore in some of the additional exercises.

3.7.3 Probability Rules

Here are a few basic probability facts that follow easily from our definition of probability and understanding of counting. While we are still under the assumption that the outcomes in the sample space are equally likely (the uniform probability distribution), these rules will hold for all probability distributions.

First, we often are interested in the probability that an event *does not* occur. We call this the **complement** of the event. Remember, events are subsets of the sample space, and not being "in" the event means you are in the complement of that subset. Using the same notation we have for sets, the complement of an event E will be written $\overline{E}$. Here is the relationship between the probability of an event and its complement.

Theorem 3.7.6

The probability of the complement of an event E is

$$P(\overline{E}) = 1 - P(E).$$

Proof. Remember that $P(E) = \frac{|E|}{|S|}$, the number of outcomes in the event E divided by the total number of outcomes. But how many outcomes are *not* in E? All the others. That is,

$$\left|\overline{E}\right| = |S| - |E|\,.$$

So for sure, we have

$$P(\overline{E}) = \frac{\left|\overline{E}\right|}{|S|} = \frac{|S| - |E|}{|S|} = \frac{|S|}{|S|} - \frac{|E|}{|S|} = 1 - P(E).$$

Let's illustrate this proof with an example.

Example 3.7.7

Suppose you flip a fair coin 10 times. What is the probability that you will get at least one heads?

Solution. There are lots of ways you can get at least one head, but only one way you can get no heads (that is, get all tails). So it makes sense to try to compute the requested probability as the complement of a probability easier to compute.

The sample space here is the set of all 10-toss sequences. How many are there? For each term in the sequence, it could be a head (H) or tail (T), so there are $2^{10} = 1024$ possible sequences.

We want to find the probability of getting at least one head. Let's think of this just as a counting question: How many 10-toss sequences have at least one head? All 1024 of them, except the one all tails sequence. So there are

1023 sequences with at least one head. Thus the probability of getting at least one head is $\frac{1023}{1024}$.

Wait, did we use Theorem 3.7.6? Not explicitly, but essentially we have. Using the theorem, we would have said that the probability of getting at least one head is

$$1 - P(\text{all tails}) = 1 - \frac{1}{1024} = \frac{1023}{1024}.$$

Note that the calculation we did required subtracting fractions:

$$1 - \frac{1}{1024} = \frac{1024}{1024} - \frac{1}{1024} = \frac{1024-1}{1024} = \frac{1023}{1024}.$$

So whether we do the subtraction to calculate the size of the complement, or use the complement formula and subtract fractions, we get the same answer.

Complementary probabilities are very useful when answering historical questions about dice.

Example 3.7.8

What is the probability that you will roll at least one 6 in four rolls of a fair 6-sided die?

Is this the same as the probability that you will roll at least one double 6 in 24 rolls of two dice?

Solution. The complementary event is rolling a die four times and *never* getting a 6. Of the 6^4 possible rolls, there are 5^4 that contain no 6. So the probability of getting at least one 6 in four rolls is

$$P(\text{at least one 6}) = 1 - P(\text{no 6}) = 1 - \frac{5^4}{6^4} \approx 0.5177.$$

For the double 6 in 24 rolls variant, we use the complementary event as well: what is the probability of not getting double 6s? That means on every roll you get one of the 35 other pairs.

$$P(\text{at least one double 6}) = 1 - P(\text{no double 6}) = 1 - \frac{35^{24}}{36^{24}} \approx 0.4914.$$

Indeed, the Chevalier de Mere noticed that when playing the game with two dice, he tended to lose money in the long run. Who did he turn to to ask for help? Blaise Pascal, of course!

Another way to think about complementary probabilities is to say that

$$P(E) + P(\overline{E}) = 1.$$

A probability of 1 means the event is certain, so perhaps we should think of this as giving the probability that event E either happened or didn't happen. This is exactly what we want to mean by adding probabilities.

Theorem 3.7.9

Suppose A and B are two disjoint *events. Then the probability of either A or B happening is,*

$$P(A \cup B) = P(A) + P(B).$$

If A and B are not disjoint, then the probability of A or B occurring is,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

The proof of this fact is one of the exercises in this section. However, it should become clear how this works with an example.

Example 3.7.10

Suppose you roll a fair 6-sided die. What is the probability of rolling a number that is even or less than 3 ?

Solution. We don't need a theorem to answer this. The sample space is $\{1, 2, 3, 4, 5, 6\}$ and the event is the subset $E = \{1, 2, 4, 6\}$. So $P(E) = \frac{4}{6}$.

To see where the 4 comes from, let A be the event of rolling an even number (so $A = \{2, 4, 6\}$) and B be the event of rolling a number less than 3 (so $B = \{1, 2\}$). Notice that the notation $P(E) = P(A \cup B)$ makes sense, since as sets, we really do have $E = A \cup B$.

If we go back to the definition of the probability of an event, we have,

$$P(E) = \frac{|E|}{|S|} = \frac{|A \cup B|}{|S|}.$$

We must find the size of the set $A \cup B$. But we know how to find the size of the union of non-disjoint sets: use PIE! So $|A \cup B| = 3 + 2 - 1 = 4$.

As the example demonstrates, we have basically translated the sum principle into the language of probability. Can we do the same for the product principle?

We use the product principle to find the number of ways two events can both happen, one after the other. Many probability questions ask for the probability of such **compound** events. Let's consider an example to see what is going on.

Example 3.7.11

What is the probability of getting an even number when rolling a 6-sided die and a heads when flipping a coin?

Solution. First we will find the probability directly from the definition. The sample space consists of all pairs of outcomes from the die and the coin, so $S = \{(1,H),(1,T),(2,H),(2,T),(3,H),(3,T),(4,H),(4,T),(5,H),(5,T),(6,H),(6,T)\}$. Without listing these, we could have calculated the size of the sample space using the product principle: $|S| = 6 \cdot 2 = 12$. The event we are interested in is the set of outcomes $E = \{(2,H),(4,H),(6,H)\}$. Obviously that is size 3, which we could have also found as $3 \cdot 1$. So the probability of this event is $P(E) = \frac{3}{12} = \frac{1}{4}$.

Now consider the two events separately. Say A is rolling an even number, and B flipping the coin and getting heads. The probability of the first event is $P(A) = \frac{3}{6}$. The probability of the second event is $P(B) = \frac{1}{2}$. It appears that the correct way to combine these probabilities is to multiply them:

$$P(E) = (A \text{ and } B) = P(A)P(B) = \frac{3}{6}\frac{1}{2} = \frac{1}{4}.$$

How convenient that multiplying fractions is done by multiplying the numerators and denominators separately, and this is the same as applying the product principle to the numerator and denominator of the fraction.

The reason the above example worked out was that the events were **independent**. Intuitively, this means that the outcome of the first event has no influence on the outcome of the second event. Actually, we use this principle like the product principle to *define* independence.

Definition 3.7.12

Given two events A and B, we say that they are **independent** provided the probability of both events happening is the product of the probabilities of each event happening:

$$P(A \cap B) = P(A)P(B).$$

Notice that in the definition we describe the event that both A and B happen as the intersection $A \cap B$. Since events are sets, it makes sense to take an intersection. The intersection of two sets contains all the elements that are in both sets, which is exactly what we want here.

This shines a light on a key difference between this definition and the product principle. We use the product principle to construct a new set of outcomes by combining the outcomes in two sets. This creates new sorts of outcomes. For example, the product principle would combine the sets

$$\{1,2,3\} \text{ and } \{H,T\} \text{ into } \{1H,1T,2H,2T,3H,3T\}.$$

This is *not* the intersection of two sets (it is actually the Cartesian product: $A \times B = \{(a,b) : a \in A; b \in B\}$). The definition of independence involves probabilities relative to a fixed set of outcomes. So the elements in A and B in the definition of independence are already sequences like we would have created using the product principle.

If we are more careful in Example 3.7.11 where we rolled a die and flipped a coin, we should describe the event A of first rolling an even number as the set

$$A = \{(2,H),(2,T),(4,H),(4,T),(6,H),(6,T)\}$$

and the event B of then flipping heads as the set

$$B = \{(1,H),(2,H),(3,H),(4,H),(5,H),(6,H)\}.$$

We then have $P(A) = \frac{6}{12}$ and $P(B) = \frac{6}{12}$, with product $P(A)P(B) = \frac{6}{12}\frac{6}{12} = \frac{36}{144} = \frac{1}{4}$. So our solution in the example was correct but misleading. The events A and B are indeed independent since

$$A \cap B = \{(2,H),(4,H),(6,H)\}$$

so $P(A \cap B) = \frac{3}{12} = \frac{1}{4}$.

Example 3.7.13

Suppose you roll a 12-sided die (numbered 1 to 12). Consider the events:

- A is the event of rolling a number that is a multiple of 3.
- B is the event of rolling a number that is a multiple of 4.
- C is the event of rolling a number less than 7.

Are the events A and B independent? What about A and C? What about B and C?

Solution. The sample space is the set $\{1,2,3,4,5,6,7,8,9,10,11,12\}$. The event A is the set $\{3,6,9,12\}$, B is the set $\{4,8,12\}$, and C is the set $\{1,2,3,4,5,6\}$. Thus the probabilities for each of these events are

$$P(A) = \frac{4}{12} = \frac{1}{3}, \quad P(B) = \frac{3}{12} = \frac{1}{4}, \quad P(C) = \frac{6}{12} = \frac{1}{2}.$$

To decide whether events A and B are independent, we find $P(A \cap B)$. The intersection of events A and B (meaning the number rolled is both a multiple of 3 and 4) is $\frac{1}{12}$ (the only element of the intersection is 12). We compare this to $P(A)P(B) = \frac{1}{3}\frac{1}{4} = \frac{1}{12}$. Since these are equal, the events are independent.

Events B and C are *not* independent though. Since $B \cap C = \{4\}$, we have $P(B \cap C) = \frac{1}{12}$. But $P(B)P(C) = \frac{1}{4}\frac{1}{2} = \frac{1}{8}$. Since these are not equal, the events are not independent. This makes sense since there are fewer multiples of 4 less than 7 than not.

Finally, A and C are independent: $P(A \cap C) = \frac{2}{12} = \frac{1}{6}$ and $P(A)P(C) = \frac{1}{3}\frac{1}{2} = \frac{1}{6}$.

When events are not independent, we get a new interesting question we can ask: What is the probability of one event *given* that another event has occurred? This is called ...

3.7.4 Conditional Probability

The famous probability problem, known as the Monty Hall problem, presents the following conundrum. You are on the game show *Let's Make a Deal* and will win whatever is behind one of three doors you decide to open. Behind one door is a car; behind the other two are goats. You pick a door, but before opening it, the host (Monty Hall) reveals one of the other doors that has a goat behind it. You then have the opportunity to switch doors. Should you switch? What is the probability of getting the car if you do?

By the way... This problem was perhaps one of the first math problems to "go viral," although it did so when it appeared in the Sunday newspaper magazine *Parade*. After its publication, around 10,000 readers (including close to 1000 with PhDs) wrote in complaining that the author, Marilyn vos Savant, was wrong. She wasn't.

You might be tempted to say that the probability of getting the car when you switch is $\frac{1}{2}$. After all, there are two doors left, and the car is behind one of them. However, we must ask what the probability of getting the car is *given* that Monty has revealed a goat behind another, unpicked door.

Definition 3.7.14

Given two events A and B, the **conditional probability** of A given B is,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Does this definition agree with our intuition for what conditional probability should mean? Let's think about the sample space. We want to know the chances of A occurring under the assumption that B has already occurred. In other words, we only care about the elements of the sample space that belong to B.

If B becomes the sample space, then the only outcomes from A that can possibly occur are the outcomes that are in A *and* B. So perhaps the definition of conditional probability really should be,

$$P(A|B) = \frac{|A \cap B|}{|B|}.$$

Unfortunately, I'm not in charge of probability definitions. It turns out that the standard definition is just as good though. This is because,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{|A \cap B|}{|S|}}{\frac{|B|}{|S|}} = \frac{|A \cap B|}{|B|}.$$

Phew. Another crisis averted.

Example 3.7.15

Suppose you roll two 6-sided dice with your eyes closed. Your friend says, "Hey look, at least one of your dice is a 4." What is the probability that you rolled a sum of 7?

Solution. First note that the probability of rolling a sum of 7 is $\frac{6}{36} = \frac{1}{6}$, since of the 36 pairs of numbers that can appear, there are six pairs that sum to 7: $\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$. However, we are now in a situation where at least one die is a 4. This limits the sample space to the 11 pairs that contain a 4 (we can count this using PIE as $6 + 6 - 1$). Of these, only two sum to 7: $\{(3,4), (4,3)\}$. So the probability of rolling a sum of 7 given that one die is a 4 is $\frac{2}{11}$.

If we use the definition of conditional probability, we would compute this slightly differently but arrive at the same answer. We have events A (the sum is 7) and B (at least one die is a 4). Then $P(B) = \frac{11}{36}$ and $P(A \cap B) = \frac{2}{36}$. So $P(A|B) = \frac{2}{11}$.

Notice that the probability of rolling a sum of 7 given that *the red* die is a 4 (say they are different colors) will be different! That would be $\frac{1}{6}$, since we are really just asking for the probability that the other die is a 3.

Example 3.7.16

Suppose you draw two cards from a standard deck of 52 cards. What is the probability that the second card is a face card given that the first card is red?

What is the probability that the first card is red given that the second card is a face card?

Solution. We have a sample space consisting of the $52 \cdot 51$ sequences of two cards. Event A will be those pairs that have a red card as the first in the sequence. Event B will be those pairs that have a face card second in the sequence.

We are looking for both $P(A|B)$ and $P(B|A)$, so we will need to find $P(A)$, $P(B)$, and $P(A \cap B)$. There are $26 \cdot 51$ pairs in A (select one of the 26 red cards, and then any of the remaining 51 cards), so

$$P(A) = \frac{26 \cdot 51}{52 \cdot 51} = \frac{26}{52} = \frac{1}{2}.$$

There are $12 \cdot 51$ pairs in B (select one of the 12 face cards, and then any of the remaining 51 cards), so

$$P(B) = \frac{12 \cdot 51}{52 \cdot 51} = \frac{12}{52} = \frac{3}{13}.$$

Finding the size of the intersection is a little more challenging (we did so in the subsection Combining Principles). There are $20 \cdot 12$ pairs that start with a red, non-face card, and end with a face card, and another $6 \cdot 11$ pairs that start with a red face card and end with a face card. So

$$P(A \cap B) = \frac{20 \cdot 12 + 6 \cdot 11}{52 \cdot 51} = \frac{306}{2652} = \frac{3}{26}.$$

So the probability that the second card is a face card given that the first card is red is,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{3}{26}}{\frac{1}{2}} = \frac{3}{13}.$$

The probability that the first card is red given that the second card is a face card is,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{3}{26}}{\frac{3}{13}} = \frac{1}{2}.$$

Wait a second! What? The probability that the first card is red given that the second card is a face card is the same as the probability that the first card is red?? It seems that $P(A|B) = P(A)$, and that $P(B|A) = P(B)$. What could that mean?

Look what happens when you clear the denominator in the definition of conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

becomes

$$P(A \cap B) = P(B)P(A|B).$$

This looks almost like the definition of events being independent, except that instead of $P(A)$ in the product we now have $P(A|B)$. But what does $P(A|B)$ even mean if A and B are independent? If the events are independent, then it should be no more or less likely that A occurs given that B has occurred. So we should have $P(A|B) = P(A)$. This is exactly what we saw in the last example.

3.7.5 Reading Questions

1. Which of the following are true about the equation $P(A \cup B) = P(A) + P(B)$?

A. This is true as long as the events A and B are disjoint.

B. This is true as long as the events A and B are independent.

C. This is always true.

D. This is never true.

2. Which of the following relationships hold for any two events A and B?

A. $P(A \cap B) = P(A)P(B)$.

B. $P(A \cap B) = P(A)P(B|A)$

C. $P(A \cap B) = P(B)P(A|B)$

D. $P(A \cap B) = P(A)P(A|B)$

3. What questions do you have after reading this section? Ask at least one question about the material that you are curious about.

3.7.6 Practice Problems

1. You flip a fair coin three times and record whether it lands head (H) or tails (T).

(a) List all the elements in the sample space. For example, one outcome is HHT.

(b) Suppose you bet your friend that you would get more heads than tails. List all elements in this event.

(c) What is the probability of getting more heads than tails?

(d) What is the probability of getting exactly two heads?

What is the probability of getting exactly three heads?

2. Suppose you flip a fair coin 15 times.

(a) What is the size of the sample space?

(b) What is the size of the event that you get exactly 8 heads?

So then what is the probability that you get exactly 8 heads?

(c) What is the size of the event that you do NOT get exactly 8 heads?

What is the probability that you do NOT get exactly 8 heads? Use the definition of probability and the previous answer.

What probability do you get for this same event if you use the fact that the probability of an event is 1 minus the probability of the complement of the event?

3. Suppose you flip a fair coin 16 times.

(a) What is the size of the sample space?

(b) What is the size of the event that you get exactly 4 heads?

What is the size of the event that you get exactly 12 heads?

What is the size of the event that you get exactly 4 heads OR exactly 12 heads?

(c) What is the sum of the probabilities of getting exactly 4 heads and getting exactly 12 heads?

____________ + ____________ = ____________

What is the probability of getting exactly 4 heads OR getting exactly 12 heads? Compute this by finding the size of the event, divided by the size of the sample space.

____________ ÷ ____________ = ____________.

4. You have a bag of special math-themed M&M's. The bag promises that inside there are 5 blue, 4 red, 1 orange, 1 green, 2 brown, and 3 yellow M&M's.

Find the probabilities of the following events.

(a) What is the probability that if you pick a single M&M, it is blue or yellow?

(b) What is the probability that if you pick two M&M's at the same time, you will get a blue and yellow?

(c) What is the probability that if you pick two M&M's one at a time, you will get a blue first and a yellow second?

5. In a standard deck of 52 cards, 26 cards are red and 26 cards are black. Thus the probability of drawing a red card is 0.5.

(a) What is the probability that when flipping a coin twice, you get tails both times?

(b) What is the probability that if you are dealt two cards from a standard deck, both cards are red?

6. Suppose you take just eight playing cards, four red and four black. You also have a fair coin that you can flip as many times as you want.

(a) Compare the probability of getting tails when flipping a coin once to the probability of drawing a single red card.

P(tails) =__________; P(red) = __________. Difference: __________

(b) Compare the probability of getting two tails when flipping a coin twice to the probability of drawing two red cards (without replacement).

P(2 tails) =__________; P(2 reds) = __________. Difference: __________

(c) Compare the probability of getting three tails when flipping a coin thrice to the probability of drawing three red cards (without replacement).

P(3 tails) =__________; P(3 reds) = __________. Difference: __________

(d) Compare the probability of getting four tails when flipping a coin four times to the probability of drawing four red cards (without replacement).

P(4 tails) =__________; P(4 reds) = __________. Difference: __________

(e) Compare the probability of getting five tails when flipping a coin five times to the probability of drawing five red cards (without replacement).

P(5 tails) =__________; P(5 reds) = __________. Difference: __________

7. Suppose you have three dice: a 4-sided die, a 6-sided die, and an 8-sided die. Each die is fair and numbered from 1 to the number of sides it has. You roll all three dice.

(a) What is the probability that the sum of the dice is 3?

(b) What is the probability that the sum of the dice is 6?

(c) What is the probability all three dice have the same number when rolled?

(d) What is the probability all three dice have the same number when rolled *given* that the sum of the dice is 6?

8. You roll 3 fair 6-sided dice. What is the probability that all three dice show different numbers?

What is the probability that all three dice show the same number?

Is it possible for the dice to show neither all the same nor all different numbers? If so, the probability of this happening would be 0. What is the probability?

3.7.7 Additional Exercises

1. When playing 5-card poker, a **full house** is a hand that contains three cards of one rank and two cards of another rank. For example, you could have three 7s and two 4s.

Find the probability of being dealt a full house in two different ways:

(a) Assume that all five cards are dealt at once, so that the sample space has size $\binom{52}{5}$.

(b) Assume that the cards are dealt one at a time, so that the sample space has size $P(52, 5)$.

Are the two answers the same? Why does this make sense?

2. A random number generator selects single-digit numbers (0 through 9) with equal probability. Suppose the generator produces five numbers.

What is the probability that the five numbers will all be different? Answer this question in two ways:

(a) Assume the numbers come out of the generator in a sequence, so that the sample space has size 10^5.

(b) Assume the numbers come out as a multiset, or equivalently, that the numbers must appear in non-decreasing order. You will want to use sticks and stones to count the size of the sample space.

Are the two answers the same? Why does this make sense?

3. Prove Theorem 3.7.9.

4. Each of 10 friends has a deck of cards that they shuffle thoroughly. Each friend draws a card from their deck. What is the probability that at least one pair of friends draw a matching card?

5. How many people do you need to have in a room to have a 50% chance that at least two people share the same birthday (day of the year)? Assume that all birthdays are equally likely, and that nobody is born on Leap Day (February 29th).

6. At your 20th high school reunion, you meet an old friend you hadn't heard from in years. You talk about pets, specifically cats and dogs. She tells you that she has two pets, and that at least one of them is a cat. What is the probability that she has two cats? (Assume that having a cat or a dog is equally likely.)

7. Another old friend overhears your pet conversation and says that he also has two pets, and that the one he has had the longest is a cat. What is the probability that he has two cats? And why is this answer different from the previous question?

8. You are playing a shell game with three cups. Under one cup are two green balls, under another cup are two red balls, and under the third cup are one green and one red ball. You close your eyes, and your friend rearranges the cups. You then open your eyes and pick a cup at random. You see that it contains a green ball. What is the probability that the other ball under that cup is also green? Explain your answer in terms of conditional probability.

3.8 Advanced Counting Using PIE

Objectives

After completing this section, you should be able to do the following.

1. Apply the principle of inclusion/exclusion to solve counting problems involving multisets with bounded multiplicity.
2. Apply the principle of inclusion/exclusion to solve counting problems involving derangements.
3. Apply the principle of inclusion/exclusion to solve counting problems involving surjective functions.

3.8.1 Section Preview

Investigate!

You have 11 identical mini key lime pies to give to 4 children. However, you don't want any kid to get more than 3 pies. How many ways can you distribute the pies?

Sticks and stones allows us to count the number of ways to distribute 10 cookies to 3 kids and natural number solutions to $x + y + z = 10$, for example. A relatively easy modification allows us to put a *lower bound* restriction on these problems: Perhaps each kid must get at least two cookies or $x, y, z \geq 2$. This was done by first assigning each kid (or variable) 2 cookies (or units) and then distributing the rest using sticks and stones.

What if we wanted an *upper bound* restriction? For example, we might insist that no kid gets more than 4 cookies or that $x, y, z \leq 4$. It turns out this is considerably harder.

Notice that if we consider the complementary event, i.e., distributions of cookies in which kids *do* get more than 4 cookies, then we are back to a sticks and stones problem with a lower bound. If we could count this, then subtracting from the total number of distributions should give us the desired answer. However, the problem is that the complement of "no kid gets more than 4 cookies" is "at least one kid gets more than 4 cookies." We know how to take care of requiring *all* kids getting at least 4 cookies, but how do we handle the case where *one or more* kids get at least 4 cookies? We must use PIE.

Preview Activity

First, let's review some sticks and stones type questions we learned about in Section 3.5.

Then we will modify this and apply the principle of inclusion/exclusion from Section 3.3.

1. Suppose we have 10 cookies to give away to three children, Albie, Bertie, and Charlie.

   (a) How many ways can we distribute the cookies with no restrictions?

   (b) How many ways can we distribute the cookies if each child must get at least two cookies?

   Hint. Give each kid the minimum number of cookies first. How many ways are there to distribute the remaining cookies?

   (c) How many ways can you distribute the cookies if Albie gets at least 3 cookies and Bertie gets at least 2 cookies (and Charlie has no restrictions)?

2. Let's again consider the 10 cookies we want to distribute to Albie, Bertie, and Charlie. This time, we will impose some upper bound restrictions.

   (a) How many ways can we distribute the cookies if Albie *does* get more than 3 cookies (so at least 4)?

   How many ways can we distribute the cookies if Albie *does not* get more than 3 cookies?

   (b) How many ways can we distribute the cookies if Bertie *does* get more than 3 cookies?

   (c) How many ways can we distribute the cookies if both Albie and Bertie *do* get more than 3 cookies?

   (d) Using the Principle of Inclusion/Exclusion for two sets, how many ways can we distribute the cookies if *at least one* of Albie or Bertie gets more than 3 cookies? So either Albie gets more than 3 cookies, Bertie gets more than 3 cookies, or both get more than 3 cookies.

   (e) How many ways can we distribute the cookies if *neither* Albie nor Bertie gets more than 3 cookies?

3.8.2 PIE for Multisets

The Principle of Inclusion/Exclusion (PIE) gives a method for finding the cardinality of the union of not necessarily disjoint sets. We saw in Section 3.3 how this works with three sets. To find how many things are in *one or more* of the sets A, B, and C, we should just add up the number of things in each of these sets. However, if there is any overlap among the sets, those elements are counted multiple times. So we subtract the things in each intersection of a pair of sets. But doing this removes elements that are in all three sets once too often, so we need to add it back in. In terms of the cardinality of sets, we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Example 3.8.1

Three kids, Alberto, Bernadette, and Carlos, decide to share 11 cookies. They wonder how many ways they could split the cookies up provided that none of them receive more than 4 cookies (someone receiving no cookies is for some reason acceptable to these kids).

Solution. Without the "no more than 4" restriction, the answer would be $\binom{13}{2}$, using 11 stones and 2 sticks (separating the three kids). Now count the number of ways that one or more of the kids violates the condition, i.e., gets at least 4 cookies.

Let A be the set of outcomes in which Alberto gets more than 4 cookies. Let B be the set of outcomes in which Bernadette gets more than 4 cookies. Let C be the set of outcomes in which Carlos gets more than 4 cookies. We then are looking (for the sake of subtraction) for the size of the set $A \cup B \cup C$. Using PIE, we must find the sizes of $|A|$, $|B|$, $|C|$, $|A \cap B|$ and so on. Here is what we find.

- $|A| = \binom{8}{2}$. First give Alberto 5 cookies, then distribute the remaining 6 to the three kids without restrictions, using 6 stones and 2 sticks.
- $|B| = \binom{8}{2}$. Just like above, only now Bernadette gets 5 cookies at the start.
- $|C| = \binom{8}{2}$. Carlos gets 5 cookies first.
- $|A \cap B| = \binom{3}{2}$. Give Alberto and Bernadette 5 cookies each, leaving 1 (stone) to distribute to the three kids (2 sticks).
- $|A \cap C| = \binom{3}{2}$. Alberto and Carlos get 5 cookies first.
- $|B \cap C| = \binom{3}{2}$. Bernadette and Carlos get 5 cookies first.
- $|A \cap B \cap C| = 0$. It is not possible for all three kids to get 4 or more cookies.

Combining all of these we see

$$|A \cup B \cup C| = \binom{8}{2} + \binom{8}{2} + \binom{8}{2} - \binom{3}{2} - \binom{3}{2} - \binom{3}{2} + 0 = 75.$$

Thus the answer to the original question is $\binom{13}{2} - 75 = 78 - 75 = 3$. This makes sense now that we see it. The only way to ensure that no kid gets more than 4 cookies is to give two kids 4 cookies and one kid 3; there are three choices for which kid that should be. We could have found the answer much quicker through this observation, but the point of the example is to illustrate that PIE works!

For four or more sets, we do not write down a formula for PIE. Instead, we just think of the principle: Add up all the elements in single sets and then subtract out things you counted twice (elements in the intersection of a *pair* of sets); then add back in elements you removed too often (elements in the intersection of groups of three sets); then take back out elements you added back in too often (elements in the intersection of groups of four sets); then add back in, take back out, add back in, etc. This would be very difficult if it wasn't for the fact that in these problems, all the cardinalities of the single sets are equal, as are all the cardinalities of the intersections of two sets, and that of three sets, and so on. Thus we can group all of these together and multiply by how many different combinations of $1, 2, 3, \ldots$ sets there are.

Example 3.8.2

How many ways can you distribute 10 cookies to 4 kids so that no kid gets more than 2 cookies?

Solution. There are $\binom{13}{3}$ ways to distribute 10 cookies to 4 kids (using 10 stones and 3 sticks). We will subtract all the outcomes in which a kid gets 3 or more cookies. How many outcomes are there like that? We can force kid A to eat 3 or more cookies by giving him 3 cookies before we start. Doing so reduces the problem to one in which we have 7 cookies to give to 4 kids without any restrictions. In that case, we have 7 stones (the 7 remaining cookies) and 3 sticks (one less than the number of kids) so we can distribute the cookies in $\binom{10}{3}$ ways. Of course we could choose any one of the 4 kids to give too many cookies, so it would appear that there are $\binom{4}{1}\binom{10}{3}$ ways to distribute the cookies giving too many to one kid. But in fact, we have overcounted.

We must get rid of the outcomes in which two kids have too many cookies. There are $\binom{4}{2}$ ways to select 2 kids to give extra cookies. It takes 6 cookies to do this, leaving only 4 cookies. So we have 4 stones and still 3 sticks. The remaining 4 cookies can thus be distributed in $\binom{7}{3}$ ways (for each of the $\binom{4}{2}$ choices of which 2 kids to over-feed).

But now we have removed too much. We must add back in all the ways to give too many cookies to three kids. This uses 9 cookies, leaving only 1 to distribute to the 4 kids using sticks and stones, which can be done in $\binom{4}{3}$ ways. We must consider this outcome for every possible choice of which three kids we over-feed, and there are $\binom{4}{3}$ ways of selecting that set of 3 kids.

Next we would subtract all the ways to give four kids too many cookies, but in this case, that number is 0.

All together we get that the number of ways to distribute 10 cookies to 4 kids without giving any kid more than 2 cookies is:

$$\binom{13}{3} - \left[\binom{4}{1}\binom{10}{3} - \binom{4}{2}\binom{7}{3} + \binom{4}{3}\binom{4}{3}\right]$$

which is

$$286 - [480 - 210 + 16] = 0.$$

This makes sense: There is NO way to distribute 10 cookies to 4 kids and make sure that nobody gets more than 2. It is slightly surprising that

$$\binom{13}{3} = \left[\binom{4}{1}\binom{10}{3} - \binom{4}{2}\binom{7}{3} + \binom{4}{3}\binom{4}{3}\right],$$

but since PIE works, this equality must hold.

Just so you don't think that these problems always have easier solutions, consider the following example.

Example 3.8.3

Earlier (Example 3.5.6) we counted the number of solutions to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13,$$

where $x_i \geq 0$ for each x_i.

How many of those solutions have $0 \leq x_i \leq 3$ for each x_i?

Solution. We must subtract off the number of solutions in which one or more of the variables has a value greater than 3. We will need to use PIE because counting the number of solutions for which each of the five variables separately are greater than 3 counts solutions multiple times. Here is what we get:

- Total solutions: $\binom{17}{4}$.
- Solutions where $x_1 > 3$: $\binom{13}{4}$. Give x_1 4 units first; then distribute the remaining 9 units to the 5 variables.
- Solutions where $x_1 > 3$ and $x_2 > 3$: $\binom{9}{4}$. After you give 4 units to x_1 and another 4 to x_2, you only have 5 units left to distribute.
- Solutions where $x_1 > 3$, $x_2 > 3$ and $x_3 > 3$: $\binom{5}{4}$.
- Solutions where $x_1 > 3$, $x_2 > 3$, $x_3 > 3$, and $x_4 > 3$: 0.

We also need to account for the fact that we could choose any of the five variables in the place of x_1 above (so there will be $\binom{5}{1}$ outcomes like this), any pair of variables in the place of x_1 and x_2 ($\binom{5}{2}$ outcomes) and so on. It is because of this that the double counting occurs, so we need to use PIE. All together we have that the number of solutions with $0 \le x_i \le 3$ is

$$\binom{17}{4} - \left[\binom{5}{1}\binom{13}{4} - \binom{5}{2}\binom{9}{4} + \binom{5}{3}\binom{5}{4}\right] = 15.$$

3.8.3 Counting Derangements

Investigate!

For your senior prank, you decide to switch the nameplates on your favorite 5 professors' doors. So that none of them feel left out, you want to make sure that all of the nameplates end up on the wrong door. How many ways can this be accomplished?

The advanced use of PIE has applications beyond sticks and stones. A **derangement** of n elements $\{1, 2, 3, \ldots, n\}$ is a permutation in which no element is fixed. For example, there are 6 permutations of the three elements $\{1, 2, 3\}$:

$$123 \quad 132 \quad 213 \quad 231 \quad 312 \quad 321.$$

but most of these have one or more elements fixed: 123 has all three elements fixed since all three elements are in their original positions, 132 has the first element fixed (1 is in its original first position), and so on. In fact, the only derangements of three elements are

$$231 \text{ and } 312.$$

If we go up to 4 elements, there are 24 permutations (because we have 4 choices for the first element, 3 choices for the second, 2 choices for the third leaving only 1 choice for the last). How many of these are derangements? If you list out all 24 permutations and eliminate those that are not derangements, you will be left with just 9 derangements. Let's see how we can get that number using PIE.

Example 3.8.4

How many derangements are there of 4 elements?

Solution. We count all permutations and subtract those that are not derangements. There are $4! = 24$ permutations of 4 elements. Now for a permutation to *not* be a derangement, at least one of the 4 elements must be fixed. There are $\binom{4}{1}$ choices for which single element we fix. Once fixed, we need to find a permutation of the other three elements. There are 3! permutations on 3 elements.

But now we have counted too many non-derangements, so we must subtract those permutations that fix two elements. There are $\binom{4}{2}$ choices for which two elements we fix, and then for each pair, 2! permutations of the remaining elements. But this subtracts too many, so add back in permutations that fix 3 elements, all $\binom{4}{3}1!$ of them. Finally subtract the $\binom{4}{4}0!$ permutations (recall $0! = 1$) which fix all four elements. All together we get that the number of derangements of 4 elements is:

$$4! - \left[\binom{4}{1}3! - \binom{4}{2}2! + \binom{4}{3}1! - \binom{4}{4}0!\right] = 24 - 15 = 9.$$

Of course we can use a similar formula to count the derangements of any number of elements. However, the more elements we have, the longer the formula gets. Here is another example:

Example 3.8.5

Five gentlemen attend a party, leaving their hats at the door. At the end of the party, they hastily grab hats on their way out. How many different ways could this happen so that none of the gentlemen leaves with his own hat?

Solution. We are counting derangements on 5 elements. There are 5! ways for the gentlemen to grab hats in any order—but many of these permutations will result in someone getting their own hat. So we subtract all the ways in which one or more of the men get their own hat. In other words, we subtract the non-derangements. Doing so requires PIE. Thus the answer is:

$$5! - \left[\binom{5}{1}4! - \binom{5}{2}3! + \binom{5}{3}2! - \binom{5}{4}1! + \binom{5}{5}0!\right].$$

3.8.4 Counting Functions

Investigate!

1. Consider all functions $f : \{1,2,3,4,5\} \to \{1,2,3,4,5\}$. How many functions are there in total? How many of those are injective? Remember, a function is an injection if every input goes to a different output.

2. Consider all functions $f : \{1,2,3,4,5\} \to \{1,2,3,4,5\}$. How many of the *injections* have the property that $f(x) \neq x$ for any $x \in \{1,2,3,4,5\}$?

Your friend claims that the answer is:

$$5! - \left[\binom{5}{1}4! - \binom{5}{2}3! + \binom{5}{3}2! - \binom{5}{4}1! + \binom{5}{5}0!\right].$$

Explain why this is correct.

3. Recall that a *surjection* is a function for which every element of the codomain is in the range. How many of the functions $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ are surjective? Use PIE!

We have seen throughout this chapter that many counting questions can be rephrased as questions about counting functions with certain properties. This is reasonable since many counting questions can be thought of as counting the number of ways to assign elements from one set to elements of another.

Example 3.8.6

You decide to give away your video game collection so as to better spend your time studying advanced mathematics. How many ways can you do this, provided:

1. You want to distribute your 3 different PS4 games among 5 friends, so that no friend gets more than one game?
2. You want to distribute your 8 different 3DS games among 5 friends?
3. You want to distribute your 8 different SNES games among 5 friends, so that each friend gets at least one game?

In each case, model the counting question as a function counting question.

Solution.

1. We must use the three games (call them 1, 2, 3) as the domain and the 5 friends (a,b,c,d,e) as the codomain (otherwise the function would not be defined for the whole domain when a friend didn't get any game). So how many functions are there with domain $\{1, 2, 3\}$ and codomain $\{a, b, c, d, e\}$? The answer to this is $5^3 = 125$, since we can assign any of 5 elements to be the image of 1, any of 5 elements to be the image of 2 and any of 5 elements to be the image of 3.

   But this is not the correct answer to our counting problem, because one of these functions is $f = \begin{pmatrix} 1 & 2 & 3 \\ a & a & a \end{pmatrix}$; one friend can get more than one game. What we really need to do is count *injective* functions. This gives $P(5,3) = 60$ functions, which is the answer to our counting question.

2. Again, we need to use the 8 games as the domain and the 5 friends as the codomain. We are counting all functions, so the number of ways to distribute the games is 5^8.

3. This question is harder. Use the games as the domain and friends as the codomain (the reverse would not give a function). To ensure that every friend gets at least one game means that every element of the codomain is in the range. In other words, we are looking for *surjective* functions. How do you count those?

In Example 3.2.13 we saw how to count all functions (using the multiplicative principle) and in Example 3.4.7 we learned how to count injective functions (using permutations). Surjective functions are not as easily counted (unless the size of the domain is smaller than the codomain, in which case there are none).

The idea is to count the functions that are *not* surjective, and then subtract that from the total number of functions. This works very well when the codomain has two elements in it:

Example 3.8.7

How many functions $f : \{1,2,3,4,5\} \to \{a,b\}$ are surjective?

Solution. There are 2^5 functions total, two choices for where to send each of the 5 elements of the domain. Now of these, the functions that are *not* surjective must exclude one or more elements of the codomain from the range. So first, consider functions for which a is not in the range. This can only happen one way: Everything gets sent to b. Alternatively, we could exclude b from the range. Then everything gets sent to a, so there is only one function like this. These are the only ways in which a function could not be surjective (no function excludes both a and b from the range) so there are exactly $2^5 - 2$ surjective functions.

When there are three elements in the codomain, there are now three choices for a single element to exclude from the range. Additionally, we could pick pairs of two elements to exclude from the range, and we must make sure we don't overcount these. It's PIE time!

Example 3.8.8

How many functions $f : \{1,2,3,4,5\} \to \{a,b,c\}$ are surjective?

Solution. Again start with the total number of functions: 3^5 (as each of the five elements of the domain can go to any of three elements of the codomain).

Now we count the functions that are *not* surjective.

Start by excluding a from the range. Then we have two choices (b or c) for where to send each of the five elements of the domain. Thus there are 2^5 functions that exclude a from the range. Similarly, there are 2^5 functions that exclude b, and another 2^5 that exclude c. Now have we counted all functions that are not surjective? Yes, but in fact, we have counted some multiple times. For example, the function which sends everything to c was one of the 2^5 functions we counted when we excluded a from the range, and also one of the 2^5 functions we counted when we excluded b from the range. We must subtract out all the functions which specifically exclude two elements from the range. There is 1 function when we exclude a and b (everything goes to c), one function when we exclude a and c, and one function when we exclude b and c.

We are using PIE: To count the functions that are not surjective, we added up the functions that exclude a, b, and c separately; then subtracted the functions that exclude pairs of elements. We would then add back in the functions that exclude groups of three elements, except that there are no such functions. We find that the number of functions that are *not* surjective is

$$2^5 + 2^5 + 2^5 - 1 - 1 - 1 + 0.$$

Perhaps a more descriptive way to write this is

$$\binom{3}{1}2^5 - \binom{3}{2}1^5 + \binom{3}{3}0^5.$$

since each of the 2^5's was the result of choosing 1 of the 3 elements of the codomain to exclude from the range; each of the three 1^5's was the result of choosing 2 of the 3 elements of the codomain to exclude. Writing 1^5 instead of 1 makes sense too: We have 1 choice of where to send each of the 5 elements of the domain.

Now we can finally count the number of surjective functions:

$$3^5 - \left[\binom{3}{1}2^5 - \binom{3}{2}1^5\right] = 150.$$

You might worry that to count surjective functions when the codomain is larger than 3 elements would be too tedious. We need to use PIE but with more than 3 sets the formula for PIE is very long. However, we have lucked out. As we saw in the example above, the number of functions that exclude a single element from the range is the same no matter which single element is excluded. Similarly, the number of functions that exclude a pair of elements will be the same for every pair. With larger codomains, we will see the same behavior with groups of 3, 4, and more elements excluded. So instead of adding/subtracting each of these, we can simply add or subtract all of them at once, if you know how many there are. This works just like it did in for the other types of counting questions in this section, only now the size of the various combinations of sets is a number raised to a power, as opposed to a binomial coefficient or factorial. Here's what happens with 4 and 5 elements in the codomain.

Example 3.8.9

1. How many functions $f : \{1, 2, 3, 4, 5\} \to \{a, b, c, d\}$ are surjective?

2. How many functions $f : \{1, 2, 3, 4, 5\} \to \{a, b, c, d, e\}$ are surjective?

Solution.

1. There are 4^5 functions all together; we will subtract the functions that are not surjective. We could exclude any one of the four elements of the codomain, and doing so will leave us with 3^5 functions for each excluded element. This counts too many, so we subtract the functions that exclude two of the four elements of the codomain, each pair giving 2^5 functions. But this excludes too many, so we add back in the functions that exclude three of the four elements of the codomain, each triple giving 1^5 function. There are $\binom{4}{1}$ groups of functions excluding a single element, $\binom{4}{2}$ groups of functions excluding a pair of elements, and $\binom{4}{3}$ groups of functions excluding a triple of elements. This means that the number of functions that are *not* surjective is:

$$\binom{4}{1}3^5 - \binom{4}{2}2^5 + \binom{4}{3}1^5.$$

   We can now say that the number of functions that are surjective is:

$$4^5 - \left[\binom{4}{1}3^5 - \binom{4}{2}2^5 + \binom{4}{3}1^5\right].$$

2. The number of surjective functions is:

$$5^5 - \left[\binom{5}{1}4^5 - \binom{5}{2}3^5 + \binom{5}{3}2^5 - \binom{5}{4}1^5\right].$$

   We took the total number of functions 5^5 and subtracted all that were not surjective. There were $\binom{5}{1}$ ways to select a single element from the codomain to exclude from the range, and for each there were 4^5 functions. But this double counts, so we use PIE and subtract functions excluding two elements from the range: There are $\binom{5}{2}$ choices for the two elements to exclude, and for each pair, 3^5 functions. This takes out too many functions, so we add back in functions that exclude 3 elements from the range: $\binom{5}{3}$ choices for which 3 to exclude, and then 2^5 functions for each choice of elements. Finally we take back out the 1 function that excludes 4 elements for each of the $\binom{5}{4}$ choices of 4 elements.

If you happen to calculate this number precisely, you will get 120 surjections. That happens to also be the value of 5!. This might seem like an amazing coincidence until you realize that every surjective function $f : X \to Y$ with $|X| = |Y|$ finite must necessarily be a bijection. The number of bijections is always $|X|!$ in this case. What we have here is a *combinatorial proof* of the following identity:

$$n^n - \left[\binom{n}{1}(n-1)^n - \binom{n}{2}(n-2)^n + \cdots + \binom{n}{n-1}1^n\right] = n!.$$

We have seen that counting surjective functions is another nice example of the advanced use of the principle of inclusion/exclusion. Also, counting injective functions turns out to be equivalent to permutations, and counting all functions has a solution akin to those counting problems where order matters but repeats are allowed (like counting the number of words you can make from a given set of letters).

These are not just a few more examples of the techniques we have developed in this chapter. Quite the opposite: Everything we have learned in this chapter is an example of *counting functions*!

Example 3.8.10

How many 5-letter words can you make using the eight letters a through h? How many contain no repeated letters?

Solution. By now it should be no surprise that there are 8^5 words, and $P(8,5)$ words without repeated letters. The new piece here is that we are actually counting functions. For the first problem, we are counting all functions from $\{1, 2, \ldots, 5\}$ to $\{a, b, \ldots, h\}$. The numbers in the domain represent the *position* of the letter in the word; the codomain represents the letter that could be assigned to that position. If we ask for no repeated letters, we are asking for injective functions.

If A and B are *any* sets with $|A| = 5$ and $|B| = 8$, then the number of functions $f : A \to B$ is 8^5 and the number of injections is $P(8,5)$. So if you can represent your counting problem as a function counting problem, most of the work is done.

Example 3.8.11

How many subsets are there of $\{1, 2, \ldots, 9\}$? How many 9-bit strings are there (of any weight)?

Solution. We saw in Section 3.1 that the answer to both these questions is 2^9, as we can say yes or no (or 0 or 1) to each of the 9 elements in the set (positions in the bit-string). But 2^9 also looks like the answer you get from counting functions. In fact, if you count all functions $f : A \to B$ with $|A| = 9$ and $|B| = 2$, this is exactly what you get.

This makes sense! Let $A = \{1, 2, \ldots, 9\}$ and $B = \{y, n\}$. We are assigning each element of the set either a yes or a no. Or in the language of bit-strings, we would take the 9 positions in the bit string as our domain and the set $\{0, 1\}$ as the codomain.

So far we have not used a function as a model for binomial coefficients (combinations). Think for a moment about the relationship between combinations and permutations, say specifically $\binom{9}{3}$ and $P(9, 3)$. We *do* have a function model for $P(9, 3)$. This is the number of *injective* functions from a set of size 3 (say $\{1, 2, 3\}$ to a set of size 9 (say $\{1, 2, \ldots, 9\}$) since there are 9 choices for where to send the first element of the domain, then only 8 choices for the second, and 7 choices for the third. For example, the function might look like this:

$$f(1) = 5 \qquad f(2) = 8 \qquad f(3) = 4.$$

This is a different function from:

$$f(1) = 4 \qquad f(2) = 5 \qquad f(3) = 8.$$

Now $P(9, 3)$ counts these as different outcomes correctly, but $\binom{9}{3}$ will count these (among others) as just one outcome. In fact, in terms of functions $\binom{9}{3}$ just counts the number of possible ranges for injective functions. This should not be a surprise since binomial coefficients count subsets, and the range is a possible subset of the codomain.[5]

While it is possible to interpret combinations as functions, perhaps the better advice is to instead use combinations (or sticks and stones) when functions are not quite the right way to interpret the counting question.

3.8.5 PRACTICE PROBLEMS

1. The dollar menu at your favorite tax-free fast food restaurant has 7 items. You have \$16 to spend. How many different meals can you buy if you spend all your money and:

a. Purchase at least one of each item.

b. Possibly skip some items.

c. Don't get more than 2 of any particular item.

[5] A more mathematically sophisticated interpretation of combinations is that we are defining two injective functions to be *equivalent* if they have the same range, and then counting the number of equivalence classes under this notion of equivalence.

2. After a late night of math studying, you and your friends decide to go to your favorite tax-free fast food Mexican restaurant, *Burrito Chime*. You decide to order off of the dollar menu, which has 6 items. Your group has \$17 to spend (and will spend all of it).

a. How many different orders are possible? (The *order* in which the order is placed does not matter, just which and how many of each item that is ordered.)

b. How many different orders are possible if you want to get at least one of each item?

c. How many different orders are possible if you don't want to get more than 4 of any one item?

3. After another gym class you are tasked with putting the 14 identical dodgeballs away into 5 bins. This time, no bin can hold more than 6 balls. How many ways can you clean up?

4. Consider the equation $x_1 + x_2 + x_3 + x_4 = 19$. How many solutions are there with $1 \leq x_i \leq 6$ for all $i \in \{1, 2, 3, 4\}$?

5. Suppose you planned on giving 7 gold stars to some of the 12 star students in your class. Each student can receive at most one star. How many ways can you do this?

a. Use Pascal's triangle to find the numeric answer.

a. Use the principle of inclusion/exclusion.

6. How many permutations of $\{1, 2, 3, 4\}$ leave exactly 1 element fixed?

7. 12 ladies of a certain age drop off their red hats at the hat check of a museum. As they are leaving, the hat check attendant gives the hats back randomly. In how many ways can exactly 8 of the ladies receive their own hat (and the other 4 not)?

8. The Grinch sneaks into a room in which there are 8 Christmas presents for 8 different people. He proceeds to switch the name labels on the presents. How many ways could he do this if:

a. No present is allowed to end up with its original label? Explain what each term in your answer represents.

b. Exactly 4 presents keep their original labels? Explain.

c. Exactly 7 presents keep their original labels? Explain.

9. Consider functions $f : \{1, 2, 3, 4, 5, 6, 7, 8\} \to \{a, b, c, d, e, f\}$. How many functions have the property that $f(1) \neq c$ or $f(2) \neq f$, or both?

10. Consider sets A and B with $|A| = 8$ and $|B| = 5$. How many functions $f : A \to B$ are surjective?

11. Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. How many injective functions $f : A \to A$ have the property that for each $x \in A$, $f(x) \neq x$?

3.8.6 Additional Exercises

1. Based on the previous question, give a combinatorial proof for the identity:

$$\binom{n}{k} = \binom{n+k-1}{k} - \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} \binom{n+k-(2j+1)}{k-2j}.$$

2. Illustrate how the counting of derangements works by writing all permutations of $\{1, 2, 3, 4\}$ and then crossing out those which are not derangements. Keep track of the permutations you cross out more than once, using PIE.

3. Let d_n be the number of derangements of n objects. For example, using the techniques of this section, we find

$$d_3 = 3! - \left(\binom{3}{1}2! - \binom{3}{2}1! + \binom{3}{3}0!\right).$$

We can use the formula for $\binom{n}{k}$ to write this all in terms of factorials. After simplifying, for d_3 we would get

$$d_3 = 3!\left(1 - \frac{1}{1} + \frac{1}{2} - \frac{1}{6}\right).$$

Generalize this to find a nicer formula for d_n. Bonus: For large n, approximately what fraction of all permutations are derangements? Use your knowledge of Taylor series from calculus.

3.9 Chapter Summary

Investigate!

Suppose you have a huge box of animal crackers containing plenty of each of 10 different animals. For the counting questions below, carefully examine their similarities and differences, and then give an answer. The answers are all one of the following:

$$P(10,6), \quad \binom{10}{6}, \quad 10^6, \quad \binom{15}{9}.$$

1. How many animal parades containing 6 crackers can you line up?
2. How many animal parades of 6 crackers can you line up so that the animals appear in alphabetical order?
3. How many ways could you line up 6 different animals in alphabetical order?
4. How many ways could you line up 6 different animals if they can come in any order?
5. How many ways could you give 6 children one animal cracker each?
6. How many ways could you give 6 children one animal cracker each so that no two kids get the same animal?
7. How many ways could you give out 6 giraffes to 10 kids?
8. Write a question about giving animal crackers to kids that has the answer $\binom{10}{6}$.

With all the different counting techniques we have mastered in this last chapter, it might be difficult to know when to apply which technique. Indeed, it is very easy to get mixed up and use the wrong counting method for a given problem. You get better with practice. As you practice, you start to notice some trends that can help you distinguish between types of counting problems. Here are some suggestions that you might find helpful when deciding how to tackle a counting problem and checking whether your solution is correct.

- Remember that you are counting the number of items in some *list of outcomes*. Write down part of this list. Write down an element in the middle of the list – how are you deciding whether your element really is in the list? Could you get this element more than once using your proposed answer?
- If generating an element on the list involves selecting something (for example,

picking a letter or picking a position to put a letter, etc.), can the things you select be repeated? Remember, permutations and combinations select objects from a set *without* repeats.

- Does order matter? Be careful here, and be sure you know what your answer really means. We usually say that order matters when you get different outcomes when the same objects are selected in different orders. Combinations and "sticks and stones" are used when order *does not* matter.

- There are four possibilities when it comes to order and repeats. If order matters and repeats are allowed, the answer will look like n^k. If order matters and repeats are not allowed, we have $P(n,k)$. If order doesn't matter and repeats are allowed, use sticks and stones. If order doesn't matter and repeats are not allowed, use $\binom{n}{k}$. But be careful: this only applies when you are selecting things, and you should make sure you know exactly what you are selecting before determining which case you are in.

- Think about how you would represent your counting problem in terms of sets or functions. We know how to count different sorts of sets and different types of functions.

- As we saw with combinatorial proofs, you can often solve a counting problem in more than one way. Do that, and compare your numerical answers. If they don't match, something is amiss.

While we have covered many counting techniques, we have really only scratched the surface of the large subject of *enumerative combinatorics*. There are mathematicians doing original research in this area even as you read this. Counting can be really hard.

In the next chapter, we will approach counting questions from a very different direction, and in doing so, answer infinitely many counting questions at the same time. We will create *sequences* of answers to related questions.

Chapter Review

1. You have 15 presents to give to your 5 kids. How many ways can this be done if:

a. The presents are identical, and each kid gets at least one present?

b. The presents are identical, and some kids might get no presents?

c. The presents are unique, and some kids might get no presents?

d. The presents are unique and each kid gets at least one present?

2. For each of the following counting problems, say whether the answer is $\binom{10}{4}$, $P(10,4)$, or neither. If your answer is "neither," say what the answer should be instead.

(a) How many shortest lattice paths are there from $(0,0)$ to $(10,4)$?

(b) If you have 10 bow ties, and you want to select 4 of them for next week, how many choices do you have?

(c) Suppose you have 10 bow ties and you will wear a different one on each of the next 4 days. How many choices do you have?

(d) If you want to wear 4 of your 10 bow ties next week (Monday through Sunday), in how many ways can this be accomplished?

(e) Out of a group of 10 classmates, how many ways can you rank your top 4 friends?

(f) If 10 students come to their professor's office but only 4 can fit at a time, how many different combinations of 4 students can see the prof first?

(g) How many 4-letter words can be made from the first 10 letters of the alphabet?

(h) In how many ways can you make the word "cake" from the first 10 letters of the alphabet?

(i) How many ways are there to distribute 10 identical apples among 4 children?

(j) If you have 10 kids (and live in a shoe) and 4 types of cereal, in how many ways can your kids eat breakfast?

(k) In how many ways can you arrange exactly 4 ones in a string of 10 binary digits?

(l) You want to select 4 distinct, single-digit numbers as your lotto picks. How many choices do you have?

(m) 10 kids want ice cream. You have 4 varieties. How many ways are there to give the kids as much ice cream as they want?

(n) How many 1-1 functions are there from $\{1, 2, \ldots, 10\}$ to $\{a, b, c, d\}$?

(o) How many surjective functions are there from $\{1, 2, \ldots, 10\}$ to $\{a, b, c, d\}$?

(p) Each of your 10 bow ties matches 4 pairs of suspenders. How many outfits can you make?

(q) After the party, the 10 kids each choose one of 4 party-favors. How many outcomes are there?

(r) How many 6-elements subsets are there of the set $\{1, 2, \ldots, 10\}$

(s) In how many ways can you split up 11 kids into 5 named teams?

(t) How many solutions are there to $x_1 + x_2 + \cdots + x_5 = 6$ where each x_i is a non-negative integer?

(u) Your band goes on tour. There are 10 cities within driving distance, but only enough time to play 4 of them. How many choices do you have for the cities on your tour?

(v) In how many different ways can you play the 4 cities you choose?

(w) Out of the 10 breakfast cereals available, you want to have 4 bowls. In how many ways can you do this?

(x) There are 10 types of cookies available. You want to make a 4 cookie stack. How many different stacks can you make?

(y) From your home at (0,0) you want to go to either the donut shop at (5,4) or the one at (3,6). How many paths could you take?

(z) How many 10-digit numbers do not contain a sub-string of 4 repeated digits?

3. Recall, you own 9 regular ties and 5 bow ties. You realize that it would be okay to wear more than two ties to your clown college interview.

a. You must select some of your ties to wear. Everything is okay, from no ties up to all ties. How many choices do you have?

b. If you want to wear at least one regular tie and one bow tie, but are willing to wear up to all your ties, how many choices do you have for which ties to wear?

c. How many choices of which ties to wear do you have if you wear exactly 3 of the 9 regular ties and 2 of the 5 bow ties?

d. Once you have selected 3 regular and 2 bow ties, in how many orders could you put the ties on, assuming you must have one of the bow ties on top?

4. Give a counting question where the answer is $8 \cdot 3 \cdot 3 \cdot 5$. Give another question where the answer is $8 + 3 + 3 + 5$.

5. Consider numbers of the form $\alpha = a_1 a_2 a_3 \ldots a_n$, with n digits, each digit from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. For example, if $n = 4$, the number $\alpha = a_1 a_2 a_3 a_4$ will have 4 digits from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

Let $n = 11$.

a. How many such numbers are there?

b. How many such numbers are there for which the *sum* of the digits is even?

c. How many such numbers contain more even digits than odd digits?

6. In a recent small survey of airline passengers, 27 said they had flown American in the last year, 20 had flown Jet Blue, and 29 had flown Continental. Of those, 13 reported they had flown on American and Jet Blue, 10 had flown on Jet Blue and Continental, and 11 had flown on American and Continental. 6 passengers had flown on all three airlines.

How many passengers were surveyed? (Assume the results above make up the entire survey.)

7. Recall, by 10-bit strings, we mean strings of binary digits, of length 10.

a. How many 10-bit strings are there total?

b. How many 10-bit strings have weight 5?

c. How many subsets of the set {1,2,3,4,5,6,7,8,9,10} contain exactly 5 elements?

8. What is the coefficient of x^{10} in the expansion of $(x+3)^{15} + x^3(x+5)^{21}$?

9. How many 10-letter words contain exactly 3 vowels? (For example, an 8-letter word with 5 vowels is "aaioobtt"; don't consider "y" a vowel for this exercise.)

What if repeated letters were not allowed?

10. For each of the following, find the number of shortest lattice paths from $(4,4)$ to $(10,10)$ which:

a. pass through the point $(6,5)$.

b. avoid (do not pass through) the point $(9,9)$.

c. either pass through $(6,5)$ or $(9,9)$ (or both).

11. You live in Grid-Town on the corner of 2nd and 4th, and work in a building on the corner of 17th and 20th. How many routes are there which take you from home to work and then back home, but by a different route?

12. How many 11-bit strings start with 101 or end with 10 or both?

13. How many 15-bit strings of weight 5 start with 101 or end with 10 or both?

14. We are making 5-letter words from the set of letters $a, b, c, d, e, f \ldots$, in such a way that we have exactly the first 5 letters available so that we use all letters.

a. How many 5-letter words can we make from the letters without repeats that do not contain the sub-word "bad" in consecutive letters?

a. How many words don't contain the subword "bad" in not-necessarily-consecutive letters (but in order)?

15. Explain using lattice paths why $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

16. Suppose you have 16 one-dollar bills to give out as prizes to your top 6 discrete math students. How many ways can you do this if:

a. Each of the 6 students gets at least 1 dollar?

b. Some students might get nothing?

c. Each student gets at least 1 dollar but no more than 8 dollars?

17. How many functions $f : \{1,2,3,4\} \to \{1,2,3,4\}$ are there satisfying:

a. $f(1) = 1$ or $f(2) = 2$ (or both)?

b. $f(1) \neq 1$ or $f(2) \neq 2$ (or both)?

c. $f(1) \neq 1$ *and* $f(2) \neq 2$, and f is injective?

d. f is surjective, but $\forall x \in \{1, 2, \ldots, 4\}, f(x) \neq x$?

