Does $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{100}}$ diverge, converge absolutely, or converge conditionally?

## Solution

The function $f(x) = \frac{1}{x(\ln x)^{100}}$ is continuous, positive, and decreasing on $[3, \infty)$.

We do the following indefinite integral using the substitution $u = \ln x$, so $du = \frac{1}{x}\, dx$:

$$\int \frac{1}{x(\ln x)^{100}}\, dx = \int u^{-100}\, du = \frac{-1}{99u^{99}} + C = \frac{-1}{99(\ln x)^{99}} + C$$

so

$$\begin{aligned}
\int_3^{\infty} \frac{1}{x(\ln x)^{100}}\, dx &= \lim_{t\to\infty} \int_3^{t} \frac{1}{x(\ln x)^{100}}\, dx \\
&= \lim_{t\to\infty} \left[ \frac{-1}{99(\ln t)^{99}} - \frac{-1}{99(\ln 3)^{99}} \right] \\
&= \frac{1}{99(\ln 3)^{99}}
\end{aligned}$$

Since the integral $\int_3^{\infty} \frac{1}{x \ln x}\, dx$ converges, the series $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^{100}}$ converges by the Integral Test. So the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges as well.

Since all terms (except when $n = 2$) are positive, we essentially have $\sum |a_n| = \sum a_n$, so the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{100}}$ converges absolutely.