$$\int x\cos^2 x\,dx$$

Solution

Let $u = x$ and $dv = \cos^2 x\,dx$. We first find the integral of $\cos^2 x$.

- One way to evaluate $\int \cos^2 x\,dx$ is apply integration by parts with $u = \cos x$ and $dv = \cos x\,dx$. Then $du = -\sin x\,dx$ and $v = \sin x$, so

  $$\int \cos^2 x\,dx = \cos x\sin x + \int \sin^2\,dx = \cos x\sin x + \int (1-\cos^2 x)\,dx = \cos x\sin x + x - \int \cos^2 x\,dx$$

  Since

  $$\int \cos^2 x\,dx = \cos x\sin x + x - \int \cos^2 x\,dx$$

  by algebra we have

  $$\int \cos^2 x\,dx = \frac{\cos x\sin x + x}{2} + C.$$

- Alternately, you can evalute $\int \cos^2 x\,dx$ by using an identity for $\cos^2 x$.

  $$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}\int 1+\cos 2x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

We integrated $\cos^2 x$ two different ways with two different-looking answers. Recall that we did this because we started with an integration by parts and had $dv = \cos^2 x\,dx$. We can use either $v = \frac{1}{2}(\cos x\sin x + x)$ or $v = \frac{1}{2}x + \frac{1}{4}\sin 2x$ from our two methods. It is conceivable that one might work better than the other for finishing the integration by parts. In this example, both work out well, so we'll try this both ways:

- Suppose we use $v = \frac{1}{2}(\cos x\sin x + x)$. Note $du = dx$ since $u = x$. Then, our original integral is

  $$\int x\cos^2 x\,dx = \frac{x}{2}(\cos x\sin x + x) - \frac{1}{2}\int \cos x\sin x + x\,dx = \frac{x}{2}(\cos x\sin x + x) - \frac{1}{4}\sin^2 x + \frac{x^2}{2} + C.$$

- Suppose we use $v = \frac{1}{2}x + \frac{1}{4}\sin 2x$. Note $du = dx$ since $u = x$. Then our original integral is

  $$\int x\cos^2 x\,dx = \frac{x^2}{2} + \frac{x\sin 2x}{4} - \int \frac{1}{2}x + \frac{1}{4}\sin 2x\,dx = \frac{x^2}{2} + \frac{x\sin 2x}{4} - \frac{x^2}{4} + \frac{1}{8}\cos 2x + C.$$