18. How many functions map $\{1,2,3,4,5\}$ *onto* $\{1,2,3\}$ (i.e., how many *surjections* are there)?

19. To thank your math professor for doing such an amazing job all semester, you decide to bake your professor cookies. You know how to make 13 different types of cookies.

a. If you want to give your professor 8 different types of cookies, how many different combinations of cookie type can you select? Explain your answer.

b. To keep things interesting, you decide to make a different number of each type of cookie. If again you want to select 8 cookie types, how many ways can you select the cookie types and decide for which there will be the most, second most, etc. Explain your answer.

c. You change your mind again. This time you decide you will make a total of 23 cookies. Each cookie could be any one of the 13 types of cookies you know how to bake (and it's okay if you leave some types out). How many choices do you have? Explain.

d. You realize that the previous plan did not account for presentation. This time, you once again want to make 23 cookies, each one could be any one of the 13 types of cookies. However, now you plan to shape the cookies into the numerals 1, 2, ..., 23. How many choices do you have for which types of cookies to bake into which numerals? Explain.

e. The only flaw with the last plan is that your professor might not get to sample all 13 different varieties of cookies. How many choices do you have for which types of cookies to make into which numerals, given that each type of cookie should be present at least once? Explain.

20. For which of the parts of the previous problem (Exercise 3.9.19) does it make sense to interpret the counting question as counting some number of functions? Say what the domain and codomain should be, and whether you are counting all functions, injections, surjections, or something else.

Chapter 4

Sequences

We have encountered *finite* sequences already as a discrete structure and something we can count. In this chapter, we will consider possibly *infinite* sequences of numbers. When the sequence itself is infinite, it no longer makes sense to ask how many possible sequences there are, but there is still an interesting connection to counting: Each term in the sequence can represent an answer to a counting question!

4.1 Describing Sequences

Objectives

After completing this section, you should be able to do the following.

1. Use proper notation to represent a sequence.
2. Explain the difference between a closed formula and a recursive definition for a sequence.
3. Find a recursive definition for a sequence based on its description.

4.1.1 Section Preview

Investigate!

There is a monastery in Hanoi, as the legend goes, with a great hall containing three tall pillars. Resting on the first pillar were 64 giant disks (or washers), all different sizes, stacked from largest to smallest. The monks of the monastery have been moving the disks for generations, attempting to move the entire stack of disks to the third pillar. However, due to the size of the disks, the monks cannot move more than one at a time. Each disk must be placed on one of the pillars before the next disk is moved. And because the disks are so heavy and fragile, the monks may never place a larger disk on top of a smaller disk. When the monks finally complete their task, the world shall come to an end.

Your task: Figure out how long it will be before we need to start worrying about the end of the world.

This puzzle is called the *Tower of Hanoi*. You are tasked with finding the minimum number of moves to complete the puzzle. This certainly sounds like a counting problem. Perhaps you have an answer? If not, what else could we try?

The answer to the puzzle depends on the number of disks you need to move. In fact, we could answer the puzzle first for 1 disk, then 2, then 3, and so on. If we list all of the answers for each number of disks, we will get a **sequence** of numbers. The nth term in the sequence is the answer to the question, "What is the smallest number of moves required to complete the Tower of Hanoi puzzle with n disks?"

Give it a try. Find the smallest number of moves needed to transport the stack of n disks to another pillar, for different values of n.

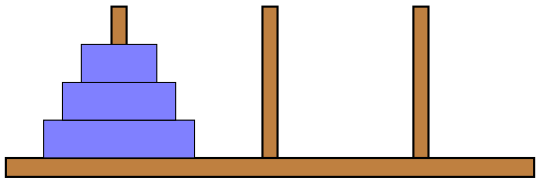


Figure 4.1.1 The Towers of Hanoi puzzle.

You might wonder why we would create such a sequence instead of just answering the question. By looking at how the sequence of numbers grows, we gain insight into the problem. It is easy to count the number of moves required for a small number of disks. We can then look for a pattern among the first few terms of the sequence. Hopefully this will suggest a method for finding the nth term of the sequence, which is the answer to our question. Of course we will also need to verify that our suspected pattern is correct, and that this correct pattern really does give us the nth term that we think it does, but it is impossible to prove that your formula is correct without having a formula to start with.

In this section, we will explore how to represent sequences of numbers in various ways, and two sorts of formulas that can be used to describe the sequence. We will see that sequences are also interesting mathematical objects to study in their own right.

Preview Activity

Let's get a feel for the sequence from the Towers of Hanoi puzzle.

1.

(a) What is the smallest number of moves required to transport 2 disks to another pillar?

(b) What is the smallest number of moves required to transport 3 disks to another pillar?

(c) What is the smallest number of moves required to transport 4 disks to another pillar?

(d) To find the smallest number of moves required to transport 5 disks to another pillar, let's try to relate this task to the task of moving 4 disks. How many moves do each of the following tasks require?

a. Move the 4 smallest disks to the second pillar: _____ moves.

b. Move the largest disk to the third pillar: _____ moves.

c. Move the 4 smallest disks to the third pillar (on top of the largest disk): _____ moves.

Therefore, the smallest number of moves required to move five disks is: _____

(e) Generalize the last observation. Let a_n represent the smallest number of moves required to transport n disks from the start pillar to another pillar. Then a_{n-1} represents the number of moves required to transport $n-1$ disks to another pillar.

Give a formula for a_n in terms of a_{n-1}.

4.1.2 Sequences and Formulas

A **sequence** is simply an ordered list of numbers. Unlike a *set* of numbers, the order of the numbers in a sequence is an essential characteristic of the sequence. For this reason, when we use variables to represent terms in a sequence they will look like this:

$$a_0, a_1, a_2, a_3, \ldots.$$

To refer to the *entire* sequence at once, we will write $(a_n)_{n\in\mathbb{N}}$ or $(a_n)_{n\geq 0}$, or sometimes if we are being sloppy, just (a_n) (in which case we assume we start the sequence with a_0).

We might replace the a with another letter, and sometimes we omit a_0, starting with a_1, in which case we would use $(a_n)_{n\geq 1}$ to refer to the sequence as a whole. The numbers in the subscripts are called **indices** (the plural of **index**).

While we often just think of a sequence as an ordered list of numbers, it is really a type of function. Specifically, the sequence $(a_n)_{n\geq 0}$ is a function with domain $\mathbb{N}$ where a_n is the image of the natural number n. Later we will manipulate sequences in much the same way you have manipulated functions in algebra or calculus. We can shift a sequence up or down, add two sequences, or ask for the rate of change of a sequence. These are done exactly as you would for functions.

That said, while keeping the rigorous mathematical definition in mind is helpful, we often describe sequences by writing out the first few terms.

Example 4.1.2

Can you find the next term in the following sequences?

1. $7, 7, 7, 7, 7, \ldots$
2. $3, -3, 3, -3, 3, \ldots$
3. $1, 5, 2, 10, 3, 15, \ldots$
4. $1, 2, 4, 8, 16, 32, \ldots$
5. $1, 4, 9, 16, 25, 36, \ldots$
6. $1, 2, 3, 5, 8, 13, 21, \ldots$
7. $1, 3, 6, 10, 15, 21, \ldots$
8. $2, 3, 5, 7, 11, 13, \ldots$
9. $3, 2, 1, 0, -1, \ldots$
10. $1, 1, 2, 6, \ldots$

Solution. No, you cannot.

You might guess that the next terms are:

1. 7
2. −3
3. 4
4. 64
5. 49
6. 34
7. 28
8. 17
9. −2
10. 24

In fact, those are the next terms of the sequences I had in mind when I made up the example, but there is no way to be sure they are correct.

Still, we will often do this. Given the first few terms of a sequence, we can ask what the pattern in the sequence suggests the next terms are.

Given that no number of initial terms in a sequence is enough to say for certain which sequence we are dealing with, we need to find another way to specify a sequence. We consider two main ways to do this:

Definition 4.1.3 Closed formula.

A **closed formula** for a sequence $(a_n)_{n\in\mathbb{N}}$ is a formula for a_n using a fixed finite number of operations on n. This is what you normally think of as a formula in n, just as if you were defining a function in terms of n (because that is exactly what you are doing).

Definition 4.1.4 Recursive definition.

A **recursive definition** (sometimes called an **inductive definition**) for a sequence $(a_n)_{n\in\mathbb{N}}$ consists of a **recurrence relation** : an equation relating a term of the sequence to previous terms (terms with smaller index), and an **initial condition**: a list of a few terms of the sequence (one less than the number of terms in the recurrence relation).

It is easier to understand what is going on here with an example:

Example 4.1.5

Here are a few closed formulas for sequences:

- $a_n = n^2$.
- $a_n = \dfrac{n(n+1)}{2}$.
- $a_n = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^{-n}}{\sqrt{5}}$.

Note in each formula, if you are given n, you can calculate a_n directly: Just plug in n. For example, to find a_3 in the second sequence, just compute $a_3 = \frac{3(3+1)}{2} = 6$.

Here are a few recursive definitions for sequences:

- $a_n = 2a_{n-1}$ with $a_0 = 1$.
- $a_n = 2a_{n-1}$ with $a_0 = 27$.
- $a_n = a_{n-1} + a_{n-2}$ with $a_0 = 0$ and $a_1 = 1$.

In these formulas, if you are given n, you cannot calculate a_n directly; you first need to find a_{n-1} (or a_{n-1} and a_{n-2}). In the second sequence, to find a_3 you would take $2a_2$, but to find $a_2 = 2a_1$ we would need to know $a_1 = 2a_0$. We do know this, so we could trace back through these equations to find $a_1 = 54$, $a_2 = 108$ and finally $a_3 = 216$.

You might wonder why we would bother with recursive definitions for sequences. After all, it is harder to find a_n with a recursive definition than with a closed formula. This is true, but it is also harder to find a closed formula for a sequence than it is to find a recursive definition. So to find a useful closed formula, we might first find the recursive definition, and then use that to find the closed formula.

Example 4.1.6

For defeating the evil dragon, the king promises you a reward of one grain of rice on the first day, and then twice as many grains the next day as the previous day, for the rest of the month.

Not to be swindled, you barter and get him to agree to add one additional grain of rice each day as well.

How many grains of rice will you receive on the 30th day?

Solution. We can write down the first few terms of the sequence $(a_n)_{n \geq 1}$

that gives the number of grains of rice you receive on day n. We get

$$1, 3, 7, 15, \ldots,$$

since, for example, on day 4, you take the number of grains of rice you had on day 3 (all seven of them), double that and add 1, to get 15. The next day you would get $2 \cdot 15 + 1 = 31$.

By actually computing these values, we see right away what the recurrence relation is:

$$a_n = 2a_{n-1} + 1.$$

This is justified because the problem the sequence is modeling says that we double the number of grains each day, and then add 1. We can read the recurrence relation right off the problem.

The original question asked about a_{30}. We could find this using the recurrence relation, but we would first need to find a_{29}, and to find that we would need a_{28}, which requires a_{27} first, and so on. While I'm sure we could work all the way back to a_4, which we already found, this sounds like a lot of work.

It would be so much nicer to have a closed formula. If we could *solve the recurrence relation* and find that closed formula, we could just substitute 30 for n.

Let's guess. It looks like the sequence is close to $2, 4, 8, 16, \ldots$ which has closed formula 2^n. That sequence has terms one greater than the terms in our sequence, so we might guess that

$$a_n = 2^n - 1.$$

To be clear, we have no good reason to believe this guess is correct, but maybe later in this chapter we will. However, if it *is* correct, then we are golden (and incredibly sick of rice):

$$a_{30} = 2^{30} - 1 = 1,073,741,823.$$

This is not to say that recursive definitions aren't useful in finding a_n. You can always calculate a_n given a recursive definition; it might just take a while.

Example 4.1.7

Find a_6 in the sequence defined by $a_n = 2a_{n-1} - a_{n-2}$ with $a_0 = 3$ and $a_1 = 4$.

Solution. We know that $a_6 = 2a_5 - a_4$. So to find a_6 we need to find a_5 and a_4. Well

$$a_5 = 2a_4 - a_3 \qquad \text{and} \qquad a_4 = 2a_3 - a_2,$$

so if we can only find a_3 and a_2, we would be set. Of course

$$a_3 = 2a_2 - a_1 \qquad \text{and} \qquad a_2 = 2a_1 - a_0,$$

so we only need to find a_1 and a_0. But we are given these. Thus

$$\begin{aligned}
a_0 &= 3\\
a_1 &= 4\\
a_2 &= 2\cdot 4 - 3 = 5\\
a_3 &= 2\cdot 5 - 4 = 6\\
a_4 &= 2\cdot 6 - 5 = 7\\
a_5 &= 2\cdot 7 - 6 = 8\\
a_6 &= 2\cdot 8 - 7 = 9.
\end{aligned}$$

Note that now we can guess a closed formula for the nth term of the sequence: $a_n = n + 3$. To be sure this will always work, we could plug in this formula into the recurrence relation:

$$\begin{aligned}
2a_{n-1} - a_{n-2} &= 2((n-1)+3) - ((n-2)+3)\\
&= 2n + 4 - n - 1\\
&= n + 3\\
&= a_n.
\end{aligned}$$

That is not quite enough though, since there can be multiple closed formulas that satisfy the same recurrence relation; we must also check that our closed formula agrees on the initial terms of the sequence. Since $a_0 = 0+3 = 3$ and $a_1 = 1+3 = 4$ are the correct initial conditions, we can now conclude that we have the correct closed formula.

Finding closed formulas, or even recursive definitions, for sequences is not trivial. There is no one method for doing this. Just as in evaluating integrals or solving differential equations, it is useful to have a bag of tricks you can apply, but sometimes there is no easy answer.

One useful method is to relate a given sequence to another sequence for which we already know the closed formula. To do this, we need a few "known sequences" to compare mystery sequences to. Here are a few that are good to know. We will verify the formulas for these in the coming sections.

Common Sequences.

$1, 4, 9, 16, 25, \ldots$ The **square numbers**. The sequence $(s_n)_{n\geq 1}$ has closed formula $s_n = n^2$

$1, 3, 6, 10, 15, 21, \ldots$	The **triangular numbers**. The sequence $(T_n)_{n\geq 1}$ has closed formula $T_n = \frac{n(n+1)}{2}$.
$1, 2, 4, 8, 16, 32, \ldots$	The **powers of 2**. The sequence $(a_n)_{n\geq 0}$ with closed formula $a_n = 2^n$.
$1, 1, 2, 3, 5, 8, 13, \ldots$	The **Fibonacci numbers** (or Fibonacci sequence), defined recursively by $F_n = F_{n-1} + F_{n-2}$ with $F_1 = F_2 = 1$.

Example 4.1.8

Use the formulas $T_n = \frac{n(n+1)}{2}$ and $a_n = 2^n$ to find closed formulas that agree with the following sequences. Assume each first term corresponds to $n = 0$.

1. (b_n): $1, 2, 4, 7, 11, 16, 22, \ldots$.
2. (c_n): $3, 5, 9, 17, 33, \ldots$.
3. (d_n): $0, 2, 6, 12, 20, 30, 42, \ldots$.
4. (e_n): $3, 6, 10, 15, 21, 28, \ldots$.
5. (f_n): $0, 1, 3, 7, 15, 31, \ldots$.
6. (g_n) $3, 6, 12, 24, 48, \ldots$.
7. (h_n): $6, 10, 18, 34, 66, \ldots$.
8. (j_n): $15, 33, 57, 87, 123, \ldots$.

Solution. We wish to compare these sequences to the triangular numbers $(0, 1, 3, 6, 10, 15, 21, \ldots)$, when we start with $n = 0$, and the powers of 2: $(1, 2, 4, 8, 16, \ldots)$.

1. $(1, 2, 4, 7, 11, 16, 22, \ldots)$. Note that if we subtract 1 from each term, we get the sequence (T_n). So we have $b_n = T_n + 1$. Therefore a closed formula is $b_n = \frac{n(n+1)}{2} + 1$. A quick check of the first few n confirms we have it right.

2. $(3, 5, 9, 17, 33, \ldots)$. Each term in this sequence is one more than a power of 2, so we might guess the closed formula is $c_n = a_n + 1 = 2^n + 1$. If we try this though, we get $c_0 = 2^0 + 1 = 2$ and $c_1 = 2^1 + 1 = 3$. We are off because the indices are shifted. What we really want is $c_n = a_{n+1} + 1$ giving $c_n = 2^{n+1} + 1$.

3. $(0, 2, 6, 12, 20, 30, 42, \ldots)$. Notice that all these terms are even. What happens if we factor out a 2? We get (T_n)! More precisely, we find that $d_n/2 = T_n$, so this sequence has closed formula $d_n = n(n+1)$.

4. $(3, 6, 10, 15, 21, 28, \ldots)$. These are all triangular numbers. However, we are starting with 3 as our initial term instead of as our third term. So if we could plug in 2 instead of 0 into the formula for T_n, we would be set. Therefore the closed formula is $e_n = \frac{(n+2)(n+3)}{2}$ (where $n+3$ came from $(n+2)+1$). Thinking about sequences as functions, we are doing a horizontal shift by 2: $e_n = T_{n+2}$ which would cause the graph to shift 2 units to the left.

5. $(0, 1, 3, 7, 15, 31, \ldots)$. Try adding 1 to each term, and we get powers of 2. You might guess this because each term is a little more than twice the previous term (the powers of 2 are *exactly* twice the previous term). Closed formula: $f_n = 2^n - 1$.

6. $(3, 6, 12, 24, 48, \ldots)$. These numbers are also doubling each time, but are also all multiples of 3. Dividing each by 3 gives $1, 2, 4, 8, \ldots$. Aha. We get the closed formula $g_n = 3 \cdot 2^n$.

7. $(6, 10, 18, 34, 66, \ldots)$. To get from one term to the next, we almost double each term. So maybe we can relate this back to 2^n. Yes, each term is 2 more than a power of 2. So we get $h_n = 2^{n+2} + 2$ (the $n+2$ is because the first term is 2 more than 2^2, not 2^0). Alternatively, we could have related this sequence to the second sequence in this example: Starting with $3, 5, 9, 17, \ldots$ we see that this sequence is twice the terms from that sequence. That sequence had closed formula $c_n = 2^{n+1} + 1$. Our sequence here would be twice this, so $h_n = 2(2^n + 1)$, which is the same as what we got before.

8. $(15, 33, 57, 87, 123, \ldots)$. Try dividing each term by 3. That gives the sequence $5, 11, 19, 29, 41, \ldots$. Now add 1 to each term: $6, 12, 20, 30, 42, \ldots$, which is (d_n) in this example, except starting with 6 instead of 0. So let's start with the formula $d_n = n(n+1)$. To start with the 6, we shift: $(n+2)(n+3)$. But this is one too many, so subtract 1: $(n+2)(n+3) - 1$. That gives us our sequence, but divided by 3. So we want $j_n = 3((n+2)(n+3) - 1)$.

4.1.3 Partial Sums and Differences

Some sequences naturally arise as the sum of terms of another sequence.

Example 4.1.9

Sam keeps track of how many push-ups she does each day of her "do lots of push-ups challenge." Let $(a_n)_{n\geq 1}$ be the sequence that describes the number of push-ups done on the nth day of the challenge. The sequence starts

$$3, 5, 6, 10, 9, 0, 12, \ldots.$$

Describe a sequence $(b_n)_{n\geq 1}$ that gives the *total number* of push-ups done by Sam after the nth day.

Solution. We can find the terms of this sequence easily enough.

$$3, 8, 14, 24, 33, 33, 45, \ldots.$$

Here b_1 is just a_1, but then

$$b_2 = 3 + 5 = a_1 + a_2,$$

$$b_3 = 3 + 5 + 6 = a_1 + a_2 + a_3,$$

and so on.

There are a few ways we might describe b_n in general. We could do so recursively as,

$$b_n = b_{n-1} + a_n,$$

since the total number of push-ups done after n days will be the number done after $n - 1$ days, plus the number done on day n.

For something closer to a closed formula, we could write

$$b_n = a_1 + a_2 + a_3 + \cdots + a_n,$$

or the same thing using *summation notation*:

$$b_n = \sum_{i=1}^{n} a_i.$$

However, note that these are not really closed formulas since even if we had a formula for a_n, we would still have an increasing number of computations to do as n increases.

Given any sequence $(a_n)_{n\in\mathbb{N}}$, we can always form a new sequence $(b_n)_{n\in\mathbb{N}}$ by

$$b_n = a_0 + a_1 + a_2 + \cdots + a_n.$$

Since the terms of (b_n) are the sums of the initial part of the sequence (a_n), we call (b_n) the **sequence of partial sums of** (a_n). Soon we will see that it is sometimes possible to find a closed formula for (b_n) from the closed formula for (a_n).

To simplify writing out these sums, we will often use notation like $\sum_{k=1}^{n} a_k$. This means add up the a_k's where k changes from 1 to n.

Example 4.1.10

Use $\sum$ notation to rewrite the sums:

1. $1 + 2 + 3 + 4 + \cdots + 100$
2. $1 + 2 + 4 + 8 + \cdots + 2^{50}$
3. $6 + 10 + 14 + \cdots + (4n - 2)$.

Solution.

1. $\displaystyle\sum_{k=1}^{100} k$
2. $\displaystyle\sum_{k=0}^{50} 2^k$
3. $\displaystyle\sum_{k=2}^{n} (4k - 2)$

It is also often useful to look at the **sequence of difference** of a sequence (a_n). By this we just mean the sequence (d_n) where

$$d_n = a_{n+1} - a_n.$$

For example, if $(a_n) = (1, 4, 9, 16, 25, \ldots)$ (the square numbers, so $a_n = n^2$), then the sequence of differences is $(d_n) = (3, 5, 7, 9, \ldots)$ since $4 - 1 = 3$, $9 - 4 = 5$, and so on. We could also find a closed formula for the sequence of differences here since we have a closed formula for a_n. We would have

$$d_n = a_{n+1} - a_n = (n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1.$$

In general, it is easy to go from a closed formula for a sequence to a closed formula for the sequence of differences. It is not always easy to go the other way around. In fact, what would it look like to start with a sequence of differences and get the "original" sequence?

Thinking about a recurrence relation is helpful here. Since

$$d_n = a_{n+1} - a_n,$$

we have that

$$a_{n+1} = a_n + d_n.$$

This makes sense: to get from one term of a sequence to the next, you add the difference between those terms. Ah! You add the differences. So the ***original sequence is the sequence of partial sums of the sequence of differences!***

In upcoming sections we will see that understanding the sequence of differences can tell us exactly where to look for a closed formula. The differences also suggest how to create a recursive definition.

Example 4.1.11

Find a recurrence relation and initial conditions that agree with the terms of this sequence: $1, 5, 17, 53, 161, 485 \ldots$.

Solution. Finding the recurrence relation would be easier if we had some context for the problem (like the Tower of Hanoi, for example). Alas, we have only the sequence. Remember, the recurrence relation tells you how to get from previous terms to future terms. What is going on here? We could look at the differences between terms: $4, 12, 36, 108, \ldots$. Notice that these are growing by a factor of 3. Is the original sequence as well? $1 \cdot 3 = 3$, $5 \cdot 3 = 15$, $17 \cdot 3 = 51$ and so on. It appears that we always end up with 2 less than the next term. Aha!

So $a_n = 3a_{n-1} + 2$ is our recurrence relation and the initial condition is $a_0 = 1$.

4.1.4 Sequences in python

Checking that a closed formula agrees with the initial terms in a sequence is easy with a calculator, or better yet, a programming language like python. One way you can do this is to define a function that returns the nth term of the sequence.

Here is an example: Suppose you wanted to check whether a formula you found for the sequence $(a_n)_{n\in\mathbb{N}} = (1, 3, 7, 15, 31, \ldots)$ is correct. Perhaps you guess that $a_n = 2^n - 1$. Try running the code below. (Note: In python, the ^ symbol means something else, so to do exponentiation, we use **.)

```
def a(n):
  return 2**n-1
print(a(3))
```

Looks promising, but we should be careful: Our sequence started with $a_0 = 1$, which makes $a_3 = 15$. Now try modifying the definition of `a(n)` (by changing the return value) to get the correct closed formula.

Perhaps you want to print out the first 20 terms of the sequence? This is easy to do with python, by putting the terms in a list:

```
def a(n):
  return 2**(n+1) - 1
sequence = []
for n in range(20):
  sequence.append(a(n))
print(sequence)
```

Note that `range(20)` starts at 0 and stops at 19 (it is the list `[0,1,2,...,19]`).

We can also use python to generate terms for a sequence given a recursive definition. Suppose we wanted to explore the sequence $a_n = 2a_{n-1} + 1$ with initial condition $a_0 = 1$. In python, we could generate the sequence as follows.

```
def a(n):
  if n == 0:
    return 1
  else:
    return 2*a(n-1)+1
# just print out the first 10 terms
for i in range(10):
  print(a(i))
```

Do you see how to translate a recurrence relation into a recursive python function? Try playing around with the code above to explore other sequences.

4.1.5 Reading Questions

1. For the formulas for sequences below, select all that are *closed* formulas (as opposed to *recursive* formulas).

A. $a_n = 3(n-1)+2$

B. $a_n = \dfrac{n^2+2n+3}{4n}$

C. $a_n = 3a_{n-1}+2$

D. $a_n = a_{n-1}+a_{n-2}+a_{n-3}$

2. Drag each sequence on the left to a recurrence relation that agrees with the sequence.

$5,5,5,5,5,\ldots$	$a_n = a_{n-1}+1$
$2,3,5,9,17,\ldots$	$a_n = 2a_{n-1}-a_{n-2}$
$1,2,3,4,5,\ldots$	$a_n = 2a_{n-1}-1$
$3,4,7,11,18,\ldots$	$a_n = a_{n-1}+a_{n-2}$

3. What questions do you have? Write at least one question about the content of this section that you or a classmate might be curious about after reading this section.

4.1.6 Practice Problems

1. For the sequence with closed formula $a_n = n^2+5n+4$, find the term a_7.

2. Consider the sequence with recurrence relation $a_n = a_{n-1}+6$ with initial term $a_0 = 3$. Find the term a_4.

3. Consider the sequence with recurrence relation $a_n = 7\cdot a_{n-1}$ with initial term $a_0 = 9$. Find the term a_6.

4. Find the closed formula for each of the following sequences $(a_n)_{n\geq 1}$ by relating them to a well known sequence. Assume the first term given is a_1.

a. $-1,2,7,14,23,\ldots$

$a_n =$ ____________

b. $3, 5, 8, 12, 17, \ldots$

$a_n =$ ____________

c. $11, 16, 22, 29, 37, \ldots$

$a_n =$ ____________

d. $25, 121, 721, 5041, 40321, \ldots$

$a_n =$ ____________

5. For each sequence given below, find a closed formula for a_n, the nth term of the sequence (assume the first terms here are always a_0) by relating it to another sequence for which you already know the formula.

a. $0, 1, 3, 7, 15, 31, \ldots$

$a_n =$ ____________

b. $-1, 0, 7, 26, 63, 124, \ldots$

$a_n =$ ____________

c. $0, 3, 12, 27, 48, 75, \ldots$

$a_n =$ ____________

d. $1, 3, 8, 17, 32, 57, \ldots$

$a_n =$ ____________

4.1.7 Additional Exercises

1. Consider the sequence $(a_n)_{n\geq 1}$ that starts $1, 3, 5, 7, 9, \ldots$ (i.e., the odd numbers in order).

(a) Give a recursive definition and closed formula for the sequence.

(b) Write out the sequence $(b_n)_{n\geq 2}$ of partial sums of (a_n). Write down the recursive definition for (b_n) and guess at the closed formula.

2. The Fibonacci sequence is $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ (where $F_0 = 0$).

(a) Write out the first few terms of the sequence of partial sums: 0, $0 + 1$, $0 + 1 + 1, \ldots$

(b) Guess a formula for the sequence of partial sums expressed in terms of a single Fibonacci number. For example, you might say $F_0 + F_1 + \cdots + F_n = 3F_{n-1}^2 + n$, although that is definitely not correct.

3. Consider the three sequences below. For each, find a recursive definition. How are these sequences related?

(a) $2, 4, 6, 10, 16, 26, 42, \ldots$.

(b) $5, 6, 11, 17, 28, 45, 73, \ldots$.

(c) $0, 0, 0, 0, 0, 0, 0, \ldots$.

4. Write out the first few terms of the sequence given by $a_1 = 3$; $a_n = 2a_{n-1} + 4$. Then find a recursive definition for the sequence $10, 24, 52, 108, \ldots$.

5. Write out the first few terms of the sequence given by $a_n = n^2 - 3n + 1$. Then find a closed formula for the sequence (starting with a_1) $0, 2, 6, 12, 20, \ldots$.

6. Show that $a_n = 3 \cdot 2^n + 7 \cdot 5^n$ is a solution to the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2}$. What would the initial conditions need to be for this to be the closed formula for the sequence?

7. Show that $a_n = 2^n - 5^n$ is also a solution to the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2}$. What would the initial conditions need to be for this to be the closed formula for the sequence?

8. Find a closed formula for the sequence with recursive definition $a_n = 2a_{n-1} - a_{n-2}$ with $a_1 = 1$ and $a_2 = 2$.

9. Give two different recursive definitions for the sequence with closed formula $a_n = 3 + 2n$. Prove you are correct. At least one of the recursive definitions should make use of two previous terms and no constants.

10. Use summation ($\sum$) or product ($\prod$) notation to rewrite the following.

(a) $2 + 4 + 6 + 8 + \cdots + 2n$.

(b) $1 + 5 + 9 + 13 + \cdots + 425$.

(c) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{50}$.

(d) $2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n$.

(e) $(\frac{1}{2})(\frac{2}{3})(\frac{3}{4}) \cdots (\frac{100}{101})$.

11. Expand the following sums and products. That is, write them out the long way.

(a) $\displaystyle\sum_{k=1}^{100}(3 + 4k)$.

(b) $\displaystyle\sum_{k=0}^{n} 2^k$.

(c) $\displaystyle\sum_{k=2}^{50} \frac{1}{(k^2 - 1)}$.

(d) $\displaystyle\prod_{k=2}^{100} \frac{k^2}{(k^2 - 1)}$.

(e) $\displaystyle\prod_{k=0}^{n}(2 + 3k)$.

12. Suppose you draw n lines in the plane so that every pair of lines cross (no lines are parallel) and no three lines cross at the same point. This will create some number of regions in the plane, including some unbounded regions. Call the number of regions R_n. Find a recursive formula for the number of regions created by n lines, and justify why your recursion is correct.

13. A **ternary** string is a sequence of 0's, 1's, and 2's. Just like a bit string, but with three symbols.

Let's call a ternary string *good* provided it never contains a 2 followed immediately by a 0. Let G_n be the number of good strings of length n. For example, $G_1 = 3$, and $G_2 = 8$ (since of the 9 ternary strings of length 2, only one is not good).

Find, with justification, a recursive formula for G_n, and use it to compute G_5.

14. Consider bit strings with length l and weight k (so strings of l 0's and 1's, including k 1's). We know how to count the number of these for a fixed l and k. Now, we will count the number of strings for which the *sum* of the length and the weight is fixed. For example, let's count all the bit strings for which $l + k = 11$.

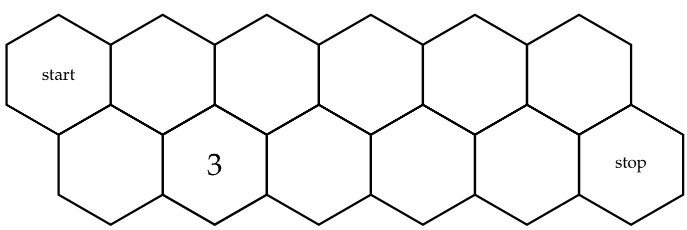
(a) Find examples of these strings of different lengths. What is the longest string possible? What is the shortest?

(b) How many strings are there of each of these lengths. Use this to count the total number of strings (with sum 11).

(c) The other approach: Let $n = l + k$ vary. How many strings have sum $n = 1$? How many have sum $n = 2$? And so on. Find and explain a recurrence relation for the sequence (a_n) that gives the number of strings with sum n.

(d) Describe what you have found above in terms of Pascal's triangle. What pattern have you discovered?

15. When bees play chess, they use a hexagonal board like the one shown below. The queen bee can move one space at a time either directly to the right or angled up-right or down-right (but can never move leftwards). How many different paths can the queen take from the top left hexagon to the bottom right hexagon? Explain your answer, and how this relates to the previous question. (As an example, there are three paths to get to the second hexagon on the bottom row.)



16. Let t_n denote the number of ways to tile a $2 \times n$ chessboard using 1×2 dominoes. Write out the first few terms of the sequence $(t_n)_{n \geq 1}$, and then give a recursive definition. Explain why your recursive formula is correct.

4.2 Rate of Growth

Objectives

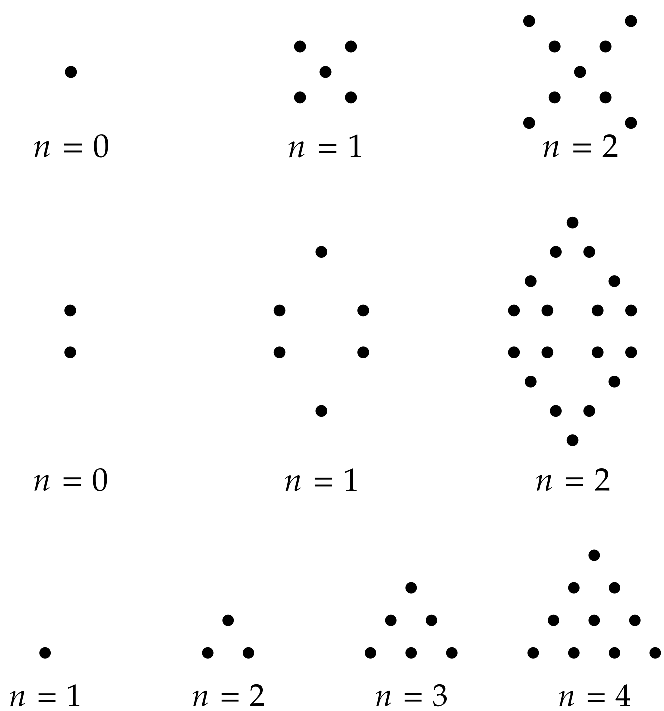
After completing this section, you should be able to do the following.

1. Identify a sequence as arithmetic or geometric based on its rate of growth.
2. Give recursive definitions and closed formulas for arithmetic and geometric sequences.

4.2.1 Section Preview

Investigate!

For each of the patterns of dots below, draw the next pattern in the sequence. Describe the rate of growth of the number of dots in the patterns. Then guess a recursive definition and a closed formula for the number of dots in the nth pattern.



Our goal is to find closed formulas for sequences. Our primary strategy will be to first determine how the sequence is changing from term to term. This will lead to a recurrence relation for the sequence, and from that recurrence relation, we will find a closed formula. We start with two types of sequences that are particularly common and useful: arithmetic and geometric sequences. Along the way, we will explore some techniques for solving recurrence relations.

Preview Activity

1. Explore the first sequence of dots from the *Investigate!* activity. We will let a_n represent the number of dots in figure n. The sequence starts $1, 5, 9, \ldots$

- **(a)** How many dots would you expect in the next two figures in the sequence?
- **(b)** How is the sequence growing? How many dots do you add to or multiply by to get the next figure?
- **(c)** Let a_n be the number of dots in figure n. Write a recursive definition for a_n.
- **(d)** Guess a closed formula for the number of dots in the nth figure.

2. Now look at the second sequence of dots from the *Investigate!* activity. We will let a_n represent the number of dots in figure n. The sequence starts $2, 6, 18, \ldots$

- **(a)** How many dots would you expect in the next two figures in the sequence?
- **(b)** How is the sequence growing? How many dots do you add to or multiply by to get the next figure?
- **(c)** Let a_n be the number of dots in figure n. Write a recursive definition for a_n.
- **(d)** Guess a closed formula for the number of dots in the nth figure.

3. Now look at the third sequence of dots from the *Investigate!* activity. We will let a_n represent the number of dots in figure n. The sequence starts $1, 3, 6, 10, \ldots$

- **(a)** How many dots would you expect in the next two figures in the sequence?
- **(b)** Let a_n be the number of dots in figure n. Write a recursive definition for a_n.
- **(c)** Guess a closed formula for the number of dots in the nth figure.

4.2.2 Arithmetic Sequences

Suppose you start a business selling prints of mathematical art. In week zero, you sell two prints. Each week after that, you sell four more prints than you did the previous week. How many prints will you sell in the nth week?

We can easily compute the first few terms of the sequence: $2, 6, 10, 14, \ldots$. How do I know this is correct? From the problem, we see that to get from one term to the next, we must add 4. It is clear then that the recurrence relation for the sequence is

$$a_n = a_{n-1} + 4.$$

The *rate of growth* for the sequence is the constant 4 since the *difference* between any two terms is 4 (note, we could write the recurrence relation as $a_n - a_{n-1} = 4$).

We call sequences with a *constant rate of change* **arithmetic sequences**.

Now let's find a closed formula for our sequence. The first term is $a_0 = 2$. To get a_1, we add 4. The next term requires us to add 4 again, which means we have added 4 to our initial term twice. Then we add 4 again, for a total of three times for a_3. In fact, to get a_n, we will have added 4 to a_0 a total of n times. Thus, the closed formula for the sequence is

$$a_n = 2 + 4n.$$

This works for any arithmetic sequence. That is, any sequence with a constant difference will have a *linear* closed formula, where the "slope" of the linear function is the common difference.

Arithmetic Sequences.

If the terms of a sequence differ by a constant, we say the sequence is **arithmetic**. If the initial term (a_0) of the sequence is a and the **common difference** is d, then we have,

Recursive definition: $a_n = a_{n-1} + d$ with $a_0 = a$.

Closed formula: $a_n = a + dn$.

As we did for our example above, for the recursive definition, we need to specify a_0. Then we need to express a_n in terms of a_{n-1}. If we call the first term a, then $a_0 = a$. For the recurrence relation, by the definition of an arithmetic sequence, the difference between successive terms is some constant, say d. So $a_n - a_{n-1} = d$, or in other words,

$$a_0 = a \qquad a_n = a_{n-1} + d.$$

Let's now argue why the closed formula is correct. One way we could do this is by using a technique sometimes called **telescoping** (a name which hopefully we become meaningful momentarily).

We write the recurrence relation in its *difference* form, $a_n - a_{n-1} = d$ for all terms starting with a_1 and going up to a_n. This gives the following:

$$\begin{aligned} a_1 - a_0 &= d \\ a_2 - a_1 &= d \\ a_3 - a_2 &= d \\ &\vdots \\ a_{n-1} - a_{n-2} &= d \end{aligned}$$

$$a_n - a_{n-1} = d.$$

Now we add all n equations together.

On the right-hand side, we have added d to itself n times, so the sum is $d \cdot n$.

On the left-hand side, we get the sum:

$$(a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}).$$

But look what happens when we regroup and cancel like terms:

$$\cancel{a_1} - a_0 + \cancel{a_2} - \cancel{a_1} + \cancel{a_3} - \cancel{a_2} + \cdots + \cancel{a_{n-1}} - \cancel{a_{n-2}} + a_n - \cancel{a_{n-1}} = a_n - a_0.$$

The sum *telescopes* down to be nice and compact for easy storage.

Putting the two sides together gives us

$$a_n - a_0 = d \cdot n$$

which becomes

$$a_n = a_0 + d \cdot n$$

as we claimed.

The telescoping we did above is useful in other contexts (see Exercise 4.2.7.6), but now that we have established a general form of the closed formula, we can apply it to any arithmetic sequence.

Example 4.2.1

Find recursive definitions and closed formulas for the arithmetic sequences below. Assume the first term listed is a_0.

1. $2, 5, 8, 11, 14, \ldots$.
2. $50, 43, 36, 29, \ldots$.

Solution. First we should check that these sequences really are arithmetic by taking differences of successive terms. Doing so will reveal the common difference d.

1. $5 - 2 = 3$, $8 - 5 = 3$, etc. To get from each term to the next, we add three, so $d = 3$. The recursive definition is therefore $a_n = a_{n-1} + 3$ with $a_0 = 2$. The closed formula is $a_n = 2 + 3n$.
2. Here the common difference is -7, since we add -7 to 50 to get 43, and so on. Thus we have a recursive definition of $a_n = a_{n-1} - 7$ with $a_0 = 50$. The closed formula is $a_n = 50 - 7n$.

4.2.3 Geometric Sequences

What about sequences like $3, 6, 12, 24, 48, \ldots$? This is not arithmetic because the difference between terms is not constant. However, the *ratio* between successive terms is constant: $\frac{6}{3} = \frac{12}{6} = \frac{24}{12} = \cdots = 2$ We call such sequences **geometric**.

Recognizing that the sequence is geometric lets us easily write down a recursive definition. $a_n = 2a_{n-1}$, with $a_0 = 3$.

A closed formula is also not difficult to reason out. How do we get the term a_3 for example? We start with 3, then multiply by 2 to get a_1, multiply by 2 again to get a_2, and multiply by 2 a third time to get a_3. So we multiplied 3 by 2 a total of three times, or $a_3 = 3 \cdot 2^3$. It looks like $a_n = 3 \cdot 2^n$.

In general, the recursive definition for the geometric sequence with initial term a and common ratio r will be

$$a_n = a_{n-1} \cdot r; a_0 = a.$$

To get the next term we multiply the previous term by r.

For the general closed formula, we could try something like telescoping again, although we would need to cancel fractions. Instead, let's illustrate another technique for solving recurrence relations called **iteration**. The idea here is that we work our way up to a_n and notice the pattern. Write

$$\begin{aligned}
a_0 &= a \\
a_1 &= a_0 \cdot r \\
a_2 &= a_1 \cdot r = a_0 \cdot r \cdot r = a_0 \cdot r^2 \\
a_3 &= a_2 \cdot r = a_0 \cdot r^2 \cdot r = a_0 \cdot r^3 \\
&\vdots \\
a_n &= a_{n-1} \cdot r = a_0 \cdot r^{n-1} \cdot r = a_0 r^n.
\end{aligned}$$

We must multiply the first term a by r a number of times, n times to be precise. We get $a_n = a \cdot r^n$.

Geometric Sequences.

A sequence is called **geometric** if the ratio between successive terms is constant. Suppose the initial term a_0 is a and the **common ratio** is r. Then we have,

Recursive definition: $a_n = ra_{n-1}$ with $a_0 = a$.
Closed formula: $a_n = a \cdot r^n$.

Example 4.2.2

Find the recursive and closed formula for the geometric sequences below. Again, the first term listed is a_0.

1. $3, 6, 12, 24, 48, \ldots$
2. $27, 9, 3, 1, 1/3, \ldots$

Solution. Start by checking that these sequences really are geometric by dividing each term by its previous term. If this ratio really is constant, we will have found r.

1. $6/3 = 2$, $12/6 = 2$, $24/12 = 2$, etc. Yes, to get from any term to the next, we multiply by $r = 2$. So the recursive definition is $a_n = 2a_{n-1}$ with $a_0 = 3$. The closed formula is $a_n = 3 \cdot 2^n$.

2. The common ratio is $r = 1/3$. So the sequence has recursive definition $a_n = \frac{1}{3}a_{n-1}$ with $a_0 = 27$ and closed formula $a_n = 27 \cdot \frac{1}{3}^n$.

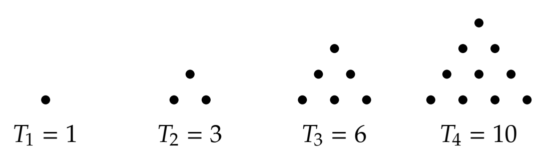
Geometric sequences are those which have a growth rate that is *proportional* to the sequence itself. Just like you might have seen in calculus, it is exactly the exponential functions that have this property.

In the examples and formulas above, we assumed that the *initial* term was a_0. If your sequence starts with a_1, you can easily find the term that would have been a_0 and use that in the formula. For example, if we want a formula for the sequence $2, 5, 8, \ldots$ and insist that $2 = a_1$, then we can find $a_0 = -1$ (since the sequence is arithmetic with common difference 3, we have $a_0 + 3 = a_1$). Then the closed formula will be $a_n = -1 + 3n$.

Remark 4.2.3 If you look at other sources, you might find that their closed formulas for arithmetic and geometric sequences differ from ours. Specifically, you might find the formulas $a_n = a + (n-1)d$ (arithmetic) and $a_n = a \cdot r^{n-1}$ (geometric). Which is correct? Both! In our case, we take a to be a_0. If instead we had a_1 as our initial term, we would get the (slightly more complicated) formulas you find elsewhere.

4.2.4 Beyond Arithmetic and Geometric Sequences

Look at the sequence $(T_n)_{n \geq 1}$ which starts $1, 3, 6, 10, 15, \ldots$. These are called the **triangular numbers** since they represent the number of dots in an equilateral triangle (think of how you arrange 10 bowling pins: a row of 4 plus a row of 3 plus a row of 2 and a row of 1).



Is this sequence arithmetic? No, since $3 - 1 = 2$ and $6 - 3 = 3 \neq 2$, so there is no common difference. Is the sequence geometric? No. $3/1 = 3$ but $6/3 = 2$, so there is no common ratio. What to do?

Notice that the *differences* between terms *do* form an arithmetic sequence: $2, 3, 4, 5, 6, \ldots$. In other words, the rate of change of this sequence is arithmetic:

$T_n - T_{n-1} = n$, which immediately gives us the recurrence relation $T_n = T_{n-1} + n$.

Another way to think of this is that the nth term of the sequence (T_n) is the *sum* of the first n terms in the sequence $1, 2, 3, 4, 5, \ldots$. Thus (T_n) is the **sequence of partial sums** of the sequence $1, 2, 3, \ldots$ (*partial* sums because we are not taking the sum of all infinitely many terms).

This should become clearer if we expand the recurrence relation to write the triangular numbers like this:

$$\begin{aligned}
T_1 &= 1 = 1 \\
T_2 &= 3 = 1 + 2 \\
T_3 &= 6 = 1 + 2 + 3 \\
T_4 &= 10 = 1 + 2 + 3 + 4 \\
&\vdots \qquad \vdots \\
T_n &= 1 + 2 + 3 + \cdots + n.
\end{aligned}$$

We are really using *iteration* here. We could also have seen this by using telescoping, taking $T_0 = 0$:

$$\begin{aligned}
T_1 - T_0 &= 1 \\
T_2 - T_1 &= 2 \\
T_3 - T_2 &= 3 \\
&\vdots \\
T_n - T_{n-1} &= n.
\end{aligned}$$

Summing these equations, the right-hand side becomes $1 + 2 + 3 + \cdots + n$; the left-hand side cancels to leave just $T_n - T_0 = T_n$.

If we know how to add up the terms of an arithmetic sequence, we can find a closed formula for a sequence whose differences are the terms of that arithmetic sequence. Consider how we could find the sum of the first 100 positive integers (that is, T_{100}). Instead of adding them in order, we regroup and add $1 + 100 = 101$. The next pair to combine is $2 + 99 = 101$. Then $3 + 98 = 101$. Keep going. This gives 50 pairs which each add up to 101, so $T_{100} = 101 \cdot 50 = 5050$.[1]

In general, using this same sort of regrouping, we find that $T_n = \frac{n(n+1)}{2}$. Incidentally, this is exactly the same as $\binom{n+1}{2}$, which makes sense if you think of the triangular numbers as counting the number of handshakes that take place at a party with $n + 1$ people: the first person shakes n hands, the next shakes an additional $n - 1$ hands and so on.

The point of all of this is that some sequences, while not arithmetic or geometric, can be interpreted as the sequence of partial sums of arithmetic and geometric

[1]This insight is usually attributed to Carl Friedrich Gauss, one of the greatest mathematicians of all time, who discovered it as a child when his unpleasant elementary teacher thought he would keep the class busy by requiring them to compute the lengthy sum.

sequences. Luckily there are methods we can use to compute these sums quickly, which we will explore in the next two sections.

4.2.5 Reading Questions

1. Match each formula on the left with the type of formula described on the right.

$a_n = 3 \cdot 2^n$	Arithmetic; closed
$a_n = 3a_{n-1}$	Arithmetic; recursive
$a_n = a_{n-1} + 3$	Geometric; closed
$a_n = 2n + 3$	Geometric; recursive

2. How can you decide whether a sequence is the sequence of partial sums of an arithmetic or geometric sequence? Describe what you would do to check, using an example.

3. What questions do you have? Write at least one question about the content of this section that you or a classmate might be curious about after reading this section.

4.2.6 Practice Problems

1. Consider the recurrence relation $a_n = a_{n-1} + 7$.

(a) Find the first five terms of the sequence defined by the recurrence relation and initial condition $a_0 = 17$.

(b) Find the closed formula for the sequence defined by the recurrence relation and initial condition $a_0 = 17$.

(c) Find the first five terms of another sequence, also defined by the same recurrence relation but this time with initial condition $a_0 = 1$.

(d) Find the closed formula for this second sequence.

2. Consider the recurrence relation $a_n = 7a_{n-1}$.

(a) Find the first five terms of the sequence defined by the recurrence relation and initial condition $a_0 = 10$.

(b) Find the closed formula for the sequence defined by the recurrence relation and initial condition $a_0 = 10$.

(c) Find the first five terms of another sequence, also defined by the same recurrence relation but this time with initial condition $a_0 = 17$.

(d) Find the closed formula for this second sequence.

3. Find a closed formula for the sequence that starts $3, 27, 243, 2187, 19683, \ldots$. Assume $a_0 = 3$.

4. Find a closed formula for the sequence that starts $16, 26, 36, 46, 56, \ldots$. Assume $a_0 = 16$.

5. Consider the sequence that starts $1, 7, 13, 19, 25, \ldots$ where $a_0 = 1$.

(a) Which of the following could be a recursive definition for the sequence?

- ☐ $a_n = 1 \cdot 6^n$
- ☐ $a_n = 6 \cdot a_{n-1}$; $a_0 = 1$
- ☐ $a_n = a_{n-1} + a_{n-2}$; $a_0 = 1$
- ☐ $a_n = a_{n-1} + 6$; $a_0 = 1$
- ☐ None of the above

(b) Find the closed formula for the sequence.

(c) Is 3007 a term of the sequence?

6. Your summer job pays you \$1000 per week, with a raise of \$50 per week as a bonus for not being a quitter. How much will you make in the 10th week?

7. Your sister's summer job pays her \$1000 per week, with a raise of 5% per week as a bonus for not being a quitter. How much will she make in the 10th week?

8. Find x and y such that $125, x, y, 1$ is part of an arithmetic sequence.
$x =$ ______, $y =$ ______
Then find x and y so that the sequence is part of a geometric sequence.
$x =$ ______, $y =$ ______
(Warning: x and y might not be integers.)

9. Find x and y such that $4, x, y, 31$ is part of an arithmetic sequence.
$x =$ ______, $y =$ ______
Then find x and y so that the sequence is part of a geometric sequence.
$x =$ ______, $y =$ ______
(Warning: x and y might not be integers.)

4.2.7 Additional Exercises

1. Suppose that the candy machine currently holds exactly 650 Skittles, and every time someone inserts a quarter, exactly 7 Skittles come out of the machine.

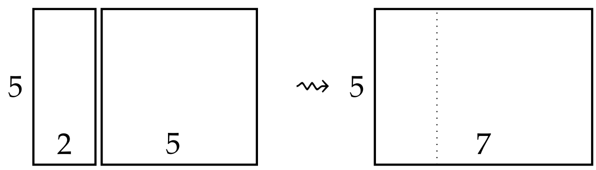
(a) How many Skittles will be left in the machine after 20 quarters have been inserted?

(b) Will there ever be exactly zero Skittles left in the machine? Explain.

2. Is there a pair of integers (a, b) such that a, x_1, y_1, b is part of an arithmetic sequence and a, x_2, y_2, b is part of a geometric sequence with x_1, x_2, y_1, y_2 all integers?

3. Are there any sequences that are both arithmetic and geometric? If so, how many can you find? If not, explain why not.

4. Starting with any rectangle, we can create a new, larger rectangle by attaching a square to the longer side. For example, if we start with a 2×5 rectangle, we would glue on a 5×5 square, forming a 5×7 rectangle:



The next rectangle would be formed by attaching a 7×7 square to the top or bottom of the 5×7 rectangle.

(a) Create a sequence of rectangles using this rule starting with a 1×2 rectangle. Then write out the sequence of *perimeters* for the rectangles (the first term of the sequence would be 6, since the perimeter of a 1×2 rectangle is 6; the next term would be 10).

(b) Repeat the above part, this time starting with a 1×3 rectangle.

(c) Find recursive formulas for each of the sequences of perimeters you found in parts (a) and (b). Don't forget to give the initial conditions as well.

(d) Are the sequences arithmetic? Geometric? If not, are they *close* to being either of these (i.e., are the differences or ratios *almost* constant)? Explain.

5. Prove that the closed formula for a geometric sequence with initial term $a \neq 0$ and common ratio r is $a_n = ar^n$, using *telescoping*.

6. **Telescoping to find a sum.** Another context in which sequences arise is calculus when you study sequences and **series** (which is the word in calculus for what we call a sequence of partial sums). Some of the techniques we have developed here can be applied there as well. This is an example of a **telescoping sum**, similar to the telescoping technique we used.

Consider the sequence $(a_n)_{n\geq 1}$ that starts

$$\frac{1}{1}, \frac{1}{3}, \frac{1}{6}, \frac{1}{10}, \frac{1}{15}, \ldots.$$

That is, each term is the reciprocal of the nth triangular number. Find the sum of the first n terms of this sequence:

$$\sum_{k=1}^{n} \frac{1}{T_k} = \frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \cdots + \frac{1}{T_n}.$$

7. A geometric sequence has a constant rate of growth in the sense that the ratio of consecutive terms is always the same. But what can we say about the *difference* of consecutive terms in an geometric sequence?

(a) Consider the geometric sequence $1, 2, 4, 8, 16, \ldots$. Find the differences between consecutive terms. That is, find its sequence of differences.

(b) Find a closed formula for the sequence of differences. Then use this closed formula to find a different recurrence relation for the original sequence (other than $a_n = 2a_{n-1}$).

(c) Repeat the two parts above for a different geometric sequence of your choice. Then explain what you found in general.

8. None of the following sequences are arithmetic or geometric:

$$1, 3, 6, 10, 15, \ldots$$

$$3, 5, 8, 12, 17, \ldots$$

$$0, 2, 5, 9, 14, \ldots$$

Explain what these sequences have in common with each other and then use that to find a closed formula for each of them. How do their closed formulas relate to each other? What can you say in general?

4.3 Polynomial Sequences

Objectives

After completing this section, you should be able to do the following.

1. Identify a sequence as having a polynomial closed formula based on its sequence of differences, and determine the polynomial's degree.
2. Fit an appropriate degree polynomial to a sequence of initial terms.
3. Explain how recurrence relations for polynomial sequences relate to their closed formulas.

4.3.1 Section Preview

Investigate!

A standard 8×8 chessboard contains 64 squares. Actually, this is just the number of unit squares. How many squares of all sizes are there on a chessboard? Start with smaller boards: 1×1, 2×2, 3×3, etc. Find a formula for the total number of squares in an $n \times n$ board.

We have seen that arithmetic sequences grow at a constant rate, and so their closed formulas are linear functions. What about sequences that grow faster? What if their rate of change (really the differences between terms) is itself growing at a constant rate?

In Section 4.2 we claimed that the triangular numbers, the sum of the first n positive integers, have closed formula

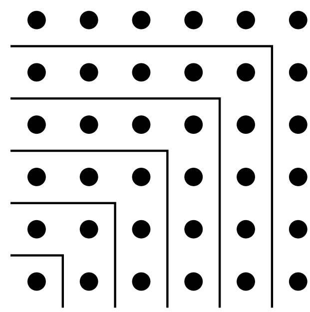
$$T_n = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}.$$

So this sequence, whose sequence of differences is arithmetic, is a degree 2 polynomial (a quadratic function).

Our goal in this section is to explore this phenomenon. We will verify that this really is the closed formula for the triangular numbers, extend it to other sequences with arithmetic differences, and then explore sequences that grow at even faster rates.

Preview Activity

1. While wandering the halls of the math department, you find yourself staring at the captivating artwork shown below.



(a) How many dots are in the figure?

The dots form a _____ by _____ square, for a total of _____ dots.

(b) We can also compute the total number of dots by summing each "hook" region, from smallest to largest:

___ + ___ + ___ + ___ + ___ + ____.

(c) Yet another way to calculate the total number of dots is to group the terms of this sum.

$1 + 11 =$ ___; $3 + 9 =$ ___; $5 + 7 =$ ___.

Since there are three pairs of sums, the total is $3\cdot$___ $=$ _____.

(d) If we generalize the diagram, so it has n hooks, how many dots will be in the largest hook?

How many dots will be in the second largest hook?

(e) What will the sum of the smallest and largest hooks be?

What will the sum of the second smallest and second largest hooks be?

(f) If we continue adding pairs of hooks (next smallest plus next largest), how many pairs will we have?

Multiplying then, the total number of dots will be: __________.

Hint. Let's assume that n is even. If it wasn't, then there would be a single "middle" hook that isn't added to anything, but this is counteracted by the fact that $n/2$ would count a half hook sum.

4.3.2 Summing Arithmetic Sequences: Reverse and Add

Let's find the sum of the first n positive integers carefully. Call that sum T_n, and write it down twice, once in the usual order and once in reverse order.

$$\begin{array}{rrcccccccccc} & T_n & = & 1 & + & 2 & + & 3 & + \cdots + & n \\ + & T_n & = & n & + & (n-1) & + & (n-2) & + \cdots + & 1 \\ \hline & 2T_n & = & n+1 & + & n+1 & + & n+1 & + \cdots + & n+1 \end{array}$$

We then added the two equations together. The left-hand side is $2T_n$. On the right-hand side, something great happens: All the terms of the sum are the same! So instead of adding up a bunch of different numbers, we now just add a bunch of the same number. That's a task that multiplication lives for! There are n terms in the sum, so we get,

$$2T_n = n(n+1).$$

Solving for T_n gives us,

$$T_n = \frac{n(n+1)}{2},$$

as expected.

This technique will work for any arithmetic sum.

Example 4.3.1

Find the sum: $2 + 5 + 8 + 11 + 14 + \cdots + 470$.

Solution. The idea is to mimic how we found the formula for triangular numbers. If we add the first and last terms, we get 472. The second term and second-to-last term also add up to 472. To keep track of everything, we might express this as follows. Call the sum S. Then,

$$\begin{array}{rrccccccccccc} & S = & 2 & + & 5 & + & 8 & + \cdots + & 467 & + & 470 \\ + & S = & 470 & + & 467 & + & 464 & + \cdots + & 5 & + & 2 \\ \hline & 2S = & 472 & + & 472 & + & 472 & + \cdots + & 472 & + & 472 \end{array}$$

To find $2S$ then, we add 472 to itself a number of times. What number? We need to decide how many terms (**summands**) are in the sum. Since the terms form an arithmetic sequence, the nth term in the sum (counting 2 as the 0th term) can be expressed as $2 + 3n$. If $2 + 3n = 470$ then $n = 156$. So n ranges from 0 to 156, giving 157 terms in the sum. This is the number of 472's in the sum for $2S$. Thus

$$2S = 157 \cdot 472 = 74104.$$

It is now easy to find S:

$$S = 74104/2 = 37052.$$

This will work for the sum of any *arithmetic* sequence. Call the sum S. Reverse and add. This produces a single number added to itself many times. Find the number of times. Multiply. Divide by 2. Done.

