

# Proving a variant of a polynomial positivity conjecture

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## Bessis-Moussa-Villani Conjecture: 1975

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In 2004, Lieb and Seiringer proved the following statement is equivalent:

## Reformulation

Let  $A$  and  $B$  be  $n \times n$  positive semidefinite matrices. Let  $p(t) = \operatorname{tr}((A + tB)^m)$ . For every  $r$ , the coefficient of  $t^r$  in the polynomial  $p(t)$  is non-negative.

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- $S_{m,r}$ : sum of all words of length  $m$  with  $r$   $B$ s and  $m - r$   $A$ s

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- Algebra  $\mathbb{R}\langle X, Y \rangle$  of polynomials in  $X, Y$  non-commuting
- When is  $S_{m,r}$  cyclically-equivalent to sums of Hermitian squares in  $\mathbb{R}\langle X, Y \rangle$ ?

Side note: the terms  $AABBAB$  and  $BBABAA$  of  $S_{6,3}$  are cyclically equivalent.

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- $m = 14$  and  $r = 6$
- $m \in \{7, 11\}$  and  $r = 3$

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$S_{m,r}$  not cyclically-equivalent to sums of Hermitian squares if

- $m \geq 12$  and  $r = 3$
- $m \in \{6, 8, 9, 10\}$  and  $r = 3$
- $m \geq 10$  and  $5 \leq r \leq m - 5$  and  $m$  or  $r$  odd
- $m = 12$  and  $r = 6$

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- Something else must change...
- After changing the conditions of the conjecture, is there something true (and interesting)?
- Instead of finding SOS (up to cyclic rotation) in the two non-commutative variables  $A$  and  $B$ , find SOS in  $2(n + \binom{n}{2})$  commutative variables.

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## Conjecture

Let  $A$  and  $B$  be  $n \times n$  symmetric matrices. If  $m \geq 4$  and  $r$  are even, the coefficient of  $t^r$  in  $p(t) = \text{tr}((A + tB)^m)$  is non-negative.

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## Theorem: Hillar 2007

If the BMV Conjecture is true for  $m = m_0$ , then the BMV Conjecture is true for all  $m \leq m_0$ .

## Question

Is the BMV Conjecture true for infinitely many  $m$ ?

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Is the alternate conjecture true for every  $m \geq 4$  a power of 2?  
with  $2m = r$ ?

## Background on algebraic formulation

For symmetric matrices  $A$  and  $B$ , if  $m \geq 4$  and  $r$  are even, the coefficient of  $t^r$  in  $p(t) = \text{tr}[(A + tB)^m]$  is non-negative.

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- Let  $A$  and  $B$  be general symmetric matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix}$$

when  $n = 2$



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## Polynomial in $a_{ij}$ and $b_{ij}$

For fixed  $n, m, r$ , the coefficient of  $t^r$  in  $p(t)$  is a polynomial in the  $2\left[\binom{n}{2} + n\right]$  variables  $a_{ij}$  and  $b_{ij}$  with  $i \leq j$ .

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## Question

Is this multivariate polynomial non-negative?

## Coefficient of $t^r$ in $p(t)$ : polynomial in $a_{i,j}$ and $b_{i,j}$

If  $n = 2$  and  $m = 4$ , the coefficient of  $t^2$  in  $p(t)$  is

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### Example

$$A = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -117 \\ -117 & 10 \end{bmatrix}$$

