Find the error

Instructions for use

- These slides are best viewed in **full screen mode** after downloading the PDF file to your computer.
- Ideally, use a PDF browser without “smooth” transitions.
- Try each exercise yourself first. Then advance forward and **find the error(s)** in the incorrect solution. The next slide has an explanation. The last slide shows an exemplary solution.

Determine the convergence of $a_n = ne^{-n^2}$

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$a_n = ne^{-n^2} = \frac{n}{e^{(n^2)}}$

$f(x) = \frac{x}{e^{(x^2)}}$ is continuous, positive, decreasing.

$\int_1^\infty \frac{x}{e^{(x^2)}}\,dx \quad \overset{\text{indef}}{\rightsquigarrow} \quad \int \frac{x}{e^{(x^2)}}\,dx = \frac{1}{2}\int e^{-u}\,du = -\frac{1}{2}e^{-x^2} + C$

$u = x^2$
$du = 2x\,dx$

$= \lim_{t\to\infty} \left(-\frac{1}{2}e^{-t^2} - \frac{-1}{2}e^{-1^2} \right)$

$= \lim_{t\to\infty} \left(\frac{-1}{2e^{t^2}} + \frac{1}{2e} \right)$

$= \frac{1}{2e}$ Converges to $\frac{1}{2e}$.

ne^{-n^2} Converges by the Integral Test.

Find the error

Determine the convergence of $a_n = ne^{-n^2}$

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$= \frac{1}{2e}$ Converges to $\frac{1}{2e}$.

The Integral Test is for series, but this is a sequence!

ne^{-n^2} Converges by the Integral Test.

Explanation

Determine the convergence of $a_n = ne^{-n^2}$

$a_n = ne^{-n^2} = \frac{n}{e^{(n^2)}}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{e^{(n^2)}} \underset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{n\to\infty} \frac{1}{e^{(n^2)} \cdot 2n} = 0$

So the sequence $\left\{\frac{n}{e^{(n^2)}}\right\}$ Converges.

Correct solution

Determine the convergence of $a_n = \dfrac{1}{2^n}$

Determine the convergence of $a_n = \dfrac{1}{2^n}$

$a_n = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$

First term $a = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$

Ratio $r = \frac{1}{2}$

$\frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1$ by the Geometric Series Test.

Find the error

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$\dfrac{1/2}{1 - 1/2} = \dfrac{1/2}{1/2} = 1$ by the Geometric Series Test.

The Geometric Series Test is for series, but this is a sequence!

Explanation

Determine the convergence of $a_n = \dfrac{1}{2^n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n} = 0.$$

The sequence $a_n = \frac{1}{2^n}$ Converges to 0.

Correct solution

Determine the convergence of $\sum_{n=5}^{\infty} \frac{n^3 + 11n}{2n^3 + 5n + 7}$

Determine the convergence of $\sum_{n=5}^{\infty} \frac{n^3+11n}{2n^3+5n+7}$

$$\lim_{n\to\infty} \frac{n^3+11n}{2n^3+5n+7} \xrightarrow[\left(\frac{\infty}{\infty}\right)]{} \lim_{n\to\infty} \frac{3n^2+11}{6n^2+5} \left(\frac{\infty}{\infty}\right) = \frac{6n}{12n} = \frac{6}{12} = \frac{1}{2}.$$

By the Test for Divergence, $\sum_{n=5}^{\infty} \frac{n^3+11n}{2n^3+5n+7}$ diverges.

Find **THREE** errors!!!!!!!!!!!!!!!

Determine the convergence of $\sum_{n=5}^{\infty} \frac{n^3 + 11n}{2n^3 + 5n + 7}$

Do not use an arrow here

missing limits

$$\lim_{n\to\infty} \frac{n^3+11n}{2n^3+5n+7} \xrightarrow[\left(\frac{\infty}{\infty}\right)]{} \lim_{n\to\infty} \underbrace{\frac{3n^2+11}{6n^2+5}\left(\frac{\infty}{\infty}\right)} = \frac{6n}{12n} = \frac{6}{12} = \frac{1}{2}.$$

The placement of $\frac{\infty}{\infty}$ makes it look like $\frac{3n^2+11}{6n^2+5}$ times $\frac{\infty}{\infty}$ ☹

By the Test for Divergence, $\sum_{n=5}^{\infty} \frac{n^3+11n}{2n^3+5n+7}$ diverges.

