Reliability Analysis Homework 5

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Problem 1.

(a)

```
library("DirichletReg")
qsev=function(p){
  log(qweibull(p,1,1))
}
psev=function(x){
  pweibull(exp(x),1,1)
}
ld=function(x,mu,sig){ #log density function
  z=(log(x)-mu)/sig
  (z-exp(z))-log(sig)-log(x)
}
lS=function(x,mu,sig){ #log survival function
  z=(log(x)-mu)/sig
  -exp(z)
}
logL=function(theta,ti,di,ri,w){
  mu=theta[1];sig=abs(theta[2])
  l=0
  for(i in 1:length(ti)){
    l=l+di[i]*w[i]*ld(ti[i],mu,sig)+(ri[i])*w[i]*lS(ti[i],mu,sig)
  }
  -l
}
```

```r
library(knitr)
library(dplyr)
bearing = read.csv("BearingCage.csv")
data1 = bearing %>% group_by(Hours) %>%
    summarise(di = ifelse(Censoring.Indicator=="Failed",Count,0),
              ri = ifelse(Censoring.Indicator=="Censored",Count,0)) %>%
    ungroup()
kable(data1, col.names = c("$t_k$","$d_k$","$r_k$"))
```

t_k	d_k	r_k
50	0	288
150	0	148
230	1	0
250	0	124
334	1	0
350	0	111
423	1	0
450	0	106
550	0	99
650	0	110
750	0	114
850	0	119
950	0	127
990	1	0
1009	1	0
1050	0	123
1150	0	93
1250	0	47
1350	0	41
1450	0	27
1510	1	0
1550	0	11
1650	0	6
1850	0	1
2050	0	2

Likelihood function for ith bootstrap sampling

$$L\left(\mu_i\ ,\ \sigma_i\ ;\ t\right)\ =\ \prod_{k=1}^{25}\left\{\left[\frac{1}{\sigma_i t_k}\phi_{\text{sev}}\left(\frac{\log(t_k)-\mu_i}{\sigma_i}\right)\right]^{d_k w_k}\left[1-\Phi_{sev}\left(\frac{\log(t_k)-\mu_i}{\sigma_i}\right)\right]^{r_k w_k}\right\}$$

