69. Problem 10.3 in Casella and Berger (2001).

70. Problem 10.4(b) in Casella and Berger (2001).

71. **(proof of consistency of MLE for 1-d parameter θ and univariable function $\tau(\theta)$)** Problems 10.8 and 10.7 in Casella and Berger (2001).

72. **(super-efficient)** Problem 10.12 in Casella and Berger (2001).

73. Let $X_1, \ldots, X_n$ i.i.d. from the Pareto distribution with density $f(x;\theta) = c^{1/\theta}\theta^{-1}x^{-(1+1/\theta)}, x > c$, where $c > 0$ is known.
    (a) If $\theta < 1$, find a Method of Moments estimate of θ using the first moment.
    (b) Compute its asymptotic efficiency with respect to the information bound.

74. **(1-d exponential family)** Problem 19 of Keener (2010) Section 9.10, p:188.

75. **(bivariate normal)** Problem 28 of Keener (2010) Section 9.10, p:191.

76. Discuss the procedure to bootstrap samples to estimate the variance of the median of the chi-square distribution with df=4. Then use software to perform your procedure by generating a random sample of size $n = 50$ and $B = 2000$ bootstrap samples. Show code and summary output and report the results.

Read Example 19.2 of Keener (2010), pp:393-394.
Practice
10.1, 10.6, 10.9(a)(b)(c) in Casella and Berger (2001).
Problems 12 and 27 of Keener (2010) Section 9.10, pp:187-191.

HW 11 110024516 統研碩一 邱繼賢

69. (a)

$$L(\theta; \underline{X}) = \prod_{i=1}^{n} f(x_i; \theta) = (2\pi\theta)^{\frac{-n}{2}} \exp\left\{ \frac{-1}{2\theta} \sum_{1}^{n} (x_i - \theta)^2 \right\}$$

$$l(\theta; \underline{X}) = \log L = \frac{-n}{2} \log(2\pi\theta) - \frac{1}{2\theta} \sum_{1}^{n} (X_i - \theta)^2$$

$$\frac{dl}{d\theta} = \frac{-n}{2\theta} - \frac{1}{2}\left[\frac{-2\theta \Sigma(X_i - \theta) - \Sigma(X_i - \theta)^2}{\theta^2} \right] \overset{set}{=\!=} 0$$

$$\Rightarrow \theta^2 + \theta - \frac{\Sigma X_i^2}{n} = 0 \Rightarrow \theta^2 + \theta - W = 0, \text{ and } \theta > 0 \quad \therefore \hat{\theta}_{MLE} = \frac{-1 + \sqrt{1 + 4W}}{2} \quad \square$$

(b)

$$\frac{dl}{d\theta} = \frac{-n}{2\theta} + \frac{\Sigma(X_i - \theta)}{\theta} + \frac{\Sigma(X_i - \theta)^2}{2\theta^2} = \frac{\Sigma X_i^2}{2\theta^2} - \frac{n}{2\theta} - n + \frac{1}{2}$$

$$\Rightarrow \frac{d^2 l}{d\theta^2} = \frac{-\Sigma X_i^2}{\theta^3} + \frac{n}{2\theta^2} = \frac{-2\Sigma X_i^2 + n\theta}{2\theta^3}$$

$$I(\theta) = -E_\theta\left[\frac{d^2 l}{d\theta^2}\right] = E_\theta\left[\frac{2\Sigma X_i^2 - n\theta}{2\theta^3} \right] = \frac{2n\theta + n}{2\theta^2}$$

$$\therefore Var(\hat{\theta}_{MLE}) = \frac{1}{I(\theta)} = \frac{2\theta^2}{2n\theta + n} \quad \square$$

70. (b)

$$\Sigma Y_i = \beta \Sigma X_i + \Sigma \varepsilon_i \Rightarrow \frac{\Sigma Y_i}{\Sigma X_i} = \beta + \frac{\Sigma \varepsilon_i}{\Sigma X_i}$$

$$E\left[\frac{\Sigma Y_i}{\Sigma X_i}\right] = \beta + E\left[\frac{\Sigma \varepsilon_i}{\Sigma X_i}\right] \approx \beta + \frac{E(\Sigma \varepsilon_i)}{E(\Sigma X_i)} = \beta + \frac{\Sigma E(\varepsilon_i)}{\Sigma E(X_i)} = \beta + \frac{0}{n\mu} = \beta$$

$$Var\left[\frac{\Sigma Y_i}{\Sigma X_i}\right] = Var\left[\frac{\Sigma \varepsilon_i}{\Sigma X_i}\right] \approx \frac{Var(\Sigma \varepsilon_i)}{[E(\Sigma X_i)]^2} = \frac{n\sigma^2}{n^2\mu^2} = \frac{\sigma^2}{n\mu^2} \quad \square$$

