

Does  $\sum_{n=2}^{\infty} \frac{n}{n^3+1}$  diverge, converge absolutely, or converge conditionally?

### Solution 1

Note

$$\frac{1}{n^3+1} \leq \frac{1}{n^3}$$

so multiplying both sides of this inequality by  $n$ , we get

$$\frac{n}{n^3+1} \leq \frac{1}{n^2}.$$

Since the series  $\sum_{n=2}^{\infty} \frac{1}{n^2}$  converges by the  $p$ -test, the series  $\sum_{n=2}^{\infty} \frac{n}{n^3+1}$  converges by the Direct Comparison

Test. Since all terms are positive, the series  $\sum_{n=2}^{\infty} \frac{n}{n^3+1}$  converges absolutely.

### Solution 2

The series  $\sum \frac{1}{n^2}$  converges by the  $p$ -test. Let  $a_n = \frac{n}{n^3+1}$  and  $b_n = \frac{1}{n^2}$ .

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{n^3}{n^3+1} \\ &= \lim_{n \rightarrow \infty} \frac{3n^2}{3n^2} \text{ by l'hospital} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1} \\ &= 1 \end{aligned}$$

The Limit Comparison Test applies, since this limit was a positive, finite number. Therefore, the series  $\sum \frac{n}{n^3+1}$  converges by the Limit Comparison Test. Since all terms are positive, the series  $\sum_{n=2}^{\infty} \frac{n}{n^3+1}$  converges absolutely.