$$\int \frac{x^3}{x^2+5}\,dx$$

Solution 1

Since the power in denominator is not larger than the degree in the numerator, we apply long division. From long division, we get

$$\frac{x^3}{x^2+5} = x - \frac{5x}{x^2+5}.$$

So

$$\begin{aligned}\int \frac{x^3}{x^2+5}\,dx &= \int x - \frac{5x}{x^2+5}\,dx \\ &= \int x\,dx - \int \frac{5x}{x^2+5}\,dx.\end{aligned}$$

Since the first integral is routine, we'll focus on the second integral by substitution of $u = x^2+5$. Thus, $du = 2x\,dx$. So

$$\int \frac{5x}{x^2+5}\,dx. = \frac{5}{2}\int \frac{1}{u}\,du = \frac{5}{2}\ln|u| + C = \frac{5}{2}\ln|x^2+5| + C.$$

Putting this together with the other short integral, we have

$$\int x\,dx - \int \frac{5x}{x^2+5}\,dx = \frac{x^2}{2} - \frac{5}{2}\ln|x^2+5| + C.$$

Solution 2

Apply substitution with $u = x^2+5$, so $du = 2x\,dx$. Then

$$\int \frac{x^3}{x^2+5}\,dx = \frac{1}{2}\int \frac{x^2}{u}\,du.$$

While the integral on the right looks hopeless, we actually solve for x^2 in our earlier equation $u = x^2+5$ to get $x^2 = u-5$. So,

$$\begin{aligned}\frac{1}{2}\int \frac{x^2}{u}\,du &= \frac{1}{2}\int \frac{u-5}{u}\,du \\ &= \frac{1}{2}\int 1 - \frac{5}{u}\,du \\ &= \frac{1}{2}(u - 5\ln|u|) + C \\ &= \frac{1}{2}(x^2+5-5\ln|x^2+5|) + C.\end{aligned}$$