Proving a variant of a polynomial positivity conjecture

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Bessis-Moussa-Villani Conjecture: 1975

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In 2004, Lieb and Seiringer proved the following statement is equivalent:

Reformulation

Let A and B be $n \times n$ positive semidefinite matrices. Let $p(t) = \operatorname{tr}((A + tB)^m)$. For every r, the coefficient of t^r in the polynomial $p(t)$ is non-negative.

Prior conjecture

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Technique:

- $S_{m,r}$: sum of all words of length m with r Bs and $m - r$ As
- Algebra $\mathbb{R}\langle X, Y\rangle$ of polynomials in X, Y non-commuting
- When is $S_{m,r}$ cyclically-equivalent to sums of Hermitian squares in $\mathbb{R}\langle X, Y\rangle$?

Side note: the terms $AABBAB$ and $BBABAA$ of $S_{6,3}$ are cyclically equivalent.

Klep, Schweighofer; Collins, Dykema, Torres-Ayala

$S_{m,r}$ cyclically-equivalent to sums of Hermitian squares in $\mathbb{R}\langle X, Y\rangle$ is sufficient for BMV in (m, r) case.

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- $r \in \{0, 1, 2, 4\}$
- $m = 14$ and $r = 6$
- $m \in \{7, 11\}$ and $r = 3$

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$S_{m,r}$ not cyclically-equivalent to sums of Hermitian squares if

- $m \geq 12$ and $r = 3$
- $m \in \{6, 8, 9, 10\}$ and $r = 3$
- $m \geq 10$ and $5 \leq r \leq m - 5$ and m or r odd
- $m = 12$ and $r = 6$

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- Something else must change...
- After changing the conditions of the conjecture, is there something true (and interesting)?
- Instead of finding SOS (up to cyclic rotation) in the two non-commutative variables A and B, find SOS in $2(n + \binom{n}{2})$ commutative variables.

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Let A and B be $n \times n$ symmetric matrices. If $m \geq 4$ and r are even, the coefficient of t^r in $p(t) = \mathrm{tr}((A + tB)^m)$ is non-negative.

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Theorem: Hillar 2007

If the BMV Conjecture is true for $m = m_0$, then the BMV Conjecture is true for all $m \leq m_0$.

Question regarding the Lieb-Seiringer formulation in light of Hillar's theorem:

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Is the alternate conjecture true for every $m \geq 4$ a power of 2? with $2m = r$?

Polynomial in $2[\binom{n}{2} + n]$ variables

For symmetric matrices A and B, if $m \geq 4$ and r are even, the coefficient of t^r in $p(t) = \mathrm{tr}[(A + tB)^m]$ is non-negative.

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- Let A and B be general symmetric matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix}$$

when $n = 2$

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Polynomial in a_{ij} and b_{ij}

For fixed n, m, r, the coefficient of t^r in $p(t)$ is a polynomial in the $2[\binom{n}{2} + n]$ variables a_{ij} and b_{ij} with $i \leq j$.

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Question

Is this multivariate polynomial non-negative?

Coefficient of t^r in $p(t)$: polynomial in $a_{i,j}$ and $b_{i,j}$

If $n = 2$ and $m = 4$, the coefficient of t^2 in $p(t)$ is

$6a_{11}^2 b_{11}^2 + 4a_{12}^2 b_{11}^2 + 16a_{11}a_{12}b_{11}b_{12} + 8a_{12}a_{22}b_{11}b_{12} + 4a_{11}^2 b_{12}^2 + 12a_{12}^2 b_{12}^2 + 4a_{11}a_{22}b_{12}^2 + 4a_{22}^2 b_{12}^2 + 4a_{12}^2 b_{11}b_{22} + 8a_{11}a_{12}b_{12}b_{22} + 16a_{12}a_{22}b_{12}b_{22} + 4a_{12}^2 b_{22}^2 + 6a_{22}^2 b_{22}^2$.

