Does $\displaystyle\sum_{n=1}^{\infty}\left(\frac{8}{9}+\frac{1}{n}\right)^n$ diverge, converge absolutely, or converge conditionally?

Note

It is tempting to compute the limit of the sequence $\left(\frac{8}{9}+\frac{1}{n}\right)^n$. However, L'Hopital's rule will not apply because you do not have an $\infty \cdot 0$ indeterminate form.

Solution

$$\begin{aligned} L &= \lim_{n\to\infty} \sqrt[n]{|a_n|} \\ &= \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{8}{9}+\frac{1}{n}\right)^n\right|} \\ &= \lim_{n\to\infty} \sqrt[n]{\left(\frac{8}{9}+\frac{1}{n}\right)^n} \\ &= \lim_{n\to\infty} \left(\frac{8}{9}+\frac{1}{n}\right) \\ &= \frac{8}{9}+0 \qquad\qquad = \frac{8}{9} \end{aligned}$$

Since $L < 1$, by the Root Test, the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{8}{9}+\frac{1}{n}\right)^n$ converges absolutely.