23. Assume that $X_1, X_2, \ldots$ i.i.d. with mean μ, positive variance σ^2, and the third and fourth central moments μ_3 and μ_4, which are all finite. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ and $S_n^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$. (a) Find the asymptotic **joint** distribution of $\bar{X}_n$ and S_n^2. (b) Find the asymptotic distribution of the sample coefficient of variation $S_n/\bar{X}_n$.

24. For $X_1, X_2, \ldots$ i.i.d. with mean μ and positive variance σ^2 which are both finite,
(a) prove that the limiting distribution of
$$n^p \left\{ \frac{(\bar{X} - \mu)^2 - \mu^*}{\sigma^*} \right\}$$
is χ^2 with 1 degree of freedom for appropriate μ^*, σ^*, and p.
(b) For $\mu \neq 0$, find the asymptotic distribution of $\bar{X}^k$, $k \geq 1$.

25. Problem 6.3 in Casella and Berger (2001)+ (b) Find a one-dimensional sufficient statistic for σ when μ is fixed.

26. Let (θ_1, θ_2) be a bivariate parameter. Suppose that $T_1(\mathbf{X})$ is sufficient for θ_1 when θ_2 is fixed and known, whereas $T_2(\mathbf{X})$ is sufficient for θ_2 whenever θ_1 is fixed and known. Assume that $\theta_1 \in \Theta_1$, $\theta_2 \in \Theta_2$ and that the set $A = \{\mathbf{x} : f(\mathbf{x}; \theta) > 0\}$ does not depend on θ.
(a) Show that if T_1 and T_2 do not depend on θ_2 and θ_1 respectively, then $(T_1(\mathbf{X}), T_2(\mathbf{X}))$ is sufficient for θ.
(b) Find an example in which $(T_1(\mathbf{X}), T_2(\mathbf{X}))$ is sufficient for θ, $T_1(\mathbf{X})$ is sufficient for θ_1 when θ_2 is fixed and known, but $T_2(X)$ is NOT sufficient for θ_2 whenever θ_1 is fixed and known.

27. Problem 6.9(a)(b)(c)(e) in Casella and Berger (2001).

28. Problem 3 of Keener (2010) Section 3.7.

29. Problem 4 of Keener (2010) Section 3.7.

30. Problem 7 of Keener (2010) Section 3.7.

Practice
Casella and Berger (2001) Eg. 6.2.4, Eg. 3.8 of Keener p:46 Uniform $(\theta, \theta + 1)$, Eg. 5.5.27 in Casella and Berger (2001), and Problems 6.2 and 6.6 in Casella and Berger (2001).

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23. (a)

Let $\left\{\begin{pmatrix} X_i-\mu \\ (X_i-\mu)^2 \end{pmatrix}\right\}_{i=1}^{n} \overset{iid}{\sim} \left(\begin{pmatrix} 0 \\ \sigma^2 \end{pmatrix}, \begin{pmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4-\sigma^4 \end{pmatrix}\right) = \left(\begin{pmatrix} 0 \\ \sigma^2 \end{pmatrix}, \Sigma\right)$

By multivariate CLT:

$$\sqrt{n}\left(\begin{pmatrix} \frac{1}{n}\sum_{1}^{n}(X_i-\mu) \\ \frac{1}{n}\sum_{1}^{n}(X_i-\mu)^2 \end{pmatrix} - \begin{pmatrix} 0 \\ \sigma^2 \end{pmatrix}\right) = \sqrt{n}\left(\begin{pmatrix} \bar{X}_n-\mu \\ S_n^{*2} \end{pmatrix} - \begin{pmatrix} 0 \\ \sigma^2 \end{pmatrix}\right) \xrightarrow{D} N(0, \Sigma)$$

$\because \lim_{n\to\infty} E(S_n^{*2}) = \sigma^2$ and $\lim_{n\to\infty} Var(S_n^{*2}) = 0$ $\therefore S_n^{*2} \xrightarrow{2nd} \sigma^2 \Rightarrow S_n^{*2} \xrightarrow{P} \sigma^2$

and $S_n^2 \xrightarrow{P} \sigma^2 \Rightarrow S_n^2 - S_n^{*2} \xrightarrow{P} 0 \Rightarrow \sqrt{n}(S_n^2 - S_n^{*2}) \xrightarrow{P} 0$

