$$\int x \arctan(4x)\, dx$$

Solution

Let $t = 4x$, so $dt = 4\, dx$. Then

$$\int x \arctan(4x)\, dx = \int \frac{t}{4} \arctan t\, dt = \frac{1}{4} \int t \arctan t\, dt$$

If we momentarily ignore the $\frac{1}{4}$, to integrate

$$\int t \arctan t\, dt$$

use integration by parts and some clever algebra after. Let $u = \arctan t$ and $dv = t\, dt$. Then $dt = \frac{1}{t^2+1}\, dt$ and $v = \frac{t^2}{2}$ so

$$\begin{aligned}
\int t \arctan t\, dt &= \frac{1}{2} t^2 \arctan t - \frac{1}{2} \int \frac{t^2}{t^2+1}\, dx \\
&= \frac{1}{2} t^2 \arctan t - \frac{1}{2} \int \frac{t^2+1-1}{t^2+1}\, dx \\
&= \frac{1}{2} t^2 \arctan t - \frac{1}{2} \int \frac{(t^2+1)-1}{t^2+1}\, dx \\
&= \frac{1}{2} t^2 \arctan t - \frac{1}{2} \int \frac{t^2+1}{t^2+1} - \frac{1}{t^2+1}\, dx \\
&= \frac{1}{2} t^2 \arctan t - \frac{1}{2} \int 1 - \frac{1}{t^2+1}\, dx \\
&= \frac{1}{2} t^2 \arctan t - \frac{1}{2} t + \frac{1}{2} \arctan t + C. \qquad = \frac{1}{2} (4x)^2 \arctan(4x) - \frac{1}{2}(4x) + \frac{1}{2} \arctan(4x) + C.
\end{aligned}$$

Now we recall the $\frac{1}{4}$ in front that we ignored, so our final answer is

$$\frac{1}{4} \left[\frac{1}{2} (4x)^2 \arctan(4x) - \frac{1}{2}(4x) + \frac{1}{2} \arctan(4x) \right] + C$$

Commentary

The point is that an input of $4x$ to the arctangent function is not so different from an input of just x, and the "just x" is played by the role of t here.