

Notes on *Analysis* by Terrence Tao

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A Appendix: the basics of mathematical logic

A.1 Notes

A.1.1 Theorems

Axiom A.1 (Reflexive axiom). *Given any object x , we have $x = x$.*

Axiom A.2 (Symmetry axiom). *Given any two objects x and y of the same type, if $x = y$, then $y = x$.*

Axiom A.3 (Transitive axiom). *Given any three objects x, y, z of the same type, if $x = y$ and $y = z$, then $x = z$.*

Axiom A.4 (Substitution axiom). *Given any two objects x and y of the same type, if $x = y$, then $f(x) = f(y)$ for all functions or operations f . Similarly, for any property $P(x)$ depending on x , if $x = y$, then $P(x)$ and $P(y)$ are equivalent statements.*

A.1.2 Remarks

I started reading Appendix A realising my lack of sophistication with Logic after finishing Chapter 2.

Discussing cases in a proof is a common example of using vacuously true implications for a non-trivial result, e.g., if we want to prove that $P(x)$ is true, we can prove the implications that if x is even, then $P(x)$ is true and that if x is odd, then $P(x)$ is true, even if one implication must have a false hypothesis because x cannot be both even and false and thus be vacuous.

Equality is also worth reviewing. I was indecisive whether it was legitimate to add a number to both sides of an equality when doing the exercises in Chapter 2. It actually follows the *substitution axiom* of equality.

A.2 Practices

Exercise A.1.1. Both X and Y are true, or both are false.

Exercise A.1.2. Either X is true, or Y is true, but not both.

Exercise A.1.3. Yes, because they are *equally* true or *equally* false, and they can only be true or false.

Exercise A.1.4. No, because that Y is true does not necessarily mean that X is true. For example, X is “ $a = 3$ ”; Y is “ $a^2 = 9$ ”.

Exercise A.1.5. Yes, because if X is true, then Y is true, then Z is true; if Z is true, then Y is true, then X is true. They are equally true or equally false.

Exercise A.1.6. Yes, likewise.

Exercise A.5.1.

(a) $\Leftrightarrow \forall (x, y) \in (\mathbb{R}^+)^2 : y^2 = x$, which is a false statement (e.g. $x = 1, y = 2$).

(b) $\Leftrightarrow \exists x \in \mathbb{R}^+ \quad \forall y \in \mathbb{R}^+ : y^2 = x$, which is a false statement.

Proof. For the sake of contradiction, suppose that the statement holds for some $x \in \mathbb{R}^+$; so $y^2 = x$. But for all $y \in \mathbb{R}^+$, $y^2 = x$, so $(y + 1)^2 = x$ where $y + 1 \in \mathbb{R}^+$; so $y^2 = x - (2y + 1) = x$; so $2y + 1 = 0$. But $2y + 1$ is positive, so there is a contradiction. \square

(c) $\Leftrightarrow \exists (x, y) \in (\mathbb{R}^+)^2 : y^2 = x$, which is a true statement (e.g. $x = 1, y = 1$).

(d) $\Leftrightarrow \forall y \in \mathbb{R}^+ \quad \exists x \in \mathbb{R}^+ : y^2 = x$, which is a true statement.

Proof. Let $a := y^2$, so $a \in \mathbb{R}^+$ because $y \in \mathbb{R}^+$. Thus, there exists some $x = a \in \mathbb{R}^+$ which satisfies the statement. \square

(e) $\Leftrightarrow \exists y \in \mathbb{R}^+ \quad \forall x \in \mathbb{R}^+ : y^2 = x$, which is a false statement (likewise).

Exercise A.7.1.

Proof. Given $a = b$ and $c = d$,

we have $b = a$	(Symmetry axiom)	
$\Rightarrow b + c = a + c$	(Substitution axiom)	
but $a = a$	(Reflexive axiom)	
$\Rightarrow a + c = a + c$	(Substitution axiom)	
$\Rightarrow a + c = a + d$	(Substitution axiom)	
so $b + c = a + d$	(Transitive axiom)	
$\Rightarrow a + d = b + c$	(Symmetry axiom)	\square

2 Starting at the beginning: the natural numbers

2.1 Notes

2.1.1 Theorems

Axiom 2.1. *0 is a natural number.*

Axiom 2.2. *If n is a natural number, then $n++$ is also a natural number.*

Axiom 2.3. *0 is not the successor of any natural number; i.e., we have $n++ \neq 0$ for every natural number n .*

