Notes on *Analysis* by Terrence Tao

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A Appendix: the basics of mathematical logic

A.1 Notes

A.1.1 Theorems

Axiom A.1 (Reflexive axiom). *Given any object x, we have $x = x$.*

Axiom A.2 (Symmetry axiom). *Given any two objects x and y of the same type, if $x = y$, then $y = x$.*

Axiom A.3 (Transitive axiom). *Given any three objects x, y, z of the same type, if $x = y$ and $y = z$, then $x = z$.*

Axiom A.4 (Substitution axiom). *Given any two objects x and y of the same type, if $x = y$, then $f(x) = f(y)$ for all functions or operations f. Similarly, for any property $P(x)$ depending on x, if $x = y$, then $P(x)$ and $P(y)$ are equivalent statements.*

A.1.2 Remarks

I started reading Appendix A realising my lack of sophistication with Logic after finishing Chapter 2.

Discussing cases in a proof is a common example of using vacuously true implications for a non-trivial result, e.g., if we want to prove that $P(x)$ is true for some integer x, we can prove the implications that if x is even, then $P(x)$ is true and that if x is odd, then $P(x)$ is true, even if one implication must have a false hypothesis because x cannot be both even and false and thus be vacuous.

Equality is also worth reviewing. I was indecisive whether it was legitimate to add a number to both sides of an equality when doing the exercises in Chapter 2. It actually follows the *substitution axiom* of equality.

A.2 Practices

Exercise A.1.1. Both X and Y are true, or both are false.

Exercise A.1.2. Either X is true, or Y is true, but not both.

Exercise A.1.3. Yes, because they are *equally* true or *equally* false, and they can only be true or false.

Exercise A.1.4. No, because that Y is true does not necessarily mean that X is true. For example, X is "$a = 3$"; Y is "$a^2 = 9$".

Exercise A.1.5. Yes, because if X is true, then Y is true, then Z is true; if Z is true, then Y is true, then X is true. They are equally true or equally false.

Exercise A.1.6. Yes, likewise.

Exercise A.5.1.

(a) $\iff \forall (x, y) \in (\mathbf{R}^+)^2 : y^2 = x$, which is a false statement (e.g. $x = 1, y = 2$).

(b) $\iff \exists x \in \mathbf{R}^+ \quad \forall y \in \mathbf{R}^+ : y^2 = x$, which is a false statement.

Proof. For the sake of contradiction, suppose that the statement holds for some $x \in \mathbf{R}^+$; so $y^2 = x$. But for all $y \in \mathbf{R}^+$, $y^2 = x$, so $(y+1)^2 = x$ where $y + 1 \in \mathbf{R}^+$; so $y^2 = x - (2y+1) = x$; so $2y + 1 = 0$. But $2y + 1$ is positive, so there is a contradiction. □

(c) $\iff \exists (x, y) \in (\mathbf{R}^+)^2 : y^2 = x$, which is a true statement (e.g. $x = 1, y = 1$).

(d) $\iff \forall y \in \mathbf{R}^+ \quad \exists x \in \mathbf{R}^+ : y^2 = x$, which is a true statement.

Proof. Let $a := y^2$, so $a \in \mathbf{R}^+$ because $y \in \mathbf{R}^+$. Thus, there exists some $x = a \in \mathbf{R}^+$ which satisfies the statement. □

(e) $\iff \exists y \in \mathbf{R}^+ \quad \forall x \in \mathbf{R}^+ : y^2 = x$, which is a false statement (likewise).

Exercise A.7.1.

Proof. Given $a = b$ and $c = d$,

we have $b = a$	(Symmetry axiom)
$\implies b + c = a + c$	(Substitution axiom)
but $a = a$	(Reflexive axiom)
$\implies a + c = a + c$	(Substitution axiom)
$\implies a + c = a + d$	(Substitution axiom)
so $b + c = a + d$	(Transitive axiom)
$\implies a + d = b + c.$	(Symmetry axiom)

□

2 Starting at the beginning: the natural numbers

2.1 Notes

2.1.1 Theorems

Axiom 2.1. 0 *is a natural number.*

Axiom 2.2. *If n is a natural number, then $n{+}{+}$ is also a natural number.*

Axiom 2.3. 0 *is not the successor of any natural number; i.e., we have $n{+}{+} \neq 0$ for every natural number n.*

Axiom 2.4. *If n, m are natural numbers and $n \neq m$, then $n{+}{+} \neq m{+}{+}$.*

Axiom 2.5 (Principle of mathematical induction)**.** *Let $P(n)$ be any property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n{+}{+})$ is also true. Then $P(n)$ is true for every natural number n.*

Definition 2.1 (Addition of natural numbers)**.** *Let m be a natural number. To add zero to m, we define $0 + m := m$. Now suppose inductively that we have defined how to add n to m. Then we can add $n{+}{+}$ to m by defining $(n{+}{+}) + m := (n + m){+}{+}$.*

Definition 2.2 (Ordering of the natural numbers)**.** *Let n and m be natural numbers. We say that n is greater than or equal to m, and write $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number a. We say that n is strictly greater than m, and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.*

2.1.2 Remarks

Axiom 2.5 and the concept of the vacuous truth need further reflection.

2.2 Practices

> Notice that we can prove easily, using Axioms 2.1, 2.2, and induction (Axiom 2.5), that the sum of two natural numbers is again a natural number (why?).

Proof. We use induction on n. $0 + m = m$ is a natural number. Suppose inductively that $n + m$ is a natural number. Then $(n{+}{+}) + m = (n + m){+}{+}$ is also a natural number. □

> As a particular corollary of Lemma 2.2.2 and Lemma 2.2.3 we see that $n{+}{+} = n + 1$ (why?).

Proof.

$$\begin{aligned} n{+}{+} &= (n+0){+}{+} && \text{(Lemma 2.2.2)} \\ &= n + (0{+}{+}) && \text{(Lemma 2.2.3)} \\ &= n + 1. \end{aligned}$$

□

Exercise 2.2.1 (Addition is associative). For any natural numbers a, b, c, we have $(a + b) + c = a + (b + c)$.

