Presenting solution to a definite integral with substitution

$$\text{Example:} \quad \int_3^5 \frac{2x}{(1+x^2)^2}\,dx$$

Solution 1 (preferred)

$$\text{indef:} \quad \int \frac{2x}{(1+x^2)^2}\,dx = \int \frac{1}{u^2}\,du = \int u^{-2}\,du = \frac{u^{-1}}{-1} + C = \frac{-1}{1+x^2} + C$$

$$\boxed{\begin{aligned} u &= 1+x^2 \\ \tfrac{du}{2x} &= dx \end{aligned}}$$

So, the definite integral is $\displaystyle\int_3^5 \frac{2x}{(1+x^2)^2}\,dx = \left.\frac{-1}{1+x^2}\right|_3^5 = \frac{-1}{26} + \frac{1}{10} = \frac{16}{260} = \frac{4}{65}$. ☺

- Usually least amount to write, especially if multiple substitutions or replacing back some x's
- Least amount of ways to possibly make a mistake

Solution 2

$$\int_3^5 \frac{2x}{(1+x^2)^2}\,dx = \int_{10}^{26} \frac{1}{u^2}\,du = \int_{10}^{26} u^{-2}\,du = \left.\frac{u^{-1}}{-1}\right|_{10}^{26} = \frac{-1}{26} + \frac{1}{10} = \frac{16}{260} = \frac{4}{65}. \text{ ☺}$$

$$\boxed{\begin{aligned} u &= 1+x^2 \\ \tfrac{du}{2x} &= dx \\ \text{if } x = 3&\text{, then } u = 10 \\ \text{if } x = 5&\text{, then } u = 26 \end{aligned}}$$

- Often involves more writing, and therefore takes up more **time**. Easier to make the mistake below.
- On rare occasion, this is the more useful way to evaluate a definite integral

Incorrect

$$\int_3^5 \frac{2x}{(1+x^2)^2}\,dx = \underbrace{\int_3^5 \frac{1}{u^2}\,du = \int_3^5 u^{-2}\,du = \left.\frac{u^{-1}}{-1}\right|_3^5}_{\text{☹}} = \left.\frac{-1}{1+x^2}\right|_3^5 = \frac{-1}{26} + \frac{1}{10} = \frac{16}{260} = \frac{4}{65}. \text{ ☹}$$

$$\boxed{\begin{aligned} u &= 1+x^2 \\ \tfrac{du}{2x} &= dx \end{aligned}}$$

☹ Give any of these three expressions to someone and they won't get $\frac{4}{65}$. ☹
☹ Instead, they will get $\frac{2}{15}$. ☹

- The second integral is **NOT** equal to the first because $\int_3^5 \frac{1}{u^2}\,du$ is the same as $\int_3^5 \frac{1}{x^2}\,dx$.
- The problem is that the 3 and 5 are x-values but a "du" integral needs u-values as endpoints!
- The same problem affects the several expressions $\int_3^5 u^{-2}\,du$ and $\frac{u^{-1}}{-1}\big|_3^5$ as well.

Acceptable, but why???

$$\int_3^5 \frac{2x}{(1+x^2)^2}\,dx = \int_{x=3}^{x=5} \frac{1}{u^2}\,du = \int_{x=3}^{x=5} u^{-2}\,du = \left.\frac{u^{-1}}{-1}\right|_{x=3}^{x=5} = \left.\frac{-1}{1+x^2}\right|_{x=3}^{x=5} = \frac{-1}{26} + \frac{1}{10} = \frac{16}{260} = \frac{4}{65}. \text{ ☺}$$

$$\boxed{\begin{aligned} u &= 1+x^2 \\ \tfrac{du}{2x} &= dx \end{aligned}}$$

- Problems fixed by specifying what's an x-value when the expression only has u's in it.
- Lots of room for errors (by forgetting the "$x =$", which are time-consuming to write!)