Axiom 2.4. *If n, m are natural numbers and $n \neq m$, then $n++ \neq m++$.*

Axiom 2.5 (Principle of mathematical induction). *Let $P(n)$ be any property pertaining to a natural number n . Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n++)$ is also true. Then $P(n)$ is true for every natural number n .*

Definition 2.1 (Addition of natural numbers). *Let m be a natural number. To add zero to m , we define $0 + m := m$. Now suppose inductively that we have defined how to add n to m . Then we can add $n++$ to m by defining $(n++) + m := (n + m)++$.*

Definition 2.2 (Ordering of the natural numbers). *Let n and m be natural numbers. We say that n is greater than or equal to m , and write $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number a . We say that n is strictly greater than m , and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.*

2.1.2 Remarks

Axiom 2.5 and the concept of the vacuous truth need further reflection.

2.2 Practices

Notice that we can prove easily, using Axioms 2.1, 2.2, and induction (Axiom 2.5), that the sum of two natural numbers is again a natural number (why?).¹

Proof. We use induction on n . $0 + m = m$ is a natural number. Suppose inductively that $n + m$ is a natural number. Then $(n++) + m = (n + m)++$ is also a natural number. □

¹24

As a particular corollary of Lemma 2.2.2 and Lemma 2.2.3 we see that $n++ = n + 1$ (why?).²

Proof.

$$\begin{aligned} n++ &= (n + 0)++ && \text{(Lemma 2.2.2)} \\ &= n + (0++) && \text{(Lemma 2.2.3)} \\ &= n + 1. \end{aligned} \quad \square$$

Exercise 2.2.1 (Addition is associative) For any natural numbers a, b, c , we have $(a + b) + c = a + (b + c)$.

Proof. We use induction on b . The base case $(a + 0) + c = a + (0 + c)$ follows as both sides equal $a + c$. Suppose inductively that $(a + b) + c = a + (b + c)$. We have to prove that $[a + (b++)] + c = a + [(b++) + c]$. The left side

$$\begin{aligned} [a + (b++)] + c &= [(a + b)++] + c \\ &= [(a + b) + c]++. \end{aligned}$$

The right side

$$\begin{aligned} a + [(b++) + c] &= a + [(b++) + c] \\ &= [a + (b + c)]++, \end{aligned}$$

which is equal to the left side by the inductive hypothesis. \square

Exercise 2.2.2 Let a be a positive number. Then there exists exactly one natural number b such that $b++ = a$.

Proof. (Existence) We use induction on a . The base case follows as 0 is not a positive number. Suppose inductively that $b++ = a$. Then $(b++)++ = a++$, where $b++$ is a natural number. (Uniqueness) Suppose for the sake of contradiction that b and c are different natural numbers such that $b++ = a$ and $c++ = a$. Because $b \neq c$, $b++ \neq c++$. There is a contradiction that $b++ = c++$. \square

Exercise 2.2.3 (Basic properties of order for natural numbers) Let a, b, c be natural numbers. Then

(a) (Order is reflexive) $a \geq a$.

Proof. $a = a + 0$. \square

(b) (Order is transitive) If $a \geq b$ and $b \geq c$, then $a \geq c$.

Proof. $a = b + m$ and $b = c + n$ for some natural numbers m, n . Then $a = (c + n) + m = c + (n + m)$, where $n + m$ is a natural number. \square

(c) (Order is anti-symmetric) If $a \geq b$ and $b \geq a$, then $a = b$.

Proof. $a = b + m$ and $b = a + n$ for some natural numbers m, n . Then $a = (a + n) + m = a + (n + m)$, which leads to that $0 = n + m$. It follows that $n = m = 0$. Therefore, $a = b + 0 = b$. \square

(d) (Addition preserves order) $a \geq b$ if and only if $a + c \geq b + c$.

Proof. (1) If $a \geq b$, $a = b + m$ for some natural number m . Then $a + c = (b + m) + c = (b + c) + m$, which means that $a + c \geq b + c$. (2) If $a + c \geq b + c$, $a + c = b + c + n$ for some natural number n . It follows that $a + c = b + n + c$, and thus that $a = b + n$, which means that $a \geq b$. \square

(e) $a < b$ if and only if $a++ \leq b$.

