$$\int \frac{\sqrt{49x^2 - 25}}{x}\, dx$$

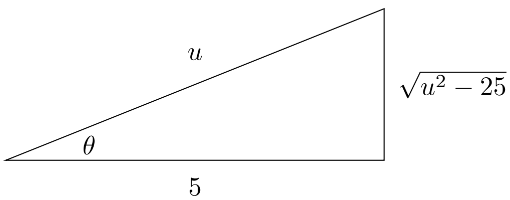
Solution 1

$$\int \frac{\sqrt{49x^2-25}}{x}\, dx = \int \frac{\sqrt{(7x)^2 - 25}}{x}\, dx$$

If $u = 7x$, then $du = 7\, dx$ so the integral becomes

$$\frac{1}{7}\int \frac{\sqrt{u^2-25}}{x}\, du$$

Let $u = 5\sec\theta$ with θ in quadrants I or III. So, $du = 5\sec\theta\tan\theta\, d\theta$ and we use $\sec\theta = \frac{u}{5}$ to draw a right triangle with u as the hypotenuse side side and 5 as the adjacent side. The side opposite to θ is $\sqrt{u^2-25}$, which is found using the Pythagorean Theorem, and we have



Then $\tan\theta = \dfrac{\sqrt{u^2-25}}{5}$. Based on this equation, we will use $5\tan\theta = \sqrt{u^2-25}$ if it is helpful. Since $\cos\theta = \dfrac{5}{u}$, we will use $u = \dfrac{5}{\cos\theta} = 5\sec\theta$ if it is helpful.

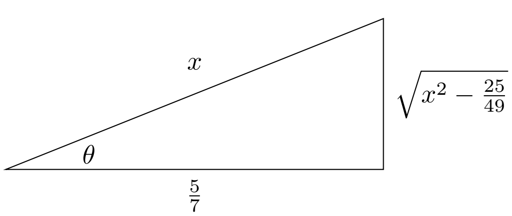
So

$$\begin{aligned}
\frac{1}{7}\int \frac{\sqrt{u^2-25}}{u}\, du &= \frac{1}{7}\int \frac{5\tan\theta}{5\sec\theta} 5\sec\theta\tan\theta\, d\theta \\
&= \frac{5}{7}\int \tan^2\theta\, d\theta \\
&= \frac{5}{7}\int (\sec^2\theta - 1)\, d\theta \\
&= \frac{5}{7}\left[\tan\theta - \theta\right] + C \\
&= \frac{5}{7}\left[\frac{\sqrt{u^2-25}}{5} - \sec^{-1}\frac{u}{5}\right] + C \\
&= \frac{5}{7}\left[\frac{\sqrt{(7x)^2-25}}{5} - \sec^{-1}\frac{7x}{5}\right] + C
\end{aligned}$$

Solution 2

$$\int \frac{\sqrt{49x^2-25}}{x}\, dx = \int \frac{\sqrt{49(x^2 - \frac{25}{49})}}{x}\, dx = 7\int \frac{\sqrt{x^2 - \frac{25}{49}}}{x}\, dx$$

Let $x = \frac{5}{7}\sec\theta$ with θ in quadrants I or III. So, $dx = \frac{5}{7}\sec\theta\tan\theta\, d\theta$ and we use $\sec\theta = \frac{x}{\frac{5}{7}}$ to draw a right triangle with x as the hypotenuse side side and $\frac{5}{7}$ as the adjacent side. The side opposite to θ is $\sqrt{x^2 - \frac{25}{49}}$, which is found using the Pythagorean Theorem, and we have



Then $\tan\theta = \dfrac{\sqrt{x^2 - \frac{25}{49}}}{\frac{5}{7}}$. Based on this equation, we will use $\frac{5}{7}\tan\theta = \sqrt{x^2 - \frac{25}{49}}$ if it is helpful. Since $\cos\theta = \dfrac{\frac{5}{7}}{x}$, we will use $x = \dfrac{\frac{5}{7}}{\cos\theta} = \dfrac{5}{7}\sec\theta$ if it is helpful.

So

$$\begin{aligned}
7\int \frac{\sqrt{x^2 - \frac{25}{49}}}{x}\,dx &= 5\int \frac{\frac{5}{7}\tan\theta}{\frac{5}{7}\sec\theta} \cdot \frac{5}{7}\sec\theta\tan\theta\,d\theta \\
&= 5^2 \int \tan^2\theta\,d\theta \\
&= 5^2 \int (\sec^2\theta - 1)\,d\theta \\
&= 5^2\left[\tan\theta - \theta\right] + C \\
&= 5^2\left[\frac{7}{5}\sqrt{x^2-1} - \sec^{-1}\frac{7x}{5}\right] + C
\end{aligned}$$