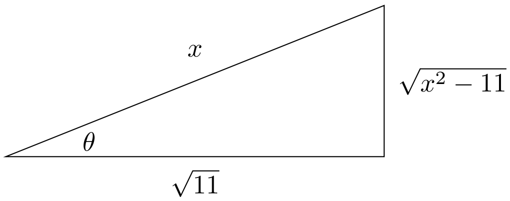
$$\int \frac{\sqrt{x^2-11}}{x}\,dx$$

Solution

Let $x = \sqrt{11}\sec\theta$ with θ in quadrants I or III. So, $dx = \sqrt{11}\sec\theta\tan\theta\,d\theta$ and we use $\sec\theta = \frac{x}{\sqrt{11}}$ to draw a right triangle with x as the hypotenuse side side and $\sqrt{11}$ as the adjacent side. The side opposite to θ is $\sqrt{x^2-11}$, which is found using the Pythagorean Theorem, and we have



Then $\tan\theta = \dfrac{\sqrt{x^2-11}}{\sqrt{11}}$. Based on this equation, we will use $\sqrt{11}\tan\theta = \sqrt{x^2-11}$ if it is helpful. Since $\cos\theta = \dfrac{\sqrt{11}}{x}$, we will use $x = \dfrac{\sqrt{11}}{\cos\theta} = \sqrt{11}\sec\theta$ if it is helpful.

So

$$\begin{aligned}
\int \frac{\sqrt{x^2-11}}{x}\,dx &= \int \frac{\sqrt{11}\tan\theta}{\sqrt{11}\sec\theta}\sqrt{11}\sec\theta\tan\theta\,d\theta \\
&= \sqrt{11}\int \tan^2\theta\,d\theta \\
&= \sqrt{11}\int (\sec^2\theta - 1)\,d\theta \\
&= \sqrt{11}\left[\tan\theta - \theta\right] + C \\
&= \sqrt{11}\left[\frac{\sqrt{x^2-11}}{\sqrt{11}} - \sec^{-1}\frac{x}{\sqrt{11}}\right] + C
\end{aligned}$$