$$\int \sin 3x \sin 5x \, dx$$

Solution 1

Let $u = \sin 3x$ and $dv = \sin 5x \, dx$. Then $du = 3 \cos 3x \, dx$ and $v = -\frac{1}{5} \cos 5x$. So

$$\int \sin 3x \sin 5x \, dx = -\frac{1}{5} \sin 3x \cos 5x + \frac{3}{5} \int \cos 3x \cos 5x \, dx.$$

We now apply a second integration by parts, for the integral $\int \cos 3x \cos 5x \, dx$. Note that we must choose $u = \cos 3x$ and $dv = \cos 5x \, dx$ to avoid going around in circles. (If you chose $u = \cos 5x$, then you'll eventually get to the equation $0 = 0$. Since $u = \cos 3x$ and $dv = \cos 5x \, dx$, we get $du = -3 \sin 3x \, dx$ and $v = \frac{1}{5} \sin 5x$. The side integral is thus

$$\int \cos 3x \cos 5x \, dx = \frac{1}{5} \cos 3x \sin 5x + \frac{3}{5} \int \sin 3x \sin 5x \, dx.$$

Placed in context with the original integral, we have

$$\int \sin 3x \sin 5x \, dx = -\frac{1}{5} \sin 3x \cos 5x + \frac{3}{5} \left(\frac{1}{5} \cos 3x \sin 5x + \frac{3}{5} \int \sin 3x \sin 5x \, dx \right).$$

We use I to denote our original integral $\int \sin 3x \sin 5x \, dx$. Then

$$I = -\frac{1}{5} \sin 3x \cos 5x + \frac{3}{5} \left(\frac{1}{5} \cos 3x \sin 5x + \frac{3}{5} I \right).$$

so

$$\begin{aligned} I &= -\frac{1}{5} \sin 3x \cos 5x + \frac{3}{25} \cos 3x \sin 5x + \frac{9}{25} I. \\ \frac{16}{25} I &= -\frac{1}{5} \sin 3x \cos 5x + \frac{3}{25} \cos 3x \sin 5x + C \end{aligned}$$

and finally

$$\int \sin 3x \sin 5x \, dx = \frac{25}{16} \left(-\frac{1}{5} \sin 3x \cos 5x + \frac{3}{25} \cos 3x \sin 5x \right) + C.$$

Solution 2

Let $u = \sin 5x$ and $dv = \sin 3x \, dx$. Then $du = 5 \cos 5x \, dx$ and $v = -\frac{1}{3} \cos 3x$. So

$$\int \sin 3x \sin 5x \, dx = -\frac{1}{3} \cos 3x \sin 5x + \frac{5}{3} \int \cos 3x \cos 5x \, dx.$$

We now apply a second integration by parts, for the integral $\int \cos 3x \cos 5x \, dx$. Note that we must choose $u = \cos 5x$ and $dv = \cos 3x \, dx$ to avoid going around in circles. (If you chose $u = \cos 3x$, then you'll eventually get to the equation $0 = 0$. Since $u = \cos 5x$ and $dv = \cos 3x \, dx$, we get $du = -5 \sin 5x \, dx$ and $v = \frac{1}{3} \sin 3x$. The side integral is thus

$$\int \cos 3x \cos 5x \, dx = \frac{1}{3} \sin 3x \cos 5x + \frac{5}{3} \int \sin 3x \sin 5x \, dx$$

Placed in context with the original integral, we have

$$\int \sin 3x \sin 5x \, dx = -\frac{1}{3} \cos 3x \sin 5x + \frac{5}{3} \left(\frac{1}{3} \sin 3x \cos 5x + \frac{5}{3} \int \sin 3x \sin 5x \, dx \right).$$

We use I to denote our original integral $\int \sin 3x \sin 5x\, dx$. Then

$$I = -\frac{1}{3}\cos 3x \sin 5x + \frac{5}{3}\left(\frac{1}{3}\sin 3x \cos 5x + \frac{5}{3}I\right).$$

so

$$I = -\frac{1}{3}\cos 3x \sin 5x + \frac{5}{9}\sin 3x \cos 5x + \frac{25}{9}I.$$

$$-\frac{16}{9}I = -\frac{1}{3}\cos 3x \sin 5x + \frac{5}{9}\sin 3x \cos 5x + C.$$

and finally

$$I = -\frac{9}{16}\left(-\frac{1}{3}\cos 3x \sin 5x + \frac{5}{9}\sin 3x \cos 5x\right) + C.$$

Solution 3

Apply the trig identity $\sin\alpha \sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

$$\begin{aligned}
\int \sin 3x \sin 5x\, dx &= \frac{1}{2}\int \cos(-2x) - \cos(8x)\, dx \\
&= \frac{1}{2}\int \cos(-2x)\, dx - \frac{1}{2}\int \cos(8x)\, dx \\
&= -\frac{1}{4}\sin(-2x) - \frac{1}{16}\sin(8x) + C.
\end{aligned}$$

where each integral is completed by substitution (using $u = -2x$ for the first integral and $u = 8x$ for the second integral).