Notes on Lie Theory (SO3, SE3)

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1 Introduction

In this post, we will discuss Lie Theory, which is used in SLAM. When studying the optimization part of SLAM, optimization methods based on Lie Theory often appear, but without prior knowledge of the content, it is difficult to understand the optimization process. Therefore, this post briefly summarizes the essential content needed to understand the optimization part of SLAM. Most of the content is written referring to Joan Solà's *Lie theory for the roboticist* YouTube video.

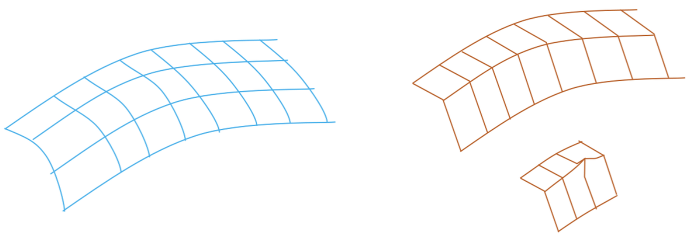
2 Group Theory

A group (Group) refers to an algebraic structure consisting of a set and a binary operation between two elements. For example, if a set is denoted as A and a binary operation as $*$, a group can be represented as $G = (A, *)$. Common sets of numbers such as integers, rational numbers, real numbers, and complex numbers, and operations such as addition and multiplication, belong to groups.

Groups generally have the following characteristics:

- Associativity: For any three elements $a, b, c \in G$ in the group, the associative law $(a*b)*c = a*(b*c)$ holds.
- Identity element: If there exists an element $e \in G$ such that $a * e = a = e * a$ for any element $a \in G$, then e is called the identity element.
- Inverse: If there exists an element $x \in G$ such that $a * x = e = x * a$ for any element $a \in G$, then x is called the inverse, and is sometimes denoted as a^{-1}.
- Composition: For any two elements $a, b \in G$, $a * b \in G$ holds. In most groups, the binary operation $*$ does not commute, meaning $a * b \neq b * a$.

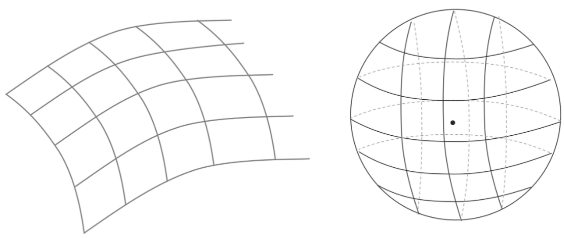
2.1 Lie Group



Smooth Manifold   Non-Smooth Manifold (Edge, Spike)

Among various groups, a Lie Group is a group that has a Smooth Manifold. Here, a Smooth Manifold refers to a manifold where all elements of the group exist without edges or spikes, as shown in the figure. All elements existing on the Smooth Manifold have the characteristic of being differentiable.

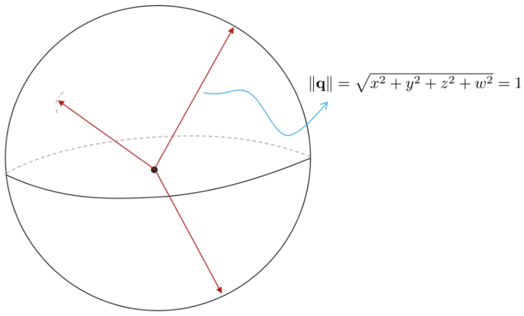
2.2 Manifold



Manifold $\mathcal{M}$

An N-dimensional Manifold $\mathcal{M}$ refers to a geometric space where any point $\mathbf{x} \in \mathcal{M}$ within $\mathcal{M}$ locally has a Euclidean structure. In other words, all points near $\mathbf{x}$ have the characteristic of being topologically homeomorphic to the $\mathbb{R}^N$ space. Intuitively, a Manifold represents the constraint space of the group.

For example, for any quaternion $\mathbf{q} = [x, y, z, w]$ to be used as a 3D rotation operator, it must satisfy the properties of a unit quaternion, where the constraint is $\|\mathbf{q}\| = 1$. This means that $\mathbf{q}$ must satisfy a point on a 4-dimensional Manifold, referred to as the unit quaternion manifold. Constraints of four dimensions or more cannot be visualized on a plane, so they are typically explained using a 3-dimensional sphere.



Unit Quaternion should meet the constraint $\|\mathbf{q}\| = 1$

In the case of the SE(3) Lie Group to be discussed later, 6-dimensional elements must satisfy specific constraints. That is, elements of the SE(3) Lie Group must exist on a 6-dimensional Manifold. Since this cannot be explained on a plane, a 3-dimensional sphere is used to illustrate it.

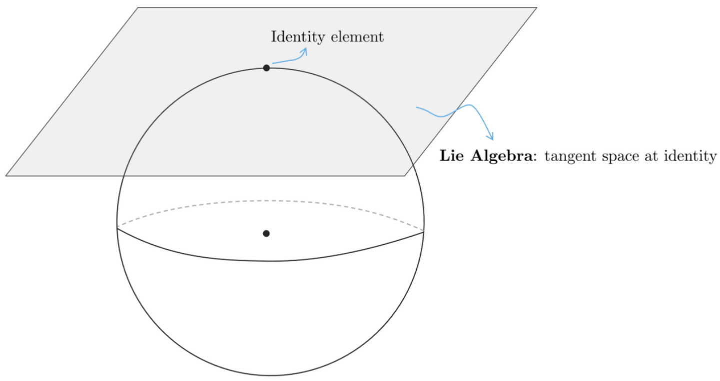
2.3 Group Action

One of the characteristics of a group is that it can transform (=act) another set or group. This means that the group can function as an operator transforming a specific set. This demonstrates that Lie Theory is a suitable tool for representing the movement of objects in 3-dimensional space.

For example, given a rotation matrix $\mathbf{R} \in SO(3)$, a 3-dimensional vector $\mathbf{x} \in \mathbb{R}^3$, and a binary operation $\cdot$: - $\mathbf{R}$ can rotate (=act on) a point in vector space $\rightarrow \mathbf{x}' = \mathbf{R} \cdot \mathbf{x}$

Given a transformation matrix $\mathbf{T} \in SE(3)$, a 4-dimensional vector $\mathbf{X} \in \mathbb{R}^4$, and a binary operation $\cdot$: - $\mathbf{T}$ can transform (=act on) a point in vector space $\rightarrow \mathbf{X}' = \mathbf{T} \cdot \mathbf{X}$

2.4 Topology of Lie Theory



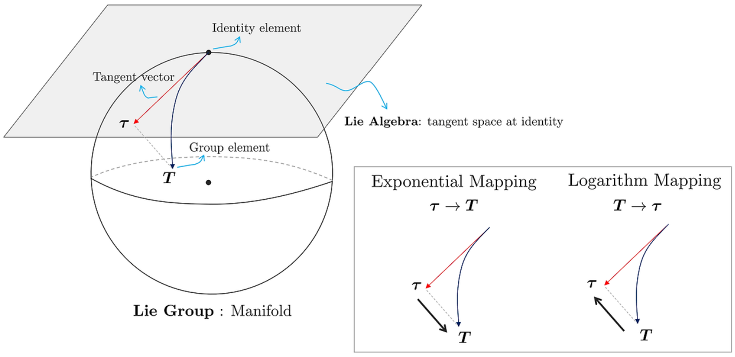
Lie Group : Manifold

The geometric structure of Lie Theory is as shown in the figure above.

- A Lie Group represents a collection of points on a manifold with constraints and has **non-linear characteristics**.

- The Lie Algebra represents the tangent space at the identity element on the manifold. This tangent space of the Lie Algebra is only valid at the identity element and has the characteristic of a **linear vector space**.

The reason why Lie Groups and Lie Algebras are important is because a 1:1 transformation is possible between the two spaces. **Therefore, instead of directly operating in the complex constraint-laden non-linear manifold space (Lie Group), operations are performed in the relatively simple linear vector space (Lie Algebra) and then transformed back to the manifold space.** The operations that make this possible are the Exponential Mapping and Logarithm Mapping.

