Does $\sum_{n=1}^{\infty} \frac{n^2-1}{n^4+7}$ diverge, converge absolutely, or converge conditionally?

Solution 1

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series which converges by the p-series test. If we use $a_n = \frac{n^2-1}{n^4+7}$ and $b_n = \frac{1}{n^2}$, then

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^2-1}{n^4+7} \cdot \frac{n^2}{1} \\
&= \lim_{n\to\infty} \frac{n^4-n}{n^4+7} \\
&= \lim_{n\to\infty} \frac{4n^3-1}{4n^3} \text{ using L'hopital} \\
&= \lim_{n\to\infty} \frac{12n^2}{12n^2} \text{ using L'hopital} \\
&= \lim_{n\to\infty} 1 \\
&= 1
\end{aligned}$$

So by the Limit Comparison Test, the series $\sum_{n=1}^{\infty} \frac{n^2-1}{n^4+7}$ converges.

Since $\sum |a_n| = \sum a_n$, the series $\sum_{n=1}^{\infty} \frac{n^2-1}{n^4+7}$ converges absolutely.

Solution 2 (comment of method)

The Direct Comparison Test will also work, but will involve finding first finding a fixed value of $K > 0$ such that

$$\frac{n^2-1}{n^3+7} \geq K \cdot \frac{1}{n}.$$

As the previous several series have shown, this will involve a bit of work to the point that the Limit Comparison Test (solution 1) will be easier/quicker.