$$\int \frac{1}{x^2+2x+3}\,dx$$

Solution

The denominator of this fraction won't factor using real numbers, so there is no need to bother with partial fractions. (We have an irreducible quadratic, so partial fractions won't work, since there would only be one "factor".) Instead, let's complete the square in the denominator:

$$\int \frac{1}{x^2+2x+3}\,dx = \int \frac{1}{x^2+2x+1+2}\,dx = \int \frac{1}{(x+1)^2+2}\,dx =$$

Let $u = x+1$. Then $du = dx$. So, the integral above is equal to

$$\int \frac{1}{u^2+2}\,du$$

While this looks okay, we cannot apply the rule for arctangent yet, because of the 2 term in the denominator. But, with some clever factoring,

$$\begin{aligned}
&\int \frac{1}{u^2+2}\,du \\
&\qquad = \int \frac{1}{2(\frac{1}{2}u^2+1)}\,du \\
&\qquad = \frac{1}{2}\int \frac{1}{\frac{1}{2}u^2+1}\,du \\
&\qquad = \frac{1}{2}\int \frac{1}{(\frac{u}{\sqrt{2}})^2+1}\,du
\end{aligned}$$

Substituting $w = \frac{u}{\sqrt{2}}$, we get $dw = \frac{1}{\sqrt{2}}\,du$, so the integral above is equal to

$$\begin{aligned}
&\frac{\sqrt{2}}{2}\int \frac{1}{w^2+1}\,dw \\
&= \frac{\sqrt{2}}{2}\tan^{-1}(w) + C \\
&= \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C \\
&= \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C.
\end{aligned}$$