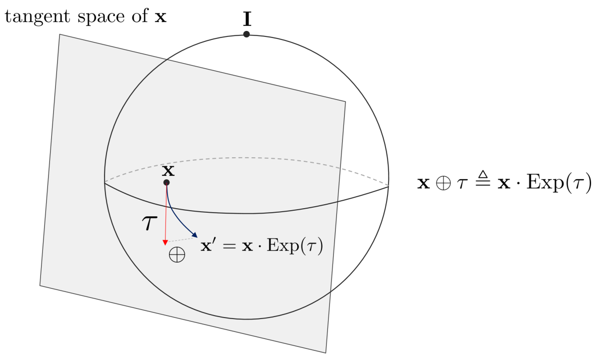


Exponential Mapping refers to the operation that transforms from Lie Algebra to Lie Group, while Logarithm Mapping is the opposite operation that transforms from Lie Group to Lie Algebra.

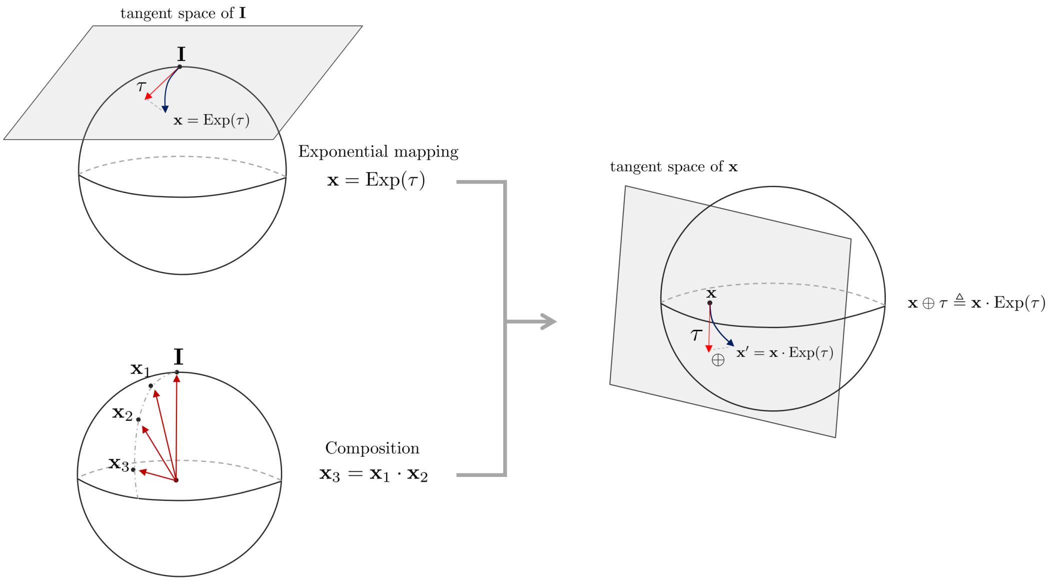
2.5 Plus and Minus Operators of Lie Group

Using the previously mentioned exponential mapping, any element τ of Lie Algebra can be used to transform an element $\mathbf{x}$ of Lie Group. Since the usual $+, -$ operators do not apply between elements of the Lie Group and Lie Algebra, new operators $\oplus, \ominus$ must be defined. The $\oplus$ operator applies an additional transformation to an arbitrary Lie Group element $\mathbf{x}$ by the amount τ.

$$\mathbf{x} \oplus \tau \triangleq \mathbf{x} \cdot \mathrm{Exp}(\tau) \tag{1}$$

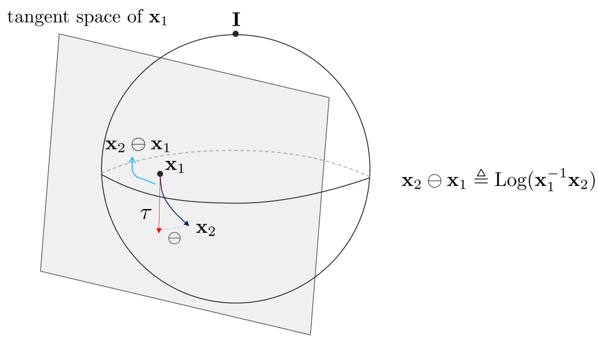


At this time, the τ vector on the tangent space of x is treated as identical to the τ vector on the tangent space of the identity element due to the composition properties of the lie group.

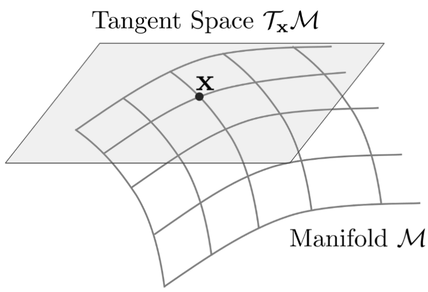


Conversely, the $\ominus$ operator is used in cases like $\mathbf{x}_2 \ominus \mathbf{x}_1$ when two Lie Group elements $\mathbf{x}_1, \mathbf{x}_2$ exist, indicating the relative change from $\mathbf{x}_1$ to $\mathbf{x}_2$.

$$\mathbf{x}_2 \ominus \mathbf{x}_1 \triangleq \mathrm{Log}(\mathbf{x}_1^{-1}\mathbf{x}_2) \tag{2}$$



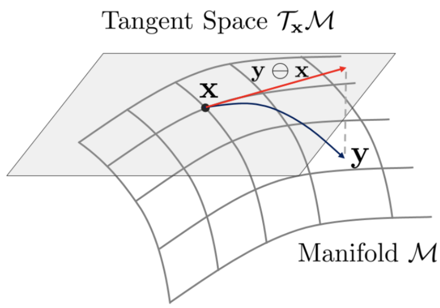
2.6 Tangent Space and Lie Algebra



When there exists a point $\mathbf{x} \in \mathcal{M}$ on an arbitrary manifold $\mathcal{M}$, its tangent space is denoted as $\mathcal{T}_{\mathbf{x}}\mathcal{M}$.

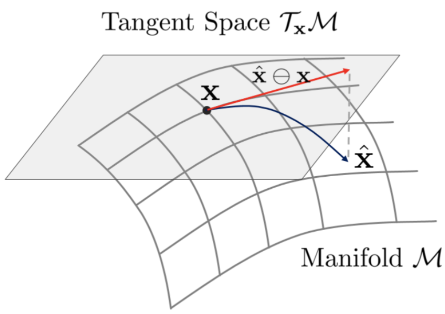
The tangent space $\mathcal{T}_{\mathbf{x}}\mathcal{M}$ is uniquely determined for each point and has the characteristic of being a vector space, enabling calculus operations. The dimension of the tangent space is determined by the number of degrees of freedom of the manifold. For instance, SO(3) has three degrees of freedom for rotation, so so(3) has a three-dimensional tangent space, while SE(3), which accounts for pose, has six degrees of freedom, making se(3) a six-dimensional tangent space. The tangent space at the identity element is specifically referred to as the Lie Algebra.

2.7 Calculus on Lie Group



Let us assume that there exist two elements $\mathbf{x}, \mathbf{y}$ on a Lie Group. In this case, the difference between the two elements can be represented on the tangent space using the $\ominus$ operator as $\mathbf{y} \ominus \mathbf{x}$. As mentioned earlier, the tangent space $\mathcal{T}_{\mathbf{x}}\mathcal{M}$ is a linear vector space, which makes it relatively easy to calculate the jacobian and covariance. This characteristic is a key reason Lie Theory can be used for optimization calculations. For example, an error function model can be used when the two elements are relatively close.

$$\mathbf{e} = \hat{\mathbf{x}} \ominus \mathbf{x} \tag{3}$$



2.8 Jacobians on Lie Group

Given a vector function $f(\mathbf{x})$ existing in a linear vector space, suppose that $f(\mathbf{x})$ satisfies $\mathbf{x} \in \mathbb{R}^n, f(\mathbf{x}) \in \mathbb{R}^m$.

$$f : \mathbb{R}^n \mapsto \mathbb{R}^m \tag{4}$$

In this case, the first partial derivative of the vector function becomes a matrix, which is specifically referred to as the jacobian.

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \tag{5}$$

When represented using a small change $\mathbf{h}$, it can be expressed as follows:

$$\mathbf{J} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \triangleq \lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})}{\mathbf{h}} \in \mathbb{R}^{m \times n} \tag{6}$$

In a linear vector space, $+, -$ operators can be used to calculate the jacobian. However, as mentioned earlier, since the elements of a Lie Group are not closed under $+, -$ operations, the jacobian cannot be expressed using traditional methods.

