Does $\displaystyle\sum_{n=2}^{\infty}\left(\frac{1}{\ln n}-\frac{1}{\ln(n+1)}\right)$ diverge, converge absolutely, or converge conditionally?

Solution

$\displaystyle\sum_{n=2}^{\infty}\left(\frac{1}{\ln n}-\frac{1}{\ln(n+1)}\right)$ appears to be a telescoping series. We look at the sequence of partial sums s_n, and note that after cancellation[1] we have

$$s_n = \frac{1}{\ln 2} - \frac{1}{\ln(n+1)}$$

Then

$$\begin{aligned}\lim_{n\to\infty} s_n &= \lim_{n\to\infty}\left(\frac{1}{\ln 2}-\frac{1}{\ln(n+1)}\right)\\ &= \frac{1}{\ln 2} - 0\\ &= \frac{1}{\ln 2}.\end{aligned}$$

It is tempting to get confused here with the Test for Divergence, but note that we found the limit of the sequence of partial sums. So, (by definition), the series $\displaystyle\sum_{n=2}^{\infty}\left(\frac{1}{\ln n}-\frac{1}{\ln(n+1)}\right)$ converges, and we furthermore know that the sum is $\frac{1}{\ln 2}$. Recall that it is unusual that we get to know the value of a convergent series.

Since $\sum|a_n| = \sum a_n$, the series $\displaystyle\sum_{n=2}^{\infty}\left(\frac{1}{\ln n}-\frac{1}{\ln(n+1)}\right)$ converges absolutely.

[1] This cancellation is much easier to describe in handwritten work than it is in typewritten work.... sorry!