Notes on *Analysis* by Terrence Tao

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A Appendix: the basics of mathematical logic

A.1 Notes

A.1.1 Theorems

Axiom A.1 (Reflexive axiom). *Given any object x, we have $x = x$.*

Axiom A.2 (Symmetry axiom). *Given any two objects x and y of the same type, if $x = y$, then $y = x$.*

Axiom A.3 (Transitive axiom). *Given any three objects x, y, z of the same type, if $x = y$ and $y = z$, then $x = z$.*

Axiom A.4 (Substitution axiom). *Given any two objects x and y of the same type, if $x = y$, then $f(x) = f(y)$ for all functions or operations f. Similarly, for any property $P(x)$ depending on x, if $x = y$, then $P(x)$ and $P(y)$ are equivalent statements.*

A.1.2 Remarks

I started reading Appendix A realising my lack of sophistication with Logic after finishing Chapter 2.

Discussing cases in a proof is a common example of using vacuously true implications for a non-trivial result, e.g., if we want to prove that $P(x)$ is true for some integer x, we can prove the implications that if x is even, then $P(x)$ is true and that if x is odd, then $P(x)$ is true, even if one implication must have a false hypothesis because x cannot be both even and false and thus be vacuous.

Equality is also worth reviewing. I was indecisive whether it was legitimate to add a number to both sides of an equality when doing the exercises in Chapter 2. It actually follows the *substitution axiom* of equality.

A.2 Practices

Exercise A.1.1. Both X and Y are true, or both are false.

Exercise A.1.2. Either X is true, or Y is true, but not both.

Exercise A.1.3. Yes, because they are *equally* true or *equally* false, and they can only be true or false.

Exercise A.1.4. No, because that Y is true does not necessarily mean that X is true. For example, X is "$a = 3$"; Y is "$a^2 = 9$".

Exercise A.1.5. Yes, because if X is true, then Y is true, then Z is true; if Z is true, then Y is true, then X is true. They are equally true or equally false.

Exercise A.1.6. Yes, likewise.

Exercise A.5.1.

(a) $\iff \forall (x, y) \in (\mathbb{R}^+)^2 : y^2 = x$, which is a false statement (e.g. $x = 1, y = 2$).

(b) $\iff \exists x \in \mathbb{R}^+ \quad \forall y \in \mathbb{R}^+ : y^2 = x$, which is a false statement.

Proof. For the sake of contradiction, suppose that the statement holds for some $x \in \mathbb{R}^+$; so $y^2 = x$. But for all $y \in \mathbb{R}^+$, $y^2 = x$, so $(y+1)^2 = x$ where $y + 1 \in \mathbb{R}^+$; so $y^2 = x - (2y+1) = x$; so $2y + 1 = 0$. But $2y + 1$ is positive, so there is a contradiction. ☐

(c) $\iff \exists (x, y) \in (\mathbb{R}^+)^2 : y^2 = x$, which is a true statement (e.g. $x = 1$, $y = 1$).

(d) $\iff \forall y \in \mathbb{R}^+ \quad \exists x \in \mathbb{R}^+ : y^2 = x$, which is a true statement.

Proof. Let $a := y^2$, so $a \in \mathbb{R}^+$ because $y \in \mathbb{R}^+$. Thus, there exists some $x = a \in \mathbb{R}^+$ which satisfies the statement. ☐

(e) $\iff \exists y \in \mathbb{R}^+ \quad \forall x \in \mathbb{R}^+ : y^2 = x$, which is a false statement (likewise).

Exercise A.7.1.

Proof. Given $a = b$ and $c = d$,

$$
\begin{aligned}
\text{we have } b &= a && \text{(Symmetry axiom)} \\
\implies b + c &= a + c && \text{(Substitution axiom)} \\
\text{but } a &= a && \text{(Reflexive axiom)} \\
\implies a + c &= a + c && \text{(Substitution axiom)} \\
\implies a + c &= a + d && \text{(Substitution axiom)} \\
\text{so } b + c &= a + d && \text{(Transitive axiom)} \\
\implies a + d &= b + c. && \text{(Symmetry axiom)}
\end{aligned}
$$