Proof. (1) If $a < b$, $a + m = b$ for some natural number m and $a \neq b$. Suppose for the sake of contradiction that $m = 0$. It follows that $a = b$, which contradicts that $a \neq b$. Then $m \neq 0$, which means it is a positive natural number. Thus, $m = n++$ for some natural number n . It follows that $a + n++ = b$, which means that $(a + n)++ = b$, which means that $a++ + n = b$. Thus, $a++ \leq b$. (2) If $a++ \leq b$, then $(a++) + m = b$ for some natural number m . Therefore, $(a + m)++ = b$, which means that $a + m++ = b$. It follows that $a \leq b$. Now we must prove that $a \neq b$. Suppose for the sake of contradiction that $a = b$, then $a + m++ = a$, which implies that $m++ = 0$, which contradicts that 0 is not the successor of any natural number. \square

(f) $a < b$ if and only if $b = a + d$ for some positive number d .

Proof. We only have to prove that $a++ \leq b$ if and only if $b = a + d$ for some positive number d by (e). (1) If $a++ \leq b$, then $a++ + m = b$ for some natural number m . Therefore, $a + d = b$, where we let $d := m++$. Suppose for the sake of contradiction that d is not positive, which means that $d = 0$, which contradicts that 0 is not the successor of any natural number. Thus, d is a positive natural number. (2) If $b = a + d$ for some positive number d , then $b = a + n++$ for some natural number n . It follows that $a++ + n = b$, which implies that $a++ \leq b$. \square

Exercise 2.2.4.

[We] have $0 \leq b$ for all b (why?).

Proof. $0 + b = b$. □

If $a > b$, then $a++ > b$ (why?). If $a = b$, then $a++ > b$ (why?).

Proof. $a = b + m$ for some m . Then $a++ = a + 1 = b + m + 1 = b + m++$. Therefore $a++ > b$. □

Exercise 2.2.5. (Strong principle of induction) Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m . Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true. (In particular, this means that $P(m_0)$ is true, since in this case the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.³

Proof. Define $Q(m)$ to be the property for any arbitrary natural number m that $P(m')$ is true for all $m_0 \leq m' < m$. For each $m \geq m_0$, if $Q(m)$ is true, $P(m)$ is also true.

We first prove that $Q(m)$ is true for all $m \geq m_0$. We use induction on m . In the base case $m = 0$, we consider three cases

- (1) $m_0 < 0$. $m_0 + k = 0$ for some k and $m_0 \neq 0$. But because $m_0 + k = 0$, $m_0 = 0$, which is a contradiction. Then m_0 cannot be less than 0.
- (2) $m_0 = 0$ or $m_0 > 0$. $m_0 \leq m' < m$, therefore $m' + l = m = 0$ for some l . Likewise, there is a contradiction that l cannot be less than 0, so $Q(0)$ is vacuously true.

We then suppose inductively that the case $m = n$ holds. Consider the case $m = n++$. We consider three cases

- (1) $m_0 < n++$. We consider three cases
 - (i) $m' < n$. Because $Q(n)$ is true, $P(m')$ is true for all $m_0 \leq m' < n$. Thus, $P(m')$ is true for $m' < n$.
 - (ii) $m' = n$. Because $Q(n)$ is true, $P(n)$ is true according to the inductive hypothesis. Thus, $P(n)$ is true for $m' = n$.

³Done with reference to Proposition 2.2.14 Strong principle of induction.

- (iii) $m' > n$. Because $m' < n++$, $m' + k = n++$ for some k and $m' \neq n++$. Suppose for the sake of contradiction that $k = 0$, then $m' = n++$, a contradiction. Thus, k is positive, so $k = l++$ for some l . We have $m' + l++ = n++$, which means $m' + l = n$, which means $m' \leq n$, which contradicts that $m' > n$. Thus, $P(n)$ is vacuously true for $m' > n$.

Therefore, $P(m')$ is true for any $m_0 \leq m' < n++$, i.e., $Q(n++)$ is true for $m_0 < n++$.

- (2) $m_0 = n++$. Then, $n++ \leq m' < n++$, i.e., we have $n++ \neq m'$, $m' \geq n++$ and $n++ \geq m'$. It follows that $n++ = m'$, which is a contradiction. Thus, $Q(n++)$ is vacuously true for $m_0 = n++$.
- (3) $m_0 > n++$. Then, $m_0 \leq m' < n++ < m_0$, which means that $m_0 \leq m' < m_0$. Likewise, there is no m' such that this case exists. Thus, $Q(n++)$ is vacuously true for $m_0 > n++$.

Combining the above cases, $Q(n++)$ is true when $Q(n)$ is true. This closes the induction.

