Does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2+1}$ diverge, converge absolutely, or converge conditionally?

Solution 1

The function $f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[2, \infty)$.

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x^2+1}\,dx &= \lim_{t\to\infty} \int_2^t \frac{1}{x^2+1}\,dx \\
&= \lim_{t\to\infty} (\arctan t - \arctan 2) \\
&= \frac{\pi}{2} - \arctan 2.
\end{aligned}$$

Since the integral $\displaystyle\int_2^{\infty} \frac{1}{x^2+1}\,dx$ converges, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2+1}$ converges by the Integral Test.

Since $\sum |a_n| = \sum a_n$, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2+1}$ converges absolutely.

Solution 2

Since $\frac{1}{n^2+1} \le \frac{1}{n^2}$ and since the series $\sum \frac{1}{n^2}$ converges by the p-test, the series $\sum \frac{1}{n^2+1}$ converges by the Direct Comparison Test.

Since $\sum |a_n| = \sum a_n$, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2+1}$ converges absolutely.

Solution 3

The series $\sum \frac{1}{n^2}$ converges by the p-test. Let $a_n = \frac{1}{n^2+1}$ and $b_n = \frac{1}{n^2}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^2}{n^2+1} \\
&= \lim_{n\to\infty} \frac{2n}{2n} \text{ by l'hopital} \\
&= \lim_{n\to\infty} \frac{1}{1} \text{ by algebra} \\
&= 1
\end{aligned}$$

The Limit Comparison Test applies, since this limit was a positive, finite number. Therefore, the series $\sum \frac{1}{n^2+1}$ converges by the Limit Comparison Test.

Since $\sum |a_n| = \sum a_n$, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2+1}$ converges absolutely.