Edward Song
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Math 138

1 Integration

1.5 The Fundamental Theorem of Calculus

This theorem provides a connection between differential and integral calculus and is expressed in 2 parts.

Definition: FTC Part 1

Assume f is continuous on an open interval I containing a point a. Let

$$G(x) = \int_a^x f(t)\,dt$$

Then $G(x)$ is differentiable at each $x \in I$ and

$$G'(x) = f(x)$$

Equivalently,

$$G'(x) = \frac{d}{dx}\int_a^x f(t)dt$$

Proof. By definition:

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^a f(t)dt\right) \\
&= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt \\
&= \lim_{h\to 0} f(c) \text{ for } c \in [x, x+h] \\
&= f(x) \text{ since } f \text{ is continuous and } c \to x \text{ as } h \to 0
\end{aligned}$$

□

Example

Problem:
For $g(x) = \int_0^x \frac{dt}{\sqrt{1+t^4}}$ for $x > 0$, find $g'(2)$.

Solution:
Let $f(t) = \frac{1}{\sqrt{1+t^4}}$, then $f(t)$ is continuous for all $t \to$ FTC 1 applies. Thus, $g'(x) = \frac{1}{\sqrt{1+x^4}}$ for $x > 0$ by FTC1.

$$g'(2) = \frac{1}{17}$$

Example

Problem:
Let $p(x) = \int_1^{x^2} \frac{1}{t}e^{-t}dt$, find $p'(x)$.
Solution:
Define $f(t) = \frac{1}{t}e^{-t}$, then $f(t)$ is continuous for all $t \neq 0$. FTC1 applies for all $t \neq 0$.

Let $u = x^2$, then

$$\begin{aligned} p(u) - \int_1^u \frac{1}{t}e^{-t}dt \to \frac{dp}{dx} = p'(x) &= \frac{dp}{du} \cdot \frac{du}{dx} \\ &= (\frac{1}{u}e^{-u})(2x) \\ &= \frac{2}{x}e^{-x^2} \text{ for } x \neq 0 \end{aligned}$$

Example

Problem:
For $H(x) = \int_{x^2}^{e^x} \cos(t^2)dt$ find $H'(x)$
Solution:
Define $f(t) = \cos(t^2)$, then $f(t)$ is continuous for all t.

Let $u = x^2$ and $v = e^x$, then

$$\begin{aligned} H = \int_u^v f(t)dt &= \int_u^a f(t)dt + \int_a^v f(t)dt \\ &= -\int_a^u f(t)dt + \int_a^v f(t)dt \end{aligned}$$

Next, apply the Chain rule and FTC1 to get:

$$\begin{aligned} H'(x) = \frac{dH}{dx} &= -f(u)\frac{du}{dx} + f(v)\frac{dv}{dx} \\ &= e^x\cos(e^{2x}) - 2x\cos(x^4) \end{aligned}$$

This leads to the generalized version of the FTC1.

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Definition: FTC1 - Extended Version

Assume that f is continuous and $g + h$ are differentiable.

Let

$$H(x) = \int_{g(x)}^{h(x)} f(t)dt$$

Then $H(x)$ is differentiable and $H'(x) = f(h(x))h'(x) - f(g(x))g'(x)$.

Definition: FTC2

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F'(x) = f(x)$

Notes:

1. A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

2. If F is an antiderivative of f, then so is $F + c$ for all $c \in \mathbb{R}$.

3. We define the evaluation symbol as:

$$F(x)|_a^b = F(b) - F(a)$$

4. Using FTC2 is more practical for evaluataing integrals than using Riemann Sums. However, FTC2 assumes that the antiderivative of f is known. We will learn some techniques of integration to help us determine the antiderivatives.