Does $\sum_{n=1}^{\infty}\left(\frac{\sin n}{9n-1}\right)^n$ diverge, converge absolutely, or converge conditionally?

Solution

$$\begin{aligned}
L &= \lim_{n\to\infty} \sqrt[n]{|a_n|} \\
&= \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{\sin n}{9n-1}\right)^n\right|} \\
&= \lim_{n\to\infty} \sqrt[n]{\left(\frac{\sin n}{9n-1}\right)^n} \\
&= \lim_{n\to\infty} \frac{\sin n}{9n-1}
\end{aligned}$$

Note that you CANNOT use L'hopital's rule on this limit problem. (The top does not go to infinity or to zero.) But, note the following inequalities:

$$-1 \le \sin n \le 1$$

$$\frac{-1}{9n-1} \le \frac{\sin n}{9n-1} \le \frac{1}{9n-1}$$

Since

$$\lim_{n\to\infty} \frac{-1}{9n-1} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{9n-1} = 0$$

by the Squeeze Theorem for Sequences,

$$\lim_{n\to\infty} \frac{\sin n}{9n-1} = 0$$

Note that this number (0) is L. Since $L < 1$, by the Root Test, the series $\sum_{n=1}^{\infty}\left(\frac{\sin n}{9n-1}\right)^n$ converges absolutely.