Does $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ diverge, converge absolutely, or converge conditionally?

Solution 1

Rewrite $\frac{1}{n(n+1)}$ using partial fraction decomposition:

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

So

$$1 = A(n+1) + Bn$$

Using $n = 0$, we get $A = 1$. Using $n = -1$, we get $B = -1$. So

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Therefore, our original series can be rewritten: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

We appear to have a telescoping series. After cancellations, the nth term of the sequence of partial sums is:

$$s_n = \frac{1}{1} - \frac{1}{n+1}$$

Thus,

$$\begin{aligned}
\lim_{n\to\infty} s_n &= \lim_{n\to\infty} \left(\frac{1}{1} - \frac{1}{n+1}\right) \\
&= \frac{1}{1} - 0 \\
&= 1
\end{aligned}$$

So, (by definition), the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges, and we furthermore know that the sum is 1. Recall that it is unusual that we get to know the value of a convergent series.

Since $\sum |a_n| = \sum a_n$, the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges absolutely.

Solution 2

Note that $n(n+1) \geq n^2$ for all $n \geq 1$. So

$$\frac{1}{n(n+1)} \leq \frac{1}{n^2}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test. Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges by the Direct Comparison Test.

Since $\sum |a_n| = \sum a_n$, the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges absolutely.