$$\int \frac{1}{x^2+2x-25}\,dx$$

Solution

The denominator of $\frac{1}{x^2+2x-25}$ won't factor using integers, but the roots can be obtained using the quadratic formula, and we'll see in a moment that this quadratic still factors. Taking $x^2+2x-25$, the two roots are

$$x = \frac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot(-25)}}{2} = \frac{-2\pm\sqrt{104}}{2}.$$

Applying partial fraction decomposition to $\frac{1}{x^2+2x-25}$, we have

$$\frac{1}{x^2+2x-25} = \frac{A}{x-\frac{-2+\sqrt{104}}{2}} + \frac{B}{x-\frac{-2-\sqrt{104}}{2}}.$$

We multiply both sides by $(x-\frac{-2+\sqrt{104}}{2})(x-\frac{-2-\sqrt{104}}{2})$, which is also $x^2+2x-25$, and the previous equation becomes:

$$1 = A\left(x - \frac{-2+\sqrt{104}}{2}\right) + B\left(x - \frac{-2-\sqrt{104}}{2}\right)$$

Starting from this equation (twice), we will substitute two different values of x.

- By substituting $x = \frac{-2+\sqrt{104}}{2}$, we get

$$1 = A\cdot 0 + B\left(\frac{-2+\sqrt{104}}{2} - \frac{-2-\sqrt{104}}{2}\right)$$

  which simpliies to $1 = B\sqrt{104}$, so $B = \frac{1}{\sqrt{104}}$.

- By substituting $x = \frac{-2-\sqrt{104}}{2}$, we get

$$1 = A\left(\frac{-2-\sqrt{104}}{2} - \frac{-2+\sqrt{104}}{2}\right) + B\cdot 0$$

  which similpfies to $1 = A(-\sqrt{104})$, so $A = \frac{-1}{\sqrt{104}}$.

Thus,

$$\frac{1}{x^2+2x-25} = \frac{-1/\sqrt{104}}{x-\frac{-2+\sqrt{104}}{2}} + \frac{1/\sqrt{104}}{x-\frac{-2-\sqrt{104}}{2}}.$$

Now, to integrate,

$$\int \frac{1}{x^2+2x-25}\,dx = \int \frac{-1}{\sqrt{104}}\frac{1}{x-\frac{-2+\sqrt{104}}{2}} + \frac{1}{\sqrt{104}}\frac{1}{x-\frac{-2-\sqrt{104}}{2}}\,dx$$

The last integral can be split into two, and each of the two integrals is obtained by a substitution (using $u = x - \frac{-2+\sqrt{104}}{2}$ in the first case and $u = x - \frac{-2-\sqrt{104}}{2}$ in the second). So, our final answer is

$$\frac{-1}{\sqrt{104}}\ln\left|x - \frac{-2+\sqrt{104}}{2}\right| + \frac{1}{\sqrt{104}}\ln\left|x-\frac{-2-\sqrt{104}}{2}\right| + C.$$