$$\int \sec^3 x\,dx$$

Solution

To integrate

$$\int \sec^3 x\,dx$$

apply integration by parts with $u = \sec x$ and $dv = \sec^2 x\,dx$. So $du = \sec x \tan x dx$ and $v = \tan x$. So

$$\begin{aligned}
\int \sec^3 x\,dx &= \sec x \tan x - \int \tan^2 x \sec x\,dx \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x\,dx \\
&= \sec x \tan x - \int \sec^3 x\,dx - \int \sec x\,dx.
\end{aligned}$$

It may seem disconcerting that we are have the integral of $\sec^3 x$, but since the sign is negative, if we could just obtain the last integral (of $\sec x$) then we'd be able to piece everything together using algebra. We do the integral of secant off to the side:

- To integrate secant,

$$\begin{aligned}
\int \sec x\,dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\,dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx
\end{aligned}$$

  Let $u = \sec x + \tan x$, so $du = \sec x \tan x + \sec^2 x\,dx$. So the integral above is equal to

$$\begin{aligned}
\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx &= \int \frac{1}{u}\,du \\
&= \ln|u| + C \\
&= \ln|\sec x + \tan x| + C.
\end{aligned}$$

Put all together,

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx - \ln|\sec x + \tan x|.$$

By adding the integral in question to both sides,

$$2\int \sec^3 x\,dx = \sec x \tan x - \ln|\sec x + \tan x| + C.$$

Dividing both sides by 2, we finally get

$$\int \sec^3 x\,dx = \frac{\sec x \tan x - \ln|\sec x + \tan x|}{2} + C.$$