

$$\int \sec^3 x \, dx$$

## Solution

To integrate

$$\int \sec^3 x \, dx$$

apply integration by parts with  $u = \sec x$  and  $dv = \sec^2 x \, dx$ . So  $du = \sec x \tan x \, dx$  and  $v = \tan x$ . So

$$\begin{aligned} \int \sec^3 x \, dx &= \sec x \tan x - \int \tan^2 x \sec x \, dx \\ &= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx \\ &= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx. \end{aligned}$$

It may seem disconcerting that we have the integral of  $\sec^3 x$ , but since the sign is negative, if we could just obtain the last integral (of  $\sec x$ ) then we'd be able to piece everything together using algebra. We do the integral of secant off to the side:

- To integrate secant,

$$\begin{aligned} \int \sec x \, dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \, dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \end{aligned}$$

Let  $u = \sec x + \tan x$ , so  $du = \sec x \tan x + \sec^2 x \, dx$ . So the integral above is equal to

$$\begin{aligned} \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx &= \int \frac{1}{u} \, du \\ &= \ln |u| + C \\ &= \ln |\sec x + \tan x| + C. \end{aligned}$$

Put all together,

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \ln |\sec x + \tan x|.$$

By adding the integral in question to both sides,

$$2 \int \sec^3 x \, dx = \sec x \tan x + \ln |\sec x + \tan x| + C.$$

Dividing both sides by 2, we finally get

$$\int \sec^3 x \, dx = \frac{\sec x \tan x + \ln |\sec x + \tan x|}{2} + C.$$