Does $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ diverge, converge absolutely, or converge conditionally?

Solution 1

Note $2 \le n^2$ for all positive integers $n \ge 2$. By adding n^2 to both sides,

$$n^2 + 2 \le 2n^2.$$

By subtracting 1 from both sides,

$$n^2 \le 2n^2 - 2$$

Factoring the right side,

$$n^2 \le 2(n^2 - 1)$$

Dividing both sides by $n^2(n^2-1)$ we get

$$\frac{1}{n^2-1} \le \frac{2}{n^2}.$$

Since the series $\sum \frac{2}{n^2} = 2\sum \frac{1}{n^2}$ converges by the p-test, the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges by the Direct Comparison Test. Since all terms of the series are positive, the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges absolutely.

Solution 2

The series $\sum \frac{1}{n^2}$ converges by the p-test. Let $a_n = \frac{1}{n^2-1}$ and $b_n = \frac{1}{n^2}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^2}{n^2-1} \\
&= \lim_{n\to\infty} \frac{2n}{2n} \text{ by L'hopital} \\
&= \lim_{n\to\infty} \frac{1}{1} \\
&= 1
\end{aligned}$$

Since this limit is a finite, positive number, by the Limit Comparison Test, the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges. Since all terms of the series are positive, the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges absolutely.

Solution 3

By partial fraction decomposition, we get

$$\frac{1}{n^2-1} = \frac{1/2}{n-1} - \frac{1/2}{n+1}.$$

So, the originally given series can be rewritten as

$$\sum_{n=2}^{\infty} \left(\frac{1/2}{n-1} - \frac{1/2}{n+1} \right)$$

After cancellations, we get the sequence of partial sums:

$$s_n = \frac{1/2}{1} + \frac{1/2}{2} - \frac{1/2}{n} - \frac{1/2}{n+1}$$

Thus,

$$
\begin{aligned}
\lim_{n\to\infty} s_n &= \lim_{n\to\infty} \left(\frac{1/2}{1} + \frac{1/2}{2} - \frac{1/2}{n} - \frac{1/2}{n+1} \right) \\
&= \frac{1/2}{1} + \frac{1/2}{2} \\
&= \frac{1}{2} + \frac{1}{4} \\
&= \frac{3}{4}.
\end{aligned}
$$

So, the series converges (by definition) to $\frac{3}{4}$.

Since all terms of the series are positive, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges absolutely.