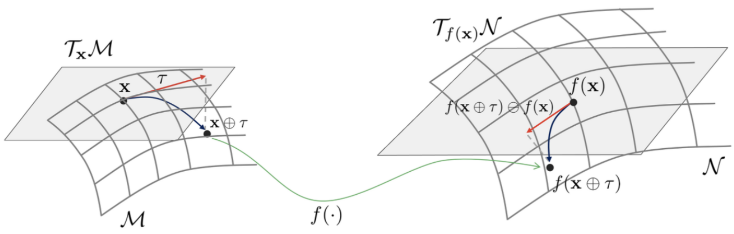
Given a vector function $f(\mathbf{x})$ existing on a Lie Group as follows:

$$f : \mathcal{M} \mapsto \mathcal{N}; \mathbf{x} \mapsto \mathbf{y} = f(\mathbf{x}) \tag{7}$$

This function maps an element $\mathbf{x}$ on the manifold $\mathcal{M}$ to another element $\mathbf{y}$ on the manifold $\mathcal{N}.Thejacobianforthiscanbe$

$$\mathbf{J} = \frac{Df(\mathbf{x})}{D\mathbf{x}} = \lim_{\tau \to 0} \frac{f(\mathbf{x} \oplus \tau) \ominus f(\mathbf{x})}{\tau} \in \mathbb{R}^{m \times n} \tag{8}$$

The jacobian $\mathbf{J}$ can be thought of as a function mapping an element on $\mathcal{T}_{\mathbf{x}}\mathcal{M}$ to an element on $\mathcal{T}_{f(\mathbf{x})}\mathcal{N}$. This can be illustrated as follows:



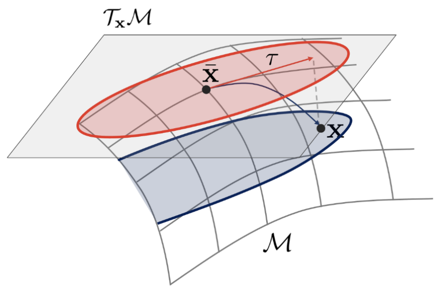
2.9 Perturbations on Lie Group

Using the property that the tangent space is a linear vector space, elements of the Lie Group can be modeled as perturbations of a random variable.

$$\mathbf{x} = \bar{\mathbf{x}} \oplus \tau \quad \text{where, } \tau = \mathbf{x} \ominus \bar{\mathbf{x}} \tag{9}$$

$$\mathbf{P_x} = \mathbb{E}[\tau \cdot \tau^{\intercal}] \tag{10}$$

$$\mathbf{P_x} = \mathbb{E}[(\mathbf{x} \ominus \bar{\mathbf{x}}) \cdot (\mathbf{x} \ominus \bar{\mathbf{x}})^{\intercal}] \tag{11}$$

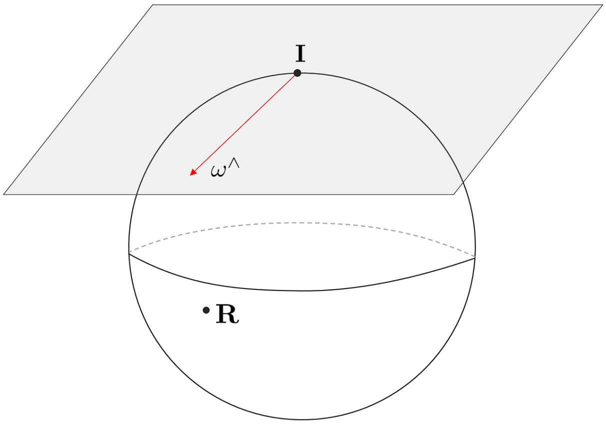


If a function $\mathbf{y} = f(\mathbf{x})$ is given, the jacobian becomes $\mathbf{J} = \frac{D\mathbf{y}}{D\mathbf{x}}$ and the covariance $\mathbf{P_y}$ can be calculated as follows:

$$\mathbf{P_y} = \mathbf{J} \cdot \mathbf{P_x} \cdot \mathbf{J}^{\intercal} \tag{12}$$

This follows the same formula for covariance propagation in vector space.

3 SO(3) Group



3.1 Lie Group SO(3)

One of the Lie groups, the Special Orthogonal 3 (SO(3)) group, consists of 3-dimensional rotation matrices and operations that are closed under these matrices. It is used to represent the rotation of 3D objects.

$$SO(3) = \{\mathbf{R} \in \mathbb{R}^{3\times3} \mid \mathbf{R}\mathbf{R}^T = \mathbf{I}, \det(\mathbf{R}) = 1\} \tag{13}$$

3.1.1 SO(3) group properties

- Associativity: Associativity holds as $(\mathbf{R}_1 \cdot \mathbf{R}_2) \cdot \mathbf{R}_3 = \mathbf{R}_1 \cdot (\mathbf{R}_2 \cdot \mathbf{R}_3)$
- Identity element: An identity matrix $\mathbf{I}$ exists such that $\mathbf{R} \cdot \mathbf{I} = \mathbf{I} \cdot \mathbf{R} = \mathbf{R}$
- Inverse: An inverse matrix exists such that $\mathbf{R}^{-1} \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{R}^{-1} = \mathbf{I}$. Due to the properties of the SO(3) group, $\mathbf{R}^{-1} = \mathbf{R}^\intercal$. Thus, $\mathbf{R} \cdot \mathbf{R}^\intercal = \mathbf{I}$.
- Composition: The composition in the SO(3) group is performed by matrix multiplication as follows
$$\mathbf{R}_1 \cdot \mathbf{R}_2 = \mathbf{R}_3 \in SO(3) \tag{14}$$
- Non-commutative: The commutative property does not hold as $\mathbf{R}_1 \cdot \mathbf{R}_2 \neq \mathbf{R}_2 \cdot \mathbf{R}_1$
- Determinant: The determinant of $\mathbf{R}$ satisfies $\det(\mathbf{R}) = 1$. (Only pure rotation without reflection or inversion is represented)
- Rotation: A point or vector $\mathbf{x} = \begin{bmatrix} x & y & z \end{bmatrix}^\intercal \in \mathbb{P}^2$ space can be rotated to another point or vector $\mathbf{x}'$.
$$\mathbf{x}' = \mathbf{R} \cdot \mathbf{x} \tag{15}$$
- Adjoint: Given any rotation matrix $\mathbf{R}$ and an angular velocity ω that exists on the tangent plane of $\mathbf{R}$, the properties of the adjoint matrix yield the following useful formulas.
$$\exp((\mathbf{R}\omega)^\wedge) = \mathbf{R}\exp(\omega^\wedge)\mathbf{R}^\intercal = \exp(\mathbf{R}\omega^\wedge\mathbf{R}^\intercal) \tag{16}$$

3.2 Lie Algebra so(3)

The Lie Algebra so(3) of the SO(3) group consists of the following $\mathbb{R}^{3\times3}$ size skew-symmetric matrices:

$$\mathrm{so}(3) = \left\{ \omega^\wedge = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \in \mathbb{R}^{3\times3} \;\middle|\; \omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \in \mathbb{R}^3 \right\} \tag{17}$$

The generators of so(3) are derived from rotations about each axis from the origin and signify orthogonal basis matrices.

$$G_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{18}$$

Each element of so(3) can be expressed as a linear combination of the generators.

$$\begin{aligned} &\omega \in \mathbb{R}^3 \\ &\omega_1 G_1 + \omega_2 G_2 + \omega_3 G_3 \in so(3) \end{aligned} \tag{19}$$

Here, ω is a 3-dimensional vector representing angular velocity about an arbitrary axis. By generating a skew-symmetric matrix through ω, it becomes so(3).

$$\omega^\wedge \in so(3) \tag{20}$$

3.3 Exponential Mapping and Logarithm Mapping

The Lie Group SO(3) and the Lie Algebra so(3) are uniquely matched through exponential mapping and logarithm mapping.

$$\begin{aligned} \exp(\omega^\wedge) &= \mathbf{R} \in SO(3) \\ \log(\mathbf{R}) &= \omega^\wedge \in so(3) \end{aligned} \tag{21}$$

Expanding $\exp(\omega^\wedge)$ according to the definition of Exponential Mapping results in:

$$\begin{aligned} \exp(\omega^\wedge) &= \exp\left(\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}\right) \\ &= \mathbf{I} + \omega^\wedge + \frac{1}{2!}(\omega^\wedge)^2 + \frac{1}{3!}(\omega^\wedge)^3 + \cdots \end{aligned} \tag{22}$$

In three-dimensional space, an arbitrary angular velocity ω can be separated into magnitude $|\omega|$ and unit vector $\mathbf{u}$.

$$\begin{aligned} \omega &= |\omega|\mathbf{u} \\ &= \theta\mathbf{u} \quad (\theta = |\omega|) \end{aligned} \tag{23}$$

