$$\int \sqrt{4-4x^2}\,dx$$

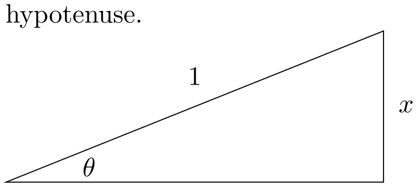
Solution

First, some algebra:

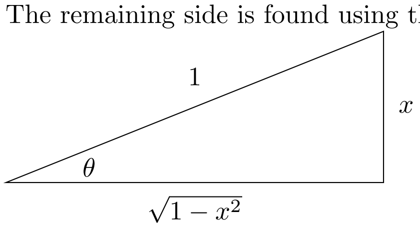
$$\int \sqrt{4-4x^2}\,dx = \int \sqrt{4(1-x^2)}\,dx = \int 2\sqrt{1-x^2}\,dx = 2\int \sqrt{1-x^2}\,dx.$$

Because of seeing $\sqrt{a^2-x^2}$ in the integral with $a=1$, we apply trig substitution with $x = 1\sin\theta$, for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

So, $dx = 1\cos\theta\,d\theta$ and we use $\sin\theta = \frac{x}{1}$ to draw a right triangle with x as the opposite side at 1 as the hypotenuse.



The remaining side is found using the Pythagorean Theorem, and we have



So our integral becomes

$$\int 2\sqrt{1-x^2}\,dx = \int \sqrt{1-x^2}\cos\theta\,d\theta$$

We still need to replace $\sqrt{1-x^2}$, so we look to the side of the triangle with that quantity (it is adjacent to θ) and the number side (which is the hypotenuse). Since $\cos\theta = \frac{\sqrt{1-x^2}}{1}$, we have $\sqrt{1-x^2} = \cos\theta$. So, our integral becomes

$$\begin{aligned}
2\int \cos\theta\cdot\cos\theta\,d\theta &= 2\int \cos^2\theta\,d\theta \\
&= 2\cdot\frac{1}{2}\int 1+\cos 2\theta\,d\theta \\
&= \int 1+\cos 2\theta\,d\theta \\
&= \theta + \frac{1}{2}\sin 2\theta + C \\
&= \theta + \sin\theta\cos\theta + C \\
&= \sin^{-1}\frac{x}{1} + \frac{x}{1}\cdot\frac{\sqrt{1-x^2}}{1} + C. \\
&= \sin^{-1}x + x\sqrt{1-x^2} + C.
\end{aligned}$$