Proof. We use induction on b. The base case $(a + 0) + c = a + (0 + c)$ follows as both sides equal $a + c$. Suppose inductively that $(a + b) + c = a + (b + c)$. We have to prove that $[a + (b{+}{+})] + c = a + [(b{+}{+}) + c]$. The left side

$$\begin{aligned} [a + (b{+}{+})] + c &= [(a + b){+}{+}] + c \\ &= [(a + b) + c]{+}{+}. \end{aligned}$$

The right side

$$\begin{aligned} a + [(b{+}{+}) + c] &= a + [(b{+}{+}) + c] \\ &= [a + (b + c)]{+}{+}, \end{aligned}$$

which is equal to the left side by the inductive hypothesis. □

Exercise 2.2.2. Let a be a positive number. Then there exists exactly one natural number b such that $b{+}{+} = a$.

Proof. (Existence) We use induction on a. The base case follows as 0 is not a positive number. Suppose inductively that $b{+}{+} = a$. Then $(b{+}{+}){+}{+} = a{+}{+}$, where $b{+}{+}$ is a natural number. (Uniqueness) Suppose for the sake of contradiction that b and c are different natural numbers such that $b{+}{+} = a$ and $c{+}{+} = a$. Because $b \neq c$, $b{+}{+} \neq c{+}{+}$. There is a contradiction that $b{+}{+} = c{+}{+}$. □

Exercise 2.2.3 (Basic properties of order for natural numbers). Let a, b, c be natural numbers. Then

(a) (Order is reflexive) $a \geq a$.

   Proof. $a = a + 0$. □

(b) (Order is transitive) If $a \geq b$ and $b \geq c$, then $a \geq c$.

Proof. $a = b+m$ and $b = c+n$ for some natural numbers m, n. Then $a = (c+n)+m = c + (n + m)$, where $n + m$ is a natural number. $\square$

(c) (Order is anti-symmetric) If $a \geq b$ and $b \geq a$, then $a = b$.

Proof. $a = b+m$ and $b = a+n$ for some natural numbers m, n. Then $a = (a+n)+m = a + (n + m)$, which leads to that $0 = n + m$. It follows that $n = m = 0$. Therefore, $a = b + 0 = b$. $\square$

(d) (Addition preserves order) $a \geq b$ if and only if $a + c \geq b + c$.

Proof. (1) If $a \geq b$, $a = b + m$ for some natural number m. Then $a + c = (b + m) + c = (b + c) + m$, which means that $a + c \geq b + c$. (2) If $a + c \geq b + c$, $a + c = b + c + n$ for some natural number n. It follows that $a + c = b + n + c$, and thus that $a = b + n$, which means that $a \geq b$. $\square$

(e) $a < b$ if and only if $a{++} \leq b$.

Proof. (1) If $a < b$, $a + m = b$ for some natural number m and $a \neq b$. Suppose for the sake of contradiction that $m = 0$. It follows that $a = b$, which contradicts that $a \neq b$. Then $m \neq 0$, which means it is a positive natural number. Thus, $m = n{++}$ for some natural number n. It follows that $a + n{++} = b$, which means that $(a + n){++} = b$, which means that $a{++} + n = b$. Thus, $a{++} \leq b$. (2) If $a{++} \leq b$, then $(a{++}) + m = b$ for some natural number m. Therefore, $(a + m){++} = b$, which means that $a + m{++} = b$. It follows that $a \leq b$. Now we must prove that $a \neq b$. Suppose for the sake of contradiction that $a = b$, then $a + m{++} = a$, which implies that $m{++} = 0$, which contradicts that 0 is not the successor of any natural number. $\square$

(f) $a < b$ if and only if $b = a + d$ for some positive number d.

Proof. We only have to prove that $a{++} \leq b$ if and only if $b = a + d$ for some positive number d by (e). (1) If $a{++} \leq b$, then $a{++} + m = b$ for some natural number m. Therefore, $a + d = b$, where we let $d := m{++}$. Suppose for the sake of contradiction that d is not positive, which means that $d = 0$, which contradicts that 0 is not the successor of any natural number. Thus, d is a positive natural number. (2) If $b = a + d$ for some positive number d, then $b = a + n{++}$ for some natural number n. It follows that $a{++} + n = b$, which implies that $a{++} \leq b$. $\square$

Exercise 2.2.4.

[We] have $0 \le b$ for all b (why?).

Proof. $0 + b = b$. □

If $a > b$, then $a{+}{+} > b$ (why?). If $a = b$, then $a{+}{+} > b$ (why?).

Proof. $a = b + m$ for some m. Then $a{+}{+} = a + 1 = b + m + 1 = b + m{+}{+}$. Therefore $a{+}{+} > b$. □

Exercise 2.2.5 (Strong principle of induction). Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \ge m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \le m' < m$, then $P(m)$ is also true. (In particular, this means that $P(m_0)$ is true, since in this case the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \ge m_0$.[1]

Proof. Define $Q(m)$ to be the property for any arbitrary natural number m that $P(m')$ is true for all $m_0 \le m' < m$. For each $m \ge m_0$, if $Q(m)$ is true, $P(m)$ is also true.

We first prove that $Q(m)$ is true for all $m \ge m_0$. We use induction on m. In the base case $m = 0$, we consider three cases

(1) $m_0 < 0$. $m_0 + k = 0$ for some k and $m_0 \ne 0$. But because $m_0 + k = 0$, $m_0 = 0$, which is a contradiction. Then m_0 cannot be less than 0.

(2) $m_0 = 0$ or $m_0 > 0$. $m_0 \le m' < m$, therefore $m' + l = m = 0$ for some l. Likewise, there is a contradiction that l cannot be less than 0, so $Q(0)$ is vacuously true.

We then suppose inductively that the case $m = n$ holds. Consider the case $m = n{+}{+}$. We consider three cases

(1) $m_0 < n{+}{+}$. We consider three cases

  (i) $m' < n$. Because $Q(n)$ is true, $P(m')$ is true for all $m_0 \le m' < n$. Thus, $P(m')$ is true for $m' < n$.

  (ii) $m' = n$. Because $Q(n)$ is true, $P(n)$ is true according to the inductive hypothesis. Thus, $P(n)$ is true for $m' = n$.

