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January 7, 2022
Math 138

1 Integration

Definitions that our prof briefly mentioned but did not go over carefully:

Definition: Right-hand Riemann Sum

The *right-hand Riemann Sum* for f with respect to the partition P is the Riemann sum R obtained from P by choosing c_i to be t_i, the right-hand endpoint of $[t_{i-1}, t_i]$. That is

$$R = \sum_{i=1}^{n} f(t_i)\Delta t_i$$

If $P^{(n)}$ is the regular n-partition, we denote the right-hand Riemann sum by:

$$\begin{aligned} R_n = \sum_{i=1}^{n} f(t_i)\Delta t_i &= \sum_{i=1}^{n} f(t_i)\frac{b-a}{n} \\ &= \sum_{i=1}^{n} f(a + i(\frac{b-a}{n}))(\frac{b-a}{n}) \end{aligned}$$

Definition: Left-hand Riemann Sum

The *right-hand Riemann Sum* for f with respect to the partition P is the Riemann sum R obtained from P by choosing c_i to be t_i, the right-hand endpoint of $[t_{i-1}, t_i]$. That is

$$L = \sum_{i=1}^{n} f(t_{i-1})\Delta t_i$$

If $P^{(n)}$ is the regular n-partition, we denote the right-hand Riemann sum by:

$$\begin{aligned} R_n = \sum_{i=1}^{n} f(t_{i-1})\Delta t_i &= \sum_{i=1}^{n} f(t_{i-1})\frac{b-a}{n} \\ &= \sum_{i=1} nf(a + (i-1)(\frac{b-a}{n}))(\frac{b-a}{n}) \end{aligned}$$

1.2 Riemann Sums and the Definite Integral

Definition

The definite integral of a function f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x; \quad \Delta x = \frac{b-a}{n}, x_i \in [x_{i-1}, x_i]$$

If this limit exists, then f is said to be integrable on the inteveral $[a, b]$

Compare to textbook definition:

Definition

We say that a bounded function f is *integrable* on $[a, b]$ if there exists a unique number $I \in \mathbb{R}$ such that if whenever $\{P_n\}$ is a sequence of partitions with $\lim_{n\to\infty} ||P_n|| = 0$ and $\{S_n\}$ is any sequence of Riemann sums associated with the P_n's, we have

$$\lim_{n\to\infty} S_n = I$$

In this case, we call the I the integral of f over $[a, b]$ and denote it by:

$$\int_a^b f(t)\,dt$$

Important notes:

- the width Δx is not the same
- the sum $\sum_{i=1}^{n} f(x_i)\Delta x$ is called a Riemann sum
- the symbol $\int$ is called an integral sign
- the definite integral represnts a number, does not depend on x, called a <u>dummy variable</u> since it can be replaced by another variable
- if $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx$ can be interprested as the area A

Theorem 1.1: Why does our prof not mention this?

Let f be continuous on $[a, b]$. Then f is integrable on $[a, b]$. Moreover

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} S_n$$

where

$$S_n = \sum_{i=1}^{n} f(c_i)\Delta t_i$$

is any Riemann summ associated with the regular n-partitions. In particular:

$$\int_a^b f(t)\,dt = \overset{n\to\infty}{\lim} R_n = \lim_n \to \infty \sum_{i=1}^{n} f(t_i)\frac{b-a}{n}$$

and

$$\int_a^b f(t)\,dt = \overset{n\to\infty}{\lim} L_n = \lim_n \to \infty \sum_{i=1}^{n} f(t_{i-1})\frac{b-a}{n}$$

Theorem 1.2: No name

If f is continuous on $[a, b]$, then f is integrable, that is, the definite integral $\int_a^b f(x)\,dx$ exists.

<ins>**Note:**</ins> If f is discontinuous, then $\int_a^b f(x)\,dx$ may exist. For example if f has a jump discontinuity as shown, then $\int_a^b f(x)\,dx$ exists.

1.3 Properties of the Definite Integral

Theorem 1.3: Properties of Integrals

Assume that f and g are integrable on the interval $[a, b]$. Then:

1. For any $c \in \mathbb{R}$, $\int_a^b c \cdot f(t)\,dt = c\int_a^b f(t)\,dt$
2. $\int_a^b (f+g)(t)\,dt = \int_a^b f(t)\,dt + \int_a^b f(t)\,dt$
3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
4. $\int_a^a f(x)\,dx = 0$
5. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
6. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$
7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq g(x)$
8. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$ (Integral Squeeze Theorem)
9. $|\int_a^b f(x)\,dx| \leq \int_a^b |f(x)|\,dx$ (Integral Triangle Inequality)
10. If $f(x)$ is an odd function, that is, $f(-x) = -f(x)$, then
$$\int_{-a}^{a} f(x)\,dx$$
11. If $f(x)$ is an even function, that is, $f(-x) = f(x)$, then
$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

1.4 Average Value of a Function

Recall

Recall that the average value of n real numbers, $\alpha_1, \alpha_2, \cdots, \alpha_n$ is given by:

$$\frac{\alpha_1, \alpha_2, \cdots, \alpha_n}{n} = \frac{\sum_{i=1}^{n} \alpha_i}{n}$$

If we denote the average value of $f(x)$ by $\tilde{f}$, then we expect that

$$\tilde{f} \approx \frac{\sum_{i=1}^{n} f(x_i)(\frac{b-a}{n})}{b-a}$$

We can prove this.

Proof. Suppose $f(x)$ has a min value of m and a max value of M on $[a, b]$. Then we expect that $m \leq \tilde{f} \leq M$. We can show this as follows:

$$\begin{aligned} m \leq & f(x) \leq M, a \leq x \leq b \\ m(b-a) \leq & \int_a^b f(x)\,dx \leq M(b-a) \quad 1.3 \\ m \leq & \frac{1}{b-a} \int_a^b f(x)\,dx \leq M \quad \text{Integral MVT} \end{aligned}$$

Since $f(x)$ is continuous, then by IVT $\exists c \in [a, b]$ such that

$$\tilde{f} = f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$$

or

$$f(c)(b-a) = \int_a^b f(x)\,dx$$

□