By Slutsky's thm:

$$\sqrt{n}\left(\begin{pmatrix} \bar{X}-\mu \\ S_n^{*2}-\sigma^2 \end{pmatrix} + \begin{pmatrix} 0 \\ S_n^2 - S_n^{*2} \end{pmatrix}\right) = \sqrt{n}\begin{pmatrix} \bar{X}-\mu \\ S_n^2-\sigma^2 \end{pmatrix} \xrightarrow{D} N(0, \Sigma)$$

$$\Rightarrow \begin{pmatrix} \bar{X} \\ S_n^2 \end{pmatrix} \xrightarrow{D} N\left(\begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix}, \frac{1}{n}\begin{pmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4-\sigma^4 \end{pmatrix}\right) \square$$

(b) Define $g(\alpha,\beta) = \dfrac{\sqrt{\beta}}{\alpha}$

$$\Rightarrow \nabla g = \begin{pmatrix} \dfrac{\partial g}{\partial \alpha} \\ \dfrac{\partial g}{\partial \beta} \end{pmatrix} = \begin{pmatrix} \dfrac{-\sqrt{\beta}}{\alpha^2} \\ \dfrac{1}{2\alpha\sqrt{\beta}} \end{pmatrix}$$

By multivariate Delta method

$$\sqrt{n}\left(g(\bar{X}, S_n^2) - g(\mu, \sigma^2) \right) = \sqrt{n}\left(\frac{S_n}{\bar{X}_n} - \frac{\sigma}{\mu} \right)$$

$$\xrightarrow{D} N\left(0, \nabla g^T \Sigma \nabla g\right) = N\left(0, \left(\frac{-\sigma}{\mu^2}, \frac{1}{2\mu\sigma}\right) \begin{pmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{pmatrix} \begin{pmatrix} \frac{-\sigma}{\mu^2} \\ \frac{1}{2\mu\sigma} \end{pmatrix} \right)$$

$$= N\left(0, \frac{\sigma^4}{\mu_4} - \frac{\mu_3}{\mu^3} + \frac{\mu_4 - \sigma^4}{4\mu^2\sigma^2}\right)$$

$$\therefore \frac{S_n}{\bar{X}_n} \xrightarrow{D} N\left(\frac{\sigma}{\mu}, \frac{1}{n}\left(\frac{\sigma^4}{\mu_4} - \frac{\mu_3}{\mu^3} + \frac{\mu_4 - \sigma^4}{4\mu^2\sigma^2} \right) \right)$$

$\square$

24. (a) $\{X_i\}_1^n \overset{iid}{\sim} E(X_i) = \mu$, $Var(X_i) = \sigma^2 < \infty$

By CLT: $\dfrac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \xrightarrow{D} N(0, 1)$

By Continuous mapping Theorem:

$$\left[\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma}\right]^2 = \frac{n(\bar{X} - \mu)^2}{\sigma^2} \xrightarrow{D} \chi_1^2$$

$\therefore$ For $\mu^* = 0$, $\sigma^* = \sigma^2$, $p = 1$

$$n^p \left\{\frac{(\bar{X} - \mu)^2 - \mu^*}{\sigma^*}\right\} \xrightarrow{D} \chi_1^2 \quad \square$$

(b) By CLT: $\sqrt{n}(\bar{X}-\mu) \xrightarrow{D} N(0, \sigma^2)$

Define $g(\mu) = \mu^k$,

$$g'(\mu) = k\mu^{k-1} \text{ exists and } \neq 0 \; (\because \mu \neq 0)$$

By Delta method:

$$\sqrt{n}(g(\bar{X}) - g(\mu)) \xrightarrow{D} N(0, [g'(\mu)]^2 \sigma^2)$$

$$\Rightarrow \sqrt{n}(\bar{X}^k - \mu^k) \xrightarrow{D} N(0, k^2 \mu^{2k-2} \sigma^2)$$

$$\therefore \bar{X}^k \xrightarrow{D} N\left(\mu^k, \frac{k^2 \mu^{2k-2} \sigma^2}{n}\right) \square$$

