$$\int \sin 3x \cos 5x \, dx$$

Solution 1

Let $u = \sin 3x$ and $dv = \cos 5x \, dx$. Then $du = 3\cos 3x \, dx$ and $v = \frac{1}{5}\sin 5x$. So

$$\int \sin 3x \cos 5x \, dx = \frac{1}{5}\sin 3x \sin 5x - \frac{3}{5}\int \cos 3x \sin 5x \, dx.$$

We now apply a second integration by parts, for the integral $\int \cos 3x \sin 5x \, dx$. Note that we must choose $u = \cos 3x$ and $dv = \sin 5x \, dx$ to avoid going around in circles. (If you chose $u = \sin 5x$, then you'll eventually get to the equation $0 = 0$. Since $u = \cos 3x$ and $dv = \sin 5x \, dx$, we get $du = -3\sin 3x \, dx$ and $v = -\frac{1}{5}\cos 5x$. The side integral is thus

$$\int \cos 3x \sin 5x \, dx = -\frac{1}{5}\cos 3x \cos 5x + \frac{3}{5}\int \sin 3x \cos 5x \, dx.$$

Placed in context with the original integral, we have

$$\int \sin 3x \cos 5x \, dx = \frac{1}{5}\sin 3x \sin 5x - \frac{3}{5}\left(-\frac{1}{5}\cos 3x \cos 5x + \frac{3}{5}\int \sin 3x \cos 5x \, dx\right).$$

We use I to denote our original integral $\int \sin 3x \cos 5x \, dx$. Then

$$I = \frac{1}{5}\sin 3x \sin 5x - \frac{3}{5}\left(-\frac{1}{5}\cos 3x \cos 5x + \frac{3}{5}I\right).$$

so

$$\begin{aligned} I &= \frac{1}{5}\sin 3x \sin 5x + \frac{3}{25}\cos 3x \cos 5x - \frac{9}{25}I \\ \frac{34}{25}I &= \frac{1}{5}\sin 3x \sin 5x + \frac{3}{25}\cos 3x \cos 5x + C \end{aligned}$$

and finally

$$\int \sin 3x \cos 5x \, dx = \frac{25}{34}\left(\frac{1}{5}\sin 3x \sin 5x + \frac{3}{25}\cos 3x \cos 5x\right) + C.$$

Solution 2

Let $u = \cos 5x$ and $dv = \sin 3x \, dx$. Then $du = -5\sin 5x \, dx$ and $v = -\frac{1}{3}\cos 3x$. So

$$\int \sin 3x \cos 5x \, dx = -\frac{1}{3}\cos 3x \cos 5x - \frac{5}{3}\int \cos 3x \sin 5x \, dx.$$

We now apply a second integration by parts, for the integral $\int \cos 3x \sin 5x \, dx$. Note that we must choose $u = \sin 5x$ and $dv = \cos 3x \, dx$ to avoid going around in circles. (If you chose $u = \cos 3x$, then you'll eventually get to the equation $0 = 0$. Since $u = \sin 5x$ and $dv = \cos 3x \, dx$, we get $du = 5\cos 5x \, dx$ and $v = \frac{1}{3}\sin 3x$. The side integral is thus

$$\int \cos 3x \sin 5x \, dx = \frac{1}{3}\sin 3x \sin 5x - \frac{5}{3}\int \sin 3x \cos 5x \, dx$$

Placed in context with the original integral, we have

$$\int \sin 3x \cos 5x \, dx = -\frac{1}{3}\cos 3x \cos 5x - \frac{5}{3}\left(\frac{1}{3}\sin 3x \sin 5x - \frac{5}{3}\int \sin 3x \cos 5x \, dx\right).$$

We use I to denote our original integral $\int \sin 3x \cos 5x \, dx$. Then

$$I = -\frac{1}{3} \cos 3x \cos 5x - \frac{5}{3} \left(\frac{1}{3} \sin 3x \sin 5x - \frac{5}{3} I \right).$$

so

$$I = -\frac{1}{3} \cos 3x \cos 5x - \frac{5}{9} \sin 3x \sin 5x + \frac{25}{9} I.$$

$$\frac{34}{9} I = -\frac{1}{3} \cos 3x \cos 5x - \frac{5}{9} \sin 3x \sin 5x + C.$$

and finally

$$\int \sin 3x \cos 5x \, dx = \frac{9}{34} \left(-\frac{1}{3} \cos 3x \cos 5x - \frac{5}{9} \sin 3x \sin 5x \right) + C.$$

Solution 3

Apply the trig identity $\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)]$, with $\alpha = 3x$ and $\beta = 5x$.

$$\begin{aligned} \int \sin 3x \cos 5x \, dx &= \frac{1}{2} \int \sin(-2x) + \sin(8x) \, dx \\ &= \frac{1}{2} \int \sin(-2x) \, dx + \frac{1}{2} \int \sin(8x) \, dx \\ &= \frac{1}{4} \cos(-2x) - \frac{1}{16} \cos(8x) + C. \end{aligned}$$

where each integral is completed by substitution (using $u = -2x$ for the first integral and $u = 8x$ for the second integral).