Does $\displaystyle\sum_{n=1}^{\infty} n^7 e^{-n^8}$ diverge, converge absolutely, or converge conditionally?

Solution

The function $f(x) = x^7 e^{-x^8} = \frac{x^7}{e^{(x^8)}}$ is continuous, positive, and decreasing on $[1, \infty)$.

Note that using the substitution $u = x^8$, we can do the indefinite integral

$$\int \frac{x^7}{e^{(x^8)}}\,dx = \int \frac{8}{e^u}\,du = \int 8e^{-u}\,du = -8e^{-u} + C = \frac{-8}{e^{(x^8)}} + C$$

So

$$\begin{aligned}\int_1^{\infty} \frac{x^7}{e^{(x^8)}}\,dx &= \lim_{t\to\infty} \int_1^t \frac{x^7}{e^{(x^8)}}\,dx \\ &= \lim_{t\to\infty} \left(\frac{-8}{e^{(t^8)}} - \frac{-8}{e^{(1^8)}} \right) \\ &= \frac{8}{e}\end{aligned}$$

Since the integral $\displaystyle\int_1^{\infty} \frac{x^7}{e^{(x^8)}}\,dx$ converges, the series $\displaystyle\sum_{n=1}^{\infty} n^7 e^{-n^8}$ converges by the integral test.

Since $\sum |a_n| = \sum a_n$, the series $\displaystyle\sum_{n=1}^{\infty} n^7 e^{-n^8}$ converges absolutely.