Discrete Mathematics

An Open Introduction, 4th Edition

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4th Edition

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4th Edition

A current version can always be found for free at
http://discrete.openmathbooks.org/

Cover image: *Tiling with Fibonacci and Pascal*.

For Madeline and Teagan

Acknowledgements

This book would not exist if not for "Discrete and Combinatorial Mathematics," by Richard Grassl and Tabitha Mingus. It is the book I learned discrete math out of, and that I taught with the semester before I began writing this text. I wanted to maintain the inquiry-based feel of their book but update, expand, and rearrange some of the material. Some of the best exposition and exercises here were graciously donated from this source.

Thanks go to the graduate students who have co-taught the Discrete Mathematics course with me over the years, including Evan Czysz, Alees Lee, and Sarah Sparks, who helped develop new activities and exercises that have been incorporated into this text. Michelle Morgan provided copy-editing support, and Jennifer Zakotnik-Gutierrez helped code many of the interactive exercises in the online version of the book. Thanks also to Katie Morrison, Nate Eldredge, and Richard Grassl (again) for their suggestions after using parts of this text in their classes.

The online version of the book is written in PreTeXt and hosted on Runestone Academy thanks to the tremendous development work of Rob Beezer, Brad Miller, David Farmer, and Alex Jordan along with the rest of the participants of the pretext-support group (`groups.google.com/g/pretext-support`).

Finally, a thank you to the numerous students who have pointed out typos and made suggestions over the years, and a thanks in advance to those who will do so in the future.

Preface

This text aims to introduce select topics in discrete mathematics at a level appropriate for first- or second-year undergraduate math and computer science majors, especially those who intend to teach middle and high school mathematics. The book began as a set of notes for the Discrete Mathematics course at the University of Northern Colorado. This course serves both as a survey of the topics in discrete math and as the "bridge" course for math majors, as UNC does not offer a separate "introduction to proofs" course. As this course has evolved to support our computer science major, so has the text. The current version of the book is intended to support inquiry-based teaching for understanding that is so crucial for future teachers, while also providing the necessary mathematical foundation and application-based motivation for computer science students. While teaching the course in Spring 2024 using an early version of this edition, I was pleasantly surprised by how many students reported that they, for the first time, saw how useful math could be in the "real world." I hope that this experience can be replicated in other classes using this text.

This book is intended to be used in a class taught using problem-oriented or inquiry-based methods. Each section begins with a preview of the content that includes an open-ended *Investigate!* motivating question, as well as a structured preview activity. The preview activities are carefully scaffolded to provide an entry-point to the section's topic and to prime students to engage deeply in the material. Depending on the pace of the class, I have found success assigning only the section preview before class, using the preview activity as in-class group work, or assigning the entire section to be read before class (each section concludes with a small set of reading questions that can be assigned to encourage students to actually read). For those readers using this book for self-study, the organization of the sections will hopefully mimic the style of a rich inquiry-based classroom.

The topics covered in this text were chosen to match the needs of the students I teach at UNC. The main areas of study are logic and proof, graph theory, combinatorics, and sequences. Induction is covered at the end of the chapter on sequences. Discrete structures are introduced "as needed", but a more thorough treatment of sets and functions is included as a separate chapter, which can be studied independent of the other content. The final chapter covers two additional topics: generating functions and number theory.

While I believe this selection and order of topics is optimal, you should feel free to skip around to what interests you. There are occasionally examples and exercises that rely on earlier material, but I have tried to keep these to a minimum, and they usually can either be skipped or understood without too much additional study. If you are an instructor, you can also create a custom version by editing the PreTeXt source to fit your needs.

Improvements to the 4th Edition. Many of the sections have been rewritten to improve the clarity of the exposition.

- Nearly 300 new exercises, bringing the total to more than 750. These are better divided into preview activity questions, reading questions, practice problems, and additional exercises. Most of the new exercises are interactive for the online version.
- New sections on probability, relations, and discrete structures and their proofs. Some other sections have been split up to make it more likely that a single class period can be devoted to a single topic.
- Improved presentation for the counting chapter with a focus on considering sets of outcomes more than following rules.
- The *Investigate!* activities of the 3rd Edition have been split into two types: *Investigate!* questions and Preview Activities. The former are open-ended questions designed to engage you with the topic soon to be discussed. The latter are structured preview activities that you should be able to completely answer before reading the section.

The previous editions (3rd Edition, released in 2019, 2nd Edition, released in 2016, and the Fall 2015 Edition) will still be available for instructors who wish to use those versions due to familiarity.

I plan to continue improving the book. Some of this will happen in real-time by updating the online versions to include new content (numbering will remain consistent). Thus I encourage you to send along any suggestions and comments as you have them.

Oscar Levin, Ph.D.
University of Northern Colorado, 2024

How to Use This Book

In addition to expository text, this book has a few features designed to encourage you to interact with the mathematics.

***Investigate!* questions.** Sprinkled throughout the sections (usually at the very beginning of a topic) you will find open-ended questions designed to engage you with the topic soon to be discussed. You really should spend some time thinking about, or even working through, these problems before reading the section. However, don't worry if you cannot find a satisfying solution right away. The goal is to pique your interest, so you will read what is next looking for answers.

Preview Activities. Most sections include a structured preview activity. These contain leading questions that you should be able to completely answer before reading the section. The idea is that the questions prime you to engage meaningfully with the new content ahead. If you are using the online version, most of these questions will provide you with immediate feedback so you can be confident moving forward.

Examples. I have tried to include the "correct" number of examples. For those examples that include *problems*, full solutions are included. Before reading the solution, try to at least have an understanding of what the problem is asking. Unlike some textbooks, the examples are not meant to be all-inclusive for problems you will see in the exercises. They should not be used as a blueprint for solving other problems. Instead, use the examples to deepen your understanding of the concepts and techniques discussed in each section. Then use this understanding to solve the exercises at the end of each section.

Exercises. You get good at math through practice. Each section concludes with practice problems meant to solidify concepts and basic skills presented in that section; the online version provides immediate feedback on these problems. There are then additional exercises that are more challenging and open-ended. These might be assigned as written homework or used in class as group work. Some of the additional exercises have hints or solutions in the back of the book, but use these as little as possible. Struggle is good for you. At the end of each chapter, a larger collection of similar exercises is included (as a sort of "chapter review") which might bridge the material of different sections in that chapter.

Interactive Online Version. For those of you reading this in print or as a PDF, I encourage you to also check out the interactive online version. Many of the preview activities and exercises are interactive and can give you immediate feedback. Some of these have randomized components, allowing you to practice many similar versions of the same problems until you master the topic.

Hints and solutions to examples are also hidden away behind an extra click to encourage you to think about the problem before reading the solution. There is a good search feature available as well, and the index has expandable links to see the content without jumping to the page immediately. There is also a python scratch pad (the pencil icon) so you can try out some code if you feel so inclined.

Additional interactivity is planned. These "bonus" features will be added on a rolling basis, so keep an eye out!

You can view the interactive version for free at `discrete.openmathbooks.org` or by scanning the QR code below.

Contents

Chapter 0

Introduction and Preliminaries

Welcome to Discrete Mathematics. If this is your first time encountering the subject, you will probably find discrete mathematics quite different from other math subjects. You might not even know what discrete math is! Hopefully this short introduction will shed some light on what the subject is about and what you can expect as you move forward in your studies.

0.1 What is Discrete Mathematics?

dis·crete / dis′krët.

Adjective: Individually separate and distinct.

Synonyms: separate - detached - distinct - abstract.

Defining *discrete mathematics* is hard because defining *mathematics* is hard. What is mathematics? The study of numbers? In part yes, but you also study functions and lines and triangles and parallelepipeds and vectors and .... Or perhaps you want to say that mathematics is a collection of tools that allow you to solve problems. What sort of problems? Well, those that involve numbers, functions, lines, triangles, ... Whatever your conception of what mathematics is, try applying the concept of "discrete" to it, as defined above. Some math fundamentally deals with *stuff* that is individually separate and distinct.

In an algebra or calculus class, you might have found a particular set of numbers (perhaps they constitute the range of a function). You would represent this set as an interval: $[0, \infty)$ is the range of $f(x) = x^2$ since the set of outputs of the function are all real numbers 0 and greater. This set of numbers is NOT discrete. The numbers in the set are not separated by much at all. In fact, take any two numbers in the set and there are infinitely many more between them that are also in the set.

Discrete math could still ask about the range of a function, but the set would not be an interval. Consider the function that gives the number of children of each person reading this. What is the range? I'm guessing it is something like $\{0, 1, 2, 3, 4\}$. Maybe 5 or 6 is in there too.[1] But certainly nobody reading this has 1.32419 children. This output set *is* discrete because the elements are separate. The inputs to the function also form a discrete set because each input is an individual person.

There are many discrete mathematical objects besides sets of numbers; we will introduce some of these in Section 0.2. Studying these discrete **structures** is the main

[1] Even larger natural numbers for old ladies who live in shoes.

focus of discrete mathematics and this book. However, the reason we want to study these structures is because they provide a way to model "real-world" problems.[2]

To get a feel for the subject, let's consider the types of problems you solve in discrete math. Here are a few simple examples:

Investigate!

Note: Throughout the book you will see Investigate! *activities like this one. Answer the questions in these as best you can to give yourself a feel for what is coming next.*

1. The most popular mathematician in the world is throwing a party for all of his friends. To kick things off, they decide that everyone should shake hands. Assuming all 10 people at the party each shake hands with every other person (but not themselves, obviously) exactly once, how many handshakes take place?

2. At the warm-up event for Oscar's All-Star Hot Dog Eating Contest, Al ate one hot dog. Bob then showed him up by eating three hot dogs. Not to be outdone, Carl ate five. This continued with each contestant eating two more hot dogs than the previous contestant. How many hot dogs did Zeno (the 26th and final contestant) eat? How many hot dogs were eaten in total?

3. After excavating for weeks, you finally arrive at the burial chamber. The room is empty except for two large chests. On each is carved a message (strangely in English):

   Exactly one of these chests contains a treasure, while the other is filled with deadly immortal scorpions.

   For either chest, if the chest's message is true, then the chest contains treasure.

   The problem is, you don't know whether the messages are true or false. What do you do?

4. Back in the days of yore, five small towns decided they wanted to build roads directly connecting each pair of towns. While the towns had plenty of money to build roads as long and as winding as they wished, it was very important that the roads not intersect with each other (as stop signs had not yet been invented). Also, tunnels and bridges were not allowed, for moral reasons. Is it possible for each of these towns to build a road to each of the four other towns without creating any intersections?

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[2]Many of the problems discussed in this book are admittedly contrived and clearly fictional, but hopefully you will see how these toy problems can be generalized to actually represent problems that people would care about in reality.

As you consider the problems above, don't worry if it is not obvious to you what the solutions are. We are more interested here in what sort of information we need to be able to answer the questions. How can we represent the situation using individually separate and distinct objects? Don't read on until you have thought about at least this for each of the questions.

Ready? Here are some things you might have thought about:

1. The people at the party are individuals. We can consider the *set* of people. We can also consider sets of pairs of people, since it takes exactly two people to shake hands. So the question is really, how many pairs can you make using elements from a 10-element set?

   For example, if there were three people at the party, conveniently named 1, 2, and 3, then the pairs would be $(1,2)$, $(1,3)$, and $(2,3)$. Or should we include $(2,1)$, $(3,1)$, and $(3,2)$ as well?

2. To count the number of hot dogs eaten, either by an individual or in total, we could use a **sequence** of integers (whole numbers). The nth term in the sequence might represent the number of hot dogs eaten by the nth contestant. We can consider a second sequence, also of integers, that gives the total number of hot dogs eaten by the first n contestants combined.

   The solution to the problem will then be the value of the 26th term in the sequence. To help us find this, we could consider the rate of growth of the sequences, as well as how these two sequences relate to each other.

3. Logic questions also belong under the discrete math umbrella: Each statement can have a *value* of True or False (and there is nothing in-between). To answer questions like that of the chests of scorpions, we must understand the structure of the statements, and how the truth values of the parts of the statements interact to determine the truth value of the whole statement.

4. The last question is about a discrete structure called a **graph**, not to be confused with a graph of a function or set of points. We can use a graph to represent which elements of a set (or towns) are related to each other (or connected by a road). In this case, the question becomes, can we draw a graph with five vertices (towns) and ten edges (roads) such that no two edges intersect?

The four problems above illustrate the four main topics of this book: **combinatorics** (the theory of ways things *combine*; in particular, how to count these ways), **sequences**, **symbolic logic**, and **graph theory**. However, there are other topics that are also considered part of discrete mathematics, including computer science, abstract algebra, number theory, game theory, probability, and geometry (some of these, particularly the last two, have both discrete and non-discrete variants).

Ultimately the best way to learn what discrete math is about is to *do* it. Let's get started! Before we can begin answering more complicated (and fun) problems, we will consider a very brief overview of the types of discrete structures we will be using.

Reading Questions

Each section of the book will end with a small number of *Reading Questions* like the ones below. These are designed to help you reflect on what you have read. In particular, the final reading question asks you to ask a question of your own. Thinking about what you don't yet know is a wonderful way to further your understanding of what you do.

1. Right now, how would you describe what **discrete** mathematics is about, if you were telling your friends about the class you are in? Write one or two sentences.

2. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

0.2 Discrete Structures

0.2.1 Introduction

Investigate!

A double-six domino set consists of tiles containing pairs of numbers, each from 0 to 6. How many tiles are in a double-six domino set? How many dominoes are in a double-nine domino set? How many dominoes are in a double-n domino set?

Try it 0.2.1

Spend a few minutes thinking about the questions above. Then write 2-3 sentences describing your thoughts. You do not need to find a complete solution, but you should describe what you could try and what you think you might need to do to find a solution.

We are taking a problem-solving approach to discrete mathematics: We will consider a large variety of questions that have a discrete feel to them, and consider how to answer those questions (and prove that our answers are correct). This is not the only way to study discrete math. Another approach would be to study the tools used to solve the problems. If we were art students, we could study paintbrushes and easels and the composition of paint, which would be interesting for sure, but I think it is more enjoyable to actually paint those happy little trees.

That said, understanding your tools does help you use them, so in this section, we will consider some basic tools used in discrete mathematics. We will come back to these throughout our studies and understand them better as we need to.

The tools in our subject are called **discrete structures**. They are the mathematical objects that we use to represent parts of the problems we are solving. "Structure" is a good word here, since these "things" have fairly rigid constraints that make them what they are, just like an apartment building is going to have different characteristics than an airplane hangar or a suspension bridge (these are types of physical structures, not mathematical structures, just to be overly clear and destroy the metaphor).

The structures we will use most in discrete math are **sets**, **functions**, **sequences**, **relations**, and **graphs**. We now briefly preview each of these. As we progress through our studies, each will be explored in more detail.

0.2.2 Sets

A **set** is an *unordered* collection of elements. This is fairly vague, but unless we want to spend a whole book trying to understand sets more precisely, it will be good enough for us. It is possible to define all of mathematics using just sets (even numbers can be thought of as sets themselves), but this is also not what we will do. Rather, we want to be able to talk about collections of numbers and other objects, and we will collect them in something we call a **set**.

We can describe sets by saying exactly what elements are members of the set. We could specify this membership in words (e.g., Let A be the set of all natural numbers less than 10), or by explicitly listing all the elements (e.g., $A = \{3, 5, 7\}$), or using something called **set comprehension** (also called **set builder notation**). An example of this is $A = \{x \in \mathbb{N} : x < 10\}$. We would read that as "A is the set of natural numbers that satisfy the property that they are less than 10." More precisely, the $\mathbb{N}$ symbol represents the natural numbers, which is itself a set: $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ (this shows another way to describe a set). The $\in$ symbol means "is an element of". The : is read "such that" and tells us that what comes next is the *condition* that must be true of the set's elements.

> **By the way...** In this book, we define the **natural numbers** to be the whole numbers starting with 0. Not every book includes 0 in this set. It largely depends on what area of mathematics you study.

Since there are multiple ways to describe the same set, we should be careful about what it means for sets to be the same or different. A set is *determined* by its membership, so all four of the following describe exactly the same set:

1. $\{1, 2, 3, 4\}$.
2. $\{1, 2, 1+1, 1+2, 2+2\}$. (How many numbers belong to this set? It's not 5.)
3. $\{2, 4, 1, 3\}$. (All that matters is what elements are in the collection, not what order they were written down in.)
4. $\{x \in \mathbb{N} : x < 5 \text{ and } x \geq 1\}$.

There are lots of things you can *do* with sets, which we will consider in more detail as we need to. We will see that it is often helpful to build new sets from ones we already have (by taking the **union** or **intersection** of sets, for example), to compare sets (asking if one set is a **subset** of another), and to find the number of elements of a set (called its **cardinality**). We might also want to match up elements of one set with another: To do this, we might use a **function**, which we will discuss next. Awesomely, we can also use sets themselves to describe functions. Let's check it out.

0.2.3 Functions

One way to define a **function** is as a rule that assigns each input exactly one output. The output is called the **image** of the input. Functions also come equipped with a **domain**, the set of all inputs, and a **codomain**, the set of all allowable outputs. You might also speak of the **range** of the function, which is the set of all *actual* outputs, or put another way, the set of all *images* of elements from the domain.

We write $f : X \to Y$ to describe a function with name f, domain X and codomain Y. This does not tell us *which* function f is though. To define the function, we must describe the rule. Often this is done with a formula (for example, $f(x) = x^2$ says that each element of the domain is mapped to its square), or in words (like how we just described the squaring function). We could also define a function with a table or a graph.

The key thing that makes a rule a *function* is that there is *exactly one* output for each input. That is, the rule must be a good rule. What output do we assign to the input 7? There can only be one answer for any particular function.

Since a function maps one set (the domain) to another set (the codomain), there is an obvious connection between sets and functions. There is another connection worth considering though: The graph of a function is often described as a ***set*** of points. Here is an example.

Example 0.2.2

Consider the function $f : \{1, 2, 3\} \to \{2, 4, 6\}$ defined by $f(x) = 2x$. If we wanted to plot a graph of this function, we would draw the points $(1, 2)$, $(2, 4)$, and $(3, 6)$ (but we would not connect these points with lines, since we are studying discrete math; the domain only contains three elements, not the infinitely many between them).

So associated with the function is the set of points $\{(1, 2), (2, 4), (3, 6)\}$. In fact, this set of points is associated exactly with this (and only this) function. So we can think of this set of points *as the function itself*.

Of course, we are using a mathematical object here: an **ordered pair**. This is not a set (since sets are unordered). How should we talk about ordered things? We will take this question up in the next section.

There is one more important consideration about how we define a function with a rule. A **closed formula** is one in which each output is given by an explicit rule based solely on its input. This is what most of us think of as a formula. For example, $f(n) = 3n + 1$ is a closed formula, since to find $f(5)$ (say) we take the input 5 and do something to it: Multiply it by 3 and then add 1.

What else could a formula possibly be? A **recursive definition** of a function tells us how to compute the output for a given input *based on other outputs of the function*. For example, we might insist that $f(n) = 2 \cdot f(n-1)$. If we also specify an **initial condition** that $f(0) = 3$, then we can find $f(1) = 2 \cdot 3 = 6$, and then $f(2) = 2 \cdot 6 = 12$, and so on. What is $f(5)$ here? The only way to answer that is to find $f(4)$, which means we need to find $f(3)$, which we could do, since we have computed $f(2)$.

Recursive definitions of functions might be less useful for finding a particular output, but they are often easier to specify for a particular application. We will explore this phenomenon more when we study sequences. Speaking of which...

0.2.4 Sequences

Sometimes we are interested not just in a collection of numbers, but in what *order* those numbers appear. In such cases, we cannot use *sets*, since they do not distinguish between the order of their elements. Instead, we consider **sequences**.

We will consider both **finite** and **infinite** sequences. A finite sequence may be something as simple as $(1, 2, 3)$; that is a sequence with 3 elements, in that particular order. We might also call this an **ordered triple**, the same way that $(7, 3)$ is an ordered pair. In general, we could call this an n**-tuple** if it has n elements (we assume that tuples are ordered).

The key difference between the sequence $(1, 2, 3)$ and the set $\{1, 2, 3\}$ is that we "care" about the order. That is, the sequence $(1, 2, 3)$ is different from the sequence $(2, 3, 1)$, while the set $\{1, 2, 3\}$ is identical to $\{2, 3, 1\}$.

We will often use sequences as a counting tool. For example, a very simple counting question is, "How many wheels do 100 cars have?" Instead of answering just this one question, we could generalize and ask how many wheels n cars have, and get a sequence of answers. This yields the infinite sequence $(4, 8, 12, \ldots)$. The order these multiples of 4 appear in is important, since each number in the sequence corresponds to a specific version of the question.

It is fine to think of a sequence of numbers as an ordered list. We can refer to the **terms** simply as

$$a_0, a_1, a_2, \ldots$$

and might refer to the entire sequence as $(a_n)_{n\geq 0}$ or $(a_n)_{n\in\mathbb{N}}$.

If we want to be a little more precise and more abstract, we can think of a sequence as a *function*. The domain is the natural numbers or some subset of consecutive natural numbers (like $\{1, 2, 3, 4\}$). The codomain is some set. We think of the domain as the positions in the sequence, and the image of those inputs as the terms in the sequence.

For example, we might consider the Fibonacci sequence $(f_n)_{n\geq 1}$, which starts $1, 1, 2, 3, 5, 8, \ldots$. What is the 4th term in the sequence? We might say $f_4 = 3$ (this is assuming the first 1 is the first term and not the 0th term). Note that there can only be one 4th term. There can only be one n-th term for any particular value of n. So for any input (the position of the term), there is only one output (the term). It would be perfectly reasonable to write $f(4) = 3$, and that really does look like a function. But we like to use subscripts.

We can also describe the terms in a sequence using a table. We might write something like the following:

n	1	2	3	4	...
a_n	1	3	6	10	...

This looks exactly like how you would represent a function, even though this table describes the sequence of **triangular numbers** (we will see why they are called this later).

Since sequences are functions, we can use any of the techniques to describe functions to describe sequences. In particular, we might give a **closed formula** for a sequence by explicitly giving the function for the n-th term. For example,

$$a_n = \frac{n(n+1)}{2}.$$

Alternatively, we could define a sequence recursively by saying how to get from one term to the next. This is especially useful for the Fibonacci sequence:

$$f_n = f_{n-1} + f_{n-2}; f_1 = 1, f_2 = 1.$$

Much of our effort in understanding sequences will go into taking a recursive definition and finding a closed formula for the sequence. We will study this, and everything else sequence related in `[cross-reference to target(s) "ch_sequences" missing or not unique]`.

0.2.5 Relations

How are the numbers 2 and 6 related? Oh, I know: $2 < 6$. Also, 6 is a multiple of 2. The two numbers are also both even. And here is another fact: They are *not equal*. All four of these are examples of **relations**: less than, multiple of, both even, not equal. And there are many more (infinitely many) relations for pairs of numbers.

The examples above are all **binary relations** in that they relate two elements. You could also consider relations between more than two elements. For example, we could consider the relation "Pythagorean triple" on three numbers that holds precisely if they are the side lengths of a right triangle. So the relation is true of the triple $(3, 4, 5)$, but not of $(4, 5, 6)$ (since $3^2 + 4^2 = 5^2$, but $4^2 + 5^2 \neq 6^2$).

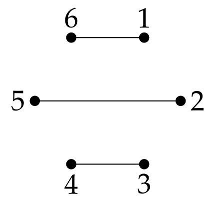
Notice that we can talk about a pair or triple *satisfying* a relation. We might say that a pair *belongs* to the relation. The careful and formal way to define a relation is as a *set* of ordered pairs (or triples, etc.). Consider the (infinite) set of all ordered pairs (a, b) such that $a < b$. Every element of this set contains numbers for which the relation "less than" is true, and every pair of numbers for which the relation is true is a pair in the set. So we can say that this set of pairs ***is*** the relation.

Relations can have some standard properties, and deciding whether a particular relation has a given property can often help us understand the relation better. The less-than relation is, for example, **irreflexive** because there are no elements that are less than themselves. It is also **antisymmetric** since there are no distinct numbers a and b such that $a < b$ and $b < a$. It is also **transitive** since if $a < b$ and $b < c$ then it must follow that $a < c$. These are just a few examples of relation properties though.

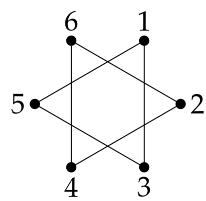
Why would we care about these properties? It turns out that some groups of properties happen together frequently, and for such collections of properties, we can prove general results about the relations that satisfy them. So if we can prove that a given relation is **reflexive**, **symmetric**, and **transitive** (whatever those mean), then we know the relation is an **equivalence relation**, and therefore we know it has a bunch of other properties. A large portion of discrete mathematics is about studying particular types of relations. One of my favorites is a relation that gives us a graph.

