$$\int \frac{x^7}{\sqrt{11+3x^2}}\,dx$$

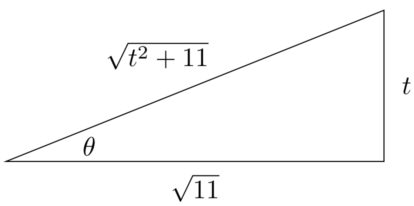
Solution 1

$$\int \frac{x^7}{\sqrt{11+3x^2}}\,dx = \int \frac{x^7}{\sqrt{11+(\sqrt{3}x)^2}}\,dx$$

and say $t=\sqrt{3}x$, so $dt=\sqrt{3}\,dx$ and x can be replaced with $\frac{t}{\sqrt{3}}$, so the integral becomes

$$\int \frac{(\frac{t}{\sqrt{3}})^7}{\sqrt{11+t^2}}\cdot\frac{1}{\sqrt{3}}\,dt = \frac{1}{3^4}\int \frac{t^7}{\sqrt{11+t^2}}\,dt$$

Let $t=\sqrt{11}\tan\theta$ with $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. So, $dt=\sqrt{11}\sec^2\theta\,d\theta$ and we use $\tan\theta=\frac{t}{\sqrt{11}}$ to draw a right triangle with t as the opposite side at $\sqrt{11}$ as the adjacent side. The hypotenuse is found using the Pythagorean Theorem, and we have



Then $\cos\theta=\dfrac{\sqrt{11}}{\sqrt{t^2+11}}$. Based on this equation, we will use either $\sqrt{t^2+11}=\dfrac{\sqrt{11}}{\cos\theta}=\sqrt{11}\sec\theta$ or we will use $\dfrac{\cos\theta}{\sqrt{11}}=\dfrac{1}{\sqrt{t^2+11}}$ if they are helpful.

So

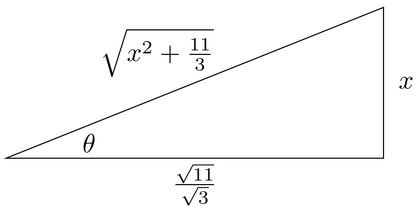
$$\frac{1}{3^4}\int\frac{t^7}{\sqrt{11+t^2}}\,dt=\frac{1}{3^4}\int(\sqrt{11}\tan\theta)^7\cdot\frac{\cos\theta}{\sqrt{11}}\sqrt{11}\sec^2\theta\,d\theta=\frac{\sqrt{11}^7}{3^4}\int\tan^7\theta\sec\theta\,d\theta$$

$$\begin{aligned}
\frac{\sqrt{11}^7}{3^4}\int\tan^7\theta\sec\theta\,d\theta &= \frac{\sqrt{11}^7}{3^4}\int\tan^6\theta\cdot\sec\theta\tan\theta\,d\theta\\
&= \frac{\sqrt{11}^7}{3^4}\int(\tan^2\theta)^3\cdot\sec\theta\tan\theta\,d\theta\\
&= \frac{\sqrt{11}^7}{3^4}\int(\sec^2\theta-1)^3\cdot\sec\theta\tan\theta\,d\theta \qquad u=\sec\theta \quad du=\sec\theta\tan\theta\,d\theta\\
&= \frac{\sqrt{11}^7}{3^4}\int(u^2-1)^3\,du\\
&= \frac{\sqrt{11}^7}{3^4}\int(u^6-3u^4+3u^2-1)\,du\\
&= \frac{\sqrt{11}^7}{3^4}\left[\frac{u^7}{7}-3\cdot\frac{u^5}{5}+3\cdot\frac{u^3}{3}-u\right]+C\\
&= \frac{\sqrt{11}^7}{3^4}\left[\frac{(\frac{1}{\sqrt{11}}\sqrt{t^2+11})^7}{7}-3\cdot\frac{(\frac{1}{\sqrt{11}}\sqrt{t^2+11})^5}{5}+3\cdot\frac{(\frac{1}{\sqrt{11}}\sqrt{t^2+11})^3}{3}-\frac{1}{\sqrt{11}}\sqrt{t^2+11}\right]+C\\
&= \frac{\sqrt{11}^7}{3^4}\left[\frac{(\frac{1}{\sqrt{11}}\sqrt{(\sqrt{3}x)^2+11})^7}{7}-3\cdot\frac{(\frac{1}{\sqrt{11}}\sqrt{(\sqrt{3}x)^2+11})^5}{5}+3\cdot\frac{(\frac{1}{\sqrt{11}}\sqrt{(\sqrt{3}x)^2+11})^3}{3}-\frac{1}{\sqrt{11}}\sqrt{(\sqrt{3}x)^2+11}\right]
\end{aligned}$$