  (iii) $m' > n$. Because $m' < n{+}{+}$, $m' + k = n{+}{+}$ for some k and $m' \ne n{+}{+}$. Suppose for the sake of contradiction that $k = 0$, then $m' = n{+}{+}$, a contradiction. Thus, k is positive, so $k = l{+}{+}$ for some l. We have $m' + l{+}{+} = n{+}{+}$, which means $m' + l = n$, which means $m' \le n$, which contradicts that $m' > n$. Thus, $P(n)$ is vacuously true for $m' > n$.

[1] Done with reference to Proposition 2.2.14 Strong principle of induction.

Therefore, $P(m')$ is true for any $m_0 \le m' < n{+\!+}$, i.e., $Q(n{+\!+})$ is true for $m_0 < n{+\!+}$.

(2) $m_0 = n{+\!+}$. Then, $n{+\!+} \le m' < n{+\!+}$, i.e., we have $n{+\!+} \ne m'$, $m' \ge n{+\!+}$ and $n{+\!+} \ge m'$. It follows that $n{+\!+} = m'$, which is a contradiction. Thus, $Q(n{+\!+})$ is vacuously true for $m_0 = n{+\!+}$.

(3) $m_0 > n{+\!+}$. Then, $m_0 \le m' < n{+\!+} < m_0$, which means that $m_0 \le m' < m_0$. Likewise, there is no m' such that this case exists. Thus, $Q(n{+\!+})$ is vacuously true for $m_0 > n{+\!+}$.

Combining the above cases, $Q(n{+\!+})$ is true when $Q(n)$ is true. This closes the induction.

Because $Q(m)$ is true for all $m \ge m_0$, $P(m)$ is also true for all $m \ge m_0$. □

Exercise 2.2.6. Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m{+\!+})$ is true, then $P(m)$ is true. Suppose that $P(n)$ is also true. Prove that $P(m)$ is true for all natural numbers $m \le n$; this is known as the principle of backwards induction.

Proof. We use induction on n. The base case is $n = 0$. Because $m \le n$, $m + k = n = 0$ for some natural number k. This means that $m = 0 = n$, which means that $P(m)$ is true.

We assume inductively that the case $n = l$ holds for some natural number l, then consider the case $n = l{+\!+}$. From the inductive hypothesis, $P(m)$ is true for all natural numbers $m \le l$. $m \le l$ iff $m + a = l$ for some natural number a, iff $m + a{+\!+} = l{+\!+}$.

(1) Suppose for the sake of contradiction that $m = l{+\!+}$, so $a{+\!+} = 0$. But $a{+\!+} \ne 0$ as 0 is not the successor of any natural number. So $m < l{+\!+}$. (2) If $m < l{+\!+}$, likewise, then $m + a{+\!+} = l{+\!+}$.

Therefore, $m + a{+\!+} = l{+\!+}$ iff $m < l{+\!+}$. This means that $P(m)$ is true for all natural numbers $m < l{+\!+}$ if $P(l)$ is true.

Because $P(l{+\!+})$ is true, $P(l)$ is true from the inductive hypothesis. Thus, $P(m)$ is true for all natural numbers $m < l{+\!+}$. Combining with that $P(l{+\!+})$ is true, $P(m)$ is true for all natural numbers $m \le l{+\!+}$. □

Exercise 2.3.1 (Multiplication is commutative).

Proof. We use induction on n. The base case is $0 \times m = m \times 0$. The left side equals 0. We use another induction on m to show that the right side also equals 0. The base case $0 \times 0 = 0$ by definition. Suppose inductively that $k \times 0 = 0$ for some k. Then $(k{+\!+}) \times 0 = k \times 0 + 0 = 0 + 0 = 0$. Thus, the second induction is closed; the base case of the first induction is true.

We suppose inductively that $0 \times l = l \times 0$ for some l. Thus, $l \times 0 = 0$. Likewise, $(l{+\!+}) \times 0 = 0$. Because $0 \times (l{+\!+})$, $0 \times (l{+\!+}) = (l{+\!+}) \times 0$. □

Exercise 2.3.2 (Positive natural numbers have no zero divisors).

Proof. (1) If one of n, m is equal to 0, then, without loss of generality, we let $m = 0$, so $nm = n0 = 0$. (2) If $nm = 0$, we suppose for the sake of contradiction that none of n, m is 0. That is, $n = l{+}{+}$ and $m = k{+}{+}$ for some natural numbers l, k. $nm = (l{+}{+})(k{+}{+}) = k(l{+}{+}) + (l{+}{+}) = 0$. Thus, $l{+}{+} = 0$, which is a contradiction. $\square$

Exercise 2.3.3 (Multiplication is associative).

Proof. We use induction on b. The base case $(a0)c = a(0c)$ holds as both sides equal 0. We suppose inductively that $(ab)c = a(bc)$, and need to prove that $(a(b{+}{+}))c = a((b{+}{+})c)$. The left side equals $(ab + a)c = abc + ac$; the right side equals $a(bc + c) = abc + ac$. $\square$

Exercise 2.3.4. Prove the identity $(a + b)^2 = a^2 + 2ab + b^2$ for all natural numbers a, b.

Proof. The left side equals $(a + b)(a + b) = (a + b)a + (a + b)b = aa + ba + ab + bb$. The right side equals $aa + ab + ab + bb$. $\square$

Exercise 2.3.5 (Euclidean algorithm). Let n be a natural number, and let q be a positive number. Then there exist natural numbers m, r such that $0 \leq r < q$ and $n = mq + r$.

Proof. We use induction on n. The base case $n = 0$ holds as we can find $m = 0$, $r = 0$ such that $0 = 0q + 0$. We suppose inductively that $n = mq + r$, and want to prove that $n{+}{+} = m'q + r'$ for some m', r'. From the inductive hypothesis, $n{+}{+} = mq + r{+}{+}$. We discuss the cases

(1) $r{+}{+} < q$. Then $m' = m$, $r' = r{+}{+}$ satisfies that $n{+}{+} = m'q + r'$.

(2) $r{+}{+} = q$. Then $n{+}{+} = mq + r{+}{+} = mq + q = (m{+}{+})q + 0$. We have $m' = m{+}{+}$ and $r' = 0$ satisfying this case.