0.2.6 Graphs

Consider the set $V = \{1, 2, 3, 4, 5, 6\}$. Which pairs of numbers from that set add up to 7? We could have $\{1, 6\}$ or $\{2, 5\}$ or $\{3, 4\}$. We can picture the set and the interesting (unordered) pairs (i.e., two-element subsets) as a picture called a graph:



On the other hand, we might want to consider pairs of numbers whose sum is even. Then, we would get the following picture.



We call these discrete structures **graphs**. A graph is a type of relation, one that is **symmetric** (if a is related to b, then b is related to a) and **irreflexive** (no element is related to itself).

However, we mostly think of graphs as the drawings of dots and lines, or more precisely as a set of **vertices** together with a set of **edges**, where each edge is a two-element subset of the vertices. Notice that even here, we are using the structure *set* to define the structure *graph*.

Graphs show up in all sorts of real-world applications: In a class, some students are friends with each other, so take the students to be the vertices and the edges to be the friendships. In geography, some countries share a border, so take the countries to be the vertices and connect a pair of vertices with an edge if the countries share a border. Perhaps you are planning a trip and want to fly from Denver to Paris. Is there a direct flight, or must you stop in Newark? That is, does the graph of flights have an edge between Denver and Paris or only between Denver and Newark and between Newark and Paris? When your Amazon driver delivers packages to 10 houses in your neighborhood, how does her app know in which order to deliver the packages? Graph theory!

The study of graphs is a subject in its own right, in which many mathematicians hold doctorate degrees and write hundreds of papers each year. We will scratch the surface of this fascinating subject in `[cross-reference to target(s) "ch_graphtheory" missing or not unique]`

0.2.7 Even More Structures

Our list of structures could go on and on, but we will stop here. We will spend just a little time looking at **multisets**, which are just like sets except that they can have repeated elements. Since this is not a geometry class, we will not consider **finite geometries**, or **designs** (which are somewhere between graphs and geometries). Discrete structures are useful in computer science, but we will stop short of studying **linked lists** or **red-black trees**. Although abstract algebra is a fascinating subject, we will not get to **groups** or **rings** or **metroids** or **POSets** or **Boolean algebras** or .... These are examples of sets on which we define additional operations and study the algebraic structure of how the sets and operations interact.

The point is, discrete mathematics is awesome, and you can spend multiple lifetimes studying it. So what are we waiting for? Let's dive in and solve some problems.

0.2.8 Reading Questions

1. Think back to the domino problem at the beginning of this section. We asked how many dominoes are in a double-six domino *set*. Is this really a set, in our mathematical sense? What discrete structure would you use to represent each domino individually?

2. A double-zero domino set would contain only one domino (both sides showing 0). A double-one set would contain this plus the dominoes $(1,0)$ and $(1,1)$. We can continue in this way, creating a sequence of domino sets. Find the next three terms of the sequence.

1, 3, _ , _ , _ , ...

3. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

Chapter 1

Expressions and Equations

Text before the first section.

1.1 Section Title

Text of section.

1.2 Section Title

Text of section.

1.3 Section Title

Text of section.

1.4 Section Title

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1.5 Section Title

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1.8 Section Title

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1.9 Section Title

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Chapter 2

Logic and Proofs

Logic is the study of consequence. Given a few mathematical statements or facts, we would like to be able to draw some conclusions. For example, if I told you that a particular real-valued function was continuous on the interval $[0, 1]$, and $f(0) = -1$ and $f(1) = 5$, can we conclude that there is some point between $[0, 1]$ where the graph of the function crosses the x-axis? Yes, we can, thanks to the Intermediate Value Theorem from calculus. Can we conclude that there is exactly one point? No. Whenever we find an "answer" in math, we really have a (perhaps hidden) argument.

Mathematics is really about establishing general statements (like the Intermediate Value Theorem). This is done via an argument called a proof. We start with some given conditions, the *premises* of our argument, and from these, we find a consequence of interest, our *conclusion*.

The problem is, as you no doubt know from arguing with friends, not all arguments are *good* arguments. A bad argument is one in which the conclusion does not follow from the premises; i.e., the conclusion is not a consequence of the premises. Logic is the study of what makes an argument good or bad. In other words, logic aims to determine in which cases a conclusion is, or is not, a consequence of a set of premises.

We will start in Section 2.1 by considering statements, the building blocks of arguments. Understanding what counts as a statement and what form statements can take is the first step in understanding arguments. We will take a closer look at how statements can be combined in Section 2.2. Then we will see what mathematical tools we can develop to better analyze these statements and how they interact in Section 2.3. Finally, we will put all of this together in Section 2.4 and Section 2.5 to see how we can use these tools to construct arguments and prove statements.

2.1 Mathematical Statements

Objectives

After completing this section, you should be able to do the following.

1. Identify the logical structure of statements to determine their truth value in terms of the truth values of their parts.
2. Identify the use of quantifiers in a statement, and determine the truth value of the statement based on those quantifiers.
3. Translate between statements in natural language and logical symbols.

2.1.1 Section Preview

Investigate!

While walking through a fictional forest, you encounter three trolls guarding a bridge. Each is either a *knight*, who always tells the truth, or a *knave*, who always lies. The trolls will not let you pass until you correctly identify each as either a knight or a knave. Each troll makes a single statement:

Troll 1: If I am a knave, then there are exactly two knights here.

Troll 2: Troll 1 is lying.

Troll 3: Either we are all knaves, or at least one of us is a knight.

Which troll is which?

Try it 2.1.1

Spend a few minutes thinking about the Investigate problem above. What could you conclude if you knew Troll 1 really was a knave (i.e., their statement was false)? Share your initial thoughts on this.

In order to *do* mathematics, we must be able to *talk* and *write* about mathematics. Perhaps your experience with mathematics so far has mostly involved finding numerical answers to problems. As we embark towards more advanced and abstract mathematics, writing will play a more prominent role in the mathematical process.

In fact, the primary goal of mathematics, as an academic discipline in its own right, is to establish general mathematical truths. How can we know whether these facts, perhaps called *theorems* or *propositions*, are true? We construct valid arguments, called *proofs*, which establish the truth of the statements. Here, an argument is not the sort of thing you have with your Mom when you disagree about what to have for dinner. Rather, we have a technical definition of the term.

Definition 2.1.2 Argument.

An **argument** is a sequence of statements, the last of which is called the **conclusion** and the rest of which are called **premises**.

An argument is said to be **valid** provided the conclusion must be true whenever the premises are all true. An argument is **invalid** if it is not valid; that is, all the premises can be true, and the conclusion could still be false.

An argument is **sound** provided it is valid and all the premises are true. A **proof** of a statement is a sound argument whose conclusion is the statement.

By the way... Our definitions of **argument**, **valid argument**, and **sound argument** are the same ones used in philosophy, the other primary academic discipline concerned with logic and reasoning.

To determine whether we have a proof of a statement, we must decide both whether every premise is true, and whether the argument is valid: whether the conclusion *follows from* the premises. How can we do this?

Example 2.1.3

Consider the following two arguments:

If Edith eats her vegetables, then she can have a cookie.
Edith eats her vegetables.

∴ Edith gets a cookie.

Florence must eat her vegetables to get a cookie.
Florence eats her vegetables.

∴ Florence gets a cookie.

(The symbol "∴" means "therefore")

Are these arguments valid?

Solution. Do you agree that the first argument is valid but the second argument is not? We will soon develop a better understanding of the logic involved in this analysis, but if your intuition agrees with this assessment, then you are in good shape.

Notice the two arguments look almost identical. Edith and Florence both eat their vegetables. In both cases, there is a connection between the eating of vegetables and cookies. Yet we claim that it is valid to conclude that Edith gets a cookie, but not that Florence does. The difference must be in the connection between eating vegetables and getting cookies. We need to be skilled at reading and comprehending these sentences. Do the two sentences mean the same thing?

Unfortunately, in everyday language we are often sloppy, and you might be tempted to say they are equivalent. But notice that just because Florence *must* eat her vegetables, we have not claimed that doing so would be *enough* (she might also need to clean her room, for example). In everyday (non-mathematical) practice, you might be tempted to say this "other direction" is implied. In mathematics, we never get that luxury.

Remark 2.1.4 The arguments in the example above illustrate another important point: Even if you don't care about the advancement of human knowledge in the field of mathematics, becoming skilled at analyzing arguments is useful. And even if you don't want to give your grandmother a cookie. If you are *using* mathematics to solve problems in some other discipline, it is still necessary to demonstrate that your solution is correct. You better have a good argument that it is!

Since arguments are built up of statements, we must agree on what counts as a statement.

Definition 2.1.5

A **statement** is a declarative sentence that is either true or false.

If the sentences in an argument could not be true or false, there would be no way to determine whether the argument was valid, since validity describes a relationship between the truth values of the premises and conclusions.

The goal of this section is to explore the different "shapes" a statement can take. We will see that more complicated statements can be built up from simpler ones, in ways that entirely determine their truth value based on the truth values of their parts.

Preview Activity

Before reading on to the main content of the section, complete this preview activity to start thinking about the types of questions this section will address.

1. Which of the following sentences should count as statements? That is, for which of the sentences below could you *potentially* claim the sentence was either true or false? Select all that apply.

A. The sum of the first 100 positive integers.

B. What is the sum of the first 100 positive integers?

C. The sum of the first 100 positive integers is 5050.

D. Is the sum of the first 100 positive integers 5050?

E. The sum of the first 100 positive integers is 17.

2. You and your roommate are arguing, and they make the audacious claim that pineapple is good both on pizza and in smoothies. Which of the following are reasonable responses to this claim, from a logical point of view?

A. The statement is false because even though pineapple is good in smoothies, it is NOT good on pizza.

B. The statement is false because while pineapple is good on pizza and pineapple is good in smoothies, a pizza smoothie is never good.

C. The statement is half true because regardless of what you think about pineapple on pizza, we can all agree at least that pineapple is good in smoothies.

D. The statement is false because everyone who likes pineapple on pizza does NOT like pineapple in smoothies.

3. Your roommate now makes an even more outrageous claim: If a superhero movie is part of the Marvel Cinematic Universe, then it is good. Which of the following are reasonable responses to this claim, from a logical point of view?

A. This is false because there are good superhero movies, like Wonder Woman and Dark Knight, that are based on DC Comics, and so not part of the Marvel Cinematic Universe.

B. The statement is false because there is at least one superhero movie that is part of the Marvel Cinematic Universe that is also not good.

C. The statement is false because, for example, Green Lantern is neither Marvel nor good.

D. The statement is true because more than half of the Marvel movies are good.

4. Your roommate just won't let up with their outrageous claims. Now they claim that either every troll is a knave, or there is at least one troll that is a knight. What can you say to this?

A. Yes, this is true because every troll is either a knight or a knave. If it is not the case that *all* trolls are knaves, then there must be *some* troll that is a knight.

B. This is false because some trolls are knights and some other trolls are knaves.

C. The statement is false because there is no way to verify which of the two options is the case.

D. The statement is false because no troll could say that all trolls are knaves, since knaves always lie.

2.1.2 Atomic and Molecular Statements

A **statement** is any declarative sentence which is either true or false. A statement is **atomic** if it cannot be divided into smaller statements, otherwise it is called **molecular**.

Example 2.1.6

These are statements (in fact, *atomic* statements):

- Telephone numbers in the USA have 10 digits.
- The moon is made of cheese.
- 42 is a perfect square.
- Every even number greater than 2 can be expressed as the sum of two primes.
- $3 + 7 = 12$

And these are not statements:

- Would you like some cake?
- The sum of two squares.
- $1 + 3 + 5 + 7 + \cdots + 2n + 1$.
- Go to your room!
- $3 + x = 12$

The reason the sentence "$3 + x = 12$" is not a statement is that it contains a variable. Depending on what x is, the sentence is either true or false, but right now it is neither. One way to make the *sentence* into a *statement* is to specify the value of the variable in some way. This could be done by setting a specific substitution, for example, "$3 + x = 12$ where $x = 9$," which is a true statement. Or you could *capture* the free variable by *quantifying* over it, as in, "For all values of x, $3 + x = 12$," which is false. We will discuss quantifiers in more detail in the subsection Quantifiers and Predicates below.

You can build more complicated (molecular) statements out of simpler (atomic or molecular) ones using **logical connectives**. For example, this is a molecular statement:

Telephone numbers in the USA have 10 digits, and 42 is a perfect square.

Note that we can break this down into two smaller statements. The two shorter statements are *connected* by an "and." We will consider 5 connectives: "and" (Sam is a man, and Chris is a woman), "or" (Sam is a man, or Chris is a woman), "if. . . , then. . . " (if Sam is a man, then Chris is a woman), "if and only if" (Sam is a man if and only if Chris is a woman), and "not" (Sam is not a man). The first four are called **binary connectives** (because they connect two statements) while "not" is an example of a **unary connective** (since it applies to a single statement).

These molecular statements are, of course, still statements, so they must be either true or false. The crucial observation here is that which **truth value** the molecular statement achieves is completely determined by the type of connective and the truth values of the parts. We do not need to know what the parts actually say or whether they have some material connection to each other, only whether those parts are true or false.

To analyze logical connectives, it is enough to consider **propositional variables** (sometimes called *sentential* variables), usually capital letters in the middle of the alphabet: $P, Q, R, S, \ldots$. We think of these as standing in for (usually atomic) statements, but there are only two *values* the variables can achieve: true or false.[1] We also have symbols for the logical connectives: $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\neg$.

Definition 2.1.7 Logical Connectives.

We define the following **logical connectives**.

- $P \wedge Q$ is read "P and Q," and is called a **conjunction**.
- $P \vee Q$ is read "P or Q," and is called a **disjunction**.
- $P \rightarrow Q$ is read "if P then Q," and is called an **implication** or **conditional**.
- $P \leftrightarrow Q$ is read "P if and only if Q," and is called a **biconditional**.
- $\neg P$ is read "not P," and is called a **negation**.

The **truth value** of a statement is determined by the truth value(s) of its part(s), depending on the connectives:

Definition 2.1.8 Truth Conditions for Connectives.

The **truth conditions** for the logical connectives are defined as follows.

- $P \wedge Q$ is true when both P and Q are true.
- $P \vee Q$ is true when P or Q or both are true.
- $P \rightarrow Q$ is true when P is false or Q is true (or both).
- $P \leftrightarrow Q$ is true when P and Q are both true, or both false.
- $\neg P$ is true when P is false.

Each of the above definitions can be represented in a table, called a **truth table**. We simply list what the truth value of the statement is for each possible combination of truth values of the parts.

[1] In computer programming, we should call such variables **Boolean variables**.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

P	$\neg P$
T	F
F	T

Figure 2.1.9 Truth tables for logical connectives.

For example, we can use the truth table for $P \rightarrow Q$ to decide whether the statement, "If 5 is even, then 6 is even," is true or false. Here P is the statement "5 is even," and Q is the statement "6 is even." Since 5 is not even, the statement P is false. Since 6 is even, the statement Q is true. The truth table tells us that the statement $P \rightarrow Q$ is true when P is false and Q is true (the 3rd row). So the statement, "If 5 is even, then 6 is even," is true. (If you don't like that the statement is true, hold on to that thought, and we will hopefully resolve it soon.)

Note that for us, *or* is the **inclusive or** (and not the sometimes used *exclusive or*) meaning that $P \vee Q$ is true when P or Q *or both* P *and* Q are true. As for the other connectives, "and" behaves as you would expect, as does negation. The biconditional (if and only if) might seem a little strange, but you should think of this as saying the two parts of the statements are *equivalent* in that they have the same truth value.

This leaves only the implication $P \rightarrow Q$ which has a slightly different meaning in mathematics than in ordinary usage. However, implications are so common and useful in mathematics that we must develop a level of fluency with their use which warrants a whole section (Section 2.2).

Example 2.1.10

Using the truth conditions for the logical connectives, determine which statements below are true and which are false.

1. 17 is prime, and 17 is odd.
2. 17 is prime, and 18 is prime.
3. 17 is prime, or 18 is prime.
4. 17 is prime, or 19 is prime.
5. If 17 is prime, then 19 is prime.
6. If 18 is prime, then my favorite number is 17.
7. 17 is prime if and only if 19 is prime.
8. 17 is not prime if and only if 19 is not prime.

Solution. First, let's agree on some facts: 17 really is prime and odd, 18 is not either, and 19 is prime.

1. True. Both parts of the conjunction are true, so the entire statement is true.

2. False. The first part is true, but the second part is false, so the entire statement is false.

3. True. The first part is true, so the entire statement is true. As soon as we see one true statement in a disjunction, we can stop checking and declare the entire statement true.

4. True. Since we use the inclusive or, the statement is true when both parts are true.

5. True. Don't be worried that there isn't a good reason that 17 being prime *causes* 19 to be prime. That is not what we mean by a conditional statement. Since the "then" part is true, we know that the statement overall is true.

6. True. The "if" part of the statement is false. That's all we need. I bet you don't even know what my favorite number is, and you don't need to. The statement is true.

7. True. Do both parts have the same truth value? Yes, since they are both true. So the entire statement is true.

8. True as well. Now both parts are false (since both are the negation of a true statement), so the entire statement is true.

The way we define logical connectives and their truth value is very precise and technical. Often, language is not. Part of learning how to communicate mathematics is learning the cultural norms of mathematical language and how to translate statements in ordinary language into these technical statements. This will get easier with practice, so make sure you are talking to lots of people about the math you are studying.

Here are a few examples of how ordinary language might be difficult to translate.

Example 2.1.11

Identify the logical structure of each of the following statements.

1. 4 and 5 are both prime.

2. Only one of 4 or 5 is prime.

3. You must attend every day and do the homework to pass this class.

4. Every number is even or odd.

Solution.

1. Do you agree this is the same statement as "4 is prime, *and* 5 is prime"? Notice that it would not make sense to write this as $P \wedge Q$ where P is "4" and Q is "5 is prime". But if we let P be the statement, "4 is prime," then both parts of the conjunction are statements.

2. Again, we can't just put what is on one side of the "or" as a statement. But if we let P be "4 is prime" and Q be "5 is prime,", then we can write this as $(P \vee Q) \wedge \neg(P \wedge Q)$. That is, either 4 is prime or 5 is prime, and it is not the case that both 4 is prime and 5 is prime.

3. Here is another way you could phrase the same statement: If you pass the class, then it must be the case that you attended every day and that you did the homework. If we agree that this is just a clearer way to state the original statement, then we could illustrate its structure as $P \rightarrow (Q \wedge R)$.

4. Notice that this is not the same as saying, "Every number is even, or every number is odd." Of course, saying, "3 is even or odd," *is* the same as saying, "3 is even, or 3 is odd." Language is confusing!

   We don't yet have the logical technology to translate this statement as anything more than P, where P is the statement, "Every number is even or odd." Luckily, that technology is available, starting... now!

2.1.3 Quantifiers and Predicates

Did you know that all mammals have hair? That every integer is even or odd? That some odd numbers are not prime?

Our goal is to explore how to write statements such as these in mathematical notation to highlight the logical structure of the statements.

This will require considering a new sort of basic sentence called a **predicate**, which is like a statement, but contains a **free variable**. When you replace that variable with a constant of some sort, then the sentence becomes a statement proper. Think of a predicate as making a claim about the values that are substituted for the "placeholder" variable(s).

A predicate can be made into a (true or false) statement by evaluating it at some constant(s), or we can claim that some or all possible constants would make the resulting statement true or false. This is done using **quantifiers**.

Definition 2.1.12 Quantifiers.

The **universal quantifier** is written $\forall$ and is read, "for all." The **existential quantifier** is written $\exists$ and is read, "there exists" or "for some."

We usually write predicates similar to how you write a function, although with capital letters. For example, we might use the predicate $P(x)$ to represent "x is prime". We can then say that $P(7)$ is true (since 7 is prime) and that $P(8)$ is false. Or using quantifiers, we can (falsely) claim that all numbers are prime by writing $\forall x P(x)$ or (truthfully) claim that there is at least one prime number, by writing $\exists x P(x)$.

Example 2.1.13

Translate the statement, "Every number is even or odd," into symbols.

Solution. Before we even start using symbols, it is helpful to rephrase this in a way that captures the logical structure of the statement. What is the claim saying? Given any number, it will either be the case that the number is even, or that the number is odd. In particular, we are not claiming that either all numbers are even or all numbers are odd.

Let's use $E(x)$ to say that x is even, and $O(x)$ to say that x is odd. Then we can write,

$$E(x) \vee O(x)$$

to say that x is even or x is odd. Which x is that true for (according to the claim)? *All* of them. So we write the statement as,

$$\forall x(O(x) \vee E(x)).$$

We added some parentheses to emphasize that the **scope** of the universal quantifier includes both predicates.

Note that if we incorrectly interpreted the statement as claiming that either all numbers are even or all numbers are odd, we could write that as $\forall x O(x) \vee \forall x E(x)$. This is not the same!

Just like we did for propositional logic and the logical connectives, we should decide what it means for a quantified predicate to be true or false. We say $\forall x P(x)$ is true if $P(a)$ is true no matter what constant a we substitute for x. And similarly, $\exists x P(x)$ is true if there is at least one value a for which $P(a)$ is true.

However, we must be careful here. Consider the statement

$$\forall x \exists y (y < x).$$

You would read this, "For every x there is some y such that y is less than x." Note that $<$ is a predicate with two free variables; we have chosen to write it with the symbol between the variables instead of the funky-looking $L(y, x)$ or $<(y, x)$.

Is this statement true? The answer depends on our **domain of discourse**. When we say "for all" x, do we mean all positive integers or all real numbers or all elephants or...? Usually, this information is implied by the context of the statement. In discrete mathematics, we almost always quantify over the *natural numbers*, $0, 1, 2, \ldots$, so let's take that for our domain of discourse here.

For the statement to be true, we need, no matter what natural number we select, for there to be some natural number that is strictly smaller. Perhaps we could let y be $x - 1$? But here is the problem: what if $x = 0$? Then $y = -1$, and that is *not a number!* (in our domain of discourse). Thus we see that the statement is false because there is a number less than or equal to all other numbers. In symbols,

$$\exists x \forall y (y \geq x).$$

We will explore some rules for working with quantifiers and other connectives in Section 2.3. For now, we will focus on translating between informal statements in ordinary language and the more precise language of logic. There is no perfect algorithm for doing this translation, but here are a few useful rules of thumb.

Every blank is blank.

Any statement of the form, "Every P-thing is a Q-thing" can be written as

$$\forall x (P(x) \rightarrow Q(x)).$$

Example: all mammals have hair, becomes $\forall x (M(x) \rightarrow H(x))$, where $M(x)$ means x is a mammal, and $H(x)$ means x has hair.

To make sense of this, think about what we mean by statements like these in terms of sets. We claim that the set of mammals is contained in, or is a subset of, the set of hairy things. What we mean by "A is a subset of B" is precisely that every element of x is an element of y. This can also be expressed by saying that "if x is an element of A, then x is also an element of B."

Some blanks are blank.

Any statement of the form, "Some P-things are Q-things," can be written as

$$\exists x (P(x) \wedge Q(x)).$$

Example: Some cats can swim, becomes $\exists x (C(x) \wedge S(x))$, where $C(x)$ means x is a cat, and $S(x)$ means x can swim.

Again, it is helpful to think of how to express such statements in terms of sets. To say that some cats can swim is to say that there are things that belong both to the set of cats and to the set of swimming things. Such animals belong to the *intersection* of these two sets, which you can describe as belonging to the first set *and* the second set. Existential statements of this form claim that the intersection of the two sets is not empty.

Implicit Quantifiers. It is always a good idea to be precise in mathematics. Sometimes though, we can relax a bit, as long as we all agree on a convention. An example of such a convention is to assume that sentences containing predicates with free variables are intended as statements, where the variables are universally quantified.

For example, do you believe that if a shape is a square, then it is a rectangle? But how can that be true if it is not a statement? To be a little more precise, we have two predicates: $S(x)$ for "x is a square" and $R(x)$ for "x is a rectangle". The *sentence* we are looking at is

$$S(x) \to R(x).$$

This is neither true nor false, as it is not a statement. But come on! We all know that we meant to consider the statement,

$$\forall x(S(x) \to R(x)),$$

and this is what our convention tells us to consider. We call the resulting statement the **universal generalization** of the original sentence.

Definition 2.1.14

Given a sentence with free variables, the **universal generalization** of that sentence is the statement obtained by adding enough universal quantifiers to the beginning of the sentence so that all free variables become bound.

Similarly, we will often be a bit sloppy about the distinction between a predicate and a statement. For example, we might write, *let $P(n)$ be the* statement, *"n is prime,"* which is technically incorrect. It is implicit that we mean that we are defining $P(n)$ to be a predicate, which for each n becomes the statement, n is prime.

2.1.4 Reading Questions

1. Match each statement in symbols with its type of statement.

$P \to Q$	P and Q (conjunction)
$P \vee Q$	If P, then Q, (implication)
$P \wedge Q$	P or Q (disjunction)
$\neg P$	Not P (negation)

2. Consider the sentence, "If $x > 3$, then x is even."

Which of the following statements are true about the sentence? Select all that apply.