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Example

$$A = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -117 \\ -117 & 10 \end{bmatrix}$$

$p(t) = 381577202t^4 - 12044936t^3 + 371308t^2 - 4360t + 50$

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If $n = 3$ and $m = 4$, the coefficient of t^2 in $p(t)$ is

$$\begin{aligned}\nu = {} & 6a_{11}^2b_{11}^2 + 4a_{12}^2b_{11}^2 + 4a_{13}^2b_{11}^2 + 16a_{11}a_{12}b_{11}b_{12} + 8a_{12}a_{22}b_{11}b_{12} + 8a_{13}a_{23}b_{11}b_{12} + \\ & 4a_{11}^2b_{12}^2 + 12a_{12}^2b_{12}^2 + 4a_{13}^2b_{12}^2 + 4a_{11}a_{22}b_{12}^2 + 4a_{22}^2b_{12}^2 + 4a_{23}^2b_{12}^2 + 16a_{11}a_{13}b_{11}b_{13} + \\ & 8a_{12}a_{23}b_{11}b_{13} + 8a_{13}a_{33}b_{11}b_{13} + 16a_{12}a_{13}b_{12}b_{13} + 8a_{11}a_{23}b_{12}b_{13} + 8a_{22}a_{23}b_{12}b_{13} + \\ & 8a_{23}a_{33}b_{12}b_{13} + 4a_{11}^2b_{13}^2 + 4a_{12}^2b_{13}^2 + 12a_{13}^2b_{13}^2 + 4a_{23}^2b_{13}^2 + 4a_{11}a_{33}b_{13}^2 + 4a_{33}^2b_{13}^2 + 4a_{12}^2b_{11}b_{22} + \\ & 8a_{11}a_{12}b_{12}b_{22} + 16a_{12}a_{22}b_{12}b_{22} + 8a_{13}a_{23}b_{12}b_{22} + 8a_{12}a_{23}b_{13}b_{22} + 4a_{12}^2b_{22}^2 + 6a_{22}^2b_{22}^2 + \\ & 4a_{23}^2b_{22}^2 + 8a_{12}a_{13}b_{11}b_{23} + 8a_{11}a_{13}b_{12}b_{23} + 8a_{13}a_{22}b_{12}b_{23} + 16a_{12}a_{23}b_{12}b_{23} + 8a_{13}a_{33}b_{12}b_{23} + \\ & 8a_{11}a_{12}b_{13}b_{23} + 8a_{12}a_{22}b_{13}b_{23} + 16a_{13}a_{23}b_{13}b_{23} + 8a_{12}a_{33}b_{13}b_{23} + 8a_{12}a_{13}b_{22}b_{23} + \\ & 16a_{22}a_{23}b_{22}b_{23} + 8a_{23}a_{33}b_{22}b_{23} + 4a_{12}^2b_{23}^2 + 4a_{13}^2b_{23}^2 + 4a_{22}^2b_{23}^2 + 12a_{23}^2b_{23}^2 + 4a_{22}a_{33}b_{23}^2 + \\ & 4a_{33}^2b_{23}^2 + 4a_{13}^2b_{11}b_{33} + 8a_{13}a_{23}b_{12}b_{33} + 8a_{11}a_{13}b_{13}b_{33} + 8a_{12}a_{23}b_{13}b_{33} + 16a_{13}a_{33}b_{13}b_{33} + \\ & 4a_{23}^2b_{22}b_{33} + 8a_{12}a_{13}b_{23}b_{33} + 8a_{22}a_{23}b_{23}b_{33} + 16a_{23}a_{33}b_{23}b_{33} + 4a_{13}^2b_{33}^2 + 4a_{23}^2b_{33}^2 + 6a_{33}^2b_{33}^2.\end{aligned}$$