Because $Q(m)$ is true for all $m \geq m_0$, $P(m)$ is also true for all $m \geq m_0$. \square

Exercise 2.2.6. Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m++)$ is true, then $P(m)$ is true. Suppose that $P(n)$ is also true. Prove that $P(m)$ is true for all natural numbers $m \leq n$; this is known as the principle of backwards induction.

Proof. We use induction on n . The base case is $n = 0$. Because $m \leq n$, $m + k = n = 0$ for some natural number k . This means that $m = 0 = n$, which means that $P(m)$ is true.

We assume inductively that the case $n = l$ holds for some natural number l , then consider the case $n = l++$. From the inductive hypothesis, $P(m)$ is true for all natural numbers $m \leq l$. $m \leq l$ iff $m + a = l$ for some natural number a , iff $m + a++ = l++$.

(1) Suppose for the sake of contradiction that $m = l++$, so $a++ = 0$. But $a++ \neq 0$ as 0 is not the successor of any natural number. So $m < l++$. (2) If $m < l++$, likewise, then $m + a++ = l++$.

Therefore, $m + a++ = l++$ iff $m < l++$. This means that $P(m)$ is true for all natural numbers $m < l++$ if $P(l)$ is true.

Because $P(l++)$ is true, $P(l)$ is true from the inductive hypothesis. Thus, $P(m)$ is true for all natural numbers $m < l++$. Combining with that $P(l++)$ is true, $P(m)$ is true for all natural numbers $m \leq l++$. \square

Exercise 2.3.1. (Multiplication is commutative)

Proof. We use induction on n . The base case is $0 \times m = m \times 0$. The left side equals 0. We use another induction on m to show that the right side also equals 0. The base case $0 \times 0 = 0$ by definition. Suppose inductively that $k \times 0 = 0$ for some k . Then $(k++) \times 0 = k \times 0 + 0 = 0 + 0 = 0$. Thus, the second induction is closed; the base case of the first induction is true.

We suppose inductively that $0 \times l = l \times 0$ for some l . Thus, $l \times 0 = 0$. Likewise, $(l++) \times 0 = 0$. Because $0 \times (l++)$, $0 \times (l++) = (l++) \times 0$. \square

Exercise 2.3.2. (Positive natural numbers have no zero divisors)

Proof. (1) If one of n, m is equal to 0, then, without loss of generality, we let $m = 0$, so $nm = n0 = 0$. (2) If $nm = 0$, we suppose for the sake of contradiction that none of n, m is 0. That is, $n = l++$ and $m = k++$ for some natural numbers l, k . $nm = (l++)(k++) = k(l++) + (l++) = 0$. Thus, $l++ = 0$, which is a contradiction. \square

Exercise 2.3.3. (Multiplication is associative)

Proof. We use induction on b . The base case $(a0)c = a(0c)$ holds as both sides equal 0. We suppose inductively that $(ab)c = a(bc)$, and need to prove that $(a(b++))c = a((b++)c)$. The left side equals $(ab + a)c = abc + ac$; the right side equals $a(bc + c) = abc + ac$. \square

Exercise 2.3.4. Prove the identity $(a+b)^2 = a^2 + 2ab + b^2$ for all natural numbers a, b .

Proof. The left side equals $(a+b)(a+b) = (a+b)a + (a+b)b = aa + ba + ab + bb$. The right side equals $aa + ab + ab + bb$. \square

Exercise 2.3.5. (Euclidean algorithm) Let n be a natural number, and let q be a positive number. Then there exist natural numbers m, r such that $0 \leq r < q$ and $n = mq + r$.

Proof. We use induction on n . The base case $n = 0$ holds as we can find $m = 0$, $r = 0$ such that $0 = 0q + 0$. We suppose inductively that $n = mq + r$, and want to prove that $n++ = m'q + r'$ for some m', r' . From the inductive hypothesis, $n++ = mq + r++$. We discuss the cases

(1) $r++ < q$. Then $m' = m$, $r' = r++$ satisfies that $n++ = m'q + r'$.

- (2) $r++ = q$. Then $n++ = mq + r++ = mq + q = (m++)q + 0$. We have $m' = m++$ and $r' = 0$ satisfying this case.
- (3) $r++ > q$. Then $r > q$. But $r < q$, so $n++ = m'q + r'$ is true vacuously. \square

3 Set Theory

3.1 Notes

3.1.1 Theorems

3.1.2 Remarks

3.2 Practices