Does $\sum_{n=1}^{\infty}\left(\frac{8n+1}{9n-1}+\frac{1}{n}\right)^n$ diverge, converge absolutely, or converge conditionally?

Solution

$$\begin{aligned}
L &= \lim_{n\to\infty} \sqrt[n]{|a_n|} \\
&= \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{8n+1}{9n-1}+\frac{1}{n}\right)^n\right|} \\
&= \lim_{n\to\infty} \sqrt[n]{\left(\frac{8n+1}{9n-1}+\frac{1}{n}\right)^n} \\
&= \lim_{n\to\infty} \left(\frac{8n+1}{9n-1}+\frac{1}{n}\right) \\
&= \lim_{n\to\infty} \frac{8n+1}{9n-1}+\lim_{n\to\infty}\frac{1}{n} \\
&= \lim_{n\to\infty} \frac{8n+1}{9n-1}+0 \\
&= \lim_{n\to\infty} \frac{8}{9}+0 \qquad \text{by L'Hopital's rule} \\
&= \frac{8}{9}
\end{aligned}$$

Since $L<1$, by the Root Test, the series $\sum_{n=1}^{\infty}\left(\frac{8n+1}{9n-1}+\frac{1}{n}\right)^n$ converges absolutely.