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\begin{aligned}
\nu = {} & 70a_{11}^4 b_{11}^4 + 120a_{11}^2 a_{12}^2 b_{11}^4 + 20a_{12}^4 b_{11}^4 + 40a_{11} a_{12}^2 a_{22} b_{11}^4 + 8a_{12}^2 a_{22}^2 b_{11}^4 + 320a_{11}^3 a_{12} b_{11}^3 b_{12} + 320a_{11} a_{12}^3 b_{11}^3 b_{12} + \\
& 160a_{11}^2 a_{12} a_{22} b_{11}^3 b_{12} + 128a_{12}^3 a_{22} b_{11}^3 b_{12} + 64a_{11} a_{12} a_{22}^2 b_{11}^3 b_{12} + 16a_{12} a_{22}^3 b_{11}^3 b_{12} + 120a_{11}^4 b_{11}^2 b_{12}^2 + \\
& 720a_{11}^2 a_{12}^2 b_{11}^2 b_{12}^2 + 240a_{12}^4 b_{11}^2 b_{12}^2 + 80a_{11}^3 a_{22} b_{11}^2 b_{12}^2 + 576a_{11} a_{12}^2 a_{22} b_{11}^2 b_{12}^2 + 48a_{11}^2 a_{22}^2 b_{11}^2 b_{12}^2 + 216a_{12}^2 a_{22}^2 b_{11}^2 b_{12}^2 + \\
& 24a_{11} a_{22}^3 b_{11}^2 b_{12}^2 + 8a_{22}^4 b_{11}^2 b_{12}^2 + 320a_{11}^3 a_{12} b_{11} b_{12}^3 + 640a_{11} a_{12}^3 b_{11} b_{12}^3 + 384a_{11}^2 a_{12} a_{22} b_{11} b_{12}^3 + 480a_{12}^3 a_{22} b_{11} b_{12}^3 + \\
& 288a_{11} a_{12} a_{22}^2 b_{11} b_{12}^3 + 128a_{12} a_{22}^3 b_{11} b_{12}^3 + 20a_{11}^4 b_{12}^4 + 240a_{11}^2 a_{12}^2 b_{12}^4 + 140a_{12}^4 b_{12}^4 + 32a_{11}^3 a_{22} b_{12}^4 + 360a_{11} a_{12}^2 a_{22} b_{12}^4 + \\
& 36a_{11}^2 a_{22}^2 b_{12}^4 + 240a_{12}^2 a_{22}^2 b_{12}^4 + 32a_{11} a_{22}^3 b_{12}^4 + 20a_{22}^4 b_{12}^4 + 80a_{11}^2 a_{12}^2 b_{11}^3 b_{22} + 32a_{12}^4 b_{11}^3 b_{22} + 64a_{11} a_{12}^2 a_{22} b_{11}^3 b_{22} + \\
& 24a_{12}^2 a_{22}^2 b_{11}^3 b_{22} + 160a_{11}^3 a_{12} b_{11}^2 b_{12} b_{22} + 384a_{11} a_{12}^3 b_{11}^2 b_{12} b_{22} + 192a_{11}^2 a_{12} a_{22} b_{11}^2 b_{12} b_{22} + 288a_{12}^3 a_{22} b_{11}^2 b_{12} b_{22} + \\
& 144a_{11} a_{12} a_{22}^2 b_{11}^2 b_{12} b_{22} + 64a_{12} a_{22}^3 b_{11}^2 b_{12} b_{22} + 40a_{11}^4 b_{11} b_{12}^2 b_{22} + 576a_{11}^2 a_{12}^2 b_{11} b_{12}^2 b_{22} + 360a_{12}^4 b_{11} b_{12}^2 b_{22} + \\
& 64a_{11}^3 a_{22} b_{11} b_{12}^2 b_{22} + 864a_{11} a_{12}^2 a_{22} b_{11} b_{12}^2 b_{22} + 72a_{11}^2 a_{22}^2 b_{11} b_{12}^2 b_{22} + 576a_{12}^2 a_{22}^2 b_{11} b_{12}^2 b_{22} + 64a_{11} a_{22}^3 b_{11} b_{12}^2 b_{22} + \\
& 40a_{22}^4 b_{11} b_{12}^2 b_{22} + 128a_{11}^3 a_{12} b_{12}^3 b_{22} + 480a_{11} a_{12}^3 b_{12}^3 b_{22} + 288a_{11}^2 a_{12} a_{22} b_{12}^3 b_{22} + 640a_{12}^3 a_{22} b_{12}^3 b_{22} + \\
& 384a_{11} a_{12} a_{22}^2 b_{12}^3 b_{22} + 320a_{12} a_{22}^3 b_{12}^3 b_{22} + 48a_{11}^2 a_{12}^2 b_{11}^2 b_{22}^2 + 36a_{12}^4 b_{11}^2 b_{22}^2 + 72a_{11} a_{12}^2 a_{22} b_{11}^2 b_{22}^2 + 48a_{12}^2 a_{22}^2 b_{11}^2 b_{22}^2 + \\
& 64a_{11}^3 a_{12} b_{11} b_{12} b_{22}^2 + 288a_{11} a_{12}^3 b_{11} b_{12} b_{22}^2 + 144a_{11}^2 a_{12} a_{22} b_{11} b_{12} b_{22}^2 + 384a_{12}^3 a_{22} b_{11} b_{12} b_{22}^2 + 192a_{11} a_{12} a_{22}^2 b_{11} b_{12} b_{22}^2 + \\
& 160a_{12} a_{22}^3 b_{11} b_{12} b_{22}^2 + 8a_{11}^4 b_{12}^2 b_{22}^2 + 216a_{11}^2 a_{12}^2 b_{12}^2 b_{22}^2 + 240a_{12}^4 b_{12}^2 b_{22}^2 + 24a_{11}^3 a_{22} b_{12}^2 b_{22}^2 + 576a_{11} a_{12}^2 a_{22} b_{12}^2 b_{22}^2 + \\
& 48a_{11}^2 a_{22}^2 b_{12}^2 b_{22}^2 + 720a_{12}^2 a_{22}^2 b_{12}^2 b_{22}^2 + 80a_{11} a_{22}^3 b_{12}^2 b_{22}^2 + 120a_{22}^4 b_{12}^2 b_{22}^2 + 24a_{11}^2 a_{12}^2 b_{11} b_{22}^3 + 32a_{12}^4 b_{11} b_{22}^3 + \\
& 64a_{11} a_{12}^2 a_{22} b_{11} b_{22}^3 + 80a_{12}^2 a_{22}^2 b_{11} b_{22}^3 + 16a_{11}^3 a_{12} b_{12} b_{22}^3 + 128a_{11} a_{12}^3 b_{12} b_{22}^3 + 64a_{11}^2 a_{12} a_{22} b_{12} b_{22}^3 + 320a_{12}^3 a_{22} b_{12} b_{22}^3 + \\
& 160a_{11} a_{12} a_{22}^2 b_{12} b_{22}^3 + 320a_{12} a_{22}^3 b_{12} b_{22}^3 + 8a_{11}^2 a_{12}^2 b_{22}^4 + 20a_{12}^4 b_{22}^4 + 40a_{11} a_{12}^2 a_{22} b_{22}^4 + 120a_{12}^2 a_{22}^2 b_{22}^4 + 70a_{22}^4 b_{22}^4.
\end{aligned}
$$

