Solution

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+6x+13}}\,dx &= \int \frac{1}{\sqrt{x^2+6x+9-9+13}}\,dx\\ &= \int \frac{1}{\sqrt{(x+3)^2-9+13}}\,dx\\ &= \int \frac{1}{\sqrt{(x+3)^2+4}}\,dx\end{aligned}$$

If $u = x+3$, then $du = dx$ so the integral above is equal to

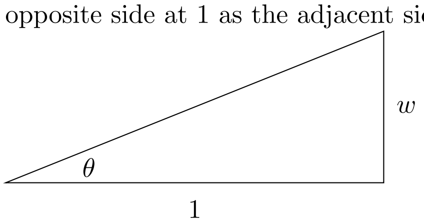
$$\int \frac{1}{\sqrt{u^2+4}}\,du$$

The integral above can be completed several ways:

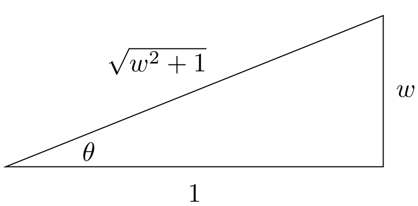
1. Using the substitution $w = \frac{u}{2}$,

$$\begin{aligned}\int \frac{1}{\sqrt{u^2+4}}\,du &= \int \frac{1}{\sqrt{4(\frac{1}{4}u^2+1)}}\,du\\ &= \frac{1}{2}\int \frac{1}{\sqrt{\frac{1}{4}u^2+1}}\,du\\ &= \frac{1}{2}\int \frac{1}{\sqrt{(\frac{u}{2})^2+1}}\,du\\ &= \frac{1}{2}\cdot 2\int \frac{1}{\sqrt{w^2+1}}\,dw\\ &= \int \frac{1}{\sqrt{w^2+1}}\,dw\end{aligned}$$

Let $w = 1\tan\theta$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. So, $dw = 1\sec^2\theta\,d\theta$ and we use $\tan\theta = \frac{w}{1}$ to draw a right triangle with w as the opposite side at 1 as the adjacent side.



The hypotenuse is found using the Pythagorean Theorem, and we have



Then $\cos\theta = \dfrac{1}{\sqrt{w^2+1}}$. Based on this equation, we will use either $\sqrt{w^2+1} = \dfrac{1}{\cos\theta} = 1\sec\theta$ or we will use $\dfrac{\cos\theta}{1} = \dfrac{1}{\sqrt{w^2+1}}$ if they are helpful.

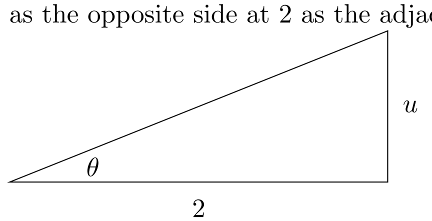
So the integral becomes

$$
\begin{aligned}
\int \frac{1}{\sqrt{w^2+1}}\,dw &= \int \cos\theta \sec^2\theta d\theta \\
&= \int \sec\theta\,d\theta \\
&= \ln|\sec\theta+\tan\theta| + C \\
&= \ln\left|\sqrt{w^2+1}+w\right| + C \\
&= \ln\left|\sqrt{(u/2)^2+1}+\frac{u}{2}\right| + C \\
&= \ln\left|\sqrt{((x+3)/2)^2+1}+\frac{x+3}{2}\right| + C
\end{aligned}
$$

2. An alternate method (probably the more typical method) to find

$$\int \frac{1}{\sqrt{u^2+4}}\,du$$

is as follows: Let $u = 2\tan\theta$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. So, $du = 2\sec^2\theta\,d\theta$ and we use $\tan\theta = \frac{u}{2}$ to draw a right triangle with u as the opposite side at 2 as the adjacent side.



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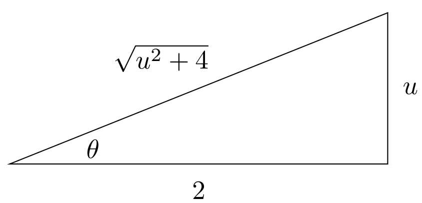

Then $\cos\theta = \dfrac{2}{\sqrt{u^2+4}}$. Based on this equation, we will use either $\sqrt{u^2+4} = \dfrac{2}{\cos\theta} = 2\sec\theta$ or we will use $\dfrac{\cos\theta}{2} = \dfrac{1}{\sqrt{u^2+4}}$ if they are helpful.

So, the integral we have becomes

$$
\begin{aligned}
\int \frac{1}{\sqrt{u^2+4}}\,du &= \int \frac{1}{2\sec\theta} 2\sec^2\theta\,d\theta \\
&= \int \sec\theta\,d\theta \\
&= \ln|\sec\theta+\tan\theta| + C \\
&= \ln\left|\frac{\sqrt{u^2+4}}{2}+\frac{u}{2}\right| + C \\
&= \ln\left|\frac{\sqrt{(x+3)^2+4}}{2}+\frac{x+3}{2}\right| + C
\end{aligned}
$$