☐

2 Starting at the beginning: the natural numbers

2.1 Notes

2.1.1 Theorems

Axiom 2.1. 0 *is a natural number.*

Axiom 2.2. *If n is a natural number, then $n{+}{+}$ is also a natural number.*

Axiom 2.3. 0 *is not the successor of any natural number; i.e., we have $n{+}{+} \neq 0$ for every natural number n.*

Axiom 2.4. *If n, m are natural numbers and $n \neq m$, then $n{+}{+} \neq m{+}{+}$.*

Axiom 2.5 (Principle of mathematical induction)**.** *Let $P(n)$ be any property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n{+}{+})$ is also true. Then $P(n)$ is true for every natural number n.*

Definition 2.1 (Addition of natural numbers)**.** *Let m be a natural number. To add zero to m, we define $0 + m := m$. Now suppose inductively that we have defined how to add n to m. Then we can add $n{+}{+}$ to m by defining $(n{+}{+}) + m := (n + m){+}{+}$.*

Definition 2.2 (Ordering of the natural numbers)**.** *Let n and m be natural numbers. We say that n is greater than or equal to m, and write $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number a. We say that n is strictly greater than m, and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.*

2.1.2 Remarks

Axiom 2.5 and the concept of the vacuous truth need further reflection.

2.2 Practices

> Notice that we can prove easily, using Axioms 2.1, 2.2, and induction (Axiom 2.5), that the sum of two natural numbers is again a natural number (why?).[1]

Proof. We use induction on n. $0 + m = m$ is a natural number. Suppose inductively that $n + m$ is a natural number. Then $(n{+}{+}) + m = (n + m){+}{+}$ is also a natural number. □

> As a particular corollary of Lemma 2.2.2 and Lemma 2.2.3 we see that $n{+}{+} = n + 1$ (why?).[2]

Proof.

$$\begin{aligned} n{+}{+} &= (n + 0){+}{+} && \text{(Lemma 2.2.2)} \\ &= n + (0{+}{+}) && \text{(Lemma 2.2.3)} \\ &= n + 1. \end{aligned}$$

□

[1] 24
[2] 26

Exercise 2.2.1 (Addition is associative) For any natural numbers a, b, c, we have $(a + b) + c = a + (b + c)$.

Proof. We use induction on b. The base case $(a + 0) + c = a + (0 + c)$ follows as both sides equal $a + c$. Suppose inductively that $(a + b) + c = a + (b + c)$. We have to prove that $[a + (b{++})] + c = a + [(b{++}) + c]$. The left side

$$\begin{aligned}[a + (b{++})] + c &= [(a + b){++}] + c \\ &= [(a + b) + c]{++}.\end{aligned}$$

The right side

$$\begin{aligned}a + [(b{++}) + c] &= a + [(b{++}) + c] \\ &= [a + (b + c)]{++},\end{aligned}$$

which is equal to the left side by the inductive hypothesis. □

Exercise 2.2.2 Let a be a positive number. Then there exists exactly one natural number b such that $b{++} = a$.

Proof. (Existence) We use induction on a. The base case follows as 0 is not a positive number. Suppose inductively that $b{++} = a$. Then $(b{++}){++} = a{++}$, where $b{++}$ is a natural number. (Uniqueness) Suppose for the sake of contradiction that b and c are different natural numbers such that $b{++} = a$ and $c{++} = a$. Because $b \neq c$, $b{++} \neq c{++}$. There is a contradiction that $b{++} = c{++}$. □

Exercise 2.2.3 (Basic properties of order for natural numbers) Let a, b, c be natural numbers. Then

(a) (Order is reflexive) $a \geq a$.

   Proof. $a = a + 0$. □

(b) (Order is transitive) If $a \geq b$ and $b \geq c$, then $a \geq c$.

   Proof. $a = b + m$ and $b = c + n$ for some natural numbers m, n. Then $a = (c + n) + m = c + (n + m)$, where $n + m$ is a natural number. □

(c) (Order is anti-symmetric) If $a \geq b$ and $b \geq a$, then $a = b$.

   Proof. $a = b + m$ and $b = a + n$ for some natural numbers m, n. Then $a = (a + n) + m = a + (n + m)$, which leads to that $0 = n + m$. It follows that $n = m = 0$. Therefore, $a = b + 0 = b$. □

(d) (Addition preserves order) $a \geq b$ if and only if $a + c \geq b + c$.

