Does $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ diverge, converge absolutely, or converge conditionally?

Solution 1

Rewrite $\frac{1}{n(n+2)}$ using partial fraction decomposition:

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

So

$$1 = A(n+2) + Bn$$

Using $n = 0$, we get $A = \frac{1}{2}$. Using $n = -2$, we get $B = -\frac{1}{2}$. So

$$\frac{1}{n(n+2)} = \frac{\frac{1}{2}}{n} - \frac{\frac{1}{2}}{n+2}$$

Therefore, our original series can be rewritten: $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \left(\frac{1/2}{n} - \frac{1/2}{n+2} \right)$

We appear to have a telescoping series. After cancellations, the nth term of the sequence of partial sums is:

$$s_n = \frac{1/2}{1} + \frac{1/2}{2} - \frac{1/2}{n+1} - \frac{1/2}{n+2}$$

Thus,

$$\begin{aligned}
\lim_{n\to\infty} s_n &= \lim_{n\to\infty} \left(\frac{1/2}{1} + \frac{1/2}{2} - \frac{1/2}{n+1} - \frac{1/2}{n+2} \right) \\
&= \frac{1}{2} + \frac{1}{4} - 0 - 0 \\
&= \frac{3}{4}
\end{aligned}$$

So, (by definition), the series $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ converges, and we furthermore know that the sum is $\frac{3}{4}$. Recall that it is unusual that we get to know the value of a convergent series.

Since $\sum |a_n| = \sum a_n$, the series $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ converges absolutely.

Solution 2

Note that $n(n+2) \geq n^2$ for all $n \geq 1$. So

$$\frac{1}{n(n+2)} \leq \frac{1}{n^2}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test. Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ converges by the Direct Comparison Test.

Since $\sum |a_n| = \sum a_n$, the series $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ converges absolutely.