$$\int \tan^3 x\, dx$$

Solution 1

$$\begin{aligned}
\int \tan^3 x\, dx &= \int \tan^2 x \tan x\, dx \\
&= \int (\sec^2 x - 1) \tan x\, dx \\
&= \int \sec^2 x \tan x\, dx - \int \tan x\, dx.
\end{aligned}$$

We will complete the two integrals separately:

- Substitute $u = \tan x$, so $du = \sec^2 x\, dx$. We get:

$$\int \sec^2 x \tan x\, dx = \int u\, du = \frac{u^2}{2} + C = \frac{1}{2} \tan^2 x + C.$$

- For the second integral,

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$$

  Let $u = \cos x$, so $du = -\sin x\, dx$. So the integral above is equal to

$$-\int \frac{1}{u}\, du = -\ln|u| + C = -\ln|\cos x| + C.$$

Putting together the two integrals, the answer to our original question is

$$\begin{aligned}
\int \sec^2 x \tan x\, dx - \int \tan x\, dx &= \frac{1}{2} \tan^2 x - (-\ln|\cos x|) + C \\
&= \frac{1}{2} \tan^2 x + \ln|\cos x| + C.
\end{aligned}$$

Solution 2

$$\begin{aligned}
\int \tan^3 x\, dx &= \int \tan^2 x \tan x\, dx \\
&= \int (\sec^2 x - 1) \tan x\, dx \\
&= \int \sec^2 x \tan x\, dx - \int \tan x\, dx.
\end{aligned}$$

We will complete the two integrals separately:

- Rewrite the integral

$$\int \sec^2 x \tan x\, dx = \int \sec x \cdot \sec x \tan x\, dx$$

  Substitute $u = \sec x$, so $du = \sec x \tan x\, dx$. We get:

$$\int \sec^2 x \tan x\, dx = \int \sec x \cdot \sec x \tan x\, dx = \int u\, du = \frac{u^2}{2} + C = \frac{1}{2} \sec^2 x + C.$$

- For the second integral,

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

Let $u = \cos x$, so $du = -\sin x \, dx$. So the integral above is equal to

$$-\int \frac{1}{u} \, du = -\ln |u| + C = -\ln |\cos x| + C.$$

Putting together the two integrals, the answer to our original question is

$$\begin{aligned} \int \sec^2 x \tan x \, dx - \int \tan x \, dx &= \frac{1}{2} \sec^2 x - (-\ln |\cos x|) + C \\ &= \frac{1}{2} \sec^2 x + \ln |\cos x| + C. \end{aligned}$$

Comparing the two solutions

Using the first method ($u = \tan x$) our final answer was

$$\frac{1}{2} \tan^2 x + \ln |\cos x| + C.$$

Using the second method ($u = \sec x$) our final answer was

$$\frac{1}{2} \sec^2 x + \ln |\cos x| + C.$$

If I leave off the C's in both for a moment, use a graphing calculator (or the website Desmos.com) to graph

$$\frac{1}{2} \tan^2 x + \ln |\cos x|$$

and

$$\frac{1}{2} \sec^2 x + \ln |\cos x|$$

and you'll see that one graph is just a vertical shift of the other graph (by a half unit). That accounts for different values of C.