$$\int_0^1 x\ln(x)\,dx = \lim_{t\to 0^+} \int_t^1 x\ln(x)\,dx$$

Indef $\int x\ln(x)\,dx$

$u = \ln(x) \quad dv = x\,dx$

$du = \frac{1}{x}\,dx \quad v = \frac{x^2}{2}$

$$= \frac{x^2}{2}\ln(x) - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\ln(x) - \frac{1}{4}x^2 + C$$

$$\boxed{\begin{aligned}\left(\tfrac{2}{t^2}\right)' &= \left(2t^{-2}\right)' \\ &= -4t^{-3} \\ &= \tfrac{-4}{t^3}\end{aligned}}$$

$$= \lim_{t\to 0^+}\left(\left(\frac{1^2}{2}\underbrace{\ln(1)}_{=0} - \frac{1}{4}\cdot 1^2\right) - \left(\frac{1}{2}t^2\ln(t) - \frac{1}{4}t^2\right)\right)$$

Side $\lim_{t\to 0^+} \frac{1}{2}t^2\ln(t) = \lim_{t\to 0^+} \frac{\ln(t)}{\frac{2}{t^2}} \overset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{t\to 0^+} \frac{\frac{1}{t}}{-\frac{4}{t^3}} = \lim_{t\to 0^+} \frac{1}{t}\cdot\frac{-t^3}{4}$

$$= \lim_{t\to 0^+} -\frac{t^2}{4}$$

$$= 0$$

$$= 0 - \frac{1}{4} - \left(0 - \frac{1}{4}\cdot 0^2\right)$$