Econ 123 Review Handout 2: Binomial, Normal and t- Distributions

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This handout reviews Bernoulli and Binomial distributions, and connects them to indicator/dummy variables. It also reviews the normal distribution, the distribution of the absolute value of a normal random variable, and the t-distribution. We will use these distributions extensively for inference.

Contents

Bernoulli Distribution

EXAMPLE 1 OF HANDOUT 1 considered the random variable for a weighted coin toss, with $X = 1$ if heads and $X = 0$ if tails, and $\Pr[X = 1] = p$, $\Pr[X = 0] = 1 - p$. Such a random variable is called a Bernoulli random variable, and its distribution is called the Bernoulli distribution:

Definition 1: Bernoulli Random Variable

A random variable X is distributed *Bernoulli(p)*, $X \sim \text{Bernoulli}(p)$, if:

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p. \end{cases}$$

For a Bernoulli random variable, we call $X = 1$ a "*success*" and $X = 0$ a "*failure*".

The terminology of "success" and "failure" need not correspond to good and bad outcomes. For example, $X = 1$ might indicate that a patient dies and $X = 0$ that the patient lives, without implying that death is a good outcome. Likewise, a machine failing or a dam failing might be a "success" despite the tension in the terminology.

While we motivated the Bernoulli distribution using a coin toss, it is the distribution for any *binary variable* taking the values 0 and 1, and is how we encode as a variable yes-no and true-false answers. It is often productive in probability theory (as in economics, as in coding) to abstract from details. Thus, instead of trying to separately consider random variables for whether an individual is employed or not, or a person tests positive for a virus or not, or a machine fails or not, or a dam bursts or not, we instead study Bernoulli random variables and think of hypothetical experiments involving weighted coin flips with our abstract model of weighted coin flips describing any yes-no, true-false type variable, just with different probabilities of "success" or "failure."

In Handout 1, Example 1, we already derived the expectation and variance of a Bernoulli random variable, which we restate here in the following theorem:

Theorem 1: Expected Value and Variance of a Bernoulli r.v.

Suppose $X \sim \text{Bernoulli}(p)$. Then

$$\begin{aligned} \mathbb{E}[X] &= p, \\ \text{Var}[X] &= p \cdot (1-p). \end{aligned}$$

Logical Indicator Function and Indicator Variables

We will often refer to variables representing whether some event is true or not as indicator variables, where the variable equals 1 if the event is true, and equals 0 otherwise. Indicator variables are also called dummy variables. We will often construct indicator variables using logical indicator functions, defined as follows:

Definition 2: Logical Indicator Function

Let $\mathbb{1}$ denote the *logical indicator function*, defined for any statement (event) A as

$$\mathbb{1}[A] = \begin{cases} 1 & \text{if } A \text{ is true,} \\ 0 & \text{if } A \text{ is false.} \end{cases}$$

We define indicator variables using logical indicator functions:

Definition 3: Indicator Variables

X is an *indicator variable*, also called a *dummy variable*, if, for some given event A, $X = \mathbb{1}[A]$.

If X is an indicator variable for some event A, $X = \mathbb{1}[A]$, then X equals 1 when A is true and equals 0 when A is false. Thus, the indicator variable $X = \mathbb{1}[A]$ is a Bernoulli(p) random variable with $p = \Pr[A]$. For example, we might define $X = \mathbb{1}[\text{employed}]$ as a indicator variable for the event that an individual is employed, which is distributed Bernoulli(p) where p equals probability of being employed. From Theorem 1, we see that the expected value of a indicator variable is the probability that the event occurs, and that the variance of an indicator variable is relatively large when the probability of the event is close to a half and is small when the probability of the event is close to 0 or 1.

We will also often use logical indicator functions to construct indicator variables from some other non-binary random variable, as illustrated by the following examples.

Example 1 (Positive Returns). *Let r_A denote the return on an asset, and define X by $X = \mathbb{1}[r_A > 0]$. Then X is a indicator variable for the asset having a positive return and $\mathbb{E}[X] = \Pr[r_A > 0]$.*

Example 2 (Schooling). *Let S denote an individual's years of schooling. Then $X = \mathbb{1}[S \geq 12]$ is a indicator variable for having completed high school, and $\mathbb{E}[X] = \Pr[S \geq 12]$.*

Dummy variables are pervasive in econometrics, representing, for example, an individual working or not working, having graduated from college or not, being retired or not, being married or not, having children or not, having positive savings or not, having investments in the stock market or not, and so forth.

Binomial Distribution

CONSIDER FLIPPING A (POSSIBLY WEIGHTED) COIN n times, each flip independent of every other, and counting the number of heads. More formally, suppose $X_1, ..., X_n$ denotes i.i.d. Bernoulli(p) random variables so that $S_n = \sum_{i=1}^{n} X_i$ is number of successes in the n tosses. Then S_n is called a Binomial random variable, and its distribution is called the Binomial(n,p) distribution:

Definition 4: Binomial Distribution

Suppose $X_1, ..., X_n$ are i.i.d. Bernoulli(p) random variables, and let $S_n = \sum_{i=1}^{n} X_i$. We will refer to each X_i as a Binomial trial, and to S_n as the number of successes in n trials. Then S_n is distributed *Binomial(n, p)*, $S_n \sim \text{Binomial}(n, p)$, with

$$\Pr[S_n = k] = \binom{n}{k} p^k (1-p)^{n-k} \tag{1}$$

for $k = 0, ..., n$, where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{2}$$

By their definitions, a Bernoulli(p) random variable is the same as a Binomial(1,p) random variable..

