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Math 138

1 Integration

1.7 Method of Substitution - Change of Variables

The integration technique corresponds to the integral version of the chain rule.

Definition: Recall

Recall that: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

In integral form, this becomes :$\int f'(g(x))g'(x)dx = f(g(x)) + C$

Another way of showing this is by introducing a change of variable. Let $u = g(x)$ then $\frac{du}{dx} = g'(x)$ or in differential form we have $du = g'dx$:

$$\int f'(g(x))g'(x)dx = \int f'(u)du = f(u) + C = f(g(x)) + C$$

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Problem 1

1. $\int \frac{x}{x^2+1} dx$:

   Let $u = x^2 + 1$, then $du = 2xdx \rightarrow xdx = \frac{1}{2}du$. This,

   $$\int \frac{xdx}{x^2+1} = \frac{1}{2}\frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C$$

   Note: The substitution $u = x^2$ also works.

2. $\int \frac{\sin(3\ln x)}{x} dx$:

   Let $u = 3\ln x$, then $du = \frac{3}{x}dx \rightarrow \frac{dx}{x} = \frac{1}{3}du$. This gives:

   $$\begin{aligned}\int \frac{\sin(3\ln x)}{x} dx &= \frac{1}{3}\int \sin u du \\ &= -\frac{1}{3}\cos u + C \\ &= -\frac{1}{3}\cos(3\ln x) + C\end{aligned}$$

   Note: $u = \ln x$ also works.

3. $\int (x^2 e^{x^3}) dx$ ::

   Let $u = x^3$, then $du = 3x^2dx \rightarrow x^2dx = \frac{1}{3}du$. This gives:

   $$\begin{aligned}\int x^2 e^{x^3} dx = \frac{1}{3}\int e^u du &= \frac{1}{3}e^u + C \\ &= \frac{1}{3}e^{x^3} + C\end{aligned}$$

Problem 2

1. $\int \sqrt{1+e^x}dx$:
   Let $u = 1 + e^x$, then $du = e^x dx$. This gives:
$$\begin{aligned}\int e^x\sqrt{1+e^x}dx = \int \sqrt{u}du &= \frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{3}(1+e^x)^{\frac{3}{2}} + C\end{aligned}$$

2. $\int \frac{dx}{x^2+4x+5}$:
   Completing the square is sometimes helpful. Rewrite $x^2 + 4x + 5 = (x + 2)^2 + 1$. Let $u = x + 2$, then $du = dx$:
$$\begin{aligned}\int \frac{dx}{x^2+4x+5} &= \int \frac{dx}{1+(1+x)^2} \\ &= \int \frac{du}{1+u^2} \\ &= \tan^{-1} u + C = \tan^{-1}(x+2) + C\end{aligned}$$

3. $\frac{dx}{1+e^x}$:
   Rewrite integrand as $\frac{1}{1+e^x} = \frac{e^{-x}}{1+e^{-x}}$. Then:
$$\begin{aligned}\int \frac{dx}{\sqrt{e^{2x}-1}} &= \int \frac{e^{-x}dx}{\sqrt{1-u^2}} \\ &= -\int \frac{du}{1-u^2} \\ &= -\sin^{-1} u + C \\ &= -\sin^{-1} e^{-x} + C\end{aligned}$$

4. $\int \frac{dx}{\sqrt{e^{-2x}-1}}$:
   Rewrite integrand as
$$\frac{1}{\sqrt{e^{2x}-1}} = \frac{1}{e^x\sqrt{1-e^{-2x}}} = \frac{e^{-x}}{1-\sqrt{1-e^{-2x}}}$$
   Let $u = e^{-x}, du = -e^{-x}dx$. Then:
$$\begin{aligned}\int \frac{dx}{e^{2x}-1} &= \int \frac{e^{-x}dx}{\sqrt{1-e^{-2x}}} \\ &= -\int \frac{du}{\sqrt{1-u^2}} = -\sin^{-1} u + C \\ &= -\sin^{-1} e^{-x} + C\end{aligned}$$

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Problem 3

1. $\int \tan x dx$. Let $u = \cos x, du = -\sin x dx$

$$\begin{aligned}\int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\ &= -\int \frac{du}{u} \\ &= -\ln|u| + C \\ &= -\ln|\cos x| + C \\ &= \ln|\sec x| + C\end{aligned}$$

   Note: For trig integrals, it is sometimes helpful to express the integrand in terms of sine and cosine.

2. $\int \sin^3 x dx$: Rewrite integrand as $\sin^3 x = \sin x(1-\cos^2 x)$. Then:

$$\begin{aligned}\int \sin^3 xd &= \int \sin x(1-\cos^2 x)dx \\ &= -\int (1-u^2)du \\ &= -u + \frac{1}{3}u^3 + C = -\cos x + \frac{1}{3}\cos^3 x + C\end{aligned}$$

   Note: This approach works for odd powers of $\sin x, \cos x$.

3. $\int \sec^2 \theta \tan^2 \theta d\theta$: Let $u = \tan\theta, du = \sec^2\theta d\theta$. Then:

$$\begin{aligned}\int \sec^2\theta\tan^2\theta d\theta = \int u^2 du &= \frac{1}{3}u^3 + C \\ &= \frac{1}{3}\tan^3\theta + C\end{aligned}$$

   Note: This approach works for odd powers of $\sin x, \cos x$.

4. $\int (\frac{x+2}{x^2+1})dx$.

$$\begin{aligned}\int \frac{x+2}{x^2+1}dx &= \int \frac{x}{x^2+1}dx + 2\int \frac{dx}{1+x^2} \\ &= \frac{1}{2}\ln|1+x^2| + 2\tan^{-1} x + C\end{aligned}$$

More on Substitution

Consider the definite integral:

$$\int_a^b f(g(x))g'(x)d$$

then the substitution $u = g(x), du = g'dx$ transforms the integral to:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

When making a substitution to evaluate a definite integral remember to change the limits of integration.

Problem 3

1. Evaluate $\int_0^8 \frac{\cos(\sqrt{x+1})}{\sqrt{x+1}}dx$: Let $u = \sqrt{x+1}, du = \frac{dx}{2\sqrt{x+1}}$.

$$\int_0^8 \frac{\cos(\sqrt{x+1})}{\sqrt{x+1}}dx = 2\int_1^3 \cos u du = 2\sin u\Big|_1^3 = 2(\sin 3 - \sin 1)$$

2. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x dx}{1+\sin x}$. Let $u = \sin x, du = \cos x dx$:

$$\int_0^{\frac{\pi}{2}} dx = \int_0^1 \frac{du}{1+u} = \ln|1+u|\Big|_0^1 = \ln 2 - \ln 1 = \ln 2$$

3. Evaluate $\int_{-\pi}^{\pi} \frac{t^4 \sin t}{1+t^8}dt$. Note that $f(t) = \frac{t^4 \sin t}{1+t^8}$ is an odd function since $f(-t) = -f(t)$. Hence,

$$\int_{-\pi}^{\pi} \frac{t^4 \sin t}{1+t^8}dt = 0$$

since limits are symmetric, recall properties of the definite integral.