   Proof. (1) If $a \geq b$, $a = b + m$ for some natural number m. Then $a + c = (b + m) + c = (b + c) + m$, which means that $a + c \geq b + c$. (2) If $a + c \geq b + c$, $a + c = b + c + n$ for some natural number n. It follows that $a + c = b + n + c$, and thus that $a = b + n$, which means that $a \geq b$. □

(e) $a < b$ if and only if $a{++} \leq b$.

Proof. (1) If $a < b$, $a + m = b$ for some natural number m and $a \neq b$. Suppose for the sake of contradiction that $m = 0$. It follows that $a = b$, which contradicts that $a \neq b$. Then $m \neq 0$, which means it is a positive natural number. Thus, $m = n{+}{+}$ for some natural number n. It follows that $a + n{+}{+} = b$, which means that $(a + n){+}{+} = b$, which means that $a{+}{+} + n = b$. Thus, $a{+}{+} \leq b$. (2) If $a{+}{+} \leq b$, then $(a{+}{+}) + m = b$ for some natural number m. Therefore, $(a + m){+}{+} = b$, which means that $a + m{+}{+} = b$. It follows that $a \leq b$. Now we must prove that $a \neq b$. Suppose for the sake of contradiction that $a = b$, then $a + m{+}{+} = a$, which implies that $m{+}{+} = 0$, which contradicts that 0 is not the successor of any natural number. □

(f) $a < b$ if and only if $b = a + d$ for some positive number d.

Proof. We only have to prove that $a{+}{+} \leq b$ if and only if $b = a + d$ for some positive number d by (e). (1) If $a{+}{+} \leq b$, then $a{+}{+} + m = b$ for some natural number m. Therefore, $a + d = b$, where we let $d := m{+}{+}$. Suppose for the sake of contradiction that d is not positive, which means that $d = 0$, which contradicts that 0 is not the successor of any natural number. Thus, d is a positive natural number. (2) If $b = a + d$ for some positive number d, then $b = a + n{+}{+}$ for some natural number n. It follows that $a{+}{+} + n = b$, which implies that $a{+}{+} \leq b$. □

Exercise 2.2.4.

[We] have $0 \leq b$ for all b (why?).

Proof. $0 + b = b$. □

If $a > b$, then $a{+}{+} > b$ (why?). If $a = b$, then $a{+}{+} > b$ (why?).

Proof. $a = b + m$ for some m. Then $a{+}{+} = a + 1 = b + m + 1 = b + m{+}{+}$. Therefore $a{+}{+} > b$. □

Exercise 2.2.5. (Strong principle of induction) Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true. (In particular, this means that $P(m_0)$ is true, since in this case the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.[3]

Proof. Define $Q(m)$ to be the property for any arbitrary natural number m that $P(m')$ is true for all $m_0 \leq m' < m$. For each $m \geq m_0$, if $Q(m)$ is true, $P(m)$ is also true.

We first prove that $Q(m)$ is true for all $m \geq m_0$. We use induction on m. In the base case $m = 0$, we consider three cases

(1) $m_0 < 0$. $m_0 + k = 0$ for some k and $m_0 \neq 0$. But because $m_0 + k = 0$, $m_0 = 0$, which is a contradiction. Then m_0 cannot be less than 0.

(2) $m_0 = 0$ or $m_0 > 0$. $m_0 \leq m' < m$, therefore $m' + l = m = 0$ for some l. Likewise, there is a contradiction that l cannot be less than 0, so $Q(0)$ is vacuously true.

We then suppose inductively that the case $m = n$ holds. Consider the case $m = n{+}{+}$. We consider three cases

(1) $m_0 < n{+}{+}$. We consider three cases

[3] Done with reference to Proposition 2.2.14 Strong principle of induction.

(i) $m' < n$. Because $Q(n)$ is true, $P(m')$ is true for all $m_0 \le m' < n$. Thus, $P(m')$ is true for $m' < n$.

(ii) $m' = n$. Because $Q(n)$ is true, $P(n)$ is true according to the inductive hypothesis. Thus, $P(n)$ is true for $m' = n$.