Example 4.3.2

Find a closed formula for $6 + 10 + 14 + \cdots + (4n - 2)$.

Solution. Again, we have a sum of an arithmetic sequence. How many terms are in the sequence? Clearly each term in the sequence has the form $4k - 2$ (as evidenced by the last term). For which values of k though? To get 6, $k = 2$. To get $4n - 2$ take $k = n$. So to find the number of terms, we must count the number of integers in the range $2, 3, \ldots, n$. This is $n - 1$. (There are n numbers from 1 to n, so one less if we start with 2.)

Now reverse and add:

$$\begin{array}{rcccccccc} & S = & 6 & + & 10 & +\cdots+ & 4n-6 & + & 4n-2 \\ + & S = & 4n-2 & + & 4n-6 & +\cdots+ & 10 & + & 6 \\ \hline & 2S = & 4n+4 & + & 4n+4 & +\cdots+ & 4n+4 & + & 4n+4 \end{array}$$

Since there are $n - 1$ terms, we get

$$2S = (n-1)(4n+4) \qquad \text{so} \qquad S = \frac{(n-1)(4n+4)}{2}.$$

Besides finding sums, we can use this technique to find closed formulas for sequences we recognize as sequences of partial sums.

Example 4.3.3

Use partial sums to find a closed formula for $(a_n)_{n\geq 0}$ which starts $2, 3, 7, 14, 24, 37, \ldots\ldots$. Assume a recurrence relation for the sequence is $a_n = a_{n-1} + 3n - 2$.

Solution. First, if you look at the differences between terms, you get a sequence of differences $(d_n)_{n\geq 1}$: $1, 4, 7, 10, 13, \ldots$, which is an arithmetic sequence. Indeed, we notice that $d_n = 3n - 2$, which agrees with the recurrence relation. Written another way:

$$\begin{aligned} a_0 &= 2 \\ a_1 &= 2 + 1 = 2 + d_1 \\ a_2 &= 2 + 1 + 4 = 2 + d_1 + d_2 \\ a_3 &= 2 + 1 + 4 + 7 = 2 + d_1 + d_2 + d_3 \end{aligned}$$

and so on. We can write the general term of (a_n) in terms of the arithmetic sequence as follows:

$$a_n = 2 + 1 + 4 + 7 + 10 + \cdots + 3n - 2.$$

We can reverse and add, but the initial 2 does not fit our pattern. This just means we need to keep the 2 out of the reverse part:

$$\begin{array}{rrcccccccc} & a_n = & 2 & + & 1 & + & 4 & +\cdots+ & 3n-2 \\ + & a_n = & 2 & + & 3n-2 & + & 3(n-1)-2 & +\cdots+ & 1 \\ \hline & 2a_n = & 4 & + & 2+3n-3 & + & 2+3n-3 & +\cdots+ & 2+3n-3 \end{array}$$

Not counting the first term (the 4) there are n summands of $2+3n-3 = 3n-1$ so the right-hand side becomes $4+(3n-1)n$.

Finally, solving for a_n we get

$$a_n = \frac{4+(3n-1)n}{2}.$$

Just to be sure, we check $a_0 = \frac{4}{2} = 2$, $a_1 = \frac{4+2}{2} = 3$, etc. We have the correct closed formula.

Notice that the closed formula for a sequence that has an arithmetic (i.e., linear) rate of change is a quadratic function. Interesting....

4.3.3 Higher Degree Polynomials

Since we know how to compute the sum of the first n terms of arithmetic sequences, we can compute the closed formulas for sequences that have an arithmetic sequence of differences between terms. But what if we consider a sequence that is the sum of the first n terms of a sequence that is itself the sum of an arithmetic sequence?

How many squares (of all sizes) are there on a chessboard? A chessboard consists of 64 squares, but we also want to consider squares of longer side length. Even though we are only considering an 8×8 board, there is already a lot to count. So instead, let us build a sequence: the first term will be the number of squares on a 1×1 board, the second term will be the number of squares on a 2×2 board, and so on. After a little thought, we arrive at the sequence

$$1, 5, 14, 30, 55, \ldots.$$

This sequence is not arithmetic (or geometric for that matter), but perhaps its sequence of differences is. For differences we get

$$4, 9, 16, 25, \ldots.$$

Not a huge surprise: One way to count the number of squares in a 4×4 chessboard is to notice that there are 16 squares with side length 1, 9 with side length 2, 4 with side length 3 and 1 with side length 4. So the original sequence is just the sum of squares. Now this sequence of differences is not arithmetic since its sequence of differences (the differences of the differences of the original sequence) is not constant. In fact, this sequence of **second differences** is

$$5, 7, 9, \ldots,$$

which *is* an arithmetic sequence (with constant difference 2). Notice that our original sequence had **third differences** (that is, differences of differences of differences of the original) constant. We will call such a sequence Δ^3-constant. The sequence $1, 4, 9, 16, \ldots$ has second differences constant, so it will be a Δ^2-constant sequence. In general, we will say a sequence is a **Δ^k-constant** sequence if the kth differences are constant.

Example 4.3.4

Which of the following sequences are Δ^k-constant for some value of k?

1. $2, 3, 7, 14, 24, 37, \ldots$.
2. $1, 8, 27, 64, 125, 216, \ldots$.
3. $1, 2, 4, 8, 16, 32, 64, \ldots$.

Solution.

1. This is the sequence from Example 4.3.3, in which we found a closed formula by recognizing the sequence as the sequence of partial sums of an arithmetic sequence. Indeed, the sequence of first differences is $1, 4, 7, 10, 13, \ldots$, which itself has differences $3, 3, 3, 3, \ldots$. Thus $2, 3, 7, 14, 24, 37, \ldots$ is a Δ^2-constant sequence.
2. These are the perfect cubes. The sequence of first differences is $7, 19, 37, 61, 91, \ldots$; the sequence of second differences is $12, 18, 24, 30, \ldots$; the sequence of third differences is constant: $6, 6, 6, \ldots$. Thus the perfect cubes are a Δ^3-constant sequence.
3. If we take first differences, we get $1, 2, 4, 8, 16, \ldots$. Wait, what? That's the sequence we started with. So taking second differences will give us the same sequence again. No matter how many times we repeat this we will always have the same sequence, which in particular means no finite number of differences will be constant. Thus this sequence is not Δ^k-constant for any k.

The Δ^0-constant sequences are themselves constant, so a closed formula for them is easy to compute (it's just the constant). The Δ^1-constant sequences are arithmetic, and we have a method for finding closed formulas for them as well. Every Δ^2-constant sequence is the sum of an arithmetic sequence, so we can find formulas for these as well. But notice that the format of the closed formula for a Δ^2-constant sequence is always quadratic. For example, the square numbers are Δ^2-constant with closed formula $a_n = n^2$. The triangular numbers (also Δ^2-constant) have closed formula $a_n = \frac{n(n+1)}{2}$, which when multiplied out gives you an n^2 term as well. It appears that every time we increase the complexity of the sequence, that is, increase the number of differences before we get constants, we also increase the degree of the polynomial used for the closed formula. We go from constant to linear to quadratic. The sequence of differences between terms tells us something about the rate of growth of the sequence. If a sequence is growing at a constant rate, then the formula for the sequence will be linear. If the sequence is growing at a rate which itself is growing at a constant rate, then the formula is quadratic. You might have seen this elsewhere: If a function has a constant second derivative (rate of change), then the function must be quadratic.

This works in general:

Theorem 4.3.5 Polynomial Fitting.

The closed formula for a sequence will be a degree k polynomial if and only if the sequence is Δ^k-constant (i.e., the kth sequence of differences is constant).

This tells us that the sequence of numbers of squares on a chessboard, $1, 5, 14, 30, 55, \ldots$, which we saw to be Δ^3-constant, will have a cubic (degree 3 polynomial) for its closed formula.

Now once we know what format the closed formula for a sequence will take, it is much easier to actually find the closed formula. In the case that the closed formula is a degree k polynomial, we just need $k+1$ data points to "fit" the polynomial to the data.

Example 4.3.6

Find a formula for the sequence $3, 7, 14, 24, \ldots$. Assume $a_1 = 3$.

Solution. First, check to see if the formula has constant differences at some level. The sequence of first differences is $4, 7, 10, \ldots$ which is arithmetic, so the sequence of second differences is constant. The sequence is Δ^2-constant, so the formula for a_n will be a degree 2 polynomial. That is, we know that for some constants a, b, and c,

$$a_n = an^2 + bn + c.$$

Now to find a, b, and c. First, it would be nice to know what a_0 is, since plugging in $n = 0$ simplifies the above formula greatly. In this case, $a_0 = 2$ (work backward from the sequence of constant differences). Thus

$$a_0 = 2 = a \cdot 0^2 + b \cdot 0 + c,$$

so $c = 2$. Now plug in $n = 1$ and $n = 2$. We get

$$a_1 = 3 = a + b + 2$$

$$a_2 = 7 = a4 + b2 + 2.$$

At this point, we have two (linear) equations and two unknowns, so we can solve the system for a and b (using substitution or elimination or even matrices). We find $a = \frac{3}{2}$ and $b = \frac{-1}{2}$, so $a_n = \frac{3}{2}n^2 - \frac{1}{2}n + 2$.

Example 4.3.7

Find a closed formula for the number of squares on an $n \times n$ chessboard.

Solution. We have seen that the sequence $1, 5, 14, 30, 55, \ldots$ is Δ^3-constant, so we are looking for a degree 3 polynomial. That is,

$$a_n = an^3 + bn^2 + cn + d.$$

We can find d if we know what a_0 is. Working backward from the third differences, we find $a_0 = 0$ (unsurprisingly, since there are no squares on a 0×0 chessboard). Thus $d = 0$. Now plug in $n = 1$, $n = 2$, and $n = 3$:

$$\begin{aligned} 1 &= a + b + c \\ 5 &= 8a + 4b + 2c \\ 14 &= 27a + 9b + 3c. \end{aligned}$$

If we solve this system of equations, we get $a = \frac{1}{3}$, $b = \frac{1}{2}$ and $c = \frac{1}{6}$. Therefore the number of squares on an $n \times n$ chessboard is $a_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{1}{6}n(n+1)(2n+1)$.

Note: Since the squares-on-a-chessboard problem is really asking for the sum of squares, we now have a nice formula for $\displaystyle\sum_{k=1}^{n} k^2$.

Example 4.3.8

Find a closed formula for $(a_n)_{n \geq 0}$ which starts $2, 3, 7, 14, 24, 37, \ldots$. Assume a recurrence relation for the sequence is $a_n = a_{n-1} + 3n - 2$

Solution. Note that we have already done this in Example 4.3.3, but now we can solve this using polynomial fitting.

The sequence of (first) differences is $1, 4, 7, 10, 13, \ldots$ (which agrees with what is given in the recurrence relation). The sequence of second differences is $3, 3, 3, 3, \ldots$ constant! So we expect that the closed formula for a_n will be a degree 2 polynomial. That is, we guess,

$$a_n = an^2 + bn + c.$$

Since $a_0 = 2$, we know that $c = 2$ (as $a \cdot 0^2 + b \cdot 0^2 + c = 2$). Then, we can see what happens with $n = 1$ and $n = 2$:

$$\begin{aligned} a_1 = 3 &= a \cdot 1^2 + b \cdot 1 + 2 \\ a_2 = 7 &= a \cdot 2^2 + b \cdot 2 + 2. \end{aligned}$$

Simplifying this, we must find a and b which satisfy the equations

$$
\begin{aligned}
1 &= a + b \\
5 &= 4a + 2b.
\end{aligned}
$$

Using a computer algebra system or substitution or elimination, we find that $a = \frac{3}{2}$ and $b = -\frac{1}{2}$. Therefore the closed formula is,

$$a_n = \frac{3}{2}n^2 - \frac{1}{2}n + 2.$$

This is the same as we found in Example 4.3.3, once you multiply out that solution.

Not all sequences will have polynomials as their closed formula. We can use the theory of finite differences to identify these.

Example 4.3.9

Determine whether the following sequences can be described by a polynomial, and if so, of what degree.

1. $1, 2, 4, 8, 16, \ldots$
2. $0, 7, 50, 183, 484, 1055, \ldots$
3. $1, 1, 2, 3, 5, 8, 13, \ldots$

Solution.

1. As we saw in Example 4.3.4, this sequence is not Δ^k-constant for any k. Therefore the closed formula for the sequence is not a polynomial. In fact, we know the closed formula is $a_n = 2^n$, which grows faster than any polynomial (so is not a polynomial).
2. The sequence of first differences is $7, 43, 133, 301, 571, \ldots$. The second differences are: $36, 90, 168, 270, \ldots$. Third differences: $54, 78, 102, \ldots$. Fourth differences: $24, 24, \ldots$. As far as we can tell, this sequence of differences is constant so the sequence is Δ^4-constant, and as such the closed formula is a degree 4 polynomial.
3. This is the Fibonacci sequence. The sequence of first differences is $0, 1, 1, 2, 3, 5, 8, \ldots$, the second differences are $1, 0, 1, 1, 2, 3, 5 \ldots$. We notice that after the first few terms, we get the original sequence back. So there will never be constant differences, so the closed formula for the Fibonacci sequence is not a polynomial.

Warning 4.3.10 A degree n polynomial is completely determined by its $n + 1$ coefficients (the +1 is because of the constant term). Therefore we can always find a degree n polynomial when given $n + 1$ terms of a sequence.

If we take the $n + 1$ terms, we can take differences of differences of differences of... until (after n steps) we are left with just a single number. As far as we can tell, this nth difference is constant. This doesn't mean we have found the closed formula for the *right* sequence. This is why it is so important to work with sequences in a particular context.

4.3.4 Solving Systems of Equations with Technology

The point of polynomial fitting is that if we can be sure that a sequence has a polynomial as its closed formula, then we can find that formula. Since we know the degree of the polynomial, all we need is to find its coefficients, and with enough terms of the sequence, we can find a system of enough linear equations whose solution will be those coefficients. However, this requires solving a system of linear equations.

For a degree 2 polynomial, we need to find three coefficients (the constant term, the coefficient of n, and the coefficient of n^2). A system of three linear equations will be enough to find these three unknowns. In fact, since a_0 will be the constant term, we can really get away with just two equations and two unknowns, and this is not difficult to solve by hand.

For higher degree polynomials, the number of equations is larger, and solving by hand can be tedious. Luckily, it is easy for computers to solve these equations. Below we demonstrate how to use the free computer algebra system SageMath, as well as python, to solve these systems of equations. Besides these two choices, pretty much any computer algebra system (including Wolfram Alpha) can solve these systems of equations.

Suppose we have the following system of three equations and three unknowns, as in the chess board example above:

$$\begin{aligned} 1 =& a + b + c \\ 5 =& 8a + 4b + 2c \\ 14 =& 27a + 9b + 3c \end{aligned}$$

In SageMath, we can use the solve method to solve the system of equations. Here is the code:

```
var('a b c')
solve(
  [
    a+b+c==1,
    8*a+4*b+2*c==5,
    27*a+9*b+3*c==14
  ], a,b,c)
```

```
[[a == (1/3), b == (1/2), c == (1/6)]]
```

This is easier than in python, but python might be more readily available. One way you can solve the system in python is to use the numpy library. In this case, you would create a matrix of coefficients and a vector of constants, and then use the solve method. Here is the code:

```
import numpy as np
A = np.array([[1,1,1],[8,4,2],[27,9,3]])
b = np.array([1,5,14])
x = np.linalg.solve(A,b)
print(x)
```

An explanation of what is going on here: We create a matrix `A` of coefficients of the system of equations (not the coefficients of the closed formula we are looking for),

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ 27 & 9 & 3 \end{bmatrix}$$

and a vector `b` for the constants,

$$b = \begin{bmatrix} 1 \\ 5 \\ 14 \end{bmatrix}.$$

What numpy does is solve the matrix equation

$$Ax = b.$$

The vector x that satisfies this matrix equation will be the values of the unknowns in the system (so the vector $[a, b, c]$).

Of course, once you find the coefficients of the polynomial, you should still write out the closed formula using those coefficients. It is always a good idea to check that the formula appears to work by using an n that you did not use to get your system of equations.

4.3.5 Reading Questions

1. Match each sequence on the left with the type of closed formula it might have, on the right.

$3, 7, 11, 15, 19, \ldots$	Linear formula
$3, 4, 7, 13, 23, \ldots$	Quadratic formula
$3, 4, 7, 11, 18, 29, \ldots$	Cubic formula
$3, 5, 8, 12, 17, \ldots$	Exponential (not a polynomial)

2. Suppose (a_n) is a sequence whose sequence of differences has a degree 2 polynomial as its closed formula. What can you say about the sequence of partial sums of (a_n)? Explain.

3. What questions do you have? Write at least one question about the content of this section that you or a classmate might be curious about after reading this section.

4.3.6 Practice Problems

1. Consider the sequence $10, 13, 16, 19, 22 \ldots$ where $a_1 = 10$.

(a) What is the recursive definition for the sequence?

(b) Give a closed formula for the nth term of the sequence.

(c) Is 1555 a term in the sequence?

(d) How many terms does the finite sequence $10, 13, 16, \ldots, 433$ have?

(e) Find the sum: $10 + 13 + 16 + \ldots + 433$.

(f) Use what you found above to find b_n, the nth term of $3, 13, 26, 42, 61, \ldots$

2. Consider the sequence $(a_n)_{n\geq 0}$ which starts $5, 10, 15, 20, \ldots$.

a. What is the next term in the sequence?

b. Find a formula for the nth term of this sequence.

$a_n =$ ____________

c. Find the sum of the first 100 terms of the sequence: $\sum_{k=0}^{99} a_k$.

3. Consider the sum $9 + 14 + 19 + 24 + \cdots + 229$.

a. How many terms (summands) are in the sum?

b. Compute the sum using a technique discussed in this section.

4. Consider the sequence $-18, -6, 6, 18, \ldots, 12n + 6$.

a. How many terms are there in the sequence? Your answer will be in terms of n.

b. What is the second-to-last term?

c. Find the sum of all the terms in the sequence, in terms of n.

5. Find $2 + 8 + 14 + 20 + \cdots + 1808$ using a technique from this section.

6. Use polynomial fitting to find the formula for the nth term of the sequence $(a_n)_{n\geq 0}$ which starts,
$0, 9, 20, 33, 48, 65, \ldots$
$a_n =$ __________

7. Use polynomial fitting to find the formula for the nth term of the sequence $(a_n)_{n\geq 0}$ which starts,
$-1, 4, 11, 20, 31, 44, \ldots$
$a_n =$ ________

8. Use polynomial fitting to find the formula for the nth term of the sequence $(a_n)_{n\geq 0}$ which starts,

$$5, 5, 17, 53, 125, 245, \ldots$$

$a_n =$ ________

9. Use polynomial fitting to find the formula for the nth term of the sequence $(a_n)_{n\geq 1}$ which starts,
$9, 43, 117, 243, 433, \ldots$ Note the first term above is a_1, not a_0.
$a_n =$ ________

10. Suppose $a_n = 2n^2 + 3n + 2$. Find a closed formula for the sequence of differences by computing $a_n - a_{n-1}$. Simplify your answer as much as possible.

11. Use polynomial fitting to find the formula for the nth term of the sequence $(a_n)_{n\geq 1}$ which starts,
$7, 32, 87, 184, 335, \ldots$ Note the first term above is a_1, not a_0.
$a_n =$ ________

4.3.7 Additional Exercises

1. Your friendly neighborhood bodega has a candy machine that gives 7 Skittles to the first customer who puts in a quarter, 10 to the second, 13 to the third, 16 to the fourth, etc. How many candies has the machine given out in total after 20 quarters are put into the machine? After n quarters?

2. Not to be outdone, the mega-mart across the street has installed a candy machine that gives 4 Skittles to the first customer, 7 to the second, 12 to the third, 19 to the fourth, etc. How many Skittles has the machine given out in total after 20 quarters are put into the machine? After n quarters?

3. Make up sequences that have

(a) $3, 3, 3, 3, \ldots$ as its second differences.

(b) $1, 2, 3, 4, 5, \ldots$ as its third differences.

(c) $1, 2, 4, 8, 16, \ldots$ as its 100th differences.

4. Consider the sequence $1, 3, 7, 13, 21, \ldots$. Explain how you know the closed formula for the sequence will be quadratic. Then "guess" the correct formula by comparing this sequence to the squares $1, 4, 9, 16, \ldots$ (do not use polynomial fitting).

5. Use a similar technique as in the previous exercise to find a closed formula for the sequence $2, 11, 34, 77, 146, 247, \ldots$.

6. Consider the sequence $2, 7, 15, 26, 40, 57, \ldots$ (with $a_0 = 2$). By looking at the differences between terms, express the sequence as a sequence of partial sums. Then find a closed formula for the sequence by computing the nth partial sum.

7. If you have enough toothpicks, you can make a large triangular grid. Below, are the triangular grids of size 1 and of size 2. The size 1 grid requires 3 toothpicks, the size 2 grid requires 9 toothpicks.

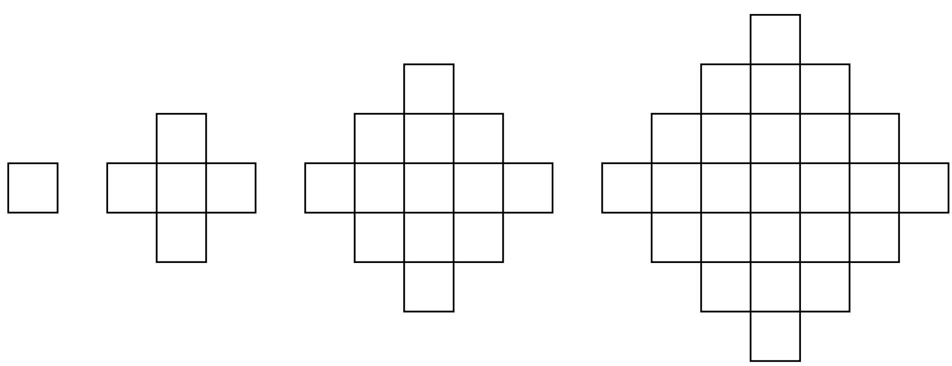
(a) Let t_n be the number of toothpicks required to make a size n triangular grid. Write out the first 5 terms of the sequence $t_1, t_2, \ldots$.

(b) Find a recursive definition for the sequence. Explain why you are correct.

(c) Is the sequence arithmetic or geometric? If not, is it the sequence of partial sums of an arithmetic or geometric sequence? Explain why your answer is correct.

(d) Use your results from part (c) to find a closed formula for the sequence. Show your work.

8. If you were to shade in an $n \times n$ square on graph paper, you could do it the boring way (with sides parallel to the edge of the paper) or the interesting way, as illustrated below:



The interesting thing here is that a 3×3 square now has area 13. Our goal is to find a formula for the area of an $n \times n$ (diagonal) square.

(a) Write out the first few terms of the sequence of areas (assume $a_1 = 1$, $a_2 = 5$, etc). Is the sequence arithmetic or geometric? If not, is it the sequence of partial sums of an arithmetic or geometric sequence? Explain why your answer is correct, referring to the diagonal squares.

(b) Use your results from part (a) to find a closed formula for the sequence. Show your work. Note that while there are lots of ways to find a closed formula here, you should use partial sums specifically.

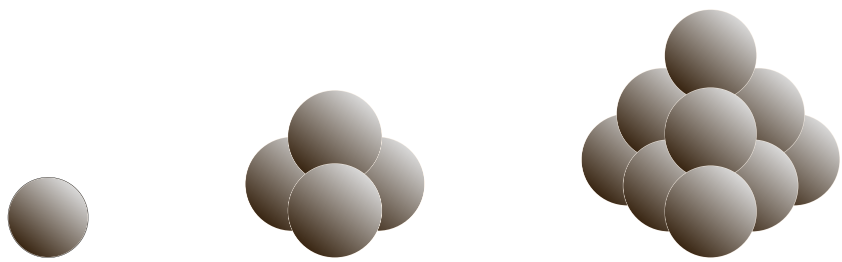
(c) Find the closed formula in as many other interesting ways as you can.

9. Generalize Practice Problem 5: Find a closed formula for the sequence of differences of $a_n = an^2 + bn + c$. That is, prove that every quadratic sequence has arithmetic differences.

10. Can you use polynomial fitting to find the formula for the nth term of the sequence $4, 7, 11, 18, 29, 47, \ldots$? Explain why or why not.

11. Will the nth sequence of differences of $2, 6, 18, 54, 162, \ldots$ ever be constant? Explain.

12. In their down time, ghost pirates enjoy stacking cannonballs in triangular based pyramids (aka, tetrahedrons), like those pictured here:



Note: These are solid tetrahedrons, so there will be some cannonballs obscured from view (the picture on the right has one cannonball in the back not shown in the picture, for example).

The pirates wonder how many cannonballs would be required to build a pyramid 15 layers high (thus breaking the world cannonball stacking record). Can you help?

(a) Let $P(n)$ denote the number of cannonballs needed to create a pyramid n layers high. So $P(1) = 1$, $P(2) = 4$, and so on. Calculate $P(3)$, $P(4)$, and $P(5)$.

(b) Use polynomial fitting to find a closed formula for $P(n)$. Show your work.

(c) Answer the pirate's question: How many cannonballs do they need to make a pyramid 15 layers high?

(d) Bonus: Locate this sequence in Pascal's triangle. Why does that make sense?

4.4 Exponential Sequences

Objectives

After completing this section, you should be able to do the following.

1. Identify a sequence as exponential based on its recurrence relation.
2. Apply the characteristic root technique to solve appropriate recurrence relations.

4.4.1 Section Preview

Investigate!

You have a large collection of 1×1 squares and 1×2 dominoes. You want to arrange these to make a 1×15 strip. How many ways can you do this?

What if the squares come in three different colors and the dominos come in four different colors? And why is this second question easier than the first?

Hint. Start by creating a recurrence relation. How are the different 1×3 strips and 1×4 strips related to the 1×5 strips?

In Section 4.3 we saw that if a sequence has some sequence of differences that is constant, then the sequence has a polynomial closed formula. Are there sequences that never have constant differences? And what would their closed formulas look like?

Consider the Fibonacci sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots.$$

If we look at the first differences, we get this sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots,$$

which is the Fibonacci sequence itself. This is not surprising, since the Fibonacci sequence is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$. That is saying precisely that to get the next turn of the sequence, we take the current term and add... a term in the sequence!

Of course, if we take another difference, we will get the same sequence back, and again and again, so no nth differences will be constant.

Another sequence that has this behavior is the powers of 2:

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, \ldots$$

which has differences

$$1, 2, 4, 8, 16, 32, 64, 128, 256, \ldots.$$

We can also see this from the recurrence relation, since

$$a_n = 2a_{n-1} = a_{n-1} + a_{n-1}.$$

The rate of growth for this, and in fact any geometric sequence, is the sequence itself.

If you have studied calculus, you may recall that the functions that have themselves (or close) as their rate of change (derivative, in the calculus context) are exactly the exponential functions. Here too, geometric sequences, which have exponential closed formulas, have themselves as their rate of change.

In this section, we will explore sequences that are changing at a rate proportional to the sequence itself, and see how these all have an exponential closed formula, or some variation of that.

Preview Activity

1. Consider the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}.$$

Since the nth term is given as a combination of the two previous terms, we will need two initial terms to determine the sequence. Different initial terms will give different sequences.

(a) What sequence do you get if the initial conditions are $a_0 = 1$, $a_1 = 2$? Give the first five terms (including 1 and 2).

(b) Based on the first few terms, what is a closed formula for this sequence?

(c) What sequence do you get if the initial conditions are $a_0 = 1$, $a_1 = 3$? Give the first five terms.

(d) Based on the first few terms, what is a closed formula for this sequence?

(e) What sequence do you get if the initial conditions are $a_0 = 2$, $a_1 = 5$? Give the first five terms.

(f) Based on the first few terms, what is a closed formula for this sequence?

Hint. How do the terms in this sequence relate to the terms in the previous two sequences?

4.4.2 Summing Geometric Sequences: Multiply, Shift, and Subtract

Suppose a candy machine dispenses candy in a geometric sequence by first giving 1 candy, then 2 candies, then 4, then 8, and so on. How many candies will you have received in total after 10 turns of the machine?

We can create the sequence of partials sum as $1, 1+2, 1+2+4, 1+2+4+8, \ldots$ that gives

$$1, 3, 7, 15, 31, 63, \ldots$$

This is not a geometric sequence, but is almost. In fact, if we add 1 to each term, we get what sure looks like the geometric sequence $2, 4, 8, 16, 32, 64, \ldots$, so we might guess that the closed formula for the sequence of sums is $2^{n+1} - 1$. If this is correct, then the answer to the candy question would be $2^{11} - 1 = 2047$.

More intriguing though is the observation that the sequence of partial sums of a geometric sequence is again geometric-ish. Let's consider how to find the sum of a geometric sequence in general.

We cannot just reverse and add as we did for the sum of an arithmetic sequence. Do you see why? The reason we got the same term added to itself many times is because there was a constant difference. So as we added that difference in one direction, we subtracted the difference going the other way, leaving a constant total. For geometric sums; we have a different technique.

Example 4.4.1

What is $3 + 6 + 12 + 24 + \cdots + 12288$?

Solution. Multiply each term by 2, the common ratio. We get $2S = 6 + 12 + 24 + \cdots + 24576$. Now subtract: $2S - S = -3 + 24576 = 24573$. Since $2S - S = S$, we have our answer.

To better see what happened in the above example, we can write it this way:

$$\begin{array}{rrcll} & S = & 3+ & 6 + 12 + 24 + \cdots + 12288 & \\ - & 2S = & & 6 + 12 + 24 + \cdots + 12288 & +24576 \\ \hline & -S = & 3+ & 0 + 0 + 0 + \cdots + 0 & -24576 \end{array}$$

Then divide both sides by -1 and we have the same result for S. The idea is, by multiplying the sum by the common ratio, each term becomes the next term. We shift over the sum to get the subtraction to mostly cancel out, leaving just the first term and the new last term.

Example 4.4.2

Find a closed formula for $S(n) = 2 + 10 + 50 + \cdots + 2 \cdot 5^n$.

Solution. The common ratio is 5. So we have

$$\begin{array}{rrcl} & S & = 2 + 10 + 50 + \cdots + 2 \cdot 5^n & \\ - & 5S & = \quad 10 + 50 + \cdots + 2 \cdot 5^n & + 2 \cdot 5^{n+1} \\ \hline & -4S & = 2 - 2 \cdot 5^{n+1} & \end{array}$$

Thus $S = \dfrac{2 - 2 \cdot 5^{n+1}}{-4}$

Even though this might seem like a new technique, you have probably used it before.

Example 4.4.3

Express $0.464646\ldots$ as a fraction.

Solution. Let $N = 0.46464646\ldots$. Consider $0.01N$. We get:

$$\begin{array}{rrl} & N = & 0.4646464\ldots \\ - & 0.01N = & 0.00464646\ldots \\ \hline & 0.99N = & 0.46 \end{array}$$

So $N = \frac{46}{99}$. What have we done? We viewed the repeating decimal $0.464646\ldots$ as a sum of the geometric sequence $0.46, 0.0046, 0.000046, \ldots$ The common ratio is 0.01. The only real difference is that we are now computing an *infinite* geometric sum, we do not have the extra "last" term to consider. Really, this is the result of taking a limit as we would in calculus when we compute *infinite* geometric sums.

To summarize, we now can find a closed formula for a sequence a_n that has a rate of growth that is an exponential function: $a_n - a_{n-1} = b_n$, where b_n is a geometric sequence (i.e., an exponential function). What sort of closed formula do we get here? It's *another* exponential function!

4.4.3 The Characteristic Root Technique

Suppose we want to solve a recurrence relation expressed as a combination of the two previous terms, such as $a_n = a_{n-1} + 6a_{n-2}$. In other words, we want to find a function of n which satisfies $a_n - a_{n-1} - 6a_{n-2} = 0$. Think about how we build up this sequence iteratively.

$$\begin{aligned} a_2 &= a_1 + 6a_0 \\ a_3 &= a_2 + 6a_1 = a_1 + 6a_0 + 6a_1 \\ a_4 &= a_3 + 6a_2 = a_1 + 6a_0 + 6a_1 + 6^2a_0 + 6a_1 \end{aligned}$$

Let's stop there and agree this is getting very complicated. However, we do notice that in each step, we would, among other things, multiply a previous iteration by 6. So our closed formula would include 6 multiplied some number of times. Thus it is reasonable to guess the solution will contain parts that look geometric. Perhaps the solution will take the form r^n for some constant r.

The nice thing is, we know how to check whether a formula is actually a solution to a recurrence relation: plug it in. What happens if we plug in r^n into the recursion above? We get

$$r^n - r^{n-1} - 6r^{n-2} = 0.$$

Now solve for r:

$$r^{n-2}(r^2 - r - 6) = 0,$$

so by factoring, $r = -2$ or $r = 3$ (or $r = 0$, although this does not help us). This tells us that $a_n = (-2)^n$ is a solution to the recurrence relation, as is $a_n = 3^n$. Which one is correct? They both are, unless we specify initial conditions. Notice we could also have $a_n = (-2)^n + 3^n$. Or $a_n = 7(-2)^n + 4 \cdot 3^n$. In fact, for any a and b, $a_n = a(-2)^n + b3^n$ is a solution (try plugging this into the recurrence relation). To find the values of a and b, use the initial conditions.

This points us in the direction of a more general technique for solving recurrence relations. Notice we will always be able to factor out the r^{n-2} as we did above. So we really only care about the other part. We call this other part the **characteristic equation** for the recurrence relation. We are interested in finding the roots of the characteristic equation, which are called (surprise) the **characteristic roots**.

Characteristic Roots.

Given a recurrence relation $a_n + \alpha a_{n-1} + \beta a_{n-2} = 0$, the **characteristic polynomial** is

$$x^2 + \alpha x + \beta$$

giving the **characteristic equation**:

$$x^2 + \alpha x + \beta = 0.$$

If r_1 and r_2 are two distinct roots of the characteristic polynomial (i.e., solutions to the characteristic equation), then the solution to the recurrence relation is

$$a_n = ar_1^n + br_2^n,$$

where a and b are constants determined by the initial conditions.

Example 4.4.4

Solve the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2}$ with $a_0 = 2$ and $a_1 = 3$.

Solution. Rewrite the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$. Now form the characteristic equation:

$$x^2 - 7x + 10 = 0$$

and solve for x:

$$(x - 2)(x - 5) = 0,$$

so $x = 2$ and $x = 5$ are the characteristic roots. We therefore know that the solution to the recurrence relation will have the form

$$a_n = a2^n + b5^n.$$

To find a and b, plug in $n = 0$ and $n = 1$ to get a system of two equations with two unknowns:

$$2 = a2^0 + b5^0 = a + b$$

$$3 = a2^1 + b5^1 = 2a + 5b.$$

Solving this system gives $a = \frac{7}{3}$ and $b = -\frac{1}{3}$, so the solution to the recurrence relation is

$$a_n = \frac{7}{3}2^n - \frac{1}{3}5^n.$$

Perhaps the most famous recurrence relation is $F_n = F_{n-1} + F_{n-2}$, which together with the initial conditions $F_0 = 0$ and $F_1 = 1$ defines the Fibonacci sequence. But notice that this is precisely the type of recurrence relation on which we can use the characteristic root technique. When we do, the only thing that changes is that the characteristic equation does not factor, so we must use the quadratic formula to find the characteristic roots. In fact, doing so gives the third most famous irrational number, φ, the **golden ratio**.

Before leaving the characteristic root technique, we should think about what might happen when solving the characteristic equation. We have an example above in which the characteristic polynomial has two distinct roots. These roots can be integers, or perhaps irrational numbers (requiring the quadratic formula to find them). In these cases, we know what the solution to the recurrence relation looks like.

However, it is possible for the characteristic polynomial to have only one root. This can happen if the characteristic polynomial factors as $(x - r)^2$. It is still the case that r^n would be a solution to the recurrence relation, but we won't be able to find solutions for all initial conditions using the general form $a_n = ar_1^n + br_2^n$, since we can't distinguish between r_1^n and r_2^n. We are in luck though:

Characteristic Root Technique for Repeated Roots.

Suppose the recurrence relation $a_n = \alpha a_{n-1} + \beta a_{n-2}$ has a characteristic polynomial with only one root r. Then the solution to the recurrence relation is

$$a_n = ar^n + bnr^n$$

where a and b are constants determined by the initial conditions.

Notice the extra n in bnr^n. This allows us to solve for the constants a and b from the initial conditions.

Example 4.4.5

Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 4$.

Solution. The characteristic polynomial is $x^2 - 6x + 9$. We solve the characteristic equation

$$x^2 - 6x + 9 = 0$$

by factoring:

$$(x-3)^2 = 0,$$

so $x = 3$ is the only characteristic root. Therefore we know that the solution to the recurrence relation has the form

$$a_n = a3^n + bn3^n$$

for some constants a and b. Now use the initial conditions:

$$\begin{aligned} a_0 &= 1 = a3^0 + b \cdot 0 \cdot 3^0 = a \\ a_1 &= 4 = a \cdot 3 + b \cdot 1 \cdot 3 = 3a + 3b. \end{aligned}$$

Since $a = 1$, we find that $b = \frac{1}{3}$. Therefore the solution to the recurrence relation is

$$a_n = 3^n + \frac{1}{3}n3^n.$$

Although we will not consider examples more complicated than these, this characteristic root technique can be applied to much more complicated recurrence relations. For example, $a_n = 2a_{n-1} + a_{n-2} - 3a_{n-3}$ has characteristic polynomial $x^3 - 2x^2 - x + 3$. Assuming we see how to factor such a degree 3 (or more) polynomial, we can easily find the characteristic roots and as such solve the recurrence relation (the solution would look like $a_n = ar_1^n + br_2^n + cr_3^n$ if there were 3 distinct roots). It is also possible that the characteristic roots are complex numbers.

However, the characteristic root technique is only useful for solving recurrence relations in a particular form: a_n is given as a linear combination of some number of previous terms. These recurrence relations are called **linear homogeneous recurrence relations with constant coefficients**. The "homogeneous" refers to the fact that there is no additional term in the recurrence relation other than a multiple of a_j terms. For example, $a_n = 2a_{n-1} + 1$ is *non-homogeneous* because of the additional constant 1. There are general methods of solving such things, but we will not consider them here, other than through the use of telescoping or iteration described above.

4.4.4 Reading Questions

1. Which of the following recurrence relations would be good candidates to try the characteristic root technique on? Select all that apply

A. $a_n = 3a_{n-1} + a_{n-2}$

B. $a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3}$

C. $a_n = \frac{1}{3} \cdot 2^n + \frac{2}{3}(-1)^n$

D. $a_n = a_{n-1} + 3a_{n-2} + 5$

E. $x^2 - 3x - 1 = 0$

2. At what step do you need to refer to the initial conditions when completing the characteristic root technique? What would happen if you didn't use these? Explain.

3. What questions do you have? Write at least one question about the content of this section that you or a classmate might be curious about.

4.4.5 Practice Problems

1. Find $6 + 30 + 150 + \cdots + 6 \cdot 5^{19}$.

2. Find $1 - \frac{7}{4} + \frac{49}{16} - \cdots + (-1)^{37}\frac{7^{37}}{4^{37}}$.

3. Solve the recurrence relation $a_n = a_{n-1} + 2^n$ with $a_0 = -2$.

$a_n =$ ____________________

4. Find the solution to the recurrence relation $a_n = 2a_{n-1} + 24a_{n-2}$ with initial terms $a_0 = 2$ and $a_1 = 2$.

$a_n =$ __________

5. Find the solution to the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with initial terms $a_0 = 2$ and $a_1 = 1$.

$a_n =$ ____________________________

Find the solution to the recurrence relation $b_n = b_{n-1} + 2b_{n-2}$ with initial terms $b_0 = 9$ and $b_1 = 14$.

$b_n =$ ___________________________

6. Find the solution to the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with initial terms $a_0 = 4$ and $a_1 = 17$.

$a_n =$ __________

4.4.6 Additional Exercises

1. Find the next two terms in $(a_n)_{n\geq 0}$ beginning $3, 5, 11, 21, 43, 85 \ldots$. Then give a recursive definition for the sequence. Finally, use the characteristic root technique to find a closed formula for the sequence.

2. Consider the sequences $2, 5, 12, 29, 70, 169, 408, \ldots$ (with $a_0 = 2$).

(a) Describe the rate of growth of this sequence.

(b) Find a recursive definition for the sequence.

(c) Find a closed formula for the sequence.

(d) If you look at the sequence of differences between terms, and then the sequence of second differences, the sequence of third differences, and so on, will you ever get a constant sequence? Explain how you know.

3. Show that 4^n is a solution to the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$.

4. Suppose that r^n and q^n are both solutions to a recurrence relation of the form $a_n = \alpha a_{n-1} + \beta a_{n-2}$. Prove that $c \cdot r^n + d \cdot q^n$ is also a solution to the recurrence relation, for any constants c, d.

5. Think back to the magical candy machine at your neighborhood grocery store. Suppose that the first time a quarter is put into the machine 1 Skittle comes out. The second time, 4 Skittles, the third time 16 Skittles, the fourth time 64 Skittles, etc.

(a) Find both a recursive and closed formula for how many Skittles the nth customer gets.

(b) Check your solution for the closed formula by solving the recurrence relation using the characteristic root technique.

6. Let a_n be the number of $1 \times n$ tile designs you can make using 1×1 squares available in 4 colors and 1×2 dominoes available in 5 colors.

(a) First, find a recurrence relation to describe the problem. Explain why the recurrence relation is correct (in the context of the problem).

(b) Write out the first 6 terms of the sequence $a_1, a_2, \ldots$.

(c) Solve the recurrence relation. That is, find a closed formula for a_n.

7. You have access to 1×1 tiles which come in 2 different colors and 1×2 tiles which come in 3 different colors. We want to figure out how many different $1 \times n$ path designs we can make out of these tiles.

(a) Find a recursive definition for the sequence a_n of paths of length n.

(b) Solve the recurrence relation using the characteristic root technique.

8. Solve the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$.

(a) What is the solution if the initial terms are $a_0 = 1$ and $a_1 = 2$?

(b) What do the initial terms need to be in order for $a_9 = 30$?

(c) For which x are there initial terms which make $a_9 = x$?

9. Consider the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$.

(a) Find the general solution to the recurrence relation (beware the repeated root).

(b) Find the solution when $a_0 = 1$ and $a_1 = 2$.

(c) Find the solution when $a_0 = 1$ and $a_1 = 8$.

10. Here is a surprising use of sequences to answer a counting question: How many license plates consist of 6 symbols, using only the three numerals 1, 2, and 3 and the four letters a, b, c, and d, so that no numeral appears after any letter? For example, "31ddac", "123321", and "ababab" are each acceptable license plates, but "13ba2c" is not.

(a) First answer this question by considering different cases: how many of the license plates contain no numerals? How many contain one numeral, etc.

(b) Now use the techniques of this section to show why the answer is $4^7 - 3^7$.

4.5 Proof by Induction

Objectives

After completing this section, you should be able to do the following.

1. Identify the parts of a proof by mathematical induction and how they relate to the statement being proved.
2. Prove statements using mathematical induction.
3. Explain why a proof by mathematical induction is valid.

4.5.1 Section Preview

Investigate!

What is the unit digit (the right-most digit) of 6^n? Does the answer depend on n?

Mathematical induction is a powerful proof technique that can be used to prove statements are true for a *sequence* of statements, as long as that sequence of statements has some starting place. For example, if we are trying to say something about the unit digit of 6^n, we are making that claim for $n = 1$, then $n = 2$, then $n = 3$, and so on.

Induction is closely related to recursive definitions; the main idea in a proof by induction is to explain how you can get from one statement in the sequence to the next.

Preview Activity

1. Suppose that 6^{472} had a 2 for its unit digit. That is, suppose $6^{472} = 19,381,6\ldots\ldots 2$. What would the unit digit of 6^{473} be?

Hint. $6^{473} = 6 \cdot 6^{472}$.

2. What is the unit digit of 6^2, of 6^3, and of 6^4?

3. Which of the following are true? Select all that apply.

A. If the unit's digit of 6^k is a 6, then the unit's digit of 6^{k+1} is a 6.

B. If the unit's digit of 6^k is a 2, then the unit's digit of 6^{k+1} is a 2.

C. The unit's digit of 6^{472} is a 2.

D. The unit's digit of 6^{472} is a 6.

4. Explain your answer to the previous question.

4.5.2 Recursive Reasoning

We have seen that describing a sequence recursively can often be easier than describing the sequence with a closed formula. We will now see how using similar recursive reasoning can help us prove statements using a proof technique called **mathematical induction**. This style of proof is especially useful when the different instances of the statement (for different values of n, say) are related recursively.

For example, suppose we wanted to prove a fact about all the terms in a sequence for which we have a recursive definition. Consider the sequence $(a_n)_{n\geq 0}$ defined recursively by $a_n = 3a_{n-1} - 2$ with $a_0 = 5$. Could we prove that every term in this sequence is odd?

Let's start by writing out the first few terms of the sequence:

$$5, 13, 37, 109, \ldots.$$

So far, all these numbers look odd. Will the next number be odd? Of course, we could just compute it using the recurrence relation. We would take $3 \cdot 109 - 2$. We don't actually care *which* odd number this is, just that it is, in fact, odd. We know it will be odd because the product of two odd numbers is odd, and subtracting 2 from an odd number results in an odd number.

Great, so a_4 is odd. Will a_5 be odd too? Yes, use the same argument as above: $a_5 = 3a_4 - 2$. We just convinced ourselves that a_4 is odd (without finding its actual value), so $3a_4$ is odd, and 2 less than it will be odd too.

What about a_6? Do the same thing. In fact, why are we using any particular number as the index? If it is the same argument each time, we should be able to just give this argument once and say it always works.

Suppose we have found that a_k is odd (where k is some arbitrary natural number). From this, we can find that a_{k+1} is odd, since $a_{k+1} = 3a_k - 2$, and 3 times the odd number a_k will be odd, and subtracting 2 will result in an odd number. Yay. Let's put this all together as a proof.

Proof. We claim that for any $n \geq 0$, the number a_n is odd, where $a_n = 3a_{n-1} - 2$ and $a_0 = 5$.

When $n = 0$, the claim is true, since $a_0 = 5$ is an odd number.

Further, we can prove that every larger n has a_n odd because as long as a_k is odd, so is a_{k+1} (since $a_{k+1} = 3a_k - 2$, and 3 times an odd number minus 2 is always odd).

Therefore a_n is odd for all $n \geq 0$.

Soon we will give a more rigid structure for proofs by induction, but the basic idea is exactly what we have above.

4.5.3 Formalizing Proofs

Induction can prove many statements that hold for all natural numbers, not just statements about sequences. In particular, induction should be used when there is some way to go from one case to the next – when you can see how to always "do one more."

Thinking about how we write statements in logical symbols, we will use induction to prove statements of the form

$$\forall n P(n),$$

where the domain of discourse (the values of n we quantify over) has some least element. Say that domain of discourse is the natural numbers. We are then proving this *sequence* of statements:

$$P(0), P(1), P(2), P(3), \ldots.$$

The way we do this with induction is to prove a **base case**, that $P(0)$ is true (or $P(a)$ where a is the least element of our domain of discourse). Next, we prove the **inductive case**, that $P(k) \rightarrow P(k+1)$ for all $k \geq 0$ (or $k \geq a$).

Together, these are enough to prove $P(n)$ is true for all n. How do we know? That is, why is this style of proof valid? Well, let's convince ourselves that $P(3)$ is true. We know $P(0)$ is true. And because we know that $P(0) \rightarrow P(1)$, we then also know that $P(1)$ is true. Because $P(1) \rightarrow P(2)$, we then get that $P(2)$ is true. Finally, because $P(2) \rightarrow P(3)$, we have that $P(3)$. There is nothing special about 3 here. We could have gone up as far as we like, ***to any n value!***

Think of a row of dominoes set up standing on their edges. We want to argue that in a minute, all the dominoes will have fallen. For this to happen, you will need to push the first domino. That is the base case. It will also have to be that the dominoes are close enough together that when any particular domino falls, it will cause the next domino to fall. That is the inductive case. If both of these conditions are met – you push the first domino over, and each domino will cause the next to fall – then all the dominoes will fall.

Induction is powerful! Think how much easier it is to knock over dominoes when you don't have to push over each domino yourself. You just start the chain reaction and then rely on the relative nearness of the dominoes to take care of the rest.

When writing a proof by induction, we will follow a standard style. Writing in this style allows us to keep our ideas organized and might even help us formulate the proof.

Here is the general structure of a proof by mathematical induction:

Induction Proof Structure.

Start by saying what the statement is that you want to prove: "Let $P(n)$ be the statement. . . ." To prove that $P(n)$ is true for all $n \geq 0$, you must prove two facts:

1. Base case: Prove that $P(0)$ is true. You do this directly. This is often easy.

2. Inductive case: Prove that $P(k) \rightarrow P(k+1)$ for all $k \geq 0$. That is, prove that for any $k \geq 0$ if $P(k)$ is true, then $P(k+1)$ is true as well. This is the proof of an if . . . then . . . statement, so you can assume $P(k)$ is

true ($P(k)$ is called the *inductive hypothesis*). You must then explain why $P(k+1)$ is also true, given that assumption.

Assuming you are successful on both parts above, you can conclude, "Therefore by the principle of mathematical induction, the statement $P(n)$ is true for all $n \geq 0$."

Sometimes the statement $P(n)$ will only be true for values of $n \geq 4$, for example, or some other value. In such cases, replace all the 0's above with 4's (or the other value).

Before attempting to prove a statement by mathematical induction, first think about *why* the statement is true using inductive reasoning. Explain why induction is the right thing to do, and roughly why the inductive case will work. Then, sit down and write out a careful, formal proof using the structure above.

4.5.4 Examples

Here are some examples of proof by mathematical induction.

Example 4.5.1

Prove for each natural number $n \geq 1$ that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Solution. First, let's think inductively about this equation. In fact, we know this is true for other reasons (reverse and add comes to mind). But why might induction be applicable? The left-hand side adds up the numbers from 1 to n. If we know how to do that, adding just one more term $(n+1)$ would not be that hard. For example, if $n = 100$, suppose we know that the sum of the first 100 numbers is 5050 (so $1 + 2 + 3 + \cdots + 100 = 5050$, which is true). Now to find the sum of the first 101 numbers, it makes more sense to just add 101 to 5050, instead of computing the entire sum again. We would have $1 + 2 + 3 + \cdots + 100 + 101 = 5050 + 101 = 5151$. In fact, it would always be easy to add just one more term. This is why we should use induction.

Now the formal proof:

Proof. Let $P(n)$ be the statement $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$. We will show that $P(n)$ is true for all natural numbers $n \geq 1$.

Base case: $P(1)$ is the statement $1 = \frac{1(1+1)}{2}$ which is clearly true.

Inductive case: Let $k \geq 1$ be a natural number. Assume (for induction) that $P(k)$ is true. That means $1+2+3+\cdots+k = \frac{k(k+1)}{2}$. We will prove that $P(k+1)$ is true as well. That is, we must prove that $1+2+3+\cdots+k+(k+1) = \frac{(k+1)(k+2)}{2}$. To prove this equation, start by adding $k+1$ to both sides of the inductive hypothesis:

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1).$$

Now, simplifying the right side we get:

$$\begin{aligned} \frac{k(k+1)}{2} + k + 1 &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+2)(k+1)}{2}. \end{aligned}$$

Thus $P(k+1)$ is true, so by the principle of mathematical induction, $P(n)$ is true for all natural numbers $n \geq 1$. ■

Note that in the part of the proof where we proved $P(k+1)$ from $P(k)$, we used the equation $P(k)$. This was the inductive hypothesis. Seeing how to use the inductive hypotheses is usually straightforward when proving a fact about a sum like this. In other proofs, it can be less obvious where it fits in.

Example 4.5.2

Prove that for all $n \in \mathbb{N}$, $6^n - 1$ is a multiple of 5.

Solution. Again, start by understanding the dynamics of the problem. What does increasing n do? Let's try with a few examples. If $n = 1$, then yes, $6^1 - 1 = 5$ is a multiple of 5. What does incrementing n to 2 look like? We get $6^2 - 1 = 35$, which again is a multiple of 5. Next, $n = 3$: But instead of just finding $6^3 - 1$, what did the increase in n do? We will still subtract 1, but now we are multiplying by another 6 first. Viewed another way, we are multiplying a number that is one more than a multiple of 5 by 6 (because $6^2 - 1$ is a multiple of 5, so 6^2 is one more than a multiple of 5). What do numbers that are one more than a multiple of 5 look like? They must have last digit 1 or 6. What happens when you multiply such a number by 6? It depends on the number, but in any case, the last digit of the new number must be a 6. And then if you subtract 1, you get last digit 5, so a multiple of 5.

The point is, every time we multiply by just one more six, we still get a number with last digit 6, so subtracting 1 gives us a multiple of 5. Now the formal proof:

Proof. Let $P(n)$ be the statement, "$6^n - 1$ is a multiple of 5." We will prove that $P(n)$ is true for all $n \in \mathbb{N}$.

Base case: $P(0)$ is true: $6^0 - 1 = 0$, which is a multiple of 5.

Inductive case: Let k be an arbitrary natural number. Assume, for induction, that $P(k)$ is true. That is, $6^k - 1$ is a multiple of 5. Then $6^k - 1 = 5j$ for some integer j. This means that $6^k = 5j + 1$. Multiply both sides by 6:

$$6^{k+1} = 6(5j + 1) = 30j + 6.$$

But we want to know about $6^{k+1} - 1$, so subtract 1 from both sides:

$$6^{k+1} - 1 = 30j + 5.$$

Of course $30j + 5 = 5(6j + 1)$, so is a multiple of 5.

Therefore $6^{k+1} - 1$ is a multiple of 5, or in other words, $P(k + 1)$ is true. Thus, by the principle of mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$. ■

We had to be a little bit clever (i.e., use some algebra) to locate the $6^k - 1$ inside of $6^{k+1} - 1$ before we could apply the inductive hypothesis. This is what can make inductive proofs challenging.

In the two examples above, we started with $n = 1$ or $n = 0$. We can start later if we need to.

Example 4.5.3

Prove that $n^2 < 2^n$ for all integers $n \geq 5$.

Solution. First, the idea of the argument. What happens when we increase n by 1? On the left-hand side, we increase the base of the square and go to the next square number. On the right-hand side, we increase the power of 2. This means we double the number. So the question is, how does doubling a number relate to increasing to the next square? Think about what the difference of two consecutive squares looks like. We have $(n + 1)^2 - n^2$. This factors:

$$(n + 1)^2 - n^2 = (n + 1 - n)(n + 1 + n) = 2n + 1.$$

But doubling the right-hand side increases it by 2^n, since $2^{n+1} = 2^n + 2^n$. When n is large enough, $2^n > 2n + 1$.

What we are saying here is that each time n increases, the left-hand side grows by less than the right-hand side. So if the left-hand side starts smaller (as it does when $n = 5$), it will never catch up. Now the formal proof:

Proof. Let $P(n)$ be the statement $n^2 < 2^n$. We will prove $P(n)$ is true for all integers $n \geq 5$.

Base case: $P(5)$ is the statement $5^2 < 2^5$. Since $5^2 = 25$ and $2^5 = 32$, we see that $P(5)$ is indeed true.

Inductive case: Let $k \geq 5$ be an arbitrary integer. Assume, for induction, that $P(k)$ is true. That is, assume $k^2 < 2^k$. We will prove that $P(k + 1)$ is true, i.e., $(k + 1)^2 < 2^{k+1}$. To prove such an inequality, start with the left-hand side and work towards the right-hand side:

$$\begin{aligned}(k+1)^2 &= k^2 + 2k + 1 \\ &< 2^k + 2k + 1 && \text{...by the inductive hypothesis.} \\ &< 2^k + 2^k && \text{... since } 2k+1 < 2^k \text{ for } k \geq 5. \\ &= 2^{k+1}.\end{aligned}$$

Following the equalities and inequalities through, we get $(k+1)^2 < 2^{k+1}$, in other words, $P(k+1)$. Therefore by the principle of mathematical induction, $P(n)$ is true for all $n \geq 5$. ■

The previous example might remind you of the *racetrack principle* from calculus, which says that if $f(a) < g(a)$, and $f'(x) < g'(x)$ for $x > a$, then $f(x) < g(x)$ for $x > a$. Same idea: the larger function is increasing more than the smaller function, so the larger function will stay larger. In discrete math, we don't have derivatives, so we look at differences. Thus induction is the way to go.

A Warning. With great power, comes great responsibility. Induction isn't magic. It seems very powerful to be able to assume $P(k)$ is true. After all, we are trying to prove $P(n)$ is true, and the only difference is in the variable: k vs. n. Are we assuming that what we want to prove is true? Not really. We assume $P(k)$ is true only for the sake of proving that $P(k+1)$ is true.

Still you might start to believe that you can prove anything with induction. Consider this incorrect "proof" that every Canadian has the same eye color: Let $P(n)$ be the statement that any n Canadians have the same eye color. $P(1)$ is true, since everyone has the same eye color as themselves. Now assume $P(k)$ is true. That is, assume that in any group of k Canadians, everyone has the same eye color. Now consider an arbitrary group of $k+1$ Canadians. The first k of these must all have the same eye color, since $P(k)$ is true. Also, the last k of these must have the same eye color, since $P(k)$ is true. So in fact, everyone in the group must have the same eye color. Thus $P(k+1)$ is true. So by the principle of mathematical induction, $P(n)$ is true for all n.

Clearly something went wrong. The problem is that the proof that $P(k)$ implies $P(k+1)$ assumes that $k \geq 2$. We have only shown $P(1)$ is true. In fact, $P(2)$ is false. Try this: read through the previous paragraph again, substituting 1 for each k. Can you spot the error in that argument?

4.5.5 Reading Questions

1. Suppose you wanted to prove, using mathematical induction, that $1+3+5+\cdots+2n-1 = n^2$ for all values of $n \geq 1$. Which of the following would be an appropriate *first line* of the proof? Select all that apply.

A. Let $P(n)$ be the statement "$1+3+5+\cdots+2n-1 = n^2$."

B. For each $n \geq 1$, let $P(n)$ be the statement, "the sum of the first n odd numbers is n^2."

C. Assume $1 + 3 + \cdots + 2n - 1 = n^2$ for all $n \geq 1$.

D. Let $P(n)$ be the statement, "$1 + 3 + \cdots + 2n - 1 = n^2$ for all $n \geq 1$."

E. Since $P(1) = 1 = 1^2$, the base case is true.

2. Suppose you wanted to prove that $P(n, 3) \geq \binom{n}{3}$ for all $n \geq 4$. Write the first line of a proof by induction.

3. What questions do you have? Write at least one question about the content of this section that you or a classmate might be curious about after reading this section.

4.5.6 Practice Problems

1. Suppose you are trying to prove, by mathematical induction, that a statement $P(n)$ is true for all $n \geq 0$. What would you attempt to prove in the *induction step* of the proof? (Select all that apply.)