$$p(t) = 381577202t^4 - 12044936t^3 + 371308t^2 - 4360t + 50$$

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& 160a_{11}^2a_{12}a_{22}b_{11}^3b_{12} + 128a_{12}^3a_{22}b_{11}^3b_{12} + 64a_{11}a_{12}a_{22}^2b_{11}^3b_{12} + 16a_{12}a_{22}^3b_{11}^3b_{12} + 120a_{11}^4b_{11}^2b_{12}^2 + \\
& 720a_{11}^2a_{12}^2b_{11}^2b_{12}^2 + 240a_{12}^4b_{11}^2b_{12}^2 + 80a_{11}^3a_{22}b_{11}^2b_{12}^2 + 576a_{11}a_{12}^2a_{22}b_{11}^2b_{12}^2 + 48a_{11}^2a_{22}^2b_{11}^2b_{12}^2 + 216a_{12}^2a_{22}^2b_{11}^2b_{12}^2 + \\
& 24a_{11}a_{22}^3b_{11}^2b_{12}^2 + 8a_{22}^4b_{11}^2b_{12}^2 + 320a_{11}^3a_{12}b_{11}b_{12}^3 + 640a_{11}a_{12}^3b_{11}b_{12}^3 + 384a_{11}^2a_{12}a_{22}b_{11}b_{12}^3 + 480a_{12}^3a_{22}b_{11}b_{12}^3 + \\
& 288a_{11}a_{12}a_{22}^2b_{11}b_{12}^3 + 128a_{12}a_{22}^3b_{11}b_{12}^3 + 20a_{11}^4b_{12}^4 + 240a_{11}^2a_{12}^2b_{12}^4 + 140a_{12}^4b_{12}^4 + 32a_{11}^3a_{22}b_{12}^4 + 360a_{11}a_{12}^2a_{22}b_{12}^4 + \\
& 36a_{11}^2a_{22}^2b_{12}^4 + 240a_{12}^2a_{22}^2b_{12}^4 + 32a_{11}a_{22}^3b_{12}^4 + 20a_{22}^4b_{12}^4 + 80a_{11}^2a_{12}^2b_{11}^3b_{22} + 32a_{12}^4b_{11}^3b_{22} + 64a_{11}a_{12}^2a_{22}b_{11}^3b_{22} + \\
& 24a_{12}^3a_{22}b_{11}^3b_{22} + 160a_{11}^3a_{12}b_{11}b_{12}b_{22} + 384a_{11}a_{12}^3b_{11}b_{12}b_{22} + 192a_{11}^2a_{12}a_{22}b_{11}b_{12}b_{22} + 288a_{12}^3a_{22}b_{11}b_{12}b_{22} + \\
& 144a_{11}a_{12}a_{22}^2b_{11}b_{12}b_{22} + 64a_{12}a_{22}^3b_{11}b_{12}b_{22} + 40a_{11}^4b_{11}b_{12}^2b_{22} + 576a_{11}^2a_{12}^2b_{11}b_{12}^2b_{22} + 360a_{12}^4b_{11}b_{12}^2b_{22} + \\
& 64a_{11}^3a_{22}b_{11}b_{12}^2b_{22} + 864a_{11}a_{12}^2a_{22}b_{11}b_{12}^2b_{22} + 72a_{11}^2a_{22}^2b_{11}b_{12}^2b_{22} + 576a_{12}^2a_{22}^2b_{11}b_{12}^2b_{22} + 64a_{11}a_{22}^3b_{11}b_{12}^2b_{22} + \\
& 40a_{22}^4b_{11}b_{12}^2b_{22} + 128a_{11}^3a_{12}b_{12}^3b_{22} + 480a_{11}a_{12}^3b_{12}^3b_{22} + 288a_{11}^2a_{12}a_{22}b_{12}^3b_{22} + 640a_{12}^3a_{22}b_{12}^3b_{22} + \\
& 384a_{11}a_{12}a_{22}^2b_{12}^3b_{22} + 320a_{12}a_{22}^3b_{12}^3b_{22} + 48a_{11}^2a_{12}^2b_{11}^2b_{22}^2 + 36a_{12}^4b_{11}^2b_{22}^2 + 72a_{11}a_{12}^2a_{22}b_{11}^2b_{22}^2 + 48a_{12}^2a_{22}^2b_{11}^2b_{22}^2 + \\
& 64a_{11}^3a_{12}b_{11}b_{12}b_{22}^2 + 288a_{11}a_{12}^3b_{11}b_{12}b_{22}^2 + 144a_{11}^2a_{12}a_{22}b_{11}b_{12}b_{22}^2 + 384a_{12}^3a_{22}b_{11}b_{12}b_{22}^2 + 192a_{11}a_{12}a_{22}^2b_{11}b_{12}b_{22}^2 + \\
& 160a_{12}a_{22}^3b_{11}b_{12}b_{22}^2 + 8a_{11}^4b_{12}^2b_{22}^2 + 216a_{11}^2a_{12}^2b_{12}^2b_{22}^2 + 240a_{12}^4b_{12}^2b_{22}^2 + 24a_{11}^3a_{22}b_{12}^2b_{22}^2 + 576a_{11}a_{12}^2a_{22}b_{12}^2b_{22}^2 + \\
& 48a_{11}^2a_{22}^2b_{12}^2b_{22}^2 + 720a_{12}^2a_{22}^2b_{12}^2b_{22}^2 + 80a_{11}a_{22}^3b_{12}^2b_{22}^2 + 120a_{22}^4b_{12}^2b_{22}^2 + 24a_{11}^2a_{12}^2b_{11}b_{22}^3 + 32a_{12}^4b_{11}b_{22}^3 + \\
& 64a_{11}a_{12}^2a_{22}b_{11}b_{22}^3 + 80a_{12}^2a_{22}^2b_{11}b_{22}^3 + 16a_{11}^3a_{12}b_{12}b_{22}^3 + 128a_{11}a_{12}^3b_{12}b_{22}^3 + 64a_{11}^2a_{12}a_{22}b_{12}b_{22}^3 + 320a_{12}^3a_{22}b_{12}b_{22}^3 + \\
& 160a_{11}a_{12}a_{22}^2b_{12}b_{22}^3 + 320a_{12}a_{22}^3b_{12}b_{22}^3 + 8a_{11}^2a_{12}^2b_{22}^4 + 20a_{12}^4b_{22}^4 + 40a_{11}a_{12}^2a_{22}b_{22}^4 + 120a_{12}^2a_{22}^2b_{22}^4 + 70a_{22}^4b_{22}^4.
\end{aligned}$$

