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Math 136

1 Vectors in $\mathbb{R}^n$

1.6 Projection, Components, and Perpendicular

Definition

Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ with $\vec{w} \neq 0$. The **projection of** $\vec{v}$ **onto** $\vec{w}$ is defined by

$$\text{proj}_{\vec{w}} = (\vec{v}) - \frac{(\vec{w} \cdot \vec{v})}{||\vec{w}||^2} = \frac{(\vec{v} \cdot \vec{w})}{\vec{w} \cdot \vec{w}}\vec{w}$$

Example

Suppose: $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ What is $\text{proj}_{e_1} \vec{v}$?

Solution:

$$\begin{aligned} proj_{\vec{e^1}} \vec{v} &= \frac{\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{|| \begin{bmatrix} 1 \\ 0 \end{bmatrix} ||^2} \\ &= \frac{2}{1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 0 \end{bmatrix} \end{aligned}$$

Example

Suppose: $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ What is $\text{proj}_{-e_1} \vec{v}$?

Solution:

$$\begin{aligned} prof_{-\vec{e_1}} \vec{v} &= \frac{\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix}}{|| \begin{bmatrix} -1 \\ 0 \end{bmatrix} ||^2} \\ &= \frac{-2}{1} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 0 \end{bmatrix} \end{aligned}$$

Definition

Let $\overrightarrow{v}, \overrightarrow{w} \in \mathbb{R}^n$ with $\overrightarrow{w} \neq 0$. The **projection of** $\overrightarrow{v}$ **onto** $\overrightarrow{w}$ is defined by

$$\text{perp}_{\overrightarrow{w}} = \overrightarrow{v} - proj_{\overrightarrow{w}}(\overrightarrow{v})$$

Properties:

1. $proj_{\overrightarrow{w}(\overrightarrow{v})}$ is perpendicular to $perp_{\overrightarrow{w}(\overrightarrow{v})}$
2. $proj_{\overrightarrow{w}}(c\overrightarrow{v}) = c \cdot proj_{\overrightarrow{w}}(\overrightarrow{v})$
3. $proj_{\overrightarrow{w}}(\overrightarrow{v} + \overrightarrow{u}) = proj_{\overrightarrow{w}}(\overrightarrow{v}) + proj_{\overrightarrow{w}}(\overrightarrow{v})$
4. $proj_{\overrightarrow{w}}(proj_{\overrightarrow{w}}(\overrightarrow{v})) = proj_{\overrightarrow{w}}(\overrightarrow{v})$

Proof of 4:

Proof.

$$\begin{aligned} prof_{\overrightarrow{w}}(proj_{\overrightarrow{w}}(\overrightarrow{v})) &= proj_{\overrightarrow{w}}(\frac{\overrightarrow{v} \cdot \overrightarrow{w}}{||\overrightarrow{w}||^2 \overrightarrow{w}}) \\ &= \frac{\overrightarrow{v} \cdot \overrightarrow{w}}{||\overrightarrow{w}||^2} \cdot \frac{\overrightarrow{w} \cdot \overrightarrow{w}}{||\overrightarrow{w}||^2} \overrightarrow{w} \\ &= \frac{\overrightarrow{v} \cdot \overrightarrow{w}}{||\overrightarrow{w}||^2} \overrightarrow{w} \\ &= proj_{\overrightarrow{w}}(\overrightarrow{v}) \end{aligned}$$

□

Standard Inner Project in $\mathbb{C}^n$

Instead of dot product, we define the **Standard inner product.**

Definition

The **standard inner product** of $\overrightarrow{v} = \begin{bmatrix} v_1 \\ v_2 \\ vdots \\ v_n \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{C}^n$ is

$$\langle \overrightarrow{v}, \overrightarrow{w} \rangle = v_1 \overrightarrow{w_1} + v_2 \overrightarrow{w_2} + \cdots + v_n \overrightarrow{w_n}$$

Definition

The **length** of the vector $\overrightarrow{v} \in \mathbb{C}^n$ is $||\overrightarrow{v}|| = \sqrt{\overrightarrow{v} \cdot \overrightarrow{v}}$

