Does $\displaystyle\sum_{n=1}^{\infty} \frac{2n + \cos n}{n^2 + \sin n}$ diverge, converge absolutely, or converge conditionally?

Solution

The function $f(x) = \frac{2x+\cos x}{x^2+\sin x}$ is continuous, positive, and decreasing. We do the following indefinite integral with the substitution $u = x^2 + \sin x$ to get

$$\int \frac{2x + \cos x}{x^2 + \sin x}\, dx = \int \frac{1}{u}\, du = \ln|u| + C = \ln|x^2 + \sin x| + C.$$

So the definite, improper integral evaluates

$$\begin{aligned}\int_1^{\infty} \frac{2x + \cos x}{x^2 + \sin x}\, dx &= \lim_{t \to \infty} \int_1^t \frac{2x + \cos x}{x^2 + \sin x}\, dx \\ &= \lim_{t \to \infty} \left(\ln|t^2 + \sin t| - \ln|1^2 + \sin 1|\right) \\ &= \infty\end{aligned}$$

Since the integral $\displaystyle\int \frac{2x + \cos x}{x^2 + \sin x}\, dx$ diverges, the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n + \cos n}{n^2 + \sin n}$ diverges.