Edward Song
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Math 138

1 Integration

Recall

Differential calculus focused on the tangent line problem. The lesson will concentrate on integral calculus concerning the area problem and the connection between differential and integral calculus.

1.1 Areas Under Curves

High school review.

1.2 Riemann Sums and the Definite Integral

Riemann Sum in Textbook

Given a bounded function f on $[a,b]$, a partition P

$$a = t_0 < t_1 < t_2 < ... < t_{i-1} < t_i < ... < t_{n-1} < t_n = b$$

of $[a,b]$, and a set $\{c_1, c_2, ..., c_n\}$, where $c_i \in [t_{i-1}t_i]$, then a *Riemann Sum* for f with respect to P is a sm of the form.

$$\sum_{i=1}^{n} f(c_i)\Delta t_i$$

Remark: If each subinterval is of equal length, then $\Delta x = \frac{b-a}{n}$ and $x_j = a + (\frac{b-a}{n})j, j = 0, 1, 2, ..., n$. Also, $\Delta x \to 0$ as $n \to \infty$
Similarly from the textbook:

Regular n-Partition

Given an interval $[a,b]$ and an $n \in \mathbb{N}$, *regular n-partition* of $[a,b]$ is the partition $P^{(n)}$ with

$$a = t_0 < t_1 < t_2 < ... < t_{i-1} < t_i < ... < t_{n-1} < t_n = b$$

of $[a,b]$ where each subinterval has the *same* length $\Delta t_i = \frac{b-a}{n}$

Example

Problem: Estimate the area between $y = f(x) = \sqrt{1-x^2}$ and the x-axis between $x = 0, x = 1$

Solution: The area represents $\frac{1}{4}$ of the area of a circle of radius 1, so $A = \frac{pi}{4} \approx .785$. Using Riemann sums, the value of the area can be bounded by a lower estimate and an upper estimate, e.g left-most and right-most y values. Choosing the subintervals to be of equal length gives $\Delta x = \frac{1}{n}$ so:

$$A \approx \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n} \frac{1}{n}\sqrt{1-c_i^2}, \leq c_i \leq x_i$$

If we choose $c_i = x_i = \frac{1}{n}$, this will yield a lower estimate, A_L where:

$$A_L = \frac{1}{n}\sqrt{1 - \frac{1-(i-1)^2}{n^2}}$$

Thus, $A_L < A < A_u$ for all n. You can also numerically verify this. I do not want to.

The Connection Between Differential and Integral Calculus

Consider an arbitrary function $f(x)$ that is continuous on the closed interval $[a, b]$. Define the area function, $A(x)$ as the area under the curve from point a to an arbitary point x in the interval $[a, b]$. Newton and Leibniz independently discovered the relationship between derivative of the area function $A(x)$ and the curve $f(x)$. They found that the area could be obtained by reversing the process of differentiation.

Consider:

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h}$$

Then for $h > 0$, $A(x+h) - A(x)$ represents the difference of two areas: the area between a and $x + h$ minus the area between a and x as shown above, This difference of areas can be approximated by the area of a rectangle with base h and height $f(c)$ where $x \leq c \leq x + h$. Thus, $\frac{A(x+h)-A(x)}{h} \approx \frac{f(c)h}{h} = f(c)$; in the limit as $h \to 0$ we have

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h\to 0} f(c)$$