

$$\int \sin 3x \sin 5x \, dx$$


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### Solution 1

Let  $u = \sin 3x$  and  $dv = \sin 5x \, dx$ . Then  $du = 3 \cos 3x \, dx$  and  $v = -\frac{1}{5} \cos 5x$ . So

$$\int \sin 3x \sin 5x \, dx = -\frac{1}{5} \sin 3x \cos 5x + \frac{3}{5} \int \cos 3x \cos 5x \, dx.$$

We now apply a second integration by parts, for the integral  $\int \cos 3x \cos 5x \, dx$ . Note that we must choose  $u = \cos 3x$  and  $dv = \cos 5x \, dx$  to avoid going around in circles. (If you chose  $u = \cos 5x$ , then you'll eventually get to the equation  $0 = 0$ . Since  $u = \cos 3x$  and  $dv = \cos 5x \, dx$ , we get  $du = -3 \sin 3x \, dx$  and  $v = \frac{1}{5} \sin 5x$ . The side integral is thus

$$\int \cos 3x \cos 5x \, dx = \frac{1}{5} \cos 3x \sin 5x + \frac{3}{5} \int \sin 3x \sin 5x \, dx.$$

Placed in context with the original integral, we have

$$\int \sin 3x \sin 5x \, dx = -\frac{1}{5} \sin 3x \cos 5x + \frac{3}{5} \left( \frac{1}{5} \cos 3x \sin 5x + \frac{3}{5} \int \sin 3x \sin 5x \, dx \right).$$

We use  $I$  to denote our original integral  $\int \sin 3x \sin 5x \, dx$ . Then

$$I = -\frac{1}{5} \sin 3x \cos 5x + \frac{3}{5} \left( \frac{1}{5} \cos 3x \sin 5x + \frac{3}{5} I \right).$$

so

$$\begin{aligned} I &= -\frac{1}{5} \sin 3x \cos 5x + \frac{3}{25} \cos 3x \sin 5x + \frac{9}{25} I. \\ \frac{34}{25} I &= -\frac{1}{5} \sin 3x \cos 5x + \frac{3}{25} \cos 3x \sin 5x + C \end{aligned}$$

and finally

$$\int \sin 3x \sin 5x \, dx = \frac{25}{34} \left( -\frac{1}{5} \sin 3x \cos 5x + \frac{3}{25} \cos 3x \sin 5x \right) + C.$$


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### Solution 2

Let  $u = \sin 5x$  and  $dv = \sin 3x \, dx$ . Then  $du = 5 \cos 5x \, dx$  and  $v = -\frac{1}{3} \cos 3x$ . So

$$\int \sin 3x \sin 5x \, dx = -\frac{1}{3} \cos 3x \sin 5x + \frac{5}{3} \int \cos 3x \cos 5x \, dx.$$

We now apply a second integration by parts, for the integral  $\int \cos 3x \cos 5x \, dx$ . Note that we must choose  $u = \cos 5x$  and  $dv = \cos 3x \, dx$  to avoid going around in circles. (If you chose  $u = \cos 3x$ , then you'll eventually get to the equation  $0 = 0$ . Since  $u = \cos 5x$  and  $dv = \cos 3x \, dx$ , we get  $du = -5 \sin 5x \, dx$  and  $v = \frac{1}{3} \sin 3x$ . The side integral is thus

$$\int \cos 3x \cos 5x \, dx = \frac{1}{3} \sin 3x \cos 5x + \frac{5}{3} \int \sin 3x \sin 5x \, dx$$

Placed in context with the original integral, we have

$$\int \sin 3x \sin 5x \, dx = -\frac{1}{3} \cos 3x \sin 5x + \frac{5}{3} \left( \frac{1}{3} \sin 3x \cos 5x + \frac{5}{3} \int \sin 3x \sin 5x \, dx \right).$$

We use  $I$  to denote our original integral  $\int \sin 3x \sin 5x \, dx$ . Then

$$I = -\frac{1}{3} \cos 3x \sin 5x + \frac{5}{3} \left( \frac{1}{3} \sin 3x \cos 5x + \frac{5}{3} I \right).$$

so

$$\begin{aligned} I &= -\frac{1}{3} \cos 3x \sin 5x + \frac{5}{9} \sin 3x \cos 5x + \frac{25}{9} I. \\ -\frac{16}{9} I &= -\frac{1}{3} \cos 3x \sin 5x + \frac{5}{9} \sin 3x \cos 5x + C. \end{aligned}$$

and finally

$$I = -\frac{9}{16} \left( -\frac{1}{3} \cos 3x \sin 5x + \frac{5}{9} \sin 3x \cos 5x \right) + C.$$

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### Solution 3

Apply the trig identity  $\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

$$\begin{aligned} \int \sin 3x \sin 5x \, dx &= \frac{1}{2} \int \cos(-2x) - \cos(8x) \, dx \\ &= \frac{1}{2} \int \cos(-2x) \, dx - \frac{1}{2} \int \cos(8x) \, dx \\ &= -\frac{1}{4} \sin(-2x) - \frac{1}{16} \sin(8x) + C. \end{aligned}$$

where each integral is completed by substitution (using  $u = -2x$  for the first integral and  $u = 8x$  for the second integral).