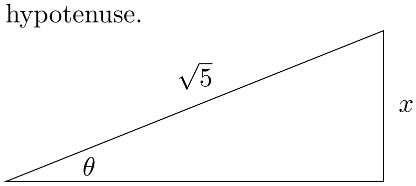
$$\int \sqrt{5-x^2}\,dx$$

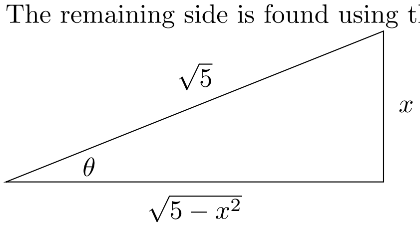
Solution

Because of seeing $\sqrt{a^2-x^2}$ in the integral with $a=\sqrt{5}$, we apply trig substitution with $x=\sqrt{5}\sin\theta$, for $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$.

So, $dx=\sqrt{5}\cos\theta\,d\theta$ and we use $\sin\theta=\frac{x}{\sqrt{5}}$ to draw a right triangle with x as the opposite side at $\sqrt{5}$ as the hypotenuse.



The remaining side is found using the Pythagorean Theorem, and we have



So our integral becomes

$$\int \sqrt{5-x^2}\,dx=\int \sqrt{5-x^2}\,\sqrt{5}\cos\theta\,d\theta$$

We still need to replace $\sqrt{5-x^2}$, so we look to the side of the triangle with that quantity (it is adjacent to θ) and the number side (which is the hypotenuse). Since $\cos\theta=\frac{\sqrt{5-x^2}}{\sqrt{5}}$, we have $\sqrt{5-x^2}=\sqrt{5}\cos\theta$. So, our integral becomes

$$\begin{aligned}
\int \sqrt{5}\cos\theta\,\sqrt{5}\cos\theta\,d\theta &= 5\int \cos^2\theta\,d\theta\\
&= \frac{5}{2}\int 1+\cos 2\theta\,d\theta\\
&= \frac{5}{2}\theta+\frac{5}{2}\cdot\frac{1}{2}\sin 2\theta+C\\
&= \frac{5}{2}\theta+\frac{5}{2}\cdot\frac{1}{2}\cdot 2\sin\theta\cos\theta+C\\
&= \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}}+5\cdot\frac{1}{2}\cdot\frac{x}{\sqrt{5}}\cdot\frac{\sqrt{5-x^2}}{\sqrt{5}}+C.
\end{aligned}$$