A. The sentence is a false statement since it has a free variable.

B. The universal generalization of the sentence is a statement.

C. If you substitute 10 for x, the resulting statement is true.

D. The sentence becomes a true statement no matter what natural number you substitute for x.

3. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

2.1.5 Practice Problems

1. For each sentence below, decide whether it is an atomic statement, a molecular statement, or not a statement at all.

(a) Some say the end is near, and some say we'll see Armageddon soon.

(b) Mom's coming 'round to put it back the way it ought to be.

(c) Learn to swim.

2. Classify each of the sentences below as an atomic statement, a molecular statement, or not a statement at all. If the statement is molecular, say what kind it is (conjuction, disjunction, conditional, biconditional, negation).

(a) Everybody can be fooled sometimes.

(b) Every natural number greater than 1 is either prime or composite.

(c) Go to your room!

(d) The Broncos will win the Super Bowl, or I'll eat my hat.

(e) This shirt is not black.

3. Determine whether each molecular statement below is true or false, or whether it is impossible to determine. Assume you do not know what my favorite number is (but you do know which numbers are prime).

(a) If 4 is my favorite number, then $4 + 1$ is my favorite number.

(b) 8 is my favorite number, and 3 is not prime.

(c) 4 is my favorite number, or 4 is prime.

(d) If 4 is prime, then $2 \cdot 4$ is prime.

(e) If 3 is prime, then 3 is my favorite number.

(f) 8 is my favorite number, and 4 is not prime.

4. Let $P(x, y)$ be the predicate, "person x can be fooled at time y."
Match each statement with its representation in symbols.

It is always true that some people can be fooled.	$\exists x \forall y P(x, y)$
Sometimes everyone can be fooled.	$\forall x \exists y P(x, y)$
Everyone can be fooled sometimes.	$\forall y \exists x P(x, y)$
Some people can be fooled all of the time.	$\exists y \forall x P(x, y)$

5. Your friend believes that you cannot fool everyone at the same time. What is another way of saying this, and how would you write that in symbols (using $P(x,y)$ to say you can fool x at time y).

A. Someone is never fooled. $\exists x \forall y \neg P(x,y)$

B. Everyone is never fooled. $\forall x \forall y \neg P(x,y)$

C. Someone is not fooled sometimes. $\exists x \exists y \neg P(x,y)$

D. Everyone is not fooled sometimes. $\forall x \exists y \neg P(x,y)$

6. Regardless of your beliefs of how many people can be fooled at various times, what could you conclude if we reinterpret $P(x,y)$ to mean $x < y$ and only quantify over the natural numbers (so $\forall x$ means "For all natural numbers," and $\exists x$ means "There exists a natural number")? Select all of the following that apply.

A. $\forall x \exists y P(x,y)$ is true.

B. $\exists x \forall y P(x,y)$ is true.

C. $\forall y \exists x P(x,y)$ is true.

D. $\exists y \forall x P(x,y)$ is true.

E. No matter what $P(x,y)$ means, we can conclude that $\forall x \exists y P(x,y)$ and $\exists y \forall x$ are NOT *logically equivalent*

7. Let $P(x)$ be the predicate, "$17x + 1$ is even."

(a) Is $P(15)$ true or false?

(b) What, if anything, can you conclude about $\exists x P(x)$ from the truth value of $P(15)$?

(c) What, if anything, can you conclude about $\forall x P(x)$ from the truth value of $P(15)$?

8. Let $P(x)$ be the predicate, "$18x + 1$ is even."

(a) Is $P(15)$ true or false?

(b) What, if anything, can you conclude about $\exists x P(x)$ from the truth value of $P(15)$?

(c) What, if anything, can you conclude about $\forall x P(x)$ from the truth value of $P(15)$?

9. Consider the sentence, $\exists x P(x,y) \rightarrow \forall x P(x,y)$. What can we say about this sentence? Select all that apply.

A. The sentence is a statement because it contains quantifiers.

B. The sentence is not a statement because x and z are free variables.

C. The sentence is not a statement because y is a free variable.

D. The universal generalization of the sentence is a statement.

10. Suppose $P(x, y)$ is some binary predicate defined on a very small domain of discourse: just the integers 1, 2, 3, and 4. For each of the 16 pairs of these numbers, $P(x, y)$ is either true or false, according to the following table (x values are rows, y values are columns).

	1	**2**	**3**	**4**
1	T	F	F	F
2	F	T	T	F
3	T	T	T	T
4	F	F	F	F

For example, $P(1, 3)$ is false, as indicated by the F in the first row, third column.
Use the table to decide whether the following statements are true or false.

(a) $\forall y \exists x P(x, y)$.

(b) $\exists x \forall y P(x, y)$.

(c) $\forall x \exists y P(x, y)$.

(d) $\exists y \forall x P(x, y)$.

2.1.6 Additional Exercises

1. Suppose P and Q are the statements: P: Jack passed math. Q: Jill passed math.

(a) Translate "Jack and Jill both passed math" into symbols.

(b) Translate "If Jack passed math, then Jill did not" into symbols.

(c) Translate "$P \vee Q$" into English.

(d) Translate "$\neg(P \wedge Q) \rightarrow Q$" into English.

(e) Suppose you know that if Jack passed math, then so did Jill. What can you conclude if you know that:

i. Jill passed math?
ii. Jill did not pass math?

2. Translate into symbols. Use $E(x)$ for "x is even" and $O(x)$ for "x is odd."

(a) No number is both even and odd.

(b) One more than any even number is an odd number.

(c) There is a prime number that is even.

(d) Between any two numbers there is a third number.

(e) There is no number between a number and one more than that number.

3. For each of the statements below, give a domain of discourse for which the statement is true, and a domain for which the statement is false.

(a) $\forall x \exists y (y^2 = x)$.

(b) $\forall x \forall y (x < y \rightarrow \exists z (x < z < y))$.

(c) $\exists x \forall y \forall z (y < z \rightarrow y \leq x \leq z)$.

2.2 Implications

Objectives

After completing this section, you should be able to do the following.

1. Explain the conditions under which an implication is true.
2. Identify statements as equivalent to a given implication or its converse.
3. Explain the relationship between the truth values of an implication, its converse, and its contrapositive.

2.2.1 Section Preview

Investigate!

Little Timmy's Mom tells him, "If you don't eat all your broccoli, then you will not get any ice cream." Of course, Timmy loves his ice cream, so he quickly eats all his broccoli (which actually tastes pretty good).

After dinner, when Timmy asks for his ice cream, he is told no! Does Timmy have a right to be upset? Why or why not?

By far, the most important type of statement in mathematics is the implication. It is also the least intuitive of our basic molecular statement types. Our goal in this section is to become more familiar with this key concept.

To see why this sort of statement is so prevalent, consider the *Pythagorean Theorem*. Despite what social media might claim, the Pythagorean Theorem is not

$$a^2 + b^2 = c^2.$$

Okay, sure, that has a variable in it, so we must be using the convention to take the universal generalization,

$$\forall a, b, c \in \mathbb{R} \left(a^2 + b^2 = c^2\right).$$

So $1^2 + 5^2 = 2^2$??? Okay, fine. The equation is true as long as a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse. In other words:

> *If* a and b are the lengths of the legs of a right triangle with hypotenuse of length c, *then* $a^2 + b^2 = c^2$.

Math is about making general claims, but a claim is rarely going to be true of absolutely *every* mathematical object. The way we *restrict* our claims to a particular type of object is with an implication: "Take any object you like, *if* it is of the right type, *then* this thing is true about it."

Similarly, as we saw in the Quantifiers and Predicates subsection, when we make claims like "Every square is a rectangle," we really have an implication: "If something is a square, then it is a rectangle."

Here is a reminder of what we mean by an implication.

Definition 2.2.1 Implication.

An **implication** (or **conditional**) is a molecular statement of the form

$$P \rightarrow Q$$

where P and Q are statements. We say that

- P is the **hypothesis** (or **antecedent**).
- Q is the **conclusion** (or **consequent**).

An implication is *true* provided P is false or Q is true (or both), and *false* otherwise. In particular, the only way for $P \rightarrow Q$ to be false is for P to be true *and* Q to be false.

The definition of truth of an implication can also be represented as a truth table:

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Figure 2.2.2 The truth table for $P \rightarrow Q$.

Does this truth table make sense? Should we believe it? Look in particular at the third row: F, T, T, and consider the implication, "If $5 < 3$ then $5 + 3 = 8$." Does that statement *feel* true? The truth table says it should be (since $5 < 3$ is false, and $5 + 3 = 8$ is true).

Much of what we will do in the remainder of this section is convince ourselves that this truth table makes sense.

PREVIEW ACTIVITY

1. Consider the statement, "If Tommy doesn't eat his broccoli, then he will not get any ice cream." Which of the following statements mean the same thing (i.e., will be true in the same situations)? Select all that apply.

A. If Tommy does eat his broccoli, then he will get ice cream.

B. If Tommy gets ice cream, then he ate his broccoli.

C. If Tommy doesn't get ice cream, then he didn't eat his broccoli.

D. Tommy ate his broccoli and still didn't get any ice cream.

2. Suppose that your shady uncle offers you the following deal: If you loan him your car, then he will bring you tacos. In which of the following situations would it be fair to say that your uncle is a liar (i.e., that his statement was false)? Select all that apply.

A. You loan him your car. He brings you tacos.

B. You loan him your car. He never buys you tacos.

C. You don't loan him your car. He still brings you tacos.

D. You don't loan him your car. He never brings you tacos.

3. Consider the *sentence*, "If $x \geq 10$, then $x^2 \geq 25$." This sentence becomes a statement when we replace x by a value, or "capture" the x in the scope of a quantifier. Which of the following claims are true (select all that apply)?

A. If we replace x by 15, then the resulting statement is true. (Note, $15^2 = 225$.)

B. If we replace x by 3, then the resulting statement is true.

C. If we replace x by 6, then the resulting statement is true.

D. The universal generalization ("for all x, if $x \geq 10$ the $x^2 \geq 25$") is true.

E. There is a number we could replace x with that makes the statement false.

4. Consider the statement, "If I see a movie, then I eat popcorn" (which happens to be true). Based solely on your intuition of English, which of the following statements mean the same thing? Select all that apply.

A. If I eat popcorn, then I see a movie.

B. If I don't eat popcorn, then I don't see a movie.

C. It is necessary that I eat popcorn when I see a movie.

D. To see a movie, it is sufficient for me to eat popcorn.

E. I only watch a movie if I eat popcorn.

2.2.2 Understanding the Truth Table

The truth value of the implication is determined by the truth values of its two parts. Our definition of the truth conditions for an implication says that there is only one way for an implication to be false: when the hypothesis is true and the conclusion is false.

Example 2.2.3

Consider the statement:

> If Bob gets a 90 on the final, then Bob will pass the class.

This is definitely an implication: P is the statement "Bob gets a 90 on the final," and Q is the statement "Bob will pass the class."

Suppose I made that statement to Bob. In what circumstances would it be fair to call me a liar? What if Bob really did get a 90 on the final, and he did pass the class? Then I have not lied; my statement is true. However, if Bob did get a 90 on the final and did not pass the class, then I lied, making the statement false. The tricky case is this: What if Bob did not get a 90 on the final? Maybe he passes the class, maybe he doesn't. Did I lie in either case? I think not. In these last two cases, P was false, and the statement $P \rightarrow Q$ was true. In the first case, Q was true, and so was $P \rightarrow Q$. So $P \rightarrow Q$ is true when either P is false or Q is true.

Just to be clear, although we sometimes read $P \rightarrow Q$ as "P *implies* Q", we are not insisting that there is some *causal* relationship between the statements P and Q (although there might be). "If $x < y$, then $x + 1 < y + 1$," is a true statement (or at least, its universal generalization is). We know it is true because we understand how the two parts interact. If you add 1 to two numbers x and y, then their order does not change. But the statement, "if $1 < 2$, then Euclid studied geometry" is also a true implication.

Example 2.2.4

Decide which of the following statements are true and which are false. Briefly explain.

1. If $1 = 1$, then most horses have 4 legs.
2. If $0 = 1$, then $1 = 1$.
3. If 8 is a prime number, then the 7624th digit of π is an 8.
4. If the 7624th digit of π is an 8, then $2 + 2 = 4$.

Solution. All four of the statements are true. Remember, the only way for an implication to be false is for the *if* part to be true and the *then* part to be false.

1. Here both the hypothesis and the conclusion are true, so the implication is true. It does not matter that there is no meaningful connection between the true mathematical fact and the fact about horses.

2. Here the hypothesis is false and the conclusion is true, so the implication is true.

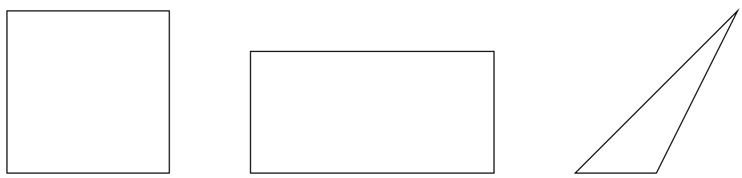
3. I have no idea what the 7624th digit of π is, but this does not matter. Since the hypothesis is false, the implication is automatically true.

4. Regardless of the truth value of the hypothesis, the conclusion is true, making the implication true.

This is a strange example and isn't really how we use implications anyway. This strangeness is not just mathematicians being stubborn though. The truth conditions for implications *must* be like they are for mathematics to make sense. Let's see why.

Example 2.2.5

Consider the statement, "All squares are rectangles," which can also be phrased as, "For all shapes, if the shape is a square, then it is a rectangle." Is this statement true or false? Are we sure? What about the following three shapes?



Solution. Of course the statement is true. A square is a 4-sided plane figure with 4 right angles and 4 equal-length sides, while a rectangle is a 4-sided plane figure with 4 right angles.

However, what we mean when we consider a universal statement like this is that, no matter what we "plug in" for the variable ("the shape" in this case), the resulting statement is true. When the statement is about a particular shape, we have an implication $P \to Q$. This means it must be true that, if the actual shape on the left is a square, then it is a rectangle. Great. The shape is a square (P is true) and is a rectangle (Q is true), so yes, the implication is true.

Is the implication true of the rectangle in the middle? Well, that shape is not a square (P is false), and it is a rectangle (Q is true). But look, we believe that all squares are rectangles, so the statement must be true. Even of a rectangle. The only way this works is if "false implies true" is true!

Similarly, all squares are rectangles is a true statement, even when we look at a triangle. P is false (the triangle is not a square), and Q is false (the triangle is not a rectangle). Thankfully, we defined implications to be true in this case as well.

We have given shapes that illustrate lines 1, 3, and 4 of the truth table for implications (Figure 2.2.2). What shape illustrates line 2? That would need to be a shape that was a square and was not a rectangle.... Of course we can't find one, precisely because the statement is true!

2.2.3 Related Statements

An implication is a way of expressing a relationship between two statements. It is often interesting to ask whether there are other relationships between the statements. Here we introduce some common language to address this question.

Definition 2.2.6 Converse, Contrapositive, and Inverse.

Given an implication $P \to Q$, we say,

- The **converse** is the statement $Q \to P$.
- The **contrapositive** is the statement $\neg Q \to \neg P$.
- The **inverse** is the statement, $\neg P \to \neg Q$.

Example 2.2.7

Consider the implication, "If you clean your room, then you can go to the party." Give the converse, contrapositive, and inverse of this statement

Solution. The converse is, "If you can go to the party, then you clean your room."

The contrapositive is, "If you can't go to the party, then you don't clean your room."

The inverse is, "If you don't clean your room, then you can't go to the party."

Symbolically, both the converse and the contrapositive *switch* the order of the two parts of the statement (or alternatively, think about turning the arrow to point in the other direction). The contrapositive and the inverse take the *negation* of both of the statements. Notice that if you take the converse (switch the order) and then take the contrapositive *of* that converse (switch the order back and negate both parts) you get the inverse. So the inverse is nothing more than the contrapositive of the converse. Or the converse of the contrapositive, which is a fun fact to mention at parties.

When considering statements with quantifiers, we ignore the outside quantifiers when forming the converse, contrapositive, and inverse.

Quantifiers and the Converse, Contrapositive, and Inverse.

A quantified implication $\forall x(P(x) \to Q(x))$ has:

Converse	$\forall x(Q(x) \to P(x))$
Contrapositive	$\forall x(\neg Q(x) \to \neg P(x))$
Inverse	$\forall x(\neg P(x) \to \neg Q(x))$

Note 2.2.8 It is unlikely that we would encounter a statement of the form $\exists x(P(x) \to Q(x))$, since this would be automatically true if there was any x that made $P(x)$ false. But if we did, the same rules would apply to the converse, contrapositive, and inverse as above: Just ignore the quantifier when swapping and/or negating the parts of the implication.

For example, "For all shapes, if the shape is a square, then it is a rectangle" (i.e., all squares are rectangles) has the converse, "For all shapes, if the shape is a rectangle, then it is a square" (so all rectangles are squares).

Well, that's not true! There exist shapes that are rectangles and are NOT squares. Indeed, this is an example of a statement that is true with a false converse. There are lots of examples of this throughout mathematics. There are also examples of true implications that have true converses. You just can't know from the logic.[2]

The contrapositive of "For all shapes, if it is a square, then it is a rectangle" is "For all shapes, if the shape is not a rectangle, then it is not a square." This is true. In fact, ***the contrapositive of a true statement is always true***!

Since the contrapositive of an implication always has the same truth value as its original implication, it can often be helpful to analyze the contrapositive to decide whether an implication is true.

Example 2.2.9

True or false: If you draw any nine playing cards from a regular deck, then you will have at least three cards all of the same suit. Is the converse true?

Solution. True. The original implication is a little hard to analyze because there are so many combinations of nine cards. But consider the contrapositive: if you *don't* have at least three cards all of the same suit, then you don't have nine cards. It is easy to see why this is true. If you don't have at least three cards in a suit, you can have at most two cards of each of the four suits, for a total of at most eight cards.

[2]It turns out the Pythagorean Theorem is one such statement. It is also true that *if* $a^2 + b^2 = c^2$, *then* there is a right triangle with legs of lengths a and b and hypotenuse of length c. So we could have also written the theorem as a biconditional: "a and b are the lengths of the legs of a right triangle with hypotenuse of length c *if and only if* $a^2 + b^2 = c^2$."

The converse: If you have at least three cards of the same suit, then you have nine cards. This is false. You could have three spades and nothing else. Note that to demonstrate that the converse (an implication) is false, we provided an example where the hypothesis is true (you do have three cards of the same suit), but where the conclusion is false (you do not have nine cards). In other words, we find some example that puts us in row 2 of the implication's truth table.

Understanding converses and contrapositives can help understand implications and their truth values:

Example 2.2.10

Suppose I tell Sue that if she gets a 93% on her final, then she will get an A in the class. Assuming that what I said is true, what can you conclude in the following cases:

1. Sue gets a 93% on her final.
2. Sue gets an A in the class.
3. Sue does not get a 93% on her final.
4. Sue does not get an A in the class.

Solution. Note first that whenever $P \to Q$ and P are both true statements, Q must be true as well. For this problem, take P to mean "Sue gets a 93% on her final" and Q to mean "Sue will get an A in the class."

1. We have $P \to Q$ and P, so Q follows. Sue gets an A.
2. You cannot conclude anything. Sue could have gotten the A because she did extra credit, for example. Notice that we do not know that if Sue gets an A, then she gets a 93% on her final. That is the converse of the original implication, so it might or might not be true.
3. The contrapositive of the converse of $P \to Q$ is $\neg P \to \neg Q$, which states that if Sue does not get a 93% on the final, then she will not get an A in the class. But this does not follow from the original implication. Again, we can conclude nothing. Sue could have done extra credit.
4. What would happen if Sue did not get an A but *did* get a 93% on the final? Then P would be true, and Q would be false. This makes the implication $P \to Q$ false! It must be that Sue did not get a 93% on the final. Notice we now have the implication $\neg Q \to \neg P$ which is the contrapositive of $P \to Q$. Since $P \to Q$ is assumed to be true, we know $\neg Q \to \neg P$ is true as well.

As we said above, an implication is not logically equivalent to its converse, but it is possible that both the implication and its converse are true. In this case, when both $P \to Q$ and $Q \to P$ are true, we say that P and Q are equivalent and write $P \leftrightarrow Q$. This is the biconditional we mentioned in Section 2.1.

You can think of "if and only if" statements as having two parts: an implication and its converse. We might say one is the "if" part, and the other is the "only if" part. We also sometimes say that "if and only if" statements have two directions: a forward direction ($P \to Q$) and a backward direction ($P \leftarrow Q$, which is really just sloppy notation for $Q \to P$).

Let's think a little about which part is which. Is $P \to Q$ the "if" part or the "only if" part? Consider an example.

Example 2.2.11

Suppose it is true that I sing if and only if I'm in the shower. We know this means both that if I sing, then I'm in the shower, and also the converse, that if I'm in the shower, then I sing. Let P be the statement, "I sing," and Q be, "I'm in the shower." So $P \to Q$ is the statement "if I sing, then I'm in the shower." Which part of the if and only if statement is this?

What we are really asking for is the meaning of "I sing *if* I'm in the shower" and "I sing *only if* I'm in the shower." When is the first one (the "if" part) *false*? When I am in the shower but not singing. That is the same condition for being false as the statement, "If I'm in the shower, then I sing." So the "if" part is $Q \to P$. On the other hand, to say, "I sing only if I'm in the shower" is equivalent to saying "If I sing, then I'm in the shower," so the "only if" part is $P \to Q$.

It is not especially important to know which part is the "if" or "only if" part, but this does illustrate something very, very important: *There are many ways to state an implication!*

Example 2.2.12

Rephrase the implication, "If I dream, then I am asleep" in as many ways as possible. Then do the same for the converse.

Solution. The following are all equivalent to the original implication:

1. I am asleep if I dream.
2. I dream only if I am asleep.
3. In order to dream, I must be asleep.
4. To dream, it is necessary that I am asleep.
5. To be asleep, it is sufficient to dream.

6. I am not dreaming unless I am asleep.

The following are equivalent to the converse (if I am asleep, then I dream):

1. I dream if I am asleep.
2. I am asleep only if I dream.
3. It is necessary that I dream in order to be asleep.
4. It is sufficient that I be asleep in order to dream.
5. If I don't dream, then I'm not asleep.

Hopefully you agree with the above example. We include the "necessary and sufficient" versions because those are common when discussing mathematics. Let's agree once and for all what they mean.

Definition 2.2.13 Necessary and Sufficient.

- "P is necessary for Q" means $Q \to P$.
- "P is sufficient for Q" means $P \to Q$.
- If P is necessary and sufficient for Q, then $P \leftrightarrow Q$.

To be honest, I have trouble with these if I'm not very careful. I find it helps to keep a standard example for reference.

Example 2.2.14

In a regular deck of cards, the red suits are hearts and diamonds. The black suits are clubs and spades. Thus it is true that, after picking a card, if my card is a spade, then my card is black.

Restate this fact using necessary and sufficient phrasing.

Solution. For my card to be a spade, it is necessary that it is black. However, it is not sufficient for it to be black to say that I am holding a spade (since I could have a club).

I can also say that to have a black card, it is sufficient to have a spade. It is not necessary that I have a spade.

It is helpful to think about the amount of evidence you need. Is knowing that the card is a spade enough evidence to conclude that it is a black card? Yes, that is sufficient! Being a spade is a sufficient condition for the card to be black.

Thinking about the necessity and sufficiency of conditions can also help when writing proofs and justifying conclusions. If you want to establish some mathematical fact, it is helpful to think what other facts would *be enough* (be sufficient) to prove your fact. If you have an assumption, think about what must also be necessary if that hypothesis is true.

2.2.4 Reading Questions

1. It happens to be true that all mammals have hair. Which of the following are also true?

A. Having hair is a necessary condition for being a mammal.

B. Having hair is a sufficient condition for being a mammal.

C. If an animal doesn't have hair, then it is not a mammal.

D. An animal is a mammal only if it has hair.

2. Give an example of a *true* implication (written out in words) that has a *false* converse. Explain why your implication is true and why the converse is false.

3. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

2.2.5 Practice Problems

1. In my safe is a sheet of paper with two shapes drawn on it in colored crayon. One is a circle, and the other is a pentagon. Each shape is drawn in a single color. Suppose you believe me when I tell you that, "If the circle is purple, then the pentagon is orange.

What do you therefore know about the truth value of the following statements?

(a) The circle and the pentagon are both purple.

(b) The circle and the pentagon are both orange.

(c) The circle is not purple, or the pentagon is orange.

(d) If the pentagon is orange, then the circle is purple.

(e) If the pentagon is not orange, then the circle is not purple.

2. Suppose the statement, *"If the square is yellow, then the circle is purple,"* is true. Assume also that the converse is false. Classify each statement below as true or false (if possible).

(a) The circle is purple.

(b) The square is yellow if and only if the circle is not purple.

(c) The square is yellow.

(d) The square is yellow if and only if the circle is purple.

3. Consider the statement, *"If you will give me magic beans, then I will give you a cow."* Decide whether each statement below is the converse, the contrapositive, or neither.

(a) If I will give you a cow, then you will not give me magic beans.

(b) If I will give you a cow, then you will give me magic beans.

(c) If you will not give me magic beans, then I will not give you a cow.

