Does $\sum_{n=2}^{\infty} \frac{n^2+2}{n^3-1}$ diverge, converge absolutely, or converge conditionally?

Solution 1

Consider the following string of equalities/inequalities:

$$\frac{1}{n} = \frac{n^2}{n^3} \leq \frac{n^2}{n^3-1} \leq \frac{n^2+2}{n^3-1}$$

Why is each expression larger than the next? Either the numerator stayed the same with the denominator decreasing (by 1) or the denominator stayed the same with the numerator increasing (by 2). In summary,

$$\frac{1}{n} \leq \frac{n^2+2}{n^3-1}.$$

Since the series $\sum \frac{1}{n}$ diverges by the p-series test, the series $\sum_{n=2}^{\infty} \frac{n^2+2}{n^3-1}$ diverges by the Direct Comparison Test.

Solution 2

The series $\sum \frac{1}{n}$ diverges by the p-test. Let $a_n = \frac{n^2+2}{n^3-1}$ and $b_n = \frac{1}{n}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^3+2n}{n^3-1} \\
&= \lim_{n\to\infty} \frac{3n^2+2}{3n^2} \text{ by l'hopital} \\
&= \lim_{n\to\infty} \frac{6n}{6n} \text{ by l'hopital} \\
&= \lim_{n\to\infty} \frac{6}{6} \text{ by l'hopital} \\
&= 1
\end{aligned}$$

The Limit Comparison Test applies, since this limit was a positive, finite number. Therefore, the series $\sum \frac{n^2+2}{n^3-1}$ diverges by the Limit Comparison Test.