$$\int \frac{1}{x^2-9}\,dx$$

Solution

First, we ignore the calculus. The denominator factors:

$$\frac{1}{x^2-9} = \frac{1}{(x+3)(x-3)},$$

which we work to rewrite using partial fractions:

$$\frac{1}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3}.$$

Multiply both sides by $(x+3)(x-3)$ to get

$$1 = A(x-3) + B(x+3).$$

If we set $x = 3$ in the equation above, we learn $B = \frac{1}{6}$. If we set $x = -3$ in the equation above, we learn $A = -\frac{1}{6}$. So

$$\frac{1}{x^2-9} = \frac{1}{(x+3)(x-3)} = \frac{-1/6}{x+3} + \frac{1/6}{x-3}$$

Back to the integral,

$$\int \frac{1}{x^2-9}\,dx = \int \left(\frac{-1/6}{x+3} + \frac{1/6}{x-3} \right) dx = -\frac{1}{6}\ln|x+3| + \frac{1}{6}\ln|x-3| + C,$$

where the work for each of

$$\int \frac{-1/6}{x+3}\,dx = -\frac{1}{6}\ln|x+3| + C$$

$$\int \frac{1/6}{x-3}\,dx = \frac{1}{6}\ln|x-3| + C$$

using substitutions (of $u = x+3$ and $u = x-3$ respectively) have been skipped here.