Solution 2

$$\int\frac{x^7}{\sqrt{11+3x^2}}\,dx=\int\frac{x^7}{\sqrt{3(\frac{11}{3}+x^2)}}\,dx=\frac{1}{\sqrt{3}}\int\frac{x^7}{\sqrt{\frac{11}{3}+x^2}}\,dx$$

Let $x = \frac{\sqrt{11}}{\sqrt{3}} \tan\theta$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. So, $dx = \frac{\sqrt{11}}{\sqrt{3}} \sec^2\theta \, d\theta$ and we use $\tan\theta = \frac{x}{\frac{\sqrt{11}}{\sqrt{3}}}$ to draw a right triangle with x as the opposite side at $\frac{\sqrt{11}}{\sqrt{3}}$ as the adjacent side. The hypotenuse is found using the Pythagorean Theorem, and we have



Then $\cos\theta = \dfrac{\frac{\sqrt{11}}{\sqrt{3}}}{\sqrt{x^2 + \frac{11}{3}}}$. Based on this equation, we will use either $\sqrt{x^2 + \dfrac{11}{3}} = \dfrac{\frac{\sqrt{11}}{\sqrt{3}}}{\cos\theta} = \dfrac{\sqrt{11}}{\sqrt{3}} \sec\theta$ or we will use $\dfrac{\cos\theta}{\frac{\sqrt{11}}{\sqrt{3}}} = \dfrac{1}{\sqrt{x^2 + \frac{11}{3}}}$ if they are helpful.

So

$$\frac{1}{\sqrt{3}} \int \frac{x^7}{\sqrt{\frac{11}{3} + x^2}} \, dx = \frac{1}{\sqrt{3}} \int \left(\frac{\sqrt{11}}{\sqrt{3}} \tan\theta \right)^7 \cdot \frac{\cos\theta}{\frac{\sqrt{11}}{\sqrt{3}}} \frac{\sqrt{11}}{\sqrt{3}} \sec^2\theta \, d\theta = \frac{\sqrt{11}^7}{3^4} \int \tan^7\theta \sec\theta \, d\theta$$

$$\begin{aligned}
\frac{\sqrt{11}^7}{3^4} \int \tan^7\theta \sec\theta \, d\theta &= \frac{\sqrt{11}^7}{3^4} \int \tan^6\theta \cdot \sec\theta \tan\theta \, d\theta \\
&= \frac{\sqrt{11}^7}{3^4} \int (\tan^2\theta)^3 \cdot \sec\theta \tan\theta \, d\theta \\
&= \frac{\sqrt{11}^7}{3^4} \int (\sec^2\theta - 1)^3 \cdot \sec\theta \tan\theta \, d\theta \qquad u = \sec\theta \quad du = \sec\theta \tan\theta \, d\theta \\
&= \frac{\sqrt{11}^7}{3^4} \int (u^2 - 1)^3 \, du \\
&= \frac{\sqrt{11}^7}{3^4} \int (u^6 - 3u^4 + 3u^2 - 1) \, du \\
&= \frac{\sqrt{11}^7}{3^4} \left[\frac{u^7}{7} - 3 \cdot \frac{u^5}{5} + 3 \cdot \frac{u^3}{3} - u \right] + C \\
&= \frac{\sqrt{11}^7}{3^4} \left[\frac{(\frac{\sqrt{3}}{\sqrt{11}} \sqrt{x^2 + \frac{11}{3}})^7}{7} - 3 \cdot \frac{(\frac{\sqrt{3}}{\sqrt{11}} \sqrt{x^2 + \frac{11}{3}})^5}{5} + 3 \cdot \frac{(\frac{\sqrt{3}}{\sqrt{11}} \sqrt{x^2 + \frac{11}{3}})^3}{3} - \frac{\sqrt{3}}{\sqrt{11}} \sqrt{x^2 + \frac{11}{3}} \right] + C
\end{aligned}$$