There is a simple intuition for Equation 1. By independence of the Bernoulli trials, any given sequence with k successes (and thus $n-k$ failures) has probability $p^k(1-p)^{n-k}$. For a sequence of size n, there are $\binom{n}{k}$ sequences that result in k succeses. Equation 1 then follows.

The Binomial(n,p) is the number of "successes" of n independent Binomial trials. We will often abstract from any particular context and think of a hypothetical experiment of flipping a weighted coin n times, each flip independent of every other flip, and counting the number of heads, but that hypothetical experiment is a model of any true-false, yes-no outcomes, for example, number of individuals in our sample who are employed, number of investors who invest in the stock market, number of patients who survive, or number of subjects who test positive for an illness as long as the underlying events are independent and have the same probability of success.

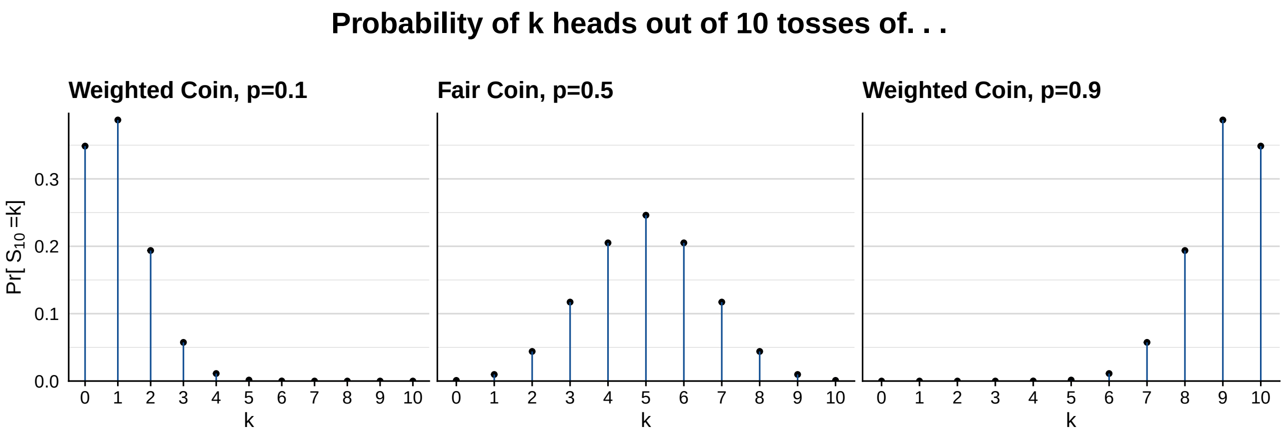


Figure 1: Plotting $\Pr[S_{10} = k] = \binom{10}{k} p^k (1-p)^{10-k}$ for $p = 0.2, 0.5$, and 0.8.

Using rules for expectations of sums (Handout 1, Theorem 7) and variance of the sum of independent and thus uncorrelated random variables (Handout 1, Theorem 8), we have the following result:

Theorem 2: Expected Value and Variance of a Bernoulli r.v.

Suppose $S_n \sim \text{Binomial}(n, p)$. Then

$$\mathbb{E}[S_n] = n \cdot p$$
$$\text{Var}[S_n] = n \cdot p \cdot (1 - p)$$

The result of Theorem 2 is intuitive. For example, consider flipping a possibly weighted coin ten times. If it is a fair coin, the expected number of heads is 5 heads out of 10. If the probability of heads is 0.2, then the expected number of heads is 2 heads out of 10. Note that our use of the word "expected" is different from the normal dictionary use of the word. For example, if you flip a fair coin 3 times, your expected number of heads is 1.5 heads out of 3, even though you will never flip exactly 1.5 heads out of 3.

Remark 1 (Sample Mean of Independent Bernoulli Trials)**.** *Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, the sample mean of the n Bernoulli trials which is equivalent to the fraction of the n Bernoulli trials that are a success. For example, if X_i is an indicator variable for being employed, then $\bar{X}_n$ is the fraction of the sample that is employed. Since $\bar{X}_n = S_n/n$, we can use Theorem 1 along with Handout 1, Theorem 1 to obtain*

$$\mathbb{E}[\bar{X}_n] = p.$$
$$Var[\bar{X}_n] = p \cdot (1-p)/n.$$

Furthermore, we can obtain the (exact) distribution of the sample mean of n i.i.d. Bernoulli(p) random variables from the Binomial(n,p) distribution:

$$\Pr[\bar{X}_n = t] = \Pr[S_n = n \cdot t] \textit{ where } S_n \sim \textit{Binomial}(n,p). \tag{3}$$

While we can also obtain a normal approximation to (standardized) $\bar{X}_n$ as n going to infinity using the Central Limit Theorem (CLT), the normal approximation from the CLT will be poor when n is small or p is close to 0 or 1, while the distribution for $\bar{X}_n$ from equation (3) is exact for any size n.

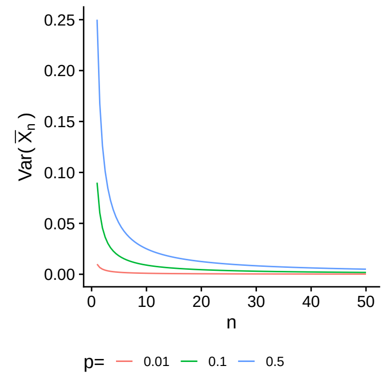


Figure 2: Var($\bar{X}_n$) for n independent Bernoulli(p) trials with p=0.01, 0.10, and 0.50 .