(d) If you will give me magic beans, then I will not give you a cow.

(e) You will give me magic beans, and I will not give you a cow.

(f) If I will not give you a cow, then you will not give me magic beans.

4. You have discovered an old paper on graph theory that discusses the *viscosity* of a graph (which for all you know, is something completely made up by the author). A theorem in the paper claims that "if a graph satisfies *condition (V)*, then the graph is *viscous*." Which of the following are equivalent ways of stating this claim? Which are equivalent to the *converse* of the claim?

(a) Only viscous graphs satisfy condition (V).

(b) For a graph to be viscous, it is necessary that it satisfies condition (V).

(c) A graph is viscous only if it satisfies condition (V).

(d) Satisfying condition (V) is a necessary condition for a graph to be viscous.

(e) A graph is viscous if it satisfies condition (V).

5. Which of the following statements are equivalent to the implication, "*if you win the lottery, then you will be rich,*" and which are equivalent to the converse of the implication?

(a) If you are not rich, then you did not win the lottery.

(b) It is sufficient to win the lottery to be rich.

(c) Either you win the lottery, or else you are not rich.

(d) If you are rich, you must have won the lottery.

(e) You will win the lottery if and only if you are rich.

2.2.6 Additional Exercises

1. Translate into English:

(a) $\forall x(E(x) \to E(x+2))$.

(b) $\forall x \exists y(\sin(x) = y)$.

(c) $\forall y \exists x(\sin(x) = y)$.

(d) $\forall x \forall y(x^3 = y^3 \to x = y)$.

2. Consider the statement, "If Oscar eats Chinese food, then he drinks milk."

(a) Write the converse of the statement.

(b) Write the contrapositive of the statement.

(c) Is it possible for the contrapositive to be false? If it was, what would that tell you?

(d) Suppose the original statement is true, and that Oscar drinks milk. Can you conclude anything (about his eating Chinese food)? Explain.

(e) Suppose the original statement is true, and that Oscar does not drink milk. Can you conclude anything (about his eating Chinese food)? Explain.

3. Write each of the following statements in the form, "If . . . , then . . . ." Careful, some statements may be false (which is fine for the purposes of this question).

(a) To lose weight, you must exercise.

(b) To lose weight, all you need to do is exercise.

(c) Every American is patriotic.

(d) You are patriotic only if you are American.

(e) The set of rational numbers is a subset of the real numbers.

(f) A number is prime if it is not even.

(g) Either the Broncos will win the Super Bowl, or they won't play in the Super Bowl.

4. Consider the implication, "If you clean your room, then you can watch TV." Rephrase the implication in as many ways as possible. Then do the same for the converse.

5. Recall from calculus, if a function is differentiable at a point c, then it is continuous at c, but that the converse of this statement is not true (for example, $f(x) = |x|$ at the point 0). Restate this fact using "necessary and sufficient" language.

6. Consider the statement, "For all natural numbers n, if n is prime, then n is solitary." You do not need to know what *solitary* means for this problem, just that it is a property that some numbers have and others do not.

(a) Write the converse and the contrapositive of the statement, saying which is which. Note: the original statement claims that an implication is true for all n, and it is that implication that we are taking the converse and contrapositive of.

(b) Write the negation of the original statement. What would you need to show to prove that the statement is false?

(c) Even though you don't know whether 10 is solitary (in fact, nobody knows this), is the statement, "If 10 is prime, then 10 is solitary" true or false? Explain.

(d) It turns out that 8 is solitary. Does this tell you anything about the truth or falsity of the original statement, its converse or its contrapositive? Explain.

(e) Assuming that the original statement is true, what can you say about the relationship between the *set* P of prime numbers and the *set* S of solitary numbers. Explain.

2.3 Rules of Logic

Objectives

After completing this section, you should be able to do the following.

1. Use truth tables to determine whether two statements are logically equivalent.
2. Use truth tables to determine whether a deduction rule is valid.
3. Use logical equivalence and deduction rules to simplify statements and make deductions.

2.3.1 Section Preview

Investigate!

Holmes always wears one of the two vests he owns: one tweed and one mint green. He always wears either the green vest or red shoes. Whenever he wears a purple shirt and the green vest, he chooses to not wear a bow tie. He never wears the green vest unless he is also wearing either a purple shirt or red shoes. Whenever he wears red shoes, he also wears a purple shirt. Today, Holmes wore a bow tie. What else did he wear?

Try it 2.3.1

Spend a few minutes thinking about the *Investigate!* question above. Of the six statements in the puzzle, only one is atomic. Use this atomic statement and one other statement to deduce a new statement about what Holmes might (or might not) be wearing. Explain why you think your new statement is true.

Hint. The atomic statement is, "Holmes wore a bow tie." Only one of the molecular statements has this as one of its *atoms*.

Logic studies the ways statements can interact with each other. More precisely, we consider the way the logical form statements can interact. The study of logic does not care about the content of the atomic statements or the meaning of predicates. For example, the claims, "If spiders have six legs, then Sam walks with a limp," and, "If the moon is made of cheese, then cheddar is a type of cheese," are identical from a logical perspective. Logic doesn't care about whether Sam is a spider or the culinary makeup of the moon. Both statements have the same form: They are implications, $P \to Q$.

Of course, in mathematics we often *do* know some relationship between various atomic statements. For example, we know a relationship between being even and being a multiple of 10. That relationship allows us to make claims such as, "If the number I'm thinking of is a multiple of 10, then it is even." Suppose I also told you that I am now thinking of a number that is not even. We can deduce that I am not thinking of a multiple of 10! Crucially, if we accept the truth of the statements here, we can make this deduction without thinking about the nature of numbers. It can feel very liberating and provide much-needed clarity when trying to understand complicated reasoning if we can separate the content from the logical form of arguments.

Our goal in this section is to establish some procedures for analyzing how the truth or falsity of statements interact, based on their logical form. We will see that some molecular statements must be true regardless of whether their atomic parts are true or false, while some statements must always be false. For other statements, it can be that two statements are always true or false together, or that whenever one statement is true, another statement must also be true.

The main method for establishing these relationships will be **truth tables**. There is a very clear procedure for constructing and analyzing truth tables, but for complicated arguments that contain many atomic statements, the truth tables become very large and unwieldy. We will therefore use truth tables to understand some basic equivalences and deductions that can be applied in a sequence of reasoning to construct larger arguments.

Preview Activity

1. Consider the statement, "Whenever Holmes wears a purple shirt and the green vest, he chooses to not wear a bow tie." Let P be the statement, "Holmes wears a purple shirt," G be the statement, "Holmes wears the green vest," and B be the statement, "Holmes wears a bow tie." Which of the following is the best translation of the statement into propositional logic?

A. $(P \wedge G) \rightarrow \neg B$

B. $(P \wedge G) \rightarrow B$

C. $(P \vee G) \rightarrow \neg B$

D. $P \wedge (G \rightarrow B)$

2. Consider the statement, "Holmes never wears the green vest unless he is also wearing either a purple shirt or red shoes." With P and G as in the previous question, and R being the statement, "Holmes wears red shoes," which of the following is the best translation of the statement into propositional logic?

A. $G \rightarrow (P \vee R)$

B. $\neg G \rightarrow (P \vee R)$

C. $(P \vee R) \rightarrow G$

D. $(P \vee R) \rightarrow \neg G$

3. Consider the statement, "If you major in math, then you will get a high-paying job," and the statement, "Either you don't major in math, or you will get a high-paying job." In which of the following cases are *both* statements true? Select all that apply.

A. You major in math and get a high-paying job.

B. You major in math and don't get a high-paying job.

C. You don't major in math and do get a high-paying job.

D. You don't major in math and don't get a high-paying job.

2.3.2 Truth Tables

Here's a question about playing Monopoly:

> If you get more doubles than any other player, then you will lose, or if you lose, then you must have bought the most properties.

True or false? We will answer this question and won't need to know anything about Monopoly. Instead, we will look at the logical *form* of the statement.

We need to decide when the statement $(P \rightarrow Q) \vee (Q \rightarrow R)$ is true. Using the definitions of the connectives in Definition 2.1.8, we see that for this to be true, either $P \rightarrow Q$ must be true or $Q \rightarrow R$ must be true (or both). Those are true if either P is false or Q is true (in the first case) and Q is false or R is true (in the second case). So—yeah, it gets a bit messy. Luckily, we can make a chart to keep track of all the possibilities with a **truth table**.

The idea is this: On each row, we list a possible combination of T's and F's (Trues and Falses) for each of the propositional variables, and then mark down whether the (molecular) statement in question is true or false in that case. We do this for every possible combination of T's and F's. Then we can clearly see the cases in which the statement is true or false. For complicated statements, we will first fill in values for each part of the statement, as a way of breaking up our task into smaller, more manageable pieces.

Since the truth value of a statement is completely determined by the truth values of its parts and how they are connected, all you need to know is the truth tables for each of the logical connectives, which we have already seen in Figure 2.1.9

The truth tables we consider here all build off the basic ones, applying the basic rules multiple times.

Example 2.3.2

Make a truth table for the statement $\neg P \vee Q$.

Solution. Note that this statement is not $\neg(P \vee Q)$; the negation belongs to P alone. The **main connective** here is the $\vee$, which means we will use that truth table *last*. First, we apply the truth table for $\neg$, and then apply the truth table for $\vee$ using "inputs" from $\neg P$ and Q.

Since there are two variables, there are four possible combinations of T's and F's. Putting this all together gives us the following truth table.

P	Q	$\neg P$	$\neg P \vee Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

We added a column for $\neg P$ to make filling out the last column easier. The entries in the $\neg P$ column were determined by the entries in the P column. Then to fill in the final column, look only at the column for Q and the column for $\neg P$ and use the rule for $\vee$.

Now let's answer our question about Monopoly.

Example 2.3.3

Analyze the statement, "If you get more doubles than any other player, then you will lose, or if you lose, then you must have bought the most properties," using truth tables.

Solution. Represent the statement in symbols as $(P \to Q) \vee (Q \to R)$, where P is the statement, "You get more doubles than any other player," Q is the statement, "You will lose," and R is the statement, "You must have bought the most properties." Now make a truth table.

The truth table must contain 8 rows to account for every possible combination of truth and falsity among the three statements. Here is the full truth table:

P	Q	R	$P \to Q$	$Q \to R$	$(P \to Q) \vee (Q \to R)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

The first three columns are simply a systematic listing of all possible combinations of T and F for the three statements (do you see how you would list the 16 possible combinations for four statements?). The next two columns are determined by the values of P, Q, and R and the definition of implication. Then, the last column is determined by the values in the previous two columns and the definition of $\vee$. It is this final column we care about.

Notice that in each of the eight possible cases, the statement in question is true. So our statement about monopoly is true (regardless of how many properties you own, how many doubles you roll, or whether you win or lose).

The statement about monopoly is an example of a **tautology**, a statement that is necessarily true based on its logical form alone. Tautologies are always true, but they don't tell us much about the world. No knowledge about monopoly was required to determine that the statement was true, and thus knowing that the statement is true tells us nothing about monopoly. It is equally true that "if the moon is made of cheese, then Elvis is still alive, or if Elvis is still alive, then unicorns have 5 legs."

2.3.3 Logical Equivalence

You might have noticed in Example 2.3.2 that the final column in the truth table for $\neg P \vee Q$ is identical to the final column in the truth table for $P \to Q$:

P	Q	$P \to Q$	$\neg P \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

This says that no matter what P and Q are, the statements $\neg P \vee Q$ and $P \to Q$ are either both true or both false. We therefore say these statements are **logically equivalent**.

Definition 2.3.4 Logical Equivalence.

Two (molecular) statements P and Q are **logically equivalent** provided P is true precisely when Q is true. That is, P and Q have the same truth value under any assignment of truth values to their atomic parts.

We write this as $P \equiv Q$.

To verify that two statements are logically equivalent, you can make a truth table for each and check whether the columns for the two statements are identical.

In Section 2.2 we claimed that whenever an implication is true, so is its contrapositive. We can now make this claim as the following theorem.

Theorem 2.3.5

An implication is logically equivalent to its contrapositive. That is,

$$P \to Q \equiv \neg Q \to \neg P.$$

Proof. We simply examine the truth tables.

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$\neg Q$	$\neg P$	$\neg Q \to \neg P$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

(Note that we have the truth value combinations in the same order in both tables, so we can easily see that the final columns are identical.)

Recognizing two statements as logically equivalent can be quite helpful. Rephrasing a mathematical statement can often lend insight into what it is saying, or how to prove or refute it. By using truth tables we can systematically verify that two statements are indeed logically equivalent.

Example 2.3.6

Are the statements, "It will not rain or snow," and, "It will not rain and it will not snow," logically equivalent?

Solution. We want to know whether $\neg(P \vee Q)$ is logically equivalent to $\neg P \wedge \neg Q$. Make a truth table which includes both statements:

P	Q	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

Since the truth values for the two statements are equal in every row, the two statements are logically equivalent.

Notice that this example gives us a way to "distribute" a negation over a disjunction (an "or"). We have a similar rule for distributing over conjunctions ("and"s):

Theorem 2.3.7 De Morgan's Laws.

The negation of a disjunction or conjunction is logically equivalent to a conjunction or disjunction of negations, respectively. That is,

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$

and,

$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q.$$

This suggests there might be a sort of "algebra" you could apply to statements (okay, there is: It is called *Boolean algebra*) to transform one statement into another. We can start collecting useful examples of logical equivalence and apply them in succession to a statement, instead of writing out a complicated truth table.

De Morgan's laws do not directly help us with implications, but as we saw above, every implication can be written as a disjunction:

Implications are Disjunctions.

$$P \to Q \equiv \neg P \vee Q.$$

Example: "If a number is a multiple of 4, then it is even" is equivalent to, "A number is not a multiple of 4, or (else) it is even."

With this and De Morgan's laws, you can take any statement and *simplify* it to the point where negations are only being applied to atomic propositions. Well, except that you could get multiple negations stacked up. But this can be easily dealt with:

Double Negation.

$$\neg\neg P \equiv P.$$

Example: "It is not the case that c is not odd" means "c is odd."

Let's see how we can apply the equivalences we have encountered.

Example 2.3.8

Prove that the statements $\neg(P \to Q)$ and $P \wedge \neg Q$ are logically equivalent without using truth tables.

Solution. We want to start with one of the statements and transform it into the other through a sequence of logically equivalent statements. Start with $\neg(P \to Q)$. We can rewrite the implication as a disjunction, so this is logically equivalent to

$$\neg(\neg P \vee Q).$$

Now apply De Morgan's law to get

$$\neg\neg P \wedge \neg Q.$$

Finally, use double negation to arrive at $P \wedge \neg Q$

Notice that the above example illustrates that the negation of an implication is NOT an implication: It is a conjunction! We saw this before, in Section 2.1, but it is so important and useful, it warrants stating as a theorem.

Theorem 2.3.9 Negation of an Implication.

The negation of an implication is a conjunction:

$$\neg(P \rightarrow Q) \equiv P \wedge \neg Q.$$

*That is, the only way for an implication to be false is for the hypothesis to be true **AND** the conclusion to be false.*

To verify that two statements are logically equivalent, you can use truth tables or a sequence of logically equivalent replacements. The truth table method, although cumbersome, has the advantage that it can verify that two statements are NOT logically equivalent.

Example 2.3.10

Are the statements $(P \vee Q) \rightarrow R$ and $(P \rightarrow R) \vee (Q \rightarrow R)$ logically equivalent?

Solution. Note that while we could start rewriting these statements with logically equivalent replacements in the hopes of transforming one into another, we will never be sure that our failure is due to their lack of logical equivalence rather than our lack of imagination. So instead, let's make a truth table:

P	Q	R	$(P \vee Q) \rightarrow R$	$(P \rightarrow R) \vee (Q \rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

Look at the fourth (or sixth) row. In this case, $(P \rightarrow R) \vee (Q \rightarrow R)$ is true, but $(P \vee Q) \rightarrow R$ is false. Therefore the statements are not logically equivalent.

While we don't have logical equivalence, it is the case that whenever $(P \vee Q) \rightarrow R$ is true, so is $(P \rightarrow R) \vee (Q \rightarrow R)$. This tells us that we can *deduce* $(P \rightarrow R) \vee (Q \rightarrow R)$ from $(P \vee Q) \rightarrow R$, just not the reverse direction.

2.3.4 Equivalence for Quantified Statements

All the examples we have looked at so far have only involved *propositional* logic, where the basic units of logic are statements that are either true or false. It is also possible to say that two statements involving quantifiers and predicates are logically equivalent.

Sometimes the quantifiers have nothing to do with the equivalence. For example,

$$\forall x(P(x) \to Q(x)) \equiv \forall x(\neg P(x) \vee Q(x)).$$

As soon as we replace the x with a constant, we are left with two statements that are logically equivalent based on their propositional form.

Other times, the more interesting times, it is exactly the logic of the quantifiers that makes the statements logically equivalent. What is especially interesting here is that we cannot use truth tables to verify these equivalences!

Instead, we need to reason about the domain of discourse as a set. For example, let's consider how negation interacts with quantifiers.

Consider the claim that "all odd numbers are prime." We might represent this symbolically as $\forall x(O(x) \to P(x))$. The statement clearly is not true, so what *is* true is that "not all odd numbers are prime" (i.e., $\neg\forall x(O(x) \to P(x))$). How do we know? Easy: 9. Yes, 9 is odd but not prime. But is it enough that just one odd number isn't prime?

To dispute a universal claim, you just need *one* single counterexample. You just need to show *there exists* a number for which the claim is false. In our case, we have the equivalence,

$$\neg\forall x(O(x) \to P(x)) \equiv \exists x(O(x) \wedge \neg P(x)).$$

If we ignore the quantifiers for a minute, we are left with

$$\neg(O \to P) \equiv O \wedge \neg P$$

which is exactly an example of Theorem 2.3.9. The new, interesting part is that when we negated the universal quantifier, we got an existential quantifier.

Negating an existential quantifier results in a universal quantifier. This makes sense. If there does not exist something with a property, then everything does not have that property.

Quantifiers and Negation.

$\neg\forall x P(x)$ is equivalent to $\exists x \neg P(x)$.

$\neg\exists x P(x)$ is equivalent to $\forall x \neg P(x)$.

Symbolically, we can pass the negation symbol over a quantifier, but that causes the quantifier to switch type.

Another way to see why this makes sense: Universal quantifiers are like (possibly infinite) conjunctions since they claim that the property is true of this thing, and that thing, and the other thing,... all things. Existential quantifiers are like (possibly infinite) disjunctions: The property is true of at least one thing, maybe this, or that, or the other, or.... De Morgan's laws tell us that when we negate a conjunction, we get a disjunction, and when we negate a disjunction, we get a conjunction. Isn't it great when everything works out as it should?

Example 2.3.11

Suppose we claim that there is no smallest number. We can translate this into symbols as

$$\neg\exists x\forall y(x \leq y).$$

("It is not true that there is a number x such that for all numbers y, x is less than or equal to y.")

However, we know how negation interacts with quantifiers: We can pass a negation over a quantifier by switching the quantifier type (between universal and existential). So the statement above should be *logically equivalent* to

$$\forall x\exists y(y < x).$$

Notice that $y < x$ is the negation of $x \leq y$. This reads, "For every number x there is a number y which is smaller than x." We see that this is another way to make our original claim.

It is important to stress that predicate logic *extends* propositional logic (much like how quantum mechanics extends classical mechanics). Everything that we learned about logical equivalence and deductions still applies. However, predicate logic allows us to analyze statements at a higher resolution, digging down into the individual propositions P, Q, etc.

To do this, we need to understand how quantifiers and connectives interact. We have already seen something about negations and quantifiers. What about the other connectives? Let's look at an example exploring how the universal quantifier and disjunctions can (or cannot) work together.

Example 2.3.12

Consider the two statements,

$$\forall x(P(x) \vee Q(x)) \qquad\qquad \forall xP(x) \vee \forall xQ(x).$$

Are these logically equivalent?

Solution. These statements are NOT logically equivalent. Intuitively, the statement on the left claims that everything is either a P-thing or a Q-thing. The statement on the right claims that either everything is a P-thing or that everything is a Q-thing. These *feel* different.

To be sure, we would like to think of predicates $P(x)$ and $Q(x)$ and some domain of discourse such that one of the statements is true and the other is false. How about we let $P(x)$ be, "x is even" and $Q(x)$ be, "x is odd." Our domain of discourse will be all integers (as that is the set of numbers for which even and odd make sense).

The statement on the left is true! Every number is either even or odd. But is every number even? No. Is every number odd? No. So the statement on the right is false (it is a *false or false*).

Interestingly, the statement on the right implies the statement on the left. That is,

$$(\forall x P(x) \vee \forall x E(x)) \rightarrow \forall x (P(x) \vee Q(x))$$

is always true.

This is similar to a tautology, although we reserve that term for necessary truths in propositional logic. A statement in predicate logic that is necessarily true gets the more prestigious designation of a **law of logic** (or sometimes **logically valid**, but that is less fun).

We can also consider how quantifiers interact with each other.

Example 2.3.13

Can you switch the order of quantifiers? For example, consider the two statements:

$$\forall x \exists y P(x, y) \qquad \text{and} \qquad \exists y \forall x P(x, y).$$

Are these logically equivalent?

Solution. These statements are NOT logically equivalent. To see this, we should provide an interpretation of the predicate $P(x, y)$ which makes one of the statements true and the other false.

Let $P(x, y)$ be the predicate $x < y$. It is true, in the natural numbers, that for all x there is some y greater than that x (since there are infinitely many numbers). However, there is no natural number y which is greater than every number x. Thus it is possible for $\forall x \exists y P(x, y)$ to be true while $\exists y \forall x P(x, y)$ is false.

We cannot do the reverse of this though. If there is some y for which every x satisfies $P(x, y)$, then certainly for every x there is some y which satisfies $P(x, y)$. The first is saying we can find one y that works for every x. The second allows different y's to work for different x's, but nothing is preventing us from using the same y that works for every x. In other words, while we don't have logical equivalence between the two statements, we do have a valid deduction rule:

$$\begin{array}{rl} & \exists y \forall x P(x, y) \\ \hline \therefore & \forall x \exists y P(x, y) \end{array}$$

Put yet another way, this says that the single statement

$$\exists y \forall x P(x, y) \rightarrow \forall x \exists y P(x, y)$$

is always true; it is a law of logic.

2.3.5 Deductions

Earlier, we claimed that the following was a valid argument:

> If Edith eats her vegetables, then she can have a cookie. Edith ate her vegetables. Therefore Edith gets a cookie.

How do we know this is valid? Let's look at the form of the statements. Let P denote, "Edith eats her vegetables" and Q denote, "Edith can have a cookie." The logical form of the argument is then:

$$\begin{array}{rc} & P \to Q \\ & P \\ \hline \therefore & Q \end{array}$$

This is an example of a **deduction rule**, an argument form that is always valid. This one is a particularly famous rule called *modus ponens*. Are you convinced that it is a valid deduction rule? If not, consider the following truth table:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

This is just the truth table for $P \to Q$, but what matters here is that all the lines in the deduction rule have their own column in the truth table. Remember that an argument is valid provided the conclusion must be true given that the premises are true. The premises in this case are $P \to Q$ and P. Which *rows* of the truth table correspond to both of these being true? P is true in the first two rows, and of those, only the first row has $P \to Q$ true as well. And lo-and-behold, in this one case, Q is also true. So if $P \to Q$ and P are both true, we see that Q must be true as well.

Think of deduction rules as a sort of *one-way* form of logical equivalence. Two statements are logically equivalent provided that in every row of the truth table in which the first statement is true, so is the second, and in every row in which the second statement is true, so is the first. A deduction only requires the first of these two parts.

Here are a few more examples.

Example 2.3.14

Show that the following is a valid deduction rule.

$$\begin{array}{rc} & P \to Q \\ & \neg P \to Q \\ \hline \therefore & Q \end{array}$$

Solution. We make a truth table which contains all the lines of the argument form:

P	Q	$P \to Q$	$\neg P$	$\neg P \to Q$
T	T	T	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	F

(we include a column for $\neg P$ just as a helping step to get the column for $\neg P \to Q$).

Now look at all the rows for which both $P \to Q$ and $\neg P \to Q$ are true. This happens only in rows 1 and 3. Hey! In those rows Q is true as well, so the argument form is valid (it is a valid deduction rule).

Example 2.3.15

Decide whether the following is a valid deduction rule.

$$\begin{array}{rc} & P \to R \\ & Q \to R \\ & R \\ \hline \therefore & P \vee Q \end{array}$$

Solution. Let's make a truth table containing all four statements.

P	Q	R	$P \to R$	$Q \to R$	$P \vee Q$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	F
F	F	F	T	T	F

Look at the second-to-last row. Here all three premises of the argument are true, but the conclusion is false. Thus this is not a valid deduction rule.