A. That assuming $P(k)$ is true for an arbitrary $k \geq 0$, we can prove that $P(k+1)$ is true.

B. That $P(k)$ implies $P(k+1)$ for all $k \geq 0$.

C. That assuming $P(k+1)$ is true for an arbitrary $k \geq 0$, we can prove that $P(k)$ is true.

D. That $P(k+1)$ implies $P(k)$ for all $k \geq 0$.

E. That $P(k)$ implies $P(k+1)$ for at least one $k \geq 0$.

2. Suppose you wanted to prove the following statement:

$$2 + 4 + 6 + \cdots + 2n = n(n+1) \text{ for all } n \geq 1.$$

What would the first line of a proof by induction be?

A. Let $P(n)$ be the statement "$2 + 4 + 6 + \cdots + 2n = n(n+1)$."

B. Let $P(n)$ be the statement "$2 + 4 + 6 + \cdots + 2n = n(n-1)$ for all $n \geq 1$."

C. Assume $P(n)$ is true for all $n \geq 1$.

D. Let $P(n) = 2 + 4 + 6 + \cdots + 2n$.

E. Suppose $P(n) = n(n+1)$ for all $n \geq 1$.

3. Suppose you were proving the following statement by mathematical induction:

$$2 + 4 + 6 + \cdots + 2n = n(n+1) \text{ for all } n \geq 1.$$

What would you need to show to establish the base case?

A. Show that $P(1)$ is true. That is, show that $2 = 1(1 + 1)$.

B. Show that $P(2)$ is true. That is, show that $2 + 4 = 2(2 + 1)$.

C. Show that $P(1)$ and $P(2)$ are both true.

D. Show that $P(1)$ implies $P(2)$.

E. Nothing, since the sum already has more than $n = 1$ terms.

4. Suppose you were proving the following statement by mathematical induction:

$$2 + 4 + 6 + \cdots + 2n = n(n + 1) \text{ for all } n \geq 1.$$

What would the first line of the inductive case be?

A. Assume $P(k)$ is true for some arbitrary $k \geq 1$, that is, assume $2 + 4 + 6 + \cdots + 2k = k(k + 1)$.

B. Assume $P(k)$ is true for all $k \geq 1$, that is, assume $2+4+6+\cdots+2k = k(k+1)$.

C. Assume $P(k)$ implies $P(k + 1)$ for an arbitrary $k \geq 1$.

D. Assume $P(k)$ is true for some large $k \geq 1$; say, assume $2+4+6+\cdots+432 = 216(217)$.

5. Arrange some of the statements below to create a correct proof by induction that the recurrence relation $a_n = 5a_{n-1} + 4$, with initial condition $a_0 = 0$ has closed formula $a_n = 5^n - 1$.

- This simplifies to $a_{k+1} = 5^{k+1} - 5 + 4 = 5^{k+1} - 1$, so $P(k + 1)$ is true.
- Therefore, by the principle of mathematical induction, $P(n)$ is true for all $n \geq 0$.
- Now assume that $P(k + 1)$ is true for an arbitrary integer $k \geq 0$.
- Then $a_{k+1} = 5a_k + 4$, so $P(k + 1)$ is true.
- Then $a_k = 5^k - 1$.
- Let $P(n)$ be the statement, "$a_n = 5^n - 1$".
- By the recurrence relation, we have $a_{k+1} = 5a_k + 4 = 5(5^k - 1) + 4$.
- Note that $a_0 = 5^0 - 1 = 0$, so $P(0)$ is true.
- Now assume that $P(k)$ is true for an arbitrary integer $k \geq 0$.

6. Arrange some of the statements below to create a correct proof by induction that for all $n \geq 1$, the number $14^n - 1$ is a multiple of 13.

- Then $14^k - 1 = 13 \cdot j$ for some integer j.
- Therefore, by the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$.
- Note that $14^2 - 1 = (14 - 1)(14 + 1)$ by difference of squares, so this is a multiple of 13.
- Now assume that $P(n)$ is true for all $n \geq 1$.
- Thus $14^k = \frac{14^{k+1}-1}{14}$, which must also be a multiple of 13.
- Since $14^{k+1} - 1 = 14(14^k - 1) + 14 - 1 = 14(13 \cdot j) + 13$, we see that $14^{k+1} - 1$ is a multiple of 13.
- Let $P(n)$ be the statement, "$14^n - 1$ is a multiple of 13."
- Thus $P(k + 1)$ is true.
- Note that $14^1 - 1 = 13$, so this is definitely a multiple of 13.
- Now assume that $P(k)$ is true for an arbitrary integers $k \geq 1$.

7. Arrange some of the statements below to create a correct proof by induction that for all $n \geq 1$, $1 + 1 + 2 + 3 + 5 + \cdots + F_n = F_{n+2} - 1$, where F_n is the nth Fibonacci number.

- By the definition of Fibonacci numbers, $F_{k+1} + F_{k+2} = F_{k+3}$, so the right-hand side simplifies to $F_{k+3} - 1$.
- Then by the inductive hypothesis, $1 + 1 + 2 + 3 + \cdots + F_{k+1} = F_{k+2} - 1$.
- That is, assume $1 + 1 + 2 + 3 + 5 + F_k = F_{k+2} - 1$.
- For the base case, note that $1 + 1 + 2 = 4 = F_5 - 1$.
- Let $P(n)$ be the statement, "$1 + 1 + 2 + 3 + 5 + \cdots + F_n = F_{n+2} - 1$."
- Thus $P(k + 1)$ is true, and therefore by the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$.
- Then adding F_{k+1} to both sides, we get $1 + 1 + 2 + 3 + 5 + \cdots + F_k + F_{k+1} = F_{k+1} + F_{k+2} - 1$.
- For the base case, note that $P(1)$ and $P(2)$ are true, because $1 = 2 - 1$ and $1 + 1 = 3 - 1$.
- Now assume that $P(k)$ is true for an arbitrary integer $k \geq 2$.
- Subtracting F_{k+1} from both sides gives us $P(k)$, which we also assumed to be true.

4.5.7 Additional Exercises

1. On the way to the market, you exchange your cow for some magic dark chocolate espresso beans. These beans have the property that every night at midnight, each bean splits into two, effectively doubling your collection. You decide to take advantage of this, and each morning (around 8 am) you eat 5 beans.

(a) Explain why it is true that *if* at noon on day n you have a number of beans ending in a 5, then at noon on day $n+1$ you will still have a number of beans ending in a 5.

(b) Why is the previous fact not enough to conclude that you will always have a number of beans ending in a 5? What additional fact would you need?

(c) Assuming you have the additional fact in part (b), and have successfully proved the fact in part (a), how do you know that you will always have a number of beans ending in a 5? Illustrate what is going on by carefully explaining how the two facts above prove that you will have a number of beans ending in a 5 on *day 4* specifically. In other words, explain why induction works in this context.

2. Use induction to prove for all $n \in \mathbb{N}$ that $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$.

3. Prove that $7^n - 1$ is a multiple of 6 for all $n \in \mathbb{N}$.

4. Prove that $1 + 3 + 5 + \cdots + (2n-1) = n^2$ for all $n \geq 1$.

5. Prove that $F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$ where F_n is the nth Fibonacci number.

6. Prove that $2^n < n!$ for all $n \geq 4$. (Recall, $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$.)

7. Prove, by mathematical induction, that $F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$, where F_n is the nth Fibonacci number ($F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$).

8. Zombie Euler and Zombie Cauchy, two famous zombie mathematicians, have just signed up for Myspace accounts. After one day, Zombie Cauchy has more followers than Zombie Euler. Each day after that, the number of new followers of Zombie Cauchy is exactly the same as the number of new followers of Zombie Euler (and neither lose any followers). Explain how a proof by mathematical induction can show that on every day after the first day, Zombie Cauchy will have more followers than Zombie Euler. That is, explain what the base case and inductive case are, and why they together prove that Zombie Cauchy will have more followers on the 4th day.

9. Find the largest number of points that it is impossible for a football team to get exactly, using just 3-point field goals and 7-point touchdowns (ignore the possibilities of safeties, missed extra points, and two-point conversions). Prove your answer is correct by mathematical induction.

10. Prove that the sum of n squares can be found as follows

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

11. Prove that the sum of the interior angles of a convex n-gon is $(n-2) \cdot 180°$. (A convex n-gon is a polygon with n sides for which each interior angle is less than $180°$.)

12. What is wrong with the following "proof" of the "fact" that $n + 3 = n + 7$ for all values of n (besides of course that the thing it is claiming to prove is false)?

Proof. Let $P(n)$ be the statement that $n + 3 = n + 7$. We will prove that $P(n)$ is true for all $n \in \mathbb{N}$. Assume, for induction, that $P(k)$ is true. That is, $k+3 = k+7$. We must show that $P(k+1)$ is true. Now since $k+3 = k+7$, add 1 to both sides. This gives $k+3+1 = k+7+1$. Regrouping $(k+1)+3 = (k+1)+7$. But this is simply $P(k+1)$. Thus by the principle of mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$.

13. The proof in the previous problem does not work. But if we modify the "fact," we can get a working proof. Prove that $n + 3 < n + 7$ for all values of $n \in \mathbb{N}$. You can do this proof with algebra (without induction), but the goal of this exercise is to write out a valid induction proof.

14. Find the flaw in the following "proof" of the "fact" that $n < 100$ for every $n \in \mathbb{N}$.

Proof. Let $P(n)$ be the statement $n < 100$. We will prove $P(n)$ is true for all $n \in \mathbb{N}$. First we establish the base case: when $n = 0$, $P(n)$ is true, because $0 < 100$. Now for the inductive step, assume $P(k)$ is true. That is, $k < 100$. Now if $k < 100$, then k is some number, like 80. Of course $80 + 1 = 81$ which is still less than 100. So $k + 1 < 100$ as well. But this is what $P(k+1)$ claims, so we have shown that $P(k) \to P(k+1)$. Thus by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

15. While the above proof does not work (it better not since the statement it is trying to prove is false!) we can prove something similar. Prove that there is a strictly increasing sequence $a_1, a_2, a_3, \ldots$ of numbers (not necessarily integers) such that $a_n < 100$ for all $n \in \mathbb{N}$. (By **strictly increasing** we mean $a_n < a_{n+1}$ for all n. So each term must be larger than the last.)

16. What is wrong with the following "proof" of the "fact" that for all $n \in \mathbb{N}$, the number $n^2 + n$ is odd?

Proof. Let $P(n)$ be the statement "$n^2 + n$ is odd." We will prove that $P(n)$ is true for all $n \in \mathbb{N}$. Suppose, for induction, that $P(k)$ is true, that is, that $k^2 + k$ is odd. Now consider the statement $P(k+1)$. Now $(k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1 = k^2 + k + 2k + 2$. By the inductive hypothesis, $k^2 + k$ is odd, and of course $2k + 2$ is even. An odd plus an even is always odd, so therefore $(k+1)^2 + (k+1)$ is odd. Therefore by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

17. Now give a valid proof (by induction, even though you might be able to do so without using induction) of the statement, "For all $n \in \mathbb{N}$, the number $n^2 + n$ is even."

18. Prove that there is a sequence of positive real numbers $a_0, a_1, a_2, \ldots$ such that the partial sum $a_0 + a_1 + a_2 + \cdots + a_n$ is strictly less than 2 for all $n \in \mathbb{N}$. Hint: Think about how you could define what a_{k+1} is to make the induction argument work.

19. Use induction to prove that if n people all shake hands with each other, that the total number of handshakes is $\frac{n(n-1)}{2}$.

20. Use induction to prove that $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$. That is, the sum of the nth row of Pascal's triangle is 2^n.

21. Use induction to prove $\binom{4}{0} + \binom{5}{1} + \binom{6}{2} + \cdots + \binom{4+n}{n} = \binom{5+n}{n}$. (This is an example of the hockey stick theorem.)

22. Use the product rule for logarithms ($\log(ab) = \log(a) + \log(b)$) to prove, by induction on n, that $\log(a^n) = n\log(a)$, for all natural numbers $n \geq 2$.

23. Let $f_1, f_2, \ldots, f_n$ be differentiable functions. Prove, using induction, that

$$(f_1 + f_2 + \cdots + f_n)' = f_1' + f_2' + \cdots + f_n'.$$

You may assume $(f + g)' = f' + g'$ for any differentiable functions f and g.

24. Suppose $f_1, f_2, \ldots, f_n$ are differentiable functions. Use mathematical induction to prove the generalized product rule:

$$(f_1 f_2 f_3 \cdots f_n)' = f_1' f_2 f_3 \cdots f_n + f_1 f_2' f_3 \cdots f_n + f_1 f_2 f_3' \cdots f_n + \cdots + f_1 f_2 f_3 \cdots f_n'.$$

You may assume the product rule for two functions is true.

25. In Exercises 1.3.8 we proved that the following is a valid deduction rule:

$$\begin{array}{rc} & P \to Q \\ & Q \to R \\ \hline \therefore & P \to R \end{array}$$

Now use mathematical induction to prove you can chain together any number of statements like this. That is, prove for any n that the following is a valid deduction rule:

$$\begin{array}{rc}
 & P_1 \to P_2 \\
 & P_2 \to P_3 \\
 & \vdots \\
 & P_{n-1} \to P_n \\
\hline
\therefore & P_1 \to P_n.
\end{array}$$

4.6 Strong Induction

Objectives

After completing this section, you should be able to do the following.

1. Explain the difference between proof by induction and proof by strong induction.
2. Use strong induction to prove statements.

4.6.1 Section Preview

Investigate!

Start with a square piece of paper. You want to cut this square into smaller squares, leaving no waste (every piece of paper you end up with must be a square). Obviously it is possible to cut the square into 4 squares. You can also cut it into 9 squares. It turns out you can cut the square into 7 squares (although not all the same size). What other numbers of squares could you end up with?

Sometimes, to prove that $P(k+1)$ is true, it would be helpful to know that $P(k)$ *and* $P(k-1)$ *and* $P(k-2)$ are all true. This is certainly the case when proving something about a recurrence relation that is given as a combination of two previous terms.

Example 4.6.1

Prove that 2^n is a solution to the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 2$.

Solution.

Proof. Let $P(n)$ be the statement, "$a_n = 2^n$." We will show this is true for all $n \geq 0$.

Base cases: $a_0 = 2^0 = 1$ and $a_1 = 2^1 = 2$ both agree with the initial conditions.

Inductive case: Let $k \geq 2$ be arbitrary. Assume $P(k)$ and $P(k-1)$ are both true. That is, assume $a_k = 2^k$ and $a_{k-1} = 2^{k-1}$. We will show that $P(k+1)$ is true. Consider a_{k+1}. We have

$$\begin{aligned} a_{k+1} =& 5a_k - 6a_{k-1} \\ =& 5 \cdot 2^k - 6 \cdot 2^{k-1} \end{aligned}$$

$$
\begin{aligned}
&= 10 \cdot 2^{k-1} - 6 \cdot 2^{k-1} \\
&= 4 \cdot 2^{k-1} \\
&= 2^{k+1}.
\end{aligned}
$$

Therefore, by the principle of mathematical induction, $P(n)$ is true for all $n \geq 0$. ■

Well, almost the principle of mathematical induction. Is what we did okay?

There are also times when we might want to go even farther back to use an assumption that $P(j)$ is true for j much smaller than $k + 1$. This is the idea behind **strong induction** and the topic of this short section.

Preview Activity

1. Consider the following puzzle:

You have a rectangular chocolate bar, made up of n identical squares of chocolate. You can take such a bar and break it along any row or column. How many times will you have to break the bar to reduce it to n single chocolate squares?

At first, this question might seem impossible. Perhaps we meant to ask for the *smallest* number of breaks needed. Let's investigate.

(a) Suppose you started with a 1×3 bar. How many breaks would you need to reduce it to single squares?

(b) If you had a 1×4 bar, how many breaks are required?

If you had 4 squares arranged in a 2×2 square, your first break would require you to break the chocolate into two 1×2 bars. Then each of these would require _____ more break(s), for a total of _____ breaks to go from the 2×2 to single squares.

(c) A 6-square bar could either be a 1×6 bar, requiring _____ breaks, or a 2×3 bar.

There are two ways to proceed now.

a. Break the bar into two 1×3 bars, each requiring _____ more breaks, for a total of _____ breaks.

b. Break the bar into a 1×2 bar and a 2×2 bar. The 1×2 bar takes _____ more break(s) and the 2×2 bar takes _____ more break(s), for a total of _____ breaks.

(d) Based on the above data, what should our conjecture be for the number of breaks to reduce an n-square bar to single squares, in terms of n?

It will take __________ breaks to reduce an n-square bar to single squares.

(e) Do we believe this? Suppose you used one break to reduce the bar into two smaller bars, with a and b squares respectively. If the conjecture is correct, how many more breaks will it take to reduce the size a bar?

How many more breaks will it take to reduce the size b bar?

How many breaks is this all together, in terms of a and b, including the initial break?

But what is $a + b$? We got a and b by breaking the n squares in two pieces, so $a + b =$_____. This gives us a total number of breaks as __________.

4.6.2 Divide and Conquer

Think of recursive definitions as instructions for building a ladder. You can build the ladder as tall as you like because you have instructions for building the next rung, as long as you are standing on the rung before it.

Induction is the corresponding proof technique. To prove that you can climb the ladder as high as you like, you prove that you can step onto the ladder (the base case) and then prove that, from any rung, you can get to the next rung (the inductive step).

More specifically, suppose you were trying to prove that you can get to rung 4 on the ladder. You have successfully proved that you can get to rung 1, and that from any rung, you can get to the next. So you can get to rung 1, and from 1 you can get to 2. From 2 you can get to 3, and from 3 you can get to 4. Therefore, you can get to 4.

But notice that along the way, you know you have visited rungs 1 through 3. We might as well assume that we have visited all the rungs below the next one we are trying to reach. This is the idea behind **strong induction**.

A better ladder metaphor for strong induction is to think of ladders as things we can stack on top of each other. We want to argue that it is possible to climb 20 rungs of a ladder. Let's divide that into two smaller ladders, say a 12-rung ladder and an 8-rung ladder. We can assume that we can climb both of these since 20 is the least size we are not yet convinced of. Well, put those two ladders together, and you get $12 + 8 = 20$ rungs.

We better climb down from our shaky metaphor before we hurt ourselves. Let's look at a formal definition of strong induction.

Strong Induction Proof Structure.

Start by saying what we want to prove: "Let $P(n)$ be the statement...." Then establish two facts:

1. Base case: Prove that $P(0)$ is true. (Perhaps also prove other needed base cases.)

2. Inductive case: Assume $P(j)$ is true for all $j \leq k$. Prove that $P(k + 1)$ is true.

Conclude, "Therefore, by strong induction, $P(n)$ is true for all $n > 0$."

Of course, it is acceptable to replace 0 with a larger base case if needed.[2]

To illustrate strong induction, let's return to the chocolate bar problem.

Proposition 4.6.2

Given an n-square rectangular chocolate bar, it always takes $n - 1$ breaks to reduce the bar to single squares.

Proof. Let $P(n)$ be the statement, "It takes $n - 1$ breaks to reduce a n-square chocolate bar to single squares."

Base case: Consider $P(2)$. The squares must be arranged into a 1×2 rectangle, and we require $2 - 1 = 1$ breaks to reduce this to single squares.

Inductive case: Fix an arbitrary $k \geq 2$ and assume $P(j)$ is true for all $j \leq k$. Consider a $(k + 1)$-square rectangular chocolate bar. Break the bar once along any row or column. This results in two chocolate bars, say of sizes a and b . That is, we have an a-square rectangular chocolate bar, a b-square rectangular chocolate bar, and $a + b = k + 1$.

We also know that $a \leq k$ and $b \leq k$, so by our inductive hypothesis, $P(a)$ and $P(b)$ are true. To reduce the a-square bar to single squares takes $a - 1$ breaks; to reduce the b-square bar to single squares takes $b - 1$ breaks. Doing this results in our original bar being reduced to single squares. All together it took the initial break, plus the $a - 1$ and $b - 1$ breaks, for a total of

$$1 + a - 1 + b - 1 = a + b - 1 = k + 1 - 1 = k$$

breaks. Thus $P(k + 1)$ is true.

Therefore, by strong induction, $P(n)$ is true for all $n \geq 2$.

Here is a more mathematically relevant example:

Example 4.6.3

Prove that any natural number greater than 1 is either prime or can be written as the product of primes.

Solution. First, the idea: If we take some number n, maybe it is prime. If so, we are done. If not, then it is composite, so it is the product of two smaller numbers. Each of these factors is smaller than n (but at least 2), so we can repeat the argument with these numbers. We have reduced to a smaller case.

Now the formal proof:

Proof. Let $P(n)$ be the statement, "n is either prime or can be written as the product of primes." We will prove $P(n)$ is true for all $n \geq 2$.

[2]Technically, strong induction does not require you to prove a separate base case. This is because when proving the inductive case, you must show that $P(0)$ is true, assuming $P(k)$ is true for all $k < 0$. But this is not any help so you end up proving $P(0)$ anyway. To be on the safe side, we will always include the base case separately.

Base case: $P(2)$ is true because 2 is indeed prime.

Inductive case: assume $P(j)$ is true for all $j \leq k$. We want to show that $P(k+1)$ is true. That is, we want to show that $k+1$ is either prime or is the product of primes. If $k+1$ is prime, we are done. If not, then $k+1$ has more than 2 divisors, so we can write $k+1 = m_1 \cdot m_2$, with m_1 and m_2 less than $k+1$ (and greater than 1). By the inductive hypothesis, m_1 and m_2 are each either prime or can be written as the product of primes. In either case, we have that $m_1 \cdot m_2 = k+1$ can be written as the product of primes.

Thus by strong induction, $P(n)$ is true for all $n \geq 2$. ∎

Whether you use regular induction or strong induction depends on the statement you want to prove. If you wanted to be safe, you could always use strong induction. It really is *stronger*, so can accomplish everything "weak" induction can. That said, using regular induction is often easier since there is only one place you can use the induction hypothesis. There is also something to be said for *elegance* in proofs. If you can prove a statement using simpler tools, it is nice to do so.

4.6.3 Reading Questions

1. True or false: To prove the inductive case of a proof by strong induction, you should assume $P(k+1)$ is true and prove that $P(j)$ is true for all $j \leq k$.

2. Which of the following claims about the relationship between proof by induction and proof by strong induction are true?

A. Any proof by induction can be written as a proof by strong induction.

B. Any proof by strong induction can be written as a proof by induction.

C. Strong induction is "stronger" because the base case is stronger.

D. Strong induction is "stronger" because the inductive hypothesis is stronger.

3. What questions do you have? Write at least one question about the content of this section that you or a classmate might be curious about after reading this section.

4.6.4 Practice Problems

1. Suppose you are trying to prove, by strong induction, that a statement $P(n)$ is true for all $n \geq 0$. What would you attempt to prove in the *induction step* of the proof? (Select all that apply.)

A. That assuming $P(j)$ is true for all $j \leq k$, for an arbitrary $k \geq 0$, we can prove that $P(k+1)$ is true.

B. That $(P(0) \wedge P(1) \wedge \cdots \wedge P(k))$ implies $P(k+1)$ for all $k \geq 0$.

C. That assuming $P(k+1)$ is true for an arbitrary $k \geq 0$, we can prove that $P(j)$ is true for all $j \leq k$.

D. That $P(k+1)$ implies $P(j)$ for all $j \leq k$.

E. That $P(k-2)$ implies $P(k+1)$ for at least one $k \geq 0$.

2. A simpler version of the chocolate bar problem is as follows: Suppose you have a chocolate bar that is n squares long. You can break the chocolate bar into two pieces by making a single straight break across the bar. No matter where you make the breaks, you will break the chocolate bar into n pieces by making $n-1$ breaks.

Arrange some of the following statements in the correct order to form a proof of this claim by strong induction.

- Assume that $P(j)$ is true for all $j \leq k$ for an arbitrary $k \geq 1$.
- Therefore, by the principle of strong induction, $P(n)$ is true for all $n \geq 1$.
- $P(1)$ is true because a chocolate bar that is 1 square long is already in one piece.
- Consider a chocolate bar that is $k+1$ squares long.
- Let $P(n)$ be the statement that a chocolate bar that is n squares long can be broken into n pieces by making $n-1$ breaks.
- Anywhere you break this bar will result in two smaller bars, say of length a and b.
- The total number of breaks is therefore $a-1+b-1+1 = a+b-1$, which is $k+1-1 = k$.
- Since a and b are no more than k, it will be possible to break these smaller bars into single squares using $a-1$ and $b-1$ breaks, respectively.

4.6.5 ADDITIONAL EXERCISES

1. Suppose a football team only scores 3-point field goals and 7-point touchdowns (ignore the possibilities of safeties, missed extra points, and two-point conversions). Prove, using *strong* induction, that the team can get any number of points, 12 points or greater.

2. Prove using *strong* induction that the sum of the interior angles of a convex n-gon is $(n-2) \cdot 180°$. (A convex n-gon is a polygon with n sides for which each interior angle is less than $180°$.)

3. Prove that every positive integer is either a power of 2 or can be written as the sum of distinct powers of 2.

4. Prove, using strong induction, that every natural number is either a Fibonacci number or can be written as the *sum* of *distinct* Fibonacci numbers.

5. We have previously proved that for any tree, the number of edges is always one less than the number of vertices. That is, a tree with v vertices and e edges satisfies $v = e + 1$.

Give an alternate proof of this fact using strong induction on the number of vertices. Do so by taking a non-leaf vertex and "splitting" it into two vertices, each belonging to a separate tree.

6. Suppose that a particular real number x has the property that $x + \frac{1}{x}$ is an integer. Prove that $x^n + \frac{1}{x^n}$ is an integer for all natural numbers n.

7. Here is an example of a more complicated induction technique called **double induction**.

You will prove that the Fibonacci numbers satisfy the identity $F_n^2 + F_{n+1}^2 = F_{2n+1}$. One way to do this is to prove the more general identity,

$$F_m F_n + F_{m+1} F_{n+1} = F_{m+n+1},$$

and realize that when $m = n$ we get our desired result.

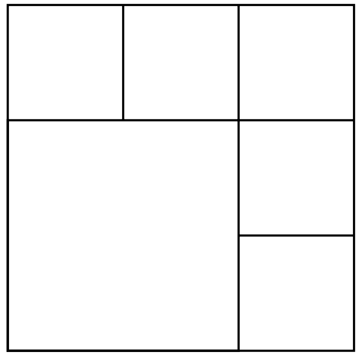
Note that we now have two variables, so we want to prove this for all $m \geq 0$ *and* all $n \geq 0$ at the same time. For each such pair (m, n), let $P(m, n)$ be the statement $F_m F_n + F_{m+1} F_{n+1} = F_{m+n+1}$

(a) First fix $m = 0$, and give a proof by mathematical induction that $P(0, n)$ holds for all $n \geq 0$. Note that this proof will be very easy.

(b) Now fix an arbitrary n, and give a proof by *strong* mathematical induction that $P(m, n)$ holds for all $m \geq 0$.

(c) You can now conclude that $P(m, n)$ holds for all $m, n \geq 0$. Do you believe that? Explain why this sort of induction is valid. For example, why do your proofs above guarantee that $P(2, 3)$ is true?

8. Given a square, you can cut the square into smaller squares by cutting along lines parallel to the sides of the original square (these lines do not need to travel the entire side length of the original square). For example, by cutting along the lines below, you will divide a square into 6 smaller squares:



Prove, using strong induction, that it is possible to cut a square into n smaller squares for any $n \geq 6$.

4.7 Chapter Summary

Investigate!

Each day your supply of magic chocolate-covered espresso beans doubles (each one splits in half), but then you eat 5 of them. You have 10 at the start of day 0.

1. Write out the first few terms of the sequence. Then give a recursive definition for the sequence, and explain how you know it is correct.

2. Prove, using induction, that the last digit of the number of beans you have on the nth day is always a 5 for all $n \geq 1$.

3. Find a closed formula for the nth term of the sequence, and prove it is correct by induction.

In this chapter, we explored sequences and mathematical induction. At first, these might not seem entirely related, but there is a link: recursive reasoning. When we have many cases (maybe infinitely many), it is often easier to describe a particular case by saying how it relates to other cases, instead of describing it from scratch. For sequences, we can describe the nth term in the sequence by saying how it is related to the *previous* term. When showing a statement involving the variable n is true for all values of n, we can describe why the case for $n = k$ is true based on why the case for $n = k - 1$ is true.

While thinking of problems recursively is often easier than thinking of them absolutely (at least after you get used to thinking in this way), our ultimate goal is to move beyond this recursive description. For sequences, we want to find *closed formulas* for the nth term of the sequence. For proofs, we want to know that the statement is true for a particular n (not only under the assumption that the statement is true for the previous value of n). In this chapter, we saw some methods for moving from recursive descriptions to absolute descriptions.

- If the terms of a sequence increase by a constant difference or constant ratio (these are both recursive descriptions), then the sequence is arithmetic or geometric, respectively, and we have closed formulas for each of these based on the initial terms and common difference or ratio.

- If the terms of a sequence increase at a polynomial rate (that is, if the differences between terms form a sequence with a polynomial closed formula), then the sequence is given by a polynomial closed formula (of degree one more than the sequence of differences).

- If the terms of a sequence increase at an exponential rate, then we expect the closed formula for the sequence to be exponential. These sequences often have relatively nice recursive formulas, and the *characteristic root technique* allows us to find the closed formula for these sequences.

- If we want to prove that a statement is true for all values of n (greater than some first small value), and we can describe why the statement for $n = k$ implies the statement for $n = k + 1$, then the *principle of mathematical induction* gives us that the statement is true for all values of n (greater than the base case).

Throughout the chapter we tried to understand *why* these facts listed above are true. In part, that is what proofs, by induction or not, attempt to accomplish: They explain why mathematical truths are, in fact, truths. As we develop our ability to reason about mathematics, it is a good idea to make sure that the methods of our reasoning are sound. The branch of mathematics that deals with deciding whether reasoning is good or not is *mathematical logic*, the subject of the next chapter.

Chapter Review

1. Find $7 + 13 + 19 + \cdots + 1243$.

2. Consider the sequence $28, 38, 48, 58, \ldots, 10n - 2$.

a. How many terms are there in the sequence?

b. What is the second-to-last term?

c. Find the sum of all the terms in the sequence.

3. Consider the sequence given by $a_n = 6 \cdot 5^{n-1}$.

a. Find the first 4 terms of the sequence.

$a_1 =$ ___, $a_2 =$ ___, $a_3 =$ ___, $a_4 =$ ___ ,...

What sort of sequence is this?

(□ arithmetic □ geometric □ neither)

b. Find the *sum* of the first 22 terms. That is, compute $\sum_{k=1}^{22} a_k$.

4. Consider the sequence $5, 11, 19, 29, 41, 55, \ldots$. Assume $a_1 = 5$.

(a) Find a closed formula for a_n, the nth term of the sequence, by writing each term as a sum of a sequence. Hint: first find a_0, but ignore it when collapsing the sum.

(b) Find a closed formula again, this time using either polynomial fitting or the characteristic root technique (whichever is appropriate). Show your work.

(c) Find a closed formula once again, this time by recognizing the sequence as a modification of some well-known sequence(s). Explain.

5. Use polynomial fitting to find the formula for the nth term of the sequence $(a_n)_{n\geq 1}$ which starts,

$$11, 22, 35, 50, 67, \ldots$$

Note the first term above is a_1, not a_0.

$a_n =$ ________

6. Suppose the closed formula for a particular sequence is a degree 3 polynomial. What can you say about the closed formula for:

(a) The sequence of partial sums?

(b) The sequence of second differences?

7. Consider the sequence given recursively by $a_1 = 4$, $a_2 = 6$, and $a_n = a_{n-1} + a_{n-2}$.

(a) Write out the first 6 terms of the sequence.

(b) Could the closed formula for a_n be a polynomial? Explain.

8. The sequence $(a_n)_{n\geq 1}$ starts $-1, 0, 2, 5, 9, 14 \ldots$ and has closed formula

$$a_n = \frac{(n+1)(n-2)}{2}.$$

Use this fact to find a closed formula for the sequence $(b_n)_{n\geq 1}$ which starts $4, 10, 18, 28, 40, \ldots$.

9. In the song *The Twelve Days of Christmas*, my true love gave to me first 1 gift; then 2 gifts and 1 gift; then 3 gifts, 2 gifts, and 1 gift; and so on. How many gifts did my true love give me all together during the twelve days?

10. Consider the recurrence relation $a_n = a_{n-1} + 20a_{n-2}$ with first two terms $a_0 = 4$ and $a_1 = 9$.

a. Write out the first 5 terms of the sequence defined by this recurrence relation.

$a_2 =$ ____, $a_3 =$ ____, $a_4 =$ ____, ...

b. Solve the recurrence relation. That is, find a closed formula for a_n.

$a_n =$ __________

11. Consider the recurrence relation $a_n = 3a_{n-1} + 18a_{n-2}$ with first two terms $a_0 = 5$ and $a_1 = 7$.

a. Find the next two terms of the sequence (a_2 and a_3):

$a_2 =$ ____

$a_3 =$ ____

b. Solve the recurrence relation. That is, find a closed formula for a_n.

$a_n =$ __________

12. Your magic chocolate bunnies reproduce like rabbits: every large bunny produces 2 new mini bunnies each day, and each day every mini bunny born the previous day grows into a large bunny. Assume you start with 2 mini bunnies and no bunny ever dies (or gets eaten).

(a) Write out the first few terms of the sequence.

(b) Give a recursive definition of the sequence, and explain why it is correct.

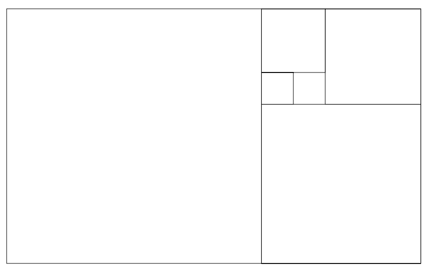
(c) Find a closed formula for the nth term of the sequence.

13. Consider the sequence of partial sums of *squares* of Fibonacci numbers: F_1^2, $F_1^2 + F_2^2$, $F_1^2 + F_2^2 + F_3^2$, ... The sequence starts $1, 2, 6, 15, 40, \ldots$

(a) Guess a formula for the nth partial sum, in terms of Fibonacci numbers. Hint: Write each term as a product.

(b) Prove your formula is correct by mathematical induction.

(c) Explain what this problem has to do with the following picture:



14. Prove the following statements by mathematical induction:

(a) $n! < n^n$ for $n \geq 2$

(b) $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n \cdot (n+1)} = \dfrac{n}{n+1}$ for all $n \in \mathbb{Z}^+$.

(c) $4^n - 1$ is a multiple of 3 for all $n \in \mathbb{N}$.

(d) The *greatest* amount of postage you *cannot* make exactly using 4 and 9 cent stamps is 23 cents.

(e) Every even number squared is divisible by 4.

15. Prove $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ holds for all $n \geq 1$, by mathematical induction.

16. Suppose $a_0 = 1$, $a_1 = 1$ and $a_n = 3a_{n-1} - 2a_{n-2}$. Prove, using strong induction, that $a_n = 1$ for all n.

17. Prove using induction that every set containing n elements has 2^n different subsets for any $n \geq 1$.

Discrete Structures Revisited

Throughout the previous chapters, we have seen many examples of discrete structures and their properties and uses. The following sections are self-contained overviews of each of the main structures we have explored. These can either be studied prior to other material in the book or as a review and reference after the fact.

5.1 Sets

Note that this section contains a lot of detail on what we can say about sets. If you have never seen anything about unions or Venn diagrams, it is worth reading this carefully.

The most fundamental objects we will use in our studies (and really in all of math) are *sets*. Much of what follows might be review, but it is very important that you are fluent in the language of set theory. Most of the notation we use below is standard, although some might be a little different than what you have seen before.

For us, a **set** will simply be an unordered collection of objects. Two examples: We could consider the set of all actors who have played *The Doctor* on *Doctor Who*, or the set of natural numbers between 1 and 10 inclusive. In the first case, Tom Baker is an element (or member) of the set, while Idris Elba, among many others, is not an element of the set. Also, the two examples are of different sets. Two sets are equal exactly if they contain the exact same elements. For example, the set containing all of the vowels in the Declaration of Independence is precisely the same set as the set of vowels in the word "questionably" (namely, all of them); we do not care about order or repetitions, just whether the element is in the set or not.

5.1.1 Notation

We need some notation to make talking about sets easier. Consider,

$$A = \{1, 2, 3\}.$$

This is read, "A is the set containing the elements 1, 2, and 3." We use curly braces "$\{,\ \}$" to enclose elements of a set. Some more notation:

$$a \in \{a, b, c\}.$$

The symbol "$\in$" is read "is in" or "is an element of." Thus the above means that a is an element of the set containing the letters a, b, and c. Note that this is a true statement. It would also be true to say that d is not in that set:

$$d \notin \{a, b, c\}.$$

Be warned: We write "$x \in A$" when we wish to express that one of the elements of the set A is x. For example, consider the set,

$$A = \{1, b, \{x, y, z\}, \emptyset\}.$$

This is a strange set, to be sure. It contains four elements: the number 1, the letter b, the set $\{x, y, z\}$, and the empty set $\emptyset = \{\}$, the set containing no elements. Is x in A? The answer is no. None of the four elements in A are the letter x, so we must conclude that $x \notin A$. Similarly, consider the set $B = \{1, b\}$. Even though the elements of B are elements of A, we cannot say that the *set* B is one of the elements of A. Therefore $B \notin A$. (Soon we will see that B is a *subset* of A, but this is different from being an *element* of A.)

We have described the sets above by listing their elements. Sometimes this is hard to do, especially when there are a lot of elements in the set (perhaps infinitely many). For instance, if we want A to be the set of all even natural numbers, would could write,

$$A = \{0, 2, 4, 6, \ldots\},$$

but this is a little imprecise. A better way would be

$$A = \{x \in \mathbb{N} : \text{ there exists } n \in \mathbb{N} \text{ such that } x = 2n\}.$$

Let's look at this carefully. First, there are some new symbols to digest: "$\mathbb{N}$" is the symbol usually used to denote the **natural numbers**, which we will take to be the set $\{0, 1, 2, 3, \ldots\}$. Next, the colon, ":", is read *such that*; it separates the elements that are in the set from the condition that the elements in the set must satisfy. So putting this all together, we would read the set as, "the set of all x in the natural numbers, such that there exists some n in the natural numbers for which x is twice n." In other words, the set of all natural numbers that are even. Here is another way to write the same set.

$$A = \{x \in \mathbb{N} : x \text{ is even}\}.$$

Note: Sometimes mathematicians use | or ∍ for the "such that" symbol instead of the colon. Also, there is a fairly even split between mathematicians about whether 0 is an element of the natural numbers, so be careful there.

This notation is usually called **set builder notation**. It tells us how to *build* a set by telling us precisely the condition elements must meet to gain access (the condition is the logical statement after the ":" symbol). Reading and comprehending sets written in this way takes practice. Here are some more examples:

Example 5.1.1

Describe each of the following sets both in words and by listing out enough elements to see the pattern.

1. $\{x : x + 3 \in \mathbb{N}\}$.
2. $\{x \in \mathbb{N} : x + 3 \in \mathbb{N}\}$.

3. $\{x : x \in \mathbb{N} \text{ or } -x \in \mathbb{N}\}$.

4. $\{x : x \in \mathbb{N} \text{ and } -x \in \mathbb{N}\}$.

Solution.

1. This is the set of all numbers that are 3 less than a natural number (i.e., that if you add 3 to them, you get a natural number). The set could also be written as $\{-3, -2, -1, 0, 1, 2, \ldots\}$ (note that 0 is a natural number, so -3 is in this set because $-3 + 3 = 0$).

2. This is the set of all natural numbers that are 3 less than a natural number. So here we just have $\{0, 1, 2, 3 \ldots\}$.

3. This is the set of all integers (positive and negative whole numbers, written $\mathbb{Z}$). In other words, $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

4. Here we want all numbers x such that x and $-x$ are natural numbers. There is only one: 0. So we have the set $\{0\}$.

There is also a subtle variation on set builder notation. While the condition is generally given after the "such that", sometimes it is hidden in the first part. Here is an example.

Example 5.1.2

List a few elements in the sets below and describe them in words. The set $\mathbb{Z}$ is the set of **integers**; positive and negative whole numbers.

1. $A = \{x \in \mathbb{Z} : x^2 \in \mathbb{N}\}$

2. $B = \{x^2 : x \in \mathbb{N}\}$

Solution.

1. The set of integers that pass the condition that their square is a natural number. Well, every integer, when you square it, gives you a non-negative integer, so a natural number. Thus $A = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$.

2. Here we are looking for the set of all x^2s where x is a natural number. So this set is simply the set of perfect squares. $B = \{0, 1, 4, 9, 16, \ldots\}$.

   Another way we could have written this set, using more strict set builder notation, would be as $B = \{x \in \mathbb{N} : x = n^2 \text{ for some } n \in \mathbb{N}\}$.

We already have a lot of notation, and there is more yet. Below is a handy chart of symbols. Some of these will be discussed in greater detail as we move forward.

Special sets.

$\emptyset$	The **empty set** is the set that contains no elements.
$\mathbb{N}$	The set of natural numbers. That is, $\mathbb{N} = \{0, 1, 2, 3 \ldots\}$.
$\mathbb{Z}$	The set of integers. That is, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$.
$\mathbb{Q}$	The set of rational numbers.
$\mathbb{R}$	The set of real numbers.
$\mathcal{P}(A)$	The **power set** of any set A is the set of all subsets of A.

Set Theory Notation.

$\{,\}$	We use these **braces** to enclose the elements of a set. So $\{1, 2, 3\}$ is the set containing 1, 2, and 3.
$:$	$\{x : x > 2\}$ is the set of all x **such that** x is greater than 2.
$\in$	$2 \in \{1, 2, 3\}$ asserts that 2 is **an element of** the set $\{1, 2, 3\}$.
$\notin$	$4 \notin \{1, 2, 3\}$ because 4 **is not an element of** the set $\{1, 2, 3\}$.
$\subseteq$	$A \subseteq B$ asserts that A **is a subset of** B: Every element of A is also an element of B.
$\subset$	$A \subset B$ asserts that A **is a proper subset of** B: Every element of A is also an element of B, but $A \neq B$.
$\cap$	$A \cap B$ is the **intersection of** A **and** B: the set containing all elements that are elements of both A and B.
$\cup$	$A \cup B$ is the **union of** A **and** B: the set containing all elements that are elements of A or B or both.
$\times$	$A \times B$ is the **Cartesian product of** A **and** B: the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$.
$\setminus$	$A \setminus B$ is **set difference between** A **and** B: the set containing all elements of A that are not elements of B.
$\overline{A}$	The **complement of** A is the set of everything that is not an element of A.
$\lvert A\rvert$	The **cardinality (or size) of** A is the number of elements in A.

Investigate!

1. Find the cardinality of each set below.

(a) $A = \{3, 4, \ldots, 15\}$.

(b) $B = \{n \in \mathbb{N} : 2 < n \leq 200\}$.

(c) $C = \{n \leq 100 : n \in \mathbb{N} \text{ and for some } m \in \mathbb{N}, (n = 2m + 1)\}$.

2. Find two sets A and B for which $|A| = 5$, $|B| = 6$, and $|A \cup B| = 9$. What is $|A \cap B|$?

3. Find sets A and B with $|A| = |B|$ such that $|A \cup B| = 7$ and $|A \cap B| = 3$. What is $|A|$?

4. Let $A = \{1, 2, \ldots, 10\}$. Define $\mathcal{B}_2 = \{B \subseteq A : |B| = 2\}$. Find $|\mathcal{B}_2|$.

5. For any sets A and B, define $AB = \{ab : a \in A \text{ and } b \in B\}$. If $A = \{1, 2\}$ and $B = \{2, 3, 4\}$, what is $|AB|$? What is $|A \times B|$?

5.1.2 Relationships between Sets

We have already said what it means for two sets to be equal: They have exactly the same elements. Thus, for example,

$$\{1, 2, 3\} = \{2, 1, 3\}.$$

(Remember, the order the elements are written down in does not matter.) Also,

$$\{1, 2, 3\} = \{1, 1 + 1, 1 + 1 + 1\} = \{I, II, III\} = \{1, 2, 3, 1 + 2\}$$

since these are all ways to write the set containing the first three positive integers (how we write them doesn't matter, just what they are).

What about the sets $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$? Clearly $A \neq B$, but notice that every element of A is also an element of B. Because of this we say that A is a *subset* of B, or in symbols, $A \subset B$ or $A \subseteq B$. Both symbols are read "is a subset of." The difference is that sometimes we want to say that A is either equal to or is a subset of B, in which case we use $\subseteq$. This is analogous to the difference between $<$ and $\leq$.

Example 5.1.3

Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6\}$, $C = \{1, 2, 3\}$, and $D = \{7, 8, 9\}$. Determine which of the following are true, false, or meaningless.

1. $A \subset B$.
2. $B \subset A$.
3. $B \in C$.
4. $\emptyset \in A$.
5. $\emptyset \subset A$.
6. $A < D$.
7. $3 \in C$.
8. $3 \subset C$.
9. $\{3\} \subset C$.

Solution.

1. False. For example, $1 \in A$ but $1 \notin B$.
2. True. Every element in B is an element in A.
3. False. The elements in C are 1, 2, and 3. The *set* B is not equal to 1, 2, or 3.
4. False. A has exactly 6 elements, and none of them are the empty set.
5. True. Everything in the empty set (nothing) is also an element of A. Notice that the empty set is a subset of every set.
6. Meaningless. A set cannot be less than another set.
7. True. 3 is one of the elements of the set C.
8. Meaningless. 3 is not a set, so it cannot be a subset of another set.
9. True. 3 is the only element of the set $\{3\}$, and is an element of C, so every element in $\{3\}$ is an element of C.

In the example above, B is a subset of A. You might wonder what other sets are subsets of A. If you collect all these subsets of A into a new set, we get a set of sets. We call the set of all subsets of A the **power set** of A, and write it $\mathcal{P}(A)$.

Example 5.1.4

Let $A = \{1, 2, 3\}$. Find $\mathcal{P}(A)$.

Solution. $\mathcal{P}(A)$ is a set of sets, all of which are subsets of A. So

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

Notice that while $2 \in A$, it is wrong to write $2 \in \mathcal{P}(A)$ since none of the elements in $\mathcal{P}(A)$ are numbers! On the other hand, we do have $\{2\} \in \mathcal{P}(A)$ because $\{2\} \subseteq A$.

What does a subset of $\mathcal{P}(A)$ look like? Notice that $\{2\} \nsubseteq \mathcal{P}(A)$ because not everything in $\{2\}$ is in $\mathcal{P}(A)$. But we do have $\{\{2\}\} \subseteq \mathcal{P}(A)$. The only element of $\{\{2\}\}$ is the set $\{2\}$, which is also an element of $\mathcal{P}(A)$. We could take the collection of all subsets of $\mathcal{P}(A)$ and call that $\mathcal{P}(\mathcal{P}(A))$. Or even the power set of that set of sets of sets.

Another way to compare sets is by their *size*. Notice that in the example above, A has 6 elements, and B, C, and D all have 3 elements. The size of a set is called the set's **cardinality**. We would write $|A| = 6$, $|B| = 3$, and so on. For sets that have a finite number of elements, the cardinality of the set is simply the number of elements in the set. Note that the cardinality of $\{1, 2, 3, 2, 1\}$ is 3. We do not count repeats (in fact, $\{1, 2, 3, 2, 1\}$ is exactly the same set as $\{1, 2, 3\}$). There are sets with infinite cardinality, such as $\mathbb{N}$, the set of rational numbers (written $\mathbb{Q}$), the set of even natural numbers, and the set of real numbers ($\mathbb{R}$). It is possible to distinguish between different infinite cardinalities, but that is beyond the scope of this text. For us, a set will either be infinite or finite; if it is finite, then we can determine its cardinality by counting elements.

Example 5.1.5

1. Find the cardinality of $A = \{23, 24, \ldots, 37, 38\}$.
2. Find the cardinality of $B = \{1, \{2, 3, 4\}, \emptyset\}$.
3. If $C = \{1, 2, 3\}$, what is the cardinality of $\mathcal{P}(C)$?

Solution.

1. Since $38 - 23 = 15$, we can conclude that the cardinality of the set is $|A| = 16$ (you need to add one since 23 is included).
2. Here $|B| = 3$. The three elements are the number 1, the set $\{2, 3, 4\}$, and the empty set.
3. We wrote out the elements of the power set $\mathcal{P}(C)$ above, and there are 8 elements (each of which is a set). So $|\mathcal{P}(C)| = 8$. (You might wonder if there is a relationship between $|A|$ and $|\mathcal{P}(A)|$ for all sets A. This is a good question that we explore in Chapter 3.)

5.1.3 Operations on Sets

Is it possible to add two sets? Not really, however there is something similar. If we want to combine two sets to get the collection of objects that are in either set, then we can take the **union** of the two sets. Symbolically,

$$C = A \cup B,$$

read, "C is the union of A and B," means that the elements of C are exactly the elements that are either an element of A or an element of B (or an element of both). For example, if $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, then $A \cup B = \{1, 2, 3, 4\}$.

The other common operation on sets is **intersection**. We write,

$$C = A \cap B$$

and say, "C is the intersection of A and B," when the elements in C are precisely those both in A and in B. So if $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, then $A \cap B = \{2, 3\}$.

Often when dealing with sets, we will have some understanding as to what "everything" is. Perhaps we are only concerned with natural numbers. In this case we would say that our **universe** is $\mathbb{N}$. Sometimes we denote this universe by $\mathcal{U}$.

Given this context, we might wish to speak of all the elements that are *not* in a particular set. We say B is the **complement** of A, and write,

$$B = \overline{A}$$

when B contains every element not contained in A. So, if our universe is $\{1, 2, \ldots, 9, 10\}$, and $A = \{2, 3, 5, 7\}$, then $\overline{A} = \{1, 4, 6, 8, 9, 10\}$.

Of course we can perform more than one operation at a time. For example, consider

$$A \cap \overline{B}.$$

This is the set of all elements that are both elements of A and not elements of B. What have we done? We've started with A and removed all of the elements that were in B. Another way to write this is the **set difference**:

$$A \cap \overline{B} = A \setminus B.$$

It is important to remember that these operations (union, intersection, complement, and difference) on sets produce other sets. Don't confuse these with the symbols from the previous section (element of and subset of). $A \cap B$ is a set, while $A \subseteq B$ is true or false. This is the same difference as between $3 + 2$ (which is a number) and $3 \le 2$ (which is false).

Example 5.1.6

Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6\}$, $C = \{1, 2, 3\}$, and $D = \{7, 8, 9\}$. If the universe is $\mathcal{U} = \{1, 2, \ldots, 10\}$, find:

1. $A \cup B$.
2. $A \cap B$.
3. $B \cap C$.
4. $A \cap D$.
5. $\overline{B \cup C}$.
6. $A \setminus B$.
7. $(D \cap \overline{C}) \cup \overline{A \cap B}$.
8. $\emptyset \cup C$.
9. $\emptyset \cap C$.

Solution.

1. $A \cup B = \{1, 2, 3, 4, 5, 6\} = A$ since everything in B is already in A.
2. $A \cap B = \{2, 4, 6\} = B$ since everything in B is in A.
3. $B \cap C = \{2\}$ as the only element of both B and C is 2.
4. $A \cap D = \emptyset$ since A and D have no common elements.
5. $\overline{B \cup C} = \{5, 7, 8, 9, 10\}$. First we find that $B \cup C = \{1, 2, 3, 4, 6\}$, and then we take everything not in that set.

6. $A \setminus B = \{1, 3, 5\}$ since the elements 1, 3, and 5 are in A but not in B. This is the same as $A \cap \overline{B}$.

7. $(D \cap \overline{C}) \cup \overline{A \cap B} = \{1, 3, 5, 7, 8, 9, 10\}$. The set contains all elements that are either in D but not in C (i.e., $\{7, 8, 9\}$), or not in both A and B (i.e., $\{1, 3, 5, 7, 8, 9, 10\}$).

8. $\emptyset \cup C = C$ since nothing is added by the empty set.

9. $\emptyset \cap C = \emptyset$ since nothing can be both in a set and in the empty set.

Having notation like this is useful. We will often want to add or remove elements from sets, and our notation allows us to do so precisely.

Example 5.1.7

If $A = \{1, 2, 3\}$, then we can describe the set we get by adding the number 4 as $A \cup \{4\}$. If we want to express the set we get by removing the number 2 from A, we can do so by writing $A \setminus \{2\}$.

Careful though. If you add an element to the set, you get a new set! So you would have $B = A \cup \{4\}$ and then correctly say that B contains 4, but A does not.

There is one more way to combine sets that will be useful for us: the **Cartesian product**, $A \times B$. This sounds fancy but is nothing you haven't seen before. When you graph a function in calculus, you graph it in the Cartesian plane. This is the set of all ordered pairs of real numbers (x, y). We can do this for *any* pair of sets, not just the real numbers with themselves.

Put another way, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$. The first coordinate comes from the first set, and the second coordinate comes from the second set. Sometimes we will want to take the Cartesian product of a set with itself, and this is fine: $A \times A = \{(a, b) : a, b \in A\}$ (we might also write A^2 for this set). Notice that in $A \times A$, we still want *all* ordered pairs, not just the ones where the first and second coordinate are the same. We can also take products of 3 or more sets, getting ordered triples, or quadruples, and so on.

Example 5.1.8

Let $A = \{1, 2\}$ and $B = \{3, 4, 5\}$. Find $A \times B$ and $A \times A$. How many elements do you expect to be in $B \times B$?

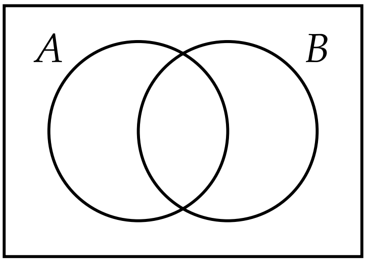
Solution. $A \times B = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)\}$.

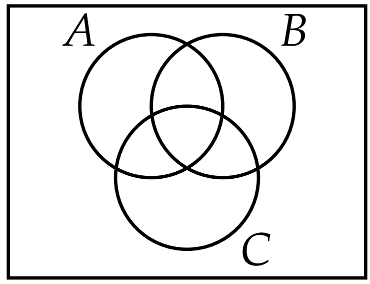
$A \times A = A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.

$|B \times B| = 9$. There will be 3 pairs with first coordinate 3, three more with first coordinate 4, and a final three with first coordinate 5.

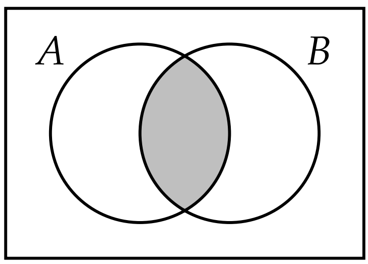
5.1.4 Venn Diagrams

There is a very nice visual tool we can use to represent operations on sets. A **Venn diagram** displays sets as intersecting circles. We can shade the region we are talking about when we carry out an operation. We can also represent the cardinality of a particular set by putting the number in the corresponding region.

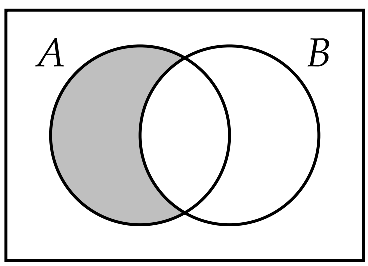




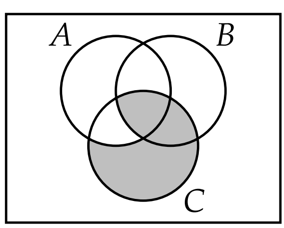
Each circle represents a set. The rectangle containing the circles represents the universe. To represent combinations of these sets, we shade the corresponding region. For example, we could draw $A \cap B$ as:



Here is a representation of $A \cap \overline{B}$, or equivalently $A \setminus B$:



A more complicated example is $(B \cap C) \cup (C \cap \overline{A})$, as seen below.



Notice that the shaded regions above could also be arrived at in another way.

We could have started with all of C and then excluded the region where C and A overlap outside of B. That region is $(A \cap C) \cap \overline{B}$. So the above Venn diagram also represents $C \cap \overline{\left((A \cap C) \cap \overline{B}\right)}$. So using just the picture, we have determined that

$$(B \cap C) \cup (C \cap \overline{A}) = C \cap \overline{\left((A \cap C) \cap \overline{B}\right)}.$$

5.1.5 Exercises

1. $A = \{2, 4, 5, 6, 7\}$ and $B = \{2, 4, 5, 6, 8\}$ Find each of the following sets. Your answers should include the curly braces

a. $A \cup B$.

b. $A \cap B$.

c. $A \setminus B$.

d. $B \setminus A$.

2. Find the least element of the following sets, if there is one.

a. $\{n \in \mathbb{N} : n^2 - 3 \geq 4\}$

b. $\{n \in \mathbb{N} : n^2 - 7 \in \mathbb{N}\}$

c. $\{n^2 + 4 : n \in \mathbb{N}\}$

d. $\{n \in \mathbb{N} : n = k^2 + 4 \text{ for some } k \in \mathbb{N}\}$

3. Find the following cardinalities.

a. $|A|$ when $A = \{4, 5, 6, 7, \ldots, 33\}$.

b. $|A|$ when $A = \{x \in \mathbb{Z} : -7 \leq x \leq 91''\}$.

c. $|A \cap B|$ when $A = \{x \in \mathbb{N} : x \leq 27\}$ and $B = \{x \in \mathbb{N} : x \text{ is prime }\}$.

4. Let $A = \{8, 10, 11, 12, 13\}$ and $B = \{8, 16, 17, 24, 32\}$. Find a set of largest possible size that is a subset of both A and B.

5. Find a set of smallest possible size that has both {4,6,7,9} and {5,6,7,9,10} as subsets.

6. Let $A = \{n \in \mathbb{N} : 39 \leq n < 55\}$ and $B = \{n \in \mathbb{N} : 16 < n \leq 44\}$. Suppose C is a set such that $C \subseteq A$ and $C \subseteq B$. What is the largest possible cardinality of C?

7. Let $A = \{1, 4, 7, 12, 14\}$ and $B = \{1, 4, 14\}$. How many sets C have the property that $C \subseteq A$ and $B \subseteq C$.

8. Let $A = \{2, 3, 4, 5, 6\}$, $B = \{4, 5, 6, 7, 8\}$, and $C = \{3, 6, 8\}$.

a. Find $A \cap B$.

b. Find $A \cup B$.

c. Find $A \setminus B$.

d. Find $A \cap \overline{(B \cup C)}$.

9. Let $A = \{x \in \mathbb{N} : 2 \leq x < 14\}$ and $B = \{x \in \mathbb{N} : x \text{ is even}\}$.

a. Find $A \cap B$.

b. Find $A \setminus B$.

10. Let $A = \{x \in \mathbb{N} : 3 \leq x \leq 13\}$, $B = \{x \in \mathbb{N} : x \text{ is even}\}$, and $C = \{x \in \mathbb{N} : x \text{ is odd}\}$.

(a) Find $A \cap B$.

(b) Find $A \cup B$.

(c) Find $B \cap C$.

(d) Find $B \cup C$.

