$\int \sqrt{4-5x^2}\,dx$

Here are two solutions:

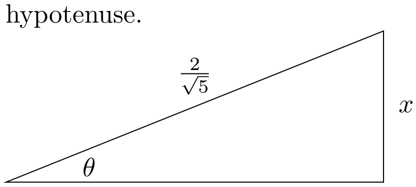
Solution 1

First, some algebra:

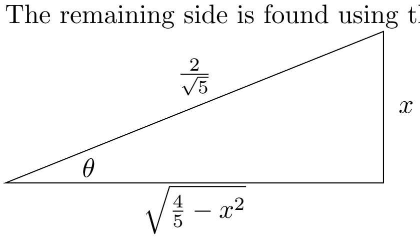
$$\int \sqrt{4-5x^2}\,dx = \int \sqrt{5(\frac{4}{5}-x^2)}\,dx = \int \sqrt{5}\sqrt{\frac{4}{5}-x^2}\,dx = \sqrt{5}\int \sqrt{\frac{4}{5}-x^2}\,dx.$$

Because of seeing $\sqrt{a^2-x^2}$ in the integral with $a = \frac{2}{\sqrt{5}}$, we apply trig substitution with $x = \frac{2}{\sqrt{5}}\sin\theta$, for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

So, $dx = \frac{2}{\sqrt{5}}\cos\theta\,d\theta$ and we use $\sin\theta = \frac{x}{\frac{2}{\sqrt{5}}}$ to draw a right triangle with x as the opposite side at $\frac{2}{\sqrt{5}}$ as the hypotenuse.



The remaining side is found using the Pythagorean Theorem, and we have



So our integral becomes

$$\sqrt{5}\int \sqrt{\frac{4}{5}-x^2}\,dx = 2\int \sqrt{\frac{4}{5}-x^2}\cos\theta\,d\theta$$

We still need to replace $\sqrt{\frac{4}{5}-x^2}$, so we look to the side of the triangle with that quantity (it is adjacent to θ) and the number side (which is the hypotenuse). Since $\cos\theta = \frac{\sqrt{\frac{4}{5}-x^2}}{\frac{2}{\sqrt{5}}}$, we have $\sqrt{\frac{4}{5}-x^2} = \frac{2}{\sqrt{5}}\cos\theta$. So, our integral becomes

$$\begin{aligned}
\frac{4}{\sqrt{5}}\int \cos\theta\cdot\cos\theta\,d\theta &= \frac{4}{\sqrt{5}}\int \cos^2\theta\,d\theta \\
&= \frac{4}{2\sqrt{5}}\int 1+\cos 2\theta\,d\theta \\
&= \frac{4}{2\sqrt{5}}\theta + \frac{4}{4\sqrt{5}}\sin 2\theta + C \\
&= \frac{4}{2\sqrt{5}}\theta + \frac{4}{4\sqrt{5}}2\sin\theta\cos\theta + C \\
&= \frac{4}{2\sqrt{5}}\sin^{-1}\frac{\sqrt{5}x}{2} + \frac{1}{4\sqrt{5}}\cdot 2\frac{x}{2/\sqrt{5}}\cdot\frac{\sqrt{\frac{4}{5}-x^2}}{2/\sqrt{5}} + C.
\end{aligned}$$

See the next page for another solution:

Solution 2

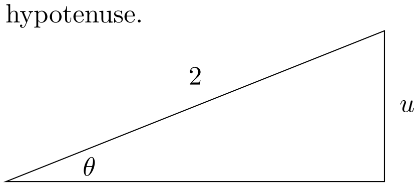
$$\int \sqrt{4-5x^2}\,dx = \int \sqrt{4-(\sqrt{5}x)^2}\,dx$$

If $u = \sqrt{5}x$, then $du = \sqrt{5}\,dx$, so the integral is equal to

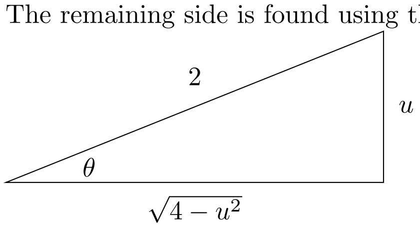
$$\frac{1}{\sqrt{5}} \int \sqrt{4-u^2}\,du.$$

Trig sub $u = \sin\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

So, $du = 2\cos\theta\,d\theta$ and we use $\sin\theta = \frac{u}{2}$ to draw a right triangle with u as the opposite side at 2 as the hypotenuse.



The remaining side is found using the Pythagorean Theorem, and we have



So our integral becomes

$$\frac{1}{\sqrt{5}} \int \sqrt{4-u^2}\,du = \frac{2}{\sqrt{5}} \int \sqrt{4-u^2}\cos\theta\,d\theta$$

We still need to replace $\sqrt{4-u^2}$, so we look to the side of the triangle with that quantity (it is adjacent to θ) and the number side (which is the hypotenuse). Since $\cos\theta = \frac{\sqrt{4-u^2}}{2}$, we have $\sqrt{4-u^2} = 2\cos\theta$. So, our integral becomes

$$\begin{aligned}
\frac{4}{\sqrt{5}} \int \cos\theta \cdot \cos\theta\,d\theta &= \frac{4}{\sqrt{5}} \int \cos^2\theta\,d\theta \\
&= \frac{4}{2\sqrt{5}} \int 1 + \cos 2\theta\,d\theta \\
&= \frac{4}{2\sqrt{5}}\theta + \frac{4}{4\sqrt{5}} \sin 2\theta + C \\
&= \frac{4}{2\sqrt{5}}\theta + \frac{4}{4\sqrt{5}} \cdot 2\sin\theta\cos\theta + C \\
&= \frac{4}{2\sqrt{5}} \sin^{-1}\frac{u}{2} + \frac{4}{4\sqrt{5}} \cdot 2 \cdot \frac{u}{2} \cdot \frac{\sqrt{1-u^2}}{2} + C. \\
&= \frac{4}{2\sqrt{5}} \sin^{-1}\frac{\sqrt{5}x}{2} + \frac{4}{4\sqrt{5}} \cdot 2 \cdot \frac{\sqrt{5}x}{2} \cdot \frac{\sqrt{1-5x^2}}{2} + C.
\end{aligned}$$