While we have the truth table in front of us, look at rows 1, 3, and 5. These are the only rows in which all of the statements $P \to R$, $Q \to R$, and $P \vee Q$ are true. It also happens that R is true in these rows as well. Thus we have discovered a new deduction rule we know *is* valid:

$$\begin{array}{rc} & P \to R \\ & Q \to R \\ & P \vee Q \\ \hline \therefore & R \end{array}$$

Quantifier deductions. There are also deduction rules we could write down for quantifiers. For example, such a rule might be:

$$\begin{array}{rl} & \forall x P(x) \\ \hline \therefore & \exists x P(x) \end{array}$$

If everything is a P-thing, then there must be something which is a P-thing.[3] These rules cannot be verified with a truth table, and a full treatment of this sort of predicate logic is beyond the scope of this text.

2.3.6 Reading Questions

1. To check whether two statements are logically equivalent, you can use a truth table. Explain what you would look for in the truth table to conclude that the two statements are logically equivalent. What would tell you they are *not* logically equivalent?

2. To check whether a deduction rule is *valid*, you can use a truth table. Explain what you would look for in the completed truth table to say that the deduction rule is valid, and what would tell you the deduction rule is *not* valid.

3. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

2.3.7 Practice Problems

1. Make a truth table for the statement $(P \wedge Q) \to (P \vee Q)$.

2. Complete a truth table for the statement $\neg Q \vee (Q \to P))$. What can you conclude about P and Q if you knew the statement above was false?

3. Construct a truth table for the statement $Q \to (\neg P \vee R)$.

4. Determine whether the statements $P \to (Q \vee R)$ and $(P \to Q) \vee (P \to R)$ are logically equivalent by completing a truth table for both statements.

5. Determine if the following is a valid deduction rule:

$$\begin{array}{rc} & P \to Q \\ & \neg Q \\ \hline \therefore & \neg P \end{array}$$

6. Determine if the following is a valid deduction rule:

$$\begin{array}{rc} & P \to (Q \vee R) \\ & \neg(P \to Q) \\ \hline \therefore & R \end{array}$$

7. Determine if the following is a valid deduction rule:

[3] Note that this does assume that your domain of discourse is non-empty.

$$\begin{array}{rc} & (P \wedge Q) \rightarrow R \\ & \neg P \vee \neg Q \\ \hline \therefore & \neg R \end{array}$$

8. Determine if the following is a valid deduction rule:

$$\begin{array}{rc} & P \rightarrow Q \\ & P \wedge \neg Q \\ \hline \therefore & R \end{array}$$

9. Which of the following statements is a *law of logic*? That is, which of the following are true no matter what your domain of discourse is and no matter what you interpret the predicates as meaning? Select all that apply.

A. $\forall x(P(x) \vee \neg P(x))$.

B. $\exists x P(x) \rightarrow \forall x P(x)$.

C. $\neg \forall x P(x) \rightarrow \exists x P(x)$.

D. $\forall x \exists y P(x, y) \leftrightarrow \exists y \forall x P(x, y)$.

2.3.8 Additional Exercises

1. You stumble upon two trolls playing Stratego®. They tell you:

Troll 1: If we are cousins, then we are both knaves.

Troll 2: We are cousins, or we are both knaves.

Could both trolls be knights? Recall that all trolls are either always-truth-telling knights or always-lying knaves. Explain your answer and how you can use truth tables to find it.

2. Next you come upon three trolls, helpfully wearing name tags. They say:

Pat	If either Quinn or I are knights, then so is Ryan.
Quinn	Ryan is a knight, and if Pat is a knight, then so am I.
Ryan	Quinn is a knave, but Pat and I share the same persuasion.

Create a truth table that includes all three statements. Then use the truth table to determine the persuasion of each troll.

3. Consider the statement about a party, "If it's your birthday or there will be cake, then there will be cake."

(a) Translate the above statement into symbols. Clearly state which statement is P and which is Q.

(b) Make a truth table for the statement.

(c) Assuming the statement is true, what (if anything) can you conclude if you know there will be cake?

(d) Assuming the statement is true, what (if anything) can you conclude if you know there will not be cake?

(e) Suppose you found out that the statement was a lie. What can you conclude?

4. Geoff Poshingten is out at a fancy pizza joint and decides to order a calzone. When the waiter asks what he would like in it, he replies, "I want either pepperoni or sausage. Also, if I have sausage, then I must also include quail. Oh, and if I have pepperoni or quail, then I must also have ricotta cheese."

(a) Translate Geoff's order into logical symbols.

(b) The waiter knows that Geoff is either a liar or a truth-teller (so either everything he says is false, or everything is true). Which is it?

(c) What, if anything, can the waiter conclude about the ingredients in Geoff's desired calzone?

5. Determine whether the following two statements are logically equivalent: $\neg(P \rightarrow Q)$ and $P \wedge \neg Q$. Explain how you know you are correct.

6. Simplify the following statements (so that negation only appears right before variables).

(a) $\neg(P \rightarrow \neg Q)$.

(b) $(\neg P \vee \neg Q) \rightarrow \neg(\neg Q \wedge R)$.

(c) $\neg((P \rightarrow \neg Q) \vee \neg(R \wedge \neg R))$.

(d) It is false that if Sam is not a man then Chris is a woman, and that Chris is not a woman.

7. Use De Morgan's Laws and any other logical equivalence facts you know to simplify the following statements. Show all your steps. Your final statements should have negations only appear directly next to the sentence variables or predicates (P, Q, $E(x)$, etc.), and no double negations. It would be a good idea to use only conjunctions, disjunctions, and negations.

(a) $\neg((\neg P \wedge Q) \vee \neg(R \vee \neg S))$.

(b) $\neg((\neg P \rightarrow \neg Q) \wedge (\neg Q \rightarrow R))$ (careful with the implications).

(c) For both parts above, verify your answers are correct using truth tables. That is, use a truth table to check that the given statement and your proposed simplification are actually logically equivalent.

8. Consider the statement, "If a number is triangular or square, then it is not prime"

(a) Make a truth table for the statement $(T \vee S) \to \neg P$.

(b) If you believed the statement was *false*, what properties would a counterexample need to possess? Explain by referencing your truth table.

(c) If the statement were true, what could you conclude about the number 5657, which is definitely prime? Again, explain using the truth table.

9. Tommy Flanagan was telling you what he ate yesterday afternoon. He tells you, "I had either popcorn or raisins. Also, if I had cucumber sandwiches, then I had soda. But I didn't drink soda or tea." Of course, you know that Tommy is the world's worst liar, and everything he says is false. What did Tommy eat?

Justify your answer by writing all of Tommy's statements using sentence variables (P, Q, R, S, T), taking their negations, and using these to deduce what Tommy actually ate.

10. Can you chain implications together? That is, if $P \to Q$ and $Q \to R$, does that means the $P \to R$? Prove that the following is a valid deduction rule:

$$\begin{array}{rl} & P \to Q \\ & Q \to R \\ \hline \therefore & P \to R \end{array}$$

11. Suppose P and Q are (possibly molecular) propositional statements. Prove that P and Q are logically equivalent if and only if $P \leftrightarrow Q$ is a tautology.

12. Suppose $P_1, P_2, \ldots, P_n$ and Q are (possibly molecular) propositional statements. Suppose further that

$$\begin{array}{rl} & P_1 \\ & P_2 \\ & \vdots \\ & P_n \\ \hline \therefore & Q \end{array}$$

is a valid deduction rule. Prove that the statement

$$(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \to Q$$

is a tautology.

13. Consider the statements below. Translate each into symbols, using the predicate $F(x, y)$ for "person x can be fooled at time y." Decide whether any of the statements are equivalent to each other, or whether any imply any others, in this context or in general.

(a) You can fool some people all of the time.

(b) You can fool everyone some of the time.

(c) You can always fool some people.

(d) Sometimes you can fool everyone.

14. Suppose $P(x)$ is some predicate for which the statement $\forall x P(x)$ is true. Is it also the case that $\exists x P(x)$ is true? In other words, is the statement $\forall x P(x) \rightarrow \exists x P(x)$ always true? Is the converse always true? Assume the domain of discourse is non-empty.

15. Simplifying negations will be especially useful when we try to prove a statement by considering what would happen if it were false. For each statement below, write the *negation* of the statement as simply as possible. Don't just say, "It is false that ..."

(a) Every number is either even or odd.

(b) There is a sequence that is both arithmetic and geometric.

(c) For all numbers n, if n is prime, then $n + 3$ is not prime.

16. We can simplify statements in predicate logic using our rules for passing negations over quantifiers before applying logical equivalence to the "inside" propositional part. Simplify the statements below (so negation appears only directly next to predicates).

(a) $\neg\exists x \forall y(\neg O(x) \vee E(y))$.

(b) $\neg\forall x \neg\forall y \neg(x < y \wedge \exists z(x < z \vee y < z))$.

(c) There is a number n for which no other number is less than or equal to n.

(d) It is false that for every number n there are two other numbers which n is between.

17. Simplify the statements below to the point that negation symbols occur only directly next to predicates.

(a) $\neg\forall x \forall y(x < y \vee y < x)$.

(b) $\neg(\exists x P(x) \rightarrow \forall y P(y))$.

2.4 Proofs

Objectives

After completing this section, you should be able to do the following.

- Identify the logical structure of, and distinguish between, *direct proofs*, a *proof by contrapositives*, and a *proof by contradictions*.
- Identify flaws in an incorrect proof and determine whether they are flaws in logic or mathematical concepts.
- Apply definitions to prove statements using basic proof styles.

2.4.1 Section Preview

Investigate!

A **mini sudoku puzzle** is a 4×4 grid of squares, divided into four 2×2 boxes. The goal is to fill each square with a digit from 1 to 4, such that no digit repeats in any row, any column, or any box.

Here is a simple mini sudoku puzzle you can try to solve.

2			1
	4		
		3	

You might notice that the solution to the above puzzle has its four outside corners all different, and its four middle squares all different.

The goal of this *Investigate!* question is to prove that this is not a coincidence: Suppose a mini sudoku puzzle has all different numbers in its four corners (marked with # below). Prove that the center four squares (marked with $*$ below) must also contain different numbers.

#			#
	$*$	$*$	
	$*$	$*$	
#			#

Try it 2.4.1

Try placing numbers into an empty mini sudoku puzzle. See if you can break the statement we were asked to prove in the *Investigate!* activity. What stops you? Briefly explain whether you think the statement is true or false, and why.

Anyone who doesn't believe there is creativity in mathematics clearly has not tried to write proofs. Finding a way to convince the world that a particular statement is necessarily true is a mighty undertaking and can often be quite challenging. There is no guaranteed path to success in the search for proofs. For example, in the summer of 1742, a German mathematician by the name of Christian Goldbach wondered whether every even integer greater than 2 could be written as the sum of two primes. Centuries later, we still don't have a proof of this apparent fact (computers have checked that Goldbach's conjecture holds for all numbers less than 4×10^{18}, but no proof that the statement holds for *all* numbers has been found).

Writing proofs is a bit of an art. Like any art, to be truly great at it, you need some sort of inspiration, as well as some foundational technique. Just as musicians can learn proper fingering, and painters can learn the proper way to hold a brush, we can look at the proper way to construct arguments.

We can view a proof through two distinct but intersecting lenses. First, we can think about the *logical* structure of a proof. Below we will consider three styles of proof that vary precisely in their logical structure. Second, we can look at the *mathematical content* of the proof. How does the proof illustrate understanding of mathematical concepts? Does it use definitions of mathematical objects correctly? How do the definitions interact with each other?

Recall that in Section 2.3 we said that a **tautology** is a necessarily true statement, but that it doesn't tell us anything interesting. Similarly, if a proof relied entirely on the logical form of the statement it was proving, it wouldn't tell us anything interesting about mathematics. Thus all the proofs we consider *must* involve some combining of mathematical concepts in addition to their logical structure.

In this section, we will see examples of how the interaction between logical and mathematical structure plays out. We will think of the logical structure as the *skeleton* or *scaffolding* of the proof and look at the different shapes this skeleton can take. We will then see how the mathematical content of the proof fills in the details of the skeleton, how it adds meat to the bones.

It is often challenging to be careful about proofs when the statements we try to prove seem too obvious or familiar. While we will definitely want to prove simple facts about numbers, like that the sum of two even numbers is even, our familiarity with numbers can make it difficult to take this task seriously. So instead, we will start by proving some facts in what is hopefully a novel setting: mini sudoku puzzles.

Preview Activity

Consider the statement:

If ab is an even number, then a or b is even.

Which of the proofs below appear to be valid proofs of this statement? Note: You can assume all the algebra below is correct (because it is).

1. Suppose a and b are odd. That is, $a = 2k + 1$ and $b = 2m + 1$ for some integers k and m. Then

$$\begin{aligned} ab &= (2k+1)(2m+1) \\ &= 4km + 2k + 2m + 1 \\ &= 2(2km + k + m) + 1. \end{aligned}$$

Therefore ab is odd.

2. Assume that a or b is even -- say it is a (the case where b is even will be identical). That is, $a = 2k$ for some integer k. Then

$$\begin{aligned} ab &= (2k)b \\ &= 2(kb). \end{aligned}$$

Thus ab is even.

3. Suppose that ab is even but a and b are both odd. Namely, $ab = 2n$, $a = 2k + 1$ and $b = 2j + 1$ for some integers n, k, and j. Then

$$\begin{aligned} 2n &= (2k+1)(2j+1) \\ 2n &= 4kj + 2k + 2j + 1 \\ n &= 2kj + k + j + \frac{1}{2}. \end{aligned}$$

But since $2kj + k + j$ is an integer, this says that the integer n is equal to a non-integer, which is impossible.

4. Let ab be an even number, say $ab = 2n$, and a be an odd number, say $a = 2k + 1$.

$$\begin{aligned} ab &= (2k+1)b \\ 2n &= 2kb + b \\ 2n - 2kb &= b \\ 2(n - kb) &= b. \end{aligned}$$

Therefore b must be even.

2.4.2 Direct Proof

The simplest style of proof is **direct proof**. Often all that is required to prove something is a systematic explanation of what everything means. You look at the definitions, carefully explain and *unpack* their meaning, until you see that the conclusion is true.

To illustrate the importance of definitions in a proof, let's give a few careful definitions about mini sudoku puzzles.

Definition 2.4.2 Mini Sudoku Definitions.

A **mini sudoku puzzle** is a partially filled in 4×4 grid of squares, divided into four 2×2 boxes. Each square can be empty or contain a digit from 1 to 4.

We say that a mini sudoku puzzle is **valid** provided no digit from 1 to 4 appears more than once in any row, any column, or any box.

A **solution** to a mini sudoku puzzle is a valid puzzle with no empty squares and every non-empty square of the puzzle unchanged.

We say that a mini sudoku puzzle is **solvable** if there is exactly one solution.

First, let's prove a useful and "obvious" fact about any valid mini sudoku puzzle.

Proposition 2.4.3

Any solution to a mini sudoku puzzle will have each digit from 1 to 4 appear exactly once in each row, in each column, and in each box, appearing a total of four times.

That's obvious, you say! Isn't that exactly what a valid puzzle is? Well, a valid completed puzzle, which is what we mean by a solution, right? Okay, not exactly, since valid means that no digit repeats... isn't that the same thing?

Yes! Exactly! Saying this is a direct proof.

Proof. Suppose you have a solution to a mini sudoku puzzle. That means that you have a 4×4 grid with each square filled in with a digit from 1 to 4 that is valid.[4] Since the puzzle is *valid*, no digit repeats in any row, any column, or any box[5]. Since a row contains four numbers that do not repeat, and there are exactly four possible digits, each of those digits must appear exactly once. This is true for every row, and for every column, and for every box.

For each digit, since it appears exactly once in four different rows, it appears exactly four times. This completes the proof.

Remark 2.4.4 The proof contained one key mathematical idea besides just explaining definitions: If four distinct numbers are chosen from a set of four numbers, then all four numbers are chosen. Perhaps you want to also explain why this is true, or just say it is an example of the pigeonhole principle (we will say what this is soon). How much you explain depends on who you are writing the proof for. A little paranoia when writing proofs is healthy.

Indeed, in most contexts, we wouldn't even need to write out any of the above proof. It would probably be sufficient to say, "Clearly this follows

[4]Definition of **solution**
[5]Definition of **valid**

from the definitions." However, we are currently trying to learn how to write proofs, and it can be useful to be overly pedantic so we can focus on the proof structure and the importance of applying definitions.

Let's prove something about a particular sudoku puzzle.

Example 2.4.5

Prove that for any solution to the mini sudoku puzzle below, if the solution has a 2 in the top-left square (r1c1), then it will contain a 2 in the bottom-right square (r4c4).

	1	3	
			1
			3

Solution. We do not know whether the puzzle is solvable (in fact, it is not), although note that it is valid right now. What we want to prove is that *if* 2 is in the top-left square in any particular solution, *then* that solution contains a 2 in the bottom-right square.

Proof. Let S be a solution to the puzzle, and assume that S contains a 2 in the top-left square. Since S is a valid puzzle, no digit repeats in any row, any column, or any box. Look first at the top row. Since the row already contains 1, 2, and 3, the remaining open square (r1c4) must be a 4.

Now look at column 4 (the right-most column). Since we now know the top-right square is a 4, this column already contains 1, 3, and 4. So the last open square (r4c4) must be a 2.

Thus S contains a 2 in the bottom-right square, which is what we needed to prove. ∎

Observe the general form of the argument above. We were trying to prove an implication $P \to Q$: If there was a 2 in r1c1, then there was a 2 in r4c4. We started by assuming P was true. From that, we deduced something, and from that something we deduced Q. This is exactly what a direct proof of an implication $P \to Q$ looks like (we could have had more steps between the P and Q as well).

Assume P. Explain, explain, . . . , explain. Therefore Q.

The one additional consideration we must make is that often we are proving a general, universal statement, a statement of the form $\forall x(P(x) \to Q(x))$. To handle the quantifier, we fix an *arbitrary* instance of x. Above, we said, "Let S be a solution to the puzzle." Since we made no additional assumptions about S besides that P

was true about it, we say that S was an *arbitrary* solution.

If we wanted to prove that all squares are rectangles, we first realize that this is the same as saying, "For any shape, if the shape is a square, then it is a rectangle." In symbols, $\forall x(S(x) \to R(x))$. We will want to assume $P(x)$ is true and deduce $Q(x)$. Which x do we use? An arbitrary one, so our proof can be applied to *all* possible x.

Example 2.4.6

Prove that for any mini sudoku puzzle with three empty squares, if the puzzle has a solution, then the puzzle is solvable.

Solution. Is this obvious? If a puzzle has a solution, then it is solvable, right? Not at all! Look again at the Mini Sudoku Definitions: Just because a puzzle has a solution doesn't mean that it has exactly one solution (i.e., it is solvable). But even if we don't think this needs a proof, let's be paranoid again and use this as an excuse to focus on the logical structure of the proof.

Notice that we are proving that the claim is true no matter what mini sudoku puzzle we start with. We might start with this puzzle:

3		4	2
2	4	3	1
1			4
4	2	1	3

Clearly there is only one way to complete the puzzle: The top row has only one open square, so we can only put one digit in it, and then columns 2 and 3 only have one open square each, so we can fill those uniquely.

Looking at a single example can often be helpful when crafting a proof, but proving a general statement with an example is ***NEVER*** a correct proof.

While there are *only* around 152.58 billion mini sudoku puzzles (and most of those are not valid, fewer have solutions, and even fewer have exactly one empty square), we don't really want to check all possible puzzles. So instead, we fix an arbitrary valid mini sudoku puzzle. We assume that it has a solution and has exactly three open squares. From this, we prove that there is only one possible solution.

Proof. Let P be an arbitrary mini sudoku puzzle. Assume P has exactly three empty squares and that S is a solution.

Since P is arbitrary, we don't know how the three empty squares are arranged. They could all be in different rows, or two could be in the same row, or all three could be in the same row.

If all the empty squares are in different rows, then in each row, there is exactly one empty square. The other three squares are filled with three different digits (since the puzzle is valid), so there is only one choice to fill the empty square. This number must be the number used in the solution S.

Now consider the case where two of the empty squares are in the same row, and the third square is in a different row. The third empty square's row has three different digits, so there is only one choice for the last square, and it must agree with S. Once this is filled in, the other two empty squares must be in two different columns, each of which has three filled-in digits. In each of these columns, the three filled-in digits are different, so there is only one choice for the empty square. So again, any solution must be exactly S.

Finally, if all three empty squares are in the same row, then they are all in different columns. So using the same argument as we did when the empty squares were in different rows, but using columns instead, we see that S is the only solution.

We have considered all possible cases, and in each case, S is the only solution, so P is solvable. ■

Direct proof can, of course, be used to prove statements in mathematics too.

Example 2.4.7

Prove: For all integers n, if n is even, then n^2 is even.

Solution. The format of the proof will be this: Let n be an arbitrary integer. Assume that n is even. Explain explain explain. Therefore n^2 is even.

To fill in the details, we explain what it means for n to be even, and then see what that means for n^2. The *definition* that is relevant here is, "An integer n is **even** if there is an integer k such that $n = 2k$." Here is a complete proof.

Proof. Let n be an arbitrary integer. Suppose n is even. Then $n = 2k$ for some integer k. Now $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Since $2k^2$ is an integer, n^2 is even. ■

Example 2.4.8

Prove: For all integers a, b, and c, if b is a multiple of a, and c is a multiple of b, then c is a multiple of a.

Solution. Even if we don't remember exactly what "is a multiple of" means, we can set up a direct proof for this statement. It will go something like this: Let a, b, and c be arbitrary integers. Assume that b is a multiple of a and that c is a multiple of b. Dot dot dot. Therefore c is a multiple of a.

How do we connect the dots? We say what our hypothesis really means and why this gives us what the conclusion really means. This is where we need the definition of b **is a multiple of** a: This means that $b = ka$ for some integers k. What are we going for? That $c = la$, for some integer l. Here is the complete proof.

Proof. Let a, b, and c be integers. Assume that b is a multiple of a and that c is a multiple of b. So there are integers k and j such that $b = ka$ and $c = jb$. Combining these (through substitution) we get that

$$c = j(ka) = (jk)a.$$

But jk is an integer, so this says that c is a multiple of a. ■

2.4.3 Proof by Contrapositive

Recall that an implication $P \to Q$ is logically equivalent to its contrapositive $\neg Q \to \neg P$. There are plenty of examples of statements that are hard to prove directly, but whose contrapositive can easily be proved using a direct proof. This is all that **proof by contrapositive** does. It gives a direct proof of the contrapositive of the implication. This is enough because the contrapositive is logically equivalent to the original implication.

The skeleton of the proof of $P \to Q$ by contrapositive will always look roughly like this:

Assume $\neg Q$. Explain, explain, ... explain. Therefore $\neg P$.

As before, if there are variables and quantifiers, we set them to be arbitrary elements of our domain.

Example 2.4.9

Prove that if a mini sudoku puzzle is solvable, then it is valid.

Solution. Remember, a puzzle is valid provided no row, column, or box contains a repeated digit. A puzzle is solvable if there is exactly one solution, where a solution is a valid puzzle with no empty squares (that doesn't change any previously filled-in square).

If we try a direct proof, we would start with an arbitrary puzzle and assume it is solvable. We could then "get" the solution, but would need to reason back in time to when the puzzle started. This seems hard. Often, when a proof seems to require breaking things apart, it is easier to try the contrapositive. That's what we will do.

Proof. Let P be an arbitrary mini sudoku puzzle and assume that it is *not* valid. This means that in at least one row, or one column, or one box, some digit appears more than once.

Now suppose we have filled in the empty squares but not changed any previously filled-in squares. The original row, column, or box that contained a duplicate digit will still contain that duplication, so the resulting completed puzzle will not be valid. Thus no solution to P exists, so P is not solvable. ■

We have proved that if a mini sudoku puzzle is not valid, then it is not solvable, which is the contrapositive of what we wanted to prove and so serves as a proof of the original statement.

Here are a couple more mathy examples.

Example 2.4.10

Is the statement, "For all integers n, if n^2 is even, then n is even," true?

Solution. This is the converse of the statement we proved in Example 2.4.7 above using a direct proof. From trying a few examples, this statement appears to be true. So let's prove it.

A direct proof of this statement would require fixing an arbitrary n and assuming that n^2 is even. But it is not at all clear how this would allow us to conclude anything about n. Just because $n^2 = 2k$ does not in itself suggest how we could write n as a multiple of 2.

Try something else: Write the contrapositive of the statement. We get, for all integers n, if n is odd, then n^2 is odd. This looks much more promising.

We need a definition of a number being odd. An integer n is **odd** provided $n = 2k + 1$ for some integer k. Our proof will look something like this:

Let n be an arbitrary integer. Suppose that n is not even. This means that .... In other words .... But this is the same as saying .... Therefore n^2 is not even.

Now we fill in the details.

Proof. We will prove the contrapositive. Let n be an arbitrary integer. Suppose that n is not even, and thus odd. Then $n = 2k + 1$ for some integer k. Now $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Since $2k^2 + 2k$ is an integer, we see that n^2 is odd and therefore not even. ■

Example 2.4.11

Prove that for all integers a and b, if $a + b$ is odd, then a is odd or b is odd.

