Edward Song
January 12, 2022
Math 136

1 Vectors in $\mathbb{R}^n$

1.8 Cross Product in $\mathbb{R}^3$

Example

Given $\overrightarrow{v}, \overrightarrow{u} \in \mathbb{R}^3$, find $\overrightarrow{w} \in \mathbb{R}^3$ that is perpendicular to both $\overrightarrow{u}$ and $\overrightarrow{v}$.

Definition

The <u>cross product</u> of $\overrightarrow{u}$ and $\overrightarrow{v}$ is:

$$\overrightarrow{u} \times \overrightarrow{v} = \begin{bmatrix} u^2v_3 - u_3v_2 \\ -(u_1v_3 - u_3v_1) \\ u_1v_2 - u_2v_1 \end{bmatrix} \in \mathbb{R}^3$$

Theorem 1.1: Properties

1. $(\overrightarrow{u} \times \overrightarrow{v}) \cdot \overrightarrow{u} = 0$
2. $(\overrightarrow{u} \times \overrightarrow{v}) \cdot \overrightarrow{v} - 0$
3. $||\overrightarrow{u} \times \overrightarrow{v} = ||\overrightarrow{u}||\,||\overrightarrow{v}||\sin\theta$

Note: Cross product satisfies the right-hand rule

Example

Suppose: $\overrightarrow{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\overrightarrow{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ What is $\text{proj}_{e_1}\overrightarrow{v}$?

Solution:

$$\begin{aligned} proj_{\overrightarrow{e1}}\overrightarrow{v} &= \frac{\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{|| \begin{bmatrix} 1 \\ 0 \end{bmatrix} ||^2} \\ &= \frac{2}{1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 0 \end{bmatrix} \end{aligned}$$

Example

Suppose: $\overrightarrow{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\overrightarrow{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ What is $\text{proj}_{-e_1}\overrightarrow{v}$?

Solution:

$$
\begin{aligned}
prof_{-\overrightarrow{e_1}}\overrightarrow{v} &= \frac{\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix}}{|| \begin{bmatrix} -1 \\ 0 \end{bmatrix} ||^2} \\
&= \frac{-2}{1} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 0 \end{bmatrix}
\end{aligned}
$$

Definition

Let $\overrightarrow{v}, \overrightarrow{w} \in \mathbb{R}^n$ with $\overrightarrow{w} \neq 0$. The **projection of** $\overrightarrow{v}$ **onto** $\overrightarrow{w}$ is defined by

$$\text{perp}_{\overrightarrow{w}} = \overrightarrow{v} - proj_{\overrightarrow{w}}(\overrightarrow{v})$$

Properties:

1. $proj_{\overrightarrow{w}(\overrightarrow{v})}$ is perpendicular to $perp_{\overrightarrow{w}(\overrightarrow{v})}$
2. $proj_{\overrightarrow{w}}(c\overrightarrow{v}) = c \cdot proj_{\overrightarrow{w}}(\overrightarrow{v})$
3. $proj_{\overrightarrow{w}}(\overrightarrow{v} + \overrightarrow{u}) = proj_{\overrightarrow{w}}(\overrightarrow{v}) + proj_{\overrightarrow{w}}(\overrightarrow{v})$
4. $proj_{\overrightarrow{w}}(proj_{\overrightarrow{w}}(\overrightarrow{v})) = proj_{\overrightarrow{w}}(\overrightarrow{v})$

Proof of 4:

Proof.

$$
\begin{aligned}
prof_{\overrightarrow{w}}(proj_{\overrightarrow{w}}(\overrightarrow{v})) &= proj_{\overrightarrow{w}}(\frac{\overrightarrow{v} \cdot \overrightarrow{w}}{||\overrightarrow{w}||^2 \overrightarrow{w}}) \\
&= \frac{\overrightarrow{v} \cdot \overrightarrow{w}}{||\overrightarrow{w}||^2} \cdot \frac{\overrightarrow{w} \cdot \overrightarrow{w}}{||\overrightarrow{w}||^2} \overrightarrow{w} \\
&= \frac{\overrightarrow{v} \cdot \overrightarrow{w}}{||\overrightarrow{w}||^2} \overrightarrow{w} \\
&= proj_{\overrightarrow{w}}(\overrightarrow{v})
\end{aligned}
$$

$\square$

Edward Song
January 12, 2022
Math 136

Standard Inner Project in $\mathbb{C}^n$

Instead of dot product, we define the **Standard inner product.**

Definition

The **standard inner product** of $\overrightarrow{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{C}^n$ is

$$\langle \overrightarrow{v}, \overrightarrow{w} \rangle = v_1\overrightarrow{w_1} + v_2\overrightarrow{w_2} + \cdots + v_n\overrightarrow{w_n}$$

Definition

The **length** of the vector $\overrightarrow{v} \in \mathbb{C}^n$ is $||\overrightarrow{v}|| = \sqrt{\overrightarrow{v} \cdot \overrightarrow{v}}$