Solution

$$
\begin{aligned}
\int \frac{1}{\sqrt{x^2+6x+5}}\,dx &= \int \frac{1}{\sqrt{x^2+6x+9-9+5}}\,dx \\
&= \int \frac{1}{\sqrt{(x+3)^2-9+5}}\,dx \\
&= \int \frac{1}{\sqrt{(x+3)^2-4}}\,dx
\end{aligned}
$$

If $u = x+3$, then $du = dx$ so the integral above is equal to

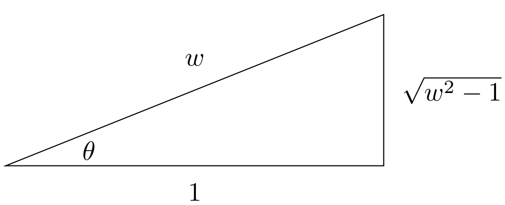
$$\int \frac{1}{\sqrt{u^2-4}}\,du$$

The integral above can be completed several ways:

1. Using the substitution $w = \frac{u}{2}$,

$$
\begin{aligned}
\int \frac{1}{\sqrt{u^2-4}}\,du &= \int \frac{1}{\sqrt{4(\frac{1}{4}u^2-1)}}\,du \\
&= \frac{1}{2}\int \frac{1}{\sqrt{\frac{1}{4}u^2-1}}\,du \\
&= \frac{1}{2}\int \frac{1}{\sqrt{(\frac{u}{2})^2-1}}\,du \\
&= \frac{1}{2}\cdot 2\int \frac{1}{\sqrt{w^2-1}}\,dw \\
&= \int \frac{1}{\sqrt{w^2-1}}\,dw
\end{aligned}
$$

Let $w = 1\sec\theta$ with θ in quadrants I or III. So, $dw = 1\sec\theta\tan\theta\,d\theta$ and we use $\sec\theta = \frac{w}{1}$ to draw a right triangle with w as the hypotenuse side side and 1 as the adjacent side. The side opposite to θ is $\sqrt{w^2-1}$, which is found using the Pythagorean Theorem, and we have



Then $\tan\theta = \dfrac{\sqrt{w^2-1}}{1}$. Based on this equation, we will use $1\tan\theta = \sqrt{w^2-1}$ if it is helpful. Since $\cos\theta = \dfrac{1}{w}$, we will use $w = \dfrac{1}{\cos\theta} = 1\sec\theta$ if it is helpful.

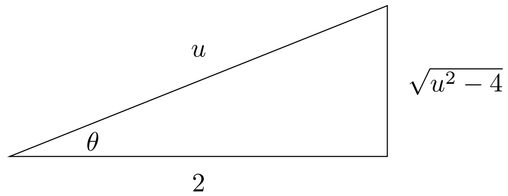
So the integral becomes

$$
\begin{aligned}
\int \frac{1}{\sqrt{w^2-1}}\,dw &= \int \frac{1}{\tan\theta}\sec\theta\tan\theta\,d\theta \\
&= \int \sec\theta\,d\theta \\
&= \ln|\tan\theta+\sec\theta| + C \\
&= \ln\left|\sqrt{w^2-1}+w\right| + C \\
&= \ln\left|\sqrt{(u/2)^2-1}+\frac{u}{2}\right| + C \\
&= \ln\left|\sqrt{((x+3)/2)^2-1}+\frac{x+3}{2}\right| + C
\end{aligned}
$$

2. An alternate method (probably the more typical method) to find

$$\int \frac{1}{\sqrt{u^2-4}}\,du$$

is as follows: Let $u = 2\sec\theta$ with θ in quadrants I or III. So, $du = 2\sec\theta\tan\theta\,d\theta$ and we use $\sec\theta = \frac{u}{2}$ to draw a right triangle with u as the hypotenuse side side and 2 as the adjacent side. The side opposite to θ is $\sqrt{u^2-4}$, which is found using the Pythagorean Theorem, and we have



Then $\tan\theta = \dfrac{\sqrt{u^2-4}}{2}$. Based on this equation, we will use $2\tan\theta = \sqrt{u^2-4}$ if it is helpful. Since $\cos\theta = \dfrac{2}{u}$, we will use $u = \dfrac{2}{\cos\theta} = 2\sec\theta$ if it is helpful.

So, the integral we have becomes

$$\begin{aligned}
\int \frac{1}{\sqrt{u^2-4}}\,du &= \int \frac{1}{2\tan\theta} 2\sec\theta\tan\theta\,d\theta \\
&= \int \sec\theta\,d\theta \\
&= \ln|\tan\theta + \sec\theta| + C \\
&= \ln\left|\frac{\sqrt{u^2-4}}{2} + \frac{u}{2}\right| + C \\
&= \ln\left|\frac{\sqrt{(x+3)^2-4}}{2} + \frac{x+3}{2}\right| + C
\end{aligned}$$