$$\int 2^x \sin x \, dx$$

Solution 1

Let $u = 2^x$ and $dv = \sin x \, dx$. Then $du = 2^x \ln 2 \, dx$ and $v = -\cos x$. So

$$\int 2^x \sin x \, dx = -2^x \cos x + \ln 2 \int 2^x \cos x \, dx.$$

To evaluate $\int 2^x \cos x \, dx$, we let $u = 2^x$ and $dv = \cos x \, dx$. (We will likely "go around in circles" if we picked u to be $\cos x$.) So $du = 2^x \ln 2 \, dx$ and $v = \sin x$. So our side integral is

$$\int 2^x \cos x \, dx = 2^x \sin x - \ln 2 \int 2^x \sin x \, dx.$$

Putting together our work,

$$\int 2^x \sin x \, dx = -2^x \cos x + \ln 2 \left(2^x \sin x - \ln 2 \int 2^x \sin x \, dx \right)$$

so

$$\int 2^x \sin x \, dx = -2^x \cos x + 2^x \ln 2 \sin x - (\ln 2)^2 \int 2^x \sin x \, dx.$$

We might optionally use I to represent our original integral $\int 2^x \sin x \, dx$ to get

$$I = -2^x \cos x + 2^x \ln 2 \sin x - (\ln 2)^2 I.$$

$$(1 + (\ln 2)^2) I = -2^x \cos x + 2^x \ln 2 \sin x + C$$

so finally

$$\int 2^x \sin x \, dx = \frac{-2^x \cos x + 2^x \ln 2 \sin x}{1 + (\ln 2)^2} + C.$$

Solution 2

Let $u = \sin x$ and $dv = 2^x \, dx$. Then $du = \cos x \, dx$ and $v = \frac{1}{\ln 2} 2^x$. So

$$\int 2^x \sin x \, dx = \frac{1}{\ln 2} 2^x \sin x - \frac{1}{\ln 2} \int 2^x \cos x \, dx.$$

The second integral is evaluated by parts using $u = \cos x$ and $dv = 2^x \, dx$, so $du = -\sin x \, dx$ and $v = \frac{1}{\ln 2} 2^x$. (Question for you: what choice of u would make the integration by parts "work" but the overall work "go around in circles" to the point where you'd just get the equation $0 = 0$?)

$$\int 2^x \cos x \, dx = \frac{1}{\ln 2} 2^x \cos x + \frac{1}{\ln 2} \int 2^x \sin x \, dx.$$

Substituting the work of this second integral into the work from earlier,

$$\int 2^x \sin x \, dx = \frac{1}{\ln 2} 2^x \sin x - \frac{1}{\ln 2} \left(\frac{1}{\ln 2} 2^x \cos x + \frac{1}{\ln 2} \int 2^x \sin x \, dx \right)$$

$$\int 2^x \sin x \, dx = \frac{1}{\ln 2} 2^x \sin x - \frac{1}{(\ln 2)^2} 2^x \cos x - \frac{1}{(\ln 2)^2} \int 2^x \sin x \, dx$$

$$\left(1 + \frac{1}{(\ln 2)^2} \right) \int 2^x \sin x \, dx = \frac{1}{\ln 2} 2^x \sin x - \frac{1}{(\ln 2)^2} 2^x \cos x + C$$

$$\int 2^x \sin x \, dx = \frac{\frac{1}{\ln 2} 2^x \sin x - \frac{1}{(\ln 2)^2} 2^x \cos x}{1 + \frac{1}{(\ln 2)^2}} + C$$