25. (a)

$$f(x;\mu,\sigma)=\frac{1}{\sigma}\exp\left\{\frac{-(x-\mu)}{\sigma}\right\}I(X>\mu)$$

$$f(\underline{X};\mu,\sigma)=\prod_{i=1}^{n}f(X_i;\mu,\sigma)$$

$$=\sigma^{-n}\exp\left\{\frac{-1}{\sigma}\left(\sum_{i=1}^{n}X_i-n\mu\right)\right\}\prod_{i=1}^{n}I(X_i>\mu)$$

$$=\sigma^{-n}\exp\left\{\frac{-1}{\sigma}\left(\sum_{i=1}^{n}X_i-n\mu\right)\right\}I(X_{(1)}>\mu)$$

Let $T(\underline{X})=\left(\sum_{i=1}^{n}X_i, X_{(1)}\right)$, $h(\underline{X})=1$

$$g(T(\underline{X});\mu,\sigma)=\sigma^{-n}\exp\left\{\frac{-1}{\sigma}\left(\sum_{i=1}^{n}X_i-n\mu\right)\right\}I(X_{(1)}>\mu)$$

By Factorization Theorem:

$T(\underline{X})=\left(\sum_{i=1}^{n}X_i, X_{(1)}\right)$ is a 2-dim sufficient statistic for (μ,σ) $\square$

(b)

$$f(\underline{X};\sigma) = \sigma^{-n} \exp\left\{\frac{-1}{\sigma}\left(\sum_{i=1}^{n} X_i - n\mu\right)\right\} I(X_{(1)} > \mu)$$

Let $T(\underline{X}) = \sum_{i=1}^{n} X_i$, $h(\underline{X}) = I(X_{(1)} > \mu)$

$$g(T(\underline{X});\sigma) = \sigma^{-n} \exp\left\{\frac{-1}{\sigma}\left(\sum_{i=1}^{n} X_i - n\mu\right)\right\}$$

By Factorization Theorem:

$T(\underline{X}) = \sum_{i=1}^{n} X_i$ is a 1-dim sufficient statistic

for σ when μ is fixed $\square$

26. (a)

$\because T_1$ is sufficient for θ_1 and does not depend on θ_2

$\therefore f(\underline{X};\theta_1,\theta_2) = g_1(T_1(\underline{X});\theta_1)\, h_1(\underline{X};\theta_2)$

And similarly to T_2

$\therefore f(\underline{X};\theta_1,\theta_2) = g_2(T_2(\underline{X});\theta_2)\, h_2(\underline{X};\theta_1)$

Make $h_2(\underline{X};\theta_1) = g_1(T_1(\underline{X});\theta_1)\, h(\underline{X})$

$$g(T_1,T_2;\theta_1,\theta_2) = g_1(T_1;\theta_1)\, g_2(T_2;\theta_2)$$

$$\Rightarrow f(\underline{X};\theta_1,\theta_2) = g_1(T_1;\theta_1)\, g_2(T_2;\theta_2)\, h(\underline{X})$$
$$= g(T_1,T_2;\theta_1,\theta_2)\, h(\underline{X})$$

$\therefore (T_1(\underline{X}), T_2(\underline{X}))$ is sufficient for θ $\square$

(b) Suppose $\{X_i\}_1^n \overset{iid}{\sim} N(\mu, \sigma^2)$

Then $(T_1(\underline{X}), T_2(\underline{X})) = \left(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2\right)$ is a

2-dim sufficient statistic for (μ, σ^2)

And when σ^2 is fixed and known,

$T_1(\underline{X}) = \sum_{i=1}^n X_i$ is still a sufficient statistic for μ

However, when μ is fixed and known,

$T_2(\underline{X}) = \sum_{i=1}^n X_i^2$ is not a sufficient statistic

for σ^2 $\square$

27. (a)

$$\frac{f(\underline{X};\theta)}{f(\underline{Y};\theta)} = \frac{\exp\left\{\frac{-1}{2}\sum_{i=1}^{n}(X_i-\theta)^2\right\}}{\exp\left\{\frac{-1}{2}\sum_{i=1}^{n}(Y_i-\theta)^2\right\}}$$

$$= \exp\left\{\frac{-1}{2}\left[\left(\sum_{i=1}^{n}X_i^2-\sum_{i=1}^{n}Y_i^2\right)-2\theta\left(\sum_{i=1}^{n}X_i-\sum_{i=1}^{n}Y_i\right)\right]\right\}$$

