7. For the gamma distribution given in Problem 5, find the mean and variance by applying differential identities discussed in Section 2.2 of Keener.

8. Let $p_\eta(x)$ be a one-parameter canonical exponential family with **non-decreasing** sufficient statistic $T(x)$, where $x \in \mathcal{X} \subseteq \mathbb{R}$:

$$p_\eta(x) = e^{\eta T(x) - A(\eta)} h(x).$$

   Let $\psi(x)$ be any non-decreasing bounded function. Show that, for $\eta \in \Xi_1^\circ$, $\frac{d}{d\eta}\mathbb{E}_\eta[\psi(X)] \geq 0$, where Ξ_1° is given in Theorem 2.4 of Keener with $f = 1$. You may apply Keener Theorem 2.4 to justify differentiating under the integral sign.

9. Let the sample space S be the interval $(-1, 1)$ with the uniform probability distribution. Define the sequence $X_1, X_2, \ldots$ as $X_n(s) = (-1)^n \times s$.
   Verify if $X_n \to X$ **in distribution** as $n \to \infty$, where $X(s) = s$.

10. $X_1, \ldots X_n$ are iid uniform $(0, 1)$ and $X_{(n)} = \max_{1 \leq i \leq n} X_i$. Derive the distribution $F_n(t)$ of $n(1 - X_{(n)})$ and examine $\lim_{n\to\infty} F_n(t)$.

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7. By problem 5.

$$A(\eta) = \log \Gamma(\eta_1 + 1) - (\eta_1 + 1)\log(-\eta_2)$$

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} k-1 \\ \frac{-1}{\theta} \end{pmatrix}, \quad T(x) = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} \log x \\ x \end{pmatrix}$$

$$E(X) = E\left(T_2(X)\right) = \frac{dA(\eta)}{d\eta_2}$$

$$= \frac{-(\eta_1 + 1)}{\eta_2} = k\theta$$

$$Var(X) = Cov\left[T_2(X), T_2(X)\right] = \frac{d^2 A(\eta)}{d\eta_2^2}$$

$$= \frac{\eta_1 + 1}{\eta_2^2} = k\theta^2 \quad \square$$

8. $p_\eta(x) = \exp[\eta T(x) - A(\eta)]\, h(x)$

$$\Rightarrow \frac{d}{d\eta} p_\eta(x) = [T(x) - A'(\eta)] \exp[\eta T(x) - A(\eta)]\, h(x)$$

$$= [T(x) - A'(\eta)]\, p_\eta(x) = [T(x) - E_\eta(T(x))]\, p_\eta(x)$$

$$E_\eta[\psi(X)] = \int \psi(x)\, p_\eta(x)\, d\mu(x)$$

$$\Rightarrow \frac{d}{d\eta} E_\eta[\psi(x)] = \int \frac{\partial}{\partial\eta}[\psi(x)\, p_\eta(x)]\, d\mu(x)$$

$$= \int \psi(x)\, T(x)\, p_\eta(x)\, d\mu(x) - E_\eta[T(x)] \int \psi(x)\, p_\eta(x)\, d\mu(x)$$

$$= E[\psi(x)\, T(x)] - E[T(x)]\, E[\psi(x)] = \mathrm{Cov}[\psi(x), T(x)]$$

$\because$ $T(x)$ and $\psi(x)$

are both non-decreasing function of x

$$\therefore \frac{d}{d\eta} E_\eta[\psi(x)] = \mathrm{Cov}[\psi(x), T(x)] \geq 0 \quad \square$$

9. $F_n(x) = P(X_n(s) \le x) = P((-1)^n s \le x)$

$\because s \sim U(-1,1) \Rightarrow -s \sim U(-1,1)$

$$\therefore F_n(x) = P((-1)^n s \le x) = \begin{cases} 0, & x < -1 \\ \frac{x+1}{2}, & -1 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

And, $F_X(x) = P(X(s) \le x) = P(s \le x)$

$$= \begin{cases} 0, & x < -1 \\ \frac{x+1}{2}, & -1 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

$\therefore X_n \longrightarrow X$ in distribution as $n \to \infty$.

10. $\{X_i\}_{i=1}^{n} \overset{iid}{\sim} U(0,1)$

$$F_n(t) = P(T_n \le t) = P\left(n(1-X_{(n)}) \le t\right)$$

$$= P\left(X_{(n)} \ge 1-\frac{t}{n}\right) = 1 - P\left(X_{(n)} < 1-\frac{t}{n}\right)$$

$$= 1 - P\left\{\left(X_1 < 1-\frac{t}{n}\right) \cap \cdots \cap \left(X_n < 1-\frac{t}{n}\right)\right\}$$

$$= 1 - \prod_{i=1}^{n} P\left(X_i < 1-\frac{t}{n}\right) = 1 - \left\{P\left(X_i < 1-\frac{t}{n}\right)\right\}^n$$

$$= 1 - \left(1-\frac{t}{n}\right)^n, \quad 0 < t < n$$

$$\lim_{n\to\infty} F_n(t) = 1 - \lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n$$

$$= 1 - e^{-t} \sim Exp(\lambda = 1) \quad \square$$