where $w\ =\ (w_1,...,w_{25})\ \sim\ 25\times$uniform Dirichlet distribution

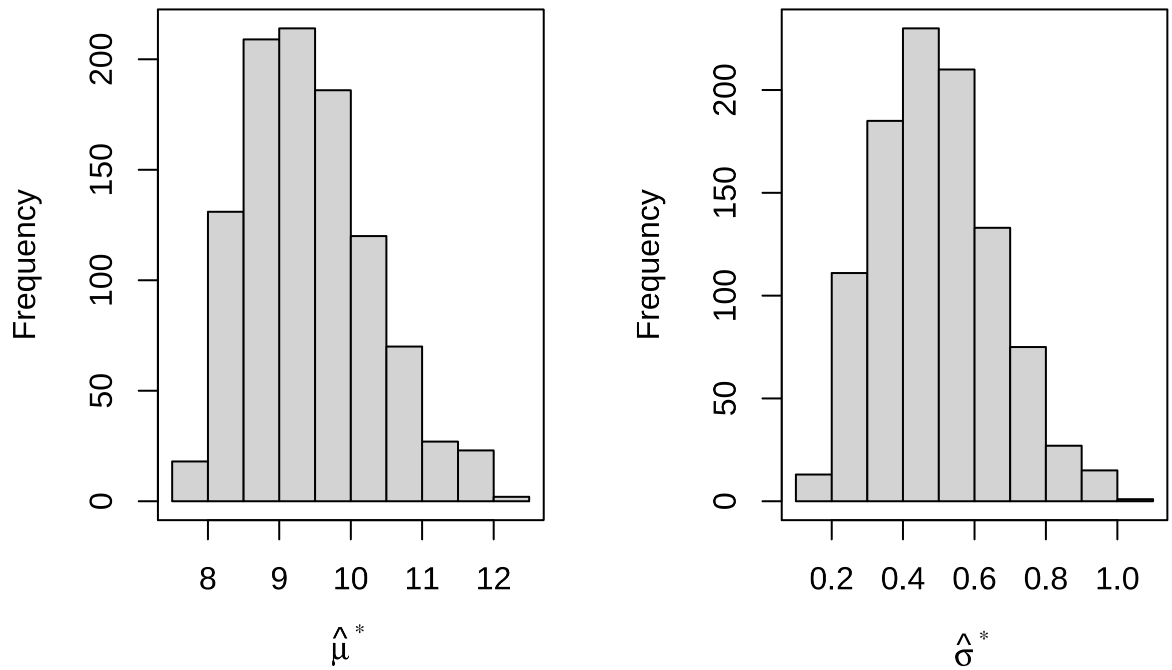
Then compute bootstrap ML estimates

$$(\hat{\mu}_i^*\ ,\ \hat{\sigma}_i^*)\ =\ \arg max\{\log L\left(\mu_i\ ,\ \sigma_i\ ;\ t\right)\}$$

Iterate the above steps for $i = 1, ..., 1000$

```
mu_star = sig_star = c()
set.seed(0231)
for (i in 1:1000) {
    w = 25*rdirichlet(1,rep(1,25))
    op=optim(c(8,0.2),ti=data1$Hours,di=data1$di,ri=data1$ri,w=w,logL,hessian=F)
    mle=op$p
    mu_star[i] = mle[1] ; sig_star[i] = abs(mle[2])
}
```

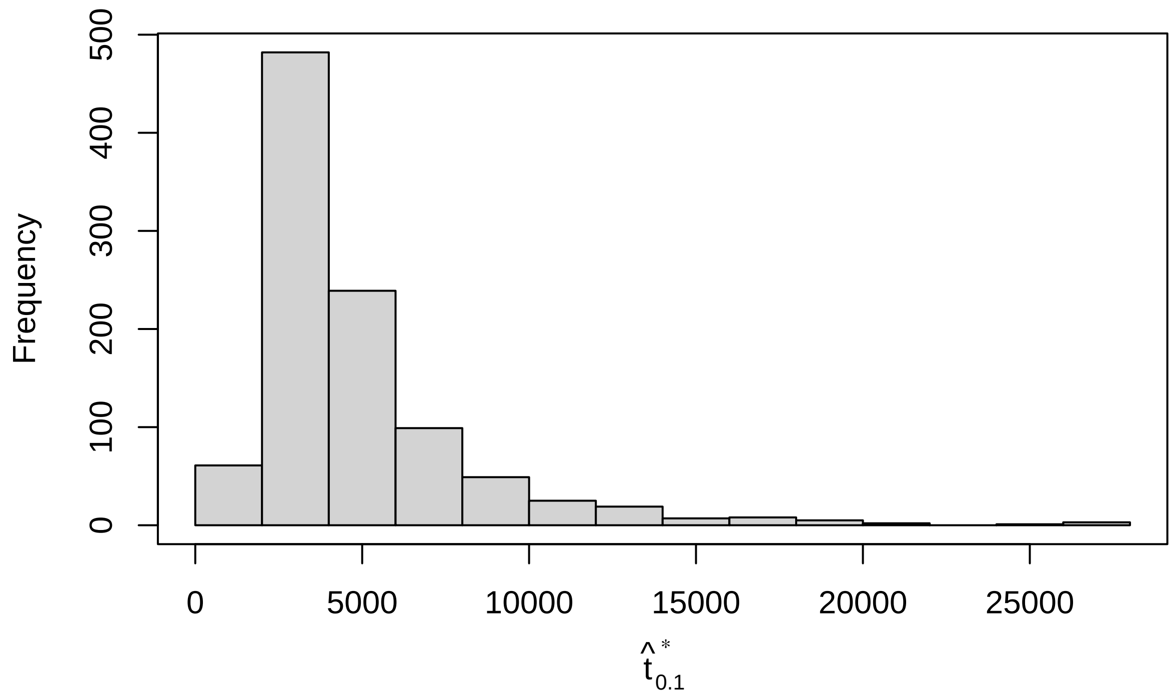
```
library(latex2exp)
par(mfrow = c(1,2))
hist(mu_star, main = "", xlab = TeX("$\\hat{\\mu}^*$")) ; box()
hist(sig_star, main = "", xlab = TeX("$\\hat{\\sigma}^*$")) ; box()
```



