General instructions

- Exam 3 will cover up to class on Wednesday, November 8. (The syllabus at `https://edward-kim-math.github.io/syllabus/2017-fall-MTH-225.html` says up to November 9, but I am not going to have counting problems from November 9 on Exam 3.) The main topics since Exam 2 which will all be a part of Exam 3 are: equivalence relations, functions, cardinality, and induction. All mathematics is cumulative. Clearly, this class has cumulative exams.
- There are no reference materials that will be made available during Exam 3.
- All definitions must include the term being defined in complete sentences. If you don't not include the term and do not write in complete sentences, the person reading the definition you write will not be able to determine if the term is a noun or an adjective or a verb. If the object being defined is a noun, your text must make clear what type of noun (set? proposition? etc.). If the object being defined is an adjective, your text must make clear what type of noun is being modified (example: even modifies integer).
- All proofs must be written in complete sentences. You may or may not include flowcharts as you wish, but they will not be examined when grading: the flowcharts are your scratch work. Since the accepted standard in the mathematical community is complete sentences in paragraphs, I will be grading any/all work that consists of complete sentences in narrative form.
- Understand the difference between proving versus using. If you need to *use* an implication but try to spend time *proving* an implication, you will waste time. (You will waste time because the task is impossible.)
- You must be completely familiar with the rules of inference (on the pink sheet). Actually, this was true going into Exam 1.
- In addition, you must be completely familiar with the rules of inference for set membership. This was true for Exam 2.
- Use our methods to prove that sets are equal or that one set is a subset of another. In every instance I have asked a student about the use of "truth" tables (truth is in quotes for a reason) or 0-1 tables, there has been a complete lack of understanding about what the authors mean in a 0-1 table. Any use of these sorts of tables does not practice our definitions and will earn an automatic zero.
- This comment is in reaction to the last test: do NOT confuse proposition and predicate. These are NOT interchangeable! (The terms predicate and propositional function ARE interchangeable.)
- Do not confuse "is a member of" with "is a subset of". Do not say that a set is "true". A set cannot be true, because a set is not a proposition. To say that $\boxed{x \text{ is an element of the set } M}$ is a proposition. However, $\boxed{M}$ itself is NOT a proposition.
- If you don't specify the word "binary" in front of "relation", that is fine: In this case, I will assume that you mean "binary relation".
- Understand the difference between proving versus using. If you need to *use* an implication but try to spend time *proving* an implication, you will waste time. (You will waste time because the task is impossible.)

Practice Problems – Key posted LATER

1. Define <u>countable</u> and <u>uncountable</u>.
2. Define <u>countably infinite</u>.

3. Suppose $P(n)$ is a predicate with universe of discourse $\mathbb{Z}_{>0}$. If one is going to prove the statement $\forall n \in \mathbb{Z}_{>0}[P(n)]$, then what must be proved to conclude this is true using the method of standard induction?
4. Suppose $P(n)$ is a predicate with universe of discourse $\mathbb{Z}_{>0}$. If one is going to prove the statement $\forall n \in \mathbb{Z}_{>0}[P(n)]$, then what must be proved to conclude this is true using the method of strong induction?
5. Prove the theorem: if B is countable and $A \subseteq B$, then A is countable.
6. Prove the theorem: if A and B are countable, then $A \cap B$ is countable. (You cannot assume that A and B are disjoint.) (Simulate the situation on a test: Prove this without using the previous theorem. Why? Because if this problem is on the test, you ought to assume the previous question is NOT on the test – and then what would you do?)
7. Prove the theorem: if A and B are countably infinite and disjoint, then $A \cup B$ is countable.
8. Prove the theorem: if A and B are disjoint and A is finite and B is countably infinite, then $A \cup B$ is countable.
9. Let $f : A \to B$ and $g : B \to C$. Prove: if $g \circ f$ is injective, then f is injective.
10. Let $f : A \to B$ and $g : B \to C$. Prove: if $g \circ f$ is surjective, then g is surjective.
11. Let $f : S \to T$. Prove: for all t and u in T, the equation $f^{-1}(t) \cup f^{-1}(u) = f^{-1}(\{t, u\})$ holds.
12. Prove for all $n > 0$ which are integer, one has
$$\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}$$
13. At a party, 25 guests mingle and shake hands with some fellow guests. Prove that at least one guest must have shaken hands with an even number of guests.
14. Prove $1 + 3 + 5 + \cdots (2n - 1) = n^2$ for all $n \in \mathbb{Z}_{>0}$.
15. Let $C = \{x \in \mathbb{R} \mid x \geq 0\}$. Let $D = \{x^2 : x \in \mathbb{R}\}$. Prove $C = D$.
16. Let $f : X \to M$. Prove that the range of f is a subset of M.
17. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x + y) = f(x) + f(y)$ for all real numbers x and y. Prove that $f(0) = 0$.
18. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x + y) = f(x) + f(y)$ for all real numbers x and y. Prove for all $x \in \mathbb{R}$, the equation $f(-x) = -f(x)$ holds.
19. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x + y) = f(x) + f(y)$ for all real numbers x and y. Fix a real number s. Prove for all $n \in \mathbb{Z}_{>0}$, the equation $f(ns) = n\,f(s)$ holds.
20. $f : A \to B$. Prove that $A = f^{-1}(B)$.
21. Let $f : X \to Y$ and $g : Y \to Z$. Prove: if $g \circ f$ is one-to-one and f is onto, then g is one-to-one.
22. Let $f : X \to Y$ and $g : Y \to Z$. Prove: if $g \circ f$ is onto and g is one-to-one, then f is onto.
23. Let A, B, C, and D be sets. Suppose A and C have the same cardinality. Suppose B and D have the same cardinality. Prove $A \times B$ has the same cardinality as $C \times D$.
24. Let A and B be sets. Prove that if A and B have the same cardinality, then $P(A)$ and $P(B)$ have the same cardinality.
25. Prove that $(3, 7)$ and $(5, 12)$ have the same cardinality.
26. Let $f : X \to Y$ be an injective function. Prove that X and $f(X)$ have the same cardinality.