$$\int \tan^3 x \, dx$$

Solution

$$\begin{aligned}
\int \tan^3 x \, dx &= \int \tan^2 x \tan x \, dx \\
&= \int (\sec^2 x - 1) \tan x \, dx \\
&= \int \sec^2 x \tan x \, dx - \int \tan x \, dx.
\end{aligned}$$

We will complete the two integrals separately:

- Substitute $u = \tan x$, so $du = \sec^2 x \, dx$. We get:

$$\int \sec^2 x \tan x \, dx = \int u \, du = \frac{u^2}{2} + C = \frac{1}{2} \tan^2 x + C.$$

- For the second integral,

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

  Let $u = \cos x$, so $du = -\sin x \, dx$. So the integral above is equal to

$$-\int \frac{1}{u} \, du = -\ln |u| + C = -\ln |\cos x| + C.$$

Putting together the two integrals, the answer to our original question is

$$\begin{aligned}
\int \sec^2 x \tan x \, dx - \int \tan x \, dx &= \frac{1}{2} \tan^2 x - (-\ln |\cos x|) + C \\
&= \frac{1}{2} \tan^2 x + \ln |\cos x| + C.
\end{aligned}$$