11. Find an example of sets A and B such that $A \cap B = \{3, 5\}$ and $A \cup B = \{2, 3, 5, 7, 8\}$.

12. Find an example of sets A and B such that $A \subseteq B$ and $A \in B$.

13. Recall $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (the integers). Let $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ be the positive integers. Let $2\mathbb{Z}$ be the even integers, $3\mathbb{Z}$ be the multiples of 3, and so on.

(a) Is $\mathbb{Z}^+ \subseteq 2\mathbb{Z}$? Explain.

(b) Is $2\mathbb{Z} \subseteq \mathbb{Z}^+$? Explain.

(c) Find $2\mathbb{Z} \cap 3\mathbb{Z}$. Describe the set in words, and using set notation.

(d) Express $\{x \in \mathbb{Z} : \exists y \in \mathbb{Z}(x = 2y \vee x = 3y)\}$ as a union or intersection of two sets already described in this problem.

14. Let A_2 be the set of all multiples of 2 except for 2. Let A_3 be the set of all multiples of 3 except for 3. And so on, so that A_n is the set of all multiples of n except for n, for any $n \geq 2$. Describe (in words) the set $\overline{A_2 \cup A_3 \cup A_4 \cup \cdots}$.

15. Draw a Venn diagram to represent each of the following:

(a) $A \cup \overline{B}$

(b) $\overline{(A \cup B)}$

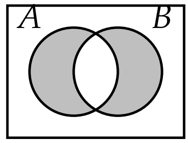
(c) $A \cap (B \cup C)$

(d) $(A \cap B) \cup C$

(e) $\overline{A} \cap B \cap \overline{C}$

(f) $(A \cup B) \setminus C$

16. Describe a set in terms of A and B (using set notation) which has the following Venn diagram:



17. Let $A = \{a, b, c, d\}$. Find $\mathcal{P}(A)$.

18. Let $A = \{1, 2, \ldots, 9\}$. How many subsets of A contain exactly one element (i.e., how many singleton subsets are there)?

How many doubleton subsets (containing exactly two elements) are there?

19. Let $A = \{1, 2, 3, 4, 5, 6\}$. Find all sets $B \in \mathcal{P}(A)$ which have the property $\{2, 3, 5\} \subseteq B$.

20. Find an example of sets A and B such that $|A| = 4$, $|B| = 5$, and $|A \cup B| = 9$.

21. Find an example of sets A and B such that $|A| = 3$, $|B| = 4$, and $|A \cup B| = 5$.

22. Are there sets A and B such that $|A| = |B|$, $|A \cup B| = 10$, and $|A \cap B| = 5$? Explain.

23. Let $A = \{2, 4, 6, 8\}$. Suppose B is a set with $|B| = 5$.

a. What are the smallest and largest possible values of $|A \cup B|$? Explain.

b. What are the smallest and largest possible values of $|A \cap B|$? Explain.

c. What are the smallest and largest possible values of $|A \times B|$? Explain.

24. Let $X = \{n \in \mathbb{N} : 10 \leq n < 20\}$. Find examples of sets with the properties below and very briefly explain why your examples work.

(a) A set $A \subseteq \mathbb{N}$ with $|A| = 10$ such that $X \setminus A = \{10, 12, 14\}$.

(b) A set $B \in \mathcal{P}(X)$ with $|B| = 5$.

(c) A set $C \subseteq \mathcal{P}(X)$ with $|C| = 5$.

(d) A set $D \subseteq X \times X$ with $|D| = 5$

(e) A set $E \subseteq X$ such that $|E| \in E$.

25. Let A, B, and C be sets.

(a) Suppose that $A \subseteq B$ and $B \subseteq C$. Does this mean that $A \subseteq C$? Prove your answer. Hint: To prove that $A \subseteq C$, you must prove the implication, "For all x, if $x \in A$, then $x \in C$."

(b) Suppose that $A \in B$ and $B \in C$. Does this mean that $A \in C$? Give an example to prove that this does NOT always happen (and explain why your example works). You should be able to give an example where $|A| = |B| = |C| = 2$.

26. In a regular deck of playing cards there are 26 red cards and 12 face cards. Explain, using sets and what you have learned about cardinalities, why there are only 32 cards which are either red or a face card.

27. Find an example of a set A with $|A| = 3$ which contains only other sets and has the following property: For all sets $B \in A$, we also have $B \subseteq A$. Explain why your example works. (FYI: Sets that have this property are called **transitive**.)

28. Consider the sets A and B, where $A = \{3, |B|\}$ and $B = \{1, |A|, |B|\}$. What are the sets?

29. Explain why there is no set A which satisfies $A = \{2, |A|\}$.

30. Find all sets A, B, and C which satisfy the following.

$$\begin{aligned} A &= \{1, |B|, |C|\} \\ B &= \{2, |A|, |C|\} \\ C &= \{1, 2, |A|, |B|\} \end{aligned}$$

.

5.2 Functions

This section contains many details about functions, much of which should be familiar from non-discrete contexts. As you read, consider how the way we talk about functions here might be different from what you know from other areas of mathematics.

A **function** is a rule that assigns each input exactly one output. We call the output the **image** of the input. The set of all inputs for a function is called the **domain**. The set of all allowable outputs is called the **codomain**. We would write $f : X \to Y$ to describe a function with name f, domain X, and codomain Y. This does not tell us *which* function f is though. To define the function, we must describe the rule. This is often done by giving a formula to compute the output for any input (although this is certainly not the only way to describe the rule).

For example, consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x^2 + 3$. Here the domain and codomain are the same set (the natural numbers). The rule: Take your input, multiply it by itself, and add 3. This works because we can apply this rule to every natural number (every element of the domain) and the result is always a natural number (an element of the codomain). Notice though that not every natural number is actually an output (there is no way to get 0, 1, 2, 5, etc.). The set of natural numbers that *are* outputs is called the **range** of the function (in this case, the range is $\{3, 4, 7, 12, 19, 28, \ldots\}$, all the natural numbers that are 3 more than a perfect square).

The key thing that makes a rule a *function* is that there is *exactly one* output for each input. That is, it is important that the rule be a good rule. What output do we assign to the input 7? There can only be one answer for any particular function.

Example 5.2.1

The following are all examples of functions:

1. $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 3n$. The domain and codomain are both the set of integers. However, the range is only the set of integer multiples of 3.

2. $g : \{1, 2, 3\} \to \{a, b, c\}$ defined by $g(1) = c$, $g(2) = a$, and $g(3) = a$. The domain is the set $\{1, 2, 3\}$, the codomain is the set $\{a, b, c\}$ and the range is the set $\{a, c\}$. Note that $g(2)$ and $g(3)$ are the same element of the codomain. This is okay since each element in the domain still has only one output.

3. $h : \{1, 2, 3, 4\} \to \mathbb{N}$ defined by the table:

x	1	2	3	4
$h(x)$	3	6	9	12

Here the domain is the finite set $\{1, 2, 3, 4\}$, and the codomain is the set of natural numbers, $\mathbb{N}$. At first you might think this function is the same as f defined above. It absolutely is not. Even though the rule is the same, the domain and codomain are different, so these are two different functions.

Example 5.2.2

Just because you can describe a rule in the same way you would write a function does not mean that the rule is a function. The following are NOT functions.

1. $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = \frac{n}{2}$. The reason this is not a function is because not every input has an output. Where does f send 3? The rule says that $f(3) = \frac{3}{2}$, but $\frac{3}{2}$ is not an element of the codomain.

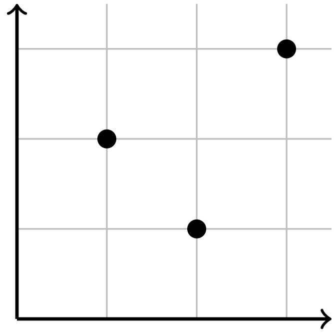
2. Consider the rule that matches each person to their phone number. If you think of the set of people as the domain and the set of phone numbers as the codomain, then this is not a function, since some people have two phone numbers. Switching the domain and codomain sets doesn't help either, since some phone numbers belong to multiple people (assuming some households still have landlines when you are reading this).

5.2.1 Describing Functions

It is worth making a distinction between a function and its description. The function is the abstract mathematical object that in some way exists whether or not anyone ever talks about it. But when we *do* want to talk about the function, we need a way to describe it. A particular function can be described in multiple ways.

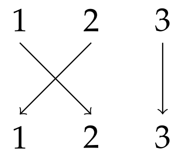
Some calculus textbooks talk about the *Rule of Four*, that every function can be described in four ways: algebraically (a formula), numerically (a table), graphically, or in words. In discrete math, we can still use any of these to describe functions, but we can also be more specific since we are primarily concerned with functions that have $\mathbb{N}$ or a finite subset of $\mathbb{N}$ as their domain.

Describing a function graphically usually means drawing the graph of the function: plotting the points on the plane. We can do this and might get a graph like the following for a function $f : \{1, 2, 3\} \to \{1, 2, 3\}$.



It would be absolutely WRONG to connect the dots or try to fit them to some curve. There are only three elements in the domain. A curve would mean that the domain contains an entire interval of real numbers.

Here is another way to represent that same function:



This shows that the function f sends 1 to 2, 2 to 1, and 3 to 3: Just follow the arrows.

The arrow diagram used to define the function above can be very helpful in visualizing functions. We will often be working with functions with *finite* domains, so this kind of picture is often more useful than a traditional graph of a function.

Note that for finite domains, finding an algebraic formula that gives the output for any input is often impossible. Of course we could use a piecewise-defined function, like

$$f(x) = \begin{cases} x+1 & \text{if } x = 1 \\ x-1 & \text{if } x = 2 \\ x & \text{if } x = 3 \end{cases}.$$

This describes exactly the same function as above, but we can all agree is a ridiculous way of doing so.

Since we will so often use functions with small domains and codomains, let's adopt some notation to describe them. All we need is some clear way of denoting the image of each element in the domain. In fact, writing a table of values would work perfectly:

x	0	1	2	3	4
$f(x)$	3	3	2	4	1

We simplify this further by writing this as a "matrix" with each input directly over its output:

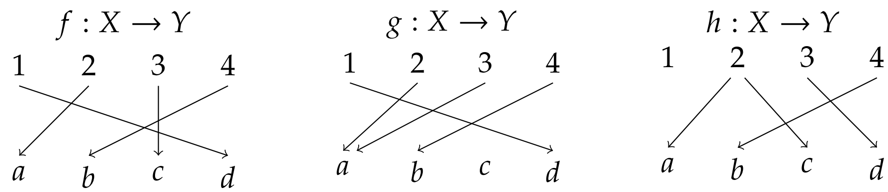
$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 3 & 3 & 2 & 4 & 1 \end{pmatrix}.$$

Note this is just notation and not the same sort of matrix you would find in a linear algebra class (it does not make sense to do operations with these matrices, or row reduce them, for example).

One advantage of the two-line notation over the arrow diagrams is that it is harder to accidentally define a rule that is not a function using two-line notation.

Example 5.2.3

Which of the following diagrams represent a function? Let $X = \{1,2,3,4\}$ and $Y = \{a,b,c,d\}$.



Solution. f is a function. So is g. There is no problem with an element of the codomain not being the image of any input, and there is no problem with a from the codomain being the image of both 2 and 3 from the domain. We could use our two-line notation to write these as

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ d & a & c & b \end{pmatrix} \qquad g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ d & a & a & b \end{pmatrix}$$

.

However, h is NOT a function. In fact, it fails for two reasons. First, the element 1 from the domain has not been mapped to any element from the codomain. Second, the element 2 from the domain has been mapped to more than one element from the codomain (a and c). Note that either one of these problems is enough to make a rule not a function. In general, neither of the following mappings are functions:

It might also be helpful to think about how you would write the two-line notation for h. We would have something like:

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & a,c? & d & b \end{pmatrix}.$$

There is nothing under 1 (bad), and we needed to put more than one thing under 2 (very bad). With a rule that is actually a function, the two-line notation will always "work."

We will also be interested in functions with domain $\mathbb{N}$. Here two-line notation is no good, but describing the function algebraically is often possible. Even tables are a little awkward since they do not describe the function completely. For example, consider the function $f : \mathbb{N} \to \mathbb{N}$ given by the table below.

x	0	1	2	3	4	5	...
$f(x)$	0	1	4	9	16	25	...

Have I given you enough entries for you to be able to determine $f(6)$? You might guess that $f(6) = 36$, but there is no way for you to *know* this for sure. Maybe I am being a jerk and intended $f(6) = 42$. In fact, for every natural number n, there is a function that agrees with the table above, but for which $f(6) = n$.

Okay, suppose I really did mean for $f(6) = 36$, and in fact, for the rule that you think is governing the function to actually be the rule. Then I should say what that rule is. $f(n) = n^2$. Now there is no confusion possible.

Giving an explicit formula that calculates the image of any element in the domain is a great way to describe a function. We will say that these explicit rules are **closed formulas** for the function.

There is another very useful way to describe functions whose domain is $\mathbb{N}$, that rely specifically on the structure of the natural numbers. We can define a function *recursively*!

Example 5.2.4

Consider the function $f : \mathbb{N} \to \mathbb{N}$ given by $f(0) = 0$ and $f(n+1) = f(n)+2n+1$. Find $f(6)$.

Solution. The rule says that $f(6) = f(5)+11$ (we are using $6 = n+1$ so $n = 5$). We don't know what $f(5)$ is though. Well, we know that $f(5) = f(4) + 9$. So we need to compute $f(4)$, which will require knowing $f(3)$, which will require knowing $f(2)$,... will it ever end?

Yes! This process will always end because we have $\mathbb{N}$ as our domain, so there is a least element. And we gave the value of $f(0)$ explicitly, so we are good. We might decide to work up to $f(6)$ instead of working down from $f(6)$:

$$\begin{aligned}
f(1) &= f(0) + 1 = & 0 + 1 &= 1\\
f(2) &= f(1) + 3 = & 1 + 3 &= 4\\
f(3) &= f(2) + 5 = & 4 + 5 &= 9\\
f(4) &= f(3) + 7 = & 9 + 7 &= 16\\
f(5) &= f(4) + 9 = & 16 + 9 &= 25\\
f(6) &= f(5) + 11 = & 25 + 11 &= 36
\end{aligned}$$

It looks like this recursively defined function is the same as the explicitly defined function $f(n) = n^2$. Is it? Later we will prove that it is.

Recursively defined functions are often easier to create from a "real-world" problem, because they describe how the values of the functions are changing. However, this comes with a price. It is harder to calculate the image of a single input, since you need to know the images of other (previous) elements in the domain.

Recursively Defined Functions.

For a function $f : \mathbb{N} \to \mathbb{N}$, a **recursive definition** consists of an **initial condition** together with a **recurrence relation**. The initial condition is the explicitly given value of $f(0)$. The recurrence relation is a formula for $f(n+1)$ in terms of $f(n)$ (and possibly n itself).

Example 5.2.5

Give recursive definitions for the functions described below.

1. $f : \mathbb{N} \to \mathbb{N}$ gives the number of snails in your terrarium n years after you built it, assuming you started with 3 snails and the number of snails doubles each year.
2. $g : \mathbb{N} \to \mathbb{N}$ gives the number of push-ups you do n days after you started your push-ups challenge, assuming you could do 7 push-ups on day 0 and you can do 2 more push-ups each day.
3. $h : \mathbb{N} \to \mathbb{N}$ defined by $h(n) = n!$. Recall that $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$ is the product of all numbers from 1 through n. We also define $0! = 1$.

Solution.

1. The initial condition is $f(0) = 3$. To get $f(n+1)$, we would double the number of snails in the terrarium the previous year, which is given by $f(n)$. Thus $f(n+1) = 2f(n)$. The full recursive definition contains both of these and would be written,

$$f(0) = 3; \ f(n+1) = 2f(n).$$

2. We are told that on day 0 you can do 7 push-ups, so $g(0) = 7$. The number of push-ups you can do on day $n+1$ is 2 more than the number you can do on day n, which is given by $g(n)$. Thus

$$g(0) = 7; \ g(n+1) = g(n) + 2.$$

3. Here $h(0) = 1$. To get the recurrence relation, think about how you can get $h(n+1) = (n+1)!$ from $h(n) = n!$. If you write out both of these as products, you see that $(n+1)!$ is just like $n!$ except that you have one more term in the product, an extra $n+1$. So we have,

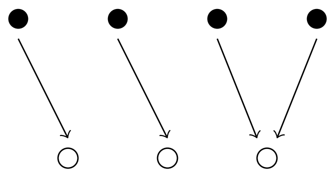
$$h(0) = 1; \ h(n+1) = (n+1) \cdot h(n).$$

5.2.2 Surjections, Injections, and Bijections

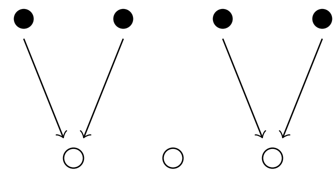
We now turn to investigating special properties functions might or might not possess.

In the examples above, you may have noticed that sometimes there are elements of the codomain that are not in the range. When this sort of thing *does not* happen (that is, when everything in the codomain is in the range), we say the function is **onto** or that the function maps the domain *onto* the codomain. This terminology should make sense: The function puts the domain (entirely) on top of the codomain. The fancy math term for an onto function is a **surjection**, and we say that an onto function is a **surjective** function.

In pictures:



Surjective



Not surjective

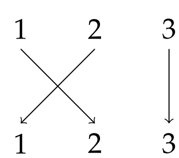
Example 5.2.6

Which functions are surjective (i.e., onto)?

1. $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 3n$.

2. $g : \{1, 2, 3\} \to \{a, b, c\}$ defined by $g = \begin{pmatrix} 1 & 2 & 3 \\ c & a & a \end{pmatrix}$.

3. $h : \{1, 2, 3\} \to \{1, 2, 3\}$ defined as follows:



Solution.

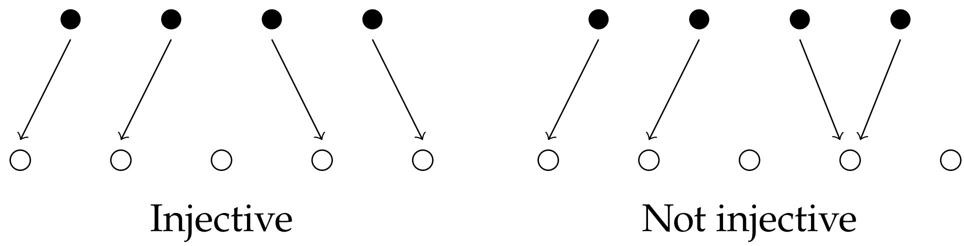
1. f is not surjective. There are elements in the codomain that are not in the range. For example, no $n \in \mathbb{Z}$ gets mapped to the number 1 (the rule would say that $\frac{1}{3}$ would be sent to 1, but $\frac{1}{3}$ is not in the domain). In fact, the range of the function is $3\mathbb{Z}$ (the integer multiples of 3), which is not equal to $\mathbb{Z}$.

2. g is not surjective. There is no $x \in \{1, 2, 3\}$ (the domain) for which $g(x) = b$, so b, which is in the codomain, is not in the range. Notice that there is an element from the codomain "missing" from the bottom row of the matrix.

3. h is surjective. Every element of the codomain is also in the range. Nothing in the codomain is missed.

To be a function, a rule cannot assign a single element of the domain to two or more different elements of the codomain. However, we have seen that the reverse *is* permissible: A function might assign the same element of the codomain to two or more different elements of the domain. When this *does not* occur (that is, when each element of the codomain is the image of at most one element of the domain), then we say the function is **one-to-one**. Again, this terminology makes sense: We are sending at most one element from the domain to one element from the codomain. One input to one output. The fancy math term for a one-to-one function is an **injection**. We call one-to-one functions **injective** functions.

In pictures:



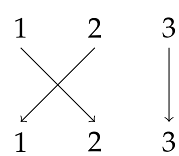
Example 5.2.7

Which functions are injective (i.e., one-to-one)?

1. $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 3n$.

2. $g : \{1, 2, 3\} \to \{a, b, c\}$ defined by $g = \begin{pmatrix} 1 & 2 & 3 \\ c & a & a \end{pmatrix}$.

3. $h : \{1, 2, 3\} \to \{1, 2, 3\}$ defined as follows:



Solution.

1. f is injective. Each element in the codomain is assigned to at *most* one element from the domain. If x is a multiple of three, then only $x/3$ is mapped to x. If x is not a multiple of 3, then there is no input corresponding to the output x.

2. g is not injective. Both inputs 2 and 3 are assigned the output a. Notice that there is an element from the codomain that appears more than once on the bottom row of the matrix.

3. h is injective. Each output is only an output once.

Be careful: "surjective" and "injective" are NOT opposites. You can see in the two examples above that there are functions that are surjective but not injective, injective but not surjective, both, or neither. In the case when a function is both one-to-one and onto (an injection and surjection), we say the function is a **bijection**, or that the function is a **bijective** function.

To illustrate the contrast between these two properties, consider a more formal definition of each, side by side.

Injective vs. Surjective.

A function is **injective** provided every element of the codomain is the image of *at most* one element from the domain.

A function is **surjective** provided every element of the codomain is the image of *at least* one element from the domain.

Notice both properties are determined by what happens to elements of the codomain: They could be repeated as images, or they could be "missed" (not be images). Injective functions do not have repeats but might or might not miss elements. Surjective functions do not miss elements, but might or might not have repeats. The bijective functions are those that do not have repeats and do not miss elements.

5.2.3 Image and Inverse Image

When discussing functions, we have notation for talking about an element of the domain (say x) and its corresponding element in the codomain (we write $f(x)$, which *is* the image of x). Sometimes we will want to talk about all the elements that are images of some subset of the domain. It would also be nice to start with some element of the codomain (say y) and talk about which element or elements (if any) from the domain it is the image of. We could write "those x in the domain such that $f(x) = y$," but this is a lot of writing. Here is some notation to make our lives easier.

To address the first situation, what we are after is a way to describe the *set* of images of elements in some subset of the domain. Suppose $f : X \to Y$ is a function and that $A \subseteq X$ is some subset of the domain (possibly all of it). We will use the notation $f(A)$ to denote the **image of A under f**, namely the set of elements in Y that are the image of elements from A. That is, $f(A) = \{f(a) \in Y : a \in A\}$.

We can do this in the other direction as well. We might ask which elements of the domain get mapped to a particular set in the codomain. Let $f : X \to Y$ be a function and suppose $B \subseteq Y$ is a subset of the codomain. Then we will write $f^{-1}(B)$

for the **inverse image of B under f**, namely the set of elements in X whose image are elements in B. In other words, $f^{-1}(B) = \{x \in X : f(x) \in B\}$.

Often we are interested in the element(s) whose image is a particular element y of in the codomain. The notation above works: $f^{-1}(\{y\})$ is the set of all elements in the domain that f sends to y. It makes sense to think of this as a set: there might not be anything sent to y (if y is not in the range), in which case $f^{-1}(\{y\}) = \emptyset$. Or f might send multiple elements to y (if f is not injective). As a notational convenience, we usually drop the set braces around the y and write $f^{-1}(y)$ instead for this set.

WARNING: $f^{-1}(y)$ is not an inverse function! Inverse functions only exist for bijections, but $f^{-1}(y)$ is defined for any function f. The point: $f^{-1}(y)$ is a *set*, not an *element* of the domain. This is just sloppy notation for $f^{-1}(\{y\})$. To help make this distinction, we would call $f^{-1}(y)$ the **complete inverse image of y under f**. It is not the image of y under f^{-1} (since the function f^{-1} might not exist).

Example 5.2.8

Consider the function $f : \{1,2,3,4,5,6\} \to \{a,b,c,d\}$ given by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ a & a & b & b & b & c \end{pmatrix}.$$

Find $f(\{1,2,3\})$, $f^{-1}(\{a,b\})$, and $f^{-1}(d)$.

Solution. $f(\{1,2,3\}) = \{a,b\}$ since a and b are the elements in the codomain to which f sends 1, 2, and 3.

$f^{-1}(\{a,b\}) = \{1,2,3,4,5\}$ since these are exactly the elements that f sends to a and b.

$f^{-1}(d) = \emptyset$ since d is not in the range of f.

Example 5.2.9

Consider the function $g : \mathbb{Z} \to \mathbb{Z}$ defined by $g(n) = n^2 + 1$. Find $g(1)$ and $g(\{1\})$. Then find $g^{-1}(1)$, $g^{-1}(2)$, and $g^{-1}(3)$.

Solution. Note that $g(1) \neq g(\{1\})$. The first is an element: $g(1) = 2$. The second is a set: $g(\{1\}) = \{2\}$.

To find $g^{-1}(1)$, we need to find all integers n such that $n^2 + 1 = 1$. Clearly only 0 works, so $g^{-1}(1) = \{0\}$ (note that even though there is only one element, we still write it as a set with one element in it).

To find $g^{-1}(2)$, we need to find all n such that $n^2 + 1 = 2$. We see $g^{-1}(2) = \{-1,1\}$.

Finally, if $n^2 + 1 = 3$, then we are looking for an n such that $n^2 = 2$. There are no such integers so $g^{-1}(3) = \emptyset$.

Since $f^{-1}(y)$ is a set, it makes sense to ask for $\left|f^{-1}(y)\right|$, the number of elements in the domain that map to y.

Example 5.2.10

Find a function $f : \{1, 2, 3, 4, 5\} \to \mathbb{N}$ such that $\left|f^{-1}(7)\right| = 5$.

Solution. There is only one such function. We need five elements of the domain to map to the number $7 \in \mathbb{N}$. Since there are only five elements in the domain, all of them must map to 7. So

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 7 & 7 & 7 & 7 & 7 \end{pmatrix}$$

.

Function Definitions. Here is a summary of all the main concepts and definitions we use when working with functions.

- A **function** is a rule that assigns each element of a set, called the **domain**, to exactly one element of a second set, called the **codomain**.
- Notation: $f : X \to Y$ is our way of saying that the function is called f, the domain is the set X, and the codomain is the set Y.
- To specify the rule for a function with small domain, use **two-line notation** by writing a matrix with each output directly below its corresponding input, as in:

  $$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 1 \end{pmatrix}$$

  .
- $f(x) = y$ means the element x of the domain (input) is assigned to the element y of the codomain. We say y is an output. Alternatively, we call y the **image of x under f** .
- The **range** is a subset of the codomain. It is the set of all elements that are assigned to at least one element of the domain by the function. That is, the range is the set of all outputs.
- A function is **injective** (an **injection** or **one-to-one**) if every element of the codomain is the image of **at most** one element from the domain.
- A function is **surjective** (a **surjection** or **onto**) if every element of the codomain is the image of **at least** one element from the domain.
- A **bijection** is a function that is both an injection and surjection. In other words, if every element of the codomain is the image of **exactly one** element from the domain.
- The **image** of an element x in the domain is the element y in the codomain that x is mapped to. That is, the image of x under f is $f(x)$.

- The **complete inverse image** of an element y in the codomain, written $f^{-1}(y)$, is the *set* of all elements in the domain that are assigned to y by the function.
- The **image** of a subset A of the domain is the set $f(A) = \{f(a) \in Y : a \in A\}$.
- The **inverse image** of a subset B of the codomain is the set $f^{-1}(B) = \{x \in X : f(x) \in B\}$.

Reading Questions

1. Explain, in your own words, the relationship between the codomain of a function and the range of a function.

2. If a function has domain and codomain of equal sizes, must the function be *surjective*? Must it be *injective*? Could it be only one of these? Briefly explain your thinking.

3. What questions do you have? Write at least one question about the content of this section that you or a classmate might be curious about after reading this section.

5.2.4 Exercises

1. Consider the function $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ given by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 4 & 2 & 5 \end{pmatrix}.$$

a. Find $f(4)$. _____

b. Find a n in the domain such that $f(n) = 4$. _____

c. Find an element n of the domain such that $f(n) = n$. _____

d. Find an element of the codomain that is not in the range. _____

2. The following functions all have $\{1, 2, 3, 4, 5\}$ as both their domain and codomain. For each, determine whether it is (only) injective, (only) surjective, bijective, or neither injective nor surjective.

(a) $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 1 \end{pmatrix}$

(b) $f(x) = \begin{cases} x & \text{if } x < 3 \\ x - 2 & \text{if } x \geq 3 \end{cases}$

(c) $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix}$

(d) $f(x) = \begin{cases} 3 - x & \text{if } x < 3 \\ x & \text{if } x \geq 3 \end{cases}$

3. Consider the following functions $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3\}$. For each, determine whether it is (only) injective, (only) surjective, bijective, or neither injective nor surjective.

(a) $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 2 & 1 & 1 \end{pmatrix}$

(b) $f(x) = \begin{cases} x & \text{if } x < 4 \\ 6 - x & \text{if } x \geq 4 \end{cases}$

(c) $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 2 & 3 & 1 \end{pmatrix}$

4. Consider the following functions $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4, 5\}$. For each, determine whether it is (only) injective, (only) surjective, bijective, or neither injective nor surjective.

(a) $f(x) = 6 - x$

(b) $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

(c) $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 5 \end{pmatrix}$

5. Write out all functions $f : \{1, 2, 3, 4\} \to \{a, b\}$ (using two-line notation).
How many functions are there?
How many are surjective?
How many are injective?
How many are bijective?

6. Write out all function $f : \{1, 2\} \to \{a, b, c, d\}$ (in two-line notation).
How many functions are there?
How many are surjective?
How many are injective?
How many are bijective?

7. Consider the function $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4\}$ given by the table below:

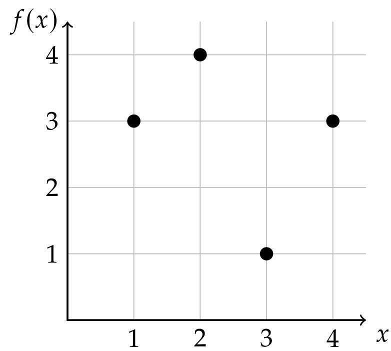
x	1	2	3	4	5
$f(x)$	3	2	4	1	2

(a) Is f injective? Explain.

(b) Is f surjective? Explain.

(c) Write the function using two-line notation.

8. Consider the function $f : \{1,2,3,4\} \to \{1,2,3,4\}$ given by the graph below.



(a) Is f injective? Explain.

(b) Is f surjective? Explain.

(c) Write the function using two-line notation.

9. Consider the function $f : \mathbb{N} \to \mathbb{N}$ given *recursively* by

$$f(0) = 1 \text{ and } f(n+1) = 3 \cdot f(n).$$

Find $f(12)$.

10. Suppose $f : \mathbb{N} \to \mathbb{N}$ satisfies the recurrence

$$f(n+1) = f(n) + 6.$$

Note that this is not enough information to define the function, since we don't have an initial condition. For each of the initial conditions below, find the value of $f(7)$.

a. If $f(0) = 1$.

b. $f(0) = 7$.

c. $f(0) = 17$.

d. $f(0) = 155$.

11. Suppose $f : \mathbb{N} \to \mathbb{N}$ satisfies the recurrence relation

$$f(n+1) = \begin{cases} \frac{f(n)}{2} & \text{if } f(n) \text{ is even} \\ 3f(n)+1 & \text{if } f(n) \text{ is odd} \end{cases}.$$

Note that with the initial condition $f(0) = 1$, the values of the function are: $f(1) = 4$, $f(2) = 2$, $f(3) = 1$, $f(4) = 4$, and so on, the images cycling through those three numbers. Thus f is NOT injective (and also certainly not surjective). Might it be under other initial conditions?[1]

(a) If f satisfies the initial condition $f(0) = 5$, is f injective? Explain why or give a specific example of two elements from the domain with the same image.

(b) If f satisfies the initial condition $f(0) = 3$, is f injective? Explain why or give a specific example of two elements from the domain with the same image.

(c) If f satisfies the initial condition $f(0) = 27$, then it turns out that $f(105) = 10$ and no two numbers less than 105 have the same image. Could f be injective? Explain.

(d) Prove that no matter what initial condition you choose, the function cannot be surjective.

12. For each function given below, determine whether or not the function is injective and whether or not the function is surjective.

(a) $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = n + 4$.

(b) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = n + 4$.

(c) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = 5n - 8$.

(d) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$

13. Let $A = \{1, 2, 3, \ldots, 10\}$. Consider the function $f : \mathcal{P}(A) \to \mathbb{N}$ given by $f(B) = |B|$. That is, f takes a subset of A as an input and outputs the cardinality of that set.

(a) Is f injective? Prove your answer.

(b) Is f surjective? Prove your answer.

(c) Find $f^{-1}(1)$.

(d) Find $f^{-1}(0)$.

(e) Find $f^{-1}(12)$.

14. Let $X = \{n \in \mathbb{N} : 0 \le n \le 999\}$ be the set of all numbers with three or fewer digits. Define the function $f : X \to \mathbb{N}$ by $f(abc) = a + b + c$, where a, b, and c are the digits of the number in X (write numbers less than 100 with leading 0's to make them three digits). In other words, f returns the sum of the digits of its input. For example, $f(253) = 2 + 5 + 3 = 10$.

(a) Let $A = \{n \in X : 306 \le x \le 323\}$. Find $f(A)$.

[1] It turns out this is a *really* hard question to answer in general. The *Collatz conjecture* is that no matter what the initial condition is, the function will eventually produce 1 as an output. This is an open problem in mathematics: nobody knows the answer.

(b) Find $f^{-1}(\{1,3\})$.

(c) Find $f^{-1}(2)$.

(d) Find $f^{-1}(139)$.

15. Consider the set $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, the set of all ordered pairs (a,b) where a and b are natural numbers. Consider a function $f : \mathbb{N}^2 \to \mathbb{N}$ given by $f((a,b)) = a+b$.

(a) Let $A = \{(a,b) \in \mathbb{N}^2 : a, b \leq 10\}$. Find $f(A)$.

(b) Find $f^{-1}(3)$ and $f^{-1}(\{0,1,2,3\})$.

(c) Give geometric descriptions of $f^{-1}(n)$ and $f^{-1}(\{0,1,\ldots,n\})$ for any $n \geq 1$.

(d) Find $\left|f^{-1}(8)\right|$ and $\left|f^{-1}(\{0,1,\ldots,8\})\right|$.

16. Let $f : X \to Y$ be some function. Suppose $3 \in Y$. What can you say about $f^{-1}(3)$ if you know,

(a) f is injective? Explain.

(b) f is surjective? Explain.

(c) f is bijective? Explain.

17. Find a set X and a function $f : X \to \mathbb{N}$ so that $f^{-1}(0) \cup f^{-1}(1) = X$.

18. What can you deduce about the sets X and Y if you know,

(a) there is an injective function $f : X \to Y$? Explain.

(b) there is a surjective function $f : X \to Y$? Explain.

(c) there is a bijective function $f : X \to Y$? Explain.

19. Suppose $f : X \to Y$ is a function. Which of the following are possible? Explain.

(a) f is injective but not surjective.

(b) f is surjective but not injective.

(c) $|X| = |Y|$ and f is injective but not surjective.

(d) $|X| = |Y|$ and f is surjective but not injective.

(e) $|X| = |Y|$, X and Y are finite, and f is injective but not surjective.

(f) $|X| = |Y|$, X and Y are finite, and f is surjective but not injective.

20. Let $f : X \to Y$ and $g : Y \to Z$ be functions. We can define the **composition** of f and g to be the function $g \circ f : X \to Z$ for which the image of each $x \in X$ is $g(f(x))$. That is, plug x into f, then plug the result into g (just like composition in algebra and calculus).

(a) If f and g are both injective, must $g \circ f$ be injective? Explain.

(b) If f and g are both surjective, must $g \circ f$ be surjective? Explain.

(c) Suppose $g \circ f$ is injective. What, if anything, can you say about f and g? Explain.

(d) Suppose $g \circ f$ is surjective. What, if anything, can you say about f and g? Explain.

21. Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-3 & \text{if } n \text{ is odd.} \end{cases}$

(a) Is f injective? Prove your answer.

(b) Is f surjective? Prove your answer.

22. At the end of the semester a teacher assigns letter grades to each of her students. Is this a function? If so, what sets make up the domain and codomain, and is the function injective, surjective, bijective, or neither?

23. In the game of *Hearts*, four players are each dealt 13 cards from a deck of 52. Is this a function? If so, what sets make up the domain and codomain, and is the function injective, surjective, bijective, or neither?

24. Seven players are playing 5-card stud. Each player initially receives 5 cards from a deck of 52. Is this a function? If so, what sets make up the domain and codomain, and is the function injective, surjective, bijective, or neither?

25. Consider the function $f : \mathbb{N} \to \mathbb{N}$ that gives the number of handshakes that take place in a room of n people assuming everyone shakes hands with everyone else. Give a recursive definition for this function.

26. Let $f : X \to Y$ be a function and $A \subseteq X$ be a finite subset of the domain. What can you say about the relationship between $|A|$ and $\left|f(A)\right|$? Consider both the general case and what happens when you know f is injective, surjective, or bijective.

27. Let $f : X \to Y$ be a function and $B \subseteq Y$ be a finite subset of the codomain. What can you say about the relationship between $|B|$ and $\left|f^{-1}(B)\right|$? Consider both the general case and what happens when you know f is injective, surjective, or bijective.

28. Let $f : X \to Y$ be a function, $A \subseteq X$ and $B \subseteq Y$.

(a) Is $f^{-1}\left(f(A)\right) = A$? Always, sometimes, never? Explain.

(b) Is $f\left(f^{-1}(B)\right) = B$? Always, sometimes, never? Explain.

(c) If one or both of the above do not always hold, is there something else you can say? Will equality always hold for particular types of functions? Is there some other relationship other than equality that would always hold? Explore.

29. Let $f : X \to Y$ be a function and $A, B \subseteq X$ be subsets of the domain.

(a) Is $f(A \cup B) = f(A) \cup f(B)$? Always, sometimes, or never? Explain.

(b) Is $f(A \cap B) = f(A) \cap f(B)$? Always, sometimes, or never? Explain.

30. Let $f : X \to Y$ be a function and $A, B \subseteq Y$ be subsets of the codomain.

(a) Is $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$? Always, sometimes, or never? Explain.

(b) Is $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$? Always, sometimes, or never? Explain.

Chapter 6

Additional Topics

6.1 Generating Functions

There is an extremely powerful tool in discrete mathematics used to manipulate sequences called the generating function. The idea is this: instead of an infinite sequence (for example: $2, 3, 5, 8, 12, \ldots$) we look at a single function which encodes the sequence. But not a function that gives the nth term as output. Instead, a function whose power series (like from calculus) "displays" the terms of the sequence. So for example, we would look at the power series $2 + 3x + 5x^2 + 8x^3 + 12x^4 + \cdots$ which displays the sequence $2, 3, 5, 8, 12, \ldots$ as coefficients.

An infinite power series is simply an infinite sum of terms of the form $c_n x^n$ were c_n is some constant. So we might write a power series like this:

$$\sum_{k=0}^{\infty} c_k x^k.$$

or expanded like this

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots.$$

When viewed in the context of generating functions, we call such a power series a *generating series*. The generating series generates the sequence

$$c_0, c_1, c_2, c_3, c_4, c_5, \ldots.$$

In other words, the sequence generated by a generating series is simply the sequence of *coefficients* of the infinite polynomial.

Example 6.1.1

What sequence is represented by the generating series $3 + 8x^2 + x^3 + \frac{x^5}{7} + 100x^6 + \cdots$?

Solution. We just read off the coefficients of each x^n term. So $a_0 = 3$ since the coefficient of x^0 is 3 ($x^0 = 1$ so this is the constant term). What is a_1? It is NOT 8, since 8 is the coefficient of x^2, so 8 is the term a_2 of the sequence. To find a_1 we need to look for the coefficient of x^1 which in this case is 0. So $a_1 = 0$. Continuing, we have $a_2 = 8$, $a_3 = 1$, $a_4 = 0$, and $a_5 = \frac{1}{7}$. So we have the sequence

$$3, 0, 8, 1, 0, \frac{1}{7}, 100, \ldots.$$

Note that when discussing generating functions, we always start our sequence with a_0.

Now you might very naturally ask why we would do such a thing. One reason is that encoding a sequence with a power series helps us keep track of which term is which in the sequence. For example, if we write the sequence $1, 3, 4, 6, 9, \ldots, 24, 41, \ldots$ it is impossible to determine which term 24 is (even if we agreed that the first term was supposed to be a_0). However, if we wrote the generating series instead, we would have $1 + 3x + 4x^2 + 6x^3 + 9x^4 + \cdots + 24x^{17} + 41x^{18} + \cdots$. Now it is clear that 24 is the 17th term of the sequence (that is, $a_{17} = 24$). Of course to get this benefit we could have displayed our sequence in any number of ways, perhaps $\boxed{1}_0 \boxed{3}_1 \boxed{4}_2 \boxed{6}_3 \boxed{9}_4 \cdots \boxed{24}_{17} \boxed{41}_{18} \cdots$, but we do not do this. The reason is that the generating series looks like an ordinary power series (although we are interpreting it differently) so we can do things with it that we ordinarily do with power series such as write down what it converges to.

For example, from calculus we know that the power series $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots + \frac{x^n}{n!} + \cdots$ converges to the function e^x. So we can use e^x as a way of talking about the sequence of coefficients of the power series for e^x. When we write down a nice compact function which has an infinite power series that we view as a generating series, then we call that function a *generating function*. In this example, we would say

$$1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \ldots, \frac{1}{n!}, \ldots \text{ has generating function } e^x.$$

6.1.1 Building Generating Functions

The e^x example is very specific. We have a rather odd sequence, and the only reason we know its generating function is because we happen to know the Taylor series for e^x. Our goal now is to gather some tools to build the generating function of a particular given sequence.

Let's see what the generating functions are for some very simple sequences. The simplest of all: $1, 1, 1, 1, 1, \ldots$. What does the *generating series* look like? It is simply $1 + x + x^2 + x^3 + x^4 + \cdots$. Now, can we find a closed formula for this power series? Yes! This particular series is really just a geometric series with common ratio x. So if we use our "multiply, shift and subtract" technique from Section 4.4, we have

$$\begin{array}{rrl} & S = & 1 + x + x^2 + x^3 + \cdots \\ - & xS = & \phantom{1 + {}} x + x^2 + x^3 + x^4 + \cdots \\ \hline & (1-x)S = & 1 \end{array}$$

Therefore we see that

$$1 + x + x^2 + x^3 \cdots = \frac{1}{1-x}.$$

You might remember from calculus that this is only true on the interval of convergence for the power series, in this case when $|x| < 1$. That is true for us, but we don't care. We are never going to plug anything in for x, so as long as there is some value of x for which the generating function and generating series agree, we are happy. And in this case we are happy.

$1, 1, 1, \ldots$.

The generating function for $1, 1, 1, 1, 1, 1, \ldots$ is $\dfrac{1}{1-x}$

Let's use this basic generating function to find generating functions for more sequences. What if we replace x by $-x$. We get

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \text{ which generates } 1, -1, 1, -1, \ldots.$$

If we replace x by $3x$ we get

$$\frac{1}{1-3x} = 1 + 3x + 9x^2 + 27x^3 + \cdots \text{ which generates } 1, 3, 9, 27, \ldots.$$

By replacing the x in $\frac{1}{1-x}$ we can get generating functions for a variety of sequences, but not all. For example, you cannot plug in anything for x to get the generating function for $2, 2, 2, 2, \ldots$. However, we are not lost yet. Notice that each term of $2, 2, 2, 2, \ldots$ is the result of multiplying the terms of $1, 1, 1, 1, \ldots$ by the constant 2. So multiply the generating function by 2 as well.

$$\frac{2}{1-x} = 2 + 2x + 2x^2 + 2x^3 + \cdots \text{ which generates } 2, 2, 2, 2, \ldots.$$

Similarly, to find the generating function for the sequence $3, 9, 27, 81, \ldots$, we note that this sequence is the result of multiplying each term of $1, 3, 9, 27, \ldots$ by 3. Since we have the generating function for $1, 3, 9, 27, \ldots$ we can say

$$\frac{3}{1-3x} = 3 \cdot 1 + 3 \cdot 3x + 3 \cdot 9x^2 + 3 \cdot 27x^3 + \cdots \text{ which generates } 3, 9, 27, 81, \ldots.$$

What about the sequence $2, 4, 10, 28, 82, \ldots$? Here the terms are always 1 more than powers of 3. That is, we have added the sequences $1, 1, 1, 1, \ldots$ and $1, 3, 9, 27, \ldots$ term by term. Therefore we can get a generating function by adding the respective generating functions:

$$\begin{aligned} 2 + 4x + 10x^2 + 28x^3 + \cdots &= (1+1) + (1+3)x + (1+9)x^2 + (1+27)x^3 + \cdots \\ &= 1 + x + x^2 + x^3 + \cdots + 1 + 3x + 9x^2 + 27x^3 + \cdots \\ &= \frac{1}{1-x} + \frac{1}{1-3x} \end{aligned}$$

The fun does not stop there: if we replace x in our original generating function by x^2 we get

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 \cdots \text{ which generates } 1,0,1,0,1,0,\ldots.$$

How could we get $0,1,0,1,0,1,\ldots$? Start with the previous sequence and *shift* it over by 1. But how do you do this? To see how shifting works, let's first try to get the generating function for the sequence $0,1,3,9,27,\ldots$. We know that $\frac{1}{1-3x} = 1 + 3x + 9x^2 + 27x^3 + \cdots$. To get the zero out front, we need the generating series to look like $x + 3x^2 + 9x^3 + 27x^4 + \cdots$ (so there is no constant term). Multiplying by x has this effect. So the generating function for $0,1,3,9,27,\ldots$ is $\frac{x}{1-3x}$. This will also work to get the generating function for $0,1,0,1,0,1,\ldots$:

$$\frac{x}{1-x^2} = x + x^3 + x^5 + \cdots \text{ which generates } 0,1,0,1,0,1\ldots.$$

What if we add the sequences $1,0,1,0,1,0,\ldots$ and $0,1,0,1,0,1,\ldots$ term by term? We should get $1,1,1,1,1,1\ldots$. What happens when we add the generating functions? It works (try it)!

$$\frac{1}{1-x^2} + \frac{x}{1-x^2} = \frac{1}{1-x}.$$

Here's a sneaky one: what happens if you take the *derivative* of $\frac{1}{1-x}$? We get $\frac{1}{(1-x)^2}$. On the other hand, if we differentiate term by term in the power series, we get $(1 + x + x^2 + x^3 + \cdots)' = 1 + 2x + 3x^2 + 4x^3 + \cdots$ which is the generating series for $1,2,3,4,\ldots$. This says

$1,2,3,\ldots$.

The generating function for $1,2,3,4,5,\ldots$ is $\dfrac{1}{(1-x)^2}$.

Take a second derivative: $\frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + 20x^3 + \cdots$. So $\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + \cdots$ is a generating function for the triangular numbers, $1,3,6,10\ldots$ (although here we have $a_0 = 1$ while $T_0 = 0$ usually).

6.1.2 Differencing

We have seen how to find generating functions from $\frac{1}{1-x}$ using multiplication (by a constant or by x), substitution, addition, and differentiation. To use each of these, you must notice a way to transform the sequence $1,1,1,1,1\ldots$ into your desired sequence. This is not always easy. It is also not really the way we have analyzed sequences. One thing we have considered often is the sequence of differences between terms of a sequence. This will turn out to be helpful in finding generating functions as well. The sequence of differences is often simpler than the original sequence. So if we know a generating function for the differences, we would like to use this to find a generating function for the original sequence.

For example, consider the sequence $2, 4, 10, 28, 82, \ldots$. How could we move to the sequence of first differences: $2, 6, 18, 54, \ldots$? We want to subtract 2 from the 4, 4 from the 10, 10 from the 28, and so on. So if we subtract (term by term) the sequence $0, 2, 4, 10, 28, \ldots$ from $2, 4, 10, 28 \ldots$, we will be set. We can get the generating function for $0, 2, 4, 10, 28, \ldots$ from the generating function for $2, 4, 10, 28 \ldots$ by multiplying by x. Use A to represent the generating function for $2, 4, 10, 28, 82, \ldots$ Then:

$$\begin{array}{rl} A &= 2 + 4x + 10x^2 + 28x^3 + 82x^4 + \cdots \\ - \quad xA &= 0 + 2x + 4x^2 + 10x^3 + 28x^4 + 82x^5 + \cdots \\ \hline (1-x)A &= 2 + 2x + 6x^2 + 18x^3 + 54x^4 + \cdots \end{array}$$

While we don't get exactly the sequence of differences, we do get something close. In this particular case, we already know the generating function A (we found it in the previous section) but most of the time we will use this differencing technique to *find* A: if we have the generating function for the sequence of differences, we can then solve for A.

Example 6.1.2

Find a generating function for $1, 3, 5, 7, 9, \ldots$.

Solution. Notice that the sequence of differences is constant. We know how to find the generating function for any constant sequence. So denote the generating function for $1, 3, 5, 7, 9, \ldots$ by A. We have

$$\begin{array}{rl} A &= 1 + 3x + 5x^2 + 7x^3 + 9x^4 + \cdots \\ - \quad xA &= 0 + x + 3x^2 + 5x^3 + 7x^4 + 9x^5 + \cdots \\ \hline (1-x)A &= 1 + 2x + 2x^2 + 2x^3 + 2x^4 + \cdots \end{array}$$

We know that $2x + 2x^2 + 2x^3 + 2x^4 + \cdots = \dfrac{2x}{1-x}$. Thus

$$(1-x)A = 1 + \frac{2x}{1-x}.$$

Now solve for A:

$$A = \frac{1}{1-x} + \frac{2x}{(1-x)^2} = \frac{1+x}{(1-x)^2}.$$

Does this make sense? Before we simplified the two fractions into one, we were adding the generating function for the sequence $1, 1, 1, 1, \ldots$ to the generating function for the sequence $0, 2, 4, 6, 8, 10, \ldots$ (remember $\frac{1}{(1-x)^2}$ generates $1, 2, 3, 4, 5, \ldots$, multiplying by $2x$ shifts it over, putting the zero out front, and doubles each term). If we add these term by term, we get the correct sequence $1, 3, 5, 7, 9, \ldots$.

Now that we have a generating function for the odd numbers, we can use that to find the generating function for the squares:

Example 6.1.3

Find the generating function for $1, 4, 9, 16, \ldots$. Note we take $1 = a_0$.

Solution. Again we call the generating function for the sequence A. Using differencing:

$$\begin{array}{rl} A &= 1 + 4x + 9x^2 + 16x^3 + \cdots \\ - \quad xA &= 0 + x + 4x^2 + 9x^3 + 16x^4 + \cdots \\ \hline (1-x)A &= 1 + 3x + 5x^2 + 7x^3 + \cdots \end{array}$$

Since $1 + 3x + 5x^2 + 7x^3 + \cdots = \dfrac{1+x}{(1-x)^2}$ we have $A = \dfrac{1+x}{(1-x)^3}$.

In each of the examples above, we found the difference between consecutive terms which gave us a sequence of differences for which we knew a generating function. We can generalize this to more complicated relationships between terms of the sequence. For example, if we know that the sequence satisfies the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$? In other words, if we take a term of the sequence and subtract 3 times the previous term and then add 2 times the term before that, we get 0 (since $a_n - 3a_{n-1} + 2a_{n-2} = 0$). That will hold for all but the first two terms of the sequence. So after the first two terms, the sequence of results of these calculations would be a sequence of 0's, for which we definitely know a generating function.

Example 6.1.4

The sequence $1, 3, 7, 15, 31, 63, \ldots$ satisfies the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$. Find the generating function for the sequence.

Solution. Call the generating function for the sequence A. We have

$$\begin{array}{rl} A &= 1 + 3x + 7x^2 + 15x^3 + 31x^4 + \cdots + a_n x^n + \cdots \\ -3xA &= 0 - 3x - 9x^2 - 21x^3 - 45x^4 - \cdots - 3a_{n-1}x^n - \cdots \\ + \quad 2x^2A &= 0 + 0x + 2x^2 + 6x^3 + 14x^4 + \cdots + 2a_{n-2}x^n + \cdots \\ \hline (1 - 3x + 2x^2)A &= 1 \end{array}$$

We multiplied A by $-3x$ which shifts every term over one spot and multiplies them by -3. On the third line, we multiplied A by $2x^2$, which shifted every term over two spots and multiplied them by 2. When we add up the corresponding terms, we are taking each term, subtracting 3 times the previous term, and adding 2 times the term before that. This will happen for each term after a_1 because $a_n - 3a_{n-1} + 2a_{n-2} = 0$. In general, we might have two terms from the beginning of the generating series, although in this case the second term happens to be 0 as well.

Now we just need to solve for A:

$$A = \frac{1}{1 - 3x + 2x^2}.$$

6.1.3 Multiplication and Partial Sums

What happens to the sequences when you multiply two generating functions? Let's see: $A = a_0 + a_1x + a_2x^2 + \cdots$ and $B = b_0 + b_1x + b_2x^2 + \cdots$. To multiply A and B, we need to do a lot of distributing (infinite FOIL?) but keep in mind we will group like terms and only need to write down the first few terms to see the pattern. The constant term is a_0b_0. The coefficient of x is $a_0b_1 + a_1b_0$. And so on. We get:

$$AB = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + \cdots.$$

Example 6.1.5

"Multiply" the sequence $1, 2, 3, 4, \ldots$ by the sequence $1, 2, 4, 8, 16, \ldots$.

Solution. The new constant term is just $1 \cdot 1$. The next term will be $1 \cdot 2 + 2 \cdot 1 = 4$. The next term: $1 \cdot 4 + 2 \cdot 2 + 3 \cdot 1 = 11$. One more: $1 \cdot 8 + 2 \cdot 4 + 3 \cdot 2 + 4 \cdot 1 = 26$. The resulting sequence is

$$1, 4, 11, 26, 57, \ldots.$$

Since the generating function for $1, 2, 3, 4, \ldots$ is $\frac{1}{(1-x)^2}$ and the generating function for $1, 2, 4, 8, 16, \ldots$ is $\frac{1}{1-2x}$, we have that the generating function for $1, 4, 11, 26, 57, \ldots$ is $\frac{1}{(1-x)^2(1-2x)}$

Consider the special case when you multiply a sequence by $1, 1, 1, \ldots$. For example, multiply $1, 1, 1, \ldots$ by $1, 2, 3, 4, 5 \ldots$. The first term is $1 \cdot 1 = 1$. Then $1 \cdot 2 + 1 \cdot 1 = 3$. Then $1 \cdot 3 + 1 \cdot 2 + 1 \cdot 1 = 6$. The next term will be 10. We are getting the triangular numbers. More precisely, we get the sequence of partial sums of $1, 2, 3, 4, 5, \ldots$. In terms of generating functions, we take $\frac{1}{1-x}$ (generating $1, 1, 1, 1, 1 \ldots$) and multiply it by $\frac{1}{(1-x)^2}$ (generating $1, 2, 3, 4, 5, \ldots$) and this gives $\frac{1}{(1-x)^3}$. This should not be a surprise as we found the same generating function for the triangular numbers earlier.

The point is, if you need to find a generating function for the sum of the first n terms of a particular sequence, and you know the generating function for *that* sequence, you can multiply it by $\frac{1}{1-x}$. To go back from the sequence of partial sums to the original sequence, you look at the sequence of differences. When you get the sequence of differences you end up multiplying by $1 - x$, or equivalently, dividing by $\frac{1}{1-x}$. Multiplying by $\frac{1}{1-x}$ gives partial sums, dividing by $\frac{1}{1-x}$ gives differences.

6.1.4 Solving Recurrence Relations with Generating Functions

We conclude with an example of one of the many reasons studying generating functions is helpful. We can use generating functions to solve recurrence relations.

Example 6.1.6

Solve the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 3$.

Solution. We saw in an example above that this recurrence relation gives the sequence $1, 3, 7, 15, 31, 63, \ldots$ which has generating function $\dfrac{1}{1-3x+2x^2}$. We did this by calling the generating function A and then computing $A - 3xA + 2x^2A$ which was just 1, since every other term canceled out.

But how does knowing the generating function help us? First, break up the generating function into two simpler ones. For this, we can use partial fraction decomposition. Start by factoring the denominator:

$$\frac{1}{1-3x+2x^2} = \frac{1}{(1-x)(1-2x)}.$$

Partial fraction decomposition tells us that we can write this faction as the sum of two fractions (we decompose the given fraction):

$$\frac{1}{(1-x)(1-2x)} = \frac{a}{1-x} + \frac{b}{1-2x} \quad \text{for some constants } a \text{ and } b.$$

To find a and b we add the two decomposed fractions using a common denominator. This gives

$$\frac{1}{(1-x)(1-2x)} = \frac{a(1-2x)+b(1-x)}{(1-x)(1-2x)}.$$

so

$$1 = a(1-2x) + b(1-x).$$

This must be true for all values of x. If $x = 1$, then the equation becomes $1 = -a$ so $a = -1$. When $x = \frac{1}{2}$ we get $1 = b/2$ so $b = 2$. This tells us that we can decompose the fraction like this:

$$\frac{1}{(1-x)(1-2x)} = \frac{-1}{1-x} + \frac{2}{1-2x}.$$

This completes the partial fraction decomposition. Notice that these two fractions are generating functions we know. In fact, we should be able to expand each of them.

$$\frac{-1}{1-x} = -1 - x - x^2 - x^3 - x^4 - \cdots \text{ which generates } -1, -1, -1, -1, -1, \ldots.$$

$$\frac{2}{1-2x} = 2 + 4x + 8x^2 + 16x^3 + 32x^4 + \cdots \text{ which generates } 2, 4, 8, 16, 32, \ldots.$$

We can give a closed formula for the nth term of each of these sequences. The first is just $a_n = -1$. The second is $a_n = 2^{n+1}$. The sequence we are interested in is just the sum of these, so the solution to the recurrence relation is

$$a_n = 2^{n+1} - 1.$$

We can now add generating functions to our list of methods for solving recurrence relations.

6.1.5 Exercises

1. Find the generating function for each of the following sequences by relating them back to a sequence with known generating function.

(a) $4, 4, 4, 4, 4, \ldots$.

(b) $2, 4, 6, 8, 10, \ldots$.

(c) $0, 0, 0, 2, 4, 6, 8, 10, \ldots$.

(d) $1, 5, 25, 125, \ldots$.

(e) $1, -3, 9, -27, 81, \ldots$.

(f) $1, 0, 5, 0, 25, 0, 125, 0, \ldots$.

(g) $0, 1, 0, 0, 2, 0, 0, 3, 0, 0, 4, 0, 0, 5, \ldots$.

2. Find the sequence generated by the following generating functions:

(a) $\dfrac{4x}{1-x}$.

(b) $\dfrac{1}{1-4x}$.

(c) $\dfrac{x}{1+x}$.

(d) $\dfrac{3x}{(1+x)^2}$.

(e) $\dfrac{1+x+x^2}{(1-x)^2}$ (Hint: multiplication).

3. Show how you can get the generating function for the triangular numbers in three different ways:

(a) Take two derivatives of the generating function for $1, 1, 1, 1, 1, \ldots$

(b) Use differencing.

(c) Multiply two known generating functions.

4. Use differencing to find the generating function for $4, 5, 7, 10, 14, 19, 25, \ldots$.

5. Find a generating function for the sequence with recurrence relation $a_n = 3a_{n-1} - a_{n-2}$ with initial terms $a_0 = 1$ and $a_1 = 5$.

