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Math 138

2 Techniques of Integration

2.1 Inverse Trigonometric Substitutions

This is a special case of the method of substitution involving trig functions. Integrals involving radicals:

$$\sqrt{a^2 - b^2x^2}, \sqrt{a^2 + b^2x^2}, \sqrt{b^2x^2 - a^2}$$

can be evaluated by making the following substitutions:

Radical	Trig Subsitution	Relevant Identity
$\sqrt{a^2 - b^2x^2}$	$bx = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{\sqrt{a^2 + b^2x^2}}$	$bx = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{b^2x^2 - a^2}$	$bx = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

This reduces the integral to a trigonometric integral which may be easier to evaluate. When using this method the following integrals arise often and hence should be added to our list of elementary integrals:

$$\int \tan\theta d\theta = \ln|\sec\theta| + C$$
$$\int \cot\theta d\theta = \ln|\csc\theta| + C$$
$$\int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$$
$$\int \csc\theta d\theta = -\ln|\csc\theta + \cot\theta| + C$$

The first two can be shown through substitution and the last two can be verified through differentiation.

Example

1. Evaluate $\int \frac{dx}{4+x^2}$.

   Let $x = 2\tan\theta$, then $dx = 2\sec^2\theta d\theta$. Thus,

   $$\int \frac{dx}{4+x^2} = \int \frac{2\sec^2\theta d\theta}{4+4\tan^2\theta} = \int \frac{sec^2\theta d\theta}{\sqrt{1+\tan^2\theta}} = \int \frac{\sec^2\theta d\theta}{|\sec\theta|}$$

   Now,

   $$\theta = \tan^{-1}\frac{x}{2} \to \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \to \sec\theta > 0 \to |\sec\theta| = \sec\theta$$

   $$\int \frac{dx}{4+4x^2} = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$$

   We need to express this in terms of x. To do this, draw a triangle. We see that

   $$\sec\theta = \frac{1}{\cos\theta} = \frac{\sqrt{4+x^2}}{2}$$

   $$\begin{aligned}\int \frac{dx}{\sqrt{4+x^2}} &= \ln|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}| + C \\ &= \ln|x + \sqrt{4+x^2}| + K\end{aligned}$$

   Recall that $\ln(\frac{A}{B}) = \ln A - \ln B$, so $K = C - \ln 2$.

2. Evaluate

   $$\int \frac{dx}{x^2\sqrt{x^2-16}} = \int \frac{4\sec\theta\tan\theta\, d\theta}{16\sec^2\theta\sqrt{16(\sec^2\theta-1)}} = \frac{1}{16}\int \frac{\tan\theta\, d\theta}{\sec\theta|\tan\theta|}$$

   Now $\theta = \sec^{-1}\frac{x}{4}$ if we restrict $\theta \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$, then $\tan\theta > 0 \to |\tan\theta| = \tan\theta$.

   $$\int \frac{dx}{x^2\sqrt{x^2-16}} = \frac{1}{16}\int \frac{d\theta}{\sec\theta} = \frac{1}{16}\cos\theta d\theta = \frac{1}{16}\sin\theta + C = \frac{\sqrt{x^2-16}}{16x} + C$$

   But from the triangle $\sin\theta = \frac{\sqrt{x^2-16}}{x}$.

   Note: If we choose $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, then we would need to consider two cases: $\tan\theta > 0, \tan\theta < 0$.

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Example

1. Evaluate $\int \frac{dx}{(5-4x-x^2)^{\frac{5}{2}}}$:

   We first complete the square: $5-4x-x^2 = 9-(x+2)^2$. Let $x+2 = 3\sin\theta \rightarrow dx = 3\cos\theta d\theta$. Then:

$$\begin{aligned}\int \frac{dx}{5-4x-x^2}^{\frac{5}{2}} &= \int \frac{dx}{[9-(x+2)^2]^{\frac{5}{2}}} \\ &= \frac{1}{3^4}\int \frac{d\theta}{\cos^2\theta} = \frac{1}{81}\int \sec^4\theta d\theta \\ &= \frac{1}{81}\int \sec^2\theta(1+\tan^2\theta)d\theta \\ &= \frac{1}{81}\int \sec^2\theta d\theta + \frac{1}{81}\int \sec^2\theta\tan^2\theta d\theta \\ &= \frac{1}{81}\tan\theta + \frac{1}{81}[\frac{1}{3}\tan^3\theta] + C \\ &= \frac{1}{81}[\frac{x+2}{9-(x+2)^2}] + \frac{1}{343}\left[\frac{(x+2^3)}{[9-(x+2)^2]}^{\frac{3}{2}}\right] + C\end{aligned}$$

2. Evaluate the definite integral $\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3dx}{(4x^2+9)^{\frac{3}{2}}}$

   Let $2x = 3\tan\theta$, then $2dx = 4\sec^2\theta\, d\theta$. Thus,

$$\begin{aligned}\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3dx}{(4x^2+9)^{\frac{3}{2}}} &= \frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{\tan^3\theta\sec^2\theta}{\sec^3\theta}d\theta \\ &= \frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{\tan^3\theta}{\sec\theta}d\theta = \frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{\sin^3\theta}{\cos^2\theta}d\theta \\ &= \frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{\sin\theta(1-\cos^2\theta)}{\cos^2\theta}d\theta\end{aligned}$$

   Now make the substitution $u = \cos\theta, du = -\sin\theta\, d\theta$.

$$\begin{aligned}&= -\frac{3}{16}\int_1^{\frac{1}{2}}]\frac{(1-u)^2}{u^2}du = \frac{3}{16}\int_{\frac{1}{2}}^1 [\frac{1}{u^2}-1]du \\ &= \frac{3}{16}[-\frac{1}{u}-u]\Big|_{\frac{1}{2}}^1 = \frac{3}{32}\end{aligned}$$