$$\int \frac{1}{x^2+2x+3}\,dx$$

Solution

The denominator of this fraction won't factor nicely, so there is no need to bother with partial fractions. Instead, let's complete the square in the denominator:

$$\int \frac{1}{x^2+2x+3}\,dx = \int \frac{1}{x^2+2x+1+2}\,dx = \int \frac{1}{(x+1)^2+2}\,dx =$$

Let $u = x+1$. Then $du = dx$. So, the integral above is equal to

$$\int \frac{1}{u^2+2}\,du$$

While this looks okay, we cannot apply the rule for arctangent yet, because of the 2 term in the denominator. But, with some clever factoring,

$$\begin{aligned}
&\int \frac{1}{u^2+2}\,du \\
&\qquad = \int \frac{1}{2(\frac{1}{2}u^2+1)}\,du \\
&\qquad = \frac{1}{2}\int \frac{1}{\frac{1}{2}u^2+1}\,du \\
&\qquad = \frac{1}{2}\int \frac{1}{(\frac{u}{\sqrt{2}})^2+1}\,du
\end{aligned}$$

Substituting $w = \frac{u}{\sqrt{2}}$, we get $dw = \frac{1}{\sqrt{2}}\,du$, so the integral above is equal to

$$\begin{aligned}
&\frac{\sqrt{2}}{2}\int \frac{1}{w^2+1}\,dw \\
&= \frac{\sqrt{2}}{2}\tan^{-1}(w) + C \\
&= \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C \\
&= \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C.
\end{aligned}$$