$$\int_0^1 \ln(x)\,dx = \lim_{t\to 0^+} \int_t^1 \ln(x)\,dx$$

Indef $\int \ln(x)\,dx$

$u = \ln(x) \quad dv = dx$

$du = \frac{1}{x}\,dx \quad v = x$

$$= x\ln x - \int 1\,dx = x\ln x - x + C$$

$$= \lim_{t\to 0^+}\Big((1\ln 1 - 1) - (t\ln t - t) \Big)$$

$$= \lim_{t\to 0^+}\big((0 - 1) - t\ln t + t \big)$$

Side $\lim_{t\to 0^+} t\ln(t) = \lim_{t\to 0^+} \frac{\ln(t)}{t^{-1}} \overset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{t\to 0^+} \frac{\frac{1}{t}}{-t^{-2}} = \lim_{t\to 0^+}(-t) = 0$

$$= 0 - 1 - 0 + 0$$