Applying the characteristic of the skew-symmetric matrix $(\omega^\wedge)^3 = -(\omega^T\omega)\cdot\omega^\wedge = -\theta^2\omega^\wedge$ results in:

$$\begin{aligned} &\theta^2 = \omega^T\omega \\ &(\omega^\wedge)^{2i+1} = (-1)^i\theta^{2i}\omega^\wedge \\ &(\omega^\wedge)^{2i+2} = (-1)^i\theta^{2i}(\omega^\wedge)^2 \end{aligned} \tag{24}$$

Reorganizing the formula through Taylor expansion results in:

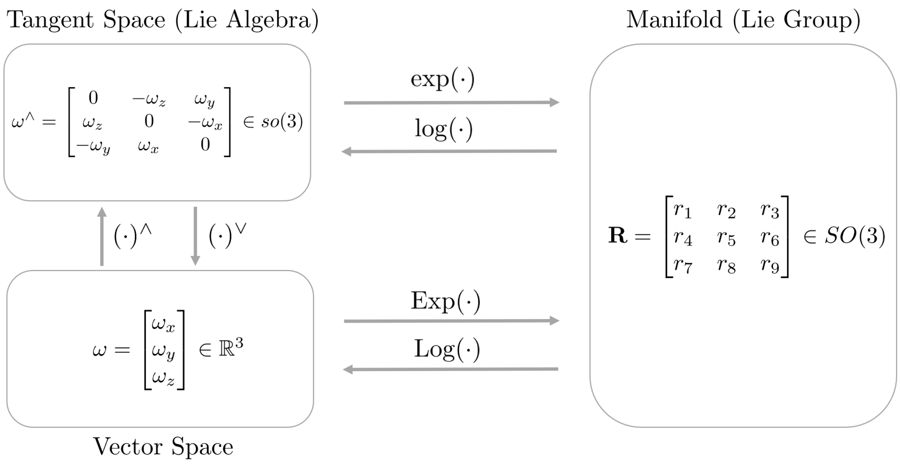
$$\begin{aligned} \exp(\omega^\wedge) &= \mathbf{I} + \omega^\wedge + \frac{1}{2!}(\omega^\wedge)^2 + \frac{1}{3!}(\omega^\wedge)^3 + \cdots \\ &= \mathbf{I} + \left(\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}}{(2i+1)!}\right)\omega^\wedge + \left(\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}}{(2i+2)!}\right)(\omega^\wedge)^2 \\ &= \mathbf{I} + \left(\frac{\sin\theta}{\theta}\right)\omega^\wedge + \left(\frac{1-\cos\theta}{\theta^2}\right)(\omega^\wedge)^2 \end{aligned} \tag{25}$$

This formula is called the Rodrigues Formula. It represents the relationship between angle-axis notation and the rotation matrix $\mathbf{R}$, acting as an axis of rotation and rotating by an angle θ. The Rodrigues Formula allows for exponential mapping from $\omega^\wedge \in so(3)$ to $\mathbf{R} \in SO(3)$.

For convenience in operation, the mapping process generally uses the $\exp(\cdot)$ operator for mapping $\omega^\wedge \to \mathbf{R}$ and the $\mathrm{Exp}(\cdot)$ operator for mapping $\omega \to \mathbf{R}$. The logarithm mapping follows similarly.

$$\begin{aligned} \exp(\cdot) &: \omega^\wedge \mapsto \mathbf{R} \\ \mathrm{Exp}(\cdot) &: \omega \mapsto \mathbf{R} \end{aligned} \tag{26}$$

The diagram summarizes this as follows.



3.4 Derivation of Exponential Mapping

An arbitrary rotation matrix satisfies the following property:

$$\mathbf{R}^\intercal \mathbf{R} = \mathbf{I} \tag{27}$$

If we consider $\mathbf{R}$ as a continuously changing camera rotation, it can be denoted as a function of time $\mathbf{R}(t)$.

$$\mathbf{R}^\intercal(t)\mathbf{R}(t) = \mathbf{I} \tag{28}$$

Differentiating both sides of the equation with respect to time, we obtain:

$$\begin{aligned} \dot{\mathbf{R}}^\intercal(t)\mathbf{R}(t) + \mathbf{R}^\intercal(t)\dot{\mathbf{R}}(t) = 0 \\ \mathbf{R}^\intercal(t)\dot{\mathbf{R}}(t) = -\Big(\mathbf{R}^\intercal(t)\dot{\mathbf{R}}(t)\Big)^\intercal \end{aligned} \tag{29}$$

From this, we can see that $\mathbf{R}^\intercal(t)\dot{\mathbf{R}}(t)$ satisfies the properties of a skew-symmetric matrix.

Any skew-symmetric matrix $\mathbf{A}$ is defined as follows. An operator that converts a 3-dimensional vector $\mathbf{a}$ into $\mathbf{A}$ is defined as $(\cdot)^\wedge$, and the reverse operation as $(\cdot)^\vee$.

$$\mathbf{a}^\wedge = \mathbf{A} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \quad \mathbf{A}^\vee = \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{30}$$

From any skew-symmetric matrix, a corresponding vector can be found. Thus, $\mathbf{R}^\intercal(t)\dot{\mathbf{R}}(t)$ can be associated with a 3-dimensional vector $\omega(t) \in \mathbb{R}^3$.

$$\mathbf{R}^\intercal(t)\dot{\mathbf{R}}(t) = \omega(t)^\wedge. \tag{31}$$

Multiplying both sides of the equation by $\mathbf{R}(t)$, and since $\mathbf{R}(t)$ is an orthogonal matrix, we obtain the following expression:

$$\dot{\mathbf{R}}(t) = \mathbf{R}(t)\omega(t)^\wedge = \mathbf{R}(t) \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{32}$$

Assuming that the rotation matrix $\mathbf{R}(0) = \mathbf{I}$ at time $t_0 = 0$, $\dot{\mathbf{R}}(0)$ can be represented as follows:

$$\dot{\mathbf{R}}(0) = \omega(0)^\wedge \tag{33}$$

Here, ω represents the tangent plane at the origin of SO(3). Assuming ω is constant near $t_0 = 0$, we have $\omega(t_0) = \omega$.

$$\dot{\mathbf{R}}(t) = \omega(0)^\wedge = \omega^\wedge \tag{34}$$

The equation being a differential equation, its solution is as follows:

$$\mathbf{R}(t) = \mathbf{R}_0 \exp(\omega^\wedge t). \tag{35}$$

Setting $\mathbf{R}_0 = \mathbf{R}(0) = \mathbf{I}$ and omitting the time function representation as $\omega^\wedge t \to \omega^\wedge$, we have:

$$\mathbf{R} = \exp(\omega^\wedge). \tag{36}$$

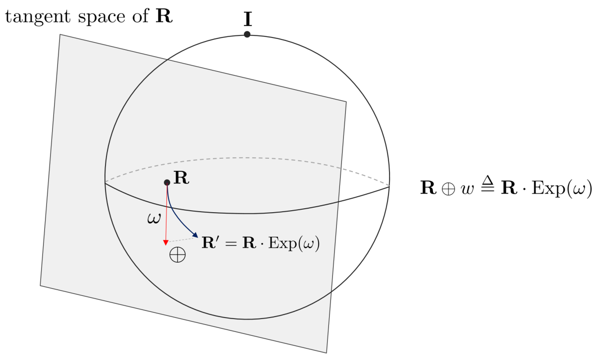
This expression means that the rotation matrix can be calculated through $\exp(\omega^\wedge)$.

