Does $\sum_{n=2}^{\infty} \frac{\arctan n}{n^2+1}$ diverge, converge absolutely, or converge conditionally?

Solution 1

The function $f(x) = \frac{\arctan x}{x^2+1}$ is continuous, positive, and decreasing. We find the following indefinite integral using the substitution $u = \arctan x$, so $du = \frac{1}{x^2+1}\,dx$:

$$\begin{aligned}\int \frac{\arctan x}{x^2+1}\,dx &= \int u\,du \\ &= \frac{1}{2}u^2 + C \\ &= \frac{1}{2}(\arctan x)^2 + C\end{aligned}$$

So the definite, improper integral evaluates:

$$\begin{aligned}\int_2^{\infty} \frac{\arctan x}{x^2+1}\,dx &= \lim_{t\to\infty} \int_2^{t} \frac{\arctan x}{x^2+1}\,dx \\ &= \lim_{t\to\infty} \left(\frac{1}{2}(\arctan t)^2 - \frac{1}{2}(\arctan 2)^2\right) \\ &= \left(\frac{1}{2}\left(\frac{\pi}{2}\right)^2 - \frac{1}{2}(\arctan 2)^2\right)\end{aligned}$$

which converges. Since the integral $\int_2^{\infty} \frac{\arctan x}{x^2+1}\,dx$ converges, the series $\sum_{n=2}^{\infty} \frac{\arctan n}{n^2+1}$ converges. Since all terms of the series are positive, the series $\sum_{n=2}^{\infty} \frac{\arctan n}{n^2+1}$ converges absolutely.

Solution 2

Note that $\arctan n \leq \frac{\pi}{2}$ for all positive n. In fact, since π is just a little bigger than three and we divide by 2, we could just state

$$\arctan n \leq 1.6$$

By dividing both sides by n^2+1, we have

$$\frac{\arctan n}{n^2+1} \leq \frac{1.6}{n^2+1}$$

and in fact, we also have

$$\frac{\arctan n}{n^2+1} \leq \frac{1.6}{n^2+1} \leq \frac{1.6}{n^2}$$

so taking the outside two expressions (and skipping the expression in the middle), we have

$$\frac{\arctan n}{n^2+1} \leq \frac{1.6}{n^2}$$

Since the series $\sum \frac{1.6}{n^2} = 1.6 \sum \frac{1}{n^2}$ converges by the p-test, the series $\sum \frac{\arctan n}{n^2+1}$ converges by the Direct Comparison Test. Since all terms of the series are positive, the series $\sum_{n=2}^{\infty} \frac{\arctan n}{n^2+1}$ converges absolutely.