$$\int \frac{x^2}{x^2+5}\,dx$$

Solution 1

Since the power in denominator is not larger than the degree in the numerator, we apply long division. From long division, we get

$$\frac{x^2}{x^2+5} = 1 - \frac{5}{x^2+5}.$$

So

$$\begin{aligned}\int \frac{x^2}{x^2+5}\,dx &= \int \left(1 - \frac{5}{x^2+5}\right)\,dx \\ &= x - \int \frac{5}{5(\frac{1}{5}x^2+1)}\,dx \\ &= x - \int \frac{1}{(\frac{x}{\sqrt{5}})^2+1}\,dx.\end{aligned}$$

By applying substitution with $u = \frac{x}{\sqrt{5}}$, we get $du = \frac{1}{\sqrt{5}}\,dx$ so the integral above is equal to

$$\begin{aligned}&= x - \int \frac{\sqrt{5}}{u^2+1}\,du \\ &= x - \tan^{-1}(u) + C \\ &= x - \tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C.\end{aligned}$$

Solution 2

In some rare (very lucky) situations, we can avoid long division with some clever algebra:

$$\frac{x^2}{x^2+5} = \frac{x^2+5-5}{x^2+5} = \frac{x^2+5}{x^2+5} - \frac{5}{x^2+5} = 1 - \frac{5}{x^2+5}.$$

So

$$\begin{aligned}\int \frac{x^2}{x^2+5}\,dx &= \int \left(1 - \frac{5}{x^2+5}\right)\,dx \\ &= x - \int \frac{5}{5(\frac{1}{5}x^2+1)}\,dx \\ &= x - \int \frac{1}{(\frac{x}{\sqrt{5}})^2+1}\,dx.\end{aligned}$$

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