Does $\sum_{n=1}^{\infty}\left(\frac{2\pi+\cos n}{n}\right)^n$ diverge, converge absolutely, or converge conditionally?

Solution

We consider the Root Test, based on the shape of the terms. (Note that all terms are positive.)

$$\begin{aligned} L &= \lim_{n\to\infty} \sqrt[n]{|a_n|} \\ &= \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{2\pi+\cos n}{n}\right)^n\right|} \\ &= \lim_{n\to\infty} \sqrt[n]{\left(\frac{2\pi+\cos n}{n}\right)^n} \\ &= \lim_{n\to\infty} \frac{2\pi+\cos n}{n} \end{aligned}$$

We cannot evaluate the limit of the sequence above using L'Hopital's rule. (Note the numerator does not go to either 0 or ∞.) Instead, we set up the following for the Squeeze Theorem. Note that

$$-1 \leq \cos n \leq 1$$

so by adding 2π to all sides,

$$2\pi - 1 \leq 2\pi + \cos n \leq 2\pi + 1$$

and by dividing,

$$\frac{2\pi-1}{n} \leq \frac{2\pi+\cos n}{n} \leq \frac{2\pi+1}{n}$$

Since

$$\lim_{n\to\infty} \frac{2\pi-1}{n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{2\pi+1}{n} = 0,$$

by the Squeeze Theorem,

$$\lim_{n\to\infty} \frac{2\pi+\cos n}{n} = 0.$$

Since $L < 1$, by the Root Test, the series $\sum_{n=1}^{\infty}\left(\frac{2\pi+\cos n}{n}\right)^n$ converges absolutely.