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Math 138

1 Integration

1.6 The Fundamental Theorem of Calculus (Part 2)

Indefinite Integrals

Recall that the definite integral, $\int_a^b f(x)dx$, is a number, where as the indefinite integral, $\int f(x)dx$, is a function and is the antiderivative of $f(x)$.

Definition: Recall

FTC2: $\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$ where $F'(x) = f(x)$.

In order to apply FTC2 or evaluate indefinite integrals, it's important to remember the antiderivative of elementary functions.

Note that there are many functions for which antiderivatives cannot be expressed in terms of elementary functions, such as

$$\int e^{-x^2}, \int \sqrt{1+x^3}dx, \int \sin(x^2)dx$$

Problem 1

(a) $\int \frac{dx}{a+x} = \ln|x+a| + c$

(b) $\int \frac{dx}{a+bx} = \frac{1}{b}\ln|a+bx| + c$

(c) $\int (a+bx)^n dx = \frac{1}{b(n+1)}(a+bx)^{n+1} + C, (n \neq 1)$

(d) $\int \cos(a+bx)dx = \frac{1}{b}\sin(a+bx) + C$

(e) $\int e^{a+bx} = \frac{1}{b}e^{a+bx} + C$

Problem 2

Evaluate $\int \cos^2 x dx$:

Even powers of $\sin x$ or $\cos x$ can be integrated by using trig identities. In this case, use

$$\cos^2 x = \frac{1}{2}(1 + 2\cos 2x)dx = \frac{1}{2}(x + \frac{1}{2}\sin 2x) + C$$

Note: $\int \cos^2 x dx \neq \frac{1}{3}\cos^3 x + C$ and is easily shown by differentiation:

$$\frac{d}{dx}(\frac{1}{3}\cos^3 x + C) = -\sin\cos^2 x \neq \cos^2 x$$

Problem 3

Evaluate $\int \frac{x}{x+1}dx$:

When integrating rational functions always perform long divsion when the degree of the numerator is greater than or equal to the degree of the denominator. Here,

$$\frac{x}{x+1} = 1 - \frac{1}{x+1}$$

$$\int \frac{x}{x+1}dx = \int (1 - \frac{1}{1+x})dx = x - \ln|x+1| + C$$

Problem 4

Evaluate $\int \frac{2+e^{-x}}{e^x}dx$

Sometimes it's helpful to manipulate/rewrite the integrand. Here, $\frac{2+e^{-x}}{2^x} = 2e^{-x} + e^{-2x}$.

$$\begin{aligned}
&\int \frac{2+e^{-x}}{e^x}dx \\
=&\int (2e^{-x} + e^{-2x})dx \\
=&-2e^{-x} - \frac{1}{2}e^{-2x} + C
\end{aligned}$$

Problem 5

Evaluate $\int_0^2 |x^2 - 3x + 2|dx$

$$\begin{aligned}
\int_0^2 |x^2 - 3x + 2|dx &= \int_0^1 (x^2 - 3x + 2)dx + \int_1^2 (-x^2 + 3x - 2)dx \\
&= (\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x)\Big|_0^1 + (-\frac{1}{3}x^3 + \frac{3}{2}x^2 - 2x)\Big|_1^2 \\
&= (\frac{1}{3} - \frac{3}{2} + 2) - 0 + (-\frac{8}{3} + 6 - 4) - (-\frac{1}{3} + \frac{3}{2} - 2) \\
&= 1
\end{aligned}$$

Example

Problem:
Let $p(x) = \int_1^{x^2} \frac{1}{t} e^{-t} dt$, find $p'(x)$.
Solution:
Define $f(t) = \frac{1}{t}e^{-t}$, then $f(t)$ is continuous for all $t \neq 0$. FTC1 applies for all $t \neq 0$.

Let $u = x^2$, then

$$\begin{aligned} p(u) - \int_1^u \frac{1}{t} e^{-t} dt \to \frac{dp}{dx} = p'(x) &= \frac{dp}{du} \cdot \frac{du}{dx} \\ &= (\frac{1}{u}e^{-u})(2x) \\ &= \frac{2}{x} e^{-x^2} \text{ for } x \neq 0 \end{aligned}$$

Example

Problem:
For $H(x) = \int_{x^2}^{e^x} \cos(t^2) dt$ find $H'(x)$
Solution:
Define $f(t) = \cos(t^2)$, then $f(t)$ is continuous for all t.

Let $u = x^2$ and $v = e^x$, then

$$\begin{aligned} H = \int_u^v f(t)dt &= \int_u^a f(t)dt + \int_a^v f(t)dt \\ &= -\int_a^u f(t)dt + \int_a^v f(t)dt \end{aligned}$$

Next, apply the Chain rule and FTC1 to get:

$$\begin{aligned} H'(x) = \frac{dH}{dx} &= -f(u)\frac{du}{dx} + f(v)\frac{dv}{dx} \\ &= e^x \cos(e^{2x}) - 2x\cos(x^4) \end{aligned}$$

This leads to the generalized version of the FTC1.

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Definition: FTC1 - Extended Version

Assume that f is continuous and $g + h$ are differentiable.

Let

$$H(x) = \int_{g(x)}^{h(x)} f(t)dt$$

Then $H(x)$ is differentiable and $H'(x) = f(h(x))h'(x) - f(g(x))g'(x)$.

Definition: FTC2

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F'(x) = f(x)$

Notes:

1. A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.
2. If F is an antiderivative of f, then so is $F + c$ for all $c \in \mathbb{R}$.
3. We define the evaluation symbol as:

$$F(x)|_a^b = F(b) - F(a)$$

4. Using FTC2 is more practical for evaluataing integrals than using Riemann Sums. However, FTC2 assumes that the antiderivative of f is known. We will learn some techniques of integration to help us determine the antiderivatives.