Solution

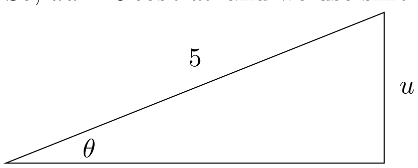
$$\begin{aligned}
\int \frac{1}{\sqrt{16-6x-x^2}}\,dx &= \int \frac{1}{\sqrt{16-(x^2+6x)}}\,dx \\
&= \int \frac{1}{\sqrt{16-(x^2+6x+9-9)}}\,dx \\
&= \int \frac{1}{\sqrt{16-(x+3)^2+9}}\,dx \\
&= \int \frac{1}{\sqrt{25-(x+3)^2}}\,dx
\end{aligned}$$

If $u = x+3$, then $du = dx$ so the integral above is equal to

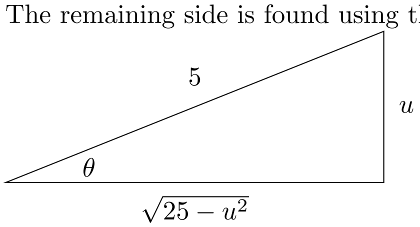
$$\int \frac{1}{\sqrt{25-u^2}}\,du$$

Let $u = 5\sin\theta$ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

So, $du = 5\cos\theta\,d\theta$ and we use $\sin\theta = \frac{u}{5}$ to draw a right triangle with u as the opposite side at 5 as the hypotenuse.



The remaining side is found using the Pythagorean Theorem, and we have



Note $\cos\theta = \dfrac{\sqrt{25-u^2}}{5}$, so $\sqrt{25-u^2} = 5\cos\theta$. So, the integral we have becomes

$$\begin{aligned}
\int \frac{1}{\sqrt{25-u^2}}\,du &= \int \frac{1}{5\cos\theta} 5\cos\theta\,d\theta \\
&= \int d\theta \\
&= \theta + C \\
&= \sin^{-1}\left(\frac{u}{5}\right) + C \\
&= \sin^{-1}\left(\frac{x+3}{5}\right) + C
\end{aligned}$$