(3) $r{+}{+} > q$. Then $r > q$. But $r < q$, so $n{+}{+} = m'q + r'$ is true vacuously. $\square$

3 Set Theory

3.1 Notes

3.1.1 Theorems

Axiom 3.1 (Sets are objects). *If A is a set, then A is also an object.*

Axiom 3.2 (Empty set). *There exists a set $\varnothing$, such that for every object x we have $x \notin \varnothing$.*

Axiom 3.3 (Singleton sets and pair sets). *If a is an object, then there exists a set $\{ a \}$, such that for every object y, we have $y \in a$ if and only if $y = a$; we refer to $\{ a \}$ as the singleton set whose element is a. Furthermore, if a and b are objects, then there exists a set $\{ a, b \}$, such that for every object y, we have $y \in \{ a, b \}$ if and only if $y = a$ or $y = b$; we refer to this set as the pair set formed by a and b.*

Axiom 3.4 (Pairwise union). *Given any two sets A, B, there exists a set $A \cup B$ such that*

$$x \in A \cup B \iff (x \in A \text{ or } x \in B).$$

Axiom 3.5 (Axiom of specification). *Let A be a set, then for any object y, there exists a set $\{ x \in A \mid P(x) \}$ such that*

$$y \in \{ x \in A \mid P(x) \} \iff (y \in A \text{ and } P(y) \text{ is true}).$$

Axiom 3.6 (Replacement). *Let A be a set. For any object $x \in A$, and any object y, suppose we have a statement $P(x, y)$ pertaining to x and y, such that for each $x \in A$ there is at most one y for which $P(x, y)$ is true. Then there exists a set $\{ y \mid P(x, y) \text{ is true for some } x \in A \}$, such that for any object z,*

$$z \in \{ y \mid P(x, y) \text{ is true for some } x \in A \} \iff P(x, z) \text{ is true for some } x \in A.$$

Axiom 3.7 (Infinity). *There exists a set $\mathbf{N}$, whose elements are called natural numbers, as well as an object 0 in $\mathbf{N}$, and an object $n{+}{+}$ assigned to every natural number $n \in \mathbf{N}$, such that the Peano axioms (Axioms 2.1 – 2.5) hold.*

Axiom 3.9 (Regularity). *If A is a non-empty set, then there is at least one element x of A which is either not a set, or is disjoint from A.*

3.1.2 Remarks

Note that some axioms listed here are redundant: we only need one axiom between the singleton sets and the pair sets axioms; the axiom of specification is implied by the axiom of replacement.

3.2 Practices

> Note that there can only be one empty set; if there were two sets $\varnothing$ and $\varnothing'$ which were both empty, then by Definition 3.1.4 they would be equal to each other (why?).

Proof. We shall prove that for every object $x \in \varnothing$ we have $x \in \varnothing'$, but this is vacuously true because for any object x we have $x \notin \varnothing$. Likewise, it is vacuously true that for every object $x \in \varnothing'$ we have $x \in \varnothing$. Thus, $\varnothing = \varnothing'$. ☐

> If A, B, A' are sets, and A is equal to A', then $A \cup B$ is equal to $A' \cup B$ (why? [...]).

Proof. $A = A'$ implies that for every object x, $x \in A$ iff $x \in A'$. Thus,

$$(x \in A \vee x \in B) \iff (x \in A' \vee x \in B).$$

This is equivalent to that

$$x \in A \cup B \iff x \in A' \cup B,$$

which implies that $A \cup B = A' \cup B$. ☐

Exercise 3.1.1 (*Equality of sets is an equivalent class).

Proof. Let A, B, C be sets. (Reflexive) For any $x \in A$, it is true that $x \in A$, so $A = A$. (Symmetric) If $A = B$, then it is true that $\forall x \in A : x \in B$ and $\forall x \in B : x \in A$. Thus, $\forall x \in B : x \in A$ and $\forall x \in A : x \in B$, so $B = A$. (Transitive) If $A = B$ and $B = C$, then ($\forall x \in A : x \in B$ and $\forall y \in B : y \in A$) and ($\forall x \in B : x \in C$ and $\forall y \in C : y \in B$) are true. Thus, $\forall x \in A : x \in B$, so $x \in C$ and $\forall y \in C : y \in B$, so $y \in A$. It implies that $A = C$. ☐

Exercise 3.1.2.

Proof. We only have to prove that $\varnothing \neq \{\varnothing\}$. This is because $\varnothing \in \{\varnothing\}$, but $\varnothing \notin \varnothing$ by definition. ☐

Exercise 3.1.3 (*The union operation).

Proof. By Axiom 3.4, we have

$$x \in A \cup B \iff (x \in A \vee x \in B)$$

and

$$x \in B \cup A \iff (x \in B \vee x \in A),$$

where

$$(x \in A \vee x \in B) \iff (x \in B \vee x \in A).$$

Thus, $x \in A \cup B$ iff $x \in B \cup A$, which implies that $A \cup B = B \cup A$. ☐

Proof. Because $x \in \varnothing$ is false for any x, we have

$$(x \in A \vee x \in A) \iff (x \in A \vee x \in \varnothing) \iff (x \in \varnothing \vee x \in A) \iff x \in A. \qquad \square$$

Exercise 3.1.4.

Proof. We have, for any object x, that

$$x \in A \implies x \in B$$

and

$$x \in B \implies x \in A.$$

Thus, $A = B$. $\square$

Proof. We have, for any object x, that

$$x \in A \implies x \in B$$

and

$$x \in B \implies x \in C,$$

thus

$$x \in A \implies x \in C,$$

so $A \subseteq C$. Because $A \neq B$, it is true that either $\exists x \in A : x \notin B$ or $\exists x \in B : x \notin A$. But $\forall x \in A : x \in B$, so $\exists x \in B : x \notin A$. Therefore, there exists some $x \in B$ such that $x \in C$ and $x \notin A$, which implies that

$$x \in A \not\Longleftrightarrow x \in C,$$

which means that $A \neq C$. Thus, $A \subsetneq C$. $\square$

Exercise 3.1.5.

