$$\int \frac{x^7}{\sqrt{25+3x^2}}\,dx$$

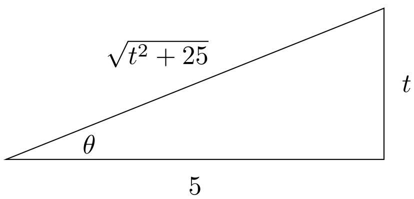
Solution 1

$$\int \frac{x^7}{\sqrt{25+3x^2}}\,dx = \int \frac{x^7}{\sqrt{25+(\sqrt{3}x)^2}}\,dx$$

and say $t = \sqrt{3}x$, so $dt = \sqrt{3}\,dx$ and x can be replaced with $\frac{t}{\sqrt{3}}$, so the integral becomes

$$\int \frac{(\frac{t}{\sqrt{3}})^7}{\sqrt{25+t^2}} \cdot \frac{1}{\sqrt{3}}\,dt = \frac{1}{3^4}\int \frac{t^7}{\sqrt{25+t^2}}\,dt$$

Let $t = 5\tan\theta$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. So, $dt = 5\sec^2\theta\,d\theta$ and we use $\tan\theta = \frac{t}{5}$ to draw a right triangle with t as the opposite side at 5 as the adjacent side. The hypotenuse is found using the Pythagorean Theorem, and we have



Then $\cos\theta = \dfrac{5}{\sqrt{t^2+25}}$. Based on this equation, we will use either $\sqrt{t^2+25} = \dfrac{5}{\cos\theta} = 5\sec\theta$ or we will use $\dfrac{\cos\theta}{5} = \dfrac{1}{\sqrt{t^2+25}}$ if they are helpful.

So

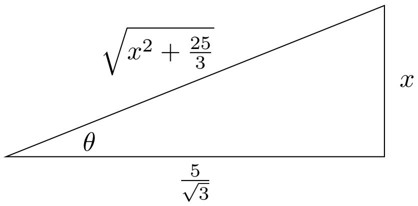
$$\frac{1}{3^4}\int \frac{t^7}{\sqrt{25+t^2}}\,dt = \frac{1}{3^4}\int (5\tan\theta)^7 \cdot \frac{\cos\theta}{5} 5\sec^2\theta\,d\theta = \frac{5^7}{3^4}\int \tan^7\theta\sec\theta\,d\theta$$

$$\begin{aligned}
\frac{5^7}{3^4}\int \tan^7\theta\sec\theta\,d\theta &= \frac{5^7}{3^4}\int \tan^6\theta\cdot\sec\theta\tan\theta\,d\theta \\
&= \frac{5^7}{3^4}\int (\tan^2\theta)^3\cdot\sec\theta\tan\theta\,d\theta \\
&= \frac{5^7}{3^4}\int (\sec^2\theta-1)^3\cdot\sec\theta\tan\theta\,d\theta \qquad u=\sec\theta \quad du=\sec\theta\tan\theta\,d\theta \\
&= \frac{5^7}{3^4}\int (u^2-1)^3\,du \\
&= \frac{5^7}{3^4}\int (u^6-3u^4+3u^2-1)\,du \\
&= \frac{5^7}{3^4}\left[\frac{u^7}{7} - 3\cdot\frac{u^5}{5} + 3\cdot\frac{u^3}{3} - u\right] + C \\
&= \frac{5^7}{3^4}\left[\frac{(\frac{1}{5}\sqrt{t^2+25})^7}{7} - 3\cdot\frac{(\frac{1}{5}\sqrt{t^2+25})^5}{5} + 3\cdot\frac{(\frac{1}{5}\sqrt{t^2+25})^3}{3} - \frac{1}{5}\sqrt{t^2+25}\right] + C \\
&= \frac{5^7}{3^4}\left[\frac{(\frac{1}{5}\sqrt{(\sqrt{3}x)^2+25})^7}{7} - 3\cdot\frac{(\frac{1}{5}\sqrt{(\sqrt{3}x)^2+25})^5}{5} + 3\cdot\frac{(\frac{1}{5}\sqrt{(\sqrt{3}x)^2+25})^3}{3} - \frac{1}{5}\sqrt{(\sqrt{3}x)^2+25}\right] + C
\end{aligned}$$

Solution 2

$$\int \frac{x^7}{\sqrt{25+3x^2}}\,dx = \int \frac{x^7}{\sqrt{3(\frac{25}{3}+x^2)}}\,dx = \frac{1}{\sqrt{3}}\int \frac{x^7}{\sqrt{\frac{25}{3}+x^2}}\,dx$$

Let $x = \frac{5}{\sqrt{3}}\tan\theta$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. So, $dx = \frac{5}{\sqrt{3}}\sec^2\theta\,d\theta$ and we use $\tan\theta = \frac{x}{\frac{5}{\sqrt{3}}}$ to draw a right triangle with x as the opposite side at $\frac{5}{\sqrt{3}}$ as the adjacent side. The hypotenuse is found using the Pythagorean Theorem, and we have



Then $\cos\theta = \dfrac{\frac{5}{\sqrt{3}}}{\sqrt{x^2+\frac{25}{3}}}$. Based on this equation, we will use either $\sqrt{x^2+\dfrac{25}{3}} = \dfrac{\frac{5}{\sqrt{3}}}{\cos\theta} = \dfrac{5}{\sqrt{3}}\sec\theta$ or we will use $\dfrac{\cos\theta}{\frac{5}{\sqrt{3}}} = \dfrac{1}{\sqrt{x^2+\frac{25}{3}}}$ if they are helpful.

So

$$\frac{1}{\sqrt{3}}\int \frac{x^7}{\sqrt{\frac{25}{3}+x^2}}\,dx = \frac{1}{\sqrt{3}}\int \left(\frac{5}{\sqrt{3}}\tan\theta\right)^7 \cdot \frac{\cos\theta}{\frac{5}{\sqrt{3}}}\frac{5}{\sqrt{3}}\sec^2\theta\,d\theta = \frac{5^7}{3^4}\int \tan^7\theta\sec\theta\,d\theta$$

$$\begin{aligned}
\frac{5^7}{3^4}\int \tan^7\theta\sec\theta\,d\theta &= \frac{5^7}{3^4}\int \tan^6\theta\cdot\sec\theta\tan\theta\,d\theta \\
&= \frac{5^7}{3^4}\int (\tan^2\theta)^3\cdot\sec\theta\tan\theta\,d\theta \\
&= \frac{5^7}{3^4}\int (\sec^2\theta-1)^3\cdot\sec\theta\tan\theta\,d\theta \qquad u=\sec\theta \quad du=\sec\theta\tan\theta\,d\theta \\
&= \frac{5^7}{3^4}\int (u^2-1)^3\,du \\
&= \frac{5^7}{3^4}\int (u^6-3u^4+3u^2-1)\,du \\
&= \frac{5^7}{3^4}\left[\frac{u^7}{7}-3\cdot\frac{u^5}{5}+3\cdot\frac{u^3}{3}-u\right]+C \\
&= \frac{5^7}{3^4}\left[\frac{(\frac{\sqrt{3}}{5}\sqrt{x^2+\frac{25}{3}})^7}{7}-3\cdot\frac{(\frac{\sqrt{3}}{5}\sqrt{x^2+\frac{25}{3}})^5}{5}+3\cdot\frac{(\frac{\sqrt{3}}{5}\sqrt{x^2+\frac{25}{3}})^3}{3}-\frac{\sqrt{3}}{5}\sqrt{x^2+\frac{25}{3}}\right]+C
\end{aligned}$$