Does $\sum_{n=2}^{\infty} \frac{2n}{n^2-1}$ diverge, converge absolutely, or converge conditionally?

Solution 1

The function $f(x) = \frac{2x}{x^2-1}$ is continuous, positive, and decreasing on $[2, \infty)$.

$$\begin{aligned}
\int_2^{\infty} \frac{2x}{x^2-1}\,dx &= \lim_{t\to\infty} \int_2^{t} \frac{2x}{x^2-1}\,dx \\
&= \lim_{t\to\infty} \left(\ln|t^2-1| - \ln|2^2-1|\right) \\
&= \infty
\end{aligned}$$

Since the integral $\int_2^{\infty} \frac{2x}{x^2-1}\,dx$ diverges, the series $\sum_{n=2}^{\infty} \frac{2n}{n^2-1}$ diverges by the Integral Test.

Solution 2

The series $\sum \frac{1}{n}$ diverges by the p-test. Let $a_n = \frac{2n}{n^2-1}$ and $b_n = \frac{1}{n}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{2n^2}{n^2-1} \\
&= \lim_{n\to\infty} \frac{4n}{2n} \text{ by L'hopital} \\
&= \lim_{n\to\infty} 2 \\
&= 2
\end{aligned}$$

Since this limit is a finite, positive number, by the Limit Comparison Test, the series $\sum_{n=2}^{\infty} \frac{2n}{n^2-1}$ diverges.

Solution 3

This series can also be examined using the Direct Comparison Test, but we skip this method for this series.