Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n - 1}$ diverge, converge absolutely, or converge conditionally?

Solution 1

We need to find a fixed number $K > 0$ such that

$$\frac{1}{5^n - 1} \leq \frac{K}{5^n}.$$

In our scratch work, by multiplying both sides by $5^n(5^n - 1)$, we want the following inequality to be true:

$$5^n \leq K(5^n - 1)$$

Now, this inequality is false when $K = 1$. However, when $K = 2$, this inequality is already true for $n \geq 1$. (Note when $K = 2$ and $n = 1$, we are looking at $5 \leq 8$.

So, with that scratch work done, let us begin:

$$5^n \leq 2(5^n - 1)$$

is true for all $n \geq 1$. Then, dividing both sides of this inequality by $5^n(5^n - 1)$, we get

$$\frac{1}{5^n - 1} \leq \frac{2}{5^n}.$$

Since

$$\frac{1}{5^n + 1} \leq \frac{2}{5^n}$$

and since $\displaystyle\sum_{n=1}^{\infty} \frac{2}{5^n}$ converges by the Geometric Series Test ($r = \frac{1}{5}$), the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n + 1}$ converges by the Direct Comparison Test.

Since $\sum |a_n| = \sum a_n$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n - 1}$ converges absolutely.

Solution 2

Let $a_n = \frac{1}{5^n - 1}$ and $b_n = \frac{1}{5^n}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{5^n - 1}{5^n} \\
&= \lim_{n\to\infty} \frac{5^n \ln 5}{5^n \ln 5} \quad \text{by L'hopital's rule} \\
&= \lim_{n\to\infty} 1 \\
&= 1
\end{aligned}$$

which is a finite positive number. Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$ converges by the Geometric Series Test ($r = \frac{1}{5}$), the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n - 1}$ converges by the Limit Comparison Test.

Since $\sum |a_n| = \sum a_n$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n - 1}$ converges absolutely.