Result when $n = 3, m = 4, r = 2$

$\nu = \mathbf{z}_1^T Q_1 \mathbf{z}_1 + \sum_{\{i,j\} \in \binom{[n]}{2}} \mathbf{z}_{2,\{i,j\}}^T Q_2 \mathbf{z}_{2,\{i,j\}}$, where

$$Q_1 = \left(\begin{array}{ccc|ccc} 6 & 0 & 0 & 6 & 6 & 0 \\ 0 & 6 & 0 & 6 & 0 & 6 \\ 0 & 0 & 6 & 0 & 6 & 6 \\ \hline 6 & 6 & 0 & 12 & 6 & 6 \\ 6 & 0 & 6 & 6 & 12 & 6 \\ 0 & 6 & 6 & 6 & 6 & 12 \end{array}\right) \qquad \mathbf{z}_1 = \left(\begin{array}{c} a_{11}b_{11} \\ a_{22}b_{22} \\ a_{33}b_{33} \\ \hline a_{12}b_{12} \\ a_{13}b_{13} \\ a_{23}b_{23} \end{array}\right)$$

and

$$Q_2 = \left(\begin{array}{ccc|ccc} 4 & 4 & 4 & 2 & 2 & 2 \\ 4 & 4 & 4 & 2 & 2 & 2 \\ 4 & 4 & 4 & 2 & 2 & 2 \\ \hline 2 & 2 & 2 & 4 & 4 & 4 \\ 2 & 2 & 2 & 4 & 4 & 4 \\ 2 & 2 & 2 & 4 & 4 & 4 \end{array}\right), \qquad \mathbf{z}_{2,\{i,j\}} = \left(\begin{array}{c} a_{i1}b_{j1} \\ a_{i2}b_{j2} \\ a_{i3}b_{j3} \\ \hline b_{i1}b_{j1} \\ b_{i2}b_{j2} \\ b_{i3}b_{j3} \end{array}\right)$$

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Q_1 and Q_2 are PSD.

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$$Q_1 = \left(\begin{array}{ccc|ccc} 6 & 0 & 0 & 6 & 6 & 0 \\ 0 & 6 & 0 & 6 & 0 & 6 \\ 0 & 0 & 6 & 0 & 6 & 6 \\ \hline 6 & 6 & 0 & 12 & 6 & 6 \\ 6 & 0 & 6 & 6 & 12 & 6 \\ 0 & 6 & 6 & 6 & 6 & 12 \end{array}\right) \qquad \mathbf{z}_1 = \left(\begin{array}{c} a_{11}b_{11} \\ a_{22}b_{22} \\ a_{33}b_{33} \\ \hline a_{12}b_{12} \\ a_{13}b_{13} \\ a_{23}b_{23} \end{array}\right)$$

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Q_1 and Q_2 are PSD. Q_1 and Q_2 have only integer entries.