6. Use the recurrence relation for the Fibonacci numbers to find the generating function for the Fibonacci sequence.

7. Use multiplication to find the generating function for the sequence of partial sums of Fibonacci numbers, $S_0, S_1, S_2, \ldots$ where $S_0 = F_0$, $S_1 = F_0 + F_1$, $S_2 = F_0 + F_1 + F_2$, $S_3 = F_0 + F_1 + F_2 + F_3$ and so on.

8. Find the generating function for the sequence with closed formula $a_n = 2(5^n) + 7(-3)^n$.

9. Find a closed formula for the nth term of the sequence with generating function $\dfrac{3x}{1-4x} + \dfrac{1}{1-x}$.

10. Find a_7 for the sequence with generating function $\dfrac{2}{(1-x)^2} \cdot \dfrac{x}{1-x-x^2}$.

11. Explain how we know that $\dfrac{1}{(1-x)^2}$ is the generating function for $1, 2, 3, 4, \ldots$.

12. Starting with the generating function for $1, 2, 3, 4, \ldots$, find a generating function for each of the following sequences.

(a) $1, 0, 2, 0, 3, 0, 4, \ldots$.

(b) $1, -2, 3, -4, 5, -6, \ldots$.

(c) $0, 3, 6, 9, 12, 15, 18, \ldots$.

(d) $0, 3, 9, 18, 30, 45, 63, \ldots$. (Hint: relate this sequence to the previous one.)

13. You may assume that $1, 1, 2, 3, 5, 8, \ldots$ has generating function $\dfrac{1}{1-x-x^2}$ (because it does). Use this fact to find the sequence generated by each of the following generating functions.

(a) $\frac{x^2}{1-x-x^2}$.

(b) $\frac{1}{1-x^2-x^4}$.

(c) $\frac{1}{1-3x-9x^2}$.

(d) $\frac{1}{(1-x-x^2)(1-x)}$.

14. Find the generating function for the sequence $1, -2, 4, -8, 16, \ldots$.

15. Find the generating function for the sequence $1, 1, 1, 2, 3, 4, 5, 6, \ldots$.

16. Suppose A is the generating function for the sequence $3, 5, 9, 15, 23, 33, \ldots$.

(a) Find a generating function (in terms of A) for the sequence of differences between terms.

(b) Write the sequence of differences between terms and find a generating function for it (without referencing A).

(c) Use your answers to parts (a) and (b) to find the generating function for the original sequence.

6.2 Introduction to Number Theory

We have used the natural numbers to solve problems. This was the right set of numbers to work with in discrete mathematics because we always dealt with a whole number of things. The natural numbers have been a tool. Let's take a moment now to inspect that tool. What mathematical discoveries can we make *about* the natural numbers themselves?

This is the main question of number theory: a huge, ancient, complex, and above all, beautiful branch of mathematics. Historically, number theory was known as the Queen of Mathematics and was very much a branch of *pure* mathematics, studied for its own sake instead of as a means to understanding real-world applications. This has changed in recent years however, as applications of number theory have been unearthed. Probably the most well-known example of this is RSA cryptography, one of the methods used to encrypt data on the internet. It is number theory that makes this possible.

What sorts of questions belong to the realm of number theory? Here is a motivating example. Recall in our study of induction, we asked:

> Which amounts of postage can be made exactly using just 5-cent and 8-cent stamps?

We were able to prove that *any* amount greater than 27 cents could be made. You might wonder what would happen if we changed the denomination of the stamps. What if we instead had 4- and 9-cent stamps? Would there be some amount after which all amounts would be possible? Well, again, we could replace two 4-cent stamps with a 9-cent stamp, or three 9-cent stamps with seven 4-cent stamps. In each case we can create one more cent of postage. Using this as the inductive case would allow us to prove that any amount of postage greater than 23 cents can be made.

What if we had 2-cent and 4-cent stamps. Here it looks less promising. If we take some number of 2-cent stamps and some number of 4-cent stamps, what can we say about the total? Could it ever be odd? Doesn't look like it.

Why does 5 and 8 work, 4 and 9 work, but 2 and 4 not work? What is it about these numbers? If I gave you a pair of numbers, could you tell me right away if they would work or not? We will answer these questions, and more, after first investigating some simpler properties of numbers themselves.

6.2.1 Divisibility

It is easy to add and multiply natural numbers. If we extend our focus to all integers, then subtraction is also easy (we need the negative numbers, so we can subtract any number from any other number, even larger from smaller). Division is the first operation that presents a challenge. If we wanted to extend our set of numbers so any division would be possible (maybe excluding division by 0), we would need to look at the rational numbers (the set of all numbers that can be written as fractions). This would be going too far, so we will refuse this option.

In fact, it is a good thing that not every number can be divided by other numbers. This helps us understand the structure of the natural numbers and opens the door to many interesting questions and applications.

If given numbers a and b, it is possible that $a \div b$ gives a whole number. In this case, we say that b *divides* a; in symbols, we write $b \mid a$. If this holds, then b is a divisor or factor of a, and a is a multiple of b. In other words, if $b \mid a$, then $a = bk$ for some integer k (this is saying a is some multiple of b).

The Divisibility Relation.

Given integers m and n, we say "m divides n" and write

$$m \mid n$$

provided $n \div m$ is an integer. Thus the following assertions mean the same thing:

1. $m \mid n$.
2. $n = mk$ for some integer k.
3. m is a factor (or divisor) of n.
4. n is a multiple of m.

Notice that $m \mid n$ is a statement. It is either true or false. On the other hand, $n \div m$ or n/m is some number. If we want to claim that n/m is not an integer, so m does not divide n, then we can write $m \nmid n$.

Example 6.2.1

Decide whether each of the statements below are true or false.

1. $4 \mid 20$
2. $20 \mid 4$
3. $0 \mid 5$
4. $5 \mid 0$
5. $7 \mid 7$
6. $1 \mid 37$
7. $-3 \mid 12$
8. $8 \mid 12$
9. $1642 \mid 136299$

Solution.

1. True. 4 "goes into" 20 five times without remainder. In other words, $20 \div 4 = 5$, an integer. We could also justify this by saying that 20 is a multiple of 4: $20 = 4 \cdot 5$.
2. False. While 20 is a multiple of 4, it is false that 4 is a multiple of 20.
3. False. $5 \div 0$ is not even defined, let alone an integer.

4. True. In fact, $x \mid 0$ is true for all x. This is because 0 is a multiple of every number: $0 = x \cdot 0$.
5. True. In fact, $x \mid x$ is true for all x.
6. True. 1 divides every number (other than 0).
7. True. Negative numbers work just fine for the divisibility relation. Here $12 = -3 \cdot 4$. It is also true that $3 \mid -12$ and that $-3 \mid -12$.
8. False. Both 8 and 12 are divisible by 4, but this does not mean that 12 is divisible by 8.
9. False. See below.

This last example raises a question: How might one decide whether $m \mid n$? Of course, if you had a trusted calculator, you could ask it for the value of $n \div m$. If it spits out anything other than an integer, you know $m \nmid n$. This seems a little like cheating though: We don't have division, so should we really use division to check divisibility?

While we don't really know how to divide, we do know how to multiply. We might try multiplying m by larger and larger numbers until we get close to n. How close? Well, we want to be sure that if we multiply m by the next larger integer, we go over n.

For example, let's try this to decide whether $1642 \mid 136299$. Start finding multiples of 1642:

$$1642 \cdot 2 = 3284 \qquad 1642 \cdot 3 = 4926 \qquad 1642 \cdot 4 = 6568 \qquad \cdots.$$

All of these are well less than 136299. I suppose we can jump ahead a bit:

$$1642 \cdot 50 = 82100 \qquad 1642 \cdot 80 = 131360 \qquad 1642 \cdot 85 = 139570.$$

Ah, so we need to look somewhere between 80 and 85. Try 83:

$$1642 \cdot 83 = 136286.$$

Is this the best we can do? How far are we from our desired 136299? If we subtract, we get $136299 - 136286 = 13$. So we know we cannot go up to 84; that will be too much. In other words, we have found that

$$136299 = 83 \cdot 1642 + 13.$$

Since $13 < 1642$, we can now safely say that $1642 \nmid 136299$.

It turns out that the process we went through above can be repeated for any pair of numbers. We can always write the number a as some multiple of the number b plus some remainder. We know this because we know about **division with remainder** from elementary school. This is just a way of saying it using multiplication. Due to the procedural nature that can be used to find the remainder, this fact is usually called the **division algorithm**:

The Division Algorithm.

Given any two integers a and b, we can always find an integer q such that

$$a = qb + r$$

where r is an integer satisfying $0 \leq r < |b|$

The idea is that we can always take a large enough multiple of b so that the remainder r is as small as possible. We do allow the possibility of $r = 0$, in which case we have $b \mid a$.

6.2.2 Remainder Classes

The division algorithm tells us that there are only b possible remainders when dividing by b. If we fix this divisor, we can group integers by the remainder. Each group is called a **remainder class modulo** b (or sometimes **residue class**).

Example 6.2.2

Describe the remainder classes modulo 5.

Solution. We want to classify numbers by what their remainder would be when divided by 5. From the division algorithm, we know there will be exactly 5 remainder classes, because there are only 5 choices for what r could be ($0 \leq r < 5$).

First consider $r = 0$. Here we are looking for all the numbers divisible by 5 since $a = 5q + 0$. In other words, the multiples of 5. We get the infinite set

$$\{\ldots, -15, -10, -5, 0, 5, 10, 15, 20, \ldots\}.$$

Notice we also include negative integers.

Next consider $r = 1$. Which integers, when divided by 5, have remainder 1? Well, certainly 1 does, as does 6, and 11. Negatives? Here we must be careful: -6 does NOT have remainder 1. We can write $-6 = -2 \cdot 5 + 4$ or $-6 = -1 \cdot 5 - 1$, but only one of these is a "correct" instance of the division algorithm: $r = 4$ since we need r to be non-negative. So in fact, to get $r = 1$, we would have -4, or -9, etc. Thus we get the remainder class

$$\{\ldots, -14, -9, -4, 1, 6, 11, 16, 21, \ldots\}.$$

There are three more to go. The remainder classes for 2, 3, and 4 are, respectively

$$\{\ldots, -13, -8, -3, 2, 7, 12, 17, 22, \ldots\}$$

$$\{\ldots, -12, -7, -2, 3, 8, 13, 18, 23, \ldots\}$$

$$\{\ldots, -11, -6, -1, 4, 9, 14, 19, 24, \ldots\}.$$

Note that in the example above, *every* integer is in exactly one remainder class. The technical way to say this is that the remainder classes modulo b form a **partition** of the integers.[1] The most important fact about partitions is that it is possible to define an **equivalence relation** from a partition: This is a relationship between pairs of numbers which acts in all the important ways like the "equals" relationship.[2]

All fun technical language aside, the idea is really simple. If two numbers belong to the same remainder class, then in some way, they are the same. That is, they are the same *up to division by* b. In the case where $b = 5$ above, the numbers 8 and 23, while not the same number, are the same when it comes to dividing by 5, because both have remainder 3.

It matters what the divisor is: 8 and 23 are the same up to division by 5, but not up to division by 7, since 8 has a remainder of 1 when divided by 7 while 23 has a remainder of 2.

With all this in mind, let's introduce some notation. We want to say that 8 and 23 are basically the same, even though they are not equal. It would be wrong to say $8 = 23$. Instead, we write $8 \equiv 23$. But this is not always true. It works if we are thinking division by 5, so we need to denote that somehow. What we will actually write is this:

$$8 \equiv 23 \pmod 5$$

which is read, "8 is congruent to 23 modulo 5" (or just "mod 5"). Of course then we could observe that

$$8 \not\equiv 23 \pmod 7.$$

Congruence Modulo n.

We say a **is congruent to** b **modulo** n, and write,

$$a \equiv b \pmod n$$

provided a and b have the same remainder when divided by n. In other words, provided a and b belong to the same remainder class modulo n.

Many books define congruence modulo n slightly differently. They say that $a \equiv b \pmod n$ if and only if $n \mid a - b$. In other words, two numbers are congruent modulo n, if their difference is a multiple of n. So which definition is correct? It turns out that it doesn't matter; they are equivalent.

To see why, consider two numbers a and b that are congruent modulo n. Then a and b have the same remainder when divided by n. We have

$$a = q_1 n + r \qquad\qquad b = q_2 n + r.$$

[1] It is possible to develop a mathematical theory of partitions, prove statements about all partitions in general, and then apply those observations to our case here.

[2] Again, there is a mathematical theory of equivalence relations which applies in many more instances than the one we look at here. See Subsection 2.6.4.

Here the two r's really are the same. Consider what we get when we take the difference of a and b:

$$a - b = q_1 n + r - (q_2 n + r) = q_1 n - q_2 n = (q_1 - q_2)n.$$

So $a - b$ is a multiple of n, or equivalently, $n \mid a - b$.

On the other hand, if we assume first that $n \mid a - b$, so $a - b = kn$, then consider what happens if we divide each term by n. Dividing a by n will leave some remainder, as will dividing b by n. However, dividing kn by n will leave 0 remainder. So the remainders on the left-hand side must cancel out. That is, the remainders must be the same.

Thus we have:

Congruence and Divisibility.

For any integers a, b, and n, we have

$$a \equiv b \pmod{n} \qquad \text{if and only if} \qquad n \mid (a - b).$$

It will also be useful to switch back and forth between congruences and regular equations. The above fact helps with this. We know that $a \equiv b \pmod{n}$ if and only if $n \mid a - b$, if and only if $a - b = kn$ for some integer k. Rearranging that equation, we get $a = b + kn$. In other words, if a and b are congruent modulo n, then a is b more than some multiple of n. This conforms with our earlier observation that all the numbers in a particular remainder class are the same amount larger than the multiples of n.

Congruence and Equality.

For any integers a, b, and n, we have

$$a \equiv b \pmod{n} \qquad \text{if and only if} \qquad a = b + kn \text{ for some integer } k.$$

6.2.3 Properties of Congruence

We said earlier that congruence modulo n behaves, in many important ways, the same way equality does. Specifically, we could prove that congruence modulo n is an **equivalence relation**, which would require checking the following three facts:

Congruence Modulo n is an Equivalence Relation.

Given any integers a, b, and c, and any positive integer n, the following hold:

1. $a \equiv a \pmod{n}$.
2. If $a \equiv b \pmod{n}$ then $b \equiv a \pmod{n}$.

3. If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.

In other words, congruence modulo n is reflexive, symmetric, and transitive, and so is an equivalence relation.

You should take a minute to convince yourself that each of the properties above actually holds for congruence. Try explaining each using both the remainder and divisibility definitions.

Next, consider how congruence behaves when doing basic arithmetic. We already know that if you subtract two congruent numbers, the result will be congruent to 0 (be a multiple of n). What if we add something congruent to 1 to something congruent to 2? Will we get something congruent to 3?

Congruence and Arithmetic.

Suppose $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$. Then the following hold:

1. $a + c \equiv b + d \pmod{n}$.
2. $a - c \equiv b - d \pmod{n}$.
3. $ac \equiv bd \pmod{n}$.

The above facts might be written a little strangely, but the idea is simple. If we have a true congruence, and we add the same thing to both sides, the result is still a true congruence. This sounds like we are saying:

If $a \equiv b \pmod{n}$ then $a + c \equiv b + c \pmod{n}$.

Of course this is true as well; it is the special case where $c = d$. But what we have works in more generality. Think of congruence as being "basically equal." If we have two numbers that are basically equal, and we add basically the same thing to both sides, the result will be basically equal.

This seems reasonable. Is it really true? Let's prove the first fact:

Proof. Suppose $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$. That means $a = b + kn$ and $c = d + jn$ for integers k and j. Add these equations:

$$a + c = b + d + kn + jn.$$

But $kn + jn = (k + j)n$, which is just a multiple of n. So $a + c = b + d + (j + k)n$, or in other words, $a + c \equiv b + d \pmod{n}$.

The other two facts can be proved in a similar way.

One of the important consequences of these facts about congruences is that we can basically replace any number in a congruence with any other number it is congruent to. Here are some examples to see how (and why) that works:

Example 6.2.3

Find the remainder of 3491 divided by 9.

Solution. We could do long division, but there is another way. We want to find x such that $x \equiv 3491 \pmod 9$. Now $3491 = 3000 + 400 + 90 + 1$. Of course $90 \equiv 0 \pmod 9$, so we can replace the 90 in the sum with 0. Why is this okay? We are actually subtracting the "same" thing from both sides:

$$\begin{aligned} x &\equiv 3000 + 400 + 90 + 1 \pmod 9 \\ - \quad 0 &\equiv 90 \pmod 9 \\ x &\equiv 3000 + 400 + 0 + 1 \pmod 9. \end{aligned}$$

Next, note that $400 = 4 \cdot 100$, and $100 \equiv 1 \pmod 9$ (since $9 \mid 99$). So we can in fact replace the 400 with simply a 4. Again, we are appealing to our claim that we can replace congruent elements, but we are really appealing to property 3 about the arithmetic of congruence: We know $100 \equiv 1 \pmod 9$, so if we multiply both sides by 4, we get $400 \equiv 4 \pmod 9$.

Similarly, we can replace 3000 with 3, since $1000 = 1 + 999 \equiv 1 \pmod 9$. So our original congruence becomes

$$x \equiv 3 + 4 + 0 + 1 \pmod 9$$

$$x \equiv 8 \pmod 9.$$

Therefore 3491 divided by 9 has remainder 8.

The above example should convince you that the well-known divisibility test for 9 is true: The sum of the digits of a number is divisible by 9 if and only if the original number is divisible by 9. In fact, we now know something more: Any number is congruent to the sum of its digits, modulo 9.[3]

Let's try another.

Example 6.2.4

Find the remainder when 3^{123} is divided by 7.

Solution. Of course, we are working with congruence because we want to find the smallest positive x such that $x \equiv 3^{123} \pmod 7$. Now first write $3^{123} = (3^3)^{41}$. We have:

$$3^{123} = 27^{41} \equiv 6^{41} \pmod 7,$$

since $27 \equiv 6 \pmod 7$. Notice further that $6^2 = 36$ is congruent to 1 modulo 7. Thus we can simplify further:

$$6^{41} = 6 \cdot (6^2)^{20} \equiv 6 \cdot 1^{20} \pmod 7.$$

[3]This works for 3 as well, but definitely not for any modulus in general.

But $1^{20} = 1$, so we are done:

$$3^{123} \equiv 6 \pmod{7}.$$

In the above example, we are using the fact that if $a \equiv b \pmod{n}$, then $a^p \equiv b^p \pmod{n}$. This is just applying property 3 a bunch of times.

So far we have seen how to add, subtract, and multiply with congruences. What about division? There is a reason we have waited to discuss it. It turns out that we cannot simply divide. In other words, even if $ad \equiv bd \pmod{n}$, we do not know that $a \equiv b \pmod{n}$. Consider, for example,

$$18 \equiv 42 \pmod{8}.$$

This is true. Now 18 and 42 are both divisible by 6. However,

$$3 \not\equiv 7 \pmod{8}.$$

While this doesn't work, note that $3 \equiv 7 \pmod{4}$. We cannot divide 8 by 6, but we can divide 8 by the greatest common factor of 8 and 6. Will this always happen?

Suppose $ad \equiv bd \pmod{n}$. In other words, we have $ad = bd + kn$ for some integer k. Of course ad is divisible by d, as is bd. So kn must also be divisible by d. Now if n and d have no common factors (other than 1), then we must have $d \mid k$. But in general, if we try to divide kn by d, we don't know that we will get an integer multiple of n. Some of the n might get divided as well. To be safe, let's divide as much of n as we can. Take the largest factor of both d and n, and cancel that out from n. The rest of the factors of d will come from k, no problem.

We will call the largest factor of both d and n the $\gcd(d,n)$, for *greatest common divisor*. In our example above, $\gcd(6,8) = 2$ since the greatest divisor common to 6 and 8 is 2.

Congruence and Division.

Suppose $ad \equiv bd \pmod{n}$. Then $a \equiv b \pmod{\frac{n}{\gcd(d,n)}}$.
If d and n have no common factors, then $\gcd(d,n) = 1$, so $a \equiv b \pmod{n}$.

Example 6.2.5

Simplify the following congruences using division: (a) $24 \equiv 39 \pmod{5}$ and (b) $24 \equiv 39 \pmod{15}$.

Solution. (a) Both 24 and 39 are divisible by 3, and 3 and 5 have no common factors, so we get

$$8 \equiv 13 \pmod{5}.$$

(b) Again, we can divide by 3. However, doing so blindly gives us $8 \equiv 13 \pmod{15}$ which is no longer true. Instead, we must also divide the modulus

15 by the greatest common factor of 3 and 15, which is 3. Again we get

$$8 \equiv 13 \pmod{5}.$$

6.2.4 Solving Congruences

Now that we have some algebraic rules to govern congruence relations, we can attempt to solve for an unknown in a congruence. For example, is there a value of x that satisfies,

$$3x + 2 \equiv 4 \pmod{5},$$

and if so, what is it?

In this example, since the modulus is small, we could simply try every possible value for x. There are really only 5 to consider, since any integer that satisfied the congruence could be replaced with any other integer it was congruent to modulo 5. Here, when $x = 4$ we get $3x + 2 = 14$, which is indeed congruent to 4 modulo 5. This means that $x = 9$ and $x = 14$ and $x = 19$ and so on will each also be a solution because, as we saw above, replacing any number in a congruence with a congruent number does not change the truth of the congruence.

So in this example, simply compute $3x + 2$ for values of $x \in \{0, 1, 2, 3, 4\}$. This gives 2, 5, 8, 11, and 14 respectively, for which only 14 is congruent to 4.

Let's also see how you could solve this using our rules for the algebra of congruences. Such an approach would be much simpler than the trial and error tactic if the modulus was larger. First, we know we can subtract 2 from both sides:

$$3x \equiv 2 \pmod{5}.$$

Then to divide both sides by 3, we first add 0 to both sides. Of course, on the right-hand side, we want that 0 to be a 10 (yes, 10 really is 0 since they are congruent modulo 5). This gives,

$$3x \equiv 12 \pmod{5}.$$

Now divide both sides by 3. Since $\gcd(3, 5) = 1$, we do not need to change the modulus:

$$x \equiv 4 \pmod{5}.$$

Notice that this in fact gives the *general solution*: Not only can $x = 4$, but x can be any number which is congruent to 4. We can leave it like this, or write "$x = 4 + 5k$ for any integer k."

Example 6.2.6

Solve the following congruences for x.

1. $7x \equiv 12 \pmod{13}$.
2. $84x - 38 \equiv 79 \pmod{15}$.

3. $20x \equiv 23 \pmod{14}$.

Solution.

1. All we need to do here is divide both sides by 7. We add 13 to the right-hand side repeatedly until we get a multiple of 7 (adding 13 is the same as adding 0, so this is legal). We get 25, 38, 51, 64, 77 – got it. So we have:
$$\begin{aligned} 7x &\equiv 12 \pmod{13} \\ 7x &\equiv 77 \pmod{13} \\ x &\equiv 11 \pmod{13}. \end{aligned}$$

2. Here, since we have numbers larger than the modulus, we can reduce them prior to applying any algebra. We have $84 \equiv 9$, $38 \equiv 8$ and $79 \equiv 4$. Thus,
$$\begin{aligned} 84x - 38 &\equiv 79 \pmod{15} \\ 9x - 8 &\equiv 4 \pmod{15} \\ 9x &\equiv 12 \pmod{15} \\ 9x &\equiv 72 \pmod{15}. \end{aligned}$$
We got the 72 by adding $0 \equiv 60 \pmod{15}$ to both sides of the congruence. Now divide both sides by 9. However, since $\gcd(9, 15) = 3$, we must divide the modulus by 3 as well:
$$x \equiv 8 \pmod{5}.$$
So the solutions are those values that are congruent to 8, or equivalently 3, modulo 5. This means that in some sense there are 3 solutions modulo 15: 3, 8, and 13. We can write the solution:
$$x \equiv 3 \pmod{15}; \quad x \equiv 8 \pmod{15}; \quad x \equiv 13 \pmod{15}.$$

3. First, reduce modulo 14:
$$20x \equiv 23 \pmod{14}$$
$$6x \equiv 9 \pmod{14}.$$
We could now divide both sides by 3 or try to increase 9 by a multiple of 14 to get a multiple of 6. If we divide by 3, we get,
$$2x \equiv 3 \pmod{14}.$$
Now try adding multiples of 14 to 3, in hopes of getting a number we can divide by 2. This will not work! Every time we add 14 to the right side, the result will still be odd. We will never get an even number, so we will never be able to divide by 2. Thus there are no solutions to the congruence.

The last congruence above illustrates the way in which congruences might not have solutions. We could have seen this immediately in fact. Look at the original congruence:

$$20x \equiv 23 \pmod{14}.$$

If we write this as an equation, we get

$$20x = 23 + 14k,$$

or equivalently $20x - 14k = 23$. We can easily see there will be no solution to this equation in integers. The left-hand side will always be even, but the right-hand side is odd. A similar problem would occur if the right-hand side was divisible by *any* number that the left-hand side was not.

So in general, given the congruence

$$ax \equiv b \pmod{n},$$

if a and n are divisible by a number by which b is not divisible, then there will be no solutions. In fact, we really only need to check one divisor of a and n: the greatest common divisor. Thus, a more compact way to say this is:

Congruences with No Solutions.

If $\gcd(a, n) \nmid b$, then $ax \equiv b \pmod{n}$ has no solutions.

6.2.5 Solving Linear Diophantine Equations

Discrete math deals with whole numbers of things. So when we want to solve equations, we usually are looking for *integer* solutions. Equations that are intended to only have integer solutions were first studied by in the third century by the Greek mathematician Diophantus of Alexandria, and as such are called *Diophantine equations*. Probably the most famous example of a Diophantine equation is $a^2+b^2 = c^2$. The integer solutions to this equation are called **Pythagorean triples**. In general, solving Diophantine equations is hard (in fact, there is provably no general algorithm for deciding whether a Diophantine equation has a solution, a result known as Matiyasevich's Theorem). We will restrict our focus to *linear* Diophantine equations, which are considerably easier to work with.

Diophantine Equations.

An equation in two or more variables is called a **Diophantine equation** if only integer solutions are of interest. A **linear** Diophantine equation takes the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ for constants $a_1, \ldots, a_n, b$.

A **solution** to a Diophantine equation is a solution to the equation consisting only of integers.

We have the tools we need to solve linear Diophantine equations. We will consider, as a main example, the equation

$$51x + 87y = 123.$$

The general strategy will be to convert the equation to a congruence, and then solve that congruence.[4] Let's work through this particular example to see how this might go.

First, check if perhaps there are no solutions because a divisor of 51 and 87 is not a divisor of 123. Really, we just need to check whether $\gcd(51, 87) \mid 123$. This greatest common divisor is 3, and yes $3 \mid 123$. At this point, we might as well factor out this greatest common divisor. So instead, we will solve:

$$17x + 29y = 41.$$

Now observe that if there are going to be solutions, then for those values of x and y, the two sides of the equation must have the same remainder as each other, no matter what we divide by. In particular, if we divide both sides by 17, we must get the same remainder. Thus we can safely write

$$17x + 29y \equiv 41 \pmod{17}.$$

We choose 17 because $17x$ will have remainder 0. This will allow us to reduce the congruence to just one variable. We could have also moved to a congruence modulo 29, although there is usually a good reason to select the smaller choice, as this will allow us to reduce the other coefficient. In our case, we reduce the congruence as follows:

$$\begin{aligned} 17x + 29y &\equiv 41 \pmod{17} \\ 0x + 12y &\equiv 7 \pmod{17} \\ 12y &\equiv 24 \pmod{17} \\ y &\equiv 2 \pmod{17}. \end{aligned}$$

Now at this point we know $y = 2 + 17k$ will work for any integer k. If we haven't made a mistake, we should be able to plug this back into our original Diophantine equation to find x:

$$\begin{aligned} 17x + 29(2 + 17k) = 41 \\ 17x &= -17 - 29 \cdot 17k \\ x &= -1 - 29k. \end{aligned}$$

We have now found all solutions to the Diophantine equation. For each k, $x = -1 - 29k$ and $y = 2 + 17k$ will satisfy the equation. We could check this for a few cases. If $k = 0$, the solution is $(-1, 2)$, and yes, $-17 + 2 \cdot 29 = 41$. If $k = 3$, the solution is $(-88, 53)$. If $k = -2$, we get $(57, -32)$.

To summarize this process, to solve $ax + by = c$, we,

[4] This is certainly not the only way to proceed. A more common technique would be to apply the **Euclidean algorithm**. Our way can be a little faster, and is presented here primarily for variety.

1. Divide both sides of the equation by $\gcd(a, b)$ (if this does not leave the right-hand side as an integer, there are no solutions). Let's assume that $ax + by = c$ has already been reduced in this way.

2. Pick the smaller of a and b (here, assume it is b), and convert to a congruence modulo b:
$$ax + by \equiv c \pmod{b}.$$
This will reduce to a congruence with one variable, x:
$$ax \equiv c \pmod{b}.$$

3. Solve the congruence as we did in the previous section. Write your solution as an equation, such as,
$$x = n + kb.$$

4. Plug this into the original Diophantine equation, and solve for y.

5. If we want to know solutions in a particular range (for example, $0 \leq x, y \leq 20$), pick different values of k until you have all required solutions.

Here is another example:

Example 6.2.7

How can you make \$6.37 using just 5-cent and 8-cent stamps? What is the smallest and largest number of stamps you could use?

Solution. First, we need a Diophantine equation. We will work in numbers of cents. Let x be the number of 5-cent stamps, and y be the number of 8-cent stamps. We have:
$$5x + 8y = 637.$$

Convert to a congruence and solve:
$$\begin{aligned} 8y &\equiv 637 \pmod{5} \\ 3y &\equiv 2 \pmod{5} \\ 3y &\equiv 12 \pmod{5} \\ y &\equiv 4 \pmod{5}. \end{aligned}$$

Thus $y = 4 + 5k$. Then $5x + 8(4 + 5k) = 637$, so $x = 121 - 8k$.

This says that one way to make \$6.37 is to take 121 of the 5-cent stamps and 4 of the 8-cent stamps. To find the smallest and largest number of stamps, try different values of k.

k	(x, y)	Stamps
-1	(129, -1)	not possible
0	(121, 4)	125
1	(113, 9)	122
2	(105, 13)	119
$\vdots$	$\vdots$	$\vdots$

This is no surprise. Having the most stamps means we have as many 5-cent stamps as possible, and to get the smallest number of stamps would require having the least number of 5-cent stamps. To minimize the number of 5-cent stamps, we want to pick k so that $121 - 8k$ is as small as possible (but still positive). When $k = 15$, we have $x = 1$ and $y = 79$.

Therefore, to make \$6.37, you can use as few as 80 stamps (1 5-cent stamp and 79 8-cent stamps) or as many as 125 stamps (121 5-cent stamps and 4 8-cent stamps).

Using this method, as long as you can solve linear congruences in one variable, you can solve linear Diophantine equations of two variables. There are times, though, that solving the linear congruence is a lot of work. For example, suppose you need to solve,

$$13x \equiv 6 \pmod{51}.$$

You *could* keep adding 51 to the right side until you get a multiple of 13: You would get 57, 108, 159, 210, 261, 312, and 312 is the first of these that is divisible by 13. This works but is really too much work. Instead we could convert *back* to a Diophantine equation:

$$13x = 6 + 51k.$$

Now solve *this* like we have in this section. Write it as a congruence modulo 13:

$$\begin{aligned} 0 &\equiv 6 + 51k \pmod{13} \\ -12k &\equiv 6 \pmod{13} \\ 2k &\equiv -1 \pmod{13} \\ 2k &\equiv 12 \pmod{13} \\ k &\equiv 6 \pmod{13}. \end{aligned}$$

so $k = 6 + 13j$. Now go back and figure out x:

$$\begin{aligned} 13x &= 6 + 51(6 + 13j) \\ x &= 24 + 51j. \end{aligned}$$

Of course you could do this switching back and forth between congruences and Diophantine equations as many times as you like. If you *only* used this technique, you would essentially replicate the Euclidean algorithm, a more standard way to solve Diophantine equations.

6.2.6 Exercises

1. Suppose a, b, and c are integers. Prove that if $a \mid b$, then $a \mid bc$.

2. Suppose a, b, and c are integers. Prove that if $a \mid b$ and $a \mid c$ then $a \mid b + c$ and $a \mid b - c$.

3. Write out the remainder classes for $n = 4$.

4. What is the largest n such that 16 and 25 are in the same remainder class modulo n? Write out the remainder class they both belong to and give an example of a number more than 100 in that class.

5. Let a, b, c, and n be integers. Prove that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a - c \equiv b - d \pmod{n}$.

6. Find the remainder of 3^{456} when divided by

(a) 2. (b) 5. (c) 7. (d) 9.

7. Repeat the previous exercise, this time dividing 2^{2019}.

8. Determine which of the following congruences have solutions, and find any solutions (between 0 and the modulus) by trial and error.

(a) $4x \equiv 5 \pmod{6}$.

(b) $6x \equiv 3 \pmod{9}$.

(c) $x^2 \equiv 2 \pmod{4}$.

9. Determine which of the following congruences have solutions, and find any solutions (between 0 and the modulus) by trial and error.

(a) $4x \equiv 5 \pmod{7}$.

(b) $6x \equiv 4 \pmod{9}$.

(c) $x^2 \equiv 2 \pmod{7}$.

10. Solve the congruence: $5x + 8 \equiv 11 \pmod{22}$. That is, describe the general solution.

11. Solve the congruence: $6x \equiv 4 \pmod{10}$.

12. Solve the congruence: $4x \equiv 24 \pmod{30}$.

13. Solve the congruence: $341x \equiv 2941 \pmod{9}$.

14. I'm thinking of a number. If you multiply my number by 7, add 5, and divide the result by 11, you will be left with a remainder of 2. What remainder would you get if you divided my original number by 11?

15. Solve the following linear Diophantine equation, using modular arithmetic (describe the general solutions).

$$6x + 10y = 32.$$

16. Solve the following linear Diophantine equation, using modular arithmetic (describe the general solutions).

$$17x + 8y = 31.$$

17. Solve the following linear Diophantine equation, using modular arithmetic (describe the general solutions).

$$35x + 47y = 1.$$

18. You have a 13 oz. bottle and a 20 oz. bottle, with which you wish to measure exactly 2 oz. However, you have a limited supply of water. If any water enters either bottle and then gets dumped out, it is gone forever. What is the least amount of water you can start with and still complete the task?

APPENDIX A

SELECTED HINTS

1 · Logic and Proofs
1.1 · Mathematical Statements
1.1.6 · Additional Exercises

1.1.6.3. First figure out what each statement is saying. For part (c), you don't need to assume the domain is an infinite set.

1.2 · Implications
1.2.6 · Additional Exercises

1.2.6.4. Of course there are many answers. It helps to assume that the statement is true and the converse is *not* true. Think about what that means in the real world, and then start saying it in different ways. Some ideas: Use "necessary and sufficient" language, use "only if," consider negations, use "or else" language.

1.3 · Rules of Logic
1.3.8 · Additional Exercises

1.3.8.1. You could probably reason through the cases by hand, but try making a truth table. Use two statements, P being "we are cousins" and Q being "we are both knaves".

1.3.8.4. You should write down three statements using the symbols P, Q, R, S. If Geoff is a truth-teller, then all three statements would be true. If he was a liar, then all three statements would be false. But in either case, we don't yet know whether the four atomic statements are true or false, since he hasn't said them by themselves.

A truth table might help, although it is probably not entirely necessary.

1.3.8.8.

(a) There will be three rows in which the statement is false.

(b) Consider the three rows that evaluate to false, and say what the truth values of T, S, and P are there.

(c) You are looking for a row in which P is true and the whole statement is true.

1.3.8.9. Write down three statements, and then take the negation of each (since he is a liar). You should find that Tommy ate one item and drank one item. (Q is for cucumber sandwiches.)

1.3.8.11. What do these concepts mean in terms of truth tables?

1.3.8.14. Try an example. What if $P(x)$ was the predicate, "x is prime"? What if it was, "If x is divisible by 4, then it is even"? Of course examples are not enough to prove something in general, but that is entirely the point of this question.

1.3.8.15. It might help to translate the statements into symbols and then use the formulaic rules to simplify negations (i.e., rules for quantifiers and De Morgan's laws). After simplifying, you should get $\forall x(\neg E(x) \wedge \neg O(x))$ for the first one, for example. Then translate this back into English.

1.4 · Proofs
1.4.8 · Additional Exercises

1.4.8.6. One of the implications will be a direct proof; the other will be a proof by contrapositive.

1.4.8.7. This is really an exercise in modifying the proof that $\sqrt{2}$ is irrational. There you proved things were even; here they will be multiples of 3.

1.4.8.8. Part (a) should be a relatively easy direct proof. Look for a counterexample for part (b).

1.4.8.10. A proof by contradiction would be reasonable here, because then you get to assume that both *a and b* are odd. Deduce that c^2 is even, and therefore a multiple of 4 (why? and why is that a contradiction?).

1.4.8.12. Use a different style of proof for each part.

1.4.8.14. Note that if $\log(7) = \frac{a}{b}$, then $7 = 10^{\frac{a}{b}}$. Can any power of 7 be the same as a power of 10?

1.4.8.15. What if there were? Deduce that x must be odd, and continue towards a contradiction.

1.4.8.16. Prove the contrapositive by cases. There will be 4 cases to consider.

1.4.8.17. Your friend's proof is a proof, but of what? What implication follows from the given proof? Is that helpful?

1.4.8.19. Consider the set of *numbers* of friends that everyone has. If everyone had a different number of friends, this set must contain 20 elements. Is that possible? Why not?

1.4.8.20. This feels like the pigeonhole principle, although a bit more complicated. At least, you could try to replicate the style of proof used by the pigeonhole principle. How would the episodes need to be spaced out so that no two of your sixty were exactly 4 apart?

1.5 · Proofs about Discrete Structures
1.5.8 · Additional Exercises

1.5.8.1. To prove that $A \subseteq B$ if and only if $A \cup B = B$, you need to prove two implications:

(a) If $A \subseteq B$, then $A \cup B = B$.

(b) If $A \cup B = B$, then $A \subseteq B$.

To prove two sets are equal, we usually prove that each is a subset of the other.

2 · Graph Theory
2.1 · Problems and Definitions
2.1.4 · Practice Problems

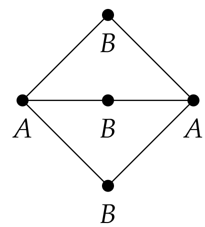
2.1.4.5. Remember that P_n is the path that contains n edges and $n + 1$ vertices.

2.1.5 · Additional Exercises

2.1.5.3. Both situations are possible. Go find some examples.

2.1.5.6. The bipartite graph is a little tricky. You will definitely want a complete bipartite graph, but it could be $K_{5,5}$ or maybe $K_{1,9}$, or . . .

2.1.5.7. The first graph is bipartite, which can be seen by labeling it as follows.



Two of the remaining three are also bipartite.

2.1.5.8. C_4 is bipartite; C_5 is not. What about all the other values of n?

2.1.5.11. You should be able to deduce everything directly from the definition. However, perhaps it would be helpful to know that the N stands for **neighborhood**.

2.1.5.12. Be careful to make sure the edges are not "directed." In a graph, if a is adjacent to b, then b is adjacent to a. In the language of relations, we say that the edge relation is **symmetric**.

2.1.5.13. You might want to answer the questions for some specific values of n to get a feel for them, but your final answers should be in terms of n.

2.1.5.14. Try a small example first: Any graph with 8 vertices must have two vertices of the same degree. If not, what would the degree sequence be?

2.1.5.15. Use the handshake lemma 2.1.8. What would happen if all the vertices had degree 2?

2.2 · Trees
2.2.7 · Additional Exercises

2.2.7.3. Careful: the graphs might not be connected.

2.2.7.4. Try Exercise 2.2.7.2.

2.2.7.5. Try a proof by contradiction, and consider a spanning tree of the graph.

2.2.7.7. For part (b), trying some simple examples should give you the formula. Then you just need to prove it is correct.

2.2.7.8. Examining the proof of Proposition 2.2.2 gives you most of what you need, but make sure to just give the relevant parts, and take care to not use proof by contradiction.

2.2.7.9. Minimality here should be in terms of the number of vertices. If you had a minimum counterexample and removed a leaf vertex, the resulting graph will be a smaller tree, so...

2.2.7.10. If e is the root, then b will have three children (a, c, and d), all of which will be siblings and have b as their parent. a will not have any children.

In general, how can you determine the number of children a vertex will have, if it is not a root?

2.2.7.14. There is an example with 7 edges.

2.2.7.15. The previous exercise will be helpful.

2.2.7.16. Note that such an edge, if removed, would disconnect the graph. We call graphs that have an edge like this **1-connected**.

2.3 · Planar Graphs
2.3.7 · Additional Exercises

2.3.7.3. What would Euler's formula tell you?

2.3.7.5. You can use the handshake lemma to find the number of edges, in terms of v, the number of vertices.

2.3.7.11. What is the length of the shortest cycle? (This quantity is usually called the **girth** of the graph.)

2.3.7.14. The girth of the graph is 4.

2.3.7.15. What has happened to the girth? Careful: We have a different number of edges as well. Better check Euler's formula.

2.4 · Euler Trails and Circuits
2.4.6 · Additional Exercises

2.4.6.7. This is harder than the previous three questions. Think about which "side" of the graph the Hamilton path would need to be on at every other step.

2.4.6.9. If you read off the names of the students in order, you would need to read each student's name exactly once, and the last name would need to be of a student who was friends with the first. What sort of a cycle is this?

2.4.6.10. Draw a graph with 6 vertices and 8 edges. What sort of walk would be appropriate?

2.5 · Coloring
2.5.6 · Additional Exercises

2.5.6.6.

- **(c)** Will you eventually color every vertex following the procedure? Will there ever be a vertex you cannot color according to the procedure?

2.5.6.7.

(a) You will want the teams to be vertices and games to be edges. Which does it make sense to color?

2.5.6.10. The chromatic number is 4. Now prove this!

Note that you cannot use the 4-color theorem, or Brooke's theorem, or the clique number here. In fact, this graph, called the *Grötzsch graph*, is the smallest graph with chromatic number 4 that does not contain any triangles.

2.5.6.13. You can color K_5 in such a way that every vertex is adjacent to exactly two blue edges and two red edges. However, there is a graph with only 5 edges that will result in a vertex incident to three edges of the same color, no matter how they are colored. What is it, and how can you generalize?

2.5.6.14. The previous exercise is useful as a starting point.

2.8 · Chapter Summary

· Chapter Review

2.8.23. You might want to give the proof in two parts. First prove by induction that the cycle C_n has $v = e$. Then consider what happens if the graph is more than just the cycle.

3 · Counting
3.2 · Combining Outcomes
3.2.6 · Practice Problems

3.2.6.9. Break the question into 4 cases.

3.2.7 · Additional Exercises

3.2.7.2. For a simpler example, there are 4 divisors of $6 = 2 \cdot 3$. They are $1 = 2^0 \cdot 3^0$, $2 = 2^1 \cdot 3^0$, $3 = 2^0 \cdot 3^1$, and $6 = 2^1 \cdot 3^1$.

3.3 · Non-Disjoint Outcomes
3.3.6 · Practice Problems

3.3.6.6. To find out how many numbers are divisible by 4 and 5, for example, take $715/(4 \cdot 5)$ and round down.

3.3.7 · Additional Exercises

3.3.7.1. For part (a) you could use the formula for PIE, but for part (b) you might be better off drawing a Venn diagram.

3.3.7.2. You could consider cases. For example, any number of the form ODD-ODD-EVEN will have an even sum. Alternatively, how many three-digit numbers have the sum of their digits even if the first two digits are 54? What if the first two digits are 19?

3.4 · Combinations and Permutations
3.4.7 · Additional Exercises

3.4.7.1. If you pick any three points, you can get a triangle, unless those three points are all on the x-axis or on the y-axis. There are other ways to start this as well, and any correct method should give the same answer.

3.5 · Counting Multisets
3.5.5 · Practice Problems

3.5.5.6. The probability will be 1 divided by however many different combinations of 9 coins your friend could have.

3.5.6 · Additional Exercises

3.5.6.3.

(b) This really requires proving four facts. That every multiset corresponds to at least one diagram, and that every diagram corresponds to at least one multiset. Then that every multiset corresponds to at most one diagram, and that every diagram corresponds to at most one multiset. In other words, we must prove that the function is well defined, injective, and surjective.

3.6 · Combinatorial Proofs
3.6.6 · Additional Exercises

3.6.6.3. There will be 185 triangles. But to find them . . .

(a) How many vertices of the triangle can be on the horizontal axis?

(b) Will *any* three dots work as the vertices?

3.6.6.4. The answer is 120.

3.6.6.6. Try Exercise 3.6.6.5.

3.6.6.7. What if you wanted a pair of co-maids-of-honor?

3.6.6.8. For the combinatorial proof: What if you don't yet know how many bridesmaids you will have?

3.6.6.9. Count handshakes.

3.6.6.13. This one might remind you of Example 3.6.6

3.6.6.14. For the lattice paths, think about what sort of paths 2^n would count. Not all the paths will end at the same point, but you could describe the set of end points as a line.

3.6.6.16. How many edges does K_n have? One of the two graphs will not be connected (unless $j = 1$).

3.7 · Applications to Probability
3.7.6 · Practice Problems

3.7.6.7.

(b) You could list out all the ways you can get a 6, or use sticks and stones.

3.7.7 · Additional Exercises

3.7.7.4. Use complementary probabilities. And don't be surprised if your answer is larger than you would have expected.

3.9 · Chapter Summary

· Chapter Review

3.9.16. Use stars and bars.

4 · Sequences
4.1 · Describing Sequences
4.1.6 · Practice Problems

4.1.6.4. Try adding or subtracting the same small number from each term to see if you recognize the sequence.

4.1.6.5. Try adding, subtracting, or dividing each term by a constant to make the sequence more recognizable.

For part (d), try expressing the sequence as the sum of two well known sequences.

4.1.7 · Additional Exercises

4.1.7.8. You will want to write out the sequence, guess a closed formula, and then verify that you are correct.

4.1.7.9. Write out the sequence, guess a recursive definition, and verify that the closed formula is a solution to that recursive definition.

4.1.7.12. Try an example: When you draw the 4th line, it will cross three other lines and so will be divided into four segments, two of which are infinite. Each segment will divide a previous region into two.

4.1.7.13. Consider three cases: The last digit is a 0, a 1, or a 2. Two of these should be easy to count, but strings ending in 0 cannot be proceeded by a 2, so they require a little more work.

4.1.7.15. Think recursively, like you did in Pascal's triangle.

4.1.7.16. There is only one way to tile a 2×1 board, and two ways to tile a 2×2 board (you can orient the dominoes in two ways). In general, consider the two ways the domino covering the top left corner could be oriented.

4.2 · Rate of Growth
4.2.7 · Additional Exercises

4.2.7.5. We can write the recurrence relation as $\frac{a_n}{a_{n-1}} = r$. What happens when you multiply all the different versions of this recurrence relation (for different values of n) together?

4.2.7.6. Telescoping to find a sum.

Using the fact that $T_n = \frac{n(n+1)}{2}$, each term in the sequence is $\frac{2}{n(n+1)}$.

What is the result of the following fraction subtraction: $\frac{2}{3} - \frac{2}{4}$, or $\frac{2}{4} - \frac{2}{5}$? What is happening in general?

4.4 · Exponential Sequences
4.4.5 · Practice Problems

4.4.5.3. Use telescoping or iteration.

4.5 · Proof by Induction
4.5.7 · Additional Exercises

4.5.7.9. It is not possible to score exactly 11 points. Can you prove that you can score n points for any $n \geq 12$?

4.5.7.11. Start with $(k+1)$-gon, and divide it up into a k-gon and a triangle.

4.5.7.15. For the inductive step, you can assume you have a strictly increasing sequence up to a_k where $a_k < 100$. Now you just need to find the next term a_{k+1} so that $a_k < a_{k+1} < 100$. What should a_{k+1} be?

4.5.7.17. For the inductive case, you will need to show that $(k+1)^2 + (k+1)$ is even. Factor this out, and locate the part of it that is $k^2 + k$. What have you assumed about that quantity?

4.5.7.18. This is similar to Exercise 4.5.7.15, although there you were showing that a sequence had all its terms less than some value, and here you are showing that the sum is less than some value. But the partial sums forms a sequence, so this is actually very similar.

4.5.7.19. We have already proved this without using induction, but looking at it inductively sheds light onto the problem (and is fun).

The question you need to answer to complete the inductive step is, how many new handshakes take place when a person $k+1$ enters the room? Why does adding this give you the correct formula?

4.5.7.20. Here's the idea: Since every entry in Pascal's triangle is the sum of the two entries above it, we can get the $k+1$st row by adding up all the pairs of entry from the kth row. But doing this uses each entry on the kth row twice. Thus each time we drop to the next row, we double the total. Of course, row 0 has sum $1 = 2^0$ (the base case). Now try to make this precise with a formal induction proof. You will use the fact that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for the inductive case.

4.5.7.21. To see why this works, try it on a copy of Pascal's triangle. We are adding up the entries along a diagonal, starting with the 1 on the left-hand side of the 4th row. Suppose we add up the first 5 entries on this diagonal. The claim is that the sum is the entry below and to the left of the last of these 5 entries. Note that if this is true, and we instead add up the first 6 entries, we will need to add the entry one spot to the right of the previous sum. But these two together give the entry below them, which is below and left of the last of the 6 entries on the diagonal. If you follow that, you can see what is going on. But it is not a great proof. A formal induction proof is needed.

4.5.7.23. You are allowed to assume the base case. For the inductive case, group all but the last function together as one sum of functions, and then apply the usual sum of derivatives rule, and then the inductive hypothesis.

4.5.7.24. For the inductive step, we know by the product rule for two functions that

$$(f_1 f_2 f_3 \cdots f_k f_{k+1})' = (f_1 f_2 f_3 \cdots f_k)' f_{k+1} + (f_1 f_2 f_3 \cdots f_k) f_{k+1}'.$$

Then use the inductive hypothesis on the first summand, and distribute.

4.5.7.25. You can inductively assume that from the first $n-2$ implications you can deduce $P_1 \to P_{n-1}$. Then you can use a truth table to verify that this simplified deduction rule is valid.

4.6 · Strong Induction
4.6.5 · Additional Exercises

4.6.5.1. If you have three base cases, can you always be sure you can get three points more?

4.6.5.2. Start with a $(k+1)$-gon, and divide it into two smaller polygons.

4.6.5.4. As with the previous question, we will want to subtract something from n in the inductive step. There we subtracted the largest power of 2 less than n. So what should you subtract here?

Note that you will still need to take care here that the sum you get from the inductive hypothesis, together with the number you subtracted, will be a sum of *distinct* Fibonacci numbers. In fact, you could prove that the Fibonacci numbers in the sum are non-consecutive!

4.6.5.6. You will need to use strong induction. For the inductive case, try multiplying $\left(x^k + \frac{1}{x^k}\right)\left(x + \frac{1}{x}\right)$, and collect which terms together are integers.

4.6.5.8. You will need three base cases. This is a very good hint actually, as it suggests that to prove $P(n)$ is true, you would want to use the fact that $P(n-3)$ is true. So somehow you need to increase the number of squares by 3.

4.7 · Chapter Summary

· Chapter Review

4.7.14.

(a) $(n+1)^{n+1} > (n+1) \cdot n^n$.

(b) This should be similar to the other sum proofs. The last bit comes down to adding fractions.

(c) Write $4^{k+1} - 1 = 4 \cdot 4^k - 4 + 3$.

(d) One 9-cent stamp is 1 more than two 4-cent stamps, and seven 4-cent stamps is 1 more than three 9-cent stamps.

(e) Be careful to actually use induction here. The base case: $2^2 = 4$. The inductive case: Assume $(2n)^2$ is divisible by 4, and consider $(2n+2)^2 = (2n)^2 + 4n + 4$. This is divisible by 4 because $4n + 4$ clearly is, and by our inductive hypothesis, so is $(2n)^2$.

4.7.15. This is a straight-forward induction proof. Note that you will need to simplify $\left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$ and get $\left(\frac{(n+1)(n+2)}{2}\right)^2$.

4.7.16. There are two base cases $P(0)$ and $P(1)$. Then, for the inductive case, assume $P(k)$ is true for all $k < n$. This allows you to assume $a_{n-1} = 1$ and $a_{n-2} = 1$. Apply the recurrence relation.

5 · Discrete Structures Revisited
5.1 · Sets
5.1.5 · Exercises

5.1.5.7. You should be able to write all of them out. Don't forget A and B, which are also candidates for C.

5.1.5.14. It might help to think about what the union $A_2 \cup A_3$ is first. Then think about what numbers are *not* in that union. What will happen when you also include A_5?

5.1.5.17. We are looking for a set containing 16 sets.

5.1.5.18. Write these out, or at least start to and look for a pattern.

5.1.5.29. It looks like you should be able to define the set A like this. But consider the two possible values for $|A|$.

5.2 · Functions
5.2.4 · Exercises

5.2.4.2. Since the domain and codomain are the same size, is it possible for a function to be injective but not surjective, or surjective but not injective?

5.2.4.20. Work with some examples. What if $f = \begin{pmatrix} 1 & 2 & 3 \\ a & a & b \end{pmatrix}$ and $g = \begin{pmatrix} a & b & c \\ 5 & 6 & 7 \end{pmatrix}$?

5.2.4.25. To find the recurrence relation, consider how many *new* handshakes occur when person $n+1$ enters the room.

5.2.4.29. One of these is not always true. Try some examples!

6 · Additional Topics
6.1 · Generating Functions
6.1.5 · Exercises

6.1.5.10. You should "multiply" the two sequences.

6.2 · Introduction to Number Theory
6.2.6 · Exercises

6.2.6.13. First reduce each number modulo 9, which can be done by adding up the digits of the numbers.

6.2.6.18. Solve the Diophantine equation $13x + 20y = 2$ (why?). Then consider which value of k (the parameter in the solution) is optimal.

Selected Solutions

1 · Logic and Proofs
1.1 · Mathematical Statements
1.1.5 · Practice Problems
1.1.5.4.

- $\exists x \forall y P(x, y)$
  - Some people can be fooled all of the time.
- $\forall x \exists y P(x, y)$
  - Everyone can be fooled sometimes.
- $\forall y \exists x P(x, y)$
  - It is always true that some people can be fooled.
- $\exists y \forall x P(x, y)$
  - Sometimes everyone can be fooled.

1.1.5.5.

A. *Correct.*

B. *Incorrect.*

C. *Incorrect.*

D. *Incorrect.*

1.1.5.6.

A. *Correct.*

B. *Incorrect.*

Careful, $P(x, y)$ means x is less than y, not x is less than *or equal* to y.

C. *Incorrect.*

D. *Incorrect.*

E. *Correct.*

1.1.5.7.

(a) $P(15)$ is true, since $17 \cdot 15 + 1 = 256$ is even.

(b) Since $P(15)$ is true, we know that $\exists x P(x)$ is true. There is at least one x for which $P(x)$ is true.

(c) Just because one value of x makes $P(x)$ true does not mean that all values of x make $P(x)$ true. But it could be. So we cannot conclude that $\forall x P(x)$ is true or false.

1.1.5.8.

(a) $P(15)$ is false, since $18 \cdot 15 + 1 = 271$ is odd.

(b) Since $P(15)$ is false, we do not know whether $\exists x P(x)$ is true. There could be some other value of x for which $P(x)$ is true.

(c) We know that there is some value of x makes $P(x)$ false so we know that $\forall x P(x)$ is false.

1.1.5.9.

A. *Incorrect.*

B. *Incorrect.*

C. *Correct.*

D. *Correct.*

1.1.6 · Additional Exercises

1.1.6.1.

(a) $P \wedge Q$.

(b) $P \rightarrow \neg Q$.

(c) Jack passed math or Jill passed math (or both).

(d) If Jack and Jill did not both pass math, then Jill did.

(e)
  i. Nothing else.
  ii. Jack did not pass math either.

1.1.6.2.

(a) $\neg \exists x (E(x) \wedge O(x))$.

(b) $\forall x (E(x) \rightarrow O(x+1))$.

(c) $\exists x (P(x) \wedge E(x))$ (where $P(x)$ means "x is prime").

(d) $\forall x \forall y \exists z (x < z < y \vee y < z < x)$.

(e) $\forall x \neg \exists y (x < y < x + 1)$.

1.2 · Implications

1.2.5 · Practice Problems

1.2.5.1. The main thing to realize is that we do not know the colors of these two shapes, but we do know that we are in one of three cases: We could have a purple circle and orange pentagon. We could have a circle that was not purple but a orange pentagon. Or we could have a circle that was not purple and a pentagon that was not orange. The case in which the circle is purple but the pentagon is not orange cannot occur, as that would make the statement false.

1.2.5.2. The only way for an implication $P \to Q$ to be true but its converse to be false is for Q to be true and P to be false. Thus we know that circle is purple and that square is not yellow.

1.2.5.3. The converse is "If I will give you a cow, then you will give me magic beans." The contrapositive is "If I will not give you a cow, then you will not give me magic beans." All the other statements are neither the converse nor contrapositive.

1.2.6 · Additional Exercises

1.2.6.1.

(a) Any even number plus 2 is an even number.

(b) For any x there is a y such that $\sin(x) = y$. In other words, every number x is in the domain of sine.

(c) For every y there is an x such that $\sin(x) = y$. In other words, every number y is in the range of sine (which is false).

(d) For any numbers, if the cubes of two numbers are equal, then the numbers are equal.

1.2.6.3.

(a) If you have lost weight, then you exercised.

(b) If you exercise, then you will lose weight.

(c) If you are American, then you are patriotic.

(d) If you are patriotic, then you are American.

(e) If a number is rational, then it is real.

(f) If a number is not even, then it is prime. (Or the contrapositive: If a number is not prime, then it is even.)

(g) If the Broncos don't win the Super Bowl, then they didn't play in the Super Bowl. Alternatively, if the Broncos play in the Super Bowl, then they will win the Super Bowl.

1.2.6.5. It is true that in order for a function to be differentiable at a point c, it is necessary for the function to be continuous at c. However, it is not necessary that a function be differentiable at c for it to be continuous at c.

It is true that to be continuous at a point c, it is sufficient that the function be differentiable at c. However, it is not the case that being continuous at c is sufficient for a function to be differentiable at c.

1.3 · Rules of Logic
1.3.7 · Practice Problems

1.3.7.1. If you think about what this statement is saying, it makes sense that it is a tautology (that it is true in every case). The complete truth table is:

P	Q	$P \wedge Q$	$P \vee Q$	$(P \wedge Q) \rightarrow (P \vee Q))$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

1.3.7.2. The truth table is:

P	Q	$\neg Q$	$Q \rightarrow P$	$\neg Q \vee (Q \rightarrow P))$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	F
F	F	T	T	T

If this statement is false, we must be in the third row, making P false and Q true.

1.3.7.3. The complete truth table is:

P	Q	R	$\neg P$	$\neg P \vee R$	$Q \rightarrow (\neg P \vee R)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

1.3.7.4. The complete truth table is:

P	Q	R	$P \to (Q \vee R)$	$(P \to Q) \vee (P \to R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

Since the two columns are identical, the statements are logically equivalent.

