Does $\sum_{n=1}^{\infty} \frac{1}{5^n+1}$ diverge, converge absolutely, or converge conditionally?

Solution 1

Since

$$\frac{1}{5^n+1} \leq \frac{1}{5^n}$$

and since $\sum_{n=1}^{\infty} \frac{1}{5^n}$ converges by the Geometric Series Test ($r = \frac{1}{5}$), the series $\sum_{n=1}^{\infty} \frac{1}{5^n+1}$ converges by the Direct Comparison Test.

Since $\sum |a_n| = \sum a_n$, the series $\sum_{n=1}^{\infty} \frac{1}{5^n+1}$ converges absolutely.

Solution 2

Let $a_n = \frac{1}{5^n+1}$ and $b_n = \frac{1}{5^n}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{5^n+1}{5^n} \\
&= \lim_{n\to\infty} \frac{5^n \ln 5}{5^n \ln 5} \quad \text{by L'hopital's rule} \\
&= \lim_{n\to\infty} 1 \\
&= 1
\end{aligned}$$

which is a finite positive number. Since the series $\sum_{n=1}^{\infty} \frac{1}{5^n}$ converges by the Geometric Series Test ($r = \frac{1}{5}$), the series $\sum_{n=1}^{\infty} \frac{1}{5^n+1}$ converges by the Limit Comparison Test.

Since $\sum |a_n| = \sum a_n$, the series $\sum_{n=1}^{\infty} \frac{1}{5^n+1}$ converges absolutely.