Does $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$ diverge, converge absolutely, or converge conditionally?

Solution

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)\cdot n!}{(2n+2)\cdot(2n+1)\cdot(2n)!} \cdot \frac{(2n)!}{n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)}{(2n+2)(2n+1)} \right| \\
&= \lim_{n\to\infty} \frac{n+1}{4n^2+6n+2} \\
&= \lim_{n\to\infty} \frac{1}{8n+6} \text{ by L'hopital} \\
&= 0
\end{aligned}$$

Since $L < 1$, the series $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$ converges absolutely by the Ratio Test.