$$\int 2^x \cos 3x \, dx$$

Solution 1

Let $u = 2^x$ and $dv = \cos 3x \, dx$. Then $du = 2^x \ln 2 \, dx$ and $v = \frac{1}{3} \sin 3x$. So

$$\int 2^x \cos 3x \, dx = \frac{1}{3} 2^x \sin 3x - \frac{\ln 2}{3} \int 2^x \sin 3x \, dx.$$

To evaluate $\int 2^x \sin 3x \, dx$, we let $u = 2^x$ and $dv = \sin 3x \, dx$. (We will likely "go around in circles" if we picked u to be $\sin 3x$.) So $du = 2^x \ln 2 \, dx$ and $v = -\frac{1}{3} \cos 3x$. So our side integral is

$$\int 2^x \sin 3x \, dx = -\frac{1}{3} 2^x \cos 3x + \frac{\ln 2}{3} \int 2^x \cos 3x \, dx.$$

Putting together our work,

$$\int 2^x \cos 3x \, dx = \frac{1}{3} 2^x \sin 3x - \frac{\ln 2}{3} \left(-\frac{1}{3} 2^x \cos 3x + \frac{\ln 2}{3} \int 2^x \cos 3x \, dx \right)$$

so

$$\int 2^x \cos 3x \, dx = \frac{1}{3} 2^x \sin 3x + 2^x \frac{\ln 2}{9} \cos 3x - \frac{(\ln 2)^2}{9} \int 2^x \cos 3x \, dx.$$

We might optionally use I to represent our original integral $\int 2^x \cos 3x \, dx$ to get

$$I = \frac{1}{3} 2^x \sin 3x + 2^x \frac{\ln 2}{9} \cos 3x - \frac{(\ln 2)^2}{9} I.$$

$$\left(1 + \frac{\ln(2)^2}{9}\right) I = \frac{1}{3} 2^x \sin 3x + 2^x \frac{\ln 2}{9} \cos 3x + C$$

so finally

$$\int 2^x \cos 3x \, dx = \frac{\frac{1}{3} 2^x \sin 3x + 2^x \frac{\ln 2}{9} \cos 3x}{1 + \frac{\ln(2)^2}{9}} + C.$$

Solution 2

Let $u = \cos 3x$ and $dv = 2^x \, dx$. Then $du = -3 \sin 3x \, dx$ and $v = \frac{1}{\ln 2} 2^x$. So

$$\int 2^x \cos 3x \, dx = \frac{1}{\ln 2} 2^x \cos 3x + \frac{3}{\ln 2} \int 2^x \sin 3x \, dx.$$

The second integral is evaluated by parts using $u = \sin 3x$ and $dv = 2^x \, dx$, so $du = 3 \cos 3x \, dx$ and $v = \frac{1}{\ln 2} 2^x$. (Question for you: what choice of u would make the integration by parts "work" but the overall work "go around in circles" to the point where you'd just get the equation $0 = 0$?)

$$\int 2^x \sin 3x \, dx = \frac{1}{\ln 2} 2^x \sin 3x - \frac{3}{\ln 2} \int 2^x \cos 3x \, dx.$$

Substituting the work of this second integral into the work from earlier,

$$\int 2^x \cos 3x \, dx = \frac{1}{\ln 2} 2^x \cos 3x + \frac{3}{\ln 2} \left(\frac{1}{\ln 2} 2^x \sin 3x - \frac{3}{\ln 2} \int 2^x \cos 3x \, dx \right).$$

$$\int 2^x \cos 3x \, dx = \frac{1}{\ln 2} 2^x \cos 3x + \frac{3}{(\ln 2)^2} 2^x \sin 3x - \frac{9}{(\ln 2)^2} \int 2^x \cos 3x \, dx.$$

$$\left(1 + \frac{9}{(\ln 2)^2}\right) \int 2^x \cos 3x \, dx = \frac{1}{\ln 2} 2^x \cos 3x + \frac{3}{(\ln 2)^2} 2^x \sin 3x + C.$$

$$\int 2^x \cos 3x \, dx = \frac{\frac{1}{\ln 2} 2^x \cos 3x + \frac{3}{(\ln 2)^2} 2^x \sin 3x}{1 + \frac{9}{(\ln 2)^2}} + C.$$