$$\int (\sin x + \cos x)^2 \, dx$$

Solution

First, FOIL to examine:

$$\int (\sin x + \cos x)^2 \, dx = \int \sin^2 x + 2 \sin x \cos x + \cos^2 x \, dx$$

Now, each term can be integrated separately. For instance, you can integrate $\sin^2 x$ using either integration by parts or by rewriting $\sin^2 x$ as $\frac{1-\cos 2x}{2}$. A similar comment can be made for integrating $\cos^2 x$. However, it would actually be easier to notice that $\sin^2 x + \cos^2 x = 1$, so our integral becomes

$$\int 1 + 2 \sin x \cos x \, dx.$$

Integrating the 1 and the $2 \sin x \cos x$ should be done separately. There are three ways to integrate $2 \sin x \cos x$.

Solution 1

Use $u = \sin x$, so $du = \cos x \, dx$. Then

$$\int 2 \sin x \cos x \, dx = 2 \int u \, du = 2 \frac{u^2}{2} + C = (\sin x)^2 + C.$$

Solution 2

Use $u = \cos x$, so $du = - \sin x \, dx$. Then

$$\int 2 \sin x \cos x \, dx = -2 \int u \, du = -2 \frac{u^2}{2} + C = -(\cos x)^2 + C.$$

Solution 3

Take the trig identity $\sin 2x = 2 \sin x \cos x$ to rewrite the integral:

$$\int 2 \sin x \cos x \, dx = \int \sin 2x \, dx.$$

Then, by substituting $u = 2x$, we have $du = 2 \, dx$, so the integral above is equal to

$$\frac{1}{2} \int \sin u \, du = -\frac{1}{2} \cos u + C = -\frac{1}{2} \cos 2x + C.$$

Putting it all together

If you had turned $\sin^2 x + \cos^2 x$ into 1 earlier and the only integration youi did was $2 \sin x \cos x$, then it is likely that one of these three lines below is your final answer:

$$\int 1 + 2 \sin x \cos x \, dx = x + \sin^2 x + C.$$

$$\int 1 + 2 \sin x \cos x \, dx = x - \cos^2 x + C.$$

$$\int 1 + 2 \sin x \cos x \, dx = x - \frac{1}{2} \cos 2x + C.$$

The reason the third answer is equal to the previous two involves trig identities. You may see that $x + \sin^2 x$ and $x - \cos^2 x$ are "off" from each other when graphed, but it is just by a vertical transformation. (The different values of "C" can account for this: when C is 1 for one of the solutions, then the C in the other solution is 2, etc.)