Does $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverge, converge absolutely, or converge conditionally?

Solution

$$
\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right| \\
&= \lim_{n\to\infty} \left| \left(\frac{n+1}{n} \right)^n \right| \\
&= \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n .
\end{aligned}
$$

This is an indeterminate form, so we let $y = \left(\frac{n+1}{n}\right)^n$. So $\ln y = n \ln \frac{n+1}{n}$. Applying limit to both sides, we have

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} n \ln \frac{n+1}{n}.$$

The limit on the right can be computed using L'Hopital's rule (with a chain rule and quotent rule).

$$
\begin{aligned}
&\lim_{n\to\infty} n \ln \frac{n+1}{n}, \text{ which was the limit on the right side above} \\
&= \lim_{n\to\infty} \frac{\ln \frac{n+1}{n}}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{\frac{n}{n+1} \cdot \frac{n\cdot 1-(n+1)\cdot 1}{n^2}}{\frac{-1}{n^2}} \text{ by L'Hopital's} \\
&= \lim_{n\to\infty} \frac{n}{n+1} \text{ after algebra simplification} \\
&= \lim_{n\to\infty} \frac{1}{1} \\
&= 1.
\end{aligned}
$$

So, $\lim y = e^1 = e$. So $L = e$.

Since $L > 1$, the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges by the Ratio Test.