$= c(\underline{X},\underline{Y})$ without θ

$$\Rightarrow T(\underline{X}) = \sum_{i=1}^{n}X_i = \sum_{i=1}^{n}Y_i = T(\underline{Y})$$

$\therefore \sum_{i=1}^{n}X_i$ is the minimal statistic for θ $\square$

(b) $f(x;\theta) = \exp\{-(x-\theta)\} I(X>\theta)$

$$\frac{f(\underline{X};\theta)}{f(\underline{Y};\theta)} = \frac{\exp\{-\sum_{i=1}^{n} X_i + n\theta\} \prod_{i=1}^{n} I(X_i > \theta)}{\exp\{-\sum_{i=1}^{n} Y_i + n\theta\} \prod_{i=1}^{n} I(Y_i > \theta)}$$

$$= \exp\left\{-\left[\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Y_i\right]\right\} \frac{I(X_{(1)} > \theta)}{I(Y_{(1)} > \theta)}$$

$= c(\underline{X}, \underline{Y})$ without θ

$\Rightarrow T(\underline{X}) = X_{(1)} = Y_{(1)} = T(\underline{Y})$

$\therefore$ $X_{(1)}$ is the minimal statistic for θ $\square$

(c) $f(x;\theta) = \dfrac{\exp\{-(x-\theta)\}}{(1+\exp\{-(x-\theta)\})^2}$

$$f(\underline{X};\theta) = \frac{\exp\{-\sum_{i=1}^{n} X_i + n\theta\}}{\left[\prod_{i=1}^{n}(1+\exp\{-(X_i-\theta)\})\right]^2}$$

$$\frac{f(\underline{X};\theta)}{f(\underline{Y};\theta)} = \exp\left\{-\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Y_i\right)\right\}\left[\prod_{i=1}^{n}\left(\frac{1+\exp\{-Y_i+\theta\}}{1+\exp\{-X_i+\theta\}}\right)\right]^2$$

$$= \exp\left\{-\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Y_i\right)\right\}\left[\prod_{i=1}^{n}\left(\frac{e^{-\theta}+e^{-Y_{(i)}}}{e^{-\theta}+e^{-X_{(i)}}}\right)\right]^2$$

$= c(\underline{X}, \underline{Y})$ without θ

$\Rightarrow T(\underline{X}) = (X_{(1)}, \ldots, X_{(n)}) = (Y_{(1)}, \ldots, Y_{(n)}) = T(\underline{Y})$

$\therefore (X_{(1)}, \ldots, X_{(n)})$ is the minimal statistic for θ $\square$

(e) $f(x;\theta) = \frac{1}{2} \exp\{-|x-\theta|\}$

$$f(\underline{X};\theta) = 2^{-n} \exp\{-\sum_{i=1}^{n} |X_i - \theta|\}$$

$$\frac{f(\underline{X};\theta)}{f(\underline{Y};\theta)} = \exp\{-[\sum_{i=1}^{n}|X_i-\theta| - \sum_{i=1}^{n}|Y_i-\theta|]\}$$

$$= \exp\{-[\sum_{i=1}^{n}|X_{(i)}-\theta| - \sum_{i=1}^{n}|Y_{(i)}-\theta|]\}$$

$= c(\underline{X},\underline{Y})$ without θ

$$\Rightarrow T(\underline{X}) = (X_{(1)}, \dots, X_{(n)}) = (Y_{(1)}, \dots, Y_{(n)}) = T(\underline{Y})$$

$\therefore (X_{(1)}, \dots, X_{(n)})$ is the minimal statistic for θ $\square$

28. $\{X_i\}_{i=1}^{n} \overset{\text{indep.}}{\sim} N(\theta, \sigma_i^2)$

$$f(X_i ; \theta, \sigma_i^2) = (2\pi\sigma_i^2)^{-\frac{1}{2}} \exp\left\{\frac{-1}{2\sigma_i^2}(X_i - \theta)^2\right\}$$

$$= \exp\left\{\frac{-X_i^2}{2\sigma_i^2} + \theta\frac{X_i}{\sigma_i^2} - \frac{\theta^2}{2\sigma_i^2} - \frac{1}{2}\log 2\pi - \frac{1}{2}\log\sigma_i^2\right\}$$

