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Math 136

2 Spans, Sections, Lines

2.1 Linear Combinations and Span

Definition: Recall

Recall definition: Let $\overrightarrow{v_1}, \overrightarrow{v_2}, \cdots, \overrightarrow{v_n} \in \mathbb{F}^n$. A linear combination of $\overrightarrow{v_1} \cdots, \overrightarrow{v_n}$ is a vector of the form:

$$c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \cdots + c_n\overrightarrow{v_n}$$

Problem 1

1. In $\mathbb{R}^2$:
$$2\begin{bmatrix}1\\1\end{bmatrix} + (-1)\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$
is a linear combination of $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$
2. In $\mathbb{F}^n$
$\overrightarrow{v} = 1 \cdot \overrightarrow{v}$
$\overrightarrow{v} = 1 \cdot \overrightarrow{v} + 0\overrightarrow{u_1} + 0\overrightarrow{u_2} + \cdots + 0\overrightarrow{u_n}$
3. In $\mathbb{F}^n$ $\overrightarrow{0} = 0\overrightarrow{v_1} + \cdots + 0\overrightarrow{v_n}$

Problem 2

Let $\overrightarrow{v_1}, \cdots, \overrightarrow{v_n} \in \mathbb{F}^n$. The span of $\overrightarrow{v_1}, \cdots, \overrightarrow{v_n} \in \mathbb{F}^n$ is the set span $\{\overrightarrow{v_1}, \cdots, \overrightarrow{v_n}\} = \{c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \cdots + c_n\overrightarrow{v_n} : c_1, \cdots, c_n \in \mathbb{F}\}$

Warning:

$$span\{\overrightarrow{v_1}, \cdots, \overrightarrow{v_n}\} = c_1\overrightarrow{v_1} + \cdots + c_n\overrightarrow{v_n}$$

The left side is a set, while the right side is a vector.

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Problem 4

1. T/F: $\mathbb{R}^2 = span\{\overrightarrow{e_1}, \overrightarrow{e_2}\} = span\{\begin{bmatrix}1\\0\end{bmatrix}\}, \begin{bmatrix}1\\0\end{bmatrix}\}$

   Proof.

$$\begin{aligned}&span\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}\\ =&\{a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} : a, b \in \mathbb{R}\}\\ =&\{\begin{bmatrix}a\\b\end{bmatrix} : a, b \in \mathbb{R}\}\\ =&\mathbb{R}^2\end{aligned}$$

   □

2. T/F: $\mathbb{R}^2 = span\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\}$:

   Proof.

$$\begin{aligned}span\{\begin{bmatrix}1\\1\end{bmatrix}\}, \begin{bmatrix}3, 4\end{bmatrix} &= \{a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}3\\4\end{bmatrix} : a, b \in \mathbb{R}\}\\ &= \{\begin{bmatrix}a+3b\\a+4b\end{bmatrix} : a, b \in \mathbb{R}\}\\ &= \mathbb{R}^2 \text{ Note this is not obvious}\end{aligned}$$

   □

Claim: $\mathbb{R}^2 = span\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\}$

Proof. We can easily prove $\supseteq$. Let's prove $\subseteq$
Let $\overrightarrow{v} \in \mathbb{R}^2$, say $\overrightarrow{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$. For $\overrightarrow{v}$ to be in $span\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\}$ we must be able to find $a, b \in \mathbb{R}$ such

that

$$\iff \overrightarrow{v} = \begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}3\\4\end{bmatrix}$$
$$\iff v_1 = a + 3b, v_2 = a + 4b$$
$$\to v_2 - v_1 = b$$
$$a = v_1 - 3b = v_1 - 3(v_2 - v)1$$
$$a = 4_1 - 3v_2$$

$\therefore \begin{bmatrix}v_1\\v_2\end{bmatrix} = (2v_1 - 3v_2)\begin{bmatrix}1\\1\end{bmatrix} + (v_2 - v_1)\begin{bmatrix}3\\4\end{bmatrix}$ $\square$

Problem 5

1. T/F: $\mathbb{R}^3 = span\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\}$

   Answer is false.

$$\begin{aligned}\mathbb{R}^3 &= span\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\}\\ &= \{a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}1\\1\\0\end{bmatrix} : a, b \in \mathbb{R}\}\\ &= \{\begin{bmatrix}a+b\\b\\0\end{bmatrix} : a, b \in \mathbb{R}\}\\ &\neq \mathbb{R}^3 \text{ because it does nto contain } \begin{bmatrix}0\\0\\1\end{bmatrix}\end{aligned}$$