Does $\sum_{n=1}^{\infty} \frac{1}{(n+1)^n}$ diverge, converge absolutely, or converge conditionally?

## Solution 1

$$\begin{aligned}
L &= \lim_{n\to\infty} \sqrt[n]{|a_n|} \\
&= \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{(n+1)^n}\right|} \\
&= \lim_{n\to\infty} \frac{1}{n+1} \\
&= 0.
\end{aligned}$$

Since $L < 1$, the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)^n}$ converges absolutely by the Root Test.

## Solution 2: much longer than Solution 1

In an attempt to use the Ratio Test, you'd have to consider the limit

$$\begin{aligned}
L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left|\frac{1}{(n+2)^{n+1}} \frac{(n+1)^n}{1}\right| \\
&= \lim_{n\to\infty} \frac{(n+1)^n}{(n+2)^{n+1}},
\end{aligned}$$

but it looks like there's not any really nice limit here, and while L'Hopital's rule applies, it will be quite a mess. Actually, it is possible to evaluate this limit using some algebra and some limit laws:

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{(n+1)^n}{(n+2)^{n+1}} \\
&= \lim_{n\to\infty} \frac{(n+1)^n}{(n+2)^n(n+2)} \\
&= \lim_{n\to\infty} \left[\left(\frac{n+1}{n+2}\right)^n \cdot \frac{1}{n+2}\right].
\end{aligned}$$

We consider the limit

$$\lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n$$

by defining $y = \left(\frac{n+1}{n+2}\right)^n$. So $\ln y = n \ln \frac{n+1}{n+2}$. So

$$\ln y = \frac{\ln \frac{n+1}{n+2}}{\frac{1}{n}}$$

By applying a limit to both sides,

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln \frac{n+1}{n+2}}{\frac{1}{n}}$$

and the limit on the right uses L'Hopital's rule. (Be sure to pay attention to the Chain Rule and the Quotient Rule.)

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{\frac{n+2}{n+1} \cdot \frac{(n+2)\cdot 1-(n+1)\cdot 1}{(n+2)^2}}{\frac{-1}{n^2}} \\
&= \lim_{n\to\infty} \frac{-n^2}{(n+1)(n+2)} \\
&= -1.
\end{aligned}$$

after two more applications of L'Hopital's rule. Thus,

$$\lim_{n\to\infty} y = e^{-1} = \frac{1}{e}.$$

We were originally considering the limit:

$$\lim_{n\to\infty} \left[\left(\frac{n+1}{n+2}\right)^n \cdot \frac{1}{n+2}\right] \qquad = \left[\lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n\right] \cdot \left[\lim_{n\to\infty} \frac{1}{n+2}\right] = \frac{1}{e} \cdot 0 \qquad = 0.$$

and we could treat the limit of the product as a product of the limits because both limits existed. Since $L < 1$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)^n}$ converges absolutely by the Ratio Test.