71. 10.8 (a)

$$\frac{1}{\sqrt{n}} l'(\theta_0|\underline{X}) = \frac{1}{\sqrt{n}} \frac{d}{d\theta}\left[\sum_1^n \log f(x_i|\theta)\right] = \frac{1}{\sqrt{n}}\left[\sum_1^n \frac{\frac{d}{d\theta} f(x_i|\theta)}{f(x_i|\theta)}\right] = \sqrt{n}\left[\frac{1}{n}\sum W_i\right]$$

By CLT:

$$\xrightarrow{D} N(E(W_i), Var(W_i)) = N(0, I(\theta_0)) \quad \square$$

(b)

$$\frac{-1}{n} l''(\theta_0|\underline{X}) = \frac{-1}{n}\frac{d}{d\theta}\left[\sum_1^n \frac{\frac{d}{d\theta} f(x_i|\theta)}{f(x_i|\theta)}\right] = \frac{-1}{n}\left\{\sum_1^n \frac{\left[\frac{d^2}{d\theta^2} f(x_i|\theta)\right] f(x_i|\theta) - \left[\frac{d}{d\theta} f(x_i|\theta)\right]^2}{f(x_i|\theta)^2}\right\}$$

$$= \frac{1}{n}\sum_1^n\left[\frac{\frac{d}{d\theta} f(x_i|\theta)}{f(x_i|\theta)}\right]^2 - \frac{1}{n}\sum_1^n \frac{\frac{d^2}{d\theta^2} f(x_i|\theta)}{f(x_i|\theta)} = \frac{1}{n}\sum W_i^2 - \frac{1}{n}\sum \frac{\frac{d^2}{d\theta^2} f(x_i|\theta)}{f(x_i|\theta)}$$

By WLLN:

$$\xrightarrow{P} I(\theta_0) + 0 = I(\theta_0) \quad \square$$

10.9

Assume that $\tau(\theta)$ is differentiable at $\theta = \theta_0$ (true value of parameter)

By Delta Method:

$$\sqrt{n}\left(\tau(\hat{\theta}) - \tau(\theta_0)\right) \xrightarrow{D} N\left(0, Var(\hat{\theta})\,\tau'(\theta_0)^2\right) = N\left(0, \frac{\tau'(\theta_0)^2}{I(\theta_0)}\right)$$

where $V(\theta_0) = \frac{\tau'(\theta_0)^2}{I(\theta_0)}$ is the CRLB

By Thm 10.1.12

$\tau(\hat{\theta})$ is a consistent and asymptotically efficient estimator of $\tau(\theta)$ $\square$

72.

Let $\{X_i\}_{i=1}^{n} \overset{iid}{\sim} N(\theta, 1)$,

$Var(\bar{X}) = \frac{1}{n}$ is the CRLB for unbiased estimator of μ

$$d_n = \begin{cases} \bar{X}, & \text{if } |\bar{X}| \geq n^{-\frac{1}{4}} \\ a\bar{X}, & \text{if } |\bar{X}| < n^{-\frac{1}{4}} \end{cases}$$

By the proof in class:

① For $\theta \neq 0$

$$P(|\bar{X}| \geq n^{-\frac{1}{4}}) \longrightarrow 1, \quad \text{as } n \to \infty$$

