$$\int x \arctan(x+4)\, dx$$

Solution

Let $t = x + 4$, so $dt = dx$. Then

$$\int x \arctan(x+4)\, dx = \int (t-4) \arctan t\, dt = \int t \arctan t\, dt - 4 \int \arctan t\, dt$$

There are two separate integrals to consider.

- To integrate
$$\int t \arctan t\, dt$$
use integration by parts and some clever algebra after. Let $u = \arctan t$ and $dv = t\, dt$. Then $dt = \frac{1}{t^2+1}\, dt$ and $v = \frac{t^2}{2}$ so
$$\begin{aligned}
\int t \arctan t\, dt &= \frac{1}{2} t^2 \arctan t - \frac{1}{2} \int \frac{t^2}{t^2+1}\, dx \\
&= \frac{1}{2} t^2 \arctan t - \frac{1}{2} \int \frac{t^2+1-1}{t^2+1}\, dx \\
&= \frac{1}{2} t^2 \arctan t - \frac{1}{2} \int \frac{(t^2+1)-1}{t^2+1}\, dx \\
&= \frac{1}{2} t^2 \arctan t - \frac{1}{2} \int \frac{t^2+1}{t^2+1} - \frac{1}{t^2+1}\, dx \\
&= \frac{1}{2} t^2 \arctan t - \frac{1}{2} \int 1 - \frac{1}{t^2+1}\, dx \\
&= \frac{1}{2} t^2 \arctan t - \frac{1}{2} t + \frac{1}{2} \arctan t + C.
\end{aligned}$$

- The **second** integral we need to consider is $\int \arctan t\, dt$, which we do using integration by parts: let $u = \arctan t$ and $dv = dt$. So $du = \dfrac{1}{1+t^2} dt$ and $v = t$. So
$$\int \arctan t\, dt = t \arctan t - \int \frac{t}{1+t^2}\, dt.$$
To do the integral
$$\int \frac{t}{1+t^2}\, dt$$
do substitution with $u = 1 + t^2$, so $du = 2t\, dt$, thus
$$\int \frac{t}{1+t^2}\, dt = \frac{1}{2} \int \frac{1}{u}\, du = \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln|1+t^2| + C.$$
Combining the work from earlier,
$$\int \arctan t\, dt = t \arctan t - \int \frac{t}{1+t^2}\, dt = t \arctan t - \frac{1}{2} \ln|1+t^2| + C.$$

Now that we have done our two side problems, let us go back to the expression we had, and make replacements using our work:

$$\int t \arctan t\, dt - 4 \int \arctan t\, dt = \left[\frac{1}{2} t^2 \arctan t - \frac{1}{2} t + \frac{1}{2} \arctan t\right] - 4 \left[t \arctan t - \frac{1}{2} \ln|1+t^2|\right] + C.$$

Commentary

The point is that an input of $x+4$ to the arctangent function is not so different from an input of just x, and the "just x" is played by the role of t here.