(b)

$$\hat{t}^*_{0.1} \;=\; \exp\left[\hat{\mu}^* \;+\; \Phi^{-1}_{\text{sev}}(0.1)\;\hat{\sigma}^*\right]$$

```
t0.1_star = exp(mu_star + qsev(0.1)*sig_star)
hist(t0.1_star, xlab = TeX("$\\hat{t}^*_{0.1}$"), main = "") ; box()
```



(c)

```
quantile(t0.1_star,c(0.025,0.975))
```

```
##      2.5%     97.5%
##  1811.622 14263.024
```

```
table = data.frame(a = c("Bootstrap","Wald","LR-based"),
                   low = c(1811.622, 160.4161, 2122.449),
                   upp = c(14263.024, 7641.4650,22185.714))
kable(table,col.names = c("", "2.5%","97.5%"))
```

	2.5%	97.5%
Bootstrap	1811.6220	14263.024
Wald	160.4161	7641.465

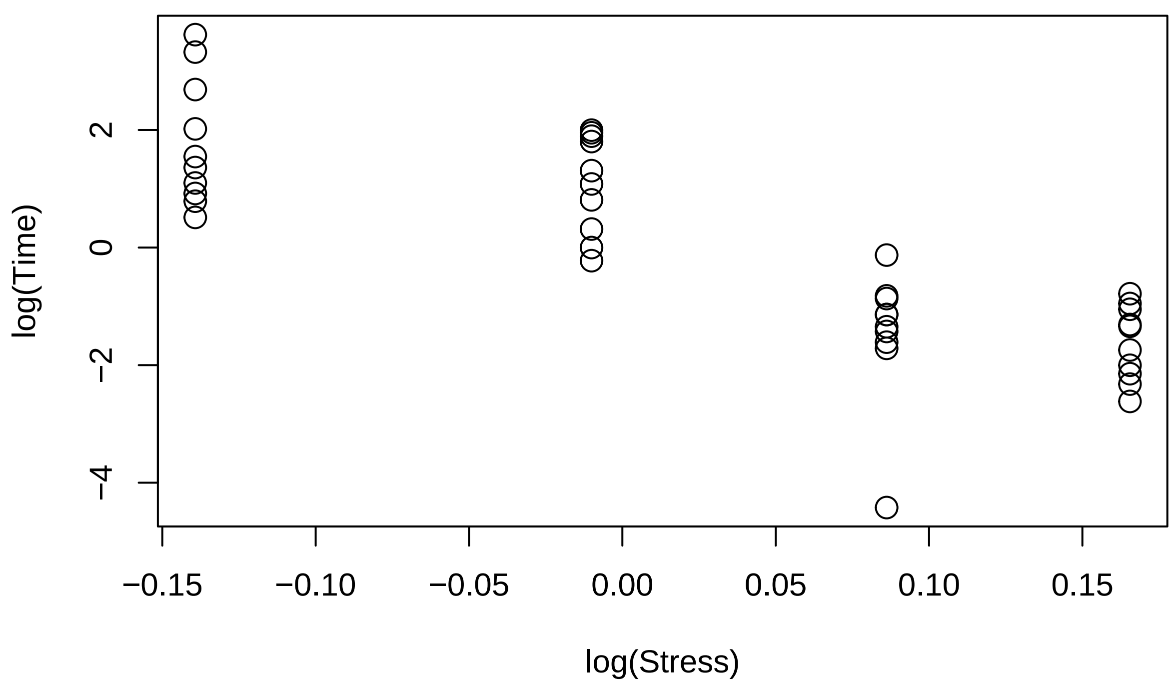
	2.5%	97.5%
LR-based	2122.4490	22185.714

We can see that the interval by bootstrap is much wider than the Wald one, and relatively closed to the LR-based one.