*Binomial Distribution in **R***

We will typically not work with equation (1) directly, but rather use **R** to calculate probabilities for the binomial distribution.

R Functions for Binomial Distribution

Function	Returns
`dbinom(k, n, p)`	$\Pr[S_n = k]$ for $S_n \sim \text{Binomial}(n,p)$.
`pbinom(k, n, p)`	$\Pr[S_n \leq k]$ for $S_n \sim \text{Binomial}(n,p)$.
`qbinom(q, n, p)`	qth quantile of Binomial(n,p)
`rbinom(m, n, p)`	m random draw of S_n, $S_n \sim \text{Binomial}(n,p)$

$\Pr[S_n > k] = 1 - \Pr[S_n \leq k]$, and thus we can compute $\Pr[S_n > k]$ by `1-pbinom(k,n,p)`. Alternatively, we can use `pbinom(k,n,p,lower.tail=FALSE)`.

```
> # Example: 10 tosses of fair coin,
>
> # prob of 8 heads out of 10
> dbinom(8,10,0.5)
[1] 0.04394531
> # prob of 8 or fewer heads
> pbinom(8,10,0.5)
[1] 0.9892578
> # prob of more then 8 heads
> # (9 or more)
> 1-pbinom(8,10,0.5)
[1] 0.01074219
```

```
> # Simulating 10 flips of a fair coin
> rbinom(10,1,.5)
 [1] 1 0 1 0 1 1 1 0 0 0
> # another 10 flips of a fair coin
> rbinom(10,1,.5)
 [1] 1 1 0 0 1 0 1 1 0 1
> # Simulating sum of 10 flips
> rbinom(1,10,.5)
[1] 7
> # Simulating sum of 10 flips, 5 times
> rbinom(5,10,.5)
[1] 4 2 7 6 5
```

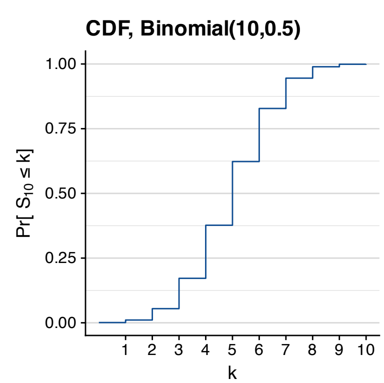


Figure 3: Plotting `pbinom(k, 10, 0.5)`

Normal Distribution

We have thus far considered the Bernoulli(p) and Binomial(n,p) distributions, and we have that many random variables of interest are exactly distributed Bernoulli(p) or Binomial(n,p). We now consider the normal distribution, which is different in that we rarely believe that random variables in practice are exactly normally distributed. However, we often believe that random variables are approximately normally distributed, and normal approximations plays a critical role in large samples due to the Central Limit Theorem.

Definition 5: Normal Distribution

A random variable X is *Normally Distributed* with mean μ and variance σ^2, written $X \sim N(\mu, \sigma^2)$, if the probability density function of X is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

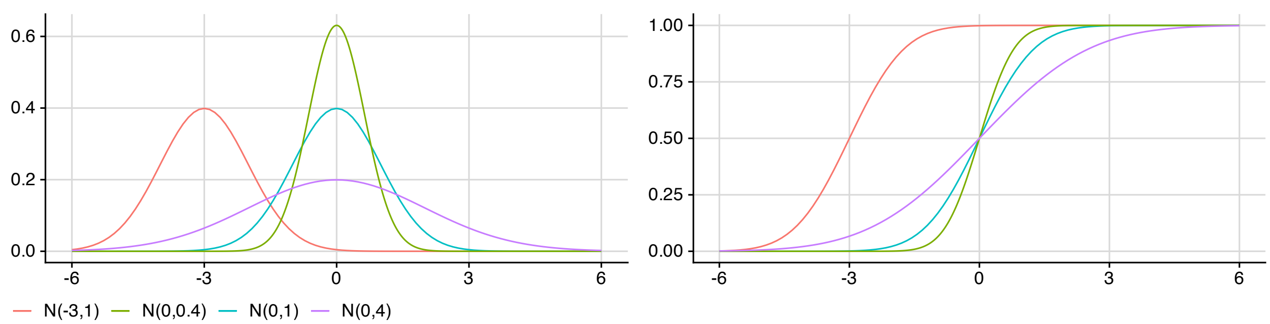


Figure 4: Plotting PDF (left) and CDF (right) of $N(\mu, \sigma^2)$. Note that μ determines where the density is centered and the density is symmetric around μ, while σ^2 determines the spread of the density. Note that — $N(0,1)$ is standard normal.

Particularly important for hypothesis testing will be the normal distribution with $\mu = 0$ and $\sigma^2 = 1$, called the standard normal distribution.

