Does $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+7}$ diverge, converge absolutely, or converge conditionally?

Solution 1

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2$. Since $p > 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test. If we use $a_n = \frac{n^2+1}{n^4+7}$ and $b_n = \frac{1}{n^2}$, then

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^2+1}{n^4+7} \cdot \frac{n^2}{1} \\
&= \lim_{n\to\infty} \frac{n^4+n^2}{n^4+7} \\
&= \lim_{n\to\infty} \frac{4n^3+2n}{4n^3} \text{ using L'hopital} \\
&= \lim_{n\to\infty} \frac{12n^2+2}{12n^2} \text{ using L'hopital} \\
&= \lim_{n\to\infty} \frac{24n}{24n} \text{ using L'hopital} \\
&= \lim_{n\to\infty} 1 \\
&= 1
\end{aligned}$$

So by the Limit Comparison Test, the series $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+7}$ converges. Since $\sum |a_n| = \sum a_n$, the series $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+7}$ converges absolutely.

Solution 2

We are looking for a fixed $K > 0$ such that

$$\frac{n^2+1}{n^4+7} \le \frac{K}{n^2}$$

By multiplying both sides by $n^2(n^4+7)$, we have

$$n^4 + n^2 \le K(n^4+7)$$

$$n^4 + n^2 \le Kn^4 + 7K$$

and it appears we can use $K = 2$, because then we'd have

$$n^4 + n^2 \le 2n^4 + 14$$

$$n^4 + n^2 \le n^4 + n^4 + 14$$

and $n^2 \le n^4$ (when comparing second terms). So we are ready to present starting from true inequalities.

With the scratch work above done, since $n^2 \le n^4$, we have

$$n^4 + n^2 \le n^4 + n^4 + 14$$

$$n^4 + n^2 \le 2n^4 + 14$$

$$n^4 + n^2 \le 2(n^4 + 7)$$

by dividing both sides by $n^2(n^4+7)$, we get

$$\frac{n^2+1}{n^4+7} \le \frac{2}{n^2}$$

Since $\sum_{n=1}^{\infty} \frac{2}{n^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test, the series $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+7}$ converges by the Direct Comparison Test. Since $\sum |a_n| = \sum a_n$, the series $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+7}$ converges absolutely.