1.3.7.5. The complete truth table is:

P	Q	$P \to Q$	$\neg Q$	$\neg P$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

There is only one row in which both premises are true (row 4). In this row, the conclusion is also true. Thus the deduction rule is valid.

1.3.7.6. The complete truth table is:

P	Q	R	$P \to (Q \vee R)$	$\neg(P \to Q)$
T	T	T	T	F
T	T	F	T	F
T	F	T	T	T
T	F	F	F	T
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	T	F

There is only one row in which both premises are true (row 3). In this row, the conclusion is also true, so the deduction rule is valid.

1.3.7.7. The complete truth table is:

P	Q	R	$(P \wedge Q) \rightarrow R$	$\neg P \vee \neg Q$	$\neg R$
T	T	T	T	F	F
T	T	F	F	F	T
T	F	T	T	T	F
T	F	F	T	T	T
F	T	T	T	T	F
F	T	F	T	T	T
F	F	T	T	T	F
F	F	F	T	T	T

In rows 3, 5 and 7 both of the premises are true, but the conclusion is false. Thus the deduction rule is not valid.

1.3.7.8. The complete truth table is:

P	Q	R	$P \rightarrow Q$	$P \wedge \neg Q$
T	T	T	T	F
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	T	F

There is no row in which both premises are true (indeed, these are contradictory premises; the second is the negation of the first). Thus every row in which both premises are true (i.e., no row), the conclusion is also true. Therefore the deduction rule is valid. (This is an example of how everything follows from a contradiction.)

1.3.7.9.

A. *Correct.*

B. *Incorrect.*

C. *Incorrect.*

D. *Incorrect.*

1.3.8 · Additional Exercises

1.3.8.3.

(a) P: It's your birthday; Q: There will be cake. $(P \vee Q) \rightarrow Q$

(b) Hint: You should get three T's and one F.

(c) Only that there will be cake.

(d) It's NOT your birthday!

(e) It's your birthday, but the cake is a lie.

1.3.8.5. Make a truth table for each and compare. The statements are logically equivalent.

1.3.8.6.

(a) $P \wedge Q$.

(b) $(\neg P \vee \neg R) \rightarrow (Q \vee \neg R)$ or, replacing the implication with a disjunction first: $(P \wedge Q) \vee (Q \vee \neg R)$.

(c) $(P \wedge Q) \wedge (R \wedge \neg R)$. This is necessarily false, so it is also equivalent to $P \wedge \neg P$.

(d) Either Sam is a woman and Chris is a man, or Chris is a woman.

1.3.8.16.

(a) $\forall x \exists y (O(x) \wedge \neg E(y))$.

(b) $\exists x \forall y (x \geq y \vee \forall z (x \geq z \wedge y \geq z))$.

(c) There is a number n for which every other number is strictly greater than n.

(d) There is a number n which is not between any other two numbers.

1.4 · Proofs

1.4.7 · Practice Problems

1.4.7.1.

- Let n be an arbitrary integer, and assume n is even.
- Since the product of any number with an even number is even,
- $7n$ must be even.

1.4.7.2.

- Let n be an arbitrary integer, and assume n is odd.
- Since 7 is odd and the product of an odd number and an odd number is odd,
- $7n$ must be odd.

1.4.7.3.

- Let a and b be integers, and assume both are even.
- The sum of two even integers must also be even.
- Therefore $a + b$ is even.

1.4.7.4.

- Let a and b be integers, and assume that $a + b$ is odd but a and b are both even.
- The sum of two even integers must also be even.

- But then $a + b$ is both even and odd, a contradiction.

1.4.7.6.

- Direct proof
    - Assume $f : A \to B$ is a bijection
- Proof by contrapositive
    - Assume $|A| \neq |B|$
- Proof by contradiction
    - Assume $f : A \to B$ is a bijection and $|A| \neq |B|$

1.4.8 · Additional Exercises

1.4.8.1.

(a) The claim that $\forall x P(x)$ means that $P(n)$ is true no matter what n you consider in the domain of discourse. Thus the only way to prove that $\forall x P(x)$ is true is to check or otherwise argue that $P(n)$ is true for all n in the domain.

(b) To prove $\forall x P(x)$ is false all you need is one example of an element in the domain for which $P(n)$ is false. This is often called a **counterexample**.

(c) We are simply claiming that there is some element n in the domain of discourse for which $P(n)$ is true. If you can find one such element, you have verified the claim.

(d) Here we are claiming that no element we find will make $P(n)$ true. The only way to be sure of this is to verify that *every* element of the domain makes $P(n)$ false. Note that the level of proof needed for this statement is the same as to prove that $\forall x P(x)$ is true.

1.4.8.2.

(a) For all integers a and b, if a or b is not even, then $a + b$ is not even.

(b) For all integers a and b, if a and b are even, then $a + b$ is even.

(c) There are numbers a and b such that $a + b$ is even but a and b are not both even.

(d) False. For example, $a = 3$ and $b = 5$. $a + b = 8$, but neither a nor b is even.

(e) False, since it is equivalent to the original statement.

(f) True. Let a and b be integers. Assume both are even. Then $a = 2k$ and $b = 2j$ for some integers k and j. But then $a + b = 2k + 2j = 2(k + j)$, which is even.

(g) True, since the statement is false.

1.4.8.3.

(a) Proof by contradiction. Start of proof: Assume, for the sake of contradiction, that there are integers x and y such that x is a prime greater than 5 and $x = 6y + 3$. End of proof: ... this is a contradiction, so there are no such integers.

(b) Direct proof. Start of proof: Let n be an integer. Assume n is a multiple of 3. End of proof: Therefore n can be written as the sum of consecutive integers.

(c) Proof by contrapositive. Start of proof: Let a and b be integers. Assume that a and b are even. End of proof: Therefore $a^2 + b^2$ is even.

1.4.8.4.

(a) Direct proof.

Proof. Let n be an integer. Assume n is even. Then $n = 2k$ for some integer k. Thus $8n = 16k = 2(8k)$. Therefore $8n$ is even. ■

(b) The converse is false. That is, there is an integer n such that $8n$ is even but n is odd. For example, consider $n = 3$. Then $8n = 24$ which is even, but $n = 3$ is odd.

1.4.8.5.

(a) This is an example of the pigeonhole principle. We can prove it by contrapositive.

Proof. Suppose that each number only came up 6 or fewer times. So there are at most six 1's, six 2's, and so on. That's a total of 36 dice, so you must not have rolled all 40 dice. ■

(b) We can have 9 dice without any four matching or any four being all different: three 1's, three 2's, three 3's. We will prove that whenever you roll 10 dice, you will always get four matching or all being different.

Proof. Suppose you roll 10 dice, but that there are NOT four matching rolls. This means that at most there are three of any given value. If we only had three different values, that would be only 9 dice, so there must be 4 different values, giving 4 dice that are all different. ■

1.5 · Proofs about Discrete Structures

1.5.7 · Practice Problems

1.5.7.1.

A. *Incorrect.*

B. *Correct.*

C. *Incorrect.*

D. *Incorrect.*

This would be a good start to a proof by contradiction or contrapositive, not a direct proof.

1.5.7.2.

A. *Incorrect.*

B. *Correct.*

C. *Incorrect.*

D. *Incorrect.*

This would be a good start to a direct proof, not a proof by contradiction.

1.5.7.3.

- Suppose $B \subseteq A \cap B$, and let b be an element of B.
- Then b is an element of $A \cap B$ since $B \subseteq A \cap B$.
- Since $A \cap B$ contains all the elements that are in both A and B, b is an element of A.
- Therefore $B \subseteq A$.

1.5.7.4.

- First we will prove that $(A \cap B) \cup A \subseteq A$.
- Let x be an element of $(A \cap B) \cup A$.
- Then x is an element of $A \cap B$, or x is an element of A.
- So in particular, x is an element of A.
- Therefore $(A \cap B) \cup A \subseteq A$.
- Second, we will prove that $A \subseteq (A \cap B) \cup A$.
- Let x be an element of A.
- Then x is an element of $(A \cap B) \cup A$, since x is in A or in the other set.
- Therefore $A \subseteq (A \cap B) \cup A$.
- Since $(A \cap B) \cup A \subseteq A$ and $A \subseteq (A \cap B) \cup A$, we have $(A \cap B) \cup A = A$.

1.5.7.5.

- Suppose $B_1 \subseteq B_2$.
- Let a be an element of $f^{-1}(B_1)$.
- This means that $f(a)$ is an element of B_1.

- Since $B_1 \subseteq B_2$, $f(a)$ is an element of B_2.
- This then means that a is an element of $f^{-1}(B_2)$.
- Therefore $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

1.6 · Chapter Summary

· Chapter Review

1.6.1.

P	Q	R	$\neg P \to (Q \wedge R)$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

1.6.2. Peter is not tall, and Robert is not skinny. You must be in row 6 in the truth table above.

1.6.3. Yes. To see this, make a truth table for each statement and compare.

1.6.4. Make a truth table that includes all three statements in the argument:

P	Q	R	$P \to Q$	$P \to R$	$P \to (Q \wedge R)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

Notice that in every row for which both $P \to Q$ and $P \to R$ is true, so is $P \to (Q \wedge R)$. Therefore, whenever the premises of the argument are true, so is the conclusion. In other words, the deduction rule is valid.

1.6.5.

(a) Negation: The power goes off, and the food does not spoil.

Converse: If the food spoils, then the power went off.

Contrapositive: If the food does not spoil, then the power did not go off.

(b) Negation: The door is closed, and the light is on.

Converse: If the light is off, then the door is closed.

Contrapositive: If the light is on, then the door is open.

(c) Negation: $\exists x(x < 1 \wedge x^2 \geq 1)$

Converse: $\forall x(x^2 < 1 \rightarrow x < 1)$

Contrapositive: $\forall x(x^2 \geq 1 \rightarrow x \geq 1)$.

(d) Negation: There is a natural number n which is prime but not solitary.

Converse: For all natural numbers n, if n is solitary, then n is prime.

Contrapositive: For all natural numbers n, if n is not solitary, then n is not prime.

(e) Negation: There is a function which is differentiable and not continuous.

Converse: For all functions f, if f is continuous, then f is differentiable.

Contrapositive: For all functions f, if f is not continuous then f is not differentiable.

(f) Negation: There are integers a and b for which $a \cdot b$ is even but a or b is odd.

Converse: For all integers a and b, if a and b are even, then ab is even.

Contrapositive: For all integers a and b, if a or b is odd, then ab is odd.

(g) Negation: There are integers x and y such that for every integer n, $x > 0$ and $nx \leq y$.

Converse: For every integer x and every integer y there is an integer n such that if $nx > y$, then $x > 0$.

Contrapositive: For every integer x and every integer y there is an integer n such that if $nx \leq y$, then $x \leq 0$.

(h) Negation: There are real numbers x and y such that $xy = 0$, but $x \neq 0$ and $y \neq 0$.

Converse: For all real numbers x and y, if $x = 0$ or $y = 0$, then $xy = 0$

Contrapositive: For all real numbers x and y, if $x \neq 0$ and $y \neq 0$, then $xy \neq 0$.

(i) Negation: There is at least one student in Math 228 who does not understand implications but will still pass the exam.

Converse: For every student in Math 228, if they fail the exam, then they did not understand implications.

Contrapositive: For every student in Math 228, if they pass the exam, then they understood implications.

1.6.6.

(a) The statement is true. If n is an even integer less than or equal to 7, then the only way it could not be negative is if n was equal to 0, 2, 4, or 6.

(b) There is an integer n such that n is even and $n \leq 7$, but n is not negative and $n \notin \{0, 2, 4, 6\}$. This is false, since the original statement is true.

(c) For all integers n, if n is not negative and $n \notin \{0, 2, 4, 6\}$, then n is odd or $n > 7$. This is true, since the contrapositive is equivalent to the original statement (which is true).

(d) For all integers n, if n is negative or $n \in \{0, 2, 4, 6\}$, then n is even and $n \leq 7$. This is false. $n = -3$ is a counterexample.

1.6.7.

(a) For any number x, if it is the case that adding any number to x gives that number back, then multiplying any number by x will give 0. This is true (of the integers or the reals). The "if" part only holds if $x = 0$, and in that case, anything times x will be 0.

(b) The converse in words is this: For any number x, if everything times x is zero, then everything added to x gives itself. Or in symbols: $\forall x(\forall z(x \cdot z = 0) \rightarrow \forall y(x + y = y))$. The converse is true: The only number which when multiplied by any other number gives 0 is $x = 0$. And if $x = 0$, then $x + y = y$.

(c) The contrapositive in words is: For any number x, if there is some number which when multiplied by x does not give zero, then there is some number which when added to x does not give that number. In symbols: $\forall x(\exists z(x \cdot z \neq 0) \rightarrow \exists y(x + y \neq y))$. We know the contrapositive must be true because the original implication is true.

(d) The negation: There is a number x such that any number added to x gives the number back again, but there is a number you can multiply x by and not get 0. In symbols: $\exists x(\forall y(x + y = y) \wedge \exists z(x \cdot z \neq 0))$. Of course since the original implication is true, the negation is false.

1.6.8.

(a) $(\neg P \vee Q) \wedge (\neg R \vee (P \wedge \neg R))$.

(b) $\forall x \forall y \forall z(z = x + y \wedge \forall w(x - y \neq w))$.

1.6.9.

(a) Direct proof.

Proof. Let n be an integer. Assume n is odd. So $n = 2k + 1$ for some integer k. Then

$$7n = 7(2k + 1) = 14k + 7 = 2(7k + 3) + 1.$$

Since $7k + 3$ is an integer, we see that $7n$ is odd. ■

(b) The converse is: For all integers n, if $7n$ is odd, then n is odd. We will prove this by contrapositive.

Proof. Let n be an integer. Assume n is not odd. Then $n = 2k$ for some integer k. So $7n = 14k = 2(7k)$ which is to say $7n$ is even. Therefore $7n$ is not odd. ■

1.6.10.

(a) Suppose you only had 5 coins of each denomination. This means you have 5 pennies, 5 nickels, 5 dimes, and 5 quarters. This is a total of 20 coins. But you have more than 20 coins, so you must have more than 5 of at least one type.

(b) Suppose you have 22 coins, including $2k$ nickels, $2j$ dimes, and $2l$ quarters (so an even number of each of these three types of coins). The number of pennies you have will then be

$$22 - 2k - 2j - 2l = 2(11 - k - j - l).$$

But this says that the number of pennies is also even (it is 2 times an integer). Thus we have established the contrapositive of the statement, "If you have an odd number of pennies, then you have an odd number of at least one other coin type."

(c) You need 10 coins. You could have 3 pennies, 3 nickels, and 3 dimes. The 10th coin must either be a quarter, giving you 4 coins that are all different, or else a 4th penny, nickel, or dime. To prove this, assume you don't have 4 coins that are all the same or all different. In particular, this says that you only have 3 coin types, and each of those types can only contain 3 coins, for a total of 9 coins, which is less than 10.

2 · Graph Theory
2.1 · Problems and Definitions
2.1.4 · Practice Problems

2.1.4.1. The graphs in (a) and (c) are isomorphic to G.

2.1.4.2. Graphs (a) and (c) are isomorphic to each other. So are graphs (b) and (d).

2.1.4.3. G_1 has 24 edges. $k = 2$ for G_2. G_3 has 8 vertices.

2.1.4.4. The graph must have 29 edges.

2.1.4.5.

(a) The largest n such that P_n is a subgraph of K_5 is 4.

(b) The largest n such that C_n is a subgraph of K_5 is 5.

(c) The largest n such that P_n is an *induced* subgraph of K_5 is 1.

(d) The largest n such that C_n is an *induced* subgraph of K_5 is 3.

2.1.5 · Additional Exercises

2.1.5.1. This is asking for the number of edges in K_{10}. Each vertex (person) has degree (shook hands with) 9 (people). So the sum of the degrees is 90. However, the degrees count each edge (handshake) twice, so there are 45 edges in the graph. That is how many handshakes took place.

2.1.5.2. It is possible for everyone to be friends with exactly 2 people. You could arrange the 5 people in a circle and say that everyone is friends with the two people on either side of them (so you get the graph C_5). However, it is not possible for everyone to be friends with 3 people. That would lead to a graph with an odd number of odd degree vertices which is impossible since the sum of the degrees must be even.

2.1.5.4. The graphs are not equal. For example, graph 1 has an edge $\{a, b\}$, but graph 2 does not have that edge. They are isomorphic. One possible isomorphism is $f : G_1 \to G_2$ defined by $f(a) = d$, $f(b) = c$, $f(c) = e$, $f(d) = b$, $f(e) = a$.

2.1.5.9.

(a) For example:

(b) This is not possible if we require the graphs to be connected. If not, we could take C_8 as one graph and two copies of C_4 as the other.

(c) Not possible. If you have a graph with 5 vertices all of degree 4, then every vertex must be adjacent to every other vertex. This is the graph K_5.

(d) This is not possible. In fact, there is not even one graph with this property (such a graph would have $5 \cdot 3/2 = 7.5$ edges).

2.1.5.10.

(a) False. (b) True. (c) True. (d) False.

2.2 · Trees

2.2.6 · Practice Problems

2.2.6.3. Every spanning tree must still contain 22 vertices. Since it is a tree, it will have $22 - 1 = 21$ edges.

Thus we must remove 8 edges to get a spanning tree.

2.2.6.5. Let x be the number of leaves. Then the sum of degrees will be $x+8+6+5+3 = x + 22$. This is twice the number of edges. Since the number of edges is one less than the number of vertices, which is $x + 4$, we also know that the number of edges is $x + 3$.

Thus we have $2(x + 3) = x + 22$. Solving for x gives $x = 16$.

2.2.7 · Additional Exercises

2.2.7.1.

(a) This is not a tree since it contains a cycle. Note also that there are too many edges to be a tree, since we know that all trees with v vertices have $v-1$ edges.

(b) This is a tree since it is connected and contains no cycles (which you can see by drawing the graph). All paths are trees.

(c) This is a tree since it is connected and contains no cycles (draw the graph). All stars are trees.

(d) This is a not a tree since it is not connected. Note that there are not enough edges to be a tree.

2.2.7.2.

(a) This must be the degree sequence for a tree. This is because the vertex of degree 4 must be adjacent to the four vertices of degree 1 (there are no other vertices for it to be adjacent to), and thus we get a star.

(b) This cannot be a tree. Each degree 3 vertex is adjacent to all but one of the vertices in the graph. Thus each must be adjacent to one of the degree 1 vertices (and not the other). That means both degree 3 vertices are adjacent to the degree 2 vertex and to each other, so that means there is a cycle.

Alternatively, count how many edges there are!

(c) This might or might not be a tree. The length 4 path has this degree sequence (this is a tree), but so does the union of a 3-cycle and a length 1 path (which is not connected, so this is not a tree).

(d) This cannot be a tree. The sum of the degrees is 28, so there are 14 edges. But there are 14 vertices as well, so we don't have $v = e + 1$, meaning this cannot be a tree.

2.2.7.6. Yes. We will prove the contrapositive. Assume G does not contain a cycle. Then G is a tree, so this would have $v = e + 1$, contrary to stipulation.

2.2.7.12.

(a) No, although there are graphs for which this is true. For example, K_4 has a spanning tree that is a path (of three edges) and also a spanning tree that is a star (with center vertex of degree 3).

(b) Yes. For a fixed graph, we have a fixed number v of vertices. Any spanning tree of the graph will also have v vertices, and since it is a tree, must have $v-1$ edges.

(c) No, although there are graphs for which this is true (note that if all spanning trees are isomorphic, then all spanning trees will have the same number of leaves). Again, K_4 is a counterexample. One spanning tree is a path, with only two leaves, and another spanning tree is a star with 3 leaves.

2.3 · Planar Graphs

2.3.7 · Additional Exercises

2.3.7.1. No. A (connected) planar graph must satisfy Euler's formula: $v - e + f = 2$. Here $v - e + f = 6 - 10 + 5 = 1$.

2.3.7.2. G has 10 edges, since $10 = \frac{2+2+3+4+4+5}{2}$. It could be planar, and then it would have 6 faces, using Euler's formula: $6 - 10 + f = 2$ means $f = 6$. To make sure that it is actually planar though, we would need to draw a graph with those vertex degrees without edges crossing. This can be done by trial and error (and is possible).

2.3.7.6. Say the last polyhedron has n edges and also n vertices. The total number of edges the polyhedron has then is $(7 \cdot 3 + 4 \cdot 4 + n)/2 = (37 + n)/2$. In particular, we know the last face must have an odd number of edges. We also have that $v = 11$. By Euler's formula, we have $11 - (37 + n)/2 + 12 = 2$, and solving for n we get $n = 5$, so the last face is a pentagon.

2.3.7.8.

Proof. Suppose there is a graph G with fewest edges that does not satisfy Euler's formula.

Note that G cannot be a tree, since for any tree, $v = e+1$ and $f = 1$, so $v-e+f = 2$. Therefore, G must contain a cycle. Pick any edge e_0 that is part of a cycle in G and consider the graph $G' = G - e_0$ that you get by removing just the edge e_0 from G.

Since G' has fewer edges than G, it must satisfy Euler's formula. That is, $v' - e' + f' = 2$, where v', e', and f' are the number of vertices, edges, and faces of G'. Since G' is obtained by removing a single edge from a cycle in G, we have $v' = v$, $e' = e - 1$, and $f' = f - 1$. Therefore, $v - (e - 1) + (f - 1) = 2$, so $v - e + f = 2$ as well. This is a contradiction, so no such graph G can exist. ∎

2.3.7.12.

Proof. We know in any planar graph the number of faces f satisfies $3f \leq 2e$ since each face is bounded by at least three edges, but each edge borders two faces. Combine this with Euler's formula:

$$v - e + f = 2$$

$$v - e + \frac{2e}{3} \geq 2$$

$$3v - e \geq 6$$

$$3v - 6 \geq e.$$

∎

2.4 · Euler Trails and Circuits

2.4.5 · Practice Problems

2.4.5.2. Only (b) has an Euler circuit. The graph in (a) is not connected, so even though every vertex has even degree, it does not have an Euler circuit. (c) has two vertices of odd degree, so it does not have an Euler circuit.

One Euler circuit for (b) is

$$(a, d, f, e, g, c, b, e, h, b, i, c, f, a).$$

2.4.5.3. The first graph has an Euler trail, but not an Euler circuit. The second graph has an Euler circuit. The third graph has neither an Euler circuit nor an Euler trail. You can see this by drawing the graphs, but also by finding the degrees of the vertices.

2.4.6 · Additional Exercises

2.4.6.1. This is a question about finding Euler trails. Draw a graph with a vertex in each state, and connect vertices if their states share a border. Exactly two vertices will have odd degree, the vertices for Nevada and Utah. Thus you must start your road trip at in one of those states and end it in the other.

2.4.6.2.

(a) K_4 does not have an Euler trail or circuit.

(b) K_5 has an Euler circuit (so also an Euler trail).

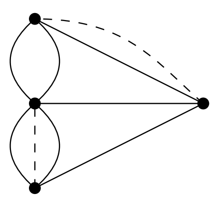
(c) $K_{5,7}$ does not have an Euler trail or circuit.

(d) $K_{2,7}$ has an Euler trail but not an Euler circuit.

(e) C_7 has an Euler circuit (it is a circuit graph!)

(f) P_7 has an Euler trail but no Euler circuit.

2.4.6.8. If we build one bridge, we can have an Euler trail. Two bridges must be built for an Euler circuit.



2.5 · Coloring

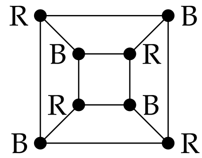
2.5.6 · Additional Exercises

2.5.6.1. 2, since the graph is bipartite. One color for the top set of vertices, another color for the bottom set of vertices.

2.5.6.2. For example, K_6. If the chromatic number is 6, then the graph is not planar; the 4-color theorem states that all planar graphs can be colored with 4 or fewer colors.

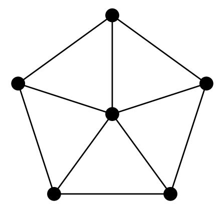
2.5.6.3. The chromatic numbers are 2, 3, 4, 5, and 3 respectively from left to right.

2.5.6.5. The cube can be represented as a planar graph and colored with two colors as follows:



Since it would be impossible to color the vertices with a single color, we see that the cube has chromatic number 2 (it is bipartite).

2.5.6.9. The wheel graph below has this property. The outside of the wheel forms an odd cycle and so requires 3 colors; the center of the wheel must be a different color from all the outside vertices.



2.5.6.12. If we drew a graph with each letter representing a vertex and each edge connecting two letters that were consecutive in the alphabet, we would have a graph containing two vertices of degree 1 (A and Z) and the remaining 24 vertices all of degree 2 (for example, D would be adjacent to both C and E). By Brooks' theorem, this graph has chromatic number at most 2, as that is the maximal degree in the graph, and the graph is not a complete graph or odd cycle. Thus only two boxes are needed.

2.5.6.13.

2.6 · Relations and Graphs

2.6.7 · Practice Problems

2.6.7.1.

(a) Not reflexive, since, for example, $0 + 0 = 0$ is not odd.

(b) Not reflexive, since, for example, $(-1) + (-1)$ is not positive.

(c) Reflexive. Any number times itself is non-negative.

(d) Not reflexive. $0 \cdot 0$ is not positive.

(e) Reflexive. Since $n - n = 0$ is a multiple of 10.

2.6.7.2. All the relations listed are symmetric.

2.6.7.3.

(a) Not transitive. For example, $1 \sim 2$ and $2 \sim 3$, but $1 \nsim 3$.

(b) Not transitive. For example, $-1 \sim 3$ and $3 \sim -2$, but $-1 \nsim -2$.

(c) Not transitive. For example, $-1 \sim 0$ and $0 \sim 1$, but $-1 \sim 1$.

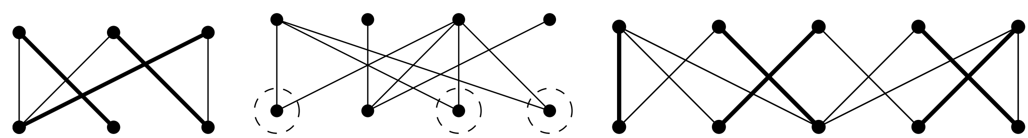
(d) Transitive. If $x \sim y$, then x and y have the same sign. And then if $y \sim z$, z has the same sign as y, so z also has the same sign as x. Thus $x \sim z$.

(e) Transitive. This is because $z - y + y - x = z - x$, so if $z - y$ and $y - x$ are both multiples of 10, then so is $z - x$.

2.7 · Matching in Bipartite Graphs

· Exercises

2.7.1. The first and third graphs have a matching, shown in bold (there are other matchings as well). The middle graph does not have a matching. If you look at the three circled vertices, you see that they only have two neighbors, which violates the matching condition $|N(S)| \geq |S|$ (the three circled vertices form the set S).



2.8 · Chapter Summary

· Chapter Review

2.8.1. The first and the third graphs are the same (try dragging vertices around to make the pictures match up), but the middle graph is different (which you can see, for example, by noting that the middle graph has only one vertex of degree 2, while the others have two such vertices).

2.8.2. The first (and third) graphs contain an Euler trail. All the graphs are planar.

2.8.3. For example, K_5.

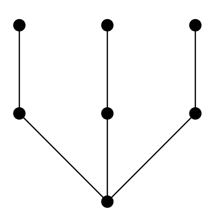
2.8.4. For example, $K_{3,3}$.

2.8.5.

(a) Yes, the graphs are isomorphic, which you can see by drawing them. One isomorphism is:

$$f = \begin{pmatrix} a & b & c & d & e & f & g \\ u & z & v & x & w & y & t \end{pmatrix}.$$

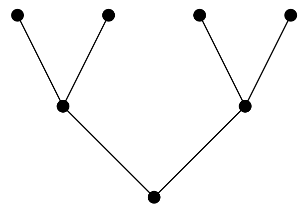
(b) This is easy to do if you draw the picture. Here is such a graph:



Any labeling of this graph will be not isomorphic to G. For example, we could take $V'' = \{a, b, c, d, e, f, g\}$ and $E'' = \{ab, ac, ad, be, cf, dg\}$.

(c) The degree sequence for G is $(3, 3, 2, 1, 1, 1, 1)$.

(d) In general this should be possible: The degree sequence does not determine the graph's isomorphism class. However, in this case, I was almost certain this was not possible. That is, until I stumbled up this:



(e) G is a tree (there are no cycles) and as such is also bipartite.

(f) Yes, all trees are planar. You can draw them in the plane without edges crossing.

(g) The chromatic number of G is 2. It shouldn't be hard to give a 2-coloring (for example, color a, d, e, g red and b, c, f blue), but we know that all bipartite graphs have chromatic number 2.

(h) It is clear from the drawing that there is no Euler trail, let alone an Euler circuit. Also, since there are more than 2 vertices of odd degree, we know for sure there is no Euler trail.

2.8.6. Yes. According to Euler's formula it would have 2 faces. It does. The only such graph is C_{10}.

2.8.7.

(a) Only if $n \geq 6$ and is even.

(b) None.

(c) 12. Such a graph would have $\frac{5n}{2}$ edges. If the graph is planar, then $n - \frac{5n}{2} + f = 2$ so there would be $\frac{4+3n}{2}$ faces. Also, we must have $3f \leq 2e$, since the graph is simple. So we must have $3\left(\frac{4+3n}{2}\right) \leq 5n$. Solving for n gives $n \geq 12$.

2.8.8.

(a) There were 24 couples: 6 choices for the girl and 4 choices for the boy.

(b) There were 45 couples: $\binom{10}{2}$ since we must choose two of the 10 people to dance together.

(c) For part (a), we are counting the number of edges in $K_{4,6}$. In part (b) we count the edges of K_{10}.

2.8.9. Yes, as long as n is even. If n were odd, then the corresponding graph would have an odd number of odd degree vertices, which is impossible.

2.8.10.

(a) No. The 9 triangles each contribute 3 edges, and the 6 pentagons contribute 5 edges. This gives a total of 57, which is exactly twice the number of edges, since each edge borders exactly 2 faces. But 57 is odd, so this is impossible.

(b) Now adding up all the edges of all the 16 polygons gives a total of 64, meaning there would be 32 edges in the polyhedron. We can then use Euler's formula $v - e + f = 2$ to deduce that there must be 18 vertices.

(c) If you add up all the vertices from each polygon separately, we get a total of 64. This is not divisible by 3, so it cannot be that each vertex belongs to exactly 3 faces. Could they all belong to 4 faces? That would mean there were $64/4 = 16$ vertices, but we know from Euler's formula that there must be 18 vertices. We can write $64 = 3x + 4y$ and solve for x and y (as integers). We get that there must be 10 vertices with degree 4 and 8 with degree 3. (Note that the number of faces joined at a vertex is equal to its degree in graph theoretic terms.)

2.8.11. No. Every polyhedron can be represented as a planar graph, and the Four Color Theorem says that every planar graph has chromatic number at most 4.

2.8.12. $K_{n,n}$ has n^2 edges. The graph will have an Euler circuit when n is even. The graph will be planar only when $n < 3$.

2.8.13. G has 8 edges (since the sum of the degrees is 16). If G is planar, then it will have 4 faces (since $6 - 8 + 4 = 2$). G does not have an Euler trail since there are more than 2 vertices of odd degree.

2.8.14. 7 colors. Thus K_7 is not planar (by the contrapositive of the Four Color Theorem).

2.8.15. The chromatic number of $K_{3,4}$ is 2, since the graph is bipartite. You cannot say whether the graph is planar based on this coloring (the converse of the Four Color Theorem is not true). In fact, the graph is *not* planar, since it contains $K_{3,3}$ as a subgraph.

2.8.16. We have that $K_{3,4}$ has 7 vertices and 12 edges (each vertex in the group of 3 has degree 4). Then by Euler's formula we have that $7 - 12 + f = 2$, so if the graph were planar, it would have $f = 7$ faces. However, since the girth of the graph is 4 (there are no cycles of length 3), we get that $4f \leq 2e$. But this would mean that $28 \leq 24$, a contradiction.

2.8.17. For all these questions, we are really coloring the vertices of a graph. You get the graph by first drawing a planar representation of the polyhedron and then taking its planar dual: Put a vertex in the center of each face (including the outside), and connect two vertices if their faces share an edge.

(a) Since the planar dual of a dodecahedron contains a 5-wheel, its chromatic number is at least 4. Alternatively, suppose you could color the faces using 3 colors without any two adjacent faces colored the same. Take any face, and color it blue. The 5 pentagons bordering this blue pentagon cannot be colored blue. Color the first one red. Its two neighbors (adjacent to the blue pentagon) get colored green. The remaining 2 cannot be blue or green, but also cannot both be red since they are adjacent to each other. Thus a 4th color is needed.

(b) The planar dual of the dodecahedron is itself a planar graph. Thus by the 4-color theorem, it can be colored using only 4 colors without two adjacent vertices (corresponding to the faces of the polyhedron) being colored identically.

(c) The cube can be properly 3-colored. Color the "top" and "bottom" red, the "front" and "back" blue, and the "left" and "right" green.

2.8.18.

(a) False. To prove this, we can give an example of a pair of graphs with the same chromatic number that are not isomorphic. For example, $K_{3,3}$ and $K_{3,4}$ both have chromatic number 2 but are not isomorphic.

(b) False. The previous example does not work, but you can easily draw two trees that have the same number of vertices and edges but are not isomorphic. Since all trees have chromatic number 2, this is a counterexample.

(c) True. If there is an isomorphism from G_1 to G_2, then we have a bijection that tells us how to match up vertices between the graph. Any proper vertex coloring of G_1 will tell us how to properly color G_2, simply by coloring $f(v_i)$ the same color as v_i, for each vertex $v_i \in V$. That is, color the vertices in G_2 exactly how you color the corresponding vertices in G_1. Similarly, any proper vertex coloring of G_2 corresponds to a proper vertex coloring of G_1. Thus the smallest number of colors needed to properly color G_1 cannot be smaller than the smallest number of colors needed to properly color G_2, and vice-versa, so the chromatic numbers must be equal.

2.8.19. G has 13 edges, since we need $7 - e + 8 = 2$.

2.8.20.

(a) The graph does have an Euler trail, but not an Euler circuit. There are exactly two vertices with odd degree. The path starts at one and ends at the other.

(b) The graph is planar. Even though as it is drawn with edges crossing, it is easy to redraw it without edges crossing.

(c) The graph is not bipartite (there is an odd cycle), nor complete.

(d) The chromatic number of the graph is 3.

2.8.21.

(a) False. For example, $K_{3,3}$ is not planar.

(b) True. The graph is bipartite, so it is possible to divide the vertices into two groups with no edges between vertices in the same group. Thus we can color all the vertices of one group red and the other group blue.

(c) False. $K_{3,3}$ has 6 vertices with degree 3 and so contains no Euler trail.

(d) False. $K_{3,3}$ again.

(e) False. The sum of the degrees of all vertices is even for *all* graphs so this property does not imply that the graph is bipartite.

2.8.22.

(a) If a graph has an Euler trail, then it is planar.

(b) If a graph does not have an Euler trail, then it is not planar.

(c) There is a graph which is planar and does not have an Euler trail.

(d) Yes. In fact, in this case it is because the original statement is false.

(e) False. K_4 is planar but does not have an Euler trail.

(f) False. K_5 has an Euler trail but is not planar.

3 · Counting
3.1 · Pascal's Arithmetical Triangle
3.1.7 · Practice Problems

3.1.7.7.

a. $\binom{10}{5} = 252$ choices, since you have to select a 5-element subset of the set of 10 toppings.

b. $\binom{9}{5} = 126$ choices, since you must select 5 of the 9 non-green pepper toppings.

c. $\binom{9}{4} = 126$ choices, since you must select 4 of the remaining 9 non-green pepper toppings to have in addition to the green pepper.

Note that $252 = 126 + 126$ choices, which makes sense because every 5-topping pizza either has green pepper or does not have green pepper as a topping.

3.1.7.9. To get an x^{14}, we must pick 14 of the 16 factors to contribute an x, leaving the other 2 to contribute a 3. There are $\binom{16}{14}$ ways to select these 14 factors. So the term containing an x^{14} will be $\binom{16}{14}x^{14}3^2$. In other words, the coefficient of x^{14} is $\binom{16}{14}3^2 = 1080$.

3.1.7.10. To get an x^9 from the first term, we must pick 9 of the 17 factors to contribute an x, leaving the other 8 to contribute a 2. There are $\binom{17}{9}$ ways to select these 9 factors, so the coefficient will be $\binom{17}{9} * 2^8$. Now we must choose 5 of the factors in the second term to contribute an x, leaving the other 16 terms to contribute a 3. This gives us $\binom{21}{5} * 3^{16}$ for the coefficient resulting from this term. In total, the term containing an x^9 will be $\binom{17}{9} * 2^8 + \binom{21}{5} * 3^{16} = 8.75964 \times 10^{11}$.

3.2 · Combining Outcomes

3.2.6 · Practice Problems

3.2.6.1. By the product principle, there are $6 \times 3 \times 15 = 270$ different outfits.

3.2.6.2.

a. $8 + 3 = 11$ ties. Use the additive principle.

b. $8 \cdot 3 = 24$ ties. Use the multiplicative principle

c. $2 \cdot (7 + 3) + 8 = 28$ outfits.

3.2.6.3.

a. There are 256 3-digit hexadecimals in which the first digit is an E (one for each choice of the remaining digits). Similarly, there are 256 hexadecimals in which the first digit is an F. We want the union of these two disjoint sets, so there are $256 + 256 = 2 \cdot 256 = 512$ 3-digit hexadecimals in which the first digit is either an E or an F.

b. We can select the first digit in 6 ways, digits 2-5 in 16 ways each, and the final digit in 10 ways. Thus there are $6 \cdot 16^4 \cdot 10 = 3932160$ hexadecimals given these restrictions.

c. The number of 4-digit hexadecimals that start with a letter is $6 \cdot 16^3 = 24576$. The number of 4-hexadecimals that end with a numeral is $16^3 \cdot 10 = 40960$. We want all the elements from both these sets. However, both sets include those 4-digit hexadecimals which *both* start with a letter and end with a numeral($6 \cdot 16^2 \cdot 10 = 15360$), so we must subtract these (once). Thus the number of 4-digit hexadecimals starting with a letter or ending with a numeral is:

   $24576 + 40960 - 15360 = 50176$

3.2.6.4.

a. $2^8 = 256$ subsets. We need to select yes/no for each of the 8 elements.

b. $2^5 = 32$ subsets. We need to select yes/no for each of the remaining 5 elements.

c. $2^8 - 2^4 = 240$ subsets. We subtract the number of subsets which do *not* contain any odd numbers (2^4-select yes or no for each even element) from the total number of possible subsets.

d. $\binom{4}{1} \cdot 2^4 = 64$ subsets. First pick the even number. Then say yes or no to each of the odd numbers.

3.2.6.5.

a. $\binom{6}{4} = 15$ subsets

b. $\binom{3}{1} = 3$ subsets. We need to select 1 of the 3 remaining elements of S to be in the subset.

c. $\binom{6}{4} = 15$ subsets. All subsets of cardinality 4 must contain at least one odd number.

d. $\binom{3}{1} = 3$ subsets. Select 1 of the 3 even numbers. The remaining 3 odd numbers of S must all be in the set.

3.2.6.7.

a. We can think of each row as a 4-bit string of weight 2 (since of the 4 coins, we require 2 to be pennies). Thus there are $\binom{4}{2} = 6$ rows possible. Each row requires 4 coins, so if we want to make all the rows at the same time, we will need 24 coins, half of them nickels and half of them pennies.

b. Now there are $2^4 = 16$ rows possible, which is also $\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \cdots + \binom{4}{4}$, if you break them up into rows containing 0, 1, 2, etc. pennies. Thus, since there are 4 coins in each row, we need $4 \cdot 16 = 64$ total coins.

3.2.6.8. Count the number of strings with each permissible number of 1's separately and then add them up. So there are $\binom{11}{4} + \binom{11}{5} + \cdots \binom{11}{11} = 1816$ strings.

3.2.6.9. This is the same as a question about 7-bit strings, since we can think of each subset as a 7-bit string with a 1 representing that we include that element in the subset. $\binom{10}{7} + \binom{10}{8} + \cdots + \binom{10}{10} = 176$ subsets.

3.2.6.10.

a. $\binom{14}{7} = 3432$ paths. The paths all have length 14 (7 steps up and 7 steps right); we just select which 7 of those 14 should be right.

b. $\binom{7}{3}\binom{7}{4} = 1225$ paths. First travel to $(8, 9)$, and then continue on to $(12, 12)$.

c. $\binom{14}{7} - \binom{7}{3}\binom{7}{4} = 2207$ paths. Remove all the paths that you found in part (b).

3.2.7 · Additional Exercises

3.2.7.1.

(a) For example, 16 is the number of choices you have if you want to watch one movie, either a comedy or horror flick.

(b) For example, 63 is the number of choices you have if you will watch two movies, first a comedy and then a horror movie.

3.3 · Non-Disjoint Outcomes

3.3.6 · Practice Problems

3.3.6.3. 49 students. Use Venn diagram or PIE: $30 + 24 + 28 - (18 + 10 + 12) + 7 = 49$.

3.3.6.4. Using the principle of inclusion/exclusion, the number of students who like their potatoes in at least one of the ways described is

$54 + 39 + 51 - (18 + 28 + 29) + 11 = 80$.

Therefore there are $100 - 80 = 20$ students who do not like potatoes. You can also do this problem with a Venn diagram.

3.3.6.5.

a. $2^{10} = 1024$. You have 2 choices for each of the remaining 10 bits.

b. $\binom{10}{6} = 210$. You need to choose 6 of the remaining 10 bits to be 1's.

c. 2816. There are 2^{10} strings that start with 011, and another 2^{11} which end with 01 (we choose 1 or 0 for 11 bits). However, we count the strings that start with 011 and end with 01 twice; there are 2^8 such strings. So using PIE, we have $2^{10} + 2^{11} - 2^8 = 2816$.

d. 484. There are $\binom{10}{6} = 210$ strings of weight 8 which start with 011, and another $\binom{11}{7} = 330$ which end with 01. We have over counted again: There are weight 8 strings which both start with 011 and end with 01, in fact $\binom{8}{5} = 56$ of them. So all together we have $210 + 330 - 56 = 484$ strings.

3.3.6.6. 429 values of n. Use PIE: $178 + 238 + 143 - (59 + 35 + 47) + 11$ or a Venn diagram. To find out how many numbers are divisible by 3 and 5, for example, take $715/(3 \cdot 5)$ and round down.

3.3.6.7. To find out how many numbers strictly less than 1400 are multiples of 4, we can divide 1400 by 4 and round down. There are 349 of these. Similarly, there are 199 multiples of 7 and 155 multiples of 9 less than 1400.

We also need the combinations of these. To be a multiple of 4 and 7 means you are a multiple of 28 , and there are 49 multiples of 4 and 7. There will be 38 multiples of 4 and 9. There will be 22 multiples of 7 and 9. Finally, there will be 5 multiples of all three.

Using PIE, we get
$349 + 199 + 155 - (49 + 38 + 22) + 5 = 599$
multiples of 4, 7, or 9 less than 1400.

3.3.6.8.

a. $12^{13} = 1.06993 \times 10^{14}$ words, since you select from 12 letters 13 times.

b. $(12) \cdot (11) \cdot (10) \ldots (12 - 13 + 1) = 0$ words. After selecting a letter, you have fewer letters to select for the next one.

c. $12^{10} = 6.19174 \times 10^{10}$ words: You need to select the letters that follow the "ade."

d. $12^{10} + 12^{11} - 12^8 = 8.04496 \times 10^{11}$ words. There are 12^{10} words which start with "ade" and another 12^{11} words that end with "be." Then we need to subtract the words that have both, which we have overcounted.

e. $(12 \cdot (12-1) \cdot (12-2) \ldots (12-13+1)) - (11 \cdot (9) \cdot (9-1) \cdot (9-2) \ldots 9-10+1) = 0$ words. All possible words without repeats minus the bad ones. The taboo word "bed" can be in any of 11 positions, and for each position we must choose the remaining 10 letters from the remaining 9 letters in our alphabet.

3.4 · Combinations and Permutations

3.4.6 · Practice Problems

3.4.6.1.

a. $\binom{8}{2} = 28$ pizzas. We must choose (in no particular order) 2 out of the 8 toppings.

b. $2^8 = 256$ pizzas. Say yes or no to each topping.

c. $P(8,4) = 1680$ ways. Assign each of the 4 spots in the left column to a unique pizza topping.

3.4.6.2. Despite its name, we are not looking for a combination here. The order in which the three numbers appear matters. There are $P(41,3) = 41 \cdot 40 \cdot 39$ different possibilities for the "combination". This is assuming you cannot repeat any of the numbers (if you could, the answer would be 41^3).

3.4.6.3.

a. This is just the multiplicative principle. There are 8 digits which we can select for each of the 7 positions, so we have $8^7 = 2.09715 \times 10^6$ such numbers.

b. Now we have 8 choices for the first number, 7 for the second, etc. So there are $8 \cdot 7 \cdot \cdots \cdot 7 = P(8,7) = 40320$ such numbers.

c. To build such a number we simply must select 7 different digits. After doing so, there will only be one way to arrange them into increasing order. Thus there are $\binom{8}{7} = 8$ such numbers.

3.4.6.4.

a. We can write the answer as $P(30,19) = 30 \cdot 29 \cdot 28 \cdot \cdots \cdot 12$, which is the same as $\frac{30!}{11!}$. Or, if you think of picking the 19 books and then arranging those 19, you can write this as $\binom{30}{19} \cdot 19!$. Note, that since any order is acceptable, we are distinguishing between different orders, so a permutation is appropriate here.

b. Here we just need to select the books, and have no choice as how to arrange them. So the answer is just $\binom{30}{19}$

3.4.6.5. Since there are 14 different letters in "ambidextrously", we have 14 choices for the first letter, 13 for the next, 12 for the next, and so on. Thus there are 14! anagrams.

3.4.6.6. After the first letter (namely, s), we must rearrange the remaining 5 letters. There are only two choices of letter now, so this is really just a bit-string question where one of the letters is 0, and the other letter is 1. Thus there are $\binom{5}{2} = 10$ anagrams starting with "s".

3.4.6.7. First, decide where to put the "g"s. There are 7 positions, and we must choose 3 of them to fill with an "g". This can be done in $\binom{7}{3}$ ways. The remaining 4 spots all get a different letter, so there are 4! ways to finish off the anagram. This gives a total of $\binom{7}{3} \cdot 4!$ anagrams. Strangely enough, this is 840, which is also equal to $P(7,4)$. To get the answer that way, start by picking one of the 7 *positions* to be filled by the first non-"g" letter, one of the remaining 7 positions to be filled by the next, and so on. Then put "g"s in the remaining 3 positions.

3.4.6.8.

a. $\binom{40}{4} \cdot \binom{36}{4} \cdot \binom{32}{4} \cdot \cdots \cdot \binom{4}{4}$ ways. Pick 4 out of 40 people to be in the first foursome, then 4 of the remaining 36 for the second foursome, and so on (use the multiplicative principle to combine).

b. $10!\binom{30}{3} \cdot \binom{27}{3} \cdot \binom{24}{3} \cdot \cdots \cdot \binom{3}{3}$ ways. First determine the tee time of the 10 board members, then select 3 of the 30 non-board members to golf with the first board member, then 3 of the remaining 27 to golf with the second, and so on.

3.4.6.9. 13!. There are 14 people seated around the table, but it does not matter where King Arthur sits, only who sits to his left, two seats to his left, and so on.

3.4.6.10.

a. 22^{13} functions. There are 22 choices for the image of each element in the domain.

b. $P(22, 13) = 22 \cdot 21 \cdot 20 \cdot \cdots \cdot 9$ injective functions. There are 22 choices for the image of the first element of the domain, then only 21 choices for the second, 20 for the third, and so on.

3.4.6.11.

a. $6^5 = 7776$ functions, since there are 6 choices of where to send each of the 5 elements of the domain.

b. $P(6, 5) = 6 \cdot 5 \cdot \cdots \cdot 2 = 720$ functions, since outputs cannot be repeated.

c. $\binom{6}{5} = 6$ functions. Since the function must be injective and increasing, we just need to select the 5 distinct elements of the range from the 6 elements of the codomain. Once these have been selected, we must put the smallest as the image of 1, the next smallest as the image of 2, and so on (doing this does not increase the number of functions, since there is one choice for how this event can occur).

3.4.7 · Additional Exercises

3.4.7.1. 120.

3.5 · Counting Multisets

3.5.5 · Practice Problems

3.5.5.1.

a. $\binom{10}{9}$ sets. We must select 9 of the 10 digits to put in the set.

b. Use sticks and stones: Each stone represents one of the 9 elements of the set; each stick represents a switch between digits. So there are 9 stones and 9 sticks, giving us $\binom{18}{9}$ sets.

3.5.5.2.

a. There are $\binom{6}{5}$ numbers. We simply choose 5 of the 6 digits and, once the choice is made, put them in increasing order.

b. This uses sticks and stones. Use a stone to represent each of the 5 digits in the number, and use their position relative to the sticks to say what numeral fills that spot. So we will have 5 stones and 5 sticks, giving $\binom{10}{5}$ numbers.

3.5.5.3.

a. $\binom{25}{16}$ ways. Each outcome can be represented by a sequence of 9 sticks and 16 stones.

b. $\binom{15}{6}$ ways. First put one ball in each bin. This leaves 9 sticks and 6 stones.

3.5.5.4.

a. $\binom{11}{9}$ solutions. After each variable gets 1 stone for free, we are left with 9 stones and 2 sticks.

b. $\binom{14}{12}$ solutions. We have 12 stones and 2 sticks.

c. $\binom{23}{2}$ solutions. This problem is equivalent to finding the number of solutions to $x' + y' + z' = 21$ where x', y', and z' are non-negative. (In fact, we really just do a substitution. Let $x = x' - 3$, $y = y' - 3$ and $z = z' - 3$).

3.5.5.5. $\binom{10}{5}$ outcomes are possible in traditional Yahtzee. Each die is a stone; its value is determined by where it is put relative to the sticks.

For Super-Yahtzee, $\binom{13}{9}$ outcomes. We have 4 stones (the 4 dice) and 9 sticks (the 9 switches between the numbers 1-10).

3.5.5.6. We must figure out how many different combinations of 9 coins are possible. Let a stone represent each coin, and a stick represent switching between type of coin. For example, if we have 7 coins, $**|*||****$ represents 2 pennies, one nickel, no dimes and 4 quarters. The number of such stone and stick diagrams for 12 total stones and sticks (with 9 stones and 3 sticks) is $\binom{12}{9} = 0.00454545$. Thus you have a 1 in 0.00454545 chance of guessing correctly.

3.5.5.7. $\binom{25}{22}$ solutions. First we guarantee the restrictions on the variables by distributing 13 units to the variables. Then we find all solutions to $x_1' + x_2' + x_3' + x_4' = 22$ in non-negative integers.

3.5.5.8.

a. $\binom{13}{6} = 1716$. Note that a strictly increasing function is automatically injective. So the 6 outputs must all be different. So let's first pick which 6 outputs we will use: there are $\binom{13}{6}$ ways to do this. Now how many ways are there to assign those outputs to the inputs 1 through 6? Only one way, since there is only one way to arrange numbers in increasing order.

b. $\binom{18}{6}$. This is in fact a sticks and stones problem. The stones are the 6 inputs, and the sticks are the 12 spots between the 13 possible outputs. Think of it this way: We will specify $f(1)$, then $f(2)$, then $f(3)$, and so on in that order. Start with the possible output 0. We can use it as the output of $f(1)$, or we can switch to 1 as a potential output. Say we put $f(1) = 1$. Now we are at 1 (can't go back to 0). Should $f(2) = 1$? If yes, then we are putting down another stone. If no, put down a bar and switch to 2. Maybe you switch to 3, then assign $f(2) = 3$ and $f(3) = 3$ (two more stones) before switching to 4 as a possible output. And so on.

3.5.5.9.

a. $\binom{28}{7}$ sodas (order does not matter, and repeats are not allowed).

b. $P(28,7) = (28) \cdot (28-1) \cdot \cdots \cdot 22$ sodas (order matters, and repeats are not allowed).

c. $\binom{34}{7}$ sodas (order does not matter, and repeats are allowed; 7 stars and 27 bars).

d. 28^7 sodas (order matters, and repeats are allowed; 28 choices 7 times).

3.5.6 · Additional Exercises

3.5.6.1.

(a) You take 3 strawberry, 1 lime, 0 licorice, 2 blueberry, and 0 bubblegum.

(b) This is backwards. We don't want the stones to represent the kids because the kids are not identical, but the stones are. Instead we should use 5 stones (for the lollipops) and use 5 sticks to switch between the 6 kids. For example,

$$\circ\circ||\circ\circ\circ|||$$

would represent the outcome with the first kid getting 2 lollipops, the third kid getting 3, and the rest of the kids getting none.

(c) This is the word AAAEOO.

(d) This doesn't represent a solution. Each stone should represent one of the 6 units that add up to 6, and the sticks should *switch* between the different variables. We have one too many sticks. An example of a correct diagram would be

$$\circ|\circ\circ||\circ\circ\circ,$$

representing that $x_1 = 1$, $x_2 = 2$, $x_3 = 0$, and $x_4 = 3$.

3.6 · Combinatorial Proofs

3.6.5 · Practice Problems

3.6.5.1.

- Consider the question, "How many two-digit numbers start with a 3 or 4?"

- The first way to answer this is $10 + 10$.
- This is because there are 10 numbers that start with 3, and another 10 that start with 4.
- A second answer to the question is $2 \cdot 10$.
- This is because you have 2 choices for the first digit, and 10 choices for the second digit.
- Since both expressions answer the same question, they must be equal. Therefore $10 + 10 = 2 \cdot 10$.

3.6.5.2.

A. *Incorrect.*

   This is not going to be helpful, since when we switch from one side of the equation to the other, we will not be counting the same thing. We need a question where the k things and the 2 things come from the same set of n things.

B. *Correct.*

C. *Incorrect.*

   If we wanted either 2 or k people, we would need to add the number of outcomes from each.

D. *Correct.*

3.6.6 · Additional Exercises

3.6.6.1.

Proof. Question: How many 2-letter words start with a, b, or c and end with either y or z?

Answer 1: There are two words that start with a, two that start with b, two that start with c, for a total of $2 + 2 + 2$.

Answer 2: There are three choices for the first letter and two choices for the second letter, for a total of $3 \cdot 2$.

Since the two answers are both answers to the same question, they are equal. Thus $2 + 2 + 2 = 3 \cdot 2$. ■

3.6.6.5.

(a) She has $\binom{15}{6}$ ways to select the 6 bridesmaids, and then for each way, has 6 choices for the maid of honor. Thus she has $\binom{15}{6}6$ choices.

(b) She has 15 choices for who will be her maid of honor. Then she needs to select 5 of the remaining 14 friends to be bridesmaids, which she can do in $\binom{14}{5}$ ways. Thus she has $15\binom{14}{5}$ choices.

(c) We have answered the question (how many wedding parties can the bride choose from) in two ways. The first way gives the left-hand side of the identity, and the second way gives the right-hand side of the identity. Therefore the identity holds.

3.6.6.7.

Proof. Question: You have a large container filled with ping-pong balls, all with a different number on them. You must select k of the balls, putting two of them in a jar and the others in a box. How many ways can you do this?

Answer 1: First select 2 of the n balls to put in the jar. Then select $k-2$ of the remaining $n-2$ balls to put in the box. The first task can be completed in $\binom{n}{2}$ different ways and the second task in $\binom{n-2}{k-2}$ ways. Thus there are $\binom{n}{2}\binom{n-2}{k-2}$ ways to select the balls.

Answer 2: First select k balls from the n in the container. Then pick 2 of the k balls you picked to put in the jar, placing the remaining $k-2$ in the box. The first task can be completed in $\binom{n}{k}$ ways and the second task in $\binom{k}{2}$ ways. Thus there are $\binom{n}{k}\binom{k}{2}$ ways to select the balls.

Since both answers count the same thing, they must be equal, and the identity is established. ■

3.8 · Advanced Counting Using PIE

3.8.5 · Practice Problems

3.8.5.1.

a. $\binom{15}{6}$ meals. First spend \$7 to buy one of each item and then use 9 stars to select items between 6 bars.

b. $\binom{22}{6}$ meals. Here you have 16 stars and 6 bars (separating the 7 items). a. $\binom{22}{6} - \left[\binom{7}{1}\binom{19}{6} - \binom{7}{2}\binom{16}{6} + \binom{7}{3}\binom{13}{6}\cdots\right]$ meals. Use PIE to subtract all the meals in which you get 3 or more of a particular item.

3.8.5.2.

a. $\binom{22}{5} = 26334$ - there are 17 stars and 5 bars.

b. $\binom{16}{5} = 4368$ - buy one of each item, using \$6. This leaves you \$11 to distribute to the 6 items, so 11 stars and 5 bars.

c. $\binom{22}{5} - \left[\binom{6}{1}\binom{17}{5} - \binom{6}{2}\binom{12}{5} + \cdots\right]$ meals. Use PIE to subtract all the meals in which you get 5 or more of a particular item.

3.8.5.3. $\binom{18}{4} - \left[\binom{5}{1}\binom{11}{4} - \binom{5}{2}\binom{4}{4}\right]$.

3.8.5.4. The easiest way to solve this is to first distribute the minimum number of units to each variable (1), and then count the solutions to $y_1 + y_2 + y_3 + y_4 = 15$ with $0 \le y_i \le 5$. By taking $x_i = y_i + 5$, each solution to this new equation corresponds to exactly one solution to the original equation.

Now all the ways to distribute the 15 units to the four y_i variables can be found using stars and bars, specifically 15 stars and 3 bars, so $\binom{18}{15}$ ways. But this includes the ways that one or more y_i variables can be assigned more than 3 units. So subtract using PIE to get $\binom{18}{3} - \left(\binom{4}{1}\binom{12}{3} - \binom{4}{2}\binom{6}{3} + \dots\right)$

The $\binom{4}{1}$ counts the number of ways to pick one variable to be over-assigned; the $\binom{12}{3}$ is the number of ways to assign the remaining units to the 4 variables, etc.

3.8.5.5.

a. $\binom{12}{7} = 792$. This makes sense because if each student can receive at most one star, you must pick which 7 of the 12 kids will get one.

b. Without any restriction, there would be $\binom{18}{11}$ ways to distribute the stars. Now we must use PIE to eliminate all distributions in which one or more students get more than one star:

$$\binom{18}{11} - \left[\binom{12}{1}\binom{16}{11} - \binom{12}{2}\binom{14}{11} + \binom{12}{3}\binom{12}{11} - \cdots\right] = 792.$$

3.8.5.6. First pick one of the 4 elements to be fixed. For each such choice, derange the remaining 3, using the standard advanced PIE formula. We get $\binom{4}{1}\left(3! - \left[\binom{3}{1}2! - \binom{3}{2}1! + \dots \binom{3}{3}0!\right]\right)$ permutations.

3.8.5.7. $\binom{12}{8}\left(4! - \left[\binom{4}{1}3! - \binom{4}{2}2! \cdots + \binom{4}{4}0!\right]\right) = 4455$ ways. We choose 8 of the 12 ladies to get their own hats, and then multiply by the number of ways the remaining hats can be deranged.

