Does $\sum_{n=1}^{\infty} \frac{15n^4+4n^3+3n^2+2n+1}{3n^5+n^4+n^3+n^2+n+1}$ diverge, converge absolutely, or converge conditionally?

Solution

The function $f(x) = \frac{15x^4+4x^3+3x^2+2x+1}{3x^5+x^4+x^3+x^2+x+1}$ is continuous, positive, and decreasing.

We will apply the Integral Test, so we integrate $f(x)$ using the substitution $u = 3x^5+x^4+x^3+x^2+x+1$ to get

$$\begin{aligned}
\int \frac{15x^4+4x^3+3x^2+2x+1}{3x^5+x^4+x^3+x^2+x+1}\,dx &= \int \frac{1}{u}\,du \\
&= \ln|u| + C \\
&= \ln|3x^5+x^4+x^3+x^2+x+1| + C.
\end{aligned}$$

Thus the definite, improper integral is

$$\begin{aligned}
\int_1^{\infty} \frac{15x^4+4x^3+3x^2+2x+1}{3x^5+x^4+x^3+x^2+x+1}\,dx &= \lim_{t\to\infty} \int_1^{t} \frac{15x^4+4x^3+3x^2+2x+1}{3x^5+x^4+x^3+x^2+x+1}\,dx \\
&= \lim_{t\to\infty} \left(\ln|3t^5+t^4+t^3+t^2+t+1| - \ln|8|\right) \\
&= \infty.
\end{aligned}$$

Since the integral $\int_1^{\infty} f(x)\,dx$ diverges, by the Integral Test, the series $\sum_{n=1}^{\infty} \frac{15n^4+4n^3+3n^2+2n+1}{3n^5+n^4+n^3+n^2+n+1}$ diverges.

Comment

This series can be studied using the Limit Comparison Test, and would be rather awful (but possible!) using the Direct Comparison Test. However, the Integral Test (above) is probably going to be much shorter!