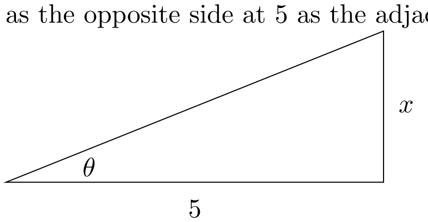
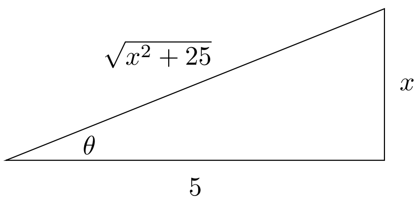
$$\int \frac{x^7}{\sqrt{25+x^2}}\,dx$$

Solution

Let $x = 5\tan\theta$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. So, $dx = 5\sec^2\theta\,d\theta$ and we use $\tan\theta = \frac{x}{5}$ to draw a right triangle with x as the opposite side at 5 as the adjacent side.



The hypotenuse is found using the Pythagorean Theorem, and we have



Then $\cos\theta = \dfrac{5}{\sqrt{x^2+25}}$. Based on this equation, we will use either $\sqrt{x^2+25} = \dfrac{5}{\cos\theta} = 5\sec\theta$ or we will use $\dfrac{\cos\theta}{5} = \dfrac{1}{\sqrt{x^2+25}}$ if they are helpful.

So
$$\int \frac{x^7}{\sqrt{25+x^2}}\,dx = \int (5\tan\theta)^7 \cdot \frac{\cos\theta}{5} 5\sec^2\theta\,d\theta = 5^7 \int \tan^7\theta\sec\theta\,d\theta$$

Since the power of tangent is odd, we rewrite the integral

$$\begin{aligned}
5^7 \int \tan^7\theta\sec\theta\,d\theta &= 5^7 \int \tan^6\theta \cdot \sec\theta\tan\theta\,d\theta \\
&= 5^7 \int (\tan^2\theta)^3 \cdot \sec\theta\tan\theta\,d\theta \\
&= 5^7 \int (\sec^2\theta - 1)^3 \cdot \sec\theta\tan\theta\,d\theta \qquad u = \sec\theta \quad du = \sec\theta\tan\theta\,d\theta \\
&= 5^7 \int (u^2-1)^3\,du \\
&= 5^7 \int (u^6 - 3u^4 + 3u^2 - 1)\,du \\
&= 5^7 \left[\frac{u^7}{7} - 3\cdot\frac{u^5}{5} + 3\cdot\frac{u^3}{3} - u\right] + C \\
&= 5^7 \left[\frac{(\frac{1}{5}\sqrt{x^2+25})^7}{7} - 3\cdot\frac{(\frac{1}{5}\sqrt{x^2+25})^5}{5} + 3\cdot\frac{(\frac{1}{5}\sqrt{x^2+25})^3}{3} - \frac{1}{5}\sqrt{x^2+25}\right] + C
\end{aligned}$$