(iii) $m' > n$. Because $m' < n{+}{+}$, $m' + k = n{+}{+}$ for some k and $m' \ne n{+}{+}$. Suppose for the sake of contradiction that $k = 0$, then $m' = n{+}{+}$, a contradiction. Thus, k is positive, so $k = l{+}{+}$ for some l. We have $m' + l{+}{+} = n{+}{+}$, which means $m' + l = n$, which means $m' \le n$, which contradicts that $m' > n$. Thus, $P(n)$ is vacuously true for $m' > n$.

Therefore, $P(m')$ is true for any $m_0 \le m' < n{+}{+}$, i.e., $Q(n{+}{+})$ is true for $m_0 < n{+}{+}$.

(2) $m_0 = n{+}{+}$. Then, $n{+}{+} \le m' < n{+}{+}$, i.e., we have $n{+}{+} \ne m'$, $m' \ge n{+}{+}$ and $n{+}{+} \ge m'$. It follows that $n{+}{+} = m'$, which is a contradiction. Thus, $Q(n{+}{+})$ is vacuously true for $m_0 = n{+}{+}$.

(3) $m_0 > n{+}{+}$. Then, $m_0 \le m' < n{+}{+} < m_0$, which means that $m_0 \le m' < m_0$. Likewise, there is no m' such that this case exists. Thus, $Q(n{+}{+})$ is vacuously true for $m_0 > n{+}{+}$.

Combining the above cases, $Q(n{+}{+})$ is true when $Q(n)$ is true. This closes the induction.

Because $Q(m)$ is true for all $m \ge m_0$, $P(m)$ is also true for all $m \ge m_0$. □

Exercise 2.2.6. Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m{+}{+})$ is true, then $P(m)$ is true. Suppose that $P(n)$ is also true. Prove that $P(m)$ is true for all natural numbers $m \le n$; this is known as the principle of backwards induction.

Proof. We use induction on n. The base case is $n = 0$. Because $m \le n$, $m + k = n = 0$ for some natural number k. This means that $m = 0 = n$, which means that $P(m)$ is true.

We assume inductively that the case $n = l$ holds for some natural number l, then consider the case $n = l{+}{+}$. From the inductive hypothesis, $P(m)$ is true for all natural numbers $m \le l$. $m \le l$ iff $m + a = l$ for some natural number a, iff $m + a{+}{+} = l{+}{+}$.

(1) Suppose for the sake of contradiction that $m = l{+}{+}$, so $a{+}{+} = 0$. But $a{+}{+} \ne 0$ as 0 is not the successor of any natural number. So $m < l{+}{+}$. (2) If $m < l{+}{+}$, likewise, then $m + a{+}{+} = l{+}{+}$.

Therefore, $m + a{+}{+} = l{+}{+}$ iff $m < l{+}{+}$. This means that $P(m)$ is true for all natural numbers $m < l{+}{+}$ if $P(l)$ is true.

Because $P(l{+}{+})$ is true, $P(l)$ is true from the inductive hypothesis. Thus, $P(m)$ is true for all natural numbers $m < l{+}{+}$. Combining with that $P(l{+}{+})$ is true, $P(m)$ is true for all natural numbers $m \le l{+}{+}$. □

Exercise 2.3.1. (Multiplication is commutative)

Proof. We use induction on n. The base case is $0 \times m = m \times 0$. The left side equals 0. We use another induction on m to show that the right side also equals 0. The base case $0 \times 0 = 0$ by definition. Suppose inductively that $k \times 0 = 0$ for some k. Then $(k{+}{+}) \times 0 = k \times 0 + 0 = 0 + 0 = 0$. Thus, the second induction is closed; the base case of the first induction is true.

We suppose inductively that $0 \times l = l \times 0$ for some l. Thus, $l \times 0 = 0$. Likewise, $(l{+}{+}) \times 0 = 0$. Because $0 \times (l{+}{+})$, $0 \times (l{+}{+}) = (l{+}{+}) \times 0$. □

Exercise 2.3.2. (Positive natural numbers have no zero divisors)

Proof. (1) If one of n, m is equal to 0, then, without loss of generality, we let $m = 0$, so $nm = n0 = 0$. (2) If $nm = 0$, we suppose for the sake of contradiction that none of n, m is 0. That is, $n = l{+}{+}$ and $m = k{+}{+}$ for some natural numbers l, k. $nm = (l{+}{+})(k{+}{+}) = k(l{+}{+}) + (l{+}{+}) = 0$. Thus, $l{+}{+} = 0$, which is a contradiction. □