$$f(\underline{X} ; \theta, \sigma_i^2)$$

$$= \exp\left\{\frac{-1}{2}\sum_{i=1}^{n}\frac{X_i^2}{\sigma_i^2} + \theta\sum_{i=1}^{n}\frac{X_i}{\sigma_i^2} - \frac{\theta^2}{2}\sum_{i=1}^{n}\frac{1}{\sigma_i^2} - \frac{n}{2}\log 2\pi - \frac{1}{2}\sum_{i=1}^{n}\log\sigma_i^2\right\}$$

By Factorization Theorem:

$T(\underline{X}) = \sum_{i=1}^{n}\frac{X_i}{\sigma_i^2}$ is a sufficient statistic for θ

$\therefore$ The weighted average $\frac{T(\underline{X})}{\sum_{j=1}^{n}\sigma_j^{-2}} = \frac{\sum_{i=1}^{n}\sigma_i^{-2}X_i}{\sum_{i=1}^{n}\sigma_i^{-2}}$

which is an estimator for θ $\square$

29. $P(X=x) = \begin{cases} p_1, & x=1 \\ p_2, & x=2 \\ p_3, & x=3 \end{cases}$

$$P(\underline{X}) = \prod_{i=1}^{n} P(X_i) = p_1^{n_1}\, p_2^{n_2}\, p_3^{n_3}$$

$$\left(\begin{array}{l} \text{where} \quad n_k = \#\{X_i = k\}, \quad k = 1, 2, 3 \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad i = 1, \ldots, n \\ \\ \text{and} \quad n_1 + n_2 + n_3 = n \end{array}\right)$$

$$= p_1^{n_1}\, p_2^{n_2}\, p_3^{n-n_1-n_2} = \left(\frac{p_1}{p_3}\right)^{n_1} \left(\frac{p_2}{p_3}\right)^{n_2} p_3^{n}$$

Let $T(\underline{X}) = (n_1, n_2) = \left(\sum_{i=1}^{n} I(X_i = 1), \sum_{i=1}^{n} I(X_i = 2)\right)$

$h(\underline{X}) = 1$, $g(T(\underline{X}); p_1, p_2, p_3) = \left(\frac{p_1}{p_3}\right)^{n_1} \left(\frac{p_2}{p_3}\right)^{n_2} p_3^{n}$

By Factorization Theorem:

$\left(\sum_{i=1}^{n} I(X_i = 1), \sum_{i=1}^{n} I(X_i = 2)\right)$ is a 2-dim. sufficient statistic $\square$

30. $\{X_i\}_{i=1}^{n} \overset{\text{indep.}}{\sim} Ber(p_i)$

$$p(\underline{X}) = \prod_{i=1}^{n} p_i^{X_i} (1-p_i)^{(1-X_i)} \quad , \quad X_i = 0, 1$$

$$= \prod_{i=1}^{n} \left(\frac{p_i}{1-p_i} \right)^{X_i} (1-p_i)$$

$$= \prod_{i=1}^{n} \exp\left\{ X_i \log\left(\frac{p_i}{1-p_i} \right) + \log(1-p_i) \right\}$$

$$= \exp\left\{ \alpha \sum_{i=1}^{n} X_i + \beta \sum_{i=1}^{n} t_i X_i + \sum_{i=1}^{n} \log(1-p_i) \right\}$$

$$\frac{p(\underline{X})}{p(\underline{Y})} = \exp\left\{ \alpha \left[\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Y_i \right] + \beta \left[\sum_{i=1}^{n} t_i X_i - \sum_{i=1}^{n} t_i Y_i \right] \right\}$$

$= C(\underline{X}, \underline{Y})$ without (α, β)

$$\Rightarrow T(\underline{X}) = \left(\sum_{i=1}^{n} X_i , \sum_{i=1}^{n} t_i X_i \right) = \left(\sum_{i=1}^{n} Y_i , \sum_{i=1}^{n} t_i Y_i \right) = T(\underline{Y})$$

$\therefore \left(\sum_{i=1}^{n} X_i , \sum_{i=1}^{n} t_i X_i \right)$ is the minimal statistic for (α, β) $\square$