Does $\sum_{n=1}^{\infty} \frac{5^n \ln 5}{5^n - 1}$ diverge, converge absolutely, or converge conditionally?

Solution 1

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{5^n \ln 5}{5^n - 1} \\
&= \lim_{n\to\infty} \frac{5^n \ln 5 \ln 5}{5^n \ln 5} \quad \text{by L'Hopital} \\
&= \lim_{n\to\infty} \frac{\ln 5}{1} \quad \text{by algebra cancellation} \\
&= \ln 5
\end{aligned}$$

so the series $\sum_{n=1}^{\infty} \frac{5^n \ln 5}{5^n - 1}$ diverges by the Test for Divergence.

Solution 2

The function $f(x) = \frac{5^x \ln 5}{5^x - 1}$ is continuous, positive, and decreasing on $[1, \infty)$.

We first find the indefinite integral by substitution $u = 5^x - 1$ so $du = 5^x \ln 5 \, dx$:

$$\begin{aligned}
\int \frac{5^x \ln 5}{5^x - 1} \, dx &= \int \frac{1}{u} \, du \\
&= \ln|u| + C \\
&= \ln|5^x - 1| + C.
\end{aligned}$$

So, going to our definite (improper) integral,

$$\begin{aligned}
\int_1^{\infty} \frac{5^x \ln 5}{5^x - 1} \, dx &= \lim_{t\to\infty} \int_1^t \frac{5^x \ln 5}{5^x - 1} \, dx \\
&= \lim_{t\to\infty} (\ln|5^t - 1| - \ln|5^1 - 1|) \\
&= \infty.
\end{aligned}$$

Since the integral $\int_1^{\infty} \frac{5^x \ln 5}{5^x - 1} \, dx$ diverges, the series $\sum_{n=1}^{\infty} \frac{5^n \ln 5}{5^n - 1}$ diverges by the Integral Test.