Exercise 2.3.3. (Multiplication is associative)

Proof. We use induction on b. The base case $(a0)c = a(0c)$ holds as both sides equal 0. We suppose inductively that $(ab)c = a(bc)$, and need to prove that $(a(b{+}{+}))c = a((b{+}{+})c)$. The left side equals $(ab + a)c = abc + ac$; the right side equals $a(bc + c) = abc + ac$. □

Exercise 2.3.4. Prove the identity $(a + b)^2 = a^2 + 2ab + b^2$ for all natural numbers a, b.

Proof. The left side equals $(a + b)(a + b) = (a + b)a + (a + b)b = aa + ba + ab + bb$. The right side equals $aa + ab + ab + bb$. □

Exercise 2.3.5. (Euclidean algorithm) Let n be a natural number, and let q be a positive number. Then there exist natural numbers m, r such that $0 \le r < q$ and $n = mq + r$.

Proof. We use induction on n. The base case $n = 0$ holds as we can find $m = 0$, $r = 0$ such that $0 = 0q + 0$. We suppose inductively that $n = mq + r$, and want to prove that $n{+}{+} = m'q + r'$ for some m', r'. From the inductive hypothesis, $n{+}{+} = mq + r{+}{+}$. We discuss the cases

(1) $r{+}{+} < q$. Then $m' = m$, $r' = r{+}{+}$ satisfies that $n{+}{+} = m'q + r'$.

(2) $r{+}{+} = q$. Then $n{+}{+} = mq + r{+}{+} = mq + q = (m{+}{+})q + 0$. We have $m' = m{+}{+}$ and $r' = 0$ satisfying this case.

(3) $r{+}{+} > q$. Then $r > q$. But $r < q$, so $n{+}{+} = m'q + r'$ is true vacuously. □

3 Set Theory

3.1 Notes

3.1.1 Theorems

Axiom 3.1 (Sets are objects). *If A is a set, then A is also an object.*

Axiom 3.2 (Empty set). *There exists a set $\varnothing$, known as the empty set, which contains no elements, i.e., for every object x we have $x \notin \varnothing$.*

Axiom 3.3 (Singleton sets and pair sets). *If a is an object, then there exists a set $\{ a \}$ whose only element is a, i.e., for every object y, we have $y \in a$ if and only if $y = a$; we refer to $\{ a \}$ as the singleton set whose element is a. Furthermore, if a and b are objects, then there exists a set $\{ a, b \}$ whose only elements are a and b; i.e., for every object y, we have $y \in \{ a, b \}$ if and only if $y = a$ or $y = b$; we refer to this set as the pair set formed by a and b.*

Axiom 3.4 (Pairwise union). *Given any two sets A, B, there exists a set $A \cup B$ such that*

$$x \in A \cup B \iff (x \in A \text{ or } x \in B).$$

Axiom 3.5 (Axiom of specification). *Let A be a set, then for any object y,*

$$y \in \{ x \in A : P(x) \} \iff (y \in A \text{ and } P(y) \text{ is true}).$$

3.1.2 Remarks

Some axioms listed here are redundant: we only need one axiom between the singleton sets and the pair sets axioms.

3.2 Practices

> Note that there can only be one empty set; if there were two sets $\varnothing$ and $\varnothing'$ which were both empty, then by Definition 3.1.4 they would be equal to each other (why?).

Proof. We shall prove that for every object $x \in \varnothing$ we have $x \in \varnothing'$, but this is vacuously true because for every object x we have $x \notin \varnothing$. Likewise, it is vacuously true that for every object $x \in \varnothing'$ we have $x \in \varnothing$. Thus, $\varnothing = \varnothing'$. □

> If A, B, A' are sets, and A is equal to A', then $A \cup B$ is equal to $A' \cup B$ (why? [...]).

Proof. $A = A'$ implies that for every object x, $x \in A$ iff $x \in A'$. Thus,

$$(x \in A \text{ or } x \in B) \iff (x \in A' \text{ or } x \in B).$$

This is equivalent to that

$$x \in A \cup B \iff x \in A' \cup B,$$

which implies that $A \cup B = A' \cup B$. □