Explanation

Determine the convergence of $\sum_{n=5}^{\infty} \frac{n^3 + 11n}{2n^3 + 5n + 7}$

$$\lim_{n\to\infty} \frac{n^3+11n}{2n^3+5n+7} \overset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{n\to\infty} \frac{3n^2+11}{6n^2+5} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{6n}{12n} = \lim_{n\to\infty} \frac{6}{12} = \frac{1}{2}.$$

By the Test for Divergence, $\sum_{n=5}^{\infty} \frac{n^3+11n}{2n^3+5n+7}$ diverges.

Correct solution

Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{3^n}$

Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{3^n}$

$p = 3$

So by p-series test, $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges.

Find the error

Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{3^n}$

$p = 3$

So by p-series test, $\sum_{n=1}^{\infty} \frac{1}{3^n}$ Converges.

This is <u>not</u> a p-series, so you cannot use the p-test!

Explanation

Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{3^n}$

$\sum \frac{1}{3^n} = \sum \left(\frac{1}{3}\right)^n$ is a geometric series with $r = \frac{1}{3}$. So $|r| < 1$.

So by Geometric Series test, $\sum_{n=1}^{\infty} \frac{1}{3^n}$ Converges.

Correct solution

Determine the convergence of $\sum_{n=1}^{\infty} \frac{2+\sin(n)}{n}$

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{2+\sin(n)}{n}$

$$-1 \leq \sin n \leq 1$$
$$(+2) \quad (+2) \quad (+2)$$

$$1 \leq 2+\sin n \leq 3$$

$$\frac{1}{n} \leq \frac{2+\sin n}{n} \leq \frac{3}{n}$$

$\lim_{n\to\infty} \frac{1}{n} = 0$ and $\lim_{n\to\infty} \frac{3}{n} = 0$. So by the Squeeze Theorem, $\lim_{n\to\infty} \frac{2+\sin n}{n} = 0$

$$\sum_{n=1}^{\infty} \frac{2+\sin n}{n} = 0 \quad \text{converges.}$$

Find the error

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$$\sum_{n=1}^{\infty} \frac{2+\sin n}{n} = 0 \quad \text{converges.}$$

Consider $\sum a_n$. It is rarely the case (if ever) that $\lim_{n\to\infty} a_n = L$ means that $\sum_{1} a_n = L$.

The limit of the sequence of terms is not the value of the sum/series.

Explanation

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{2+\sin(n)}{n}$

$$-1 \le \sin n$$
$$(+2) \qquad (+2)$$

$$1 \le 2+\sin n$$

$$\frac{1}{n} \le \frac{2+\sin n}{n}$$

The series $\sum \frac{1}{n}$ diverges by p-test.

So the series $\sum \frac{2+\sin n}{n}$ diverges by the (Direct) Comparison Test.

Correct solution

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$\lim_{n\to\infty} \frac{1}{n^2+1} = 0.$

Thus, by the Test for Divergence, $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

Find the error

Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$\lim_{n\to\infty} \frac{1}{n^2+1} = 0.$

Thus, by the Test for Divergence, $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

The Test for Divergence cannot be used when $\boxed{\lim_{n\to\infty} a_n = 0}$.

Any sentence of the form $\boxed{\text{by the Test for Divergence } \ldots \textstyle\sum \text{ converges}}$ is invalid!

Move on to another test.

This is a microcosm of a recurring type of error: just because a requirement of a test fails does not mean you get to conclude the opposite of the test's conclusion!

Explanation

Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$\frac{1}{n^2+1} \leq \frac{1}{n^2}$ for all n.

