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Math 138

2.2 Techniques of Integration

The integration technique corresponds to the integral version of the product rule.

In integral form this becomes:

$$\int udv = uv - \int vdu$$

Note that $\frac{dv}{dx} = dv$, $\frac{du}{dx}dx = du$ and the constant is ignored.
In this method, we break up the integral into a product of two parts: u and dv, where dv is easily integrated and the integral $\int vdu$ on the RHS is simpler than the original integral $\int udv$.

Example

1. $\int xe^x dx$.

   Let $u = x, dv = e^x dx$, then $du = dx, v = e^x$

   $$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

2. Evaluate $\int x^2 \sin x dx$. Let $u = x^2, dv = \sin x dx$, then $du = 2xdx, v = -\cos x$,

   $$\int x^2 \sin x dx = -x^2 \cos x + 2 \int x \cos x dx$$

   We can apply IBP again which yields: $u = x, dv = \cos x dx$ then $du = dx, v = \sin x$.

   $$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + C$$

   Hence,

   $$\begin{aligned}\int x^2 \sin x dx &= -x^2 \cos x + 2[x \sin x + \cos x + C] \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + K, K = 2C\end{aligned}$$

   to consider two cases: $\tan \theta > 0, \tan \theta < 0$.

Remark

For integrands of the form $x^n\{\sin(ax), \cos(ax), e^{ax}\}$ with n a positive number, choose $u = x^n + dv = \{\sin(ax), \cos(ax), e^{ax}\}dx$

Example

1. $\int \ln x dx$. Let $u = \ln x, dv = dx$, then $du = \frac{dx}{x}, v = x$.

$$\ln x dx = x \ln x - \int dx = x \ln x - x + C$$

Remark: For integrals involving $\ln x$, try $u = \ln x$.

2. Evaluate $\int x \tan^{-1} x dx$. Let $u = \tan^{-1} x, dv = xdx$, then $du = \frac{dx}{1+x^2}, v = \frac{1}{2}x^2$.

$$\begin{aligned}\int x \tan^{-1} x dx &= \frac{1}{2}x^2 \tan_{-1} x - \frac{1}{2}\int \frac{x^2 dx}{1+x^2} \\ &= \frac{1}{2}x^2 \tan^{-1} - \frac{1}{2}\int [1 - \frac{1}{1+x^2}]dx \text{ by long division} \\ &= \frac{1}{2}x^2 \tan^{-1} - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + C\end{aligned}$$

3. Evaluate $\int_1^e x \ln x dx$. Let $u = \ln x, dv = xdx$, then $du = \frac{dx}{x}, v = \frac{1}{2}x^2$.

$$\begin{aligned}\int_1^e \ln x dx &= \frac{1}{2}x^2 \ln x \Big|_1^e - \frac{1}{2}\int_1^e x dx \\ &= \frac{1}{2}e^2 - \frac{1}{4}x^2|_1^{e^1} = \frac{1}{4}(e^2 + 1)\end{aligned}$$

4. Evaluate $\int e^{ax} \cos(bx) dx$. Can either pick $u = e^{ax} + dx = \cos(bx)dx$ or $u = \cos(bx) + dv = e^{ax} dx$
Let's use $u = e^{ax}, dv = \cos(bx)dx$, then $du = ae^{ax}dx, v = \frac{1}{b}\sin(bx)$.

$$\int e^{ax} \cos(bx) dx = \frac{1}{b}e^{ax} \sin(bx) - \frac{a}{b}\int e^{ax} \sin(bx) dx$$

We apply IBP again so: $u = e^{ax}, dv = \sin(bx)dx$, then $du = ae^{ax}dx, v = -\frac{1}{b}\cos(bx)$.

$$\int e^{ax} \cos(bx) dx = \frac{1}{b}e^{ax} \sin(bx) - \frac{a}{b}[-\frac{1}{b}e^{ax} \cos(bx) + \frac{a}{b}\int e^{ax} \cos(bx) dx]$$

We can combine both these integrals:

$$\rightarrow (1 + \frac{a^2}{b^2} \int e^{ax} \cos(bx) dx = \frac{1}{b}e^{ax} \sin(bx) + \frac{a}{b^2}e^{ax} \cos(bx)$$

$$\begin{aligned}\rightarrow \int e^{ax} \cos(bx) dx &= \frac{\frac{1}{b}e^{ax}(\sin(bx) + \frac{a}{b}\cos(bx))}{(1 + \frac{a^2}{b^2})} + C \\ &= \frac{e^{ax}(b \sin(bx) + a \cos(bx))}{a^2 + b^2} + C\end{aligned}$$