Does $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$ diverge, converge absolutely, or converge conditionally?

Important Remark

If we were to use the Ratio Test, on this series, we would consider the limit

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n}\right| = \lim_{n\to\infty} \left|\frac{(-1)(n+1)}{n+2}\right| = \lim_{n\to\infty} \frac{n+1}{n+2} = \lim_{n\to\infty} \frac{1}{1} = 1$$

with one use of L'Hopital's above. Since $L = 1$, we get NO INFORMATION from the Ratio Test. (Note that later in the solution, we well get a limit of 1, but since we're using the Limit Comparison Test, which has different requirements, we WILL get information.)

Solution

The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$ is alternating, and $b_n = |a_n| = \frac{1}{n+1}$. The sequence b_n is decreasing and has limit 0. So by the Alternating Series Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$ converges.

To determine if $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$ converges absolutely or conditionally, we consider the series $\sum |a_n|$, which in this case is

$$\sum_{n=1}^{\infty} \frac{1}{n+1}.$$

This series can be shown to diverge using the Integral Test (with a substitution of $u = x + 1$) or using the Direct Comparison Test, or using the Limit Comparison Test. We'll use the Limit Comparison Test here. Note that the series $\sum \frac{1}{n}$ diverges by the p-test. Let $a_n = \frac{1}{n+1}$ and let $b_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1} = 1.$$

Therefore, the Limit Comparison Test applies and the series $\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges. Thus, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$ converges conditionally.

Epiloque

When using the Ratio Test, we got

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$$

while with the Limit Comparison Test, we got

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

When the limit of the sequence $\left|\frac{a_{n+1}}{a_n}\right|$ is 1 in the Ratio Test, we get NO information. However, as long as the limit of the sequence $\frac{a_n}{b_n}$ is a finite positive number (including 1) we DO get to conclude something in the Limit Comparison Test, simply because the requirements of the two tests are different. Don't let this freak you out too much: in any case, the two tests made you study DIFFERENT sequences anyway!