Solution. The problem with trying a direct proof is that it will be hard to separate a and b from knowing something about $a + b$. On the other hand, if we know something about a and b separately, then combining them might give us information about $a + b$. The contrapositive of the statement we are trying to prove is: for all integers a and b, if a and b are even, then $a + b$ is even. Thus our proof will have the following format:

Let a and b be integers. Assume that a and b are both even. la la la. Therefore $a + b$ is even.

Here is a complete proof.

Proof. Let a and b be integers. Assume that a and b are even. Then $a = 2k$ and $b = 2l$ for some integers k and l. Now $a + b = 2k + 2l = 2(k + l)$. Since $k + l$ is an integer, we see that $a + b$ is even, completing the proof. ■

Note that our assumption that a and b are even is the negation of a or b is odd. We used De Morgan's law here.

Direct proofs and proofs by contrapositive can be used when proving *implications*. Remember that some statements that are not explicitly written as an implication can be rephrased as one.

Example 2.4.12

Consider the statement, "For every prime number p, either $p = 2$, or p is odd." We can rephrase this as, "For every prime number p, if $p \neq 2$, then p is odd." Now try to prove it.

Use the following as the definition of a prime number: An integer $p > 1$ is **prime** provided it has exactly two factors, namely 1 and p.

Solution.

Proof. Let p be an arbitrary prime number. Assume p is not odd. So p is divisible by 2. Since p is prime, it must have exactly two divisors, and it has 2 as a divisor, so p must be divisible by only 1 and 2. Therefore $p = 2$. This completes the proof (by contrapositive). ■

2.4.4 Proof by Contradiction

Take a step back and consider what it would mean if the conclusion of an argument were false. There are two reasons this could happen. Either the logic in the argument is faulty, or at least one assumption must be false. A **proof by contradiction** exploits this second possibility. We start with a single assumption and construct a *valid* proof that leads to a false conclusion; the only possibility is that the single assumption was false. The false conclusion is the "contradiction," which just means a necessarily false statement (technically a statement of the form $P \wedge \neg P$).

The general form of a proof by contradiction to prove a statement P is,

> Assume $\neg P$ (that P is not true). This means that..., which tells us..., so we can say... But that is a contradiction, so P must in fact be true.

Note that if we think of this style of argument as a direct proof of something, it is a direct proof of

$$\neg P \rightarrow \text{contradiction}.$$

So we have a valid proof of this implication. How can a true implication have a false conclusion? Recall the truth table for an implication (Figure 2.2.2). The only row in which the implication is true but the conclusion is false is row 4, and here the hypothesis is also false. So $\neg P$ is false, which is to say P is true.

Once you start writing proofs by contradiction, it becomes very natural. Let's see how with a sudoku proof.

Example 2.4.13

Prove that any solution to the mini sudoku puzzle below must contain a 3 in the top-right corner (r1c4).

	2		
	3		
3			
			1

Solution. There are multiple ways we could prove this, which is often the case for proofs by contradiction. While we would always start with the same initial assumption (the opposite of what we want to prove), where the contradiction is found can vary.

Proof. Let S be a solution to the puzzle, but assume that S does *not* contain a 3 in the top-right corner. Then the top row must contain a 3 somewhere else. It cannot be in column 1, since it already has a 3. It cannot be in column 2, since that square is already filled in (with the 2). It also cannot be in column 3, since if it was, there would be no place to put a three in the bottom row. So there would be no 3 in the top row, contradicting that S is a solution.

Therefore S must contain a 3 in the top-right corner. ■

Example 2.4.14

Prove that there is no solution to the sudoku puzzle below.

4			
	1		
			4

Solution. Look at the logical format of this statement:

$$\neg\exists S(S \text{ is a solution }).$$

Using the rules for negation of quantifiers, this is the same as

$$\forall S(S \text{ is not a solution}).$$

If we were to prove this directly, we would need to consider all possible solutions and show that they are not valid. That seems quite challenging.

On the other hand, if we try a proof by contradiction, we get to assume the negation of the statement we are asked to prove. That is, we can assume that there *is* a solution. That is a single solution we can reason about. Much more manageable.

Proof. Suppose, for the sake of contradiction, that there *does* exist a solution to the puzzle. This solution must have a 4 in the bottom-left box. But the only squares that can hold a 4 are either in row 4 or column 1 (or both). In either case, this contradicts that there is a 4 already in column 1 and row 4. ■

Here are three examples of proofs by contradiction about numbers:

Example 2.4.15

Prove that $\sqrt{2}$ is irrational.

Solution.

Proof. Suppose not. Then $\sqrt{2}$ is equal to a fraction $\frac{a}{b}$. Without loss of generality, assume $\frac{a}{b}$ is in lowest terms (otherwise reduce the fraction). So,

$$2 = \frac{a^2}{b^2}$$

$$2b^2 = a^2.$$

Thus a^2 is even, and as such a is even. So $a = 2k$ for some integer k, and $a^2 = 4k^2$. We then have,

$$2b^2 = 4k^2$$

$$b^2 = 2k^2.$$

Thus b^2 is even, and as such b is even. Since a is also even, we see that $\frac{a}{b}$ is not in lowest terms, a contradiction. Thus $\sqrt{2}$ is irrational. ■

Example 2.4.16

Prove: There are no integers x and y such that $x^2 = 4y + 2$.

Solution.

Proof. We proceed by contradiction. So suppose there *are* integers x and y such that $x^2 = 4y + 2 = 2(2y + 1)$. So x^2 is even. We have seen that this implies that x is even. So $x = 2k$ for some integer k. Then $x^2 = 4k^2$. This in turn gives $2k^2 = (2y + 1)$. But $2k^2$ is even, and $2y + 1$ is odd, so these cannot be equal. Thus we have a contradiction, so there must not be any integers x and y such that $x^2 = 4y + 2$. ■

Example 2.4.17

The **Pigeonhole Principle**: If more than n pigeons fly into n pigeonholes, then at least one pigeonhole will contain at least two pigeons. Prove this!

Solution.

Proof. Suppose, contrary to stipulation, that each of the pigeonholes contains at most one pigeon. Then at most, there will be n pigeons. But we assumed that there are more than n pigeons, so this is impossible. Thus there must be a pigeonhole with more than one pigeon. ■

While we phrased this proof as a proof by contradiction, we could have also used a proof by contrapositive since our contradiction was simply the negation of the hypothesis. Sometimes this will happen, in which case you can use either style of proof. There are examples, however, where the contradiction occurs "far away" from the original statement.

2.4.5 Summary of Proof Styles

We have considered three styles of proof: direct proof, proof by contrapositive, and proof by contradiction. It can be challenging to decide which style of proof to use on a given problem, and no rule will always tell us what to do. Often, there are multiple ways you can proceed in a proof, which is one reason math is so exciting.

A good starting point when writing proofs is to consider what the initial assumption would be with each style, and what the conclusion you would be looking for is. A proof is a little like a kids-menu maze. There is a *START* and an *EXIT*, and your goal is to find your way from one to the other. Sometimes it helps to work your way in from both sides and hopefully meet in the middle.

Starts and Ends Proofs.

To prove an implication $P \to Q$:

Direct	Start: Assume P. End: Therefore Q.
Contrapositive	Start: Assume $\neg Q$. End: Therefore $\neg P$.
Contradiction	Start: Assume $\neg(P \to Q)$. End: ...which is a contradiction.

You can use a proof by contradiction even if you are not trying to prove an implication, but if the statement is an implication, then assuming $\neg(P \to Q)$ is really powerful. Remember, the only way for an implication to be false is for P to be true and Q to be false. So we are actually assuming $P \wedge \neg Q$. Aha! P is what we assume in a direct proof. $\neg Q$ is what we assume in a proof by contrapositive. So a *proof by* *contradiction is like doing the other two proofs at the same time, and meeting in the middle!*

To illustrate this, let's prove the fact from the *Investigate!* activity. We will prove that if a mini sudoku puzzle has all different numbers in its corners (marked with a through d below), then the center four squares (marked with $*$ below) must also contain different numbers (in any solution).

a			b
	$*$	$*$	
	$*$	$*$	
c			d

We will give three proofs, first a direct proof, then a proof by contrapositive, and finally a proof by contradiction.

Proof. Let P be a sudoku puzzle with all different numbers in its corners: a in the top-left, b in the top-right, c in the bottom-left, and d in the bottom-right. Let S be any solution to the puzzle.

In S, whatever digit a is must appear in row 2. Since a is already in the top-left box, it cannot appear in columns 1 or 2, so it must appear in columns 3 or 4. If it is in column 3, then a appears in one of the center squares. If not, then it is in column 4. In this latter case, we ask where a appears in row 3: It cannot be in column 1 or 4, so it must be in column 2 or 3, and thus in one of the center squares.

The same argument can now be applied to each of the other three outer squares. Thus, in any solution to the puzzle, the center four squares must contain the digits a through d, all different.

Now we will prove the same statement by contrapositive.

Proof. Let P be a mini sudoku puzzle with numbers in its corners. Let S be any solution to the puzzle. Assume that the center four squares do not contain all different numbers in S. Then there must be some common number, say n, in two of the center squares.

Since n cannot appear twice in a row, it must be that n is in two center squares diagonally from each other: either in r2c2 and r3c3, or in r2c3 and r3c2. In either case, where could n be in rows 1 and 4? It cannot be in columns 2 or 3, so it must be in columns 1 and 4. This means that n will be in opposite outer corners, meaning that the digits in the four corners are not all different.

Finally, we will prove the same statement by contradiction.

Proof. Let P be a mini sudoku puzzle and assume that the four outer corners are all different, but in some solution S, the center four squares are not all different.

Suppose a, the digit in the top-left corner, appears twice in the center four squares. The only way this can happen is if a appears in r3c2 and r2c3, as shown below.

a			b
	$*$	a	
	a	$*$	
c			d

But now, where can the fourth a in the go in the solution? It must be in r4c4, contradicting that the four corners are all different. An analogous argument leads to a contradiction for each of the other possible outer corner digits being repeated in the center. Thus, the center four squares must contain all different numbers.

2.4.6 Reading Questions

1. Which of the following would be the best first line of a *direct proof* if you wanted to prove the statement, "For all sets A of single-digit numbers, if $|A| = 6$, then A contains an even number."

A. Suppose there exists a set A of single-digit numbers with $|A| = 6$ but that contains only odd numbers.

B. Fix an arbitrary set A of single-digit numbers, and assume $|A| = 6$.

C. Suppose A is a set of single-digit numbers with $|A| \neq 6$.

D. Let A be a set of single-digit numbers that contains an even number.

E. Let A be a set of single-digit numbers, and assume that A does not contain any even numbers.

2. Which of the following would be the best first line of a *proof by contrapositive* if you wanted to prove the statement, "For all sets A of single-digit numbers, if $|A| = 6$, then A contains an even number."

A. Suppose there exists a set A of single-digit numbers with $|A| = 6$ but that contains only odd numbers.

B. Fix an arbitrary set A of single-digit numbers, and assume $|A| = 6$.

C. Suppose A is a set of single-digit numbers with $|A| \neq 6$.

D. Let A be a set of single-digit numbers that contains an even number.

E. Let A be a set of single-digit numbers, and assume that A does not contain any even numbers.

3. Which of the following would be the best first line of a *proof by contradiction* if you wanted to prove the statement, "For all sets A of single-digit numbers, if

$|A| = 6$, then A contains an even number."

A. Suppose there exists a set A of single-digit numbers with $|A| = 6$ but that contains only odd numbers.

B. Fix an arbitrary set A of single-digit numbers, and assume $|A| = 6$.

C. Suppose A is a set of single-digit numbers with $|A| \neq 6$.

D. Let A be a set of single-digit numbers that contains an even number.

E. Let A be a set of single-digit numbers, and assume that A does not contain any even numbers.

4. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

2.4.7 Practice Problems

1. Arrange some of the statements below to form a correct proof of the following statement: "For any integer n, if n is even, then $7n$ is even."

- Let n be an arbitrary integer, and assume $7n$ is even.
- Let n be an arbitrary integer, and assume $7n$ is odd.
- Since 7 is odd and the product of an odd number and an odd number is odd,
- Since an even number divided by 7 must be odd,
- n must be even.
- $7n$ must be odd.
- Let n be an arbitrary integer, and assume n is even.
- Since the product of any number with an even number is even,
- $7n$ must be even.

2. Arrange some of the statements below to form a correct proof of the following statement: "For any integer n, if $7n$ is even, then n is even."

- Let n be an arbitrary integer, and assume n is even.
- Since the 7 is odd and the product of an odd number with an even number is even,
- $7n$ must be even.
- Let n be an arbitrary integer, and assume $7n$ is even.

- Since an even number divided by 7 must be even,
- n must be even.
- Let n be an arbitrary integer, and assume n is odd.
- Since 7 is odd and the product of an odd number and an odd number is odd,
- $7n$ must be odd.

3. Consider the statement, "For any numbers a and b, if $a + b$ is odd, then either a or b is odd".

Give a valid proof of the statement using a *proof by contrapositive*. Arrange some statements below to complete the proof.

- Let a and b be integers, and assume that $a + b$ is odd.
- Let a and b be integers, and assume that if $a + b$ is odd, then either a or b is odd.
- Let a and b be integers, and assume both are even.
- The sum of two even integers must also be even.
- Therefore $a + b$ is even.
- Let a and b be integers and assume that $a + b$ is odd but a and b are both even.
- The sum of two odd integers must be even.
- But then $a + b$ is both even and odd, a contradiction.

4. Consider the same statement, "For any numbers a and b, if $a + b$ is odd, then either a or b is odd."

Give a valid proof of the statement, this time using a *proof by contradiction* using some of the statements below.

- Let a and b be integers, and assume that $a + b$ is odd.
- Let a and b be integers, and assume that if $a + b$ is odd, then either a or b is odd.
- Let a and b be integers, and assume both are even.
- Let a and b be integers, and assume that $a + b$ is odd but a and b are both even.
- Therefore $a + b$ is even.
- The sum of two even integers must also be even.

- The sum of two odd integers must be even.
- But then $a + b$ is both even and odd, a contradiction.

5. Below are three statements together with a possible first line of a proof of that statement. In each case, say whether the first line is the start of a direct proof, a proof by contrapositive, or a proof by contradiction.

(a) ***Statement:*** For every integer n, the number $7n - 1$ is divisible by 6.

First line: Suppose there were some integer n for which $7n - 1$ was not divisible by 6.

(b) ***Statement:*** For any integer n, if n is prime, then n is solitary

First line: Let n be an integer, and assume n is not solitary.

(c) ***Statement:*** If a shape is a pentagon, then its interior angles add up to 480 degrees.

First line: Consider an arbitrary shape, and assume it is a pentagon.

6. What would the first line be for a proof in each style, of the following statement: "If a function $f : A \to B$ is a bijection, then $|A| = |B|$."

Assume $f : A \to B$ is a bijection	Direct proof
Assume $f : A \to B$ is a bijection and $\lvert A\rvert \neq \lvert B\rvert$	Proof by contrapositive
Assume $\lvert A\rvert \neq \lvert B\rvert$	Proof by contradiction

2.4.8 Additional Exercises

1. For a given predicate $P(x)$, you might believe that the statements $\forall x P(x)$ or $\exists x P(x)$ are either true or false. How would you decide if you were correct in each case? You have four choices: You could give an example of an element n in the domain for which $P(n)$ is true or for which $P(n)$ is false, or you could argue that no matter what n is, $P(n)$ is true or is false.

(a) What would you need to do to prove $\forall x P(x)$ is true?

(b) What would you need to do to prove $\forall x P(x)$ is false?

(c) What would you need to do to prove $\exists x P(x)$ is true?

(d) What would you need to do to prove $\exists x P(x)$ is false?

2. Consider the statement, "For all integers a and b, if $a + b$ is even, then a and b are even."

(a) Write the contrapositive of the statement.

(b) Write the converse of the statement.

(c) Write the negation of the statement.

(d) Is the original statement true or false? Prove your answer.

(e) Is the contrapositive of the original statement true or false? Prove your answer.

(f) Is the converse of the original statement true or false? Prove your answer.

(g) Is the negation of the original statement true or false? Prove your answer.

3. For each of the statements below, say what method of proof you should use to prove them. Then say how the proof starts and how it ends. Bonus points for filling in the middle.

(a) There are no integers x and y such that x is a prime greater than 5 and $x = 6y + 3$.

(b) For all integers n, if n is a multiple of 3, then n can be written as the sum of consecutive integers.

(c) For all integers a and b, if $a^2 + b^2$ is odd, then a or b is odd.

4. Consider the statement, "For all integers n, if n is even then $8n$ is even."

(a) Prove the statement. What sort of proof are you using?

(b) Is the converse true? Prove or disprove.

5. The game TENZI comes with 40 six-sided dice (each numbered 1 to 6). Suppose you roll all 40 dice.

(a) Prove that there will be at least seven dice that land on the same number.

(b) How many dice would you have to roll before you were guaranteed that some four of them would all match or all be different? Prove your answer.

6. Prove that for all integers n, it is the case that n is even if and only if $3n$ is even. That is, prove both implications: If n is even, then $3n$ is even, and if $3n$ is even, then n is even.

7. Prove that $\sqrt{3}$ is irrational.

8. Consider the statement, "For all integers a and b, if a is even and b is a multiple of 3, then ab is a multiple of 6."

(a) Prove the statement. What sort of proof are you using?

(b) State the converse. Is it true? Prove or disprove.

9. Prove the statement, "For all integers n, if $5n$ is odd, then n is odd." Clearly state the style of proof you are using.

10. Prove the statement, "For all integers a, b, and c, if $a^2 + b^2 = c^2$, then a or b is even."

11. Suppose that you would like to prove the following implication:

For all numbers n, if n is prime then n is solitary.

Write out the beginning and end of the argument if you were to prove the statement,

(a) Directly

(b) By contrapositive

(c) By contradiction

You do not need to provide the middle parts of the proofs (since you do not know what solitary means). However, make sure that you give the first few and last few lines of the proofs so that we can see the logical structure you would follow.

12. Suppose you have a collection of rare 5-cent stamps and 8-cent stamps. You desperately need to mail a letter and, having no other stamps available, decide to dip into your collection. The question is, what amounts of postage can you make?

(a) Prove that if you only use an even number of both types of stamps, the amount of postage you make must be even.

(b) Suppose you made an even amount of postage. Prove that you used an even number of at least one of the types of stamps.

(c) Suppose you made exactly 72 cents of postage. Prove that you used at least 6 of at least one type of stamp.

13. Prove: $x = y$ if and only if $xy = \dfrac{(x+y)^2}{4}$. Note, you will need to prove two "directions" here, the "if" and the "only if" part.

14. Prove that $\log(7)$ is irrational.

15. Prove that there are no integer solutions to the equation $x^2 = 4y + 3$.

16. Prove that every prime number greater than 3 is either one more or one less than a multiple of 6.

17. Your "friend" has shown you a "proof" he wrote to show that $1 = 3$. Here is the proof:

Proof. I claim that $1 = 3$. Of course we can do anything to one side of an equation as long as we also do it to the other side. So subtract 2 from both sides. This gives $-1 = 1$. Now square both sides, to get $1 = 1$. And we all agree this is true.

What is going on here? Is your friend's argument valid? Is the argument a proof of the claim $1 = 3$? Carefully explain using what we know about logic.

18. A standard deck of 52 cards consists of 4 suits (hearts, diamonds, spades, and clubs) each containing 13 different values (Ace, 2, 3, ..., 10, J, Q, K). If you draw some number of cards at random, you might or might not have a pair (two cards with the same value) or three cards all of the same suit. However, if you draw enough cards, you will be guaranteed to have these. For each of the following, find the smallest number of cards you would need to draw to be guaranteed having the specified cards. Prove your answers.

(a) Three of a kind (for example, three 7's).

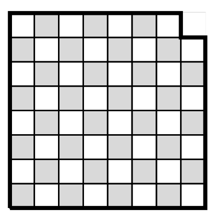
(b) A flush of five cards (for example, five hearts).

(c) Three cards that are either all the same suit or all different suits.

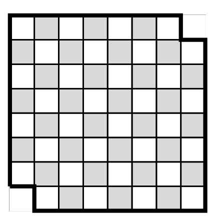
19. Suppose you are at a party with 19 of your closest friends (so including you, there are 20 people there). Explain why there must be at least two people at the party who are friends with the same number of people at the party. Assume friendship is always reciprocated.

20. Your friend has given you his list of 115 best Doctor Who episodes (in order of greatness). It turns out that you have seen 60 of them. Prove that there are at least two episodes you have seen that are exactly four episodes apart on your friend's list.

21. Suppose you have an $n \times n$ chessboard, but your dog has eaten one of the corner squares. You have dominoes that each cover exactly two squares of the board. Can you cover the remaining squares on the board with non-overlapping dominoes? What needs to be true about n? Give necessary and sufficient conditions (that is, say exactly which values of n work and which do not work). Prove your answers.



22. What if your $n \times n$ chessboard is missing two opposite corners? Prove that no matter what n is, you will not be able to cover the remaining squares with non-overlapping dominoes.



2.5 Proofs about Discrete Structures

Objectives

After completing this section, you should be able to do the following.

- Read and comprehend definitions related to discrete structures, so you can apply the definitions correctly.
- Write proofs about discrete structures.

2.5.1 Section Preview

Investigate!

Suppose there are 15 people at a party. Most people know each other already, but there are still some people who decide to shake hands. Is it possible for everyone at the party to shake hands with exactly three other people?

So far we have seen how the logical form of a statement can inform how to build the scaffolding of a proof. This can only get us so far though: To flesh out the proof skeleton requires an understanding of the mathematical objects and structures the proofs are about. Some of this can come from carefully reading definitions. Yet there is also some less concrete understanding and intuition that comes from working with the objects and structures that can lead to that "ah-ha!" moment of inspiration that suggests how to proceed with a proof.

> **By the way...** Why are we writing proofs? Besides practice in becoming better reasoners, diving into careful proofs about discrete structures is a way to learn more about the structures themselves. They are a playground for exploring mathematics, to help us build intuition for mathematical structures. So we study structures to help us write proofs about them, and we write proofs about them to help understand the structures. Bootstrapping!

Another reason to shift our focus toward proofs about discrete structures is that doing so illustrates an important feature of mathematics: abstraction. We have been proving particular facts about particular problems. We might even start to notice similarities between the proofs for some statements. This might be due to the underlying mathematical structures that the problems are (secretly) about. If we prove the general facts about these structures, then we can apply these "theorems" to many different problems.

Some discrete structures lend themselves to particular styles of proof and some "standard" proof techniques can apply to particular structures. We will see some of this here, but mostly we take this opportunity to remind ourselves of some of the basic definitions and properties for discrete structures, and use the proofs about them to help understand these better.

PREVIEW ACTIVITY

In this preview activity, we will explore some basic properties of sets and functions. Later in this section, we will write proofs about these ideas.

1. Remember that a set is just a collection of elements. Here are two definitions about sets:

a. A set A is a subset of a set B, written $A \subseteq B$, provided every element in A is also an element of B.

b. Given sets A and B, the union of A and B, written $A \cup B$, is the set containing every element that is in A or B or both.

Let's build some examples.

(a) Let $B = \{1, 3, 5, 7, 9\}$. Give an example of a set A containing 3 elements that is a subset of B.

What is $A \cup B$ for the set A you gave as an example?

(b) Give an example of two distinct sets A and B such that $A \cup B = B$.

For the example you gave, is $A \subseteq B$?

(c) Find examples, if they exist, of sets A and B such that $A \cup B \neq B$.

For the example you gave, is $A \subseteq B$?

2. Which of the following are always true?

A. For any sets A and B, $A \cup B \subseteq B$.

B. For any sets A and B, $B \subseteq A \cup B$.

C. For any sets A and B, if $A \subseteq B$, then $A \cup B \subseteq B$.

D. For any sets A and B, if $A \cup B = B$, then $A \subseteq B$.

3. For any function $f : \mathbb{N} \to \mathbb{N}$ and any set $A \subseteq \mathbb{N}$, we can define the image of A under f to be the set of all outputs of f when the input is an element of A. We write this as $f(A) = \{f(x) : x \in A\}$.

For the following tasks, let's explore the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x^2 - 3x + 8$.

(a) Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$. Find $f(A)$ and $f(B)$. Then find $f(A) \cup f(B)$.

(b) Now find $A \cup B$ and $f(A \cup B)$.

(c) Give an example, if one exists, of two distinct sets A and B such that $A \subseteq B$ and $f(A) \subseteq f(B)$.

Give an example, if one exists, of two distinct sets A and B such that $A \subseteq B$ but $f(A) \nsubseteq f(B)$.

2.5.2 Proofs about Sets

Recall that a set is an unordered collection of elements. We can describe a set by listing these elements, or by specifying a property that all elements in the set satisfy. For example,

$$A = \{1, 2, 3, 4, 5\},$$

or

$$B = \{x \in \mathbb{N} : x < 10\}.$$

The second set here is the set of natural numbers $(0, 1, 2, \ldots)$ less than 10. Notice that every element in A is also an element of B. Here is a definition that captures that idea.

Definition 2.5.1

A set A is a **subset** of a set B, written $A \subseteq B$, provided every element of A is also an element of B.