3.8.5.8.

a.
$$8! - \left[\binom{8}{1}7! - \binom{8}{2}6! \dots \binom{8}{8}0!\right] = 14833$$

b.
$$\binom{8}{4}\left(4! - \left[\binom{4}{1}3! - \binom{4}{2}2! + \dots \binom{4}{4}0!\right]\right) = 630$$

c. 0. Once 7 presents have their original label, there is only one present left and only one label left, so the 8th present must get its own label.

3.8.5.9. There are $5 \cdot 6^7$ functions for which $f(1) \neq c$ and another $5 \cdot 6^7$ functions for which $f(2) \neq f$. There are $5^2 \cdot 6^6$ functions for which both $f(1) \neq c$ and $f(2) \neq f$. So the total number of functions for which $f(1) \neq c$ or $f(2) \neq f$ or both is
$5 \cdot 6^7 + 5 \cdot 6^7 - 5^2 \cdot 6^6 = 1.63296 \times 10^6$.

3.8.5.10. $5^8 - \left[\binom{5}{1}4^8 - \binom{5}{2}3^8 + \binom{5}{3}2^8 - \cdots + \binom{5}{4}1^8\right]$ functions. The 5^8 is all the functions from A to B. We subtract those that aren't surjective. Pick one of the elements in B to not be in the range (in $\binom{5}{1}$ ways) and count all those functions (4^8). But this overcounts the functions where two elements from B are excluded from the range, so subtract those. And so on, using PIE.

3.8.5.11. 1854 functions. This is a sneaky way to ask for the number of derangements on 7 elements.

3.8.6 · Additional Exercises

3.8.6.2. The 9 derangements are: 2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321.

3.9 · Chapter Summary

· Chapter Review

3.9.1.

a. $\binom{14}{10}$ ways. After giving one present to each kid, you are left with 10 presents (stones) which need to be divided among the 5 kids (giving 4 sticks).

b. $\binom{19}{15} = 3876$ ways. You have 15 stones and 4 sticks.

c. 5^{15}. You have 5 choices for whom to give each present. This is like making a function from the set of presents to the set of kids.

d. $5^{15} - \left[\binom{5}{1}(5-1)^{15} - \binom{5}{2}(5-2)^{15} + \binom{5}{3}(5-3)^{15} \ldots\right]$ ways. Now the function from the set of presents to the set of kids must be surjective.

3.9.2.

(a) Neither. $\binom{14}{4}$ paths.

(b) $\binom{10}{4}$ bow ties.

(c) $P(10,4)$, since order is important.

(d) Neither. Assuming you will wear each of the 4 ties on just 4 of the 7 days, without repeats: $\binom{10}{4}P(7,4)$.

(e) $P(10,4)$.

(f) $\binom{10}{4}$.

(g) Neither. Since you could repeat letters: 10^4. If no repeats are allowed, it would be $P(10,4)$.

(h) Neither. Actually, "k" is the 11th letter of the alphabet, so the answer is 0. If "k" was among the first 10 letters, there would only be 1 way – write it down.

(i) Neither. Either $\binom{9}{3}$ (if every kid gets an apple) or $\binom{13}{3}$ (if appleless kids are allowed).

(j) Neither. Note that this could not be $\binom{10}{4}$ since the 10 things and 4 things are from different groups. 4^{10}, assuming each kid eats one type of cereal.

(k) $\binom{10}{4}$. Don't be fooled by the "arrange" in there. You are picking 4 out of 10 *spots* to put the 1's.

(l) $\binom{10}{4}$ (assuming order is irrelevant).

(m) Neither. 16^{10} (each kid chooses yes or no to 4 varieties).

(n) Neither. 0.

(o) Neither. $4^{10} - [\binom{4}{1}3^{10} - \binom{4}{2}2^{10} + \binom{4}{3}1^{10}]$.

(p) Neither. $10 \cdot 4$.

(q) Neither. 4^{10}.

(r) $\binom{10}{4}$ (which is the same as $\binom{10}{6}$).

(s) Neither. If all the kids were identical, and you wanted no empty teams, it would be $\binom{10}{4}$. Instead, this will be the same as the number of surjective functions from a set of size 11 to a set of size 5.

(t) $\binom{10}{4}$.

(u) $\binom{10}{4}$.

(v) Neither. 4!.

(w) Neither. It's $\binom{10}{4}$ if you won't repeat any choices. If repetition is allowed, then this becomes $x_1 + x_2 + \cdots + x_{10} = 4$, which has $\binom{13}{9}$ solutions in non-negative integers.

(x) Neither. Since repetition of cookie type is allowed, the answer is 10^4. Without repetition, you would have $P(10, 4)$.

(y) $\binom{10}{4}$ since that is equal to $\binom{9}{4} + \binom{9}{3}$.

(z) Neither. It will be a complicated (possibly PIE) counting problem.

3.9.3.

a. $2^{14} = 16384$ choices. You have two choices for each tie: wear it or don't.

b. You have 511 choices for regular ties (the 2^9 choices less the "no regular tie" option) and 31 choices for bow ties (2^5 total minus the "no bow tie" option). Thus you have $511 \cdot 31 = 15841$ total choices.

c. $\binom{9}{3}\binom{5}{2} = 840$ choices.

d. Select one of the 2 bow ties to go on top. There are then 4 choices for the next tie, 4-1 for the tie after that, and so on. Thus $2 \cdot 4! = 48$ choices.

3.9.4. You own 8 purple bow ties, 3 red bow ties, 3 blue bow ties, and 5 green bow ties. How many ways can you select one of each color bow tie to take with you on a trip? $8 \cdot 3 \cdot 3 \cdot 5$ ways. How many choices do you have for a single bow tie to wear tomorrow? $8 + 3 + 3 + 5$ choices.

3.9.5.

a. 8^{11} numbers.

b. $8^{10} \cdot 4$ numbers (choose any digits for the first 10 digits, then pick either an even or an odd last digit to make the sum even).

c. We need 6 or more even digits. 6 even digits: $\binom{11}{6}4^6 4^{11-6}$. 6+1 even digits: $\binom{11}{(6+1)}4^{6+1} * 4^{11-6-1}$, etc. By adding these together, we get the total number of ways to have 6 or more even digits.

3.9.6. 48 passengers. We are asking for the size of the union of three non-disjoint sets. Using PIE, we have $27 + 20 + 29 - (13 + 11 + 10) + 6 = 48$.

3.9.7.

a. $2^1 0$ strings.

b. $\binom{10}{5}$ strings.

c. $\binom{10}{5}$ strings.

3.9.8. $3^5 \cdot \binom{15}{10} + 5^{14} \cdot \binom{21}{7}$.

3.9.9. With repeated letters allowed, we select which 3 of the 10 letters will be vowels, then pick one of the 5 vowels for each spot, and finally pick one of the other 21 letters for each of the remaining 7 spots. Thus, $\binom{10}{3}5^3 21^7$ words.

Without repeats, we still pick the positions of the vowels, but now each time we place a vowel there, we have one fewer choice for the next one. Similarly, we cannot repeat the consonants. We get $\binom{10}{3}P(5,3)P(21,7)$ words.

3.9.10.

a. $\binom{3}{2}\binom{9}{4} = 378$ paths.

b. $\binom{12}{6} - \binom{10}{5}\binom{2}{1} = 420$ paths.

c. $\binom{3}{2}\binom{9}{4} + \binom{10}{5}\binom{2}{1} - \binom{3}{2}\binom{7}{3}\binom{2}{1} = 672$ paths.

3.9.11. $\binom{31}{15}\left(\binom{31}{15} - 1\right)$ routes. For each of the $\binom{31}{15}$ routes to work, there is exactly one less route back.

3.9.12. $2^8 + 2^9 - 2^6$ strings (using PIE).

3.9.13. $\binom{12}{3} + \binom{13}{4} - \binom{10}{2}$ strings.

3.9.14. There are 3 spots to start the word, and then there are 2! ways to arrange the other letters in the remaining three spots. Thus the number of words avoiding the sub-word "bad" in consecutive letters is $5! - 3 \cdot 2! = 114$.

If we now need to avoid words that put "b" before "a" before "d", we must choose which spots those letters go (in that order) and then arrange the remaining 2 letters. Thus, $5! - \binom{5}{3}2! = 100$ words.

3.9.15. 2^n is the number of lattice paths which have length n, since for each step you can go up or right. Such a path would end along the line $x + y = n$. So you will end at $(0, n)$, or $(1, n-1)$ or $(2, n-2)$ or ... or $(n, 0)$. Counting the paths to each of these points separately, give $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ (each time choosing which of the n steps to be to the right). These two methods count the same quantity, and so are equal.

3.9.16.

a. $\binom{15}{10} = 3003$ ways.

b. $\binom{21}{16} = 20349$ ways.

c. $\binom{15}{10} - \left[\binom{6}{1}\binom{8}{5} - \binom{6}{2}\binom{1}{5} \ldots\right] = 2877$ ways.

3.9.17.

a. $4^3 + 4^3 - 4^2 = 112$ functions.

b. $3 \cdot 4^3 + 4 \cdot 3 \cdot 4^2 - 3 \cdot 3 \cdot 4^2 = 240$ functions.

c. $4! - [3! + 3! - 2!] = 14$ functions. Note that we use factorials instead of powers because we are looking for injective functions.

d. Note that being surjective here is the same as being injective, so we can start with all 4! injective functions and subtract those which have one or more "fixed point". We get $4! - \left[\binom{4}{1}(4-1)! - \binom{4}{2}(4-2)! + \binom{4}{3}(4-3)! - \ldots \binom{4}{4}0!\right] = 9$ functions.

3.9.18. $4^6 - \left[\binom{4}{1}3^6 - \binom{4}{2}2^6 + \binom{4}{3}1^6\right]$.

3.9.19.

a. $\binom{13}{8} = 1287$ combinations. You need to choose 8 of the 13 cookie types. Order doesn't matter.

b. $P(13, 8) = 13 \cdot (13-1) \cdot (13-2) \cdot \cdots \cdot (13-8+1) = 5.18918 \times 10^7$ ways. You are choosing and arranging 8 out of 13 cookies. Order matters now.

c. $\binom{35}{23} = 8.34452 \times 10^8$ choices. You must switch between cookie type 12 times as you make your 23 cookies. The cookies are the stones; the switches between cookie types are the sticks.

d. 13^{23} choices. You have 13 choices for the "1" cookie, 13 choices for the "2" cookie, and so on.

e. $13^{23} - \left[\binom{13}{1}(13-1)^{23} - \binom{13}{2}(13-2)^{23} + \cdots - \binom{13}{13}0^{23}\right] = 2.49879 \times 10^{24}$ choices. We must use PIE to remove all the ways in which one or more cookie type is not selected.

3.9.20.

(a) You are giving your professor 4 types of cookies coming from 10 different types of cookies. This does not lend itself well to a function interpretation. We *could* say that the domain contains the 4 types you will give your professor and the codomain contains the 10 you can choose from, but then counting injections would be too much (it doesn't matter if you pick type 3 first and type 2 second, or the other way around, just that you pick those two types).

(b) We want to consider injective functions from the set {most, second most, second least, least} to the set of 10 cookie types. We want injections because we cannot pick the same type of cookie to give most and least of (for example).

(c) This is not a good problem to interpret as a function. The problem is that the domain would have to be the 12 cookies you bake, but these elements are indistinguishable (there is not a first cookie, second cookie, etc.).

(d) The domain should be the 12 shapes, the codomain the 10 types of cookies. Since we can use the same type for different shapes, we are interested in counting all functions here.

(e) Here we insist that each type of cookie be given at least once, so now we are asking for the number of surjections of those functions counted in the previous part.

4 · Sequences
4.1 · Describing Sequences
4.1.7 · Additional Exercises

4.1.7.1.

(a) The recursive definition is $a_n = a_{n-1} + 2$ with $a_1 = 1$. A closed formula is $a_n = 2n - 1$.

(b) The sequence of partial sums is $1, 4, 9, 16, 25, 36, \ldots$. A recursive definition is (as always) $b_n = b_{n-1} + a_n$ which in this case is $b_n = b_{n-1} + 2n - 1$. It appears that the closed formula is $b_n = n^2$

4.1.7.2.

(a) $0, 1, 2, 4, 7, 12, 20, \ldots$.

(b) $F_0 + F_1 + \cdots + F_n = F_{n+2} - 1$.

4.1.7.3. The sequences all have the same recurrence relation: $a_n = a_{n-1} + a_{n-2}$ (the same as the Fibonacci numbers). The only difference is the initial conditions.

4.1.7.4. $3, 10, 24, 52, 108, \ldots$. The recursive definition for $10, 24, 52, \ldots$ is $a_n = 2a_{n-1} + 4$ with $a_1 = 10$.

4.1.7.5. $-1, -1, 1, 5, 11, 19, \ldots$ Thus the sequence $0, 2, 6, 12, 20, \ldots$ has closed formula $a_n = (n+1)^2 - 3(n+1) + 2$.

4.1.7.6. This closed formula would have $a_{n-1} = 3 \cdot 2^{n-1} + 7 \cdot 5^{n-1}$ and $a_{n-2} =$

$3 \cdot 2^{n-2} + 7 \cdot 5^{n-2}$. Then we would have

$$\begin{aligned}
7a_{n-1} - 10a_{n-2} =& 7(3 \cdot 2^{n-1} + 7 \cdot 5^{n-1}) - 10(3 \cdot 2^{n-2} + 7 \cdot 5^{n-2}) \\
=& 21 \cdot 2^{n-1} + 49 \cdot 5^{n-1} - 30 \cdot 2^{n-2} - 70 \cdot 5^{n-2}) \\
=& 21 \cdot 2^{n-1} + 49 \cdot 5^{n-1} - 15 \cdot 2^{n-1} - 14 \cdot 5^{n-1}) \\
=& 6 \cdot 2^{n-1} + 35 \cdot 5^{n-1} \\
=& 3 \cdot 2^{n} + 7 \cdot 5^{n} = a_n.
\end{aligned}$$

So the closed formula agrees with the recurrence relation. The closed formula has initial terms $a_0 = 10$ and $a_1 = 41$.

4.1.7.10.

(a) $\displaystyle\sum_{k=1}^{n} 2k.$

(b) $\displaystyle\sum_{k=1}^{107} (1 + 4(k-1)).$

(c) $\displaystyle\sum_{k=1}^{50} \frac{1}{k}.$

(d) $\displaystyle\prod_{k=1}^{n} 2k.$

(e) $\displaystyle\prod_{k=1}^{100} \frac{k}{k+1}.$

4.1.7.11.

(a) $\displaystyle\sum_{k=1}^{100} (3 + 4k) = 7 + 11 + 15 + \cdots + 403.$

(b) $\displaystyle\sum_{k=0}^{n} 2^k = 1 + 2 + 4 + 8 + \cdots + 2^n.$

(c) $\displaystyle\sum_{k=2}^{50} \frac{1}{(k^2-1)} = 1 + \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \cdots + \frac{1}{2499}.$

(d) $\displaystyle\prod_{k=2}^{100} \frac{k^2}{(k^2-1)} = \frac{4}{3} \cdot \frac{9}{8} \cdot \frac{16}{15} \cdots \frac{10000}{9999}.$

(e) $\displaystyle\prod_{k=0}^{n} (2 + 3k) = (2)(5)(8)(11)(14) \cdots (2 + 3n).$

4.2 · Rate of Growth
4.2.6 · Practice Problems

4.2.6.8. For arithmetic: $x = 83.6666666666667$, $y = 42.3333333333333$. For the arithmetic sequence, we know $125 + d = x$, $x + d = y$, and $y + d = 1$. In other words, $125 + 3d = 1$ so $d = (-41.3333333333333)/3$. For geometric: $x = 25$ and $y = 5$.

Similarly we can find the common ratio for the geometric sequence by solving $125 \cdot r^3 = 1$ for r.

4.2.6.9. For arithmetic: $x = 13$, $y = 22$. We know that $4 + d = x$, $x + d = y$, and $y + d = 31$. In other words, $4 + 3d = 31$ so $d = (27)/3$.

For geometric: $x = 7.91578$ and $y = 15.6649$. We can find the common ratio for the geometric sequence by solving $4 \cdot r^3 = 31$ for r.

4.3 · Polynomial Sequences
4.3.6 · Practice Problems

4.3.6.1.

(a) $a_n = a_{n-1} + 3$; $a_1 = 10$.

(b) $a_n = 7 + 3n$.

(c) Yes, $1555 = a_{516}$.

(d) The sequence has 142 terms.

(e) The sum is 31453.

(f) $b_n = 3 + \frac{(10+7+3n)n}{2}$.

4.3.6.2.

a. 25, which is $20 + 5$.

b. The sequence is arithmetic, with $a_0 = 5$ and constant difference 5, so $a_n = 5+5n$.

c. 25250. We want $5 + 15 + \cdots + 500$. Reverse and add to get 100 sums of 505, a total of 50500, which is twice the sum we are looking for.

4.3.6.3.

a. 45.

b. $\dfrac{238 \cdot 44}{2} = 5355$.

4.3.6.4.

a. $n - (-2) + 1$ terms, since to get -18 using the formula $12n + 6$ we must use $n = -2$.

b. $12n - 6$, which is 12 less than $12n + 6$ (or plug in $n - 1$ for n).

c. $\frac{(12n-12)*(n-(-2)+1)}{2}$. Reverse and add. Each sum gives the constant $12n - 12$, and there are $n - (-2) + 1$ terms.

4.3.6.5. 273310. If we take $a_0 = 2$, the terms of the sum are an arithmetic sequence with closed formula $a_n = 2 + 6n$. Then $1808 = a_{301}$, for a total of 302 terms in the sum. Reverse and add to get 302 identical 1810 terms, which is twice the total we seek. $1810 \cdot 302/2 = 273310$.

4.3.6.6. $a_n = n^2 + 8n$. Here we know that we are looking for a quadratic because the second differences are constant. So $a_n = an^2 + bn + c$. Since $a_0 = 0$, we know $c = 0$. So just solve the system

$$9 = a + b$$

$$20 = 4a + 2b$$

to find $a = 1$ and $b = 8$.

4.3.6.7. $a_n = n^2 + 4n - 1$. Here we know that we are looking for a quadratic because the second differences are constant. So $a_n = an^2 + bn + c$. Since $a_0 = -1$, we know $c = -1$. So just solve the system

$$4 = a + b + -1$$

$$11 = 4a + 2b + -1$$

to find $a = 1$ and $b = 4$.

4.3.6.8. $a_n = 2n^3 - 2n + 5$. Here we know that we are looking for a cubic because the third differences are constant. So $a_n = an^3 + bn^2 + cn + d$. Since $a_0 = 5$, we know $d = 5$. So just solve the system

$$\begin{aligned} 5 &= a + b + c + 5 \\ 17 &= 8a + 4b + 2c + 5 \\ 53 &= 27a + 9b + 3c + 5 \end{aligned}$$

to find $a = 2$, $b = 0$, and $c = -2$.

4.3.6.9. $a_n = 2n^3 + 8n^2 - 4n + 3$. Here we know that we are looking for a cubic because the third differences are constant. So $a_n = an^3 + bn^2 + cn + d$. We can work backwards to find that $a_0 = 3$, so we know $d = 3$. Then solve the system,

$$9 = a + b + c + 3$$

$$43 = 8a + 4b + 2c + 3$$

$$117 = 27a + 9b + 3c + 3$$

to find $a = 2$, $b = 8$, and $c = -4$.

4.3.6.10. $a_{n-1} = 2(n-1)^2 + 3(n-1) + 2 = 2n^2 - n + 1$ Thus $a_n - a_{n-1} = 4n + 1$. Note that this is linear (arithmetic).

4.3.6.11. $a_n = 2n^3 + 3n^2 + 2n$. Here we know that we are looking for a cubic because the third differences are constant. So $a_n = an^3 + bn^2 + cn + d$. We can work backwards to find that $a_0 = 0$, so we know $d = 0$. Then solve the system,

$$7 = a + b + c$$

$$32 = 8a + 4b + 2c$$

$$87 = 27a + 9b + 3c$$

to find $a = 2$, $b = 3$, and $c = 2$.

4.3.7 · Additional Exercises

4.3.7.4. $a_n = n^2 - n + 1$.

4.3.7.5. $a_n = n^3 + n^2 - n + 1$

4.3.7.6. We have $2 = 2$, $7 = 2 + 5$, $15 = 2 + 5 + 8$, $26 = 2 + 5 + 8 + 11$, and so on. The terms in the sums are given by the arithmetic sequence $b_n = 2 + 3n$. In other words, $a_n = \sum_{k=0}^{n}(2 + 3k)$. To find the closed formula, we reverse and add. We get $a_n = \frac{(4+3n)(n+1)}{2}$ (we have $n + 1$ there because there are $n + 1$ terms in the sum for a_n).

4.3.7.9. $a_{n-1} = a(n-1)^2 + b(n-1) + c = an^2 - 2an + a + bn - b + c$. Therefore $a_n - a_{n-1} = 2an - a + b$, which is arithmetic. Notice that this is not quite the derivative of a_n, which would be $2an + b$, but it is close.

4.3.7.10. No. The sequence of differences is the same as the original sequence, so no differences will be constant.

4.3.7.11. No. The sequence is geometric, and in fact has closed formula $2 \cdot 3^n$. This is an exponential function, which is not equal to any polynomial of any degree. If the nth sequence of differences was constant, then the closed formula for the original sequence would be a degree n polynomial.

4.4 · Exponential Sequences

4.4.5 · Practice Problems

4.4.5.1. $\frac{6 \cdot 5^{20} - 6}{4}$. Let the sum be S, and compute $5S - S = 4S$, which causes terms except 6 and $6 \cdot 5^{20}$ to cancel. Then solve for S.

4.4.5.2. $\frac{1+(-1)^{38}\frac{7^{38}}{4^{38}}}{11/4}$. Compute $S + \frac{7}{4}S$.

4.4.5.3. $a_n = -4 + 2^{n+1}$. We should use telescoping or iteration here. For example, telescoping gives

$$\begin{aligned} a_1 - a_0 &= 2^1 \\ a_2 - a_1 &= 2^2 \\ a_3 - a_2 &= 2^3 \\ &\vdots \\ a_n - a_{n-1} &= 2^n \end{aligned}$$

which sums to $a_n - a_0 = 2^{n+1} - 2$ (using the multiply-shift-subtract technique for the geometric series on the right-hand side). Substituting $a_0 = -2$ and solving for a_n completes the solution.

4.4.5.4. By the characteristic root technique. $a_n = (-4)^n + 6^n$.

4.4.5.5. The characteristic roots for both sequences are -1 and 2 (since the two sequences have the same recurrence relation, they have the same characteristic roots). Solving for coefficients in each case gives the closed formulas. We have $a_n = (-1)^n + 2^n$ and $b_n = \left(\frac{4}{3}\right)(-1)^n + \left(\frac{23}{3}\right) \cdot 2^n$.

4.4.5.6. By the characteristic root technique. $a_n = 7 \cdot 2^n - 3(-1)^n$.

4.4.6 · Additional Exercises

4.4.6.1. 171 and 341. $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 3$ and $a_1 = 5$. Closed formula: $a_n = \frac{8}{3}2^n + \frac{1}{3}(-1)^n$. To find this solve the characteristic equation, $x^2 - x - 2 = 0$, to get characteristic roots $x = 2$ and $x = -1$. Then solve the system

$$\begin{aligned} 3 &= a + b \\ 5 &= 2a - b \end{aligned}$$

4.4.6.3. We claim $a_n = 4^n$ works. Plug it in: $4^n = 3(4^{n-1}) + 4(4^{n-2})$. This works; just simplify the right-hand side.

4.4.6.6.

(a) $a_n = 4a_{n-1} + 5a_{n-2}$.

(b) 4, 21, 104, 521, 2604, 13021

(c) $a_n = \frac{5}{6}5^n + \frac{1}{6}(-1)^n$.

4.4.6.8. We have characteristic polynomial $x^2 - 2x + 1$, which has $x = 1$ as the only repeated root. Thus, using the characteristic root technique for repeated roots, the general solution is $a_n = a + bn$ where a and b depend on the initial conditions.

(a) $a_n = 1 + n$.

(b) For example, we could have $a_0 = 21$ and $a_1 = 22$.

(c) For every x. Take $a_0 = x - 9$ and $a_1 = x - 8$.

4.5 · Proof by Induction
4.5.6 · Practice Problems

4.5.6.1.

A. *Correct.*

B. *Correct.*

C. *Incorrect.*

D. *Incorrect.*

E. *Incorrect.*

4.5.6.2.

A. *Correct.*

B. *Incorrect.*

C. *Incorrect.*

D. *Incorrect.*

E. *Incorrect.*

4.5.6.3.

A. *Correct.*

B. *Incorrect.*

Even though the sum starts with 2, we need to consider the smallest n for which the statement $P(n)$ is true.

C. *Incorrect.*

D. *Incorrect.*

E. *Incorrect.*

4.5.6.4.

A. *Correct.*

B. *Incorrect.*

Assume $P(k)$ and $P(k)$ are both true for an arbitrary $k \geq 1$; that is, assume $2 + 4 + 6 + \cdots + 2k = k(k+1)$ and $2 + 4 + 6 + \cdots + 2k + 2k + 2) = (k+1)(k+2)$.

C. *Incorrect.*

D. *Incorrect.*

4.5.6.5.

- Let $P(n)$ be the statement, "$a_n = 5^n - 1$".
- Note that $a_0 = 5^0 - 1 = 0$, so $P(0)$ is true.
- Now assume that $P(k)$ is true for an arbitrary integer $k \geq 0$.
- Then $a_k = 5^k - 1$.
- By the recurrence relation, we have $a_{k+1} = 5a_k + 4 = 5(5^k - 1) + 4$.
- This simplifies to $a_{k+1} = 5^{k+1} - 5 + 4 = 5^{k+1} - 1$, so $P(k+1)$ is true.
- Therefore, by the principle of mathematical induction, $P(n)$ is true for all $n \geq 0$.

4.5.6.6.

- Let $P(n)$ be the statement, "$14^n - 1$ is a multiple of 13."
- Note that $14^1 - 1 = 13$, so this is definitely a multiple of 13.
- Now assume that $P(k)$ is true for an arbitrary integers $k \geq 1$.
- Then $14^k - 1 = 13 \cdot j$ for some integer j.
- Since $14^{k+1} - 1 = 14(14^k - 1) + 14 - 1 = 14(13 \cdot j) + 13$, we see that $14^{k+1} - 1$ is a multiple of 13.

- Thus $P(k + 1)$ is true.
- Therefore, by the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$.

4.5.6.7.

- Let $P(n)$ be the statement, "$1 + 1 + 2 + 3 + 5 + \cdots + F_n = F_{n+2} - 1$."
- For the base case, note that $P(1)$ and $P(2)$ are true, because $1 = 2 - 1$ and $1 + 1 = 3 - 1$.
- Now assume that $P(k)$ is true for an arbitrary integer $k \geq 2$.
- That is, assume $1 + 1 + 2 + 3 + 5 + F_k = F_{k+2} - 1$.
- Then adding F_{k+1} to both sides, we get $1 + 1 + 2 + 3 + 5 + \cdots + F_k + F_{k+1} = F_{k+1} + F_{k+2} - 1$.
- By the definition of Fibonacci numbers, $F_{k+1} + F_{k+2} = F_{k+3}$, so the right-hand side simplifies to $F_{k+3} - 1$.
- Thus $P(k + 1)$ is true, and therefore by the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$.

4.5.7 · Additional Exercises

4.5.7.1.

(a) If we have a number of beans ending in a 5, and we double it, we will get a number of beans ending in a 0 (since $5 \cdot 2 = 10$). Then if we subtract 5, we will once again get a number of beans ending in a 5. Thus, if on any day we have a number ending in a 5, the next day we will also have a number ending in a 5.

(b) If you don't *start* with a number of beans ending in a 5 (on day 1), the above reasoning is still correct but not helpful. For example, if you start with a number ending in a 3, the next day you will have a number ending in a 1.

(c) Part (b) is the base case, and part (a) is the inductive case. If on day 1 we have a number ending in a 5 (by part (b)), then on day 2 we will also have a number ending in a 5 (by part (a)). Then by part (a) again, we will have a number ending in a 5 on day 3. By part (a) again, this means we will have a number ending in a 5 on day 4.

The proof by induction would say that on *every* day we will have a number ending in a 5, and this works because we can start with the base case and then use the inductive case over and over until we get up to our desired n.

4.5.7.2.

Proof. We must prove that $1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$ for all $n \in \mathbb{N}$. Thus let $P(n)$ be the statement $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$. We will prove that $P(n)$ is true for all $n \in \mathbb{N}$. First we establish the base case, $P(0)$, which claims that $1 = 2^{0+1} - 1$. Since $2^1 - 1 = 2 - 1 = 1$, we see that $P(0)$ is true. Now for the inductive case. Assume that $P(k)$ is true for an arbitrary $k \in \mathbb{N}$. That is, $1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$. We must show that $P(k+1)$ is true (i.e., that $1 + 2 + 2^2 + \cdots + 2^{k+1} = 2^{k+2} - 1$). To do this, we start with the left-hand side of $P(k+1)$ and work to the right-hand side:

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} &= 2^{k+1} - 1 + 2^{k+1} && \text{by the inductive hypothesis.} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1 \end{aligned}$$

Thus $P(k+1)$ is true, so by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. ■

4.5.7.3.

Proof. Let $P(n)$ be the statement, "$7^n - 1$ is a multiple of 6." We will show $P(n)$ is true for all $n \in \mathbb{N}$. First we establish the base case, $P(0)$. Since $7^0 - 1 = 0$, and 0 is a multiple of 6, $P(0)$ is true. Now for the inductive case. Assume $P(k)$ holds for an arbitrary $k \in \mathbb{N}$. That is, $7^k - 1$ is a multiple of 6, or in other words, $7^k - 1 = 6j$ for some integer j. Now consider $7^{k+1} - 1$:

$$\begin{aligned} 7^{k+1} - 1 &= 7^{k+1} - 7 + 6 && \text{by cleverness: } -1 = -7 + 6 \\ &= 7(7^k - 1) + 6 && \text{factor out a 7 from the first two terms} \\ &= 7(6j) + 6 && \text{by the inductive hypothesis} \\ &= 6(7j + 1) && \text{factor out a 6} \end{aligned}$$

Therefore $7^{k+1} - 1$ is a multiple of 6, or in other words, $P(k+1)$ is true. Therefore by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. ■

4.5.7.4.

Proof. Let $P(n)$ be the statement $1 + 3 + 5 + \cdots + (2n - 1) = n^2$. We will prove that $P(n)$ is true for all $n \geq 1$. First the base case, $P(1)$. We have $1 = 1^2$ which is true, so $P(1)$ is established. Now the inductive case. Assume that $P(k)$ is true for some fixed arbitrary $k \geq 1$. That is, $1 + 3 + 5 + \cdots + (2k - 1) = k^2$. We will now prove that $P(k+1)$ is also true (i.e., that $1 + 3 + 5 + \cdots + (2k + 1) = (k+1)^2$). We start with the left-hand side of $P(k+1)$ and work to the right-hand side:

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) &= k^2 + (2k + 1) && \text{by the inductive hypothesis} \\ &= (k + 1)^2 && \text{by factoring} \end{aligned}$$

Thus $P(k+1)$ holds, so by the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$. ■

4.5.7.5.

Proof. Let $P(n)$ be the statement $F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$. We will show that $P(n)$ is true for all $n \geq 0$. First the base case is easy because $F_0 = 0$ and $F_1 = 1$ so $F_0 = F_1 - 1$. Now consider the inductive case. Assume $P(k)$ is true, that is, assume

$F_0 + F_2 + F_4 + \cdots + F_{2k} = F_{2k+1} - 1$. To establish $P(k+1)$ we work from left to right:

$$\begin{aligned} F_0 + F_2 + \cdots + F_{2k} + F_{2k+2} &= F_{2k+1} - 1 + F_{2k+2} && \text{by the inductive hypothesis.} \\ &= F_{2k+1} + F_{2k+2} - 1 \\ &= F_{2k+3} - 1 && \text{by the recursive definition.} \end{aligned}$$

Therefore $F_0 + F_2 + F_4 + \cdots + F_{2k+2} = F_{2k+3} - 1$, which is to say $P(k+1)$ holds. Therefore by the principle of mathematical induction, $P(n)$ is true for all $n \geq 0$. ■

4.5.7.6.

Proof. Let $P(n)$ be the statement $2^n < n!$. We will show $P(n)$ is true for all $n \geq 4$. First, we check the base case and see that yes, $2^4 < 4!$ (as $16 < 24$), so $P(4)$ is true. Now for the inductive case. Assume $P(k)$ is true for an arbitrary $k \geq 4$. That is, $2^k < k!$. Now consider $P(k+1)$: $2^{k+1} < (k+1)!$. To prove this, we start with the left side and work to the right side.

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &< 2 \cdot k! && \text{by the inductive hypothesis} \\ &< (k+1) \cdot k! && \text{since } k+1 > 2 \\ &= (k+1)! \end{aligned}$$

Therefore $2^{k+1} < (k+1)!$, so we have established $P(k+1)$. Thus by the principle of mathematical induction $P(n)$ is true for all $n \geq 4$. ■

4.5.7.12. The only problem is that we never established the base case. Of course, when $n = 0$, $0 + 3 \neq 0 + 7$.

4.5.7.13.

Proof. Let $P(n)$ be the statement that $n + 3 < n + 7$. We will prove that $P(n)$ is true for all $n \in \mathbb{N}$. First, note that the base case holds: $0 + 3 < 0 + 7$. Now assume for induction that $P(k)$ is true. That is, $k + 3 < k + 7$. We must show that $P(k+1)$ is true. Now since $k + 3 < k + 7$, add 1 to both sides. This gives $k + 3 + 1 < k + 7 + 1$. Regrouping $(k+1)+3 < (k+1)+7$. But this is simply $P(k+1)$. Thus by the principle of mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$. ■

4.5.7.14. The problem here is that while $P(0)$ is true, and while $P(k) \to P(k+1)$ for *some* values of k, there is at least one value of k (namely $k = 99$) when that implication fails. For a valid proof by induction, $P(k) \to P(k+1)$ must be true for all values of k greater than or equal to the base case.

4.5.7.16. We once again failed to establish the base case: When $n = 0$, $n^2 + n = 0$ which is even, not odd.

4.5.7.22. The idea here is that if we take the logarithm of a^n, we can increase n by 1 if we multiply by another a (inside the logarithm). This results in adding 1 more $\log(a)$ to the total.

Proof. Let $P(n)$ be the statement $\log(a^n) = n\log(a)$. The base case, $P(2)$ is true, because $\log(a^2) = \log(a \cdot a) = \log(a) + \log(a) = 2\log(a)$, by the product rule for logarithms. Now assume, for induction, that $P(k)$ is true. That is, $\log(a^k) = k\log(a)$. Consider $\log(a^{k+1})$. We have

$$\log(a^{k+1}) = \log(a^k \cdot a) = \log(a^k) + \log(a) = k\log(a) + \log(a),$$

with the last equality due to the inductive hypothesis. But this simplifies to $(k+1)\log(a)$, establishing $P(k+1)$. Therefore by the principle of mathematical induction, $P(n)$ is true for all $n \geq 2$. ■

4.6 · Strong Induction
4.6.4 · Practice Problems

4.6.4.1.

A. *Correct.*

B. *Correct.*

C. *Incorrect.*

D. *Incorrect.*

E. *Incorrect.*

4.6.4.2.

- Let $P(n)$ be the statement that a chocolate bar that is n squares long can be broken into n pieces by making $n-1$ breaks.
- $P(1)$ is true because a chocolate bar that is 1 square long is already in one piece.
- Assume that $P(j)$ is true for all $j \leq k$ for an arbitrary $k \geq 1$.
- Consider a chocolate bar that is $k+1$ squares long.
- Anywhere you break this bar will result in two smaller bars, say of length a and b.
- Since a and b are no more than k, it will be possible to break these smaller bars into single squares using $a-1$ and $b-1$ breaks, respectively.
- The total number of breaks is therefore $a-1+b-1+1 = a+b-1$, which is $k+1-1 = k$.
- Therefore, by the principle of strong induction, $P(n)$ is true for all $n \geq 1$.

4.6.5 · Additional Exercises

4.6.5.3. The proof will be by strong induction.

Proof. Let $P(n)$ be the statement, "n is either a power of 2 or can be written as the sum of distinct powers of 2." We will show that $P(n)$ is true for all $n \geq 1$.

Base case: $1 = 2^0$ is a power of 2, so $P(1)$ is true.

Inductive case: Suppose $P(k)$ is true for all $k < n$. Now if n is a power of 2, we are done. If not, let 2^x be the largest power of 2 strictly less than n. Consider $n - 2^x$, which is a smaller number, in fact smaller than both n and 2^x. Thus $n - 2^x$ is either a power of 2 or can be written as the sum of distinct powers of 2, but none of them are going to be 2^x, so together with 2^x we have written n as the sum of distinct powers of 2.

Therefore, by the principle of (strong) mathematical induction, $P(n)$ is true for all $n \geq 1$. ∎

4.7 · Chapter Summary

· Chapter Review

4.7.1. $\frac{1250 \cdot 206}{2} = 128750$.

4.7.2.

a. $n - 3$ terms.

b. $10n + (-12)$.

c. $\dfrac{(10n + 26) * (n - 3)}{2}$.

4.7.3.

a. $6, 30, 150, 750, \ldots$ The sequence is geometric.

b. $\dfrac{6 \cdot 5^{22} - 6}{4} = 3.57628 \times 10^{15}$.

4.7.5. $a_n = n^2 + 8n + 2$. Here we know that we are looking for a quadratic because the second differences are constant. So $a_n = an^2 + bn + c$. We can work backwards to find that $a_0 = 2$, so we know $c = 2$. Then solve the system,

$$\begin{aligned} 11 &= a + b + 2 \\ 22 &= 4a + 2b + 2 \end{aligned}$$

to find $a = 1$ and $b = 8$.

4.7.6.

(a) The sequence of partial sums will be a degree 4 polynomial (its sequence of differences will be the original sequence).

(b) The sequence of second differences will be a degree 1 polynomial (an arithmetic sequence).

4.7.7.

(a) $4, 6, 10, 16, 26, 42, \ldots$.

(b) No, taking differences gives the original sequence back, so the differences will never be constant.

4.7.8. $b_n = (n+3)n$.

4.7.10. The sequence is $4, 9, 89, 269, 2049, \ldots$. It has closed formula $a_n = \left(\frac{11}{9}\right)(-4)^n + \left(\frac{25}{9}\right) \cdot 5^n$, using the characteristic root technique.

4.7.11. The sequence is $5, 7, 111, 459, 3375, \ldots$. It has closed formula $a_n = \left(\frac{23}{9}\right)(-3)^n + \left(\frac{22}{9}\right) \cdot 6^n$, using the characteristic root technique.

4.7.12.

(a) On the first day, your 2 mini bunnies become 2 large bunnies. On day 2, your 2 large bunnies produce 4 mini bunnies. On day 3, you have 4 mini bunnies (produced by your 2 large bunnies) plus 6 large bunnies (your original 2 plus the 4 newly matured bunnies). On day 4, you will have 12 mini bunnies (2 for each of the 6 large bunnies) plus 10 large bunnies (your previous 6 plus the 4 newly matured). The sequence of total bunnies is $2, 2, 6, 10, 22, 42 \ldots$ starting with $a_0 = 2$ and $a_1 = 2$.

(b) $a_n = a_{n-1} + 2a_{n-2}$. This is because the number of bunnies is equal to the number of bunnies you had the previous day (both mini and large) plus 2 times the number you had the day before that (since all bunnies you had 2 days ago are now large and producing 2 new bunnies each).

(c) Using the characteristic root technique, we find $a_n = a2^n + b(-1)^n$, and we can find a and b to give $a_n = \frac{4}{3}2^n + \frac{2}{3}(-1)^n$.

4.7.17. Let $P(n)$ be the statement, "Every set containing n elements has 2^n different subsets." We will show $P(n)$ is true for all $n \geq 1$. Base case: Any set with 1 element $\{a\}$ has exactly 2 subsets: the empty set and the set itself. Thus the number of subsets is $2 = 2^1$. Thus $P(1)$ is true. Inductive case: Suppose $P(k)$ is true for some arbitrary $k \geq 1$. Thus every set containing exactly k elements has 2^k different subsets. Now consider a set containing $k+1$ elements: $A = \{a_1, a_2, \ldots, a_k, a_{k+1}\}$. Any subset of A must either contain a_{k+1} or not. In other words, a subset of A is just a subset of $\{a_1, a_2, \ldots, a_k\}$ with or without a_{k+1}. Thus there are 2^k subsets of A which contain a_{k+1} and another 2^{k+1} subsets of A which do not contain a^{k+1}. This gives a total of $2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$ subsets of A. But our choice of A was arbitrary, so this works for any subset containing $k+1$ elements, so $P(k+1)$ is true. Therefore, by the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$.

5 · Discrete Structures Revisited

5.1 · Sets

5.1.5 · Exercises

5.1.5.1.

a. $A \cup B = \{2, 4, 5, 6, 7, 8\}$. It includes everything that is in A or B or both.

b. $A \cap B = \{2, 4, 5, 6\}$. It contains everything that is in both A and B.

c. $A \setminus B = \{7\}$. It contains everything that is in A except anything that is also in B.

We could also have written this set as $A \cap \overline{B}$.

d. $B \setminus A = \{8\}$. It contains everything in B except anything that is also in A. Another way to write this is $B \cap \overline{A}$. Note that $A \setminus B \neq B \setminus A$.

5.1.5.2.

a. This is the set $\{3, 4, 5, \ldots\}$ since we need each element to be a natural number whose square is at least 3 more than 4. Since $3^2 - 3 = 6$ but $2^2 - 3 = 1$ we see that the first such natural number is 3.

b. We get the same set as the previous part, and the smallest non-negative number for which $n^2 - 7$ is a natural number is 3.

Note that if we didn't specify $n \geq 0$ by saying that $n \in \mathbb{N}$ then any integer less than -3 would also be in the set, so there would not be a least element.

c. This is the set $\{4, 5, 8, 13, \ldots\}$, namely the set of numbers that are the *result* of squaring and adding 4 to a natural number. ($0^2 + 4 = 4$, $1^2 + 4 = 5$, $2^2 + 4 = 8$ and so on.) Thus the least element of the set is 4.

d. Now we are looking for natural numbers that are equal to taking some natural number, squaring it and adding 4. That is, $\{4, 5, 8, 13, \ldots\}$, the same set as the previous part. So again, the least element is 4.

5.1.5.4. The set of largest size that is a subset of both A and B is the *intersection* of A and B, $A \cap B = \{8\}$.

5.1.5.7. There will be exactly 4 such sets: $\{1, 4, 14\}$,$\{1, 4, 12, 14\}$, $\{1, 4, 7, 14\}$, and $\{1, 4, 7, 12, 14\}$.

5.1.5.8.

a. $A \cap B = \{4, 5, 6\}$.

b. $A \cup B = \{2, 3, 4, 5, 6, 7, 8\}$.

c. $A \setminus B = \{2, 3\}$.

d. $A \cap \overline{(B \cup C)} = \{2\}$.

5.1.5.11. For example, $A = \{2, 3, 5, 7, 8\}$ and $B = \{3, 5\}$.

5.1.5.12. For example, $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5, \{1, 2, 3\}\}$

5.1.5.13.

(a) No.

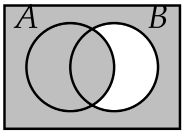
(b) No.

(c) $2\mathbb{Z} \cap 3\mathbb{Z}$ is the set of all integers which are multiples of both 2 and 3 (so multiples of 6). Therefore $2\mathbb{Z} \cap 3\mathbb{Z} = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z}(x = 6y)\}$.

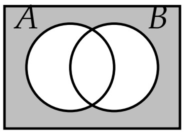
(d) $2\mathbb{Z} \cup 3\mathbb{Z}$.

5.1.5.15.

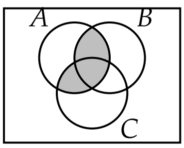
(a) $A \cup \overline{B}$:



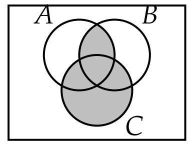
(b) $\overline{(A \cup B)}$:



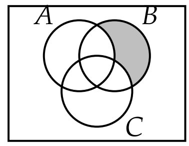
(c) $A \cap (B \cup C)$:



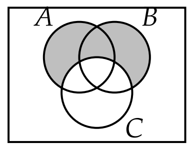
(d) $(A \cap B) \cup C$:



(e) $\overline{A} \cap B \cap \overline{C}$:



(f) $(A \cup B) \setminus C$:



5.1.5.17.

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \\ \{c,d\}\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}\}.$$

5.1.5.18. Each of the 9 elements can be in a singleton set, so there are 9 of these.

To count the number of doubletons, not that there are 8 sets that include 1, and then 7 sets that include 2 and not 1, and then 6 that include 3 and not 1 or 2, and so on. So you can find 36 by summing the numbers from 1 to 8.

5.1.5.20. For example, $A = \{1,2,3,4\}$ and $B = \{5,6,7,8,9\}$ gives $A \cup B = \{1,2,3,4,5,6,7,8,9\}$.

5.1.5.28. We need to be a little careful here. If B contains 3 elements, then A contains just the number 3 (listed twice). So that would make $|A| = 1$, which would make $B = \{1,3\}$, which only has 2 elements. Thus $|B| \neq 3$. This means that $|A| = 2$, so B contains at least the elements 1 and 2. Since $|B| \neq 3$, we must have $|B| = 2$, which agrees with the definition of B.

Therefore it must be that $A = \{2,3\}$ and $B = \{1,2\}$

5.2 · Functions
5.2.4 · Exercises

5.2.4.1.

a. $f(4) = 2$, since 2 is the number below 4 in the two-line notation.

b. Such an n is $n = 3$, since $f(3) = 4$. Note that 3 is above a 4 in the notation.

c. $n = 1$ has this property. We say that 1 is a fixed point of f. Not all functions have such a point.

d. Such an element is 3 (in fact, that is the only element in the codomain that is not in the range). In other words, 3 is not the image of any element under f; nothing is sent to 3.

5.2.4.5. There are 16 different functions. None of the functions are injective. Exactly 14 of the functions are surjective (there are 2 that are not: those that send everything to a or everything to b). No functions are both (since no functions here are injective).

5.2.4.6. There are 16 functions: you have a choice of four outputs for $f(1)$, and for each, you have four choices for the output $f(2)$. Of these functions, 12 are injective, 0 are surjective, and 0 are both (i.e., bijective):

$$f = \begin{pmatrix} 1 & 2 \\ a & a \end{pmatrix} \quad f = \begin{pmatrix} 1 & 2 \\ b & b \end{pmatrix} \quad f = \begin{pmatrix} 1 & 2 \\ c & c \end{pmatrix} \quad f = \begin{pmatrix} 1 & 2 \\ d & d \end{pmatrix}$$

$$f = \begin{pmatrix} 1 & 2 \\ a & b \end{pmatrix} \quad f = \begin{pmatrix} 1 & 2 \\ a & c \end{pmatrix} \quad f = \begin{pmatrix} 1 & 2 \\ a & d \end{pmatrix} \quad f = \begin{pmatrix} 1 & 2 \\ b & c \end{pmatrix}$$

$$f = \begin{pmatrix} 1 & 2 \\ b & a \end{pmatrix} \quad f = \begin{pmatrix} 1 & 2 \\ c & a \end{pmatrix} \quad f = \begin{pmatrix} 1 & 2 \\ d & a \end{pmatrix} \quad f = \begin{pmatrix} 1 & 2 \\ c & b \end{pmatrix}$$

$$f = \begin{pmatrix} 1 & 2 \\ b & d \end{pmatrix} \quad f = \begin{pmatrix} 1 & 2 \\ d & b \end{pmatrix} \quad f = \begin{pmatrix} 1 & 2 \\ c & d \end{pmatrix} \quad f = \begin{pmatrix} 1 & 2 \\ d & c \end{pmatrix}$$

5.2.4.7.

(a) f is not injective, since $f(2) = f(5)$; two different inputs have the same output.

(b) f is surjective, since every element of the codomain is an element of the range.

(c) $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 2 \end{pmatrix}$.

5.2.4.12.

(a) f is injective, but not surjective (since 0, for example, is never an output).

(b) f is injective and surjective. Unlike in the previous question, every integers is an output (of the integer 4 less than it).

(c) f is injective, but not surjective (10 is not 8 less than a multiple of 5, for example).

(d) f is not injective, but is surjective. Every integer is an output (of twice itself, for example) but some integers are outputs of more than one input: $f(5) = 3 = f(6)$.

5.2.4.13.

(a) f is not injective. To prove this, we must simply find two different elements of the domain which map to the same element of the codomain. Since $f(\{1\}) = 1$ and $f(\{2\}) = 1$, we see that f is not injective.

(b) f is not surjective. The largest subset of A is A itself, and $|A| = 10$. So no natural number greater than 10 will ever be an output.

(c) $f^{-1}(1) = \{\{1\}, \{2\}, \{3\}, \ldots \{10\}\}$ (the set of all the singleton subsets of A).

(d) $f^{-1}(0) = \{\emptyset\}$. Note that it would be wrong to write $f^{-1}(0) = \emptyset$; that would claim that there is no input which has 0 as an output.

(e) $f^{-1}(12) = \emptyset$, since there are no subsets of A with cardinality 12.

5.2.4.16.

(a) $|f^{-1}(3)| \leq 1$. In other words, either $f^{-1}(3)$ is the empty set or is a set containing exactly one element. Injective functions cannot have two elements from the domain both map to 3.

(b) $|f^{-1}(3)| \geq 1$. In other words, $f^{-1}(3)$ is a set containing at least one elements, possibly more. Surjective functions must have something map to 3.

(c) $|f^{-1}(3)| = 1$. There is exactly one element from X which gets mapped to 3, so $f^{-1}(3)$ is the set containing that one element.

5.2.4.17. X can really be any set, as long as $f(x) = 0$ or $f(x) = 1$ for every $x \in X$. For example, $X = \mathbb{N}$ and $f(n) = 0$ works.

5.2.4.21.

(a) f is injective.

Proof. Let x and y be elements of the domain $\mathbb{Z}$. Assume $f(x) = f(y)$. If x and y are both even, then $f(x) = x + 1$ and $f(y) = y + 1$. Since $f(x) = f(y)$, we have $x + 1 = y + 1$ which implies that $x = y$. Similarly, if x and y are both odd, then $x - 3 = y - 3$ so again $x = y$. The only other possibility is that x is even and y is odd (or vice-versa). But then $x + 1$ would be odd, and $y - 3$ would be even, so it cannot be that $f(x) = f(y)$. Therefore if $f(x) = f(y)$ we then have $x = y$, which proves that f is injective. ■

(b) f is surjective.

Proof. Let y be an element of the codomain $\mathbb{Z}$. We will show there is an element n of the domain ($\mathbb{Z}$) such that $f(n) = y$. There are two cases: First, if y is even, then let $n = y+3$. Since y is even, n is odd, so $f(n) = n-3 = y+3-3 = y$ as desired. Second, if y is odd, then let $n = y - 1$. Since y is odd, n is even, so $f(n) = n + 1 = y - 1 + 1 = y$ as needed. Therefore f is surjective. ■

5.2.4.22. Yes, this is a function, if you choose the domain and codomain correctly. The domain will be the set of students, and the codomain will be the set of possible grades. The function is almost certainly not injective, because it is likely that two students will get the same grade. The function might be surjective – it will be if there is at least one student who gets each grade.

5.2.4.24. This is not a function.

5.2.4.25. The recurrence relation is $f(n+1) = f(n) + n$.

5.2.4.26. In general, $|A| \geq \left|f(A)\right|$, since you cannot get more outputs than you have inputs (each input goes to exactly one output), but you could have fewer outputs if the function is not injective. If the function is injective, then $|A| = \left|f(A)\right|$, although you can have equality even if f is not injective (it must be injective *restricted* to A).

5.2.4.27. In general, there is no relationship between $|B|$ and $\left|f^{-1}(B)\right|$. This is because B might contain elements that are not in the range of f, so we might even have $f^{-1}(B) = \emptyset$. On the other hand, there might be lots of elements from the domain that all get sent to a few elements in B, making $f^{-1}(B)$ larger than B.

More specifically, if f is injective, then $|B| \geq \left|f^{-1}(B)\right|$ (since every element in B must come from at most one element from the domain). If f is surjective, then $|B| \leq \left|f^{-1}(B)\right|$ (since every element in B must come from at least one element of the domain). Thus if f is bijective then $|B| = \left|f^{-1}(B)\right|$.

6 · Additional Topics

6.1 · Generating Functions

6.1.5 · Exercises

6.1.5.1.

(a) $\dfrac{4}{1-x}$.

(b) $\dfrac{2}{(1-x)^2}$.

(c) $\dfrac{2x^3}{(1-x)^2}$.

(d) $\dfrac{1}{1-5x}$.

(e) $\dfrac{1}{1+3x}$.

(f) $\dfrac{1}{1-5x^2}$.

(g) $\dfrac{x}{(1-x^3)^2}$.

6.1.5.2.

(a) $0, 4, 4, 4, 4, 4, \ldots$.

(b) $1, 4, 16, 64, 256, \ldots$.

(c) $0, 1, -1, 1, -1, 1, -1, \ldots$.

(d) $0, 3, -6, 9, -12, 15, -18, \ldots$.

(e) $1, 3, 6, 9, 12, 15, \ldots$.

6.1.5.4. Call the generating function A. Compute $A - xA = 4 + x + 2x^2 + 3x^3 + 4x^4 + \cdots$.

Thus $A - xA = 4 + \frac{x}{(1-x)^2}$. Solving for A gives $\frac{4}{1-x} + \frac{x}{(1-x)^3}$.

6.1.5.5. $\frac{1+2x}{1-3x+x^2}$.

6.1.5.6. Compute $A - xA - x^2A$ and then solve for A. The generating function will be $\frac{x}{1-x-x^2}$.

6.1.5.7. $\frac{x}{(1-x)(1-x-x^2)}$.

6.1.5.8. $\frac{2}{1-5x} + \frac{7}{1+3x}$.

6.1.5.9. $a_n = 3 \cdot 4^{n-1} + 1$.

6.1.5.12.

(a) $\frac{1}{(1-x^2)^2}$.

(b) $\frac{1}{(1+x)^2}$.

(c) $\frac{3x}{(1-x)^2}$.

(d) $\frac{3x}{(1-x)^3}$. (partial sums).

6.1.5.13.

(a) $0, 0, 1, 1, 2, 3, 5, 8, \ldots$.

(b) $1, 0, 1, 0, 2, 0, 3, 0, 5, 0, 8, 0, \ldots$.

(c) $1, 3, 18, 81, 405, \ldots$.

(d) $1, 2, 4, 7, 12, 20, \ldots$.

6.1.5.15. $\frac{x^3}{(1-x)^2} + \frac{1}{1-x}$.

6.2 · Introduction to Number Theory

6.2.6 · Exercises

6.2.6.1.

Proof. Suppose $a \mid b$. Then b is a multiple of a, or in other words, $b = ak$ for some k. But then $bc = akc$, and since kc is an integer, this says bc is a multiple of a. In other words, $a \mid bc$. ∎

6.2.6.3. $\{\ldots, -8, -4, 0, 4, 8, 12, \ldots\}$, $\{\ldots, -7, -3, 1, 5, 9, 13, \ldots\}$, $\{\ldots, -6, -2, 2, 6, 10, 14, \ldots\}$, and $\{\ldots, -5, -1, 3, 7, 11, 15, \ldots\}$.

6.2.6.5.

Proof. Assume $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$. This means $a = b + kn$ and

$c = d + jn$ for some integers k and j. Consider $a - c$. We have:

$$a - c = b + kn - (d + jn) = b - d + (k - j)n.$$

In other words, $a - c$ is $b - d$ more than some multiple of n, so $a - c \equiv b - d \pmod{n}$. ■

6.2.6.6.

(a) $3^{456} \equiv 1^{456} = 1 \pmod{2}$.

(b) $3^{456} = 9^{228} \equiv (-1)^{228} = 1 \pmod{5}$.

(c) $3^{456} = 9^{228} \equiv 2^{228} = 8^{76} \equiv 1^{76} = 1 \pmod{7}$.

(d) $3^{456} = 9^{228} \equiv 0^{228} = 0 \pmod{9}$.

6.2.6.8. For all of these, just plug in all integers between 0 and the modulus to see which, if any, work.

(a) No solutions.

(b) $x = 2$, $x = 5$, $x = 8$.

(c) No solutions.

6.2.6.10. $x = 5 + 22k$ for $k \in \mathbb{Z}$.

6.2.6.12. $x = 6 + 15k$ for $k \in \mathbb{Z}$.

6.2.6.14. We must solve $7x + 5 \equiv 2 \pmod{11}$. This gives $x \equiv 9 \pmod{11}$. In general, $x = 9 + 11k$, but when you divide any such x by 11, the remainder will be 9.

6.2.6.15. Divide through by 2: $3x + 5y = 16$. Convert to a congruence, modulo 3: $5y \equiv 16 \pmod{3}$, which reduces to $2y \equiv 1 \pmod{3}$. So $y \equiv 2 \pmod{3}$ or $y = 2 + 3k$. Plug this back into $3x + 5y = 16$, and solve for x, to get $x = 2 - 5k$. So the general solution is $x = 2 - 5k$ and $y = 2 + 3k$ for $k \in \mathbb{Z}$.

List of Symbols

Symbol	Description	Page
$\therefore$	"therefore"	15
$P, Q, R, S, \ldots$	propositional (sentential) variables	19
$\wedge$	logical "and" (conjunction)	19
$\vee$	logical "or" (disjunction)	19
$\neg$	logical negation	19
K_n	the complete graph on n vertices	108
$K_{m,n}$	the complete bipartite graph of m and n vertices	109
C_n	the cycle on n vertices	109
P_n	the path on $n+1$ vertices	109
$\chi(G)$	the chromatic number of G	151
$\Delta(G)$	the maximum degree in G	156
$\chi'(G)$	the chromatic index of G	157
$N(S)$	the set of neighbors of S	182
$\mathbf{B}^n_k$	the set of length n bit strings with weight k.	198
$(a_n)_{n\in\mathbb{N}}$	the sequence $a_0, a_1, a_2, \ldots$	313
T_n	the nth triangular number	318
F_n	the nth Fibonacci number	324
Δ^k	the kth differences of a sequence	343
$\emptyset$	the empty set	392
$\mathbb{N}$	the set of natural numbers	392
$\mathbb{Z}$	the set of integers	392
$\mathbb{Q}$	the set of rational numbers	392
$\mathbb{R}$	the set of real numbers	392
$\mathcal{P}(A)$	the power set of A	392
$\{,\}$	braces, to contain set elements.	392
$:$	"such that"	392
$\in$	"is an element of"	392
$\subseteq$	"is a subset of"	392
$\subset$	"is a proper subset of"	392
$\cap$	set intersection	392
$\cup$	set union	392
$\times$	Cartesian product	392
$\setminus$	set difference	392
$\overline{A}$	the complement of A	392

(Continued on next page)

Symbol	**Description**	**Page**
$\lvert A\rvert$	cardinality (size) of A	392
$A \times B$	the Cartesian product of A and B	397
$f(A)$	the image of A under f	411
$f^{-1}(B)$	the inverse image of B under f	412

Index

Colophon

This book was authored in PreTeXt.