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Theorem 1.1: Property 1.5.3

1. $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \in \mathbb{R}$
2. $\overrightarrow{u} \cdot \overrightarrow{v} = \overrightarrow{v} \cdot \overrightarrow{u}$
3. $(\overrightarrow{u} + \overrightarrow{v}) \cdot \overrightarrow{w} = \overrightarrow{u}\overrightarrow{w} + \overrightarrow{v}\overrightarrow{w}$
4. $(\overrightarrow{u} \cdot \overrightarrow{v}) \cdot \overrightarrow{w} = \overrightarrow{v} \cdot (\overrightarrow{u} \cdot \overrightarrow{w})$
5. $\overrightarrow{v} \cdot \overrightarrow{v} \geq 0$

Geometry in $\mathbb{R}^2$

Definition

The **length** of the vector $\overrightarrow{v} \in \mathbb{R}^n$ is $||\overrightarrow{v}|| = \sqrt{\overrightarrow{v} \cdot \overrightarrow{v}}$

Aside: In $\mathbb{R}^1$:

$$||\overrightarrow{v}|| = ||[v_1]|| = \sqrt{v_1^2} = |v|$$

Theorem 1.2: Properties of Length

1. $||\overrightarrow{0}|| = 0$
2. $||c \cdot \overrightarrow{v}|| = |c| \cdot ||\overrightarrow{v}||$
3. $||\overrightarrow{v} + \overrightarrow{u}|| \neq ||\overrightarrow{v}|| + ||\overrightarrow{u}||$
4. $||\overrightarrow{v} + \overrightarrow{u}|| \leq ||\overrightarrow{v}|| + ||\overrightarrow{u}||$

Importance of dot product: It gives angles between vectors in $\mathbb{R}^2$!

Definition

$\overrightarrow{v} \in \mathbb{R}^n$ is a **unit vector** if $||\overrightarrow{v} = 1||$

Definition

When $\overrightarrow{v} \in \mathbb{R}^n$ is a non-zero vector, we can produce a unit vector

$$\hat{v} = \frac{\overrightarrow{v}}{||\overrightarrow{v}||}$$

in the direction of $\overrightarrow{v}$ by scaling $\overrightarrow{v}$. This process is called normalization.

Definition

Let $\overrightarrow{v}$ and $\overrightarrow{u}$ be non-zero vectors in $\mathbb{R}^n$. The angle θ, in radians $(0 \leq \theta\pi)$, between $\overrightarrow{u}$ and $\overrightarrow{v}$ is such that

$$\overrightarrow{v} \cdot \overrightarrow{u} = ||\overrightarrow{u}|| \cdot ||\overrightarrow{v}|| \cos\theta, \text{ that is } \theta = \arccos(\frac{\overrightarrow{u} \cdot \overrightarrow{v}}{||\overrightarrow{v}|| \cdot ||\overrightarrow{u}||})$$

Example

Problem: Given 2 vectors, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ find θ.

Solution:

$$\begin{aligned} \cos\theta &= \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{|| \begin{bmatrix} 0 \\ 1 \end{bmatrix} || \cdot || \begin{bmatrix} 1 \\ 0 \end{bmatrix} ||} \\ &= \frac{0}{1} = 0 \\ \theta &= \frac{\pi}{2} \end{aligned}$$

Definition

Let $\overrightarrow{u}, \overrightarrow{v} \in \mathbb{R}^n$. We say $\overrightarrow{u}$ and $\overrightarrow{v}$ are <u>perpendicular</u> (or <u>orthogonal</u>) if $\overrightarrow{u} \cdot \overrightarrow{v} = 0$

Example

Problem: Find a non-zero vector in $\mathbb{R}^2$ that is orthogonal to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Solution: $\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \end{bmatrix}$

Example

Problem: Find a non-zero vector perpendicular to $\begin{bmatrix} a \\ b \end{bmatrix}$

Solution:

$$\begin{aligned} &\begin{bmatrix} -b \\ a \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} \\ =& -ba + ab \\ =& 0 \end{aligned}$$

Therefore, $\begin{bmatrix} -b \\ a \end{bmatrix}$ is a solution.