Problem 2.

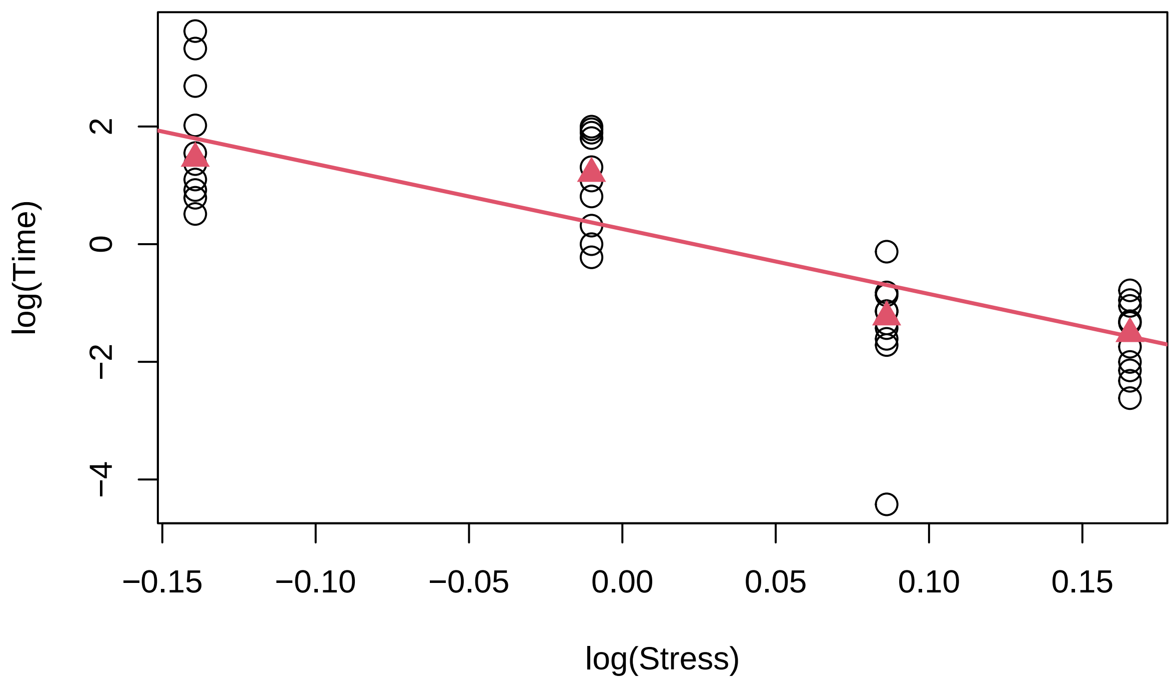
(a)

```
data2 = read.csv("CeramicBearing02.csv")
colnames(data2) = c("time","stress")
plot(log(data2$stress), log(data2$time), xlab = "log(Stress)", ylab = "log(Time)", cex = 1.5)
```



(b)

```
plot(log(data2$stress), log(data2$time), xlab = "log(Stress)", ylab = "log(Time)", cex = 1.5)
median = data2 %>% group_by(stress) %>%
    summarise(me = median(time)) %>%
    ungroup()
points(log(median$stress),log(median$me), pch = 17, col = 2, cex = 1.5)
fit = lm(log(me)~log(stress), median)
abline(fit, col = 2, lwd = 2)
```



We can see that the log median failure times are approximately fall around a straight line, so we can try to fit $\log(t_{0.50}) = \beta_0 + \beta_1 \log(\text{stress})$ in this case.