3.5 Plus and Minus Operator of SO(3)

Using the previously described exponential mapping, we can transform $\mathbf{R}$ using the angular velocity ω. As the usual $+$ and $-$ operators do not apply between SO(3) and so(3), new $\oplus$ and $\ominus$ operators must be defined. The $\oplus$ operator applies an additional rotation of ω to an arbitrary rotation matrix $\mathbf{R}$.

$$\oplus : SO(3) \times \mathbb{R}^3 \mapsto SO(3) \tag{37}$$

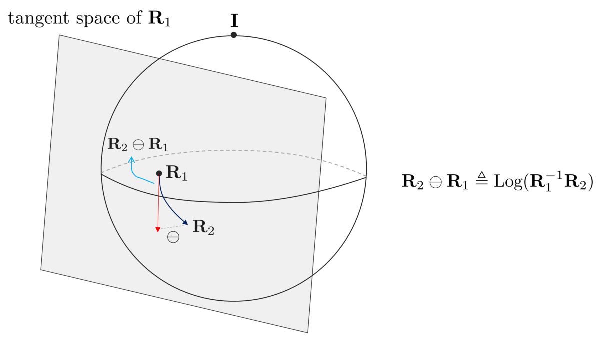
$$\mathbf{R} \oplus \omega \triangleq \mathbf{R} \cdot \mathrm{Exp}(\omega) \tag{38}$$



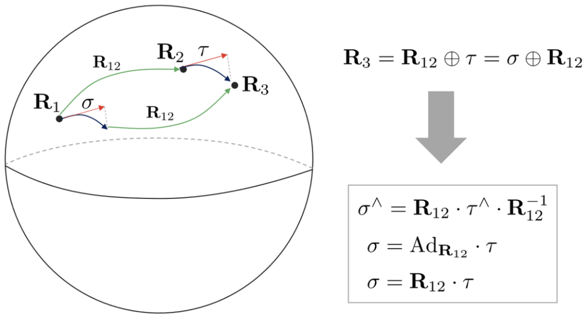
Conversely, the $\ominus$ operator is used when two rotation matrices $\mathbf{R}_1, \mathbf{R}_2$ exist, and it is used as $\mathbf{R}_2 \ominus \mathbf{R}_1$, representing the difference in rotation from $\mathbf{R}_1$ to $\mathbf{R}_2$.

$$\ominus : SO(3) \times SO(3) \mapsto \mathbb{R}^3 \tag{39}$$

$$\mathbf{R}_2 \ominus \mathbf{R}_1 \triangleq \mathrm{Log}(\mathbf{R}_1^{-1}\mathbf{R}_2) \tag{40}$$



3.6 Adjoint Matrix of SO(3)



SO(3) Manifold

The Adjoint matrix of the SO(3) group transforms an arbitrary angular velocity $\tau \in \mathbb{R}^3$ on the tangent plane of $\mathbf{R}_2 \in SO(3)$ to another angular velocity σ on the tangent plane of $\mathbf{R}_1$. If we define the Adjoint matrix with respect to $\mathbf{R}_{12} \in SO(3)$ as $\mathrm{Ad}_{\mathbf{R}_{12}}$, the following holds:

$$\sigma = \mathrm{Ad}_{\mathbf{R}_{12}} \tau \tag{41}$$

Since it transforms one angular velocity $\tau \in \mathbb{R}^3$ to another angular velocity σ, $\mathrm{Ad}_{\mathbf{R}_{12}}$ has dimensions $\mathbb{R}^{3\times 3}$.

Additionally, the Adjoint matrix satisfies the following equations:

$$\begin{aligned} \mathbf{R}_{12} \cdot \exp(\tau^\wedge) &= \exp((\mathrm{Ad}_{\mathbf{R}_{12}} \cdot \tau)^\wedge) \cdot \mathbf{R}_{12} \\ \exp((\mathrm{Ad}_{\mathbf{R}_{12}} \cdot \tau)^\wedge) &= \mathbf{R}_{12} \cdot \exp(\tau^\wedge) \cdot \mathbf{R}_{12}^{-1} \end{aligned} \tag{42}$$

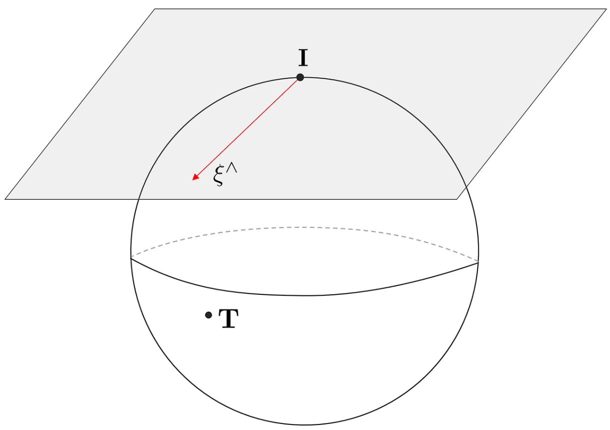
The derivation of the Adjoint matrix for the so(3) algebra is as follows:

$$\begin{aligned} (\mathrm{Ad}_{\mathbf{R}_{12}} \cdot \tau)^\wedge &= \mathbf{R}_{12} \cdot \left(\sum_{i=1}^{3} \tau_i G_i \right) \cdot \mathbf{R}_{12}^{-1} \\ &= \mathbf{R}_{12} \cdot \tau^\wedge \cdot \mathbf{R}_{12}^{-1} \\ &= (\mathbf{R}_{12} \tau)^\wedge \end{aligned} \tag{43}$$

$$\mathrm{Ad}_{\mathbf{R}_{12}} = \mathbf{R}_{12} \in \mathbb{R}^{3\times 3} \tag{44}$$

4 SE(3) Group

tangent space of se(3)



SE(3) Manifold

4.1 Lie Group SE(3)

The Special Euclidean 3 (SE(3)) group, one of the Lie groups, refers to a group composed of matrices and closed operations related to the transformation of rigid bodies in 3-dimensional space.

$$SE(3) = \left\{ \mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \mid \mathbf{R} \in SO(3), \mathbf{t} \in \mathbb{R}^3 \right\} \tag{45}$$

4.1.1 SE(3) group properties

- Associativity: Associative law holds as $(\mathbf{T}_1 \cdot \mathbf{T}_2) \cdot \mathbf{T}_3 = \mathbf{T}_1 \cdot (\mathbf{T}_2 \cdot \mathbf{T}_3)$
- Identity element: There exists a 4x4 identity matrix $\mathbf{I}$ such that $\mathbf{T} \cdot \mathbf{I} = \mathbf{I} \cdot \mathbf{T} = \mathbf{T}$
- Inverse: There exists an inverse matrix satisfying $\mathbf{T}^{-1} \cdot \mathbf{T} = \mathbf{T} \cdot \mathbf{T}^{-1} = \mathbf{I}$.

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \tag{46}$$

- Composition: The composition of the SE(3) group is performed as the multiplication of matrices

$$\begin{aligned} \mathbf{T}_1 \cdot \mathbf{T}_2 &= \begin{bmatrix} \mathbf{R}_1 & \mathbf{t}_1 \\ \mathbf{0} & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{R}_2 & \mathbf{t}_2 \\ \mathbf{0} & 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{R}_1\mathbf{R}_2 & \mathbf{R}_1\mathbf{t}_2 + \mathbf{t}_1 \\ \mathbf{0} & 1 \end{bmatrix} \in SE(3) \end{aligned} \tag{47}$$

- Non-commutative: $\mathbf{T}_1 \cdot \mathbf{T}_2 \neq \mathbf{T}_2 \cdot \mathbf{T}_1$ does not satisfy the commutative law
- Transformation: Points or vectors $\mathbf{X} = \begin{bmatrix} X & Y & Z & W \end{bmatrix}^\intercal \in \mathbb{P}^3$ space can be transformed to points or vectors $\mathbf{X}'$ with different directions and locations.

$$\begin{aligned} \mathbf{X}' = \mathbf{T} \cdot \mathbf{X} &= \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \cdot \mathbf{X} \\ &= \begin{bmatrix} \mathbf{R}(X\ Y\ Z)^T + W \cdot \mathbf{t} \\ W \end{bmatrix} \end{aligned} \tag{48}$$

4.2 Lie Algebra se(3)

The Lie algebra se(3) of the SE(3) group is defined as a matrix of size $\mathbb{R}^{4\times 4}$ as follows.

$$\text{se}(3) = \left\{ \xi^\wedge = \begin{bmatrix} \omega^\wedge & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \,\middle|\, \xi = \begin{bmatrix} \omega \\ \mathbf{v} \end{bmatrix} \in \mathbb{R}^6 \right\} \tag{49}$$

Here, ξ represents the velocity of the object in 3-dimensional space, $\omega = \begin{pmatrix} w_x & w_y & w_z \end{pmatrix}^T \in \mathbb{R}^3$ represents angular velocity, and $\mathbf{v} = \begin{pmatrix} v_x & v_y & v_z \end{pmatrix}^T \in \mathbb{R}^3$ represents velocity. The order of ω and $\mathbf{v}$ in ξ is often interchanged.

se(3) has the following 6 generators (Generators) related to rotation and translation.

$$\begin{aligned} G_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ G_4 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{50}$$

Each element of se(3) can be expressed as a linear combination of these generators.

