$$\int \frac{\sqrt{11x^2-2}}{x}\,dx$$

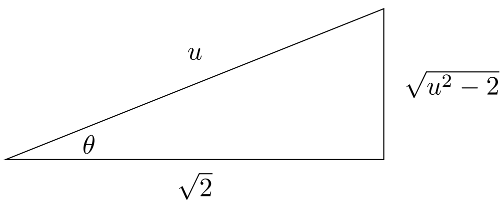
Solution 1

$$\int \frac{\sqrt{11x^2-2}}{x}\,dx = \int \frac{\sqrt{(\sqrt{11}x)^2-2}}{x}\,dx$$

If $u = \sqrt{11}x$, then $du = \sqrt{11}\,dx$ so the integral becomes

$$\frac{1}{\sqrt{11}} \int \frac{\sqrt{u^2-2}}{x}\,du$$

Let $u = \sqrt{2}\sec\theta$ with θ in quadrants I or III. So, $du = \sqrt{2}\sec\theta\tan\theta\,d\theta$ and we use $\sec\theta = \frac{u}{\sqrt{2}}$ to draw a right triangle with u as the hypotenuse side side and $\sqrt{2}$ as the adjacent side. The side opposite to θ is $\sqrt{u^2-2}$, which is found using the Pythagorean Theorem, and we have



Then $\tan\theta = \dfrac{\sqrt{u^2-2}}{\sqrt{2}}$. Based on this equation, we will use $\sqrt{2}\tan\theta = \sqrt{u^2-2}$ if it is helpful. Since $\cos\theta = \dfrac{\sqrt{2}}{u}$, we will use $u = \dfrac{\sqrt{2}}{\cos\theta} = \sqrt{2}\sec\theta$ if it is helpful.

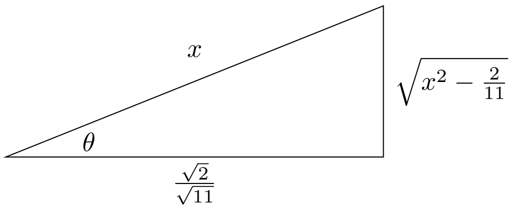
So

$$\begin{aligned}
\frac{1}{\sqrt{11}} \int \frac{\sqrt{u^2-2}}{u}\,du &= \frac{1}{\sqrt{11}} \int \frac{\sqrt{2}\tan\theta}{\sqrt{2}\sec\theta}\sqrt{2}\sec\theta\tan\theta\,d\theta \\
&= \frac{\sqrt{2}}{\sqrt{11}} \int \tan^2\theta\,d\theta \\
&= \frac{\sqrt{2}}{\sqrt{11}} \int (\sec^2\theta - 1)\,d\theta \\
&= \frac{\sqrt{2}}{\sqrt{11}} [\tan\theta - \theta] + C \\
&= \frac{\sqrt{2}}{\sqrt{11}} \left[\frac{\sqrt{u^2-2}}{\sqrt{2}} - \sec^{-1}\frac{u}{\sqrt{2}}\right] + C \\
&= \frac{\sqrt{2}}{\sqrt{11}} \left[\frac{\sqrt{(\sqrt{11}x)^2-2}}{\sqrt{2}} - \sec^{-1}\frac{\sqrt{11}x}{\sqrt{2}}\right] + C
\end{aligned}$$

Solution 2

$$\int \frac{\sqrt{11x^2-2}}{x}\,dx = \int \frac{\sqrt{11(x^2-\frac{2}{11})}}{x}\,dx = \sqrt{11}\int \frac{\sqrt{x^2-\frac{2}{11}}}{x}\,dx$$

Let $x = \frac{\sqrt{2}}{\sqrt{11}}\sec\theta$ with θ in quadrants I or III. So, $dx = \frac{\sqrt{2}}{\sqrt{11}}\sec\theta\tan\theta\,d\theta$ and we use $\sec\theta = \frac{x}{\frac{\sqrt{2}}{\sqrt{11}}}$ to draw a right triangle with x as the hypotenuse side side and $\frac{\sqrt{2}}{\sqrt{11}}$ as the adjacent side. The side opposite to θ is $\sqrt{x^2-\frac{2}{11}}$, which is found using the Pythagorean Theorem, and we have



Then $\tan\theta = \dfrac{\sqrt{x^2 - \frac{2}{11}}}{\frac{\sqrt{2}}{\sqrt{11}}}$. Based on this equation, we will use $\frac{\sqrt{2}}{\sqrt{11}}\tan\theta = \sqrt{x^2 - \frac{2}{11}}$ if it is helpful. Since $\cos\theta = \dfrac{\frac{\sqrt{2}}{\sqrt{11}}}{x}$, we will use $x = \dfrac{\frac{\sqrt{2}}{\sqrt{11}}}{\cos\theta} = \dfrac{\sqrt{2}}{\sqrt{11}}\sec\theta$ if it is helpful.

So

$$\begin{aligned}
\sqrt{11}\int \frac{\sqrt{x^2 - \frac{2}{11}}}{x}\,dx &= \sqrt{11}\sqrt{2}\int \frac{\frac{\sqrt{2}}{\sqrt{11}}\tan\theta}{\frac{\sqrt{2}}{\sqrt{11}}\sec\theta}\cdot\frac{\sqrt{2}}{\sqrt{11}}\sec\theta\tan\theta\,d\theta \\
&= (\sqrt{2})^2\int \tan^2\theta\,d\theta \\
&= 2\int(\sec^2\theta - 1)\,d\theta \\
&= 2\left[\tan\theta - \theta\right] + C \\
&= 2\left[\frac{\sqrt{11}}{\sqrt{2}}\sqrt{x^2 - 1} - \sec^{-1}\frac{\sqrt{11}x}{\sqrt{2}}\right] + C
\end{aligned}$$