$\int \sqrt{1-5x^2}\,dx$

Here are two solutions:

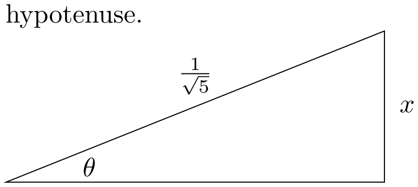
Solution 1

First, some algebra:

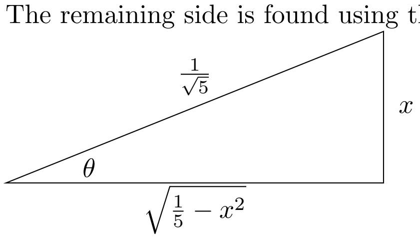
$$\int \sqrt{1-5x^2}\,dx = \int \sqrt{5(\frac{1}{5}-x^2)}\,dx = \int \sqrt{5}\sqrt{\frac{1}{5}-x^2}\,dx = \sqrt{5}\int \sqrt{\frac{1}{5}-x^2}\,dx.$$

Because of seeing $\sqrt{a^2-x^2}$ in the integral with $a=\frac{1}{\sqrt{5}}$, we apply trig substitution with $x=\frac{1}{\sqrt{5}}\sin\theta$, for $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$.

So, $dx=\frac{1}{\sqrt{5}}\cos\theta\,d\theta$ and we use $\sin\theta=\frac{x}{\frac{1}{\sqrt{5}}}$ to draw a right triangle with x as the opposite side at $\frac{1}{\sqrt{5}}$ as the hypotenuse.



The remaining side is found using the Pythagorean Theorem, and we have



So our integral becomes

$$\sqrt{5}\int \sqrt{\frac{1}{5}-x^2}\,dx = \int \sqrt{\frac{1}{5}-x^2}\cos\theta\,d\theta$$

We still need to replace $\sqrt{\frac{1}{5}-x^2}$, so we look to the side of the triangle with that quantity (it is adjacent to θ) and the number side (which is the hypotenuse). Since $\cos\theta=\frac{\sqrt{\frac{1}{5}-x^2}}{\frac{1}{\sqrt{5}}}$, we have $\sqrt{\frac{1}{5}-x^2}=\frac{1}{\sqrt{5}}\cos\theta$. So, our integral becomes

$$\begin{aligned}
\frac{1}{\sqrt{5}}\int \cos\theta\cdot\cos\theta\,d\theta &= \frac{1}{\sqrt{5}}\int \cos^2\theta\,d\theta \\
&= \frac{1}{2\sqrt{5}}\int 1+\cos 2\theta\,d\theta \\
&= \frac{1}{2\sqrt{5}}\theta + \frac{1}{4\sqrt{5}}\sin 2\theta + C \\
&= \frac{1}{2\sqrt{5}}\theta + \frac{1}{4\sqrt{5}}2\sin\theta\cos\theta + C \\
&= \frac{1}{2\sqrt{5}}\sin^{-1}(\sqrt{5}x) + \frac{1}{4\sqrt{5}}\cdot 2\frac{x}{1/\sqrt{5}}\cdot\frac{\sqrt{\frac{1}{5}-x^2}}{1/\sqrt{5}} + C.
\end{aligned}$$

See the next page for another solution:

Solution 2

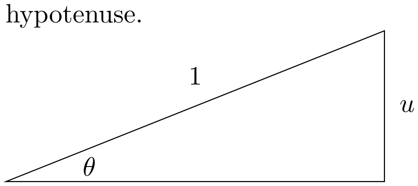
$$\int \sqrt{1-5x^2}\,dx = \int \sqrt{1-(\sqrt{5}x)^2}\,dx$$

If $u = \sqrt{5}x$, then $du = \sqrt{5}\,dx$, so the integral is equal to

$$\frac{1}{\sqrt{5}} \int \sqrt{1-u^2}\,du.$$

Trig sub $u = \sin\theta$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

So, $du = 1\cos\theta\,d\theta$ and we use $\sin\theta = \frac{u}{1}$ to draw a right triangle with u as the opposite side at 1 as the hypotenuse.



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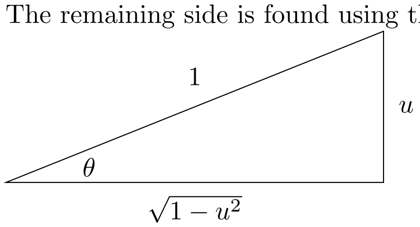

So our integral becomes

$$\frac{1}{\sqrt{5}} \int \sqrt{1-u^2}\,du = \frac{1}{\sqrt{5}} \int \sqrt{1-u^2}\cos\theta\,d\theta$$

We still need to replace $\sqrt{1-u^2}$, so we look to the side of the triangle with that quantity (it is adjacent to θ) and the number side (which is the hypotenuse). Since $\cos\theta = \frac{\sqrt{1-u^2}}{1}$, we have $\sqrt{1-u^2} = \cos\theta$. So, our integral becomes

$$\begin{aligned}
\frac{1}{\sqrt{5}} \int \cos\theta \cdot \cos\theta\,d\theta &= \frac{1}{\sqrt{5}} \int \cos^2\theta\,d\theta \\
&= \frac{1}{2\sqrt{5}} \int 1+\cos 2\theta\,d\theta \\
&= \frac{1}{2\sqrt{5}}\theta + \frac{1}{4\sqrt{5}}\sin 2\theta + C \\
&= \frac{1}{2\sqrt{5}}\theta + \frac{1}{4\sqrt{5}} \cdot 2\sin\theta\cos\theta + C \\
&= \frac{1}{2\sqrt{5}}\sin^{-1}\frac{u}{1} + \frac{1}{4\sqrt{5}} \cdot 2 \cdot \frac{u}{1} \cdot \frac{\sqrt{1-u^2}}{1} + C. \\
&= \frac{1}{2\sqrt{5}}\sin^{-1}\frac{\sqrt{5}x}{1} + \frac{1}{4\sqrt{5}} \cdot 2 \cdot \frac{\sqrt{5}x}{1} \cdot \frac{\sqrt{1-5x^2}}{1} + C.
\end{aligned}$$