$$\begin{aligned} &\xi = (\omega,\ \mathbf{v})^T \in \mathbb{R}^6 \\ &\omega_1 G_1 + \omega_2 G_2 + \omega_3 G_3 + v_1 G_4 + v_2 G_5 + v_3 G_6 \in se(3) \end{aligned} \tag{51}$$

4.3 Exponential Mapping and Logarithm Mapping

The SE(3) Lie Group and its Lie Algebra se(3) are uniquely matched through exponential mapping and logarithm mapping. Initially, the operation $\xi^\wedge$ that transforms the twist $\xi \in \mathbb{R}^6$ into the se(3) Lie algebra is defined as follows.

$$\xi^\wedge = \begin{bmatrix} \omega^\wedge & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \tag{52}$$

The exponential mapping of se(3) is defined as follows.

$$\begin{aligned} \exp(\xi^\wedge) &= \exp\left(\begin{bmatrix} \omega^\wedge & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix}\right) \\ &= \mathbf{I} + \begin{bmatrix} \omega^\wedge & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (\omega^\wedge)^2 & \omega^\wedge\mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} (\omega^\wedge)^3 & (\omega^\wedge)^2\mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} + \cdots \end{aligned} \tag{53}$$

Although the part related to rotation is identical to the SO(3) group, the part related to translation takes a separate series form.

$$\begin{aligned} \exp\left(\begin{bmatrix} \omega^\wedge & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix}\right) &= \begin{bmatrix} \sum_0^\infty \frac{1}{n!}(\omega^\wedge)^n & \sum_0^\infty \frac{1}{(n+1)!}(\omega^\wedge)^n\mathbf{v} \\ \mathbf{0} & 1 \end{bmatrix} \\ &= \begin{bmatrix} \exp(\omega^\wedge) & \mathbf{Q}\mathbf{v} \\ \mathbf{0} & 1 \end{bmatrix} \end{aligned} \tag{54}$$

$$\mathbf{Q} = \mathbf{I} + \frac{1}{2!}\omega^\wedge + \frac{1}{3!}(\omega^\wedge)^2 + \frac{1}{4!}(\omega^\wedge)^3 + \frac{1}{5!}(\omega^\wedge)^4 + \cdots \tag{55}$$

Applying the Rodrigues' formula and the principle of the antisymmetric matrix as previously explained in the SO(3) part, it can be reformulated as follows.

$$\mathbf{Q} = \mathbf{I} + \frac{1-\cos\theta}{\theta^2}\omega^\wedge + \frac{\theta - \sin\theta}{\theta^3}(\omega^\wedge)^2 \tag{56}$$

Summarizing the derivation process so far, the exponential mapping of se(3) can be expressed as follows.

$$\begin{aligned} &\xi = (\omega, \mathbf{v}) \in \mathbb{R}^6 \\ &\theta = |\omega| \\ &\theta^2 = \omega^\intercal\omega \\ &A = \frac{\sin\theta}{\theta} \\ &B = \frac{1-\cos\theta}{\theta^2} \\ &C = \frac{1-A}{\theta^2} = \frac{\theta-\sin\theta}{\theta^3} \\ &\mathbf{R} = \mathbf{I} + A\omega^\wedge + B(\omega^\wedge)^2 \\ &\mathbf{Q} = \mathbf{I} + B\omega^\wedge + C(\omega^\wedge)^2 \end{aligned} \tag{57}$$

$$\exp(\xi^\wedge) = \begin{bmatrix} \mathbf{R} & \mathbf{Q}\mathbf{v} \\ \mathbf{0} & 1 \end{bmatrix} \tag{58}$$

The logarithmic mapping of the SE(3) group uses the logarithmic mapping of the SO(3) group for the rotational part and $\mathbf{Q}^{-1}$ for the translational part.

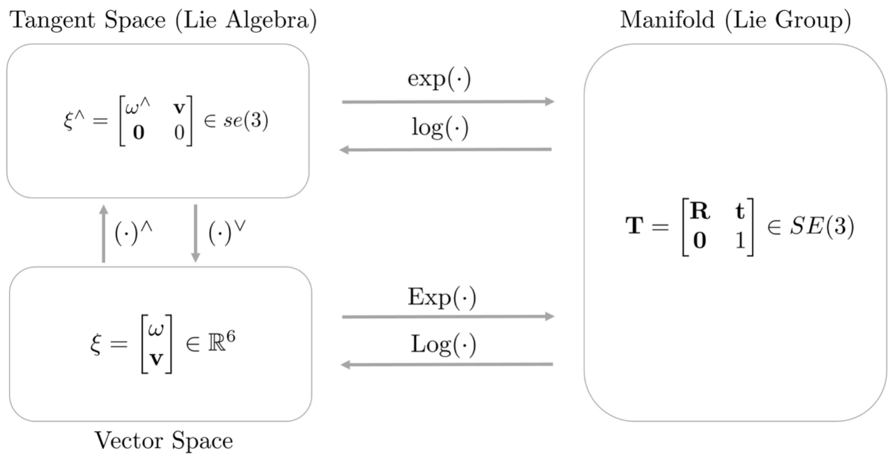
$$\mathbf{Q}^{-1} = \mathbf{I} - \frac{1}{2!}\omega^\wedge + \frac{1}{\theta^2}\left(1 - \frac{A}{2B}\right)(\omega^\wedge)^2 \tag{59}$$

$$\begin{aligned} \omega^\wedge &= \log(\mathbf{R}) \\ \mathbf{v} &= \mathbf{Q}^{-1}\mathbf{t} \end{aligned} \tag{60}$$

Just as with the SO(3), for convenience of operations, the mapping process from $\xi^\wedge$ to $\mathbf{T}$ is utilized through the $\exp(\cdot)$ operator, and the mapping process from ξ to $\mathbf{T}$ is utilized through the $\text{Exp}(\cdot)$ operator. The logarithmic mapping is the same.

$$\begin{aligned} \exp(\cdot) &: \xi^\wedge \mapsto \mathbf{T} \\ \text{Exp}(\cdot) &: \xi \mapsto \mathbf{T} \end{aligned} \tag{61}$$

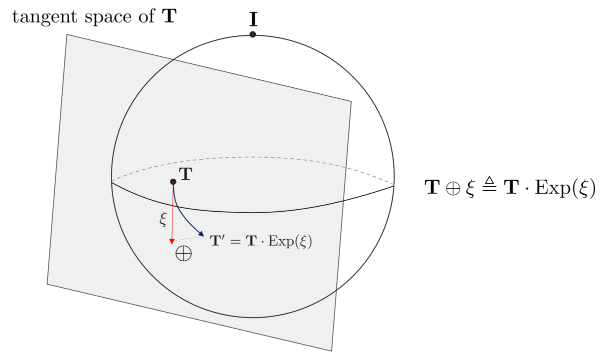
This diagram summarizes the process.



4.4 Plus and Minus Operator of SE(3)

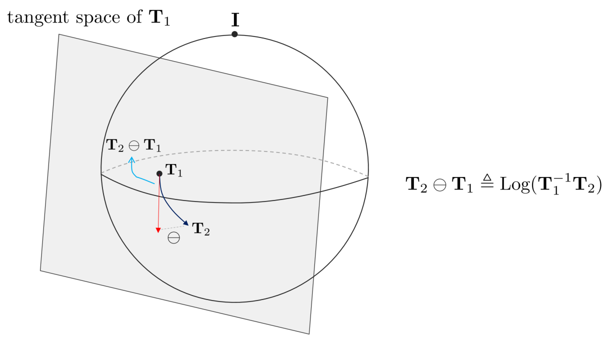
Using the described exponential mapping, transformation of $\mathbf{T}$ can be performed using twist ξ. Since general $+, -$ operators do not apply between SE(3) and se(3), new $\oplus, \ominus$ operators must be defined. First, the $\oplus$ operator applies an additional transformation of ξ to any pose $\mathbf{T}$.

$$\mathbf{T} \oplus \xi \triangleq \mathbf{T} \cdot \mathrm{Exp}(\xi) \tag{62}$$

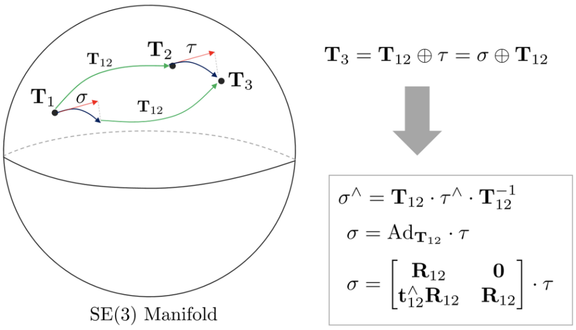


Conversely, the $\ominus$ operator is used as in $\mathbf{T}_2 \ominus \mathbf{T}_1$ when two poses $\mathbf{T}_1, \mathbf{T}_2$ exist, meaning the relative pose of $\mathbf{T}_2$ from $\mathbf{T}_1$.

