$$\int \sin^2 x \cos^2 x \, dx$$

Solution

$$\int \sin^2 x \cos^2 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx = \frac{1}{4}\int (1-\cos^2 2x)\, dx = \frac{1}{4}x - \frac{1}{4}\int \cos^2 2x \, dx.$$

The key work that is left is to integrate $\cos^2 2x$.

Ending 1

To do

$$\int \cos^2 2x \, dx$$

we might first choose to let $t = 2x$, so $dt = 2\, dx$. So the integral we have is

$$\frac{1}{2}\int \cos^2 t \, dt$$

Ignore the $\frac{1}{2}$ for a moment. Integrate by parts with $u = \cos t$ and $dv = \cos t \, dx$. So $du = -\sin t \, dt$ and $v = \sin t$. So

$$\begin{aligned} \int \cos^2 t \, dt &= \cos t \sin t + \int \sin^2 t \, dx \\ &= \cos t \sin t + \int 1 - \cos^2 t \, dt. \end{aligned}$$

So

$$\int \cos^2 t \, dx = \cos t \sin t + \int 1 - \cos^2 t \, dt$$

and using some algebra, we get

$$\int \cos^2 t \, dx = \frac{\cos t \sin t + t}{2} + C$$

So,

$$\frac{1}{2}\int \cos^2 t \, dt = \frac{\cos t \sin t + t}{4} + C$$

because we divide by 2 on both sides. (Here, we are "un-ignoring" the $\frac{1}{2}$ from earlier.) Now, replace each t with a $2x$, and our integral evaluates to

$$\frac{\cos 2x \sin 2x + 2x}{4} + C.$$

Put together with the earlier $\frac{1}{4}x$, our final answer is

$$\frac{x}{4} - \frac{\cos 2x \sin 2x + 2x}{4} + C.$$

Ending 2

Using the trigonometric identity

$$\cos^2 \theta = \frac{1+\cos(2\theta)}{2}$$

we rewrite:

$$\int \cos^2 2x \, dx = \int \frac{1}{2}\, dx + \frac{1}{2}\int \cos 4x \, dx$$

The second integral is completed using a substitution of $u = 4x$, so the integral becomes equal to

$$\frac{1}{2}x + \frac{1}{8}\sin 4x + C.$$

Put together with the earlier $\frac{1}{4}x$, our final answer is

$$\frac{1}{4}x - \left(\frac{1}{2}x + \frac{1}{8}\sin 4x\right) + C.$$