Edward Song
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Math 136

1 Vectors in $\mathbb{R}^n$

Recall

Vectors in $\mathbb{R}^n = \{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \; x_i \in \mathbb{R}\}$

We can also work with $\mathbb{C}^n = \{ \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \; z_i \in \mathbb{C}\}$

1.5 Dot Product

Definition

Let $\overrightarrow{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\overrightarrow{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$. We define their **dot product** by

$$\overrightarrow{u} \cdot \overrightarrow{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Theorem 1.1: Property 1.5.3

1. $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \in \mathbb{R}$
2. $\overrightarrow{u} \cdot \overrightarrow{v} = \overrightarrow{v} \cdot \overrightarrow{u}$
3. $(\overrightarrow{u} + \overrightarrow{v}) \cdot \overrightarrow{w} = \overrightarrow{u}\overrightarrow{w} + \overrightarrow{v}\overrightarrow{w}$
4. $(\overrightarrow{u} \cdot \overrightarrow{v}) \cdot \overrightarrow{w} = \overrightarrow{v} \cdot (\overrightarrow{u} \cdot \overrightarrow{w})$
5. $\overrightarrow{v} \cdot \overrightarrow{v} \geq 0$

Geometry in $\mathbb{R}^2$

Definition

The **length** of the vector $\overrightarrow{v} \in \mathbb{R}^n$ is $||\overrightarrow{v}|| = \sqrt{\overrightarrow{v} \cdot \overrightarrow{v}}$

Aside: In $\mathbb{R}^1$:

$$||\overrightarrow{v}|| = ||[v_1]|| = \sqrt{v_1^2} = |v|$$

Theorem 1.2: Properties of Length

1. $||\overrightarrow{0}|| = 0$
2. $||c \cdot \overrightarrow{v}|| = |c| \cdot ||\overrightarrow{v}||$
3. $||\overrightarrow{v} + \overrightarrow{u}|| \neq ||\overrightarrow{v}|| + ||\overrightarrow{u}||$
4. $||\overrightarrow{v} + \overrightarrow{u}|| \leq ||\overrightarrow{v}|| + ||\overrightarrow{u}||$

Importance of dot product: It gives angles between vectors in $\mathbb{R}^2$!

Definition

$\overrightarrow{v} \in \mathbb{R}^n$ is a **unit vector** if $||\overrightarrow{v} = 1||$

Definition

When $\overrightarrow{v} \in \mathbb{R}^n$ is a non-zero vector, we can produce a unit vector

$$\hat{v} = \frac{\overrightarrow{v}}{||\overrightarrow{v}||}$$

in the direction of $\overrightarrow{v}$ by scaling $\overrightarrow{v}$. This process is called normalization.

Definition

Let $\overrightarrow{v}$ and $\overrightarrow{u}$ be non-zero vectors in $\mathbb{R}^n$. The angle θ, in radians $(0 \leq \theta\pi)$, between $\overrightarrow{u}$ and $\overrightarrow{v}$ is such that

$$\overrightarrow{v} \cdot \overrightarrow{u} = ||\overrightarrow{u}|| \cdot ||\overrightarrow{v}|| \cos\theta, \text{ that is } \theta = \arccos(\frac{\overrightarrow{u} \cdot \overrightarrow{v}}{||\overrightarrow{v}|| \cdot ||\overrightarrow{u}||})$$

Example

Problem: Given 2 vectors, $\begin{bmatrix}0\\1\end{bmatrix}$ and $\begin{bmatrix}1\\0\end{bmatrix}$ find θ.

Solution:

$$\begin{aligned}\cos\theta &= \frac{\begin{bmatrix}0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\0\end{bmatrix}}{||\begin{bmatrix}0\\1\end{bmatrix}||\cdot||\begin{bmatrix}1\\0\end{bmatrix}||}\\ &= \frac{0}{1} = 0\\ \theta &= \frac{\pi}{2}\end{aligned}$$

Definition

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. We say $\vec{u}$ and $\vec{v}$ are <u>perpendicular</u> (or <u>orthogonal</u>) if $\vec{u}\cdot\vec{v} = 0$

Example

Problem: Find a non-zero vector in $\mathbb{R}^2$ that is orthogonal to $\begin{bmatrix}1\\2\end{bmatrix}$

Solution: $\begin{bmatrix}2\\-1\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}6\\-3\end{bmatrix}$

Example

Problem: Find a non-zero vector perpendicular to $\begin{bmatrix}a\\b\end{bmatrix}$

Solution:

$$\begin{aligned}&\begin{bmatrix}-b\\a\end{bmatrix}\cdot\begin{bmatrix}a\\b\end{bmatrix}\\ =& -ba + ab\\ =& 0\end{aligned}$$

Therefore, $\begin{bmatrix}-b\\a\end{bmatrix}$ is a solution.