$$\mathbf{T}_2 \ominus \mathbf{T}_1 \triangleq \mathrm{Log}(\mathbf{T}_1^{-1}\mathbf{T}_2) \tag{63}$$



4.5 Adjoint Matrix of SE(3)



The Adjoint matrix of the SE(3) group is a matrix that transforms any twist $\tau \in \mathbb{R}^6$ existing on the tangent plane of any $\mathbf{T}_2 \in SE(3)$ into a corresponding twist σ on the tangent plane of another $\mathbf{T}_1$. If $\mathbf{T}_{12} \in SE(3)$, the Adjoint matrix is denoted as $\mathrm{Ad}_{\mathbf{T}_{12}}$, and the following holds.

$$\xi_1 = \mathrm{Ad}_{\mathbf{T}_{12}} \xi_2 \tag{64}$$

Since it transforms one twist into another twist, the Adjoint matrix has the dimensions $\mathrm{Ad}_{\mathbf{T}_{12}} \in \mathbb{R}^{6\times 6}$. Furthermore, the Adjoint matrix satisfies the following equations.

$$\begin{aligned} \mathbf{T}_{12} \cdot \exp(\tau) &= \exp((\mathrm{Ad}_{\mathbf{T}_{12}} \cdot \tau)^{\wedge}) \cdot \mathbf{T}_{12} \\ \exp((\mathrm{Ad}_{\mathbf{T}_{12}} \cdot \tau)^{\wedge}) &= \mathbf{T}_{12} \cdot \exp(\tau) \cdot \mathbf{T}_{12}^{-1} \end{aligned} \tag{65}$$

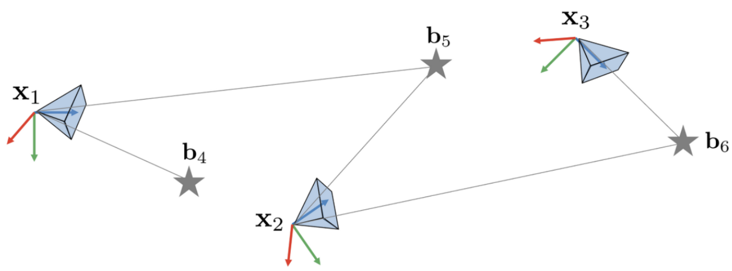
The derivation process for the Adjoint matrix of se(3) algebra is as follows.

$$\begin{aligned} (\mathrm{Ad}_{\mathbf{T}_{12}} \cdot \tau)^{\wedge} &= \mathbf{T}_{12} \cdot \left(\sum_{i=1}^{6} \xi_i G_i \right) \cdot \mathbf{T}_{12}^{-1} \\ &= \begin{pmatrix} \mathbf{R}_{12}\omega_{12} \\ \mathbf{t}_{12}^{\wedge}\mathbf{R}_{12}\omega_{12} + \mathbf{R}_{12}\mathbf{v}_{12} \end{pmatrix}^{\wedge} \\ &= \left(\begin{pmatrix} \mathbf{R}_{12} & \mathbf{0} \\ \mathbf{t}_{12}^{\wedge}\mathbf{R}_{12} & \mathbf{R}_{12} \end{pmatrix} \begin{pmatrix} \omega_{12} \\ \mathbf{v}_{12} \end{pmatrix} \right)^{\wedge} \end{aligned} \tag{66}$$

$$\mathrm{Ad}_{\mathbf{T}_{12}} = \begin{pmatrix} \mathbf{R}_{12} & \mathbf{0} \\ \mathbf{t}_{12}^{\wedge}\mathbf{R}_{12} & \mathbf{R}_{12} \end{pmatrix} \in \mathbb{R}^{6\times 6} \tag{67}$$

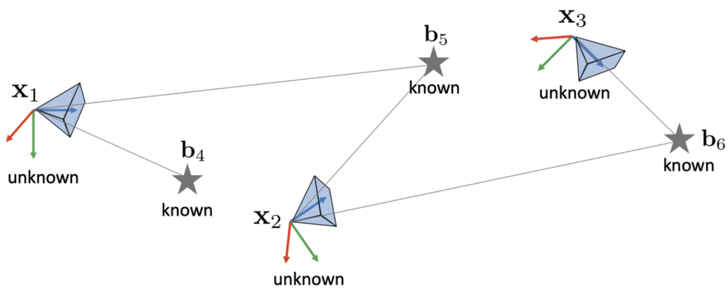
5 Applications for Estimation

Next, we explore real-world examples of state estimation using Lie Groups. Assume there are a camera and landmarks in a three-dimensional space as shown below.



Here, $\mathbf{x}_i, i = 1, 2, 3$ represent the 3D pose of the camera, and $\mathbf{b}_i, i = 4, 5, 6$ represent the coordinates of the landmarks.

5.1 EKF Map-based Localization



If only the values of $\mathbf{b}_i$ are given while $\mathbf{x}_i$ are unknown, the camera can estimate its pose $\mathbf{x}_i$ using the landmarks through EKF.

$$\begin{aligned} \mathbf{x} \sim \mathcal{N}(\bar{\mathbf{x}}, \mathbf{P}) \in SE(3) &: \text{Unknown} \\ \mathbf{P} = \mathbb{E}[(\mathbf{x} \ominus \bar{\mathbf{x}})(\mathbf{x} \ominus \bar{\mathbf{x}})^\intercal] &: \text{Unknown} \\ \mathbf{b} \in \mathbb{R}^3 &: \text{Known} \end{aligned} \tag{68}$$

The camera's motion model and measurement model are as follows.

$$\begin{aligned} &\text{motion model:} && \mathbf{x}_i = f(\mathbf{x}_{i-1}, \mathbf{u}_i) = \mathbf{x}_{i-1} \oplus (\mathbf{u_i}dt + \omega) \\ &\text{measurement model:} && \mathbf{y}_k = h(\mathbf{x}) = \mathbf{x}^{-1}\mathbf{b}_k + v, \quad \text{where, } v \sim \mathcal{N}(0, \mathbf{R}) \end{aligned} \tag{69}$$

- $\omega \sim \mathcal{N}(0, \mathbf{Q})$: perturbation
- $v \sim \mathcal{N}(0, \mathbf{R})$: noise

Since the camera's pose belongs to the SE(3) group, the $\oplus$ operation is used to update the pose. The prediction and correction steps of EKF using the above two models are as follows:

5.1.1 Prediction Step

$$\begin{aligned} \hat{\mathbf{x}} &\leftarrow \hat{\mathbf{x}} \oplus \mathbf{u}_i dt \\ \mathbf{P} &\leftarrow \mathbf{F}\mathbf{P}\mathbf{F}^\intercal + \mathbf{G}\mathbf{Q}\mathbf{G}^\intercal \end{aligned} \tag{70}$$

- $\mathbf{F} = \frac{Df}{D\mathbf{x}}$
- $\mathbf{G} = \frac{Df}{D\omega}$

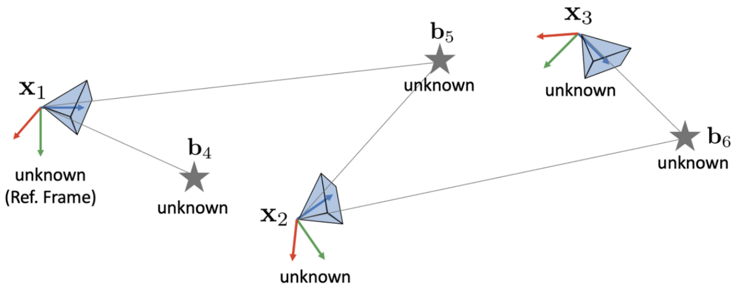
5.1.2 Correction Step

$$\begin{aligned} \mathbf{z}_k &= \mathbf{y}_k - \hat{\mathbf{x}}^{-1}\mathbf{b}_k \\ \mathbf{Z}_k &= \mathbf{H}\mathbf{P}\mathbf{H}^\intercal + \mathbf{R} \\ \mathbf{K} &= \mathbf{P}\mathbf{H}^\intercal\mathbf{Z}_k^{-1} \\ \hat{\mathbf{x}} &\leftarrow \hat{\mathbf{x}} \oplus \mathbf{K}\mathbf{z}_k \\ \mathbf{P} &\leftarrow \mathbf{P} - \mathbf{K}\mathbf{Z}_k\mathbf{K}^\intercal \end{aligned} \tag{71}$$

- $\mathbf{H} = \frac{Dh}{D\mathbf{x}}$

The above formulas are identical to the general EKF formulas. Thus, using Lie Group operators $\oplus, \ominus, \frac{D*}{D*}, \mathbf{x}^{-1}$ etc., nonlinear operations in $SE(3)$ can be performed similarly to operations in linear vector space.