## Result when $n = 3, m = 4, r = 2$

$\nu = \mathbf{z}_1^T Q_1 \mathbf{z}_1 + \sum_{\{i,j\} \in \binom{[n]}{2}} \mathbf{z}_{2,\{i,j\}}^T Q_2 \mathbf{z}_{2,\{i,j\}}$ , where

$$Q_1 = \left( \begin{array}{ccc|ccc} 6 & 0 & 0 & 6 & 6 & 0 \\ 0 & 6 & 0 & 6 & 0 & 6 \\ 0 & 0 & 6 & 0 & 6 & 6 \\ \hline 6 & 6 & 0 & 12 & 6 & 6 \\ 6 & 0 & 6 & 6 & 12 & 6 \\ 0 & 6 & 6 & 6 & 6 & 12 \end{array} \right) \quad \mathbf{z}_1 = \begin{pmatrix} a_{11}b_{11} \\ a_{22}b_{22} \\ a_{33}b_{33} \\ a_{12}b_{12} \\ a_{13}b_{13} \\ a_{23}b_{23} \end{pmatrix}$$

and

$$Q_2 = \left( \begin{array}{ccc|ccc} 4 & 4 & 4 & 2 & 2 & 2 \\ 4 & 4 & 4 & 2 & 2 & 2 \\ 4 & 4 & 4 & 2 & 2 & 2 \\ \hline 2 & 2 & 2 & 4 & 4 & 4 \\ 2 & 2 & 2 & 4 & 4 & 4 \\ 2 & 2 & 2 & 4 & 4 & 4 \end{array} \right), \quad \mathbf{z}_{2,\{i,j\}} = \begin{pmatrix} a_{i1}b_{j1} \\ a_{i2}b_{j2} \\ a_{i3}b_{j3} \\ b_{i1}b_{j1} \\ b_{i2}b_{j2} \\ b_{i3}b_{j3} \end{pmatrix}$$

