Does $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+7}$ diverge, converge absolutely, or converge conditionally?

Solution 1

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p=1$. Since $p \leq 1$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-series test. If we use $a_n = \frac{n^2+1}{n^3+7}$ and $b_n = \frac{1}{n}$, then

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^2+1}{n^3+7} \cdot \frac{n}{1} \\
&= \lim_{n\to\infty} \frac{n^3+n}{n^3+7} \\
&= \lim_{n\to\infty} \frac{3n^2+1}{3n^2} \text{ using L'hopital} \\
&= \lim_{n\to\infty} \frac{6n}{6n} \text{ using L'hopital} \\
&= \lim_{n\to\infty} 1 \\
&= 1
\end{aligned}$$

So by the Limit Comparison Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+7}$ diverges.

Solution 2

We are looking for a fixed $K > 0$ such that

$$K \cdot \frac{n^2+1}{n^3+7} \geq \frac{1}{n}$$

By multiplying both sides by $n(n^3+7)$, we have

$$K(n^3+n) \geq n^3+7$$

so

$$Kn^3 + Kn \geq n^3 + 7$$

so if we pick $K = 8$, then we'd have

$$8n^3 + 8n \geq n^3 + 7$$

is likely true because

$$n^3 + 7n^3 + 8n \geq n^3 + 7$$

with the n^3's compared together, and $7n^3$ is greater than 7. So we are ready to present starting from true inequalities.

With the scratch work above done, since $7n^3 \geq 7$ for $n \geq 1$, we have

$$n^3 + 7n^3 + 8n \geq n^3 + 7$$

$$8n^3 + 8n \geq n^3 + 7$$

$$8(n^3+n) \geq n^3 + 7$$

and dividing both sides by $8n(n^3+7)$ we get

$$\frac{n^2+1}{n^3+7} \geq \frac{1}{8n}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{8n} = \frac{1}{8} \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+7}$ diverges by the Direct Comparison Test.