5.2 Pose Graph SLAM



If both $\mathbf{x}_i$ and $\mathbf{b}_i$ are unknown, pose graph SLAM can be used to estimate both state variables.

$$\begin{aligned} \mathbf{x} \in SE(3) &: \text{Unknown} \\ \mathbf{b} \in \mathbb{R}^3 &: \text{Unknown} \end{aligned} \tag{72}$$

In this case, the state variables can be expressed together in vector form as follows:

$$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{b}_4, \mathbf{b}_5, \mathbf{b}_6) \tag{73}$$

The nonlinear optimization problem can then be defined as follows:

$$\mathbf{x}^* = \arg\min_{\mathbf{x}} \sum_p \|\mathbf{r}_p(\mathbf{x})\|^2 \tag{74}$$

The residual $\mathbf{r}$ can be defined as follows:

$$\begin{aligned} \text{prior: } \mathbf{r}_1 &= \Omega_1^{\intercal/2}(\mathbf{x}_1 \ominus \mathbf{x}_1^{ref}) \\ \text{motion: } \mathbf{r}_{ij} &= \Omega_{ij}^{\intercal/2}(\mathbf{u}_j dt - (\mathbf{x}_j \ominus \mathbf{x}_i)) \\ \text{measurement: } \mathbf{r}_{ik} &= \Omega_{ik}^{\intercal/2}(\mathbf{y}_{ik} - \mathbf{x}_i^{-1}\mathbf{b}_k) \end{aligned} \tag{75}$$

Residuals and Jacobians for all camera poses and landmarks can be defined as follows:

$$\mathbf{r} = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_{12} & \mathbf{r}_{23} & \mathbf{r}_{14} & \mathbf{r}_{15} & \mathbf{r}_{25} & \mathbf{r}_{26} & \mathbf{r}_{36} \end{bmatrix}^\intercal \tag{76}$$

$$\begin{array}{cc} & \begin{array}{cccccc} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{b}_4 & \mathbf{b}_5 & \mathbf{b}_6 \end{array} \\ \mathbf{J} = & \begin{bmatrix} \mathbf{J}^{\mathbf{r}_1}_{\mathbf{x}_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{J}^{\mathbf{r}_{12}}_{\mathbf{x}_1} & \mathbf{J}^{\mathbf{r}_{12}}_{\mathbf{x}_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}^{\mathbf{r}_{23}}_{\mathbf{x}_2} & \mathbf{J}^{\mathbf{r}_{23}}_{\mathbf{x}_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{J}^{\mathbf{r}_{14}}_{\mathbf{x}_1} & \mathbf{0} & \mathbf{0} & \mathbf{J}^{\mathbf{r}_{14}}_{\mathbf{b}_4} & \mathbf{0} & \mathbf{0} \\ \mathbf{J}^{\mathbf{r}_{15}}_{\mathbf{x}_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{J}^{\mathbf{r}_{15}}_{\mathbf{b}_5} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}^{\mathbf{r}_{25}}_{\mathbf{x}_2} & \mathbf{0} & \mathbf{0} & \mathbf{J}^{\mathbf{r}_{25}}_{\mathbf{b}_5} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}^{\mathbf{r}_{26}}_{\mathbf{x}_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{J}^{\mathbf{r}_{26}}_{\mathbf{b}_6} \\ \mathbf{0} & \mathbf{0} & \mathbf{J}^{\mathbf{r}_{36}}_{\mathbf{x}_3} & \mathbf{0} & \mathbf{0} & \mathbf{J}^{\mathbf{r}_{36}}_{\mathbf{b}_6} \end{bmatrix} & \begin{array}{c} \mathbf{r}_1 \\ \mathbf{r}_{12} \\ \mathbf{r}_{23} \\ \mathbf{r}_{14} \\ \mathbf{r}_{15} \\ \mathbf{r}_{25} \\ \mathbf{r}_{26} \\ \mathbf{r}_{36} \end{array} \end{array}$$

Using the above Jacobian, the defined nonlinear optimization problem can be solved iteratively using the Newton step.

$$\begin{aligned} \Delta\mathbf{x} &= -(\mathbf{J}^\intercal\mathbf{J})^{-1}\mathbf{J}^\intercal\mathbf{r} \\ \mathbf{x} &\leftarrow \mathbf{x} \oplus \Delta\mathbf{x} \end{aligned} \tag{77}$$

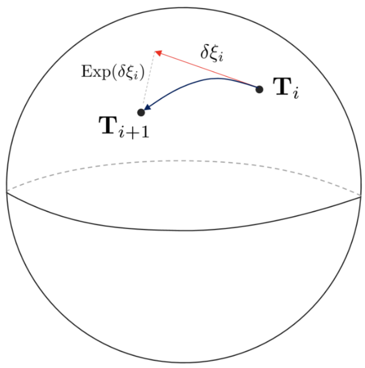
The optimization formulas used here also make it possible to perform nonlinear operations in $SE(3)$ similarly to operations in linear vector space using Lie Group operators $\oplus, \ominus, \frac{D*}{D*}, \mathbf{x}^{-1}$ etc.

6 Lie Theory-based Optimization on SLAM

When optimizing camera poses in 3D space, using Lie group (SO(3), SE(3)) representations directly can lead to over-parameterization, which poses several problems. The disadvantages of over-parameterized representations include:

- Increased computational load due to the need to compute redundant parameters.
- Potential for numerical instability due to additional degrees of freedom.
- The need to constantly check whether the parameters meet constraints upon updates.

On the other hand, using Lie algebra (so(3), se(3)) allows for constraint-free optimization by calculating the increments of Lie algebra through nonlinear optimization methods (e.g., GN, LM) and then mapping these increments to the Lie group (SO(3), SE(3)) space using the exponential map. Thus, Lie theory enables the transformation of a constrained optimization problem into an unconstrained optimization problem, offering the same benefits. Additionally, since Lie algebra is a linear vector space, it is relatively easy to calculate Jacobians and perturbations, allowing the existing optimization modeling to be directly applied.



$$\mathbf{T}_{i+1} = \text{Exp}(\delta\xi)\mathbf{T}_i$$

SE(3) Manifold

For example, in SLAM-based optimization using se(3), the optimization variables can be set as $\delta\xi = [\delta\omega, \delta\mathbf{v}]^\intercal$, and performing nonlinear optimization (e.g., GN, LM) allows each iteration's increment to be transformed to the SE(3) group through $\text{Exp}(\delta\xi)$. This can then be multiplied by the existing camera pose $\mathbf{T}$ to update the pose without any constraints.

$$\mathbf{T} \leftarrow \text{Exp}(\delta\xi) \cdot \mathbf{T} \tag{78}$$

Tip

Depending on whether the existing state $\mathbf{T}$ is multiplied on the right or left, it determines whether the pose is updated from the local coordinate system (right) or from the global coordinate system (left).

$$\begin{aligned} \mathbf{T} &\leftarrow \text{Exp}(\delta\xi) \cdot \mathbf{T} \quad \cdots \text{ globally updated (left mult)} \\ \mathbf{T} &\leftarrow \mathbf{T} \cdot \text{Exp}(\delta\xi) \quad \cdots \text{ locally updated (right mult)} \end{aligned} \tag{79}$$

7 Reference

1. (Paper) Lie Groups for 2D and 3D Transformations
2. (Paper) A tutorial on SE3 transformation parameterizations and on-manifold optimization
3. (Book) Introduction to Visual SLAM

4. (Youtube) Modern Robotics Lecture
5. (Youtube) Lie theory for the roboticist (by Joan Sola)

8 Revision log

- 1st: 2022-01-04
- 2nd: 2022-08-14
- 3rd: 2022-09-07
- 4th: 2022-11-26
- 5th: 2023-01-21
- 6th: 2023-01-23
- 7th: 2023-01-25
- 8th: 2023-01-28
- 9th: 2023-11-14
- 10th: 2024-02-24