Does $\sum_{n=2}^{\infty} \frac{2n}{n^2+1}$ diverge, converge absolutely, or converge conditionally?

Solution 1

The function $f(x) = \frac{2x}{x^2+1}$ is continuous, positive, and decreasing on $[2, \infty)$.

$$\begin{aligned}
\int_2^{\infty} \frac{2x}{x^2+1}\,dx &= \lim_{t\to\infty} \int_2^t \frac{2x}{x^2+1}\,dx \\
&= \lim_{t\to\infty} \left(\ln|t^2+1| - \ln|2^2+1|\right) \\
&= \infty
\end{aligned}$$

Since the integral $\displaystyle\int_2^{\infty} \frac{2x}{x^2+1}\,dx$ diverges, the series $\displaystyle\sum_{n=2}^{\infty} \frac{2n}{n^2+1}$ diverges by the Integral Test.

Solution 2

Since $n^2 \geq 1$, by adding n^2 to both sides of this inequality

$$2n^2 \geq n^2 + 1$$

Dividing both sides of this ineqaulity by $n(n^2+1)$, we get

$$\frac{2n}{n^2+1} \geq \frac{1}{n}$$

Since the series $\sum \frac{1}{n}$ diverges by the p-test, the series $\sum \frac{2n}{n^2+1}$ diverges by the Direct Comparison Test.

Solution 3

The series $\sum \frac{1}{n}$ diverges by the p-test. Let $a_n = \frac{2n}{n^2+1}$ and $b_n = \frac{1}{n}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{2n^2}{n^2+1} \\
&= \lim_{n\to\infty} \frac{4n}{4n} \text{ by l'hopital} \\
&= \lim_{n\to\infty} \frac{4}{4} \text{ by algebra} \\
&= 1
\end{aligned}$$

The Limit Comparison Test applies, since this limit was a positive, finite number. Therefore, the series $\sum \frac{2n}{n^2+1}$ diverges by the Limit Comparison Test.