Does $\sum_{n=2}^{\infty} \frac{n+2}{n^3+1}$ diverge, converge absolutely, or converge conditionally?

Solution 1

Note

$$\frac{1}{n^3+1} \leq \frac{1}{n^3}$$

so multiplying both sides of this inequality by n, we get

$$\frac{n}{n^3+1} \leq \frac{1}{n^2}.$$

Since the series $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges by the p-test, the series $\sum_{n=2}^{\infty} \frac{n}{n^3+1}$ converges by the Direct Comparison Test.

In addition,

$$\frac{2}{n^3+1} \leq \frac{2}{n^3}$$

so since the series $\sum_{n=2}^{\infty} \frac{1}{n^3}$ converges by the p-test, the series $\sum_{n=2}^{\infty} \frac{2}{n^3+1}$ converges by the Direct Comparison Test.

Since our series $\sum_{n=2}^{\infty} \frac{n+2}{n^3+1}$ is the sum of two convergent series $\sum_{n=2}^{\infty} \frac{n}{n^3+1}$ and $\sum_{n=2}^{\infty} \frac{2}{n^3+1}$, the series $\sum_{n=2}^{\infty} \frac{n+2}{n^3+1}$ converges.

Since all terms are positive, the series $\sum_{n=2}^{\infty} \frac{n+2}{n^3+1}$ converges absolutely.

Solution 2

The series $\sum \frac{1}{n^2}$ converges by the p-test. Let $a_n = \frac{n+2}{n^3+1}$ and $b_n = \frac{1}{n^2}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^3+2n^2}{n^3+1} \\
&= \lim_{n\to\infty} \frac{3n^2+4n}{3n^2} \text{ by l'hopital} \\
&= \lim_{n\to\infty} \frac{6n+4}{6n} \text{ by l'hopital} \\
&= \lim_{n\to\infty} \frac{6}{6} \text{ by l'hopital} \\
&= 1
\end{aligned}$$

The Limit Comparison Test applies, since this limit was a positive, finite number. Therefore, the series $\sum \frac{n+2}{n^3+1}$ converges by the Limit Comparison Test. Since all terms are positive, the series $\sum_{n=2}^{\infty} \frac{n+2}{n^3+1}$ converges absolutely.