Since $\sum \frac{1}{n^2}$ converges by p-test, $\sum \frac{1}{n^2+1}$ converges by the Comparison Test.

Other possible tests: Limit Comparison Test
Integral Test

Correct solution

Determine the convergence of $\sum_{n=1}^{\infty}[\arctan(n+1) - \arctan n]$

Determine the convergence of $\sum_{n=1}^{\infty}[\arctan(n+1) - \arctan n]$

$S_1 = (\arctan 2 - \arctan 1)$

$S_2 = (\arctan 2 - \arctan 1) + (\arctan 3 - \arctan 2)$

$\vdots$

$S_n = (\arctan 2 - \arctan 1) + (\arctan 3 - \arctan 2) + \cdots + (\arctan(n+1) - \arctan n)$

$= \arctan(n+1) - \arctan 1$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (\arctan(n+1) - \arctan 1)$

$= \frac{\pi}{2} - \frac{\pi}{4}$

$= \frac{\pi}{4}$

non-zero...

So $\sum [\arctan(n+1) - \arctan(n)]$ diverges by the Test for Divergence.

Find the error

Determine the convergence of $\sum_{n=1}^{\infty}[\arctan(n+1) - \arctan n]$

$S_1 = (\arctan 2 - \arctan 1)$

$S_2 = (\arctan 2 - \arctan 1) + (\arctan 3 - \arctan 2)$

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$S_n = (\arctan 2 - \arctan 1) + (\arctan 3 - \arctan 2) + \cdots + (\arctan(n+1) - \arctan n)$

$= \arctan(n+1) - \arctan 1$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (\arctan(n+1) - \arctan 1)$

$= \frac{\pi}{2} - \frac{\pi}{4}$

$= \frac{\pi}{4}$

non-zero...

Use the TFD if $\boxed{\lim_{n\to\infty} a_n \neq 0}$. Here, we looked at $\boxed{\lim_{n\to\infty} S_n}$ (S_n is not the same as a_n)

So $\sum [\arctan(n+1) - \arctan(n)]$ diverges by the Test for Divergence.

Explanation

Determine the convergence of $\sum_{n=1}^{\infty}[\arctan(n+1) - \arctan n]$

$S_1 = (\arctan 2 - \arctan 1)$

$S_2 = (\arctan 2 - \arctan 1) + (\arctan 3 - \arctan 2)$

$\vdots$

$S_n = (\arctan 2 - \arctan 1) + (\arctan 3 - \arctan 2) + \cdots + (\arctan(n+1) - \arctan n)$

$= \arctan(n+1) - \arctan 1$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (\arctan(n+1) - \arctan 1)$

$= \frac{\pi}{2} - \frac{\pi}{4}$

$= \frac{\pi}{4}$

So $\sum [\arctan(n+1) - \arctan(n)]$ converges (by definition) to $\frac{\pi}{4}$.

Correct solution

Determine the convergence of $\sum_{n=1}^{\infty}(-1)^n \frac{n+1}{n}$

Determine the convergence of $\sum_{n=1}^{\infty}(-1)^n \frac{n+1}{n}$

$\sum_1 (-1)^n \frac{n+1}{n}$ is an alternating series.

$b_n = |a_n| = \frac{n+1}{n}$.

b_n is decreasing.

But b_n does not have limit 0, so $\sum_1 (-1)^n \frac{n+1}{n}$ diverges by the Alternating Series Test.

Find the error

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n+1}{n}$

$\sum_1 (-1)^n \frac{n+1}{n}$ is an alternating series.

$b_n = |a_n| = \frac{n+1}{n}$.

b_n is decreasing.

But b_n does not have limit 0, so $\sum_1 (-1)^n \frac{n+1}{n}$ diverges by the Alternating Series Test.

Just because a requirement of the Alternating Series Test fails does <u>not</u> allow you to jump to the conclusion that the series diverges. Two "wrongs" don't make a "right". The Alternating Series Test will <u>never</u> validly be used to conclude that a series diverges.