The set B is sometimes called a **superset** of A.

We say A is a **proper subset** of B, written $A \subset B$, provided $A \subseteq B$ and $A \neq B$. In other words, if every element in A is an element in B, and there is at least one element in B that is *not* in A.

Example 2.5.2

Let $A = \{x \in \mathbb{N} : x < 5\}$ and $B = \{x \in \mathbb{N} : x^2 < 10\}$. Is $B \subseteq A$? Is B a *proper* subset of A?

Solution. We are asking whether every natural number less than 5 is also a natural number whose square is less than 10. Okay, we could just write out the elements of the sets: $A = \{0, 1, 2, 3, 4\}$ and $B = \{0, 1, 2, 3\}$ (since $3^2 = 9$ and $4^2 = 16$). So $B \subseteq A$. But $B \neq A$, so in fact $B \subset A$.

The sets in the example above were small, and it is easy enough to write down the elements of the sets. However, we can also prove subset relationships between sets if this isn't practical or even possible (perhaps the sets are infinite). Let's look carefully at how we could have reasoned about the example above.

We claimed that every element of B was also an element of A. Another way to say this: For all numbers n, *if* n is an element of B, *then* n is also an element of A. Recognizing this as a conditional statement, we can proceed to give a direct, contrapositive, or contradiction proof of the fact. Here a direct proof would be perfectly acceptable. Let's try it:

Proof. Let n be an element of the set B. Then $n^2 < 10$, by the definition of B. Since $4^2 = 16$, we must have that $n < 4$. By the definition of A, and the fact that $4 < 5$, we see that $n \in A$.

To be clear, this proof is *way* more than we would normally do for this example, but its format should be illuminating. Proving that one set is a subset of another is really the same as proving an implication!

To give an example of how we can apply the definition of "subset" in a more general setting, let's prove a basic fact about subsets.

Proposition 2.5.3

For any sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof. We will give a direct proof. Let A, B, and C be sets, and assume that $A \subseteq B$ and $B \subseteq C$. We will prove that $A \subseteq C$.

Let x be an element of A. Since $A \subseteq B$, we know that $x \in B$. Since $B \subseteq C$, we know that $x \in C$. Therefore, $A \subseteq C$.

Now let's prove a fact about numbers: Every multiple of 9100 is also a multiple of 13. We could factor 9100, but here is an easier way. The set of multiples of 9100 is a subset of the set of multiples of 91. And the set of multiples of 91 is a subset of the set of multiples of 13. Now apply the proposition above.

The proof of Proposition 2.5.3 is what is sometimes called an **element chasing** proof. By the definition of subset, $A \subseteq B$ means every element of A is an element of B, or equivalently, for all x, if x is an element of A, then x is an element of B. One way to prove this is to "chase" the element x from A to B.

Example 2.5.4

Prove that if $A \subseteq B$, then $A \cup B \subseteq B$. Recall that $A \cup B$ is the **union** of sets A and B, and contains all elements that are in A or B or both.

Solution. We will write a direct proof. So we will assume that $A \subseteq B$ and prove that $A \cup B \subseteq B$. Our desired conclusion is a statement about subsets, so let's do an element chasing proof for it.

Proof. Let A and B be sets and assume $A \subseteq B$. Now let x be an element in $A \cup B$. This means that x is an element of A, or x is an element of B, or both.[6]

Consider the cases. If x is an element of A, then since $A \subseteq B$, we know that x is an element of B. On the other hand, if x is not an element of A, then x must be an element of B (since x is in $A \cup B$). In either case, x is an element of B. Therefore, $A \cup B \subseteq B$. ■

We can actually prove a strong statement: $A \subseteq B$ if and only if $A \cup B = B$. You are asked to do this in the exercises.

2.5.3 Proofs about Functions

A function $f : A \to B$ is a rule that assigns each element of the set A (the domain) to exactly one element of the set B (the codomain). It is any rule: There doesn't have to be a formula or rationale for it; we just need to match up elements from A to elements in B. For example, we could let A be the set of students enrolled in a particular Discrete Math course and let B be the set of months of the year. Now define the function $f : A \to B$ to be the rule that assigns to each student the month in which their birthday falls. Since every student has an assigned month, and no more than one month, this is a function.

Here is a definition of a particular type of function.

Definition 2.5.5

A function $f : A \to B$ is **injective** (or **one-to-one**) provided every element in B is the image of at most one element in A. In other words, no element in B is the *output* for more than one *input* from A.

In the example below, we use *two-line notation* to describe a function. The top row contains the inputs, and the bottom row lists the corresponding outputs. So $f : \{1, 2, 3, 4\} \to \{a, b, c, d\}$ might be defined as,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \end{pmatrix},$$

which means that $f(1) = a$, $f(2) = b$, $f(3) = c$, and $f(4) = d$.

Example 2.5.6

Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6, 8\}$. Consider the functions $f : A \to B$ and $g : A \to B$ defined by,

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 8 & 6 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 4 \end{pmatrix}.$$

Which of these functions is injective?

Solution. The function f is injective: Each element of B is the image of at most one element of A. The function g is not injective: The element 4 in B is the image of both 1 and 3 in A.

Consider the student-to-birth-month function again. Could this possibly be injective? Or put another way, must there be two students in the course with the same birth month (which would say the function is *not* injective)? The answer seems to depend on how many students are in the class.

[6]From the definition of **union**.

But let's pause and think about the more general fact about functions we have here. Let's prove the following fact. Recall that $|A|$ denotes the **cardinality** (size) of the set A: the number of elements in A.

Proposition 2.5.7

Suppose $f : A \to B$ is a function with A and B both finite sets. If $|A| > |B|$, then f is not *injective.*

Proof. We will give a proof of the contrapositive: If f is injective, then $|A| \leq |B|$. Let $|B| = n$. Since f is injective, each element of B must be the output for *at most* one element of A. Thus there are at most n elements in A that get mapped to B by f. But the definition of a function requires that every element of the domain is mapped to exactly one element of the codomain, so there must be at most n elements in A.

Does this proof remind you of our pigeonhole-like proofs? It should, since this is precisely (one of) the careful formulations of the pigeonhole principle. We could have proved the fact about students sharing a birth month, but now we can just apply the proposition above and be done. When you apply a theorem or proposition to directly prove another result, we call the latter result a **corollary**.

Corollary 2.5.8

Suppose a class has 25 students. Then at least two students share the same birth month.

Proof. Consider the function that maps each student to their birth month. Since the domain has 25 elements and the codomain has 12 elements, the function is not injective, by Proposition 2.5.7. Therefore, at least two students share the same birth month.

Functions always have inputs from a *set* (called the **domain**) and outputs in a set as well (called the **codomain**). This naturally leads to facts to consider about the interaction between sets and functions.

Definition 2.5.9

Given a function $f : X \to Y$ and a set $A \subseteq X$, we define the **image of A under f** to be the set $f(A) = \{f(a) \in Y : a \in A\}$. That is, $f(A)$ is the set of all outputs of the function for inputs in A.

Example 2.5.10

Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = 2n$. Let $A = \{1, 2, 3\}$. Find $f(A)$.

Solution. Evaluate each element of A by f.

$$f(1) = 2; \qquad f(2) = 4; \qquad f(3) = 6.$$

We want the set of these outputs. So $f(A) = \{2, 4, 6\}$.

Now let's prove something.

Proposition 2.5.11

Let $f : X \to Y$ be a function, and let A and B be subsets of X. If $A \subseteq B$, then $f(A) \subseteq f(B)$.

Proof. Let f, A, and B be as in the proposition. Assume that $A \subseteq B$. Now consider an element $y \in f(A)$. By definition, this means that there is some $a \in A$ such that $f(a) = y$. Since $a \in A$ and $A \subseteq B$, we have that $a \in B$. Then by definition,

$$y = f(a) \in f(B).$$

Since y was an arbitrary element of $f(A)$, we have proved that $f(A) \subseteq f(B)$.

Notice that the proof above is an element chasing proof again. This makes sense as soon as you remember that $f(A)$ and $f(B)$ are just the names of sets. To prove one set is a subset of another, we chase elements from the subset to the superset.

2.5.4 Proofs about Relations

A **relation** on a set A is a set of ordered pairs of elements from A. We can think of a relation as a way to describe a type of relationship between elements of A. For example, we might have a relation on the set of people at a party that describes who is friends with whom. We might have a relation on the set of natural numbers that describes which pairs of numbers are related by the relation $x < y$.

Relations permeate all of mathematics, often without us even thinking of them. Whenever we make a statement about two elements of a set, we are implicitly defining a relation. The statement is true when the pair is in the relation. For example, the statement, "3 is less than 5," is true because the pair $(3, 5)$ is in the relation $<$. In fact, using the language we developed in the subsection Quantifiers and Predicates, we can say that a relation is just a predicate, where the variables come from the same set.

Often relations have special symbols like "$=$" or "$\leq$" or "$\perp$". When we talk about a general relation, we will either use $\sim$ and write $x \sim y$, or use a capital letter like R, and write $R(x, y)$ or xRy or even $(x, y) \in R$ (these all mean the same thing).

When we study relations, we try to identify properties that relations might have. Here is an example of a very common property.

Definition 2.5.12

A relation R on a set A is **transitive** provided for all $x, y, z \in A$, if xRy and yRz, then xRz.

Example 2.5.13

Consider the relation $\sim$ on the set of students in your Discrete Math course that holds of two students, provided they have some other class together. Is this relation transitive?

Solution. No, not necessarily (although for some sets of students it could be). For example, suppose Alice has another class with Bruce, say Introduction to Programming. Carlos is not in Intro to Programming, but he and Bruce are both in Organic Chemistry. So then Alice~Bruce and Bruce~Carlos, but it might not be the case that Alice~Carlos (since Alice need not be in Organic Chemistry with Carlos).

Proving that a relation is *not* transitive takes nothing more than finding a counterexample, which means finding three elements a, b, and c such that $a \sim b$, $b \sim c$, but $a \nsim c$ (remember, the only way for an implication to be false is for the hypothesis to be true and the conclusion to be false).

Perhaps slightly more interesting would be proof that a relation is transitive.

Example 2.5.14

Consider the set of all students in your Discrete Mathematics class, and define the relation $\sim$ that holds of students a and b (so $a \sim b$ is true), provided a is taller than b. Prove that this relation is transitive.

Solution. The definition of transitive is an implication, so we can try a direct proof.

Proof. Let a, b, and c be arbitrary students in your Discrete Math course. Assume, $a \sim b$ and $b \sim c$. That means that a is taller than b, and that b is taller than c. But then surely a must be even taller than c than they are taller than b, so we have that $a \sim c$ is true as well. Thus $\sim$ is transitive on this set. ■

2.5.5 Proofs about Graphs

We will spend all of `[cross-reference to target(s) "ch_graphtheory" missing or not unique]` studying proofs about graphs since this is such a rich area of mathematics. As a preview, here is an example of how graph proofs can go.

A **graph** is a set V of **vertices** and a set E of **edges**. The edges are two-element subsets of the vertices, and we can think of them as representing relationships between the vertices. Note, this is an abstract definition of a graph using sets, but we often draw graphs using dots for the vertices connected by lines for the edges, as this gives us a nice picture of what is going on.

Since graphs represent a type of relationship between elements (vertices), we can use graphs to represent many real-world problems. For example, the vertices of a graph might represent people at a party. Each edge can represent a handshake between two people. So if we wondered whether it is possible for the 15 people at a party to each shake hands with exactly 3 people there, we are really asking whether there is a graph with 15 vertices where each vertex belongs to 3 edges. ("Belongs to"?? Yes, because an edge is a two-element subset of the vertices, so if an edge "touches" or "comes out of" a vertex, that means the vertex belongs to that particular two-element subset.)

Here is a definition related to this idea.

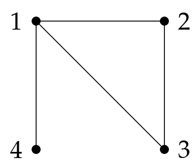
Definition 2.5.15

Let v be a vertex in a graph G. The **degree** of v, written $d(v)$, is the number of edges that contain v, i.e., the number of edges **incident** to v.

Example 2.5.16

Consider the graph G with vertices $V = \{1, 2, 3, 4\}$ and edges $E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}\}$ What is the degree of each vertex in G?

Solution. It might be helpful to picture the graph:



We have $d(1) = 3$, $d(2) = 2$, $d(3) = 2$, and $d(4) = 1$. You can see this by counting how many edges are incident to each vertex, or by counting how many edges (subsets) each vertex belongs to.

So is it possible for 15 people to each shake hands with exactly three people in their group? Well, is there a graph with 15 vertices, all of degree 3? The answer is no!

One way you can see this is if you ask how many edges such a graph would have. Each vertex is incident to three edges, so counting incidences, we get $15 \cdot 3 = 45$. But every edge is incident to two vertices, so we have counted each edge twice. So the number of edges in such a graph would be $45/2 = 22.5$. But the number of edges in a graph must be a whole number, so there is no such graph.

This suggests that we can say something more in general. The following proposition is a simple consequence of the `[cross-reference to target(s) "lem-handshake" missing or not unique]`, which we will prove in `[cross-reference to target(s) "sec_gt-intro" missing or not unique]`. Here we give a complete proof of this particular formulation of it.

Proposition 2.5.17

In any graph, the number of vertices with odd degree must be even.

Proof. We will prove this by contradiction. Suppose there was a graph with an odd number of vertices with odd degree. Consider the sum of the degrees of all the vertices. The sum for the odd-degree vertices would be odd (since the sum of an odd number of odd numbers is odd). The sum of the even-degree vertices will be even (any sum of even numbers must be even). The sum of an odd number and an even number is odd. Thus the sum of all the degrees will be odd.

However, the number of edges in a graph is half the sum of the degrees (by `[cross-reference to target(s) "lem-handshake" missing or not unique]`, or simply because each edge contributes one to the count of the degree of two vertices). Since the number of edges is a whole number, we see that the sum of the degrees must be even. This contradicts what we found in the previous paragraph.

Therefore, in any graph, the number of vertices with odd degree must be even.

2.5.6 Reading Questions

1. Which of the following is the definition of a function $f : A \to B$ being injective?

A. Every element of B is the image of at most one element of A.

B. The domain A is a larger set than the codomain B.

C. Every element of A is sent to at most one element of B.

D. The codomain B is no smaller than the domain A.

2. When would you most likely use element chasing as part of a proof?

A. When proving that one set is a subset of another.

B. When proving that a function is injective.

C. When proving that a relation is transitive.

D. When proving that a graph has an odd number of edges.

3. What questions do you have after reading this section? Write at least one question about the content of this section that you are curious about.

2.5.7 Practice Problems

1. Given sets A and B, the **intersection** of A and B, written $A \cap B$, is the set of all elements that are in both A and B.

Suppose you wanted to prove that if $A \cap B = B$ then $B \subseteq A$.

Which would be a good start to this proof if you used a direct proof?

A. Let a be an element of $A \cap B$.

B. Let b be an element of B.

C. Let a be an element of A.

D. Suppose there is an element b in B that is not in A.

2. Suppose you wanted to prove that for all sets A and B that $A \cap B \subseteq A$. Which of the following would be a good start to a proof by contradiction?

A. Suppose there is an element a in A that is not in $A \cap B$.

B. Suppose there is an element a in $A \cap B$ that is not in A.

C. Let a be an element of A.

D. Let a be an element of $A \cap B$.

3. Arrange some of the statements below to form a correct proof of the following statement: "For any sets A and B, if $B \subseteq A \cap B$ then $B \subseteq A$."

- Therefore $B \subseteq A \cap B$
- Since $A \cap B$ contains all the elements that are in both A and B, b is an element of A.
- Then b is an element of $A \cap B$ since $B \subseteq A \cap B$.
- Let b be an element of $A \cap B$.
- Suppose $B \subseteq A$.
- Suppose $B \subseteq A \cap B$, and let b be an element of B.
- Then b is an element of B since $B \subseteq A \cap B$.
- Therefore $B \subseteq A$.
- Suppose $A \subseteq B$.

4. Prove that for any sets A and B, $(A \cap B) \cup A = A$
Arrange the statements below to form a correct proof.

- So in particular, x is an element of A.
- Second, we will prove that $A \subseteq (A \cap B) \cup A$.
- Let x be an element of $(A \cap B) \cup A$.
- Therefore $A \subseteq (A \cap B) \cup A$.
- First we will prove that $(A \cap B) \cup A \subseteq A$.

- Therefore $(A \cap B) \cup A \subseteq A$.
- Since $(A \cap B) \cup A \subseteq A$ and $A \subseteq (A \cap B) \cup A$, we have $(A \cap B) \cup A = A$.
- Then x is an element of $(A \cap B) \cup A$, since x is in A or in the other set.
- Then x is an element of $A \cap B$, or x is an element of A.
- Let x be an element of A.

5. Let $f : X \to Y$ be a function and let $B \subseteq Y$ be a subset of the codomain. Define the **inverse image** of B under f to be the set $f^{-1}(B) = \{x \in X : f(x) \in B\}$. That is, it is all the elements in the domain that are mapped to elements in B.

Prove that if $B_1 \subseteq B_2$ are subsets of the codomain, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$. Arrange some of the statements below to form a correct proof.

- Thus $B_1 \subseteq B_2$.
- This means that $f(a)$ is an element of B_1.
- Therefore b is an element of B_2.
- Therefore $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.
- Suppose $B_1 \subseteq B_2$.
- Since $B_1 \subseteq B_2$, $f(a)$ is an element of B_2.
- This then means that a is an element of $f^{-1}(B_2)$.
- Let b be an element of B_1.
- Let a be an element of $f^{-1}(B_1)$.

2.5.8 Additional Exercises

1. Prove that for any two sets A and B, $A \subseteq B$ if and only if $A \cup B = B$.

2. The **intersection** of sets A and B, denoted $A \cap B$, is the set of all elements that are in both A and B.

Prove that for any two sets A and B, $A \subseteq B$ if and only if $A \cap B = A$.

3. Prove that for any sets A, B, and C, if $A \cup B \subseteq C$, then $A \subseteq C$ and $B \subseteq C$.

4. Prove that for any sets A, B, and C, if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

5. The **difference** of sets A and B, written $A \setminus B$, is the set of all elements that are in A but not in B.

The **empty set**, written $\emptyset$, is the set that contains no elements.

Prove that if $A \setminus B = A$ then $A \cap B = \emptyset$.

6. Prove that if $A \setminus B = B \setminus A$ then $A = B$.

7. Let $f : X \to Y$ be a function, and let A and B be subsets of X.

(a) Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$.

(b) Find an example of a function and two sets A and B such that $f(A \cap B) \neq f(A) \cap f(B)$.

8. Let $f : X \to Y$ be a function, and let A and B be subsets of X.

(a) Prove that $f(A \cup B) \subseteq f(A) \cup f(B)$.

(b) Prove that $f(A) \cup f(B) \subseteq f(A \cup B)$

(c) What can you conclude from the two proofs above?

9. Given a function $f : X \to Y$ and a set $B \subseteq Y$, we define the **inverse image** of B under f as the set $f^{-1}(B) = \{x \in X : f(x) \in B$. That is, it is all the elements in the domain that are mapped to elements in B.

(a) For $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n^2$, what are each of the following sets?

(a) $f^{-1}(\{1, 4, 9\})$
(b) $f^{-1}(\{2, 3, 5, 7\})$
(c) $f^{-1}(\{1, 2, \ldots, 10\})$

(b) Prove that for any set $C \subseteq X$, $C \subseteq f^{-1}(f(C))$.

(c) Give an example of a function f and a set C such that $C \neq f^{-1}(f(C))$.

(d) Prove that for any set $D \subseteq Y$, $f(f^{-1}(D)) \subseteq D$.

(e) Give an example of a function f and a set D such that $f(f^{-1}(D)) \neq D$.

10. Let $f : X \to Y$ be a function, and let A and B be subsets of Y. Prove that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

11. Let $f : X \to Y$ be a function, and let A and B be subsets of Y. Prove that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

12. For each relation below, determine whether it is transitive. If it is, prove it. If it is not, give a counterexample.

(a) The relation "$|$" (divides) on $\mathbb{Z}$ defined by $a|b$ provided b is a multiple of a.

(b) The relation "$\leq$" (less than or equal to) on $\mathbb{R}$.

(c) The relation "$\perp$" (is perpendicular to) on the set of lines in the plane.

(d) The relation "$\sim$" (is similar to) on the set of triangles in the plane (two triangles are similar if they have the same angles, but are not necessarily the same size).

2.6 Chapter Summary

We have considered logic both as its own sub-discipline of mathematics and as a means to help us better understand and write proofs. We noticed that mathematical statements have a particular logical form, and analyzing that form can help make sense of the statement.

At the most basic level, a statement might combine simpler statements using *logical connectives*. We often make use of variables and *quantify* over those variables. How to resolve the truth or falsity of a statement based on these connectives and quantifiers is what logic is all about. From this, we can decide whether two statements are logically equivalent or if one or more statements (logically) imply another.

When writing proofs (in any area of mathematics) our goal is to explain why a mathematical statement is true. Thus it is vital that our argument implies the truth of the statement. To be sure of this, we first must know what it means for the statement to be true, as well as ensure that the statements that make up the proof correctly imply the conclusion. A firm understanding of logic is required to check whether a proof is correct.

There is, however, another reason that understanding logic can be helpful. Understanding the logical structure of a statement often gives clues for how to write a proof of the statement.

This is not to say that writing proofs is always straight-forward. Consider again the *Goldbach conjecture*:

> Every even number greater than 2 can be written as the sum of two primes.

We are not going to try to prove the statement here, but we can at least say what a proof might look like, based on the logical form of the statement. Perhaps we should write the statement to highlight the quantifiers and connectives:

> For all integers n, if n is even and greater than 2, then there exist integers p and q such that p and q are prime, and $n = p + q$.

What would a direct proof look like? Since the statement starts with a universal quantifier, we would start, ``Let n be an arbitrary integer." The rest of the statement is an implication. In a direct proof we assume the "if" part, so the next line would be, "Assume n is greater than 2 and is even." I have no idea what comes next, but eventually, we would need to find two prime numbers p and q (depending on n) and explain how we know that $n = p + q$.

Or maybe we try a proof by contradiction. To do this, we first assume the negation of the statement we want to prove. What is the negation? From what we have studied we should be able to see that it is,

> There is an integer n such that n is even and greater than 2, but for all integers p and q, either p or q is not prime, or $n \neq p + q$.

Could this statement be true? A proof by contradiction would start by assuming it was and eventually conclude with a contradiction, proving that our assumption of truth was incorrect. And if you can find such a contradiction, you will have proved one of the most famous open problems in mathematics. Good luck.

Chapter Review

1. Complete a truth table for the statement $\neg P \to (Q \wedge R)$.

2. Suppose you know that the statement "if Peter is not tall, then Quincy is fat and Robert is skinny" is false. What, if anything, can you conclude about Peter and Robert if you know that Quincy is indeed fat? Explain (you may reference Question 2.6.1).

3. Are the statements $P \to (Q \vee R)$ and $(P \to Q) \vee (P \to R)$ logically equivalent? Explain your answer.

4. Is the following a valid deduction rule? Explain.

$$\begin{array}{rl} & P \to Q \\ & P \to R \\ \hline \therefore & P \to (Q \wedge R). \end{array}$$

5. Write the negation, converse and contrapositive for each of the statements below.

(a) If the power goes off, then the food will spoil.

(b) If the door is closed, then the light is off.

(c) $\forall x(x < 1 \to x^2 < 1)$.

(d) For all natural numbers n, if n is prime, then n is solitary.

(e) For all functions f, if f is differentiable, then f is continuous.

(f) For all integers a and b, if $a \cdot b$ is even, then a and b are even.

(g) For every integer x and every integer y, there is an integer n such that if $x > 0$ then $nx > y$.

(h) For all real numbers x and y, if $xy = 0$ then $x = 0$ or $y = 0$.

(i) For every student in Math 228, if they do not understand implications, then they will fail the exam.

6. Consider the statement, "For all integers n, if n is even and $n \leq 7$, then n is negative or $n \in \{0, 2, 4, 6\}$."

(a) Is the statement true? Explain why.

(b) Write the negation of the statement. Is it true? Explain.

(c) State the contrapositive of the statement. Is it true? Explain.

(d) State the converse of the statement. Is it true? Explain.

7. Consider the statement: $\forall x(\forall y(x + y = y) \rightarrow \forall z(x \cdot z = 0))$.

(a) Explain what the statement says in words. Is this statement true? Be sure to state what you are taking the universe of discourse to be.

(b) Write the converse of the statement, both in words and in symbols. Is the converse true?

(c) Write the contrapositive of the statement, both in words and in symbols. Is the contrapositive true?

(d) Write the negation of the statement, both in words and in symbols. Is the negation true?

8. Simplify the following.

(a) $\neg(\neg(P \wedge \neg Q) \rightarrow \neg(\neg R \vee \neg(P \rightarrow R)))$.

(b) $\neg\exists x \neg\forall y \neg\exists z(z = x + y \rightarrow \exists w(x - y = w))$.

9. Consider the statement, "For all integers n, if n is odd, then $7n$ is odd."

(a) Prove the statement. What sort of proof are you using?

