$$\int x^2 \sin 5x \, dx$$

Comment on solutions

There are two ways to do this integral, though both are morally the same. Please follow along both, along with the commentary at the end of the second solution.

Solution 1

Let $u = x^2$ and $dv = \sin 5x \, dx$. Then $du = 2x \, dx$ and $v = -\frac{1}{5} \cos 5x$. So

$$\int x^2 \sin x \, dx = -\frac{x^2}{5} \cos 5x + \frac{2}{5} \int x \cos 5x \, dx$$

We need to integrate $x \cos 5x$.

- To evaluate $\int x \cos 5x \, dx$, we let $u = x$ and $dv = \cos 5x \, dx$. Then $du = dx$ and $v = \frac{1}{5} \sin 5x$. So

$$\int x \cos 5x \, dx = \frac{x}{5} \sin 5x - \frac{1}{5} \int \sin 5x \, dx = \frac{x}{5} \sin 5x + \frac{1}{25} \cos 5x + C.$$

Back to our original integral, by substituting the value of our side integral, we have

$$-\frac{x^2}{5} \cos 5x + \frac{2}{5} \left(\frac{x}{5} \sin 5x + \frac{1}{25} \cos 5x \right) + C.$$

Solution 2

We start with a simple substitution of $t = 5x$. Then $dt = 5 \, dx$. So

$$\int x^2 \sin 5x \, dx = \frac{1}{5} \int x^2 \sin t \, dt$$

There is still an x^2 in the intgeral, so using $t = 5x$, we solve for x to get $x = \frac{t}{5}$. Our integral becomes

$$\frac{1}{5} \int \frac{t^2}{5^2} \sin t \, dt = \frac{1}{125} \int t^2 \sin t \, dt.$$

If we for a moment ignore the $\frac{1}{125}$, then the integral is essentially $\int t^2 \sin t \, dt$. To do that, let $u = t^2$ and $dv = \sin t \, dt$. Then $du = 2t \, dt$ and $v = -\cos t$. So

$$\int t^2 \sin t \, dt = -t^2 \cos t + 2 \int t \cos t \, dt$$

- To evaluate $\int t \cos t \, dt$, we let $u = t$ and $dv = \cos t \, dt$. Then $du = dt$ and $v = \sin t$. So

$$\int t \cos t \, dt = t \sin t - \int \sin t \, dt = t \sin t + \cos t + C.$$

So,

$$\int t^2 \sin t \, dt = -t^2 \cos t + 2(t \sin t + \cos t) + C.$$

We recall the $\frac{1}{125}$ we have ignored, and replace every t with $5x$ for our final answer:

$$\frac{1}{125} \int t^2 \sin t \, dt = \frac{1}{125} \left(-(5x)^2 \cos(5x) + 2((5x) \sin(5x) + \cos(5x)) \right) + C.$$

Commentary: The idea behind the second solution is to try to not let the 5 in front of the x cause the continued trouble it did in the first solution. Note that both solutions were done using an integration by parts with an integration by parts inside. If it is possible to substitue things like $t = 6x$ or $t = x + 6$, you might often get a cleaner form of the integral. Typically, a number next to a variable (such as $6x$) doesn't bother me, and neither does a variable plus a number (such as $x + 6$). Things usually already get crazier with somethnig like $t = 6x + 7$.