② For $\theta = 0$

$$P(|\bar{X}| < n^{-\frac{1}{4}}) \longrightarrow 1, \quad \text{as } n \to \infty$$

at this time $Var(d_n) = \frac{a^2}{n}$

if $a < 1$, $Var(d_n) <$ CRLB

then d_n is a superefficient estimator □

73. (a)

$$E(X) = \int_c^\infty c^{\frac{1}{\theta}} \theta^{-1} x^{-\frac{1}{\theta}} \, dx = c^{\frac{1}{\theta}} \theta^{-1} \left[\frac{1}{\frac{-1}{\theta}+1} x^{\frac{-1}{\theta}+1} \Big|_{x=c}^{\infty} \right]$$

$$= \frac{-c^{\frac{1}{\theta}}}{\theta-1} c^{\frac{-1}{\theta}+1} = \frac{c}{1-\theta} \stackrel{\text{set}}{=\!=\!=} \bar{X} \Rightarrow \hat{\theta}_{MoM} = 1 - \frac{c}{\bar{X}} \quad \square$$

(b) Let $Y = \log \frac{X}{c} \Rightarrow X = ce^Y \Rightarrow J = \frac{dX}{dY} = ce^Y$

$$f(y) = c^{\frac{1}{\theta}} \theta^{-1} (ce^y)^{-(1+\frac{1}{\theta})} ce^y = \frac{1}{\theta} e^{-\frac{y}{\theta}}, \; y > 0 \sim Exp(\tfrac{1}{\theta})$$

$\Rightarrow \hat{\theta}_{MLE} = \bar{Y}$ and $Var(\bar{Y}) = \frac{\theta^2}{n}$ is the CRLB

$$\therefore \sqrt{n}(\bar{Y} - \theta) \xrightarrow{D} N(0, \tfrac{\theta^2}{n})$$

And by CLT:

$$\sqrt{n}\left(\bar{X} - \frac{c}{1-\theta}\right) \xrightarrow{D} N(0, Var(X)) = N\left(0, \frac{\theta^2 c^2}{(1-\theta)^2(1-2\theta)}\right)$$

Let $g(X) = 1 - \frac{c}{X}$, and $g(X)$ is differential at $X = \bar{X}$

$\Rightarrow g'(X) = \frac{c}{X^2}$, then $g'\left(\frac{c}{1-\theta}\right)^2 = \frac{(1-\theta)^4}{c^2}$

By Delta Method:

$$\sqrt{n}\left(g(\bar{X}) - g\left(\frac{c}{1-\theta}\right)\right) \xrightarrow{D} N\left(0, g'\left(\frac{c}{1-\theta}\right)^2 Var(X)\right) = N\left(0, \frac{\theta^2(1-\theta)^2}{1-2\theta}\right)$$

$$\therefore ARE = \frac{\frac{\theta^2}{n}}{\frac{\theta^2(1-\theta)^2}{1-2\theta}} = \frac{1-2\theta}{n(1-\theta)^2}, \quad \theta < \frac{1}{2} \quad \square$$

74. $L(\theta; \underline{X}) = \prod_{i=1}^{n} h(X_i) \exp\{ \eta(\theta) \sum_{1}^{n} T(X_i) - n B(\theta) \}$

$\ell(\theta; \underline{X}) = \sum_{1}^{n} \log h(X_i) + \eta(\theta) \sum_{1}^{n} T(X_i) - n B(\theta)$

$\ell''(\theta; \underline{X}) = \eta''(\theta) \sum_{1}^{n} T(X_i) - n B''(\theta)$

$nI(\theta) = -E(\ell''(\theta)) = -\eta''(\theta) \sum_{1}^{n} E(T(X_i)) + n B''(\theta)$

$= -n\eta''(\theta) B'(\theta) \frac{d\theta}{d\eta} + n B''(\theta)$

and we known that $\ell'(\hat{\theta}) = 0 \Rightarrow n B'(\hat{\theta}) = \eta'(\hat{\theta}) \sum_{1}^{n} T(X_i)$

$\Rightarrow \frac{-\ell''(\hat{\theta})}{nI(\hat{\theta})} = \frac{-\eta''(\hat{\theta}) \Sigma T(X_i) + n B''(\hat{\theta})}{-n B'(\hat{\theta}) \eta''(\hat{\theta}) \frac{d\theta}{d\eta} + n B''(\hat{\theta})} = \frac{-\eta''(\hat{\theta}) \Sigma T(X_i) + n B''(\hat{\theta})}{-\eta'(\hat{\theta}) \Sigma T(X_i) \eta''(\hat{\theta}) \eta'(\hat{\theta})^{-1} + n B''(\hat{\theta})} = 1$ $\square$