Proof. We have $A \subseteq B$ if and only if

$$x \in A \implies x \in B.$$

We also have

$$\begin{aligned} A \cup B = B &\iff (x \in A \cup B \iff x \in B) \\ &\iff (x \in A \vee x \in B \iff x \in B) \\ &\iff (x \in A \implies x \in B). \end{aligned}$$

We also have

$$\begin{aligned} A \cap B = A &\iff (x \in A \cap B \iff x \in A) \\ &\iff (x \in A \wedge x \in B \iff x \in A) \\ &\iff (x \in A \implies x \in B). \end{aligned} \qquad \square$$

Exercise 3.1.6.

(f) (Distributivity).

Proof.

$$\begin{aligned}
&A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \\
\iff& \big(x \in A \cap (B \cup C) \iff x \in (A \cap B) \cup (A \cap C)\big) \text{ for any object } x \\
\iff& (x \in A \land x \in B \cup C \iff x \in A \cap B \lor x \in A \cap C) \\
\iff& \big(x \in A \land (x \in B \lor x \in C) \iff (x \in A \land x \in B) \lor (x \in A \land x \in C)\big) \\
\iff& \big((x \in A \land x \in B) \lor (x \in A \land x \in C) \\
&\quad \iff (x \in A \land x \in B) \lor (x \in A \land x \in C)\big).
\end{aligned}$$

□

(g) (Partition).

Proof. We shall deduce what is required for $A \cup (X \backslash A) = X$ to be true.

$$\begin{aligned}
&A \cup (X \backslash A) = X \\
\iff& \big(x \in A \cup (X \backslash A) \iff x \in X\big) \text{ for any object } x \\
\iff& (x \in A \lor x \in (X \backslash A) \iff x \in X) \\
\iff& \big(x \in A \lor (x \in X \land x \notin A) \iff x \in X\big) \\
\iff& \big((x \in A \lor x \in X) \land (x \in A \lor x \notin A) \iff x \in X\big) \\
\iff& (x \in A \lor x \in X \iff x \in X).
\end{aligned}$$

But we also have $A \subseteq X$, and

$$\begin{aligned}
&A \subseteq X \\
\iff& (x \in A \iff x \in X) \\
\iff& (x \in A \lor x \in X \iff x \in X).
\end{aligned}$$

□

Proof. We shall deduce what is required for $A \cap (X \backslash A) = \varnothing$ to be true.

$$\begin{aligned}
&A \cap (X \backslash A) = \varnothing \\
\iff& \big(x \in A \cap (X \backslash A) \iff x \in \varnothing\big) \text{ for any object } x \\
\iff& \big(x \in A \land x \in (X \backslash A) \iff x \in \varnothing\big) \\
\iff& \big(x \in A \land (x \in X \land x \notin A) \iff x \in \varnothing\big) \\
\iff& \big(x \in X \land (x \in A \land x \notin A) \iff x \in \varnothing\big) \\
\iff& (x \in X \land \bot \iff \bot) \\
\iff& (\bot \iff \bot),
\end{aligned}$$

which is a true statement. □

(h) (De Morgan laws).

Proof.

$$
\begin{aligned}
& X\backslash(A \cup B) = (X\backslash A) \cap (X\backslash B) \\
\iff & \big(x \in X\backslash(A \cup B) \iff x \in (X\backslash A) \cap (X\backslash B)\big) \text{ for any object } x \\
\iff & \big(x \in X \wedge x \notin A \cup B \iff x \in (X\backslash A) \wedge x \in (X\backslash B)\big) \\
\iff & \big(x \in X \wedge \neg(x \in A \vee x \in B) \iff (x \in X \wedge x \notin A) \wedge (x \in X \wedge x \notin B)\big) \\
\iff & \big(x \in X \wedge (x \notin A \wedge x \notin B) \iff (x \in X \wedge x \in X) \wedge (x \notin A \wedge x \notin B)\big) \\
\iff & \big(x \in X \wedge (x \notin A \wedge x \notin B) \iff x \in X \wedge (x \notin A \wedge x \notin B)\big).
\end{aligned}
$$

□

Proof.

$$
\begin{aligned}
& X\backslash(A \cap B) = (X\backslash A) \cup (X\backslash B) \\
\iff & \big(x \in X\backslash(A \cap B) \iff x \in (X\backslash A) \cup (X\backslash B)\big) \text{ for any object } x \\
\iff & \big(x \in X \wedge x \notin A \cap B \iff x \in (X\backslash A) \vee x \in (X\backslash B)\big) \\
\iff & \big(x \in X \wedge \neg(x \in A \wedge x \in B) \iff (x \in X \wedge x \notin A) \vee (x \in X \wedge x \notin B)\big) \\
\iff & \big(x \in X \wedge (x \notin A \vee x \notin B) \iff (x \in X \wedge x \notin A) \vee (x \in X \wedge x \notin B)\big) \\
\iff & \big((x \in X \wedge x \notin A) \vee (x \in X \wedge x \notin B) \\
& \quad \iff (x \in X \wedge x \notin A) \vee (x \in X \wedge x \notin B)\big).
\end{aligned}
$$

□

Exercise 3.1.7.

Proof. We shall just prove the last implication.

(1) We have $A \subseteq C$ and $B \subseteq C$, then for any object x,

$$(x \in A \implies x \in C) \wedge (x \in B \implies x \in C).$$

For any object $y \in A \cup B$, it is true that either $y \in A$ or $y \in B$. We discuss the two cases: if $y \in A$, then $y \in C$; if $y \in B$, then $y \in C$. Thus,

$$\forall y \in A \cup B : y \in C,$$

then $A \cup B \subseteq C$.

(2) We have $A \cup B \subseteq C$, then for any object x, that $x \in A \cup B$ implies that $x \in C$. Furthermore, if either $x \in A$ or $x \in B$, then $x \in A \cup B$. For the sake of contradiction, suppose that one of A and B is not a subset of C. Without loss of generality, let $A \nsubseteq C$, then there exists some $y \in A$ such that $y \notin C$. But since $y \in A$, $x \in A \cup B$, which implies that $y \in C$, a contradiction. □

Exercise 3.1.8 (*Absorption laws).

