Does $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$ diverge, converge absolutely, or converge conditionally?

Note

It is tempting to try the Root Test (and a natural choice), but you end up getting that $L = \lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ is equal to 1, so the Root Test is inconclusive. (Try it out to see.) We need to try something else instead.

Solution

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

To find the limit of a sequence like this (where the variable is in both the base and the exponent), we should let $y = \left(1+\frac{1}{n}\right)^n$, so we are trying to find the limit of y as n goes to infinity.

Then

$$\begin{aligned}
y &= \left(1+\frac{1}{n}\right)^n \\
\ln y &= \ln\left[\left(1+\frac{1}{n}\right)^n\right] \\
\ln y &= n\ln\left(1+\frac{1}{n}\right) \\
\lim_{n\to\infty} \ln y &= \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)
\end{aligned}$$

We examine the limit problem on the right:

$$\begin{aligned}
\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) &= \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \\
&= \lim_{n\to\infty}\frac{\frac{1}{1+\frac{1}{n}}\cdot\frac{-1}{n^2}}{-\frac{1}{n^2}} \quad \text{by L'hopital's rule} \\
&= \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} \quad \text{by algebra cancellation} \\
&= \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty}(1+\frac{1}{n})} \quad \text{by algebra cancellation} \\
&= \frac{1}{1+0} \qquad = 1.
\end{aligned}$$

So since

$$\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = 1$$

we also have

$$\lim_{n\to\infty} \ln y = 1$$

so

$$\ln\left(\lim_{n\to\infty} y\right) = 1$$

and using the definition of logarithm,

$$\lim_{n\to\infty} y = e^1 = e$$

Now, since

$$\lim_{n\to\infty} a_n = e$$

the series $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$ diverges by the Test for Divergence.