75. (a)

$$L(\theta; \underset{\sim}{X}, \underset{\sim}{Y}) = (2\pi)^{-n}(1-\rho^2)^{\frac{-n}{2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\sum_1^n\left[(x_i-\mu_x)^2 - 2\rho(x_i-\mu_x)(y_i-\mu_y) + (y_i-\mu_y)^2\right]\right\}$$

$$l(\theta; \underset{\sim}{X}, \underset{\sim}{Y}) = -n\log(2\pi) - \frac{n}{2}\log(1-\rho^2) - \frac{1}{2(1-\rho^2)}\sum_1^n\left[(X_i-\mu_x)^2 - 2\rho(X_i-\mu_x)(Y_i-\mu_y) + (Y_i-\mu_y)^2\right]\}$$

set $\Rightarrow$
$$\begin{cases} \frac{\partial l}{\partial \mu_x} = \frac{1}{1-\rho^2}\left[\Sigma X_i - n\mu_x + \rho\Sigma Y_i - n\rho\mu_y\right] = 0 \\ \frac{\partial l}{\partial \mu_y} = \frac{1}{1-\rho^2}\left[\Sigma Y_i - n\mu_y + \rho\Sigma X_i - n\rho\mu_x\right] = 0 \\ \frac{\partial l}{\partial \rho} = \frac{1}{(1-\rho^2)^2}\left[n\rho(1-\rho^2) + (1+\rho^2)\Sigma(X_i-\mu_x)(Y_i-\mu_y)\right] = 0 \end{cases} \Rightarrow \begin{cases} \hat{\mu}_x = \bar{X} \\ \hat{\mu}_Y = \bar{Y} \\ \hat{\rho} = \frac{1}{n}\sum_1^n (X_i-\bar{X})(Y_i-\bar{Y}) \end{cases} \square$$

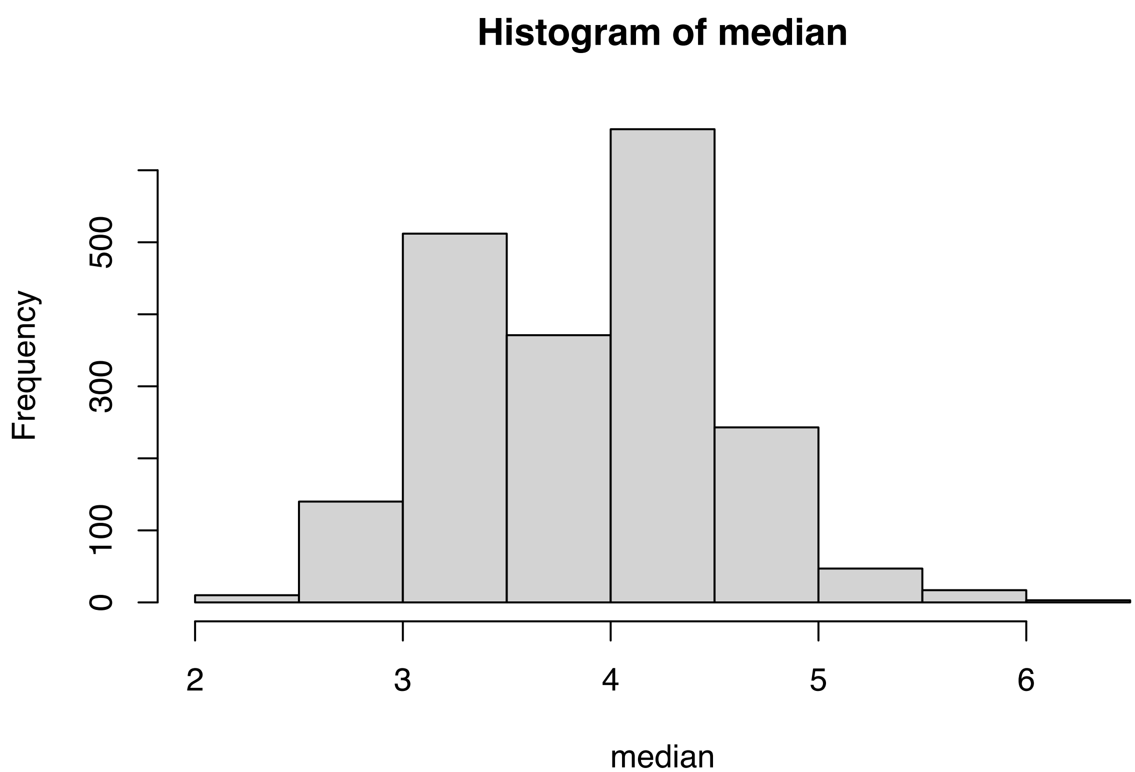
(b)
$$l_1 = -\log(2\pi) - \frac{1}{2}\log(1-\rho^2) - \frac{1}{2(1-\rho^2)}\left[(X-\mu_x)^2 - 2\rho(X-\mu_x)(y-\mu_y) + (y-\mu_y)^2\right]$$

