$$\int \frac{x^7 + x}{\sqrt{25 + x^2}}\, dx$$

Solution

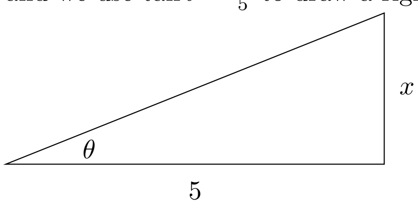
We split the integral into two:

$$\int \frac{x^7 + x}{\sqrt{25 + x^2}}\, dx = \int \frac{x^7}{\sqrt{25 + x^2}}\, dx + \int \frac{x}{\sqrt{25 + x^2}}\, dx$$

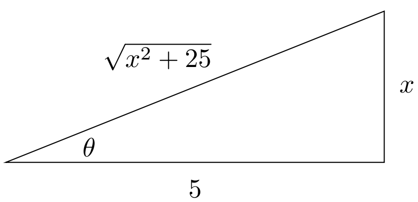
While the second integral can be completed by trigonometric substitution, it's much faster to use substitution with $u = 25 + x^2$, so that $du = 2x\, dx$. Then the second integral becomes

$$\int \frac{x}{\sqrt{25 + x^2}}\, dx = \frac{1}{2} \int \frac{1}{\sqrt{u}}\, du = \frac{1}{2} \int u^{-1/2}\, du = \frac{1}{2} \frac{u^{1/2}}{1/2} + C = \sqrt{u} + C = \sqrt{25 + x^2} + C.$$

We now need to look at the first integral, and we use trig substitution. Let $x = 5 \tan\theta$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. So, $dx = 5\sec^2\theta\, d\theta$ and we use $\tan\theta = \frac{x}{5}$ to draw a right triangle with x as the opposite side at 5 as the adjacent side.



The hypotenuse is found using the Pythagorean Theorem, and we have



Then $\cos\theta = \dfrac{5}{\sqrt{x^2 + 25}}$. Based on this equation, we will use either $\sqrt{x^2 + 25} = \dfrac{5}{\cos\theta} = 5\sec\theta$ or we will use $\dfrac{\cos\theta}{5} = \dfrac{1}{\sqrt{x^2 + 25}}$ if they are helpful.

So

$$\int \frac{x^7}{\sqrt{25 + x^2}}\, dx = \int (5\tan\theta)^7 \cdot \frac{\cos\theta}{5} 5\sec^2\theta\, d\theta = 5^7 \int \tan^7\theta \sec\theta\, d\theta$$

Since the power of tangent is odd, we rewrite the integral

$$\begin{aligned}
5^7 \int \tan^7\theta \sec\theta\, d\theta &= 5^7 \int \tan^6\theta \cdot \sec\theta \tan\theta\, d\theta \\
&= 5^7 \int (\tan^2\theta)^3 \cdot \sec\theta \tan\theta\, d\theta \\
&= 5^7 \int (\sec^2\theta - 1)^3 \cdot \sec\theta \tan\theta\, d\theta \qquad u = \sec\theta \quad du = \sec\theta\tan\theta\, d\theta \\
&= 5^7 \int (u^2 - 1)^3\, du \\
&= 5^7 \int (u^6 - 3u^4 + 3u^2 - 1)\, du \\
&= 5^7 \left[\frac{u^7}{7} - 3 \cdot \frac{u^5}{5} + 3 \cdot \frac{u^3}{3} - u \right] + C \\
&= 5^7 \left[\frac{(\frac{1}{5}\sqrt{x^2 + 25})^7}{7} - 3 \cdot \frac{(\frac{1}{5}\sqrt{x^2 + 25})^5}{5} + 3 \cdot \frac{(\frac{1}{5}\sqrt{x^2 + 25})^3}{3} - \frac{1}{5}\sqrt{x^2 + 25} \right] + C
\end{aligned}$$

Putting together this work with the other integral, by addition our final answer is

$$5^7 \left[\frac{(\frac{1}{5}\sqrt{x^2 + 25})^7}{7} - 3 \cdot \frac{(\frac{1}{5}\sqrt{x^2 + 25})^5}{5} + 3 \cdot \frac{(\frac{1}{5}\sqrt{x^2 + 25})^3}{3} - \frac{1}{5}\sqrt{x^2 + 25} \right] + \sqrt{25 + x^2} + C$$