Observation for $m = 2r = 4$

Let Q_1 be the the $s \times s$ matrix whose $s = n + \binom{n}{2}$ rows and columns are indexed by the non-empty subsets of $\{1, \ldots, n\}$ of cardinality at most 2,

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$$
\begin{array}{c c}
 & \begin{array}{cccccc} \{1\} & \{2\} & \{3\} & \{1,2\} & \{1,3\} & \{2,3\} \end{array} \\
\begin{array}{c} \{1\} \\ \{2\} \\ \{3\} \\ \{1,2\} \\ \{1,3\} \\ \{2,3\} \end{array} &
\begin{bmatrix}
6 & 0 & 0 & 6 & 6 & 0 \\
0 & 6 & 0 & 6 & 0 & 6 \\
0 & 0 & 6 & 0 & 6 & 6 \\
6 & 6 & 0 & 12 & 6 & 6 \\
6 & 0 & 6 & 6 & 12 & 6 \\
0 & 6 & 6 & 6 & 6 & 12
\end{bmatrix}
\end{array}
$$

General result for $m = 2r = 4$

Let Q_1 be defined as before. Let Q_2 be the $(2n) \times (2n)$ block matrix of all 4s on diagonal blocks and all 2s on non-diagonal blocks.

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- Coefficient matching, term counting

Re-express Q_1

Let U be the the $s \times n$ matrix whose $s = n + \binom{n}{2}$ rows are indexed by the non-empty subsets of $\{1, \ldots, n\}$ of cardinality at most 2, and the (X, w)-entry of U is 1 if $w \in X$ and 0 otherwise.

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$$
\begin{array}{c c}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{array}{c} \{1\} \\ \{2\} \\ \{3\} \\ \{1,2\} \\ \{1,3\} \\ \{2,3\} \end{array} &
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}
\end{array}
$$

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$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} \{1\} \\ \{2\} \\ \{3\} \\ \{1,2\} \\ \{1,3\} \\ \{2,3\} \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

Then $Q_1 = 6U^T U$.

General result for $m = 2r = 4$

Theorem: K.

For all $n \times n$ symmetric matrices A and B, the coefficient ν of t^2 in $\mathrm{tr}[(A+tB)^4]$ is

$$\nu = 6 \sum_{i\in[n]} \left[\left(\sum_{k\in[n]} a_{ik} b_{ik} \right)^2 \right] +$$

$$\sum_{i\neq j} \left\{ \left[\sum_{k\in[n]} (a_{ik}b_{jk} + b_{ik}a_{jk}) \right]^2 + \left[\sum_{k\in[n]} a_{ik}b_{jk} \right]^2 + \left[\sum_{k\in[n]} b_{ik}a_{jk} \right]^2 \right\},$$

which is a sum of squares.

Conjecture

Let A and B be $n \times n$ symmetric matrices. If $m \geq 4$ and r are even, the coefficient of t^r in $p(t) = \mathrm{tr}((A + tB)^m)$ is non-negative.

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Let A and B be $n \times n$ symmetric matrices. If $m \geq 4$ and r are even, the coefficient of t^r in $p(t) = \mathrm{tr}((A + tB)^m)$ is non-negative.

In the case of $m = 4$ and $r = 2$, for all $n \times n$ symmetric A and B, the coefficient of t^r in $\mathrm{tr}((A + tB)^m)$ is **a sum of squares** in the entries of A and B.

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- When $m = 8$ and $r = 4$ and $n = 2$, the coefficient of t^r is SOS.

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- When $m = 8$ and $r = 4$ and $n = 2$, the coefficient of t^r is SOS.
- When $m = 8$ and $r = 4$ and $n = 3$, the SDP stops early, but a Q_1 matrix (all non-negative integers entries, all eigenvalues integer) found.

Thank you!