Definition 6: Standard Normal Distribution

A random variable X is distributed *Standard Normal* if $X \sim N(0,1)$. We denote the pdf of a standard normal distribution by $\phi(x)$, so that

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2},$$

and the CDF of a standard normal by $\Phi(x)$, so that

$$\Phi(t) = \Pr[X \leq t] = \int_{-\infty}^{t} \phi(x)dx.$$

Remark 2 (Symmetry of Std. Normal)**.** *A $N(\mu, \sigma^2)$ density is symmetric around μ, so that the standard normal pdf, $\phi(\cdot)$, is symmetric around 0 . The symmetry of $\phi(\cdot)$ around 0 has the following implications, which will be useful for inference:*

1. $\phi(t) = \phi(-t)$ *for all* t;
2. $\Phi(t) = 1 - \Phi(-t)$ *for all* t;
3. $X \sim N(0, 1)$ *implies* $-X \sim N(0, 1)$.
4. $q_\alpha = -q_{1-\alpha}$ *for all* α*, where* q_α *is the* α *quantile of a standard normal.*

Quantiles of $N(0, 1)$

α	q_α
0.01	-2.33
0.025	-1.96
0.05	-1.64
0.95	1.64
0.975	1.96
0.99	2.33

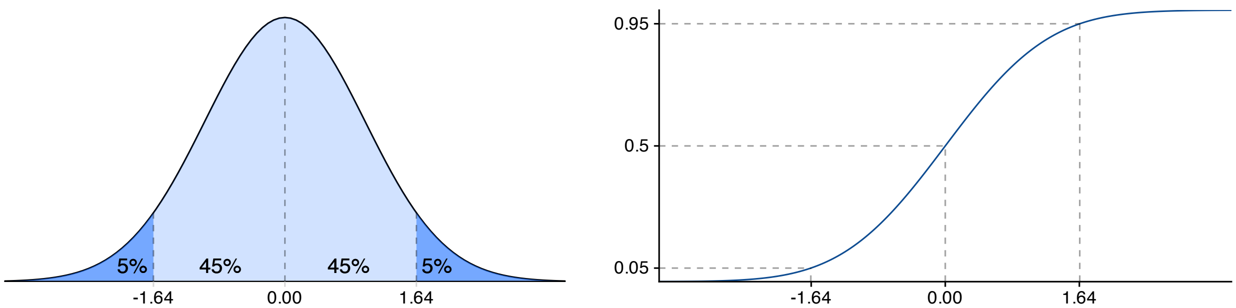


Figure 5: Standard Normal PDF (left) and CDF (right).

We will often work with linear functions of normal random variables.

Theorem 3: Linear Function of Normal

Suppose $X \sim N(\mu, \sigma^2)$. Let a and b denote constants with $b \neq 0$. Then $a + bX \sim N(a + b\mu, b^2\sigma^2)$.

Remark 3 (Converting $N(\mu, \sigma^2)$ r.v. to Std. Normal)**.** *Suppose $X \sim N(\mu, \sigma^2)$. Then, by Theorem 3*

$$\frac{X - \mu}{\sigma} \sim N(0, 1).$$

Converting a $N(\mu, \sigma^2)$ r.v. to a standard normal r.v. allows us to use Φ to find $\Pr[X \leq t]$ for $X \sim N(\mu, \sigma^2)$. In particular,

$$\begin{aligned} \Pr[X \leq t] =& \Pr\left[\frac{X - \mu}{\sigma} \leq \frac{t - \mu}{\sigma}\right] \\ =& \Phi\left(\frac{t - \mu}{\sigma}\right). \end{aligned}$$

For example, if $X \sim N(1, 4)$, then $\Pr[X \leq 2] = \Phi(\frac{2-1}{2}) = \Phi(\frac{1}{2})$.

Consider the sum of independent normal random variables.

Theorem 4: Sum of Independent Normal r.v.'s

Suppose that $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$ and X and Y are independent. Let a and b denote constants with $a \neq 0$ or $b \neq 0$. Then

$$aX + bY \sim N(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2).$$

Iterating on Theorem 4 for i.i.d. random variables leads to the following corollary:

Corollary 5: Mean of i.i.d. Normal Random Variables

Suppose $X_1, X_2, ..., X_N$ are i.i.d. with $X_i \sim N(\mu, \sigma^2)$. Then

$$\bar{X}_N \sim N(\mu, \sigma^2/N).$$

Following Remark 3 and applying Corollary 5, we have that, if $X_1, X_2, ..., X_N$ are i.i.d. with $X_i \sim N(\mu, \sigma^2)$, then

$$\frac{\bar{X}_N - \mu}{\sigma/\sqrt{N}} \sim N(0,1),$$

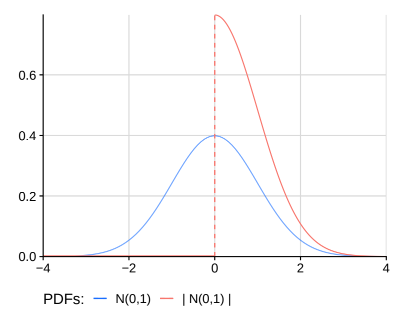
so that

$$\Pr[\bar{X}_N \leq t] = \Phi\left(\frac{t-\mu}{\sigma/\sqrt{N}}\right).$$

Remark 4 (Absolute Value of Std. Normal)**.** *The distribution of the absolute value of a standard normal r.v. is important for inference. Suppose $X \sim N(0,1)$ so that $|X| \sim |N(0,1)|$. Then, using the symmetry of $\phi(\cdot)$,*

- *for $x \geq 0$, the density of $|X|$ at x is $2 \cdot \phi(x)$;*
- *for $x \geq 0$, $\Pr[\ |X| > x\ ] = 2 \cdot \Pr[X > x] = 2\ (1 - \Phi(x))$;*
- *the $1-\alpha$ quantile of $|X|$ equals the $1-\alpha/2$ quantile of X.*

For example, if $X \sim N(0,1)$, then $\Pr[X > 1.64] = 1 - \Phi(1.64) = 0.05$, $\Pr[|X| > 1.64] = 2 \cdot (1 - \Phi(1.64)) = 0.10$, and thus 1.64 is the 0.95 quantile of $X \sim N(0,1)$ which is the 0.90 quantile of $|X|$.



