Does $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ diverge, converge absolutely, or converge conditionally?

Solution

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n. \end{aligned}$$

This is an indeterminate form, so we let $y = \left(\frac{n}{n+1}\right)^n$. So $\ln y = n \ln \frac{n}{n+1}$. Applying limit to both sides, we have

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} n \ln \frac{n}{n+1}.$$

The limit on the right can be computed using L'Hopital's rule (with a chain rule and quotent rule).

$$\begin{aligned} &\lim_{n\to\infty} n \ln \frac{n}{n+1}, \text{ which was the limit on the right side above} \\ &= \lim_{n\to\infty} \frac{\ln \frac{n}{n+1}}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{\frac{n+1}{n} \cdot \frac{(n+1)\cdot 1 - n\cdot 1}{n^2}}{\frac{-1}{n^2}} \text{ by L'Hopital's} \\ &= \lim_{n\to\infty} -\frac{n+1}{n} \text{ after algebra simplification} \\ &= \lim_{n\to\infty} -\frac{1}{1} \\ &= -1. \end{aligned}$$

So, $\lim y = e^{-1} = \frac{1}{e}$. So $L = \frac{1}{e}$.

Since $L < 1$, the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges absolutely by the Ratio Test.