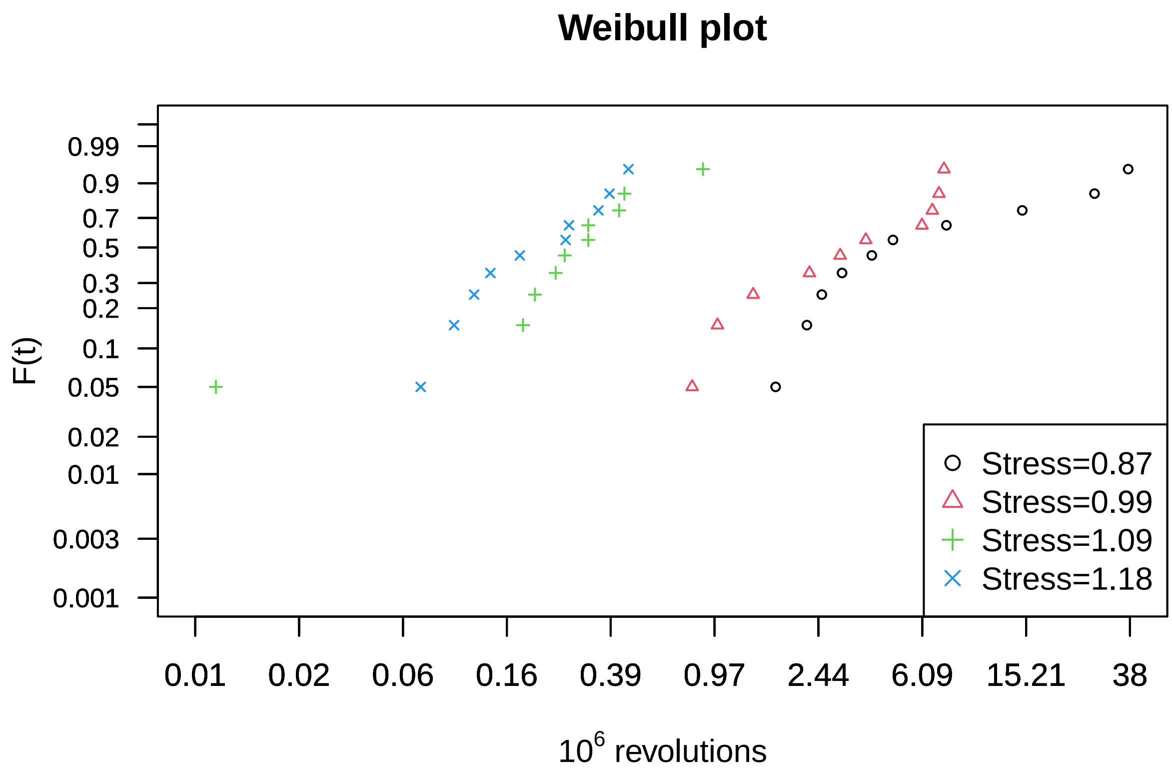
(c)

```
ti = data2[,1] ; xi = data2[,2] ; di = rep(1,40)
group=c()
```

```r
for(i in 1:length(xi)){
  group[i]=rank(c(xi[i],unique(xi)),tie="min")[1]
}
```

```r
for(k in 1:4){
  tj=ti[which(group==k)];dj=di[which(group==k)]
  dj=dj[order(tj)];tj=sort(tj)
  nj=length(tj):1
  pj=dj/nj
  Fj=1-Reduce("*",1-pj,acc=T)
  tab=cbind(tj,dj,nj,1-pj,Fj)[which(dj==1),]

  y=c(0,tab[,5])
  y=(y[-1]+y[-length(y)])/2
  if(k==1){
    plot(log(tab[,1]),qsev(y),xlim=log(c(0.01,38)),ylim=qsev(c(.001,.999)),cex=0.6,col=k,
         xlab=TeX("$10^6$ revolutions"),ylab="F(t)",yaxt="n",xaxt="n",main = "Weibull plot")
  }else{
    points(log(tab[,1]),qsev(y),cex=0.6,pch=k,col=k,
         yaxt="n",xaxt="n")
  }
  y_lab=c(.001,.003,.01,.02,.05,.1,.2,.3,.5,.7,.9,.99,.999)
  axis(2,log(-log(1-y_lab)),y_lab,cex.axis=0.8,las=1)
  x_lab=exp(seq(log(0.01),log(38),len=10))
  axis(1,log(x_lab),round(x_lab,2),las=1)
  #abline(a=-mle[2*k-1]/mle[2*k],b=1/mle[2*k],col=k)
}
legend("bottomright",
       legend = c("Stress=0.87","Stress=0.99","Stress=1.09","Stress=1.18"),
       col = 1:4, pch = 1:4)
```