$|N(0,\sigma^2)|$ is called the *half-normal* distribution, so that $|N(0,1)|$ is an example of a half-normal distribution .

Normal Distribution in ***R***

WE WILL TYPICALLY use **R** to calculate probabilities for the normal distribution.

R Functions for Normal Distribution

Function	Returns
`dnorm(x, m, s)`	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-m}{s})^2}$, the $N(m,s^2)$ density evaluated at x,
`pnorm(x, m, s)`	$\Pr[X \leq x]$ for $X \sim N(m,s^2)$.
`qnorm(q, m, s)`	qth quantile of $N(m,s^2)$,
`rnorm(n, m, p)`	n random draw of X, $X \sim N(m,s^2)$.

These functions set $m = 0$ and $s = 1$ by default if their values are not specified. Thus, `dnorm(0.5)` returns the same value as `dnorm(0.5, 0, 1)`.

```
> # 0.05 Quantile of N(0,1)
> qnorm(0.05,0,1)
[1] -1.644854
> # Std. Norm Density at -1.645
> dnorm(-1.6449,0,1)
[1] 0.1031278
> # Prob Std. Norm. Less than -1.645
> pnorm(-1.645,0,1)
[1] 0.04998491
> # Prob Std. Norm. Greater than -1.645
> 1-pnorm(-1.645,0,1)
[1] 0.9500151
> # Default mu=0, sigma=1
> pnorm(-1.645)
[1] 0.04998491
```

```
> # .95 Quantile of N(0,1)
> qnorm(0.95,0,1)
[1] 1.644854
> # Std. Norm Density at 1.645
> dnorm(1.6449,0,1)
[1] 0.1031278
> # Prob Std. Norm. Less than 1.645
> pnorm(1.645,0,1)
[1] 0.9500151
> # Prob Std. Norm. Greater than 1.645
> 1-pnorm(1.645,0,1)
[1] 0.04998491
> # Prob -1.64<Std. Norm.< 1.64
> pnorm(1.645)-pnorm(-1.645)
[1] 0.9000302
```

```
> # Simulating 1 draw from N(0,1)
> rnorm(1,0,1)
[1] 1.207962
> # Another draw, using defaults
> rnorm(1)
[1] -0.4028848
 > # Simulating 3 draws from N(0,1)
> rnorm(3,0,1)
[1]  0.55391765 -0.06191171 -0.30596266
 > # Another 3 draws  from N(0,1)
> rnorm(3)
[1] -0.46665535  0.77996512 -0.08336907
```

```
> # Consider X ~ N(1,4)
> # Pr[X<=2]
> pnorm(2,1,2)
[1] 0.6914625
> # Using Remark 3
> # Pr[X<=2]
> pnorm(0.5,0,1)
[1] 0.6914625
> # Simulating 3 draws from N(1,4)
> # Using Theorem 3
> 1 + 2 * rnorm(3)
[1]  0.7841654 -1.3307927  5.5379119
```

t-Distribution

A FAMILY OF DISTRIBUTIONS CLOSELY RELATED to the standard normal distribution is the t-distribution, which will play a key role in inference.

Definition 7: t-distribution

A random variable X has a *t-distribution* with ν degrees of freedom, written $X \sim t_\nu$, if its probability density function is given by

$$f_X(x) = \frac{\Gamma(\nu+1)/2}{\Gamma(\nu/2)\sqrt{\nu\pi}}(1+\frac{x^2}{\nu})^{-(\nu+1)/2}, \tag{4}$$

where ν is a positive integer and Γ is the gamma function.

In this course, you do not need to remember the formula for the normal or t-densities, and we will never work with them directly but rather use **R** when we need to evaluate them.

The t-distribution is a family of distributions indexed by the parameter ν, called the degrees of freedom. Like the standard normal distribution, the t-distribution is symmetric around zero and bell-shaped. However, the tails of the t-distribution are heavier than those of the normal distribution, with how much heavier depending on the parameter ν. When $\nu = 1$, the distribution is called the *cauchy distribution* and has much heavier tails and very different properties than a standard normal distribution. The larger is ν, the thinner the tails, and the closer the t_ν distribution is to a standard normal distribution. As ν goes to infinity, the t_ν distribution approaches t_∞, which is the $N(0,1)$ distribution.

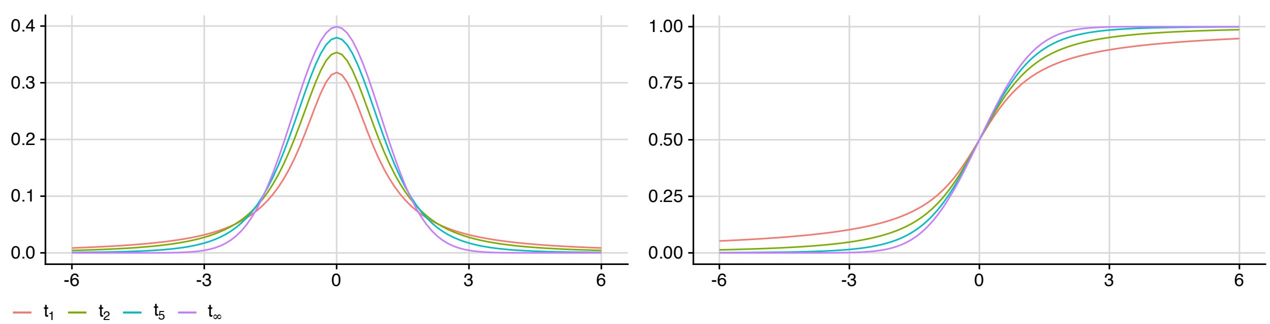
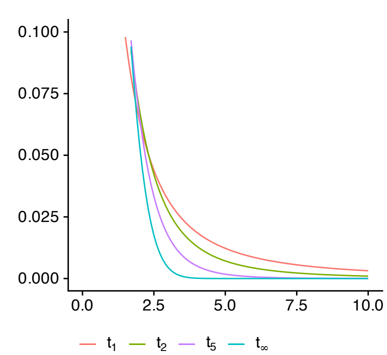


Figure 6: Plotting the t_ν PDF (above left), CDF (above right), and right tail of PDF (below). Note that — t_1 is the Cauchy density, and — t_∞ is the $N(0,1)$ density.