Proof.

$$
\begin{aligned}
& A \cap (A \cup B) = A \\
\iff & \big(x \in A \wedge x \in (A \cup B) \iff x \in A\big) \text{ for any object } x \\
\iff & \big(x \in A \wedge (x \in A \vee x \in B) \iff x \in A\big) \\
\iff & \Big(\big(x \in A \vee \big(x \in B \wedge \neg(x \in B)\big)\big) \wedge (x \in A \vee x \in B) \iff x \in A\Big) \\
\iff & \big((x \in A \vee x \in B) \wedge \big(x \in A \vee \neg(x \in B)\big) \wedge (x \in A \vee x \in B) \iff x \in A\big) \\
\iff & \big((x \in A \vee x \in B) \wedge \big(x \in A \vee \neg(x \in B)\big) \iff x \in A\big) \\
\iff & \big(x \in A \vee \big(x \in B \wedge \neg(x \in B)\big) \iff x \in A\big) \\
\iff & (x \in A \iff x \in A).
\end{aligned}
$$

□

Proof.

$$
\begin{aligned}
& A \cup (A \cap B) = A \\
\iff & \big(x \in A \vee x \in (A \cap B) \iff x \in A\big) \text{ for any object } x \\
\iff & \big(x \in A \vee (x \in A \wedge x \in B) \iff x \in A\big) \\
\iff & \Big(\big(x \in A \wedge \big(x \in B \vee \neg(x \in B)\big)\big) \vee (x \in A \wedge x \in B) \iff x \in A\Big) \\
\iff & \big((x \in A \wedge x \in B) \vee \big(x \in A \wedge \neg(x \in B)\big) \vee (x \in A \wedge x \in B) \iff x \in A\big) \\
\iff & \big((x \in A \wedge x \in B) \vee \big(x \in A \wedge \neg(x \in B)\big) \iff x \in A\big) \\
\iff & \big(x \in A \wedge \big(x \in B \vee \neg(x \in B)\big) \iff x \in A\big) \\
\iff & (x \in A \iff x \in A).
\end{aligned}
$$

□

Exercise 3.1.9.

Proof. Without loss of generality, we only prove that $B = X \backslash A$. Because $A \cup B = X$, we have, for any object x, that if either $x \in A$ or $x \in B$, then $x \in X$. Thus, if $x \in B$, then $x \in X$. Furthermore, because $A \cap B = \varnothing$, we have that if both $x \in A$ and $x \in B$, then $x \in \varnothing$, which is false for all x. Therefore, $x \in A$ and $x \in B$ cannot both be true. Thus, if $x \in B$, then $x \notin A$. To summarise, for any $x \in B$, $x \in X$ and $x \notin A$. □

Exercise 3.1.10.

Proof. Firstly, we want to show that $(A\backslash B) \cap (A \cap B) = \varnothing$. It suffices to show that for any object x,

$$x \in (A\backslash B) \wedge x \in (A \cap B) \iff x \in \varnothing.$$

The left side

$$\begin{aligned}
&x \in (A\backslash B) \wedge x \in (A \cap B) \\
\iff &(x \in A \wedge x \notin B) \wedge (x \in A \wedge x \in B) \\
\iff &x \in A \wedge \neg(x \in B) \wedge x \in B \\
\iff &\bot,
\end{aligned}$$

and the right side is false by definition. Thus, $A\backslash B$ and $A \cap B$ are disjoint.

But $A \cap B = B \cap A$, so $B\backslash A$ and $A \cap B$ are disjoint likewise.

We then want to show that $(A\backslash B) \cap (B\backslash A) = \varnothing$. It suffices to show that for any object x,

$$x \in (A\backslash B) \wedge x \in (B\backslash A) \iff x \in \varnothing.$$

The left side

$$\begin{aligned}
&x \in (A\backslash B) \wedge x \in (B\backslash A) \\
\iff &(x \in A \wedge x \notin B) \wedge (x \in B \wedge x \notin A) \\
\iff &x \in A \wedge \neg(x \in A) \wedge x \in B \wedge \neg(x \in B) \\
\iff &\bot,
\end{aligned}$$

and the right side is false by definition. Thus, $A\backslash B$ and $B\backslash A$ are disjoint.

Lastly, we want to show that $(A\backslash B) \cup (A \cap B) \cup (B\backslash A) = A \cup B$. It suffices to show that for any object x,

$$x \in (A\backslash B) \wedge x \in (A \cap B) \wedge x \in (B\backslash A) \iff x \in A \vee x \in B.$$

The left side

$$\begin{aligned}
&x \in (A\backslash B) \wedge x \in (A \cap B) \wedge x \in (B\backslash A) \\
\iff &(x \in A \wedge x \notin B) \vee (x \in A \wedge x \in B) \vee (x \in B \wedge x \notin A) \\
\iff &(x \in A \wedge \neg(x \in B)) \vee (x \in B \wedge \neg(x \in A)) \vee (x \in A \wedge x \in B) \\
\iff &\Big(\big((x \in A \vee \neg(x \in A)) \wedge (\neg(x \in B) \vee \neg(x \in A))\big) \\
&\quad \wedge \big((x \in A \vee x \in B) \wedge (\neg(x \in B) \vee x \in B)\big)\Big) \vee (x \in A \wedge x \in B) \\
\iff &\big((\neg(x \in B) \vee \neg(x \in A)) \wedge (x \in A \vee x \in B)\big) \vee (x \in A \wedge x \in B)
\end{aligned}$$

$$\iff \big(\neg(x \in A \wedge x \in B) \wedge (x \in A \vee x \in B)\big) \vee (x \in A \wedge x \in B)$$
$$\iff \big(\neg(x \in A \wedge x \in B) \vee (x \in A \wedge x \in B)\big) \wedge \big((x \in A \vee x \in B) \wedge (x \in A \vee x \in B)\big)$$
$$\iff x \in A \vee x \in B. \qquad \square$$

Exercise 3.1.11 (*Axiom of replacement implies the axiom of specification).

Proof. Let $P(x)$ be a statement pertaining to x. Let $Q(x, y)$ be a statement pertaining to x and y such that $Q(x, y)$ is true if and only if $P(x)$ is true and $x = y$. According to the axiom of replacement, we have a set $\{ y \mid Q(x, y) \text{ is true for some } x \in A \}$ such that

$$z \in \{ y \mid Q(x, y) \text{ is true for some } x \in A \} \iff Q(x, z) \text{ is true for some } x \in A.$$

The left side is true iff $z \in \{ x \mid P(x) \text{ is true for some } x \in A \}$. The right side is true iff $P(x)$ is true for some $x \in A$. Then we have

$$z \in \{ x \mid P(x) \text{ is true for some } x \in A \} \iff P(x) \text{ is true for some } x \in A,$$

which is the axiom of specification. □

Exercise 3.2.1.