$$\Rightarrow \begin{cases} \frac{\partial l_1}{\partial \mu_x} = \frac{1}{1-\rho^2}\left[x-\mu_x - \rho y + \rho\mu_y\right] \\ \frac{\partial l_1}{\partial \mu_y} = \frac{1}{1-\rho^2}\left[y-\mu_y - \rho x + \rho\mu_x\right] \\ \frac{\partial l_1}{\partial \rho} = \frac{\rho}{1-\rho^2} + \frac{(1+\rho^2)(x-\mu_x)(y-\mu_y)}{(1-\rho^2)^2} \end{cases} \Rightarrow \begin{cases} \frac{\partial^2 l_1}{\partial \mu_x^2} = \frac{-1}{1-\rho^2}, \quad \frac{\partial^2 l_1}{\partial \mu_y^2} = \frac{-1}{1-\rho^2}, \quad \frac{\partial^2 l_1}{\partial \mu_x \partial \mu_y} = \frac{\rho}{1-\rho^2} \\ \frac{\partial^2 l_1}{\partial \rho\,\partial \mu_x} = \frac{-(1+\rho^2)(y-\mu_y)}{(1-\rho^2)^2}, \quad \frac{\partial^2 l_1}{\partial \rho\,\partial \mu_y} = \frac{-(1+\rho^2)(x-\mu_x)}{(1-\rho^2)^2} \\ \frac{\partial^2 l_1}{\partial \rho^2} = \frac{1+\rho^2}{(1-\rho^2)^2} + \frac{2\rho(\rho^2+3)}{(1-\rho^2)^3}(x-\mu_x)(y-\mu_y) \end{cases}$$

$$I(\theta) = \begin{bmatrix} E\left(\frac{-\partial l_1^2}{\partial \mu_x^2}\right) & E\left(\frac{-\partial^2 l_1}{\partial \mu_y \partial \mu_x}\right) & E\left(\frac{-\partial^2 l_1}{\partial \rho\,\partial \mu_x}\right) \\ E\left(\frac{-\partial l_1}{\partial \mu_x \partial \mu_y}\right) & E\left(\frac{-\partial^2 l_1}{\partial \mu_y^2}\right) & E\left(\frac{-\partial^2 l_1}{\partial \rho\,\partial \mu_y}\right) \\ E\left(\frac{-\partial^2 l_1}{\partial \mu_x \partial \rho}\right) & E\left(\frac{-\partial^2 l_1}{\partial \mu_y \partial \rho}\right) & E\left(\frac{-\partial^2 l_1}{\partial \rho^2}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{1-\rho^2} & \frac{-\rho}{1-\rho^2} & 0 \\ \frac{-\rho}{1-\rho^2} & \frac{1}{1-\rho^2} & 0 \\ 0 & 0 & -\frac{\rho^4+6\rho^3+1}{(1-\rho^2)^3} \end{bmatrix}$$

76.

```
set.seed(1214)
data = rchisq(50, 4)
med = c()
par(mfrow = c(1,1))
set.seed(12145)
for (b in 1:2000) {
    bootstrap = sample(data, 50, replace = T)
    med[b] = median(bootstrap)
}
hist(med, xlab = "median", main = "Histogram of median")
```




```
(sd(med))^2
```

```
## [1] 0.4098483
```

The sample variance of the median of the chi-square distribution with df = 4 is

$$\hat{\sigma}^2 \;=\; 0.4098483$$