$\int e^x \cos x \, dx$

Solution 1

Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$ and $v = \sin x$. So

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

To evaluate $\int e^x \sin x \, dx$, we let $u = e^x$ and $dv = \sin x \, dx$. (We will likely "go around in circles" if we picked u to be $\sin x$.) So $du = e^x \, dx$ and $v = -\cos x$. So our side integral is

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx.$$

Putting together our work,

$$\int e^x \cos x \, dx = e^x \sin x - \left(-e^x \cos x + \int e^x \cos x \, dx \right)$$

so

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.$$

We might optionally use I to represent our original integral $\int e^x \cos x \, dx$ to get

$$I = e^x \sin x + e^x \cos x - I.$$

$$2I = e^x \sin x + e^x \cos x + C$$

so finally

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

Solution 2

Let $u = \cos x$ and $dv = e^x \, dx$. Then $du = -\sin x \, dx$ and $v = e^x$. So

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx.$$

The second integral is evaluated by parts using $u = \sin x$ and $dv = e^x \, dx$, so $du = \cos x \, dx$ and $v = e^x$. (Question for you: what choice of u would make the integration by parts "work" but the overall work "go around in circles" to the point where you'd just get the equation $0 = 0$?)

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx.$$

Substituting the work of this second integral into the work from earlier,

$$\int e^x \cos x \, dx = e^x \cos x + \left(e^x \sin x - \int e^x \cos x \, dx \right).$$

$$\int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx.$$

$$2 \int e^x \cos x \, dx = e^x \cos x + e^x \sin x + C.$$

$$\int e^x \cos x \, dx = \frac{e^x \cos x + e^x \sin x}{2} + C.$$