Proof. Obvious. □

Exercise 3.2.2.

Proof. For the sake of contradiction, suppose there exists a set A such that $A \in A$ is true. According to the axiom of regularity, for all $x \in A$, either $x \cap A = \varnothing$ or x is not a set. We shall discuss the two cases: When $x \cap A = \varnothing$, we assume for the sake of contradiction that $A \in A$ falls into this case, then $A \cap A = \varnothing$. But $A \cap A = A$, so $A = \varnothing$. But $A \in A$, so $A \neq \varnothing$, a contradiction. When x is not a set, we assume for the sake of contradiction that $A \in A$ falls into this case, but A is a set, a contradiction. Therefore, there is no set A such that $A \in A$, a contradiction. □

Proof. For the sake of contradiction, suppose there exist sets A and B such that $A \in B$ and $B \in A$ are true. According to the singleton sets axiom, there exist sets $\{ A \}$ and $\{ B \}$; according to the axiom of pairwise union, there exists a set $X = \{ A \} \cup \{ B \} = \{ A, B \}$. According to the axiom of regularity, for all $x \in X$, either $x \cap X = \varnothing$ or x is not a set. We shall discuss the two cases: When $x \cap X = \varnothing$, we assume for the sake of contradiction that $A \in X$ falls into this case, then $A \cap X = \varnothing$. But because $B \in A$ and $B \in X$, we have $B \in A \cap X$, so $B \in \varnothing$, which is a contradiction; so $A \in X$ does not fall into this case. When x is not a set, we assume for the sake of contradiction that $A \in X$ falls into this case, but A is a set, a contradiction. Therefore, $\forall x \in X : x \neq A$, which contradicts that $A \in X$. □

Exercise 3.2.3.

Proof. (1) Assume the universal specification axiom. Then there exists a set

$$\Omega = \{\, x \mid 0 = 0 \text{ is true} \,\},$$

such that for all objects x, we have $x \in \Omega \iff 0 = 0 \iff \top$. (2) Assume we have a universal set Ω such that for all objects x, we have $x \in \Omega$. Then by the axiom of specification, for any object z, there exists a set $\{\, y \mid P(y) \wedge y \in \Omega \,\}$ such that

$$z \in \{\, y \mid P(y) \wedge y \in \Omega \,\} \iff (z \in \Omega \wedge P(z)).$$

Because x and y are objects, $x \in \Omega$ and $y \in \Omega$ are always true. The above equivalence can be rewritten as

$$z \in \{\, y \mid P(y) \,\} \iff P(z),$$

which is the axiom of universal specification. □

Exercise 3.3.1.

Proof. We shall just prove the substitution property. For all $x \in X$,

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g\left(\tilde{f}(x)\right) && (x \in X) \\ &= \tilde{g}\left(\tilde{f}(x)\right) && \left(\tilde{f}(x) \in Y\right) \\ &= \tilde{g} \circ \tilde{f}(x). \end{aligned}$$ □

Exercise 3.3.2.

Proof. For any $x, x' \in X$, we have

$$f(x) = f(x') \implies x = x';$$

for any $y, y' \in Y$, we have

$$g(y) = g(y') \implies y = y'.$$

Because $f(x), f(x') \in Y$, we have, for any $x, x' \in X$, that

$$g(f(x)) = g(f(x')) \implies f(x) = f(x') \implies x = x',$$

which means that $g \circ f$ is injective. □

Proof. For any $y \in Y$, we have

$$\exists x \in X : f(x) = y;$$

for any $z \in Z$, we have

$$\exists y \in Y : g(y) = Z.$$

Because any $f(x) \in Y$, we have, for any $z \in Z$, that

$$\exists x \in X : g(f(x)) = Z,$$

which means that $g \circ f$ is surjective. □

Exercise 3.3.3.

Proof. We denote the empty function by $f : \varnothing \to A$.

When f is injective, we have

$$\forall x, x' \in \varnothing : (f(x) = f(x') \implies x = x').$$

But $x, x' \notin \varnothing$, so it is vacuously true that f is unconditionally injective.

When f is surjective, we have

$$\forall y \in A : \exists x \in \varnothing : f(x) = y.$$

We consider the case that $A \neq \varnothing$, so we can let a be an element of A. There does not exist any $x \in \varnothing$ such that $f(x) = y$, a contradiction. Then we consider the case that $A = \varnothing$. But $y \notin A$ by definition, so it is vacuously true that f is surjective.

When f is a bijection, f is defined by $f : \varnothing \to \varnothing$. □

Exercise 3.3.4.

Proof. Because g is injective, we have for any $y, y' \in Y$ that

$$g(y) = g(y') \implies y = y'.$$

But $g \circ f = g \circ \tilde{f}$, so we have for any $x \in X$ that $g(f(x)) = g(\tilde{f}(x))$, which implies that $f(x) = \tilde{f}(x)$, which means that $f = \tilde{f}$.

If g is not injective, we construct a counterexample to show that the statement no longer holds. Let $X = \{0\}$, $Y = \{0, 1\}$ and $Z = \{0\}$; let $f(0) := 0$; let $\tilde{f}(0) := 1$; let $g(0) := 0$ and $g(1) := 0$. Then $g \circ f(0) = g \circ \tilde{f}(0) = 0$. Thus, $g \circ f = g \circ \tilde{f}$. But $f \neq \tilde{f}$. □

Proof. Because f is surjective, we have for any $y \in Y$ that

$$\exists x \in X : f(x) = y.$$

But $g \circ f = \tilde{g} \circ f$, so we have for any $x \in X$ that $g(f(x)) = \tilde{g}(f(x))$. Then, for any $y \in Y$, we have that for any $x \in X$ that $g(f(x)) = \tilde{g}(f(x))$, which implies that for any $y \in Y$ we have that $g(y) = \tilde{g}(y)$. Thus, $g = \tilde{g}$.

