Does $\displaystyle\sum_{n=1}^{\infty} \frac{\cos \frac{\pi}{3} n}{n^2}$ diverge, converge absolutely, or converge conditionally?

Note

The series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos \frac{\pi}{3} n}{n^2}$ is NOT an alternating series.

Solution

We consider

$$\sum_{n=1}^{\infty} \left| \frac{\cos \frac{\pi}{3} n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos \frac{\pi}{3} n|}{n^2}$$

Note that

$$0 \leq |\cos \frac{\pi}{3} n| \leq 1$$

so by dividing all three sides by n^2, we have

$$0 \leq \frac{|\cos \frac{\pi}{3} n|}{n^2} \leq \frac{1}{n^2}.$$

The series $\displaystyle\sum \frac{1}{n^2}$ converges by the p-test. Since, terms of the series $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos \frac{\pi}{3} n|}{n^2}$ are positive, by the Direct Comparison test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos \frac{\pi}{3} n|}{n^2}$ converges.

Therefore the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos \frac{\pi}{3} n}{n^2}$ converges by the Absolute Convergence Test. In fact, the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos \frac{\pi}{3} n}{n^2}$ is absolutely convergent.