

## 2 Spans, Sections, Lines

### Lines and Planes in $\mathbb{R}^n$

#### Definition: Recall

A line in  $\mathbb{R}^n$  through the origin is a set of the form:

$$\mathbb{L} = \{0 + \vec{d} : t \in \mathbb{R}\} = \text{span}\{\vec{d}\}$$

where  $\vec{d} \in \mathbb{R}^n$  is non-zero. This is a direction vector.

#### Example

A line in  $\mathbb{R}^n$  is a set of the form:

$$l = \{\vec{p} + t\vec{d} : t \in \mathbb{R}\}$$

where  $\vec{p}, \vec{d} \in \mathbb{R}^n$  and  $\vec{d} \neq \vec{0}$ .

#### Example

$$l = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix} : t \in \mathbb{R}$$

**Convention:**  $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$  is on  $l$  if its terminal part  $(a, b)$  is on  $l$ . So for example,  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$  are on  $l \iff (1, 2)$  and  $(2, 3)$  are on  $l$ .

#### Problem 4

**True or False:** The lines  $l_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} : t \in \mathbb{R} \right\}$ ,  $l_2 = \left\{ \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix} : t \in \mathbb{R} \right\}$  are the same. Graph it and you see it does not work.

Algebraically:  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in l_1$ . Is  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in l_2 \iff$  can 1 solve for  $t$  such that  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$

#### Problem 4

**True or False:** The lines  $l_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \end{bmatrix} : t \in \mathbb{R} \right\}$ ,  $l_2 = \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -2 \\ 2 \end{bmatrix} : t \in \mathbb{R} \right\}$  are the same.

Algebraically: Suppose  $\vec{d}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  and  $\vec{d}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$   
 $\vec{d}_2 = -2\vec{d}_1$  are parallel. Now we need to check if they intersect.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 - 2t \\ 2 + 2t \end{bmatrix}$$

$$1 = -1 - 2t$$

$$0 = 2 + 2t$$

$$t = -1$$

$\therefore \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  is on  $l_2$  and they intersect.

#### Definition

A plane in  $\mathbb{R}^n$  through the origin is a set of the form:

$$p = \{s\vec{u} + t\vec{v} : s, t \in \mathbb{R}\} = \text{span}\{\vec{u}, \vec{v}\}$$

where  $\vec{u}, \vec{v} \in \mathbb{R}^n$ ,  $\vec{u} \neq c\vec{v}$  for only  $c \in \mathbb{R}$ .

#### Example

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} = p$$

#### Definition

A plane in  $\mathbb{R}^n$  is a set of the form:

$$p = \{\vec{w} + s\vec{u} + t\vec{v} : s, t \in \mathbb{R}\}$$

where  $\vec{w}, \vec{u}, \vec{v} \in \mathbb{R}^n$  and  $\vec{u} \neq c\vec{v}$  for any  $c \in \mathbb{R}$ .

Example

$$p = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\} + s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} : s, t \in \mathbb{R}$$

## 2.5 Planes in $\mathbb{R}^3$

Read by yourself.

## 3 Linear Equations

Definition

$$2x+3y+z=2$$

Definition

A linear equation is an equation of the form:

$$a_1x_1 + a_2x_2 + \cdots a_nv_n = b$$

$a$  are shown as coefficients,  $x$ : are shown as variables, and  $b$  is the RHS.

Definition

A system of linear equations is a collection of linear equation:

$$a_1x_1 + z_2x_2 + \cdots + a_nx_n = b_1$$

$$a_{k_1}x_1 + a_{k_2} + \cdots + x_{k_n}x_n = b_k$$

Definition

A solution to a system of equations is vector

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

whose component satisfies all the equations.

Example

$$x_1 + 2x_2 + x_3 = 7$$

$$x_1 - x_2 - x_3 = -4$$

vector:  $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$  whose component satisfies all the equations.