## General result for $m = 2r = 4$

Let  $U$  be the  $s \times n$   $\{0, 1\}$ -matrix whose  $s = n + \binom{n}{2}$  rows are indexed by the non-empty subsets of  $\{1, \dots, n\}$  of cardinality at most 2, and the  $U(X, t)$ -entry is 1 iff  $t \in X$ . Let  $Q_1 = 6U^T U$ . Let  $Q_2$  be the  $(2n) \times (2n)$  block matrix of all 4s on diagonal blocks and all 2s on non-diagonal blocks.

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Given  $n \times n$  matrices  $A$  and  $B$ , the coefficient of  $t^2$  in  $\text{tr}[(A + tB)^4]$  is

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The entry of  $Q_1$  indexed by  $\{i, j\}$  and  $\{i', j'\}$  is  $6|\{i, j\} \cap \{i', j'\}|$ .

## General result for $m = 2r = 4$

### Theorem: K.

For all  $n \times n$  symmetric matrices  $A$  and  $B$ , the coefficient  $\nu$  of  $t^2$  in  $\text{tr}[(A + tB)^4]$  is

$$\nu = 6 \sum_{i \in [n]} \left[ \left( \sum_{k \in [n]} a_{ik} b_{ik} \right)^2 \right] + \sum_{i \neq j} \left\{ \left[ \sum_{k \in [n]} (a_{ik} b_{jk} + b_{ik} a_{jk}) \right]^2 + \left[ \sum_{k \in [n]} a_{ik} b_{jk} \right]^2 + \left[ \sum_{k \in [n]} b_{ik} a_{jk} \right]^2 \right\},$$

which is a sum of squares.

## Conjecture

Let  $A$  and  $B$  be  $n \times n$  symmetric matrices. If  $m \geq 4$  and  $r$  are even, the coefficient of  $t^r$  in  $p(t) = \text{tr}((A + tB)^m)$  is non-negative.

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16pt

In the case of  $m = 4$  and  $r = 2$ , for all  $n \times n$  symmetric  $A$  and  $B$ , the coefficient of  $t^r$  in  $\text{tr}((A + tB)^m)$  is **a sum of squares** in the entries of  $A$  and  $B$ .



# Now what?

Future:

- Every entry of  $Q_1$  is the cardinality of the intersection of two sets. What is going on?

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- When  $m = 8$  and  $r = 4$  and  $n = 2$ , the coefficient of  $t^r$  is SOS.

# Now what?

Future:

- Every entry of  $Q_1$  is the cardinality of the intersection of two sets. What is going on?
- Obtain a general result when  $m = 2r = 8$
- When  $m = 8$  and  $r = 4$  and  $n = 2$ , the coefficient of  $t^r$  is SOS.
- When  $m = 8$  and  $r = 4$  and  $n = 3$ , the SDP stops early, but a  $Q_1$  matrix (all non-negative integers entries, all eigenvalues integer) found.

Thank you!

Thank you

# Computational evidence

◀ Return

Given  $n \times n$  symmetric matrices  $A$  and  $B$ , examine the coefficient of  $t^r$  in  $p(t) = \text{tr}[(A + tB)^m]$

```
n = 2; # Matrix size
m = 4; # Power to raise
var('t')

U = random_matrix(ZZ,n,n)
V = random_matrix(ZZ,n,n)
A = U + U.transpose()
B = V + V.transpose()
C = A + t*B;

print "A is\n", A
print "B is\n", B
p = (C^m).trace().expand();
print p
```