The heavier tails of a t-distribution results in extreme values being more likely for a t-distribution than for a standard normal distribution, especially for ν small. In the extreme case of $\nu = 1$, i.e., a

Examples of t_ν distribution

ν	$\mathbb{E}[X]$	$\text{Var}(X)$	$\Pr[\lvert X\rvert > 3]$	$\Pr[\lvert X\rvert > 5]$	name
1	Does not exist	Does not exist	0.205	0.126	Cauchy
2	0	Does not exist	0.095	0.038	
3	0	3	0.058	0.015	
4	0	2	0.040	0.007	
5	0	$1\frac{2}{3}$	0.030	0.004	
∞	0	1	0.003	0.000	Std. Normal

cauchy distribution, the tails are so heavy that that $\mathbb{E}[X]$ and $\text{Var}(X)$

do not exist. For a t_2 distribution, $\mathbb{E}[X]$ does exists and equals 0, but $\text{Var}(X)$ does not exist. For a t_ν distribution with $\nu \geq 3$, then $\mathbb{E}[X] = 0$ and $\text{Var}(X) = \frac{\nu}{\nu-2}$. For a t_∞ distribution, i.e., a $N(0,1)$ distribution, $\mathbb{E}[X] = 0$ and $\text{Var}(X) = 1$.

A normal distribution can take any value on the whole real line, and thus can take extreme values. However, the tails of a normal distribution go to zero so quickly (i.e., are so thin) that one can essentially ignore the possibility of extreme values with a normal distribution. For example, if $X \sim N(0,1)$, then $\Pr[|X| > 5] = 0.0000006$. In contrast, for a t-distribution with small ν, one cannot ignore the extreme values. For example, if $X \sim t_1$ (Cauchy), $\Pr[|X| > 5] = 0.126$ and X takes extreme values so often that $\mathbb{E}[X]$ and $\text{Var}(X)$ don't exist. Researchers sometimes use the t-distribution, especially Cauchy, when modeling variables where it is important to account for the variables taking extreme values. It is often used in physics, but is also sometimes used to model asset returns and study financial risk in the context of returns that take extreme values with too high of a probability to be ignored.

The Cauchy distribution is also used in probability theory to provide an example where standard results do not hold because $\mathbb{E}(X)$ and $\text{Var}(X)$ do not exist. For example, the LLN and CLT do not hold for the sample mean of i.i.d. draws from a Cauchy. In fact, the sample mean of n i.i.d. Cauchy random variables has the same distribution as one Cauchy random variable.

However, the most common use of a t-distribution in statistics is to model the distribution of the *studentized* mean (also called the *t-statistic* or *t-ratio*) when sampling from a normal distribution. Suppose $X_1, X_2, ..., X_n$ are i.i.d. with $X_i \sim N(\mu, \sigma^2)$. Then it follows from Corollary 5 that $\bar{X}_n \sim N(\mu, \sigma^2/n)$ so that $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$, where it is important to note that the expressions are dividing by the true, population σ which is generally unknown. Suppose we had a value of μ that we hypothesized as the true value, but don't know the value of σ^2, so that we need to estimate it. Define the studentized mean as

$$T_n = \frac{\bar{X}_n - \mu}{s_n/\sqrt{n}} \tag{5}$$

where

$$s_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2. \tag{6}$$

Then $T_n \sim t_{n-1}$, as stated in the following theorem.

Theorem 6: Studentized Mean of i.i.d. Normal r.v.s

Suppose $X_1, X_2, ..., X_N$ are i.i.d. with $X_i \sim N(\mu, \sigma^2)$. Let T_n and s_n^2 be defined by equations (5) and (6). Then $T_n \sim t_{n-1}$.

An implication of Theorem 6 is that when n is small, the Studentized statistic $\frac{\bar{X}_n - \mu}{s_n/\sqrt{n}}$ will be far from normal and will take extreme values relatively frequently because we are dividing by s_n instead of σ. On the other hand, when n is large, the distribution of $\frac{\bar{X}_n - \mu}{s_n/\sqrt{n}}$ will be close to normal and that we are substituting s_n for σ will be unimportant.

*Student-t Distribution in **R***

We will typically use **R** to calculate probabilities for the Student-t distribution.