1. 在四種 Stress 的設定值下，資料在 Weibull plot 上分布的斜率並沒有太顯著差異，與題目的假設：shape parameter $\beta = \frac{1}{\sigma}$ does not depend on stress 一致

2. 隨著 Stress 的數值上升，資料在 Weibull plot 上分布的 y 截距 $(-\frac{\mu}{\sigma})$ 隨之上升，也就是說 scale parameter μ 會隨著 Stress 的數值上升而變小

(d)

Fit the model

$$\log(t_{p,\text{Stress}}) \ = \ \mu_{\text{Stress}} \ + \ \Phi^{-1}_{\text{sev}}(p) \ \sigma \ = \ \beta_0 \ + \ \beta_1 \times \log(\text{Stress}) \ + \ \Phi^{-1}_{\text{sev}}(p) \ \sigma$$

and estimate the parameter $(\beta_0 \ , \ \beta_1 \ , \ \sigma)$ by the ML method

```
#summary(lm(log(ti)~log(xi)))#initial
logL=function(theta,ti,di,xi){
  b0=theta[1];b1=theta[2];sig=theta[3]
  if(sig<0.001)sig=0.001
```

```
  mu=b0+b1*xi
  l=0
  for(i in 1:length(ti)){
    l=l+di[i]*ld(ti[i],mu[i],sig)+(1-di[i])*lS(ti[i],mu[i],sig)
  }
  -l
}
op=optim(c(0,2,1),ti=ti,di=di,xi=log(xi),logL,hessian=T)
mle = op$p
mle
```

```
## [1]   0.7886737 -13.8886708   0.8577896
```

Therefore,

$$
\begin{aligned}
\hat{t}_{p,\text{Stress}} &= \exp\left[\hat{\beta}_0 \ + \ \hat{\beta}_1 \times \log(\text{Stress}) \ + \ \Phi^{-1}_{\text{sev}}(p)\ \hat{\sigma}\right] \\
&= \exp\left[0.7886737 \ - \ 13.8886708 \times \log(\text{Stress}) \ + \ \Phi^{-1}_{\text{sev}}(p) \times 0.8577896\right]
\end{aligned}
$$

(e)

$$\hat{t}_{0.5,1.05} \ = \ \exp\left[\hat{\beta}_0 \ + \ \hat{\beta}_1 \times \log(1.05) \ + \ \Phi^{-1}_{\text{sev}}(0.5)\ \hat{\sigma}\right]$$

```
exp(mle[1] + mle[2]*log(1.05) + qsev(0.5)*mle[3])
```

```
## [1] 0.8159944
```

$$\hat{t}_{0.01,1.05} \ = \ \exp\left[\hat{\beta}_0 \ + \ \hat{\beta}_1 \times \log(1.05) \ + \ \Phi^{-1}_{\text{sev}}(0.01)\ \hat{\sigma}\right]$$

```
exp(mle[1] + mle[2]*log(1.05) + qsev(0.01)*mle[3])
```

```
## [1] 0.02160313
```

$$\hat{t}_{0.01,0.85} \ = \ \exp\left[\hat{\beta}_0 \ + \ \hat{\beta}_1 \times \log(0.85) \ + \ \Phi^{-1}_{\text{sev}}(0.01)\ \hat{\sigma}\right]$$

```r
exp(mle[1] + mle[2]*log(0.85) + qsev(0.01)*mle[3])
```

```
## [1] 0.4065231
```