Does $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ diverge, converge absolutely, or converge conditionally?

## Solution 1

The function $f(x) = \frac{\ln x}{x}$ is continuous, positive, and decreasing on $[3, \infty)$. We consider the integral

$$\int_3^{\infty} \frac{\ln x}{x}\, dx = \lim_{t\to\infty} \int_3^t \frac{\ln x}{x}\, dx.$$

We do the indefinite integral using the substitution $u = \ln x$, so $du = \frac{1}{x}\, dx$:

$$\int \frac{\ln x}{x}\, dx = \int u\, du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$$

so, back to the improper integral,

$$\begin{aligned}\int_3^{\infty} \frac{\ln x}{x}\, dx &= \lim_{t\to\infty} \int_3^t \frac{\ln x}{x}\, dx \\ &= \lim_{t\to\infty} \left[\frac{1}{2}(\ln t)^2 - \frac{1}{2}(\ln 3)^2\right] \\ &= \infty.\end{aligned}$$

Since the integral $\int_3^{\infty} \frac{\ln x}{x}\, dx$ diverges, the series $\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges by the Integral Test. So the series $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ diverges as well (since we're only adding in one more term).

## Solution 2

Note that for $n \geq 3$, we have $\ln n \geq 1$. So,

$$\frac{\ln n}{n} \geq \frac{1}{n}$$

Since the series $\sum_{n=3} \frac{1}{n}$ diverges by the $p$-series test (with $p = 1$), the series $\sum_{n=3} \frac{\ln n}{n}$ diverges by the Direct Comparison Test. Adding on one more term does not change the convergence/divergence, so the series $\sum_{n=2} \frac{\ln n}{n}$ also diverges.