Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n-1}$ diverge, converge absolutely, or converge conditionally?

Solution 1

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series which diverges by the p-series test. If we use $a_n = \frac{1}{3n-1}$ and $b_n = \frac{1}{n}$, then

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{1}{3n-1} \cdot \frac{n}{1} \\
&= \lim_{n\to\infty} \frac{n}{3n-1} \\
&= \lim_{n\to\infty} \frac{1}{3} \text{ using L'hopital} \\
&= \frac{1}{3}
\end{aligned}$$

So by the Limit Comparison Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n-1}$ diverges.

Solution 2

The Direct Comparison Test will also work, but will involve finding first finding a fixed value of $K > 0$ such that

$$K \cdot \frac{1}{3n-1} \geq \frac{1}{n}.$$

Multiply both sides by $n(3n-1)$.

$$Kn \geq 3n - 1$$

This inequality will be true for all n if we choose $K = 3$. So we are done with the scratch work and instead begin our work:

Note that

$$3n \geq 3n - 1$$

Dividing both sides by $3n(3n-1)$, we get

$$\frac{1}{3n-1} \geq \frac{1}{3n}$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series which diverges by the p-series test. So the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n-1}$ diverges by the Direct Comparison Test.