If f is not surjective, we construct a counterexample to show that the statement no longer holds. Let $X = \{ 0 \}$, $Y = \{ 0,1 \}$ and $Z = \{ 0,1 \}$; let $f(0) := 0$; let $g(0) := 0$ and $g(1) := 0$; let $\tilde{g}(0) := 0$ and $\tilde{g}(1) := 1$. Then $g \circ f(0) = \tilde{g} \circ f(0) = 0$. Thus, $g \circ f = g \circ \tilde{f}$. But $g \neq \tilde{g}$. □

Exercise 3.3.5.

Proof. For any $x, x' \in X$, we have that

$$g(f(x)) = g(f(x')) \implies x = x'.$$

If $f(x) = f(x')$, then $g(f(x)) = g(f(x'))$, which implies that $x = x'$. Thus, f is injective.

We construct a counterexample to show that g is not necessarily injective. Let $f : \{ 0 \} \to \{ 0,1 \}$ be defined by $f(0) := 0$; let $g : \{ 0,1 \} \to \{ 0 \}$ be defined by $g(0) := 0$ and $g(1) := 0$. Then $g \circ f$ is clearly injective, but g is not injective. □

Proof. For any $z \in Z$, we have that

$$\exists x \in X : g(f(x)) = z.$$

Because $f(x) \in Y$, it is equivalent to say that

$$\exists f(x) \in Y : g(f(x)) = z.$$

Thus, g is surjective.

We construct a counterexample to show that f is not necessarily surjective. Let $f : \{ 0 \} \to \{ 0,1 \}$ be defined by $f(0) := 0$; let $g : \{ 0,1 \} \to \{ 0 \}$ be defined by $g(0) := 0$ and $g(1) := 0$. Then $g \circ f$ is clearly surjective, but f is not surjective. □

Exercise 3.3.6.

Proof. By definition, for all $y \in Y$, $f(x) = y$, where x is denoted $f^{-1}(y)$, so $f(f^{-1}(y)) = y$. Because for all $x \in X$, $f(x) \in Y$, we also have $f(f^{-1}(f(x))) = f(x)$. Since f is injective, then $f^{-1}(f(x)) = x$ for all $x \in X$.

For any $y, y' \in Y$,

$$f^{-1}(y) = f^{-1}(y') \implies f(f^{-1}(y)) = f(f^{-1}(y')) \implies y = y',$$

so f^{-1} is injective. We have that $f(x) \in Y$, and that for any $x \in X$, $f^{-1}(f(x)) = x$, which means that for any $x \in X$, $\exists y \in Y : f^{-1}(y) = x$, so f^{-1} is surjective. Therefore, f^{-1} is invertible.

Denote the inverse of f^{-1} by g. For all $x \in X$, $f^{-1}(g(x)) = x$. We also have, for all $x \in X$ that $f^{-1}(f(x)) = x$. Because f^{-1} is surjective, for all $x \in X$, $g(x) = f(x)$. Thus, $g = f$. □

Exercise 3.3.7.

Proof. For all $y \in Y$, $f(f^{-1}(y)) = y$. Because $g^{-1}(z) \in Y$, for all $z \in Z$, $f(f^{-1}(g^{-1}(z))) = g^{-1}(z)$. But for all $z \in Z$, $g(g^{-1}(z)) = z$, therefore $g(f(f^{-1}(g^{-1}(z)))) = z$, which can be rewritten as $(g \circ f)(f^{-1} \circ g^{-1})(z) = z$. Thus, $f^{-1} \circ g^{-1}$ is the inverse of $g \circ f$. □

Exercise 3.3.8.

(a) *Proof.* We have $x \in X \implies x \in Y$. For every $x \in X$, $\iota_{Y \to Z} \circ \iota_{X \to Y}(x) = \iota_{Y \to Z}(x) = x$. For every $x \in X$, $\iota_{X \to Z}(x) = x$. Furthermore, $\iota_{Y \to Z} \circ \iota_{X \to Y}$ and $\iota_{X \to Z}$ are both defined on $X \to Z$, so the two are equal. □

(b) *Proof.* We have $f(x) \in B$. For every $x \in A$, $f(x) = f(x)$; for every $x \in A$, $f \circ \iota_{A \to A}(x) = f(\iota_{A \to A}(x)) = f(x)$; for every $x \in A$, $\iota_{B \to B} \circ f(x) = \iota_{B \to B}(f(x)) = f(x)$. Furthermore, f, $f \circ \iota_{A \to A}$ and $\iota_{B \to B} \circ f$ are all defined on $A \to B$, so the three are equal. □

(c) *Proof.* Obvious from the previous results. □

(d) *Proof.* (Existence) We define

$$h(a) = \begin{cases} f(a) & a \in X, \\ g(a) & a \in Y. \end{cases}$$

$h \circ \iota_{X \to X \cup Y}$ is defined on $X \to Z$. For every $x \in X$, $h \circ \iota_{X \to X \cup Y}(a) = f(a)$. Thus, $h \circ \iota_{X \to X \cup Y} = f$. Likewise, $h \circ \iota_{Y \to X \cup Y} = g$. (Uniqueness) Now, suppose for the sake of contradiction that there exist two different functions h and $\tilde{h}$ which satisfy $h \circ \iota_{X \to X \cup Y} = f$, $h \circ \iota_{Y \to X \cup Y} = g$, $\tilde{h} \circ \iota_{X \to X \cup Y} = f$ and $\tilde{h} \circ \iota_{Y \to X \cup Y} = g$. Then for some $a \in X \cup Y$, we have $h(a) \neq \tilde{h}(a)$. Because X and Y are disjoint, $a \in X \cup Y$ implies that either $a \in X$ or $a \in Y$. We discuss the two cases: If $a \in X$, we have $h \circ \iota_{X \to X \cup Y}(a) = f(a)$ and $\tilde{h} \circ \iota_{X \to X \cup Y}(a) = f(a)$. Thus, $h \circ \iota_{X \to X \cup Y}(a) = \tilde{h} \circ \iota_{X \to X \cup Y}(a)$, which is a contradiction. If $a \in Y$, then similarly there is a contradiction. Thus, there does not exist two different functions h and $\tilde{h}$. □