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Math 136

1 Vectors in $\mathbb{R}^n$

1.8 Cross Product in $\mathbb{R}^3$

Example

Given $\overrightarrow{v}, \overrightarrow{u} \in \mathbb{R}^3$, find $\overrightarrow{w} \in \mathbb{R}^3$ that is perpendicular to both $\overrightarrow{u}$ and $\overrightarrow{v}$.

Definition

The cross product of $\overrightarrow{u}$ and $\overrightarrow{v}$ is:

$$\overrightarrow{u} \times \overrightarrow{v} = \begin{bmatrix} u^2v_3 - u_3v_2 \\ -(u_1v_3 - u_3v_1) \\ u_1v_2 - u_2v_1 \end{bmatrix} \in \mathbb{R}^3$$

Theorem 1.1: Properties

1. $(\overrightarrow{u} \times \overrightarrow{v}) \cdot \overrightarrow{u} = 0$
2. $(\overrightarrow{u} \times \overrightarrow{v}) \cdot \overrightarrow{v} - 0$
3. $||\overrightarrow{u} \times \overrightarrow{v} = ||\overrightarrow{u}|| \, ||\overrightarrow{v}|| \sin\theta$

Note: Cross product satisfies the right-hand rule

Theorem 1.2: Properties

1. $\overrightarrow{u} \times \overrightarrow{v} = -(\overrightarrow{u} \times \overrightarrow{v})$
2. $\overrightarrow{u} \times (\overrightarrow{v} + \overrightarrow{w}) = \overrightarrow{u} \times \overrightarrow{w} + \overrightarrow{u} \times \overrightarrow{w}$
3. $\overrightarrow{u} \times (c\overrightarrow{v} = c(\overrightarrow{u} \times \overrightarrow{v})) = c\overrightarrow{u} \times \overrightarrow{v}$

Example

Problem: $\overrightarrow{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \overrightarrow{e^3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \overrightarrow{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Find $\overrightarrow{e_1} \times \overrightarrow{e_2}$

1. Find using right hand rule and coordinate system.
2. Math $\overrightarrow{e_1} \times \overrightarrow{e_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \overrightarrow{e_3}$

Example

Problem: Find $\overrightarrow{e_3} \times \overrightarrow{e_2}$

1. Find using right hand rule and coordinate system.
2. Math $\overrightarrow{e_1} \times \overrightarrow{e_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \overrightarrow{e_3}$

Example

Suppose: $\overrightarrow{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\overrightarrow{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ What is $\text{proj}_{-e_1}\overrightarrow{v}$

Solution:

$$\begin{aligned} prof_{-\overrightarrow{e_1}\overrightarrow{v}} &= \frac{\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix}}{|| \begin{bmatrix} -1 \\ 0 \end{bmatrix} ||^2} \\ &= \frac{-2}{1} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 0 \end{bmatrix} \end{aligned}$$

Definition

Let $\overrightarrow{v}, \overrightarrow{w} \in \mathbb{R}^n$ with $\overrightarrow{w} \neq 0$. The **projection of** $\overrightarrow{v}$ **onto** $\overrightarrow{w}$ is defined by

$$\text{perp}_{\overrightarrow{w}} = \overrightarrow{v} - proj_{\overrightarrow{w}}(\overrightarrow{v})$$

Properties:

1. $proj_{\overrightarrow{w}(\overrightarrow{v})}$ is perpendicular to $perp_{\overrightarrow{w}(\overrightarrow{v})}$
2. $proj_{\overrightarrow{w}}(c\overrightarrow{v}) = c \cdot proj_{\overrightarrow{w}}(\overrightarrow{v})$
3. $proj_{\overrightarrow{w}}(\overrightarrow{v} + \overrightarrow{u}) = proj_{\overrightarrow{w}}(\overrightarrow{v}) + proj_{\overrightarrow{w}}(\overrightarrow{v})$
4. $proj_{\overrightarrow{w}}(proj_{\overrightarrow{w}}(\overrightarrow{v})) = proj_{\overrightarrow{w}}(\overrightarrow{v})$

Proof of 4:

Proof.

$$\begin{aligned}
prof_{\overrightarrow{w}}(proj_{\overrightarrow{w}}(\overrightarrow{v})) &= proj_{\overrightarrow{w}}(\frac{\overrightarrow{v}\cdot\overrightarrow{w}}{||\overrightarrow{w}||^2\overrightarrow{w}}) \\
&= \frac{\overrightarrow{v}\cdot\overrightarrow{w}}{||\overrightarrow{w}||^2}\cdot\frac{\overrightarrow{w}\cdot\overrightarrow{w}}{||\overrightarrow{w}||^2}\overrightarrow{w} \\
&= \frac{\overrightarrow{v}\cdot\overrightarrow{w}}{||\overrightarrow{w}||^2}\overrightarrow{w} \\
&= proj_{\overrightarrow{w}}(\overrightarrow{v})
\end{aligned}$$

□

Standard Inner Project in $\mathbb{C}^n$

Instead of dot product, we define the **Standard inner product.**

Definition

The **standard inner product** of $\overrightarrow{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{C}^n$ is

$$\langle\overrightarrow{v},\overrightarrow{w}\rangle = v_1\overrightarrow{w_1} + v_2\overrightarrow{w_2} + \cdots + v_n\overrightarrow{w_n}$$

Definition

The **length** of the vector $\overrightarrow{v} \in \mathbb{C}^n$ is $||\overrightarrow{v}|| = \sqrt{\overrightarrow{v}\cdot\overrightarrow{v}}$