$$\int \frac{x^7}{\sqrt{25+4x^2}}\,dx$$

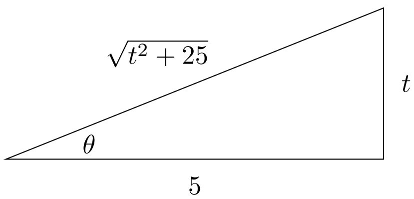
## Solution 1

$$\int \frac{x^7}{\sqrt{25+4x^2}}\,dx = \int \frac{x^7}{\sqrt{25+(2x)^2}}\,dx$$

and say $t = 2x$, so $dt = 2\,dx$ and $x$ can be replaced with $\frac{t}{2}$, so the integral becomes

$$\int \frac{(\frac{t}{2})^7}{\sqrt{25+t^2}} \cdot \frac{1}{2}\,dt = \frac{1}{2^8}\int \frac{t^7}{\sqrt{25+t^2}}\,dt$$

Let $t = 5\tan\theta$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. So, $dt = 5\sec^2\theta\,d\theta$ and we use $\tan\theta = \frac{t}{5}$ to draw a right triangle with $t$ as the opposite side at 5 as the adjacent side. The hypotenuse is found using the Pythagorean Theorem, and we have




Then $\cos\theta = \dfrac{5}{\sqrt{t^2+25}}$. Based on this equation, we will use either $\sqrt{t^2+25} = \dfrac{5}{\cos\theta} = 5\sec\theta$ or we will use $\dfrac{\cos\theta}{5} = \dfrac{1}{\sqrt{t^2+25}}$ if they are helpful.

So

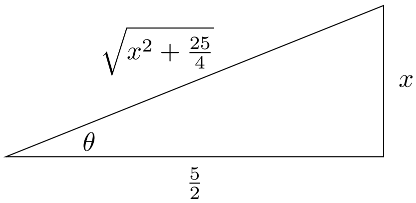
$$\frac{1}{2^8}\int \frac{t^7}{\sqrt{25+t^2}}\,dt = \frac{1}{2^8}\int (5\tan\theta)^7 \cdot \frac{\cos\theta}{5} 5\sec^2\theta\,d\theta = \frac{5^7}{2^8}\int \tan^7\theta\sec\theta\,d\theta$$

$$\begin{aligned}
\frac{5^7}{2^8}\int \tan^7\theta\sec\theta\,d\theta &= \frac{5^7}{2^8}\int \tan^6\theta\cdot\sec\theta\tan\theta\,d\theta \\
&= \frac{5^7}{2^8}\int (\tan^2\theta)^3\cdot\sec\theta\tan\theta\,d\theta \\
&= \frac{5^7}{2^8}\int (\sec^2\theta-1)^3\cdot\sec\theta\tan\theta\,d\theta \qquad u=\sec\theta \quad du=\sec\theta\tan\theta\,d\theta \\
&= \frac{5^7}{2^8}\int (u^2-1)^3\,du \\
&= \frac{5^7}{2^8}\int (u^6-3u^4+3u^2-1)\,du \\
&= \frac{5^7}{2^8}\left[\frac{u^7}{7} - 3\cdot\frac{u^5}{5} + 3\cdot\frac{u^3}{3} - u\right] + C \\
&= \frac{5^7}{2^8}\left[\frac{(\frac{1}{5}\sqrt{t^2+25})^7}{7} - 3\cdot\frac{(\frac{1}{5}\sqrt{t^2+25})^5}{5} + 3\cdot\frac{(\frac{1}{5}\sqrt{t^2+25})^3}{3} - \frac{1}{5}\sqrt{t^2+25}\right] + C \\
&= \frac{5^7}{2^8}\left[\frac{(\frac{1}{5}\sqrt{(2x)^2+25})^7}{7} - 3\cdot\frac{(\frac{1}{5}\sqrt{(2x)^2+25})^5}{5} + 3\cdot\frac{(\frac{1}{5}\sqrt{(2x)^2+25})^3}{3} - \frac{1}{5}\sqrt{(2x)^2+25}\right] + C
\end{aligned}$$

## Solution 2

$$\int \frac{x^7}{\sqrt{25+4x^2}}\,dx = \int \frac{x^7}{\sqrt{4(\frac{25}{4}+x^2)}}\,dx = \frac{1}{2}\int \frac{x^7}{\sqrt{\frac{25}{4}+x^2}}\,dx$$

Let $x = \frac{5}{2}\tan\theta$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. So, $dx = \frac{5}{2}\sec^2\theta\,d\theta$ and we use $\tan\theta = \frac{x}{\frac{5}{2}}$ to draw a right triangle with $x$ as the opposite side at $\frac{5}{2}$ as the adjacent side. The hypotenuse is found using the Pythagorean Theorem, and we have




Then $\cos\theta = \dfrac{\frac{5}{2}}{\sqrt{x^2 + \frac{25}{4}}}$. Based on this equation, we will use either $\sqrt{x^2 + \dfrac{25}{4}} = \dfrac{\frac{5}{2}}{\cos\theta} = \dfrac{5}{2}\sec\theta$ or we will use $\dfrac{\cos\theta}{\frac{5}{2}} = \dfrac{1}{\sqrt{x^2 + \frac{25}{4}}}$ if they are helpful.

So

$$\frac{1}{2}\int \frac{x^7}{\sqrt{\frac{25}{4} + x^2}}\,dx = \frac{1}{2}\int \left(\frac{5}{2}\tan\theta\right)^7 \cdot \frac{\cos\theta}{\frac{5}{2}}\frac{5}{2}\sec^2\theta\,d\theta = \frac{5^7}{2^8}\int \tan^7\theta\sec\theta\,d\theta$$

$$\begin{aligned}
\frac{5^7}{2^8}\int \tan^7\theta\sec\theta\,d\theta &= \frac{5^7}{2^8}\int \tan^6\theta\cdot\sec\theta\tan\theta\,d\theta \\
&= \frac{5^7}{2^8}\int (\tan^2\theta)^3\cdot\sec\theta\tan\theta\,d\theta \\
&= \frac{5^7}{2^8}\int (\sec^2\theta - 1)^3\cdot\sec\theta\tan\theta\,d\theta \qquad u = \sec\theta \quad du = \sec\theta\tan\theta\,d\theta \\
&= \frac{5^7}{2^8}\int (u^2-1)^3\,du \\
&= \frac{5^7}{2^8}\int (u^6 - 3u^4 + 3u^2 - 1)\,du \\
&= \frac{5^7}{2^8}\left[\frac{u^7}{7} - 3\cdot\frac{u^5}{5} + 3\cdot\frac{u^3}{3} - u\right] + C \\
&= \frac{5^7}{2^8}\left[\frac{(\frac{2}{5}\sqrt{x^2+25})^7}{7} - 3\cdot\frac{(\frac{2}{5}\sqrt{x^2+25})^5}{5} + 3\cdot\frac{(\frac{2}{5}\sqrt{x^2+25})^3}{3} - \frac{2}{5}\sqrt{x^2+25}\right] + C
\end{aligned}$$