R Functions for Student-t Distribution

Function	Returns
`dt(x, v)`	the PDF of a t_v density evaluated at x,
`pt(x, v)`	$\Pr[X \leq x]$ for $X \sim t_v$.
`qnorm(q, v)`	qth quantile of t_v,
`rt(n, v)`	n random draw of X, $X \sim t_v$.

```
> # 0.05 Quantile of Cauchy
> qt(0.05,1)
[1] -6.313752
> # 0.05 Quantile of t_5
> qt(0.05,5)
[1] -2.015048
> # 0.05 Quantile of t_infinity
> qnorm(0.05)
[1] -1.644854
> # density of Cauchy at -5
> dt(-5,1)
[1] 0.01224269
> # density of t_5 at -5
> dt(-5,5)
[1] 0.001757438
> # density ot t_infinity at -5
> dnorm(-5)
[1] 0.00000148672
```

```
> # Prob t_1 between -5 and 5
> pt(5,1)-pt(-5,1)
[1] 0.8743341
> # Prob  | t_1 | >5
> 1 - (pt(5,1)-pt(-5,1))
[1] 0.1256659
> # Prob t_5 between -5 and 5
> pt(5,5)-pt(-5,5)
[1] 0.9958953>
# Prob  | t_5 | >5
> 1 - (pt(5,5)-pt(-5,5))
[1] 0.004104716
> # Prob t_infinity between -5 and 5
> pnorm(5)-pnorm(-5)
[1] 0.9999994
> # Prob  | t_infinity | >5  3
> 1 - (pnorm(5)-pnorm(-5))
[1] 0.0000005733031
```

```
> # Simulating 6 draw from Cauchy
 > rt(6,1)
[1] -0.1055119 -3.5727334 -0.1666175
    14.4183067  2.8204201 -0.1955553
> # Another 6 draws from Cauchy
> rt(6,1)
  [1] -0.4865291 -0.4264724  1.2611932
       1.0142861 -0.6185245 -0.4247870
 > # Simulating 6 draws from t_5
> rt(3,5)
[1] -0.7905094 -0.2588564 -0.3107277
     1.3989390  0.7775500 -0.4130591
```

```
> # another 6 draws from t_5
> rt(3,5)
[1]   -2.0170308  1.0184921  1.7427500
       0.3218333  1.1276689 -0.1284361
> # Simulating 6 draws  from t_infinity
> rnorm(6)
[1] -0.6989059  0.4868481 -0.8626596
     1.5333999  0.4064118  0.1865867
> # Another 6 draws  from t_infinity
> rnorm(6)
[1] -1.5115780  2.0045310 -1.7778798
    -0.8635079 -0.1826787  0.2622432
```

Note that many draws from a Cauchy look very similar to draws from t_5 or from $N(0,1)$, though once in a while the Cauchy takes extreme values.

Summary

Table 1 provides optional reading for this handout.[1] [2]

Source	Chapters
Hogg et. al (2019)	2.4, 3.3, 5.5
Woolridge (2020)	B-5a,b,c,e

Table 1: Optional reading

[1] Hogg, R. V., E. A. Tanis, and D. L. Zimmerman (2020). *Probability and statistical inference* (10 ed.). Pearson

[2] Wooldridge, J. M. (2020). *Introductory Econometrics: A Modern Approach* (7 ed.). Cengage Learning

Important Definitions

Def 1: $X \sim$ **Bernoulli(p)** if $\Pr[X = 1] = p$, $\Pr[X = 0] = 1 - p$. We call $X = 1$ a "success" and $X = 0$ a "failure".

Def 2: $\mathbb{1}$ denotes the **logical indicator** function where, for any event A, $\mathbb{1}[A] = 1$ if A is true and $\mathbb{1}[A] = 0$ if A is false.

Def 3: X is an **indicator variable**, also called a **dummy variable**, if, for some given event A, $X = \mathbb{1}[A]$.

Def 4: If $X_1, ..., X_n$ are i.i.d. Bernoulli(p) r.v.'s, then $S_n = \sum_{i=1}^{n} X_i$ is distributed **Binomial(n,p)** with $\Pr[S_n = k] = \binom{n}{k} p^k (1-p)^{n-k}$.

Def 5: X is **Normally Distributed** with mean μ and variance σ^2, written $\boldsymbol{X \sim N(\mu, \sigma^2)}$, if the pdf of X is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

Def 6: X is distributed **Standard Normal** if $X \sim N(0, 1)$. We denote the standard normal pdf by. $\boldsymbol{\phi}(\cdot)$ and the standard normal CDF by $\boldsymbol{\Phi}(\cdot)$.

Def 7: X is distributed according to a **t-distribution** with ν degrees of freedom, written $X \sim t_\nu$, if it's pdf is given by (4).

- When $\nu = 1$, the distribution is called the **Cauchy** distribution.
- As $\nu \to \infty$, the distribution approaches t_∞, the standard normal distribution.

Important Results

Thm 1: If $X \sim$ Bernoulli(p) then $\mathbb{E}[X] = p$, $\mathrm{Var}(X) = p \cdot (1 - p)$.

Important Results (cont'd)

Thm 2: If $S_n \sim$ Binomial(n,p), then $\mathbb{E}[S_n] = n \cdot p$,
$\text{Var}[S_n] = n \cdot p \cdot (1-p)$.

Thm 3: If $X \sim N(\mu, \sigma^2)$, and a and b denote constants with $b \neq 0$, then $a + bX \sim N(a + b\mu, b^2\sigma^2)$.

Thm 4: If $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$, X and Y are independent, and a and b denote constants with $a \neq 0$ or $b \neq 0$, then
$aX + bY \sim N(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2)$.

Cor 5: Suppose $X_1, X_2, ..., X_N$ are i.i.d. with $X_i \sim N(\mu, \sigma^2)$. Then $\bar{X}_N \sim N(\mu, \sigma^2/N)$ so that $\frac{\bar{X}_N - \mu}{\sigma/\sqrt{N}} \sim N(0, 1)$.