Explanation

Determine the convergence of $\sum_{n=1}^{\infty}(-1)^n \frac{n+1}{n}$

$\lim_{n\to\infty} (-1)^n \frac{n+1}{n}$ does not exist.

Therefore, $\sum (-1)^n \frac{n+1}{n}$ diverges by the Test for Divergence.

Correct solution

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4+\sin n}{n^2}\right)^n$

Determine the convergence of $\sum_{n=1}^{\infty}\left(\frac{4+\sin n}{n^2}\right)^n$

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{4+\sin n}{n^2}\right)^n\right|} = \lim_{n\to\infty} \frac{4+\sin n}{n^2} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{\cos n}{2n} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{-\sin n}{2}$ dne

so by the Root Test, $\sum_1 \left(\frac{4+\sin n}{n^2}\right)^n$ diverges

Find the error

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4+\sin n}{n^2}\right)^n$

Not an $\frac{\infty}{\infty}$ form. L'Hôpital's does not apply

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{4+\sin n}{n^2}\right)^n\right|} = \lim_{n\to\infty}\frac{4+\sin n}{n^2} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty}\frac{\cos n}{2n} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty}\frac{-\sin n}{2} \quad \text{dne}$$

So by the Root Test, $\sum_1 \left(\frac{4+\sin n}{n^2}\right)^n$ diverges

If the limit dne, the Root Test would be inconclusive.

Explanation

Determine the convergence of $\sum_{n=1}^{\infty} \left(\frac{4+\sin n}{n^2}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{4+\sin n}{n^2}\right)^n\right|} = \lim_{n\to\infty} \frac{4+\sin n}{n^2}$$

$$-1 \le \sin n \le 1$$

$$3 \le 4+\sin n \le 5$$

$$\frac{3}{n^2} \le \frac{4+\sin n}{n^2} \le \frac{5}{n^2}$$

Since $\lim_{n\to\infty} \frac{3}{n^2} = 0$ and $\lim_{n\to\infty} \frac{5}{n^2} = 0$, by the Squeeze Theorem, $\lim_{n\to\infty} \frac{4+\sin n}{n^2} = 0$.

So by the Root Test, $\sum_1 \left(\frac{4+\sin n}{n^2}\right)^n$ converges (absolutely).

Correct solution

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n^2)}{n^3}$

Determine the convergence of $\sum_{n=1}^{\infty} \frac{\cos(n^2)}{n^3}$

$\cos(n^2) \leq 1$

$\frac{\cos(n^2)}{n^3} \leq \frac{1}{n^3}$

Since $\sum \frac{1}{n^3}$ converges by p-test, $\sum \frac{\cos(n^2)}{n^3}$ converges by the Comparison Test.

Find the error

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n^2)}{n^3}$

$$\cos(n^2) \le 1$$

$$\frac{\cos(n^2)}{n^3} \le \frac{1}{n^3}$$

Since $\sum \frac{1}{n^3}$ converges by p-test, $\underbrace{\sum \frac{\cos(n^2)}{n^3}}$ converges by the Comparison Test.

The Comparison Test may <u>not</u> be used on a series with negative terms.

What technique do we have that allows us to look at a series with <u>positive</u> terms?

Explanation

Determine the convergence of $\sum_{n=1}^{\infty} \frac{\cos(n^2)}{n^3}$

$$|\cos(n^2)| \leq 1$$

$$\frac{|\cos(n^2)|}{n^3} \leq \frac{1}{n^3} \quad \text{So} \quad \left|\frac{\cos(n^2)}{n^3}\right| \leq \frac{1}{n^3}$$

Since $\sum \frac{1}{n^3}$ converges by p-test, $\sum \left|\frac{\cos(n^2)}{n^3}\right|$ converges by the Comparison Test.

Since $\sum \left|\frac{\cos(n^2)}{n^3}\right|$ converges, we get $\sum \frac{\cos(n^2)}{n^3}$ converges by the Absolute Convergence Test.

Moral: Have negative terms and tempted to use Comparison Test?
Then look at $\sum |a_n|$ first

Correct solution

The end