(b) Prove the converse. What sort of proof are you using?

10. Suppose you break your piggy bank and scoop up a handful of 22 coins (pennies, nickels, dimes, and quarters).

(a) Prove that you must have at least 6 coins of a single denomination.

(b) Suppose you have an odd number of pennies. Prove that you must have an odd number of at least one of the other types of coins.

(c) How many coins would you need to scoop up to be sure that you either had 4 coins that were all the same or 4 coins that were all different? Prove your answer.

11. You come across four trolls playing bridge. They declare:

Troll 1: All trolls here see at least one knave.

Troll 2: I see at least one troll that sees only knaves.

Troll 3: Some trolls are scared of goats.

Troll 4: All trolls are scared of goats.

Are there any trolls that are not scared of goats? Recall, of course, that all trolls are either knights (who always tell the truth) or knaves (who always lie).

Selected Hints

2 · Logic and Proofs
2.1 · Mathematical Statements
2.1.6 · Additional Exercises

2.1.6.3. First figure out what each statement is saying. For part (c), you don't need to assume the domain is an infinite set.

2.2 · Implications
2.2.6 · Additional Exercises

2.2.6.4. Of course there are many answers. It helps to assume that the statement is true and the converse is *not* true. Think about what that means in the real world, and then start saying it in different ways. Some ideas: Use "necessary and sufficient" language, use "only if," consider negations, use "or else" language.

2.3 · Rules of Logic
2.3.8 · Additional Exercises

2.3.8.1. You could probably reason through the cases by hand, but try making a truth table. Use two statements, P being "we are cousins" and Q being "we are both knaves".

2.3.8.4. You should write down three statements using the symbols P, Q, R, S. If Geoff is a truth-teller, then all three statements would be true. If he was a liar, then all three statements would be false. But in either case, we don't yet know whether the four atomic statements are true or false, since he hasn't said them by themselves.

A truth table might help, although it is probably not entirely necessary.

2.3.8.8.

(a) There will be three rows in which the statement is false.

(b) Consider the three rows that evaluate to false, and say what the truth values of T, S, and P are there.

(c) You are looking for a row in which P is true and the whole statement is true.

2.3.8.9. Write down three statements, and then take the negation of each (since he is a liar). You should find that Tommy ate one item and drank one item. (Q is for cucumber sandwiches.)

2.3.8.11. What do these concepts mean in terms of truth tables?

2.3.8.14. Try an example. What if $P(x)$ was the predicate, "x is prime"? What if it was, "If x is divisible by 4, then it is even"? Of course examples are not enough to prove something in general, but that is entirely the point of this question.

2.3.8.15. It might help to translate the statements into symbols and then use the formulaic rules to simplify negations (i.e., rules for quantifiers and De Morgan's laws). After simplifying, you should get $\forall x(\neg E(x) \wedge \neg O(x))$ for the first one, for example. Then translate this back into English.

2.4 · Proofs

2.4.8 · Additional Exercises

2.4.8.6. One of the implications will be a direct proof; the other will be a proof by contrapositive.

2.4.8.7. This is really an exercise in modifying the proof that $\sqrt{2}$ is irrational. There you proved things were even; here they will be multiples of 3.

2.4.8.8. Part (a) should be a relatively easy direct proof. Look for a counterexample for part (b).

2.4.8.10. A proof by contradiction would be reasonable here, because then you get to assume that both *a and b* are odd. Deduce that c^2 is even, and therefore a multiple of 4 (why? and why is that a contradiction?).

2.4.8.12. Use a different style of proof for each part.

2.4.8.14. Note that if $\log(7) = \frac{a}{b}$, then $7 = 10^{\frac{a}{b}}$. Can any power of 7 be the same as a power of 10?

2.4.8.15. What if there were? Deduce that x must be odd, and continue towards a contradiction.

2.4.8.16. Prove the contrapositive by cases. There will be 4 cases to consider.

2.4.8.17. Your friend's proof is a proof, but of what? What implication follows from the given proof? Is that helpful?

2.4.8.19. Consider the set of *numbers* of friends that everyone has. If everyone had a different number of friends, this set must contain 20 elements. Is that possible? Why not?

2.4.8.20. This feels like the pigeonhole principle, although a bit more complicated. At least, you could try to replicate the style of proof used by the pigeonhole principle. How would the episodes need to be spaced out so that no two of your sixty were exactly 4 apart?

2.5 · Proofs about Discrete Structures

2.5.8 · Additional Exercises

2.5.8.1. To prove that $A \subseteq B$ if and only if $A \cup B = B$, you need to prove two implications:

(a) If $A \subseteq B$, then $A \cup B = B$.

(b) If $A \cup B = B$, then $A \subseteq B$.

To prove two sets are equal, we usually prove that each is a subset of the other.

SELECTED SOLUTIONS

2 · Logic and Proofs
2.1 · Mathematical Statements
2.1.5 · Practice Problems

2.1.5.4.

- $\exists x \forall y P(x, y)$
  - Some people can be fooled all of the time.
- $\forall x \exists y P(x, y)$
  - Everyone can be fooled sometimes.
- $\forall y \exists x P(x, y)$
  - It is always true that some people can be fooled.
- $\exists y \forall x P(x, y)$
  - Sometimes everyone can be fooled.

2.1.5.5.

A. *Correct.*

B. *Incorrect.*

C. *Incorrect.*

D. *Incorrect.*

2.1.5.6.

A. *Correct.*

B. *Incorrect.*

Careful, $P(x, y)$ means x is less than y, not x is less than *or equal* to y.

C. *Incorrect.*

D. *Incorrect.*

E. *Correct.*

2.1.5.7.

(a) $P(15)$ is true, since $17 \cdot 15 + 1 = 256$ is even.

(b) Since $P(15)$ is true, we know that $\exists x P(x)$ is true. There is at least one x for which $P(x)$ is true.

(c) Just because one value of x makes $P(x)$ true does not mean that all values of x make $P(x)$ true. But it could be. So we cannot conclude that $\forall x P(x)$ is true or false.

2.1.5.8.

(a) $P(15)$ is false, since $18 \cdot 15 + 1 = 271$ is odd.

(b) Since $P(15)$ is false, we do not know whether $\exists x P(x)$ is true. There could be some other value of x for which $P(x)$ is true.

(c) We know that there is some value of x makes $P(x)$ false so we know that $\forall x P(x)$ is false.

2.1.5.9.

A. *Incorrect.*

B. *Incorrect.*

C. *Correct.*

D. *Correct.*

2.1.6 · Additional Exercises

2.1.6.1.

(a) $P \wedge Q$.

(b) $P \to \neg Q$.

(c) Jack passed math or Jill passed math (or both).

(d) If Jack and Jill did not both pass math, then Jill did.

(e) i. Nothing else.

  ii. Jack did not pass math either.

2.1.6.2.

(a) $\neg \exists x (E(x) \wedge O(x))$.

(b) $\forall x (E(x) \to O(x+1))$.

(c) $\exists x (P(x) \wedge E(x))$ (where $P(x)$ means "x is prime").

(d) $\forall x \forall y \exists z (x < z < y \vee y < z < x)$.

(e) $\forall x \neg \exists y (x < y < x + 1)$.

2.2 · Implications
2.2.5 · Practice Problems

2.2.5.1. The main thing to realize is that we do not know the colors of these two shapes, but we do know that we are in one of three cases: We could have a purple circle and orange pentagon. We could have a circle that was not purple but a orange pentagon. Or we could have a circle that was not purple and a pentagon that was not orange. The case in which the circle is purple but the pentagon is not orange cannot occur, as that would make the statement false.

2.2.5.2. The only way for an implication $P \to Q$ to be true but its converse to be false is for Q to be true and P to be false. Thus we know that circle is purple and that square is not yellow.

2.2.5.3. The converse is "If I will give you a cow, then you will give me magic beans." The contrapositive is "If I will not give you a cow, then you will not give me magic beans." All the other statements are neither the converse nor contrapositive.

2.2.6 · Additional Exercises

2.2.6.1.

(a) Any even number plus 2 is an even number.

(b) For any x there is a y such that $\sin(x) = y$. In other words, every number x is in the domain of sine.

(c) For every y there is an x such that $\sin(x) = y$. In other words, every number y is in the range of sine (which is false).

(d) For any numbers, if the cubes of two numbers are equal, then the numbers are equal.

2.2.6.3.

(a) If you have lost weight, then you exercised.

(b) If you exercise, then you will lose weight.

(c) If you are American, then you are patriotic.

(d) If you are patriotic, then you are American.

(e) If a number is rational, then it is real.

(f) If a number is not even, then it is prime. (Or the contrapositive: If a number is not prime, then it is even.)

(g) If the Broncos don't win the Super Bowl, then they didn't play in the Super Bowl. Alternatively, if the Broncos play in the Super Bowl, then they will win the Super Bowl.

2.2.6.5. It is true that in order for a function to be differentiable at a point c, it is necessary for the function to be continuous at c. However, it is not necessary that a function be differentiable at c for it to be continuous at c.

It is true that to be continuous at a point c, it is sufficient that the function be differentiable at c. However, it is not the case that being continuous at c is sufficient for a function to be differentiable at c.

2.3 · Rules of Logic

2.3.7 · Practice Problems

2.3.7.1. If you think about what this statement is saying, it makes sense that it is a tautology (that it is true in every case). The complete truth table is:

P	Q	$P \wedge Q$	$P \vee Q$	$(P \wedge Q) \to (P \vee Q))$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

2.3.7.2. The truth table is:

P	Q	$\neg Q$	$Q \to P$	$\neg Q \vee (Q \to P))$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	F
F	F	T	T	T

If this statement is false, we must be in the third row, making P false and Q true.

2.3.7.3. The complete truth table is:

P	Q	R	$\neg P$	$\neg P \vee R$	$Q \to (\neg P \vee R)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

2.3.7.4. The complete truth table is:

P	Q	R	$P \to (Q \vee R)$	$(P \to Q) \vee (P \to R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

Since the two columns are identical, the statements are logically equivalent.

2.3.7.5. The complete truth table is:

P	Q	$P \to Q$	$\neg Q$	$\neg P$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

There is only one row in which both premises are true (row 4). In this row, the conclusion is also true. Thus the deduction rule is valid.

2.3.7.6. The complete truth table is:

P	Q	R	$P \to (Q \vee R)$	$\neg(P \to Q)$
T	T	T	T	F
T	T	F	T	F
T	F	T	T	T
T	F	F	F	T
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	T	F

There is only one row in which both premises are true (row 3). In this row, the conclusion is also true, so the deduction rule is valid.

2.3.7.7. The complete truth table is:

P	Q	R	$(P \wedge Q) \to R$	$\neg P \vee \neg Q$	$\neg R$
T	T	T	T	F	F
T	T	F	F	F	T
T	F	T	T	T	F
T	F	F	T	T	T
F	T	T	T	T	F
F	T	F	T	T	T
F	F	T	T	T	F
F	F	F	T	T	T

In rows 3, 5 and 7 both of the premises are true, but the conclusion is false. Thus the deduction rule is not valid.

2.3.7.8. The complete truth table is:

P	Q	R	$P \to Q$	$P \wedge \neg Q$
T	T	T	T	F
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	T	F

There is no row in which both premises are true (indeed, these are contradictory premises; the second is the negation of the first). Thus every row in which both premises are true (i.e., no row), the conclusion is also true. Therefore the deduction rule is valid. (This is an example of how everything follows from a contradiction.)

2.3.7.9.

A. *Correct.*

B. *Incorrect.*

C. *Incorrect.*

D. *Incorrect.*

2.3.8 · Additional Exercises

2.3.8.3.

(a) P: It's your birthday; Q: There will be cake. $(P \vee Q) \to Q$

(b) Hint: You should get three T's and one F.

(c) Only that there will be cake.

(d) It's NOT your birthday!

(e) It's your birthday, but the cake is a lie.

2.3.8.5. Make a truth table for each and compare. The statements are logically equivalent.

2.3.8.6.

(a) $P \wedge Q$.

(b) $(\neg P \vee \neg R) \rightarrow (Q \vee \neg R)$ or, replacing the implication with a disjunction first: $(P \wedge Q) \vee (Q \vee \neg R)$.

(c) $(P \wedge Q) \wedge (R \wedge \neg R)$. This is necessarily false, so it is also equivalent to $P \wedge \neg P$.

(d) Either Sam is a woman and Chris is a man, or Chris is a woman.

2.3.8.16.

(a) $\forall x \exists y (O(x) \wedge \neg E(y))$.

(b) $\exists x \forall y (x \geq y \vee \forall z (x \geq z \wedge y \geq z))$.

(c) There is a number n for which every other number is strictly greater than n.

(d) There is a number n which is not between any other two numbers.

2.4 · Proofs

2.4.7 · Practice Problems

2.4.7.1.

- Let n be an arbitrary integer, and assume n is even.
- Since the product of any number with an even number is even,
- $7n$ must be even.

2.4.7.2.

- Let n be an arbitrary integer, and assume n is odd.
- Since 7 is odd and the product of an odd number and an odd number is odd,
- $7n$ must be odd.

2.4.7.3.

- Let a and b be integers, and assume both are even.
- The sum of two even integers must also be even.
- Therefore $a + b$ is even.

2.4.7.4.

- Let a and b be integers, and assume that $a + b$ is odd but a and b are both even.
- The sum of two even integers must also be even.

- But then $a + b$ is both even and odd, a contradiction.

2.4.7.6.

- Direct proof
    - Assume $f : A \to B$ is a bijection
- Proof by contrapositive
    - Assume $|A| \neq |B|$
- Proof by contradiction
    - Assume $f : A \to B$ is a bijection and $|A| \neq |B|$

2.4.8 · Additional Exercises

2.4.8.1.

(a) The claim that $\forall x P(x)$ means that $P(n)$ is true no matter what n you consider in the domain of discourse. Thus the only way to prove that $\forall x P(x)$ is true is to check or otherwise argue that $P(n)$ is true for all n in the domain.

(b) To prove $\forall x P(x)$ is false all you need is one example of an element in the domain for which $P(n)$ is false. This is often called a **counterexample**.

(c) We are simply claiming that there is some element n in the domain of discourse for which $P(n)$ is true. If you can find one such element, you have verified the claim.

(d) Here we are claiming that no element we find will make $P(n)$ true. The only way to be sure of this is to verify that *every* element of the domain makes $P(n)$ false. Note that the level of proof needed for this statement is the same as to prove that $\forall x P(x)$ is true.

2.4.8.2.

(a) For all integers a and b, if a or b is not even, then $a + b$ is not even.

(b) For all integers a and b, if a and b are even, then $a + b$ is even.

(c) There are numbers a and b such that $a + b$ is even but a and b are not both even.

(d) False. For example, $a = 3$ and $b = 5$. $a + b = 8$, but neither a nor b is even.

(e) False, since it is equivalent to the original statement.

(f) True. Let a and b be integers. Assume both are even. Then $a = 2k$ and $b = 2j$ for some integers k and j. But then $a + b = 2k + 2j = 2(k + j)$, which is even.

(g) True, since the statement is false.

2.4.8.3.

(a) Proof by contradiction. Start of proof: Assume, for the sake of contradiction, that there are integers x and y such that x is a prime greater than 5 and $x = 6y + 3$. End of proof: ... this is a contradiction, so there are no such integers.

(b) Direct proof. Start of proof: Let n be an integer. Assume n is a multiple of 3. End of proof: Therefore n can be written as the sum of consecutive integers.

(c) Proof by contrapositive. Start of proof: Let a and b be integers. Assume that a and b are even. End of proof: Therefore $a^2 + b^2$ is even.

2.4.8.4.

(a) Direct proof.

Proof. Let n be an integer. Assume n is even. Then $n = 2k$ for some integer k. Thus $8n = 16k = 2(8k)$. Therefore $8n$ is even. ■

(b) The converse is false. That is, there is an integer n such that $8n$ is even but n is odd. For example, consider $n = 3$. Then $8n = 24$ which is even, but $n = 3$ is odd.

2.4.8.5.

(a) This is an example of the pigeonhole principle. We can prove it by contrapositive.

Proof. Suppose that each number only came up 6 or fewer times. So there are at most six 1's, six 2's, and so on. That's a total of 36 dice, so you must not have rolled all 40 dice. ■

(b) We can have 9 dice without any four matching or any four being all different: three 1's, three 2's, three 3's. We will prove that whenever you roll 10 dice, you will always get four matching or all being different.

Proof. Suppose you roll 10 dice, but that there are NOT four matching rolls. This means that at most there are three of any given value. If we only had three different values, that would be only 9 dice, so there must be 4 different values, giving 4 dice that are all different. ■

2.5 · Proofs about Discrete Structures

2.5.7 · Practice Problems

2.5.7.1.

A. *Incorrect.*

B. *Correct.*

C. *Incorrect.*

D. *Incorrect.*

This would be a good start to a proof by contradiction or contrapositive, not a direct proof.

2.5.7.2.

A. *Incorrect.*

B. *Correct.*

C. *Incorrect.*

D. *Incorrect.*

This would be a good start to a direct proof, not a proof by contradiction.

2.5.7.3.

- Suppose $B \subseteq A \cap B$, and let b be an element of B.
- Then b is an element of $A \cap B$ since $B \subseteq A \cap B$.
- Since $A \cap B$ contains all the elements that are in both A and B, b is an element of A.
- Therefore $B \subseteq A$.

2.5.7.4.

- First we will prove that $(A \cap B) \cup A \subseteq A$.
- Let x be an element of $(A \cap B) \cup A$.
- Then x is an element of $A \cap B$, or x is an element of A.
- So in particular, x is an element of A.
- Therefore $(A \cap B) \cup A \subseteq A$.
- Second, we will prove that $A \subseteq (A \cap B) \cup A$.
- Let x be an element of A.
- Then x is an element of $(A \cap B) \cup A$, since x is in A or in the other set.
- Therefore $A \subseteq (A \cap B) \cup A$.
- Since $(A \cap B) \cup A \subseteq A$ and $A \subseteq (A \cap B) \cup A$, we have $(A \cap B) \cup A = A$.

2.5.7.5.

- Suppose $B_1 \subseteq B_2$.
- Let a be an element of $f^{-1}(B_1)$.
- This means that $f(a)$ is an element of B_1.

- Since $B_1 \subseteq B_2$, $f(a)$ is an element of B_2.
- This then means that a is an element of $f^{-1}(B_2)$.
- Therefore $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

2.6 · Chapter Summary

· Chapter Review

2.6.1.

P	Q	R	$\neg P \to (Q \wedge R)$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

2.6.2. Peter is not tall, and Robert is not skinny. You must be in row 6 in the truth table above.

2.6.3. Yes. To see this, make a truth table for each statement and compare.

2.6.4. Make a truth table that includes all three statements in the argument:

P	Q	R	$P \to Q$	$P \to R$	$P \to (Q \wedge R)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

Notice that in every row for which both $P \to Q$ and $P \to R$ is true, so is $P \to (Q \wedge R)$. Therefore, whenever the premises of the argument are true, so is the conclusion. In other words, the deduction rule is valid.

2.6.5.

(a) Negation: The power goes off, and the food does not spoil.

Converse: If the food spoils, then the power went off.

Contrapositive: If the food does not spoil, then the power did not go off.

(b) Negation: The door is closed, and the light is on.

Converse: If the light is off, then the door is closed.

Contrapositive: If the light is on, then the door is open.

(c) Negation: $\exists x(x < 1 \wedge x^2 \geq 1)$

Converse: $\forall x(x^2 < 1 \rightarrow x < 1)$

Contrapositive: $\forall x(x^2 \geq 1 \rightarrow x \geq 1)$.

(d) Negation: There is a natural number n which is prime but not solitary.

Converse: For all natural numbers n, if n is solitary, then n is prime.

Contrapositive: For all natural numbers n, if n is not solitary, then n is not prime.

(e) Negation: There is a function which is differentiable and not continuous.

Converse: For all functions f, if f is continuous, then f is differentiable.

Contrapositive: For all functions f, if f is not continuous then f is not differentiable.

(f) Negation: There are integers a and b for which $a \cdot b$ is even but a or b is odd.

Converse: For all integers a and b, if a and b are even, then ab is even.

Contrapositive: For all integers a and b, if a or b is odd, then ab is odd.

(g) Negation: There are integers x and y such that for every integer n, $x > 0$ and $nx \leq y$.

Converse: For every integer x and every integer y there is an integer n such that if $nx > y$, then $x > 0$.

Contrapositive: For every integer x and every integer y there is an integer n such that if $nx \leq y$, then $x \leq 0$.

(h) Negation: There are real numbers x and y such that $xy = 0$, but $x \neq 0$ and $y \neq 0$.

Converse: For all real numbers x and y, if $x = 0$ or $y = 0$, then $xy = 0$

Contrapositive: For all real numbers x and y, if $x \neq 0$ and $y \neq 0$, then $xy \neq 0$.

(i) Negation: There is at least one student in Math 228 who does not understand implications but will still pass the exam.

Converse: For every student in Math 228, if they fail the exam, then they did not understand implications.

Contrapositive: For every student in Math 228, if they pass the exam, then they understood implications.

2.6.6.

(a) The statement is true. If n is an even integer less than or equal to 7, then the only way it could not be negative is if n was equal to 0, 2, 4, or 6.

(b) There is an integer n such that n is even and $n \leq 7$, but n is not negative and $n \notin \{0, 2, 4, 6\}$. This is false, since the original statement is true.

(c) For all integers n, if n is not negative and $n \notin \{0, 2, 4, 6\}$, then n is odd or $n > 7$. This is true, since the contrapositive is equivalent to the original statement (which is true).

(d) For all integers n, if n is negative or $n \in \{0, 2, 4, 6\}$, then n is even and $n \leq 7$. This is false. $n = -3$ is a counterexample.

2.6.7.

(a) For any number x, if it is the case that adding any number to x gives that number back, then multiplying any number by x will give 0. This is true (of the integers or the reals). The "if" part only holds if $x = 0$, and in that case, anything times x will be 0.

(b) The converse in words is this: For any number x, if everything times x is zero, then everything added to x gives itself. Or in symbols: $\forall x(\forall z(x \cdot z = 0) \rightarrow \forall y(x + y = y))$. The converse is true: The only number which when multiplied by any other number gives 0 is $x = 0$. And if $x = 0$, then $x + y = y$.

(c) The contrapositive in words is: For any number x, if there is some number which when multiplied by x does not give zero, then there is some number which when added to x does not give that number. In symbols: $\forall x(\exists z(x \cdot z \neq 0) \rightarrow \exists y(x + y \neq y))$. We know the contrapositive must be true because the original implication is true.

(d) The negation: There is a number x such that any number added to x gives the number back again, but there is a number you can multiply x by and not get 0. In symbols: $\exists x(\forall y(x + y = y) \wedge \exists z(x \cdot z \neq 0))$. Of course since the original implication is true, the negation is false.

2.6.8.

(a) $(\neg P \vee Q) \wedge (\neg R \vee (P \wedge \neg R))$.

(b) $\forall x \forall y \forall z(z = x + y \wedge \forall w(x - y \neq w))$.

2.6.9.

(a) Direct proof.

Proof. Let n be an integer. Assume n is odd. So $n = 2k + 1$ for some integer k. Then

$$7n = 7(2k + 1) = 14k + 7 = 2(7k + 3) + 1.$$

Since $7k + 3$ is an integer, we see that $7n$ is odd. ■

(b) The converse is: For all integers n, if $7n$ is odd, then n is odd. We will prove this by contrapositive.

Proof. Let n be an integer. Assume n is not odd. Then $n = 2k$ for some integer k. So $7n = 14k = 2(7k)$ which is to say $7n$ is even. Therefore $7n$ is not odd. ■

2.6.10.

(a) Suppose you only had 5 coins of each denomination. This means you have 5 pennies, 5 nickels, 5 dimes, and 5 quarters. This is a total of 20 coins. But you have more than 20 coins, so you must have more than 5 of at least one type.

(b) Suppose you have 22 coins, including $2k$ nickels, $2j$ dimes, and $2l$ quarters (so an even number of each of these three types of coins). The number of pennies you have will then be

$$22 - 2k - 2j - 2l = 2(11 - k - j - l).$$

But this says that the number of pennies is also even (it is 2 times an integer). Thus we have established the contrapositive of the statement, "If you have an odd number of pennies, then you have an odd number of at least one other coin type."

(c) You need 10 coins. You could have 3 pennies, 3 nickels, and 3 dimes. The 10th coin must either be a quarter, giving you 4 coins that are all different, or else a 4th penny, nickel, or dime. To prove this, assume you don't have 4 coins that are all the same or all different. In particular, this says that you only have 3 coin types, and each of those types can only contain 3 coins, for a total of 9 coins, which is less than 10.

APPENDIX C

LIST OF SYMBOLS

Symbol	Description	Page
$\therefore$	"therefore"	25
$P, Q, R, S, \ldots$	propositional (sentential) variables	29
$\wedge$	logical "and" (conjunction)	29
$\vee$	logical "or" (disjunction)	29
$\neg$	logical negation	29

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Colophon

This book was authored in PreTeXt.