Thm 6: Suppose $X_1, X_2, ..., X_N$ are i.i.d. with $X_i \sim N(\mu, \sigma^2)$. Then $\frac{\bar{X}_N - \mu}{s_n/\sqrt{N}} \sim t_{n-1}$, where $s_n = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}$.

Self-Study Questions

1. From equation (1), find find the probability of 0 successes in n independent Bernoulli(p) trials.
   - Optional: find the same result using basic probability theory and independence of the trials without using (1).
2. From equation (1), find find the probability of exactly 1 successes in n independent Bernoulli(p) trials.
   - Optional: find the same result using basic probability theory and independence of the trials without using (1).
3. Using your answers to questions 1 and 2 above, find the probability of *at least* 2 (i.e., 2 or more) successes in n independent Bernoulli(p) trials.
4. A flood so severe that such a severe flood happens in a given year with probability $p = 0.002$ is called a *five hundred year flood*. Suppose whether such a flood occurs in a given year is independent of whether such a flood occurs in any other year.

   (a) Using Theorem 2, explain why floods with $p = 0.002$ are called five hundred year floods.

(b) Explain why it is possible for a five hundred year flood not to happen in 500 hundred years, or to occur multiple times in 500 hundred years.

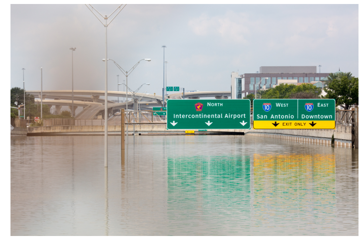


Figure 7: Flooding in Houston, 2019, in aftermath of Hurricane Harvey. (source)

(c) Using your answer to question 3, find the probability that a five hundred years occurs in two or more years out of five.

(d) Severe flooding happening more often than expected has been viewed as evidence of climate change. For example, Houston suffered what were considered five hundred year floods (based on historical data) every year for five years in a row, with some arguing that such frequency for severe floods shows that climate change has increased the probability in a given year of such floods for Houston to be far higher than 0.002 and that floods that severe should no longer be considered five hundred year floods for Houston.

This question is based on 'The "500-year" flood, explained: why Houston was so underprepared for Hurricane Harvey' from *Vox*. That article was written in 2017 after Houston had just had its third 500-year flood in three years. Houston had 500-year floods again in 2018 and 2019.

i. What is the probability of five hundred year floods happening each year for five years in a row?

ii. Argue that it is possible for a flood that occurs with probability 0.002 to occur each year for five years in a row, but is extremely unlikely, thus supporting but not proving that climate change has increased the probability of such severe floods.

5. The **R** code `-1 + 4 rnorm(1)` is simulating a random draw from $N(\mu, \sigma^2)$ for what choice of μ and σ^2?

6. Suppose you are interested in investing in a mutual fund B with return $r_{B,t}$. Suppose $r_{B,t} \sim N(\mu, \sigma^2)$ and that returns are i.i.d. over time. The fund's manager tells you that $\mathbb{E}[r_{B,t}] = 0.12$ and $\text{Var}(r_{B,t}) = 0.01$, though you worry that the actual expected return might be lower. Suppose you are interested in the asset's excess return above the risk-free rate r_f, where $r_f = 0.02$. In the following, take r_f to be a constant. Let $X_t = r_{B,t} - r_f$ denote the fund's excess return in year t.

(a) Show that, if what the manager told you is true, then:

i. $X_t \sim N(0.10, 0.01)$,

ii. $(X_1 - 0.10)/0.1 \sim N(0, 1)$.

(b) Suppose you only observe one year of returns, with a zero excess return in that year, $X_1 = 0$. How unlikely is it to have zero excess return or less in one year if what the manager told you is true? In particular, if what she said is true, what is $\Pr[X_1 \leq 0]$?

i. Express your answer in terms of $\Phi(\cdot)$.

ii. Using **R**, express your answer as a number, and discuss its magnitude.

iii. Is what the manager told you implausible given the evidence of zero excess return in one year?

(c) Suppose you observe four years of returns, and that $\bar{X}_4 = 0$. How unlikely is it to have zero average excess return across four years if what the manager told you is true? In particular, if what she said is true, what is $\Pr[\bar{X}_4 \leq 0]$?

i. Express your answer in terms of $\Phi(\cdot)$.

ii. Using **R**, express your answer as a number, and discuss its magnitude.

iii. Is what the manager told you implausible given the evidence of zero average excess return across four years?

(d) Repeat question (c), but now find probability that $\bar{X}_4$ would be that far from asserted expected excess return, i.e., find $\Pr[\ |\bar{X}_4 - 0.10| \geq 0.10]$.

i. Express your answer in terms of $\Phi(\cdot)$.

ii. Using **R**, express your answer as a number, and discuss its magnitude.

7. Let $q_{0.025}$ and $q_{0.975}$ denote the 0.025 and 0.975 quantiles of a $N(0, 1)$ distribution.

(a) Recall that, by symmetry of a $N(0, 1)$ pdf, $q_{0.975} = -q_{0.025}$. Using the `qt` function in **R**, show the analogous result for the t_1, t_5, and t_{20} distributions, that their 97.5th quantiles equals the negative of their 2.5th quantiles.

(b) Compare the 2.5th and 97.5th quantiles of the t_1, t_5, and t_{20} distributions to those of $N(0, 1)$, and discuss how the quantiles are different.

(c) Consider $\Pr[q_{0.025} < X < q_{0.975}]$. If $X \sim N(0, 1)$, then that probability equals 0.95. Using **R**, compute that probability for $X \sim t_1, t_5$ and t_{20}, and discuss how that probability changes as you vary the degrees of freedom parameter ν.