

$$\int \sqrt{1-4x^2} dx$$

Here are two solutions:

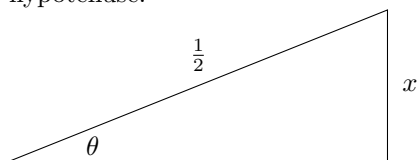
Solution 1

First, some algebra:

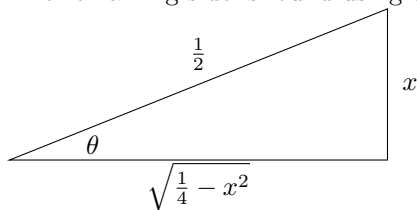
$$\int \sqrt{1-4x^2} dx = \int \sqrt{4(\frac{1}{4}-x^2)} dx = \int 2\sqrt{\frac{1}{4}-x^2} dx = 2 \int \sqrt{\frac{1}{4}-x^2} dx.$$

Because of seeing $\sqrt{a^2-x^2}$ in the integral with $a = \frac{1}{2}$, we apply trig substitution with $x = \frac{1}{2} \sin \theta$, for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

So, $dx = \frac{1}{2} \cos \theta d\theta$ and we use $\sin \theta = \frac{x}{\frac{1}{2}}$ to draw a right triangle with x as the opposite side at $\frac{1}{2}$ as the hypotenuse.



The remaining side is found using the Pythagorean Theorem, and we have



So our integral becomes

$$\int 2\sqrt{\frac{1}{4}-x^2} dx = \int \sqrt{\frac{1}{4}-x^2} \cos \theta d\theta$$

We still need to replace $\sqrt{\frac{1}{4}-x^2}$, so we look to the side of the triangle with that quantity (it is adjacent to θ) and the number side (which is the hypotenuse). Since $\cos \theta = \frac{\sqrt{\frac{1}{4}-x^2}}{\frac{1}{2}}$, we have $\sqrt{\frac{1}{4}-x^2} = \frac{1}{2} \cos \theta$. So, our integral becomes

$$\begin{aligned} 2 \int \frac{1}{2} \cos \theta \cdot \cos \theta d\theta &= \int \cos^2 \theta d\theta \\ &= \frac{1}{2} \int 1 + \cos 2\theta d\theta \\ &= \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta + C \\ &= \frac{1}{2} \theta + \frac{1}{4} 2 \sin \theta \cos \theta + C \\ &= \frac{1}{2} \sin^{-1} 2x + \frac{1}{2} \frac{x}{1/2} \cdot \frac{\sqrt{\frac{1}{4}-x^2}}{1/2} + C. \end{aligned}$$

See the next page for another solution:

Solution 2

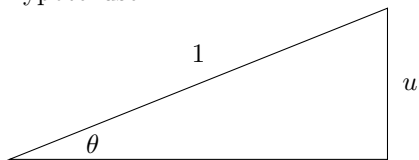
$$\int \sqrt{1-4x^2} dx = \int \sqrt{1-(2x)^2} dx$$

If $u = 2x$, then $du = 2 dx$, so the integral is equal to

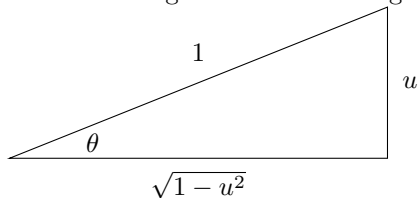
$$\frac{1}{2} \int \sqrt{1-u^2} du.$$

Trig sub $u = \sin \theta$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

So, $du = \cos \theta d\theta$ and we use $\sin \theta = \frac{u}{1}$ to draw a right triangle with u as the opposite side at 1 as the hypotenuse.



The remaining side is found using the Pythagorean Theorem, and we have



So our integral becomes

$$\frac{1}{2} \int \sqrt{1-u^2} dx = \frac{1}{2} \int \sqrt{1-u^2} \cos \theta d\theta$$

We still need to replace $\sqrt{1-u^2}$, so we look to the side of the triangle with that quantity (it is adjacent to θ) and the number side (which is the hypotenuse). Since $\cos \theta = \frac{\sqrt{1-u^2}}{1}$, we have $\sqrt{1-u^2} = \cos \theta$. So, our integral becomes

$$\begin{aligned} \frac{1}{2} \int \cos \theta \cdot \cos \theta d\theta &= \frac{1}{2} \int \cos^2 \theta d\theta \\ &= \frac{1}{4} \int 1 + \cos 2\theta d\theta \\ &= \frac{1}{4} \theta + \frac{1}{8} \sin 2\theta + C \\ &= \frac{1}{4} \theta + \frac{1}{8} \cdot 2 \sin \theta \cos \theta + C \\ &= \frac{1}{4} \sin^{-1} \frac{u}{1} + \frac{1}{4} \cdot \frac{u}{1} \cdot \frac{\sqrt{1-u^2}}{1} + C. \\ &= \frac{1}{4} \sin^{-1} \frac{2x}{1} + \frac{1}{4} \cdot \frac{2x}{1} \cdot \frac{\sqrt{1-(2x)^2}}{1} + C. \end{aligned}$$