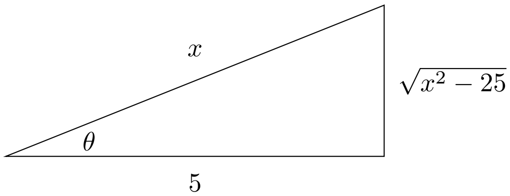
$$\int \frac{\sqrt{x^2-25}}{x}\,dx$$

Solution

Let $x = 5\sec\theta$ with θ in quadrants I or III. So, $dx = 5\sec\theta\tan\theta\,d\theta$ and we use $\sec\theta = \frac{x}{5}$ to draw a right triangle with x as the hypotenuse side side and 5 as the adjacent side. The side opposite to θ is $\sqrt{x^2-25}$, which is found using the Pythagorean Theorem, and we have



Then $\tan\theta = \dfrac{\sqrt{x^2-25}}{5}$. Based on this equation, we will use $5\tan\theta = \sqrt{x^2-25}$ if it is helpful. Since $\cos\theta = \dfrac{5}{x}$, we will use $x = \dfrac{5}{\cos\theta} = 5\sec\theta$ if it is helpful.

So

$$\begin{aligned}
\int \frac{\sqrt{x^2-25}}{x}\,dx &= \int \frac{5\tan\theta}{5\sec\theta} 5\sec\theta\tan\theta\,d\theta \\
&= 5\int \tan^2\theta\,d\theta \\
&= 5\int (\sec^2\theta - 1)\,d\theta \\
&= 5\left[\tan\theta - \theta\right] + C \\
&= 5\left[\frac{\sqrt{x^2-25}}{5} - \sec^{-1}\frac{x}{5}\right] + C
\end{aligned}$$