# Output of code on four random runs

◀ Return

$$A = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -117 \\ -117 & 10 \end{bmatrix}$$

$$p(t) = 381577202t^4 - 12044936t^3 + 371308t^2 - 4360t + 50$$

---

$$A = \begin{bmatrix} -2 & 3 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}$$

$$p(t) = 34t^4 - 40t^3 + 268t^2 - 104t + 338$$

---

$$A = \begin{bmatrix} -2 & -6 \\ -6 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -14 \\ -14 & 0 \end{bmatrix}$$

$$p(t) = 79984t^4 + 131200t^3 + 85792t^2 + 26240t + 3200$$

---

$$A = \begin{bmatrix} 2 & -2 \\ -2 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 14 \\ 14 & -4 \end{bmatrix}$$

$$p(t) = 89888t^4 - 33920t^3 + 9984t^2 - 1280t + 128$$

# Many runs at once, $m = 7$

◀ Return

```
n = 3; # Matrix size
m = 7; # Power to raise
var('t')

for i in range(10):
    U = random_matrix(ZZ,n,n)
    V = random_matrix(ZZ,n,n)
    A = U + U.transpose()
    B = V + V.transpose()
    C = A + t*B;

p = (C^m).trace().expand();
print p
```



# Many runs at once, $m = 7$

## Output

◀ Return

- $-95671736t^7 + 29943578t^6 + 1060317328t^5 - 4542979210t^4 + 10319766296t^3 - 13716905776t^2 + 9467623928t - 1691252512$
- $312057256t^7 + 1022191128t^6 + 149902256t^5 + 255541160t^4 - 1664278t^3 + 19766544t^2 - 1346282t + 531142$
- $-174012t^7 - 1611316t^6 - 8077524t^5 - 22798748t^4 - 41833204t^3 - 32404764t^2 + 9568804t + 97330764$
- $-12634864t^7 - 57997576t^6 - 72312128t^5 - 49399952t^4 - 21213920t^3 - 6139952t^2 - 1207024t - 140792$
- $1539019131802084208t^7 + 236088671889089880t^6 + 20229941786048456t^5 + 1180680267203360t^4 + 47002766369664t^3 + 1396623702264t^2 + 25768743560t + 328715088$
- $-34867607296t^7 + 91863481472t^6 - 58312727424t^5 + 29550129280t^4 - 7922606720t^3 + 1486079616t^2 - 155697024t + 8278144$
- $109135872t^7 - 20177920t^6 + 54513536t^5 - 10275328t^4 + 8433488t^3 - 1425872t^2 + 388808t - 31280$
- $-10020024t^7 - 11069128t^6 + 13630960t^5 - 29075256t^4 + 18846394t^3 - 13106800t^2 + 3874528t - 1192956$
- $-882292t^7 + 13180104t^6 - 77308980t^5 + 276714060t^4 - 607576620t^3 +$

# Many runs at once, $m = 7$

What happens when  $m$  is even?

◀ Return

```
n = 3; # Matrix size
m = 8; # Power to raise
var('t')

for i in range(5):
    U = random_matrix(ZZ,n,n)
    V = random_matrix(ZZ,n,n)
    A = U + U.transpose()
    B = V + V.transpose()
    C = A + t*B;

p = (C^m).trace().expand();
print p
```

# Many runs at once, $m = 8$

Output

[← Return](#)

- $356425679360t^8 + 397892962304t^7 + 232469246464t^6 + 88237898240t^5 + 23408069312t^4 + 4394028928t^3 + 568426016t^2 + 46354784t + 1871842$
- $1154t^8 + 10816t^7 + 233512t^6 + 1276928t^5 + 13103020t^4 + 37485376t^3 + 219837288t^2 + 205717888t + 610557282$
- $756943257986t^8 + 685492586944t^7 + 2309050799552t^6 + 1011525059072t^5 + 1641382066432t^4 + 352517873664t^3 + 433306173440t^2 + 34798141440t + 39968841728$
- $6891317142048t^8 + 29661542945280t^7 + 66321070259520t^6 + 93604309846144t^5 + 89845491346368t^4 + 58944168619008t^3 + 25688348653568t^2 + 6758373842944t + 839336551680$
- $125892343202t^8 - 10754192688t^7 + 18894255768t^6 - 3222482832t^5 + 989635820t^4 - 94745552t^3 + 19698136t^2 - 124528t + 221442$