Thank you

Computational evidence

◂ Return

Given $n \times n$ symmetric matrices A and B, examine the coefficient of t^r in $p(t) = \mathrm{tr}[(A + tB)^m]$

```
n = 2; # Matrix size
m = 4; # Power to raise
var('t')

U = random_matrix(ZZ,n,n)
V = random_matrix(ZZ,n,n)
A = U + U.transpose()
B = V + V.transpose()
C = A + t*B;

print "A is\n", A
print "B is\n", B
p = (C^m).trace().expand();
print p
```

Output of code on four random runs

◂ Return

$$A = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -117 \\ -117 & 10 \end{bmatrix}$$

$$p(t) = 381577202t^4 - 12044936t^3 + 371308t^2 - 4360t + 50$$

$$A = \begin{bmatrix} -2 & 3 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}$$

$$p(t) = 34t^4 - 40t^3 + 268t^2 - 104t + 338$$

$$A = \begin{bmatrix} -2 & -6 \\ -6 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -14 \\ -14 & 0 \end{bmatrix}$$

$$p(t) = 79984t^4 + 131200t^3 + 85792t^2 + 26240t + 3200$$

$$A = \begin{bmatrix} 2 & -2 \\ -2 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 14 \\ 14 & -4 \end{bmatrix}$$

$$p(t) = 89888t^4 - 33920t^3 + 9984t^2 - 1280t + 128$$

Many runs at once, $m = 7$

Return

```
n = 3; # Matrix size
m = 7; # Power to raise
var('t')

for i in range(10):
  U = random_matrix(ZZ,n,n)
  V = random_matrix(ZZ,n,n)
  A = U + U.transpose()
  B = V + V.transpose()
  C = A + t*B;

  p = (C^m).trace().expand();
  print p
```

Many runs at once, $m = 7$

Output

Return

- $-95671736t^7 + 29943578t^6 + 1060317328t^5 - 4542979210t^4 + 10319766296t^3 - 13716905776t^2 + 9467623928t - 1691252512$
- $312057256t^7 + 1022191128t^6 + 149902256t^5 + 255541160t^4 - 1664278t^3 + 19766544t^2 - 1346282t + 531142$
- $-174012t^7 - 1611316t^6 - 8077524t^5 - 22798748t^4 - 41833204t^3 - 32404764t^2 + 9568804t + 97330764$
- $-12634864t^7 - 57997576t^6 - 72312128t^5 - 49399952t^4 - 21213920t^3 - 6139952t^2 - 1207024t - 140792$
- $1539019131802084208t^7 + 2360886718890898880t^6 + 2022994178604845 6t^5 + 1180680267203360t^4 + 4700276636966 4t^3 + 1396623702264t^2 + 25768743560t + 328715088$
- $-34867607296t^7 + 91863481472t^6 - 58312727424t^5 + 29550129280t^4 - 7922606720t^3 + 1486079616t^2 - 155697024t + 8278144$
- $109135872t^7 - 20177920t^6 + 54513536t^5 - 10275328t^4 + 8433488t^3 - 1425872t^2 + 388808t - 31280$
- $-10020024t^7 - 11069128t^6 + 13630960t^5 - 29075256t^4 + 18846394t^3 - 13106800t^2 + 3874528t - 1192956$
- $-882292t^7 + 13180104t^6 - 77308980t^5 + 276714060t^4 - 607576620t^3 +$

Many runs at once, $m = 7$

What happens when m is even?

Return

```
n = 3; # Matrix size
m = 8; # Power to raise
var('t')

for i in range(5):
  U = random_matrix(ZZ,n,n)
  V = random_matrix(ZZ,n,n)
  A = U + U.transpose()
  B = V + V.transpose()
  C = A + t*B;

  p = (C^m).trace().expand();
  print p
```

Many runs at once, $m = 8$

Output

Return

- $356425679360t^8 + 397892962304t^7 + 232469246464t^6 + 88237898240t^5 + 23408069312t^4 + 4394028928t^3 + 568426016t^2 + 46354784t + 1871842$
- $1154t^8 + 10816t^7 + 233512t^6 + 1276928t^5 + 13103020t^4 + 37485376t^3 + 219837288t^2 + 205717888t + 610557282$
- $756943257986t^8 + 6854925869444t^7 + 2309050799552t^6 + 1011525059072t^5 + 1641382066432t^4 + 352517873664t^3 + 433306173440t^2 + 34798141440t + 39968841728$
- $6891317142048t^8 + 29661542945280t^7 + 66321070259520t^6 + 93604309846144t^5 + 89845491346368t^4 + 58944168619008t^3 + 25688348653568t^2 + 6758373842944t + 839336551680$
- $125892343202t^8 - 10754192688t^7 + 18894255768t^6 - 3222482832t^5 + 989635820t^4 - 94745552t^3 + 19698136t^2 - 124528t + 221442$