Does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverge, converge absolutely, or converge conditionally?

Solution

The function $f(x) = \frac{1}{x \ln x}$ is continuous, positive, and decreasing on $[3, \infty)$. We consider the integral

$$\int_3^{\infty} \frac{1}{x \ln x}\, dx = \lim_{t \to \infty} \int_3^t \frac{1}{x \ln x}\, dx.$$

We do the indefinite integral using the substitution $u = \ln x$, so $du = \frac{1}{x}\, dx$:

$$\int \frac{1}{x \ln x}\, dx = \int \frac{1}{u}\, du = \ln |u| + C = \ln |\ln(x)| + C$$

so, back to the improper integral,

$$\begin{aligned}
\int_3^{\infty} \frac{1}{x \ln x}\, dx &= \lim_{t \to \infty} \int_3^t \frac{1}{x \ln x}\, dx \\
&= \lim_{t \to \infty} \left[\ln |\ln(t)| - \ln |\ln(3)|\right] \\
&= \infty.
\end{aligned}$$

Since the integral $\displaystyle\int_3^{\infty} \frac{1}{x \ln x}\, dx$ diverges, the series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n}$ diverges by the Integral Test. So the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges as well.