$$\int 4x \arctan x \, dx$$

Solution

We will ignore the 4 for now and bring it back at the end. To integrate

$$\int x \arctan x \, dx$$

use integration by parts and some clever algebra after. Let $u = \arctan x$ and $dv = x\,dx$. Then $du = \frac{1}{x^2+1}\,dx$ and $v = \frac{x^2}{2}$ so

$$\begin{aligned}
\int x \arctan x \, dx &= \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int \frac{x^2}{x^2+1}\,dx \\
&= \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int \frac{x^2+1-1}{x^2+1}\,dx \\
&= \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int \frac{(x^2+1)-1}{x^2+1}\,dx \\
&= \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\,dx \\
&= \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int 1 - \frac{1}{x^2+1}\,dx \\
&= \frac{1}{2}x^2 \arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x + C.
\end{aligned}$$

Now, recall the 4 we ignored. Therefore,

$$4\int x \arctan x \, dx = 4\left(\frac{1}{2}x^2 \arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x\right) + C.$$