11. When $X_1, X_2, \ldots$ converges in distribution to X and $Y_1, Y_2, \ldots$ converges in distribution to Y, discuss whether $X_1+Y_1, X_2+Y_2, \ldots$ converges in distribution to $X+Y$. If true, give the proof, and if not true, give an example.

12. If $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$, show that $X_n Y_n \xrightarrow{p} XY$.

13. Problem 5.32 in Casella and Berger (2001).

14. Problem 5.39 (a) in Casella and Berger (2001).

15. (Weak Law of Large Numbers for pairwise un-correlated sequence)
Let $X_1, X_2, \ldots$ be pairwise un-correlated random variables with the same mean μ and positive variance σ^2 which are both finite. Show that as $n \to \infty$,
$$\frac{X_1 + \cdots + X_n}{n} \to \mu \text{ in probability.}$$

16. If $X_1, X_2, \ldots$ are independent and identically distributed random variables with finite k-th moment m_k, then what condition is needed to show
$$\frac{1}{n}(X_1^k + \cdots + X_n^k) \to m_k \text{ almost surely?}$$

17. Problem 9 is repeated here:
Let the sample space S be the interval $(-1, 1)$ with the uniform probability distribution. Define the sequence $X_1, X_2, \ldots$ as $X_n(s) = (-1)^n \times s$.
In Problem 9, you were asked to verify if $X_n \to X$ in distribution as $n \to \infty$, where $X(s) = s$.
Now does $X_n \to X$ in probability? Explain.

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1). It is true only when X_n and Y_n are independent random variable

(we don't consider about cases that X_n or Y_n converge to a constant)

① Prove for independent cases

Suppose $X_n \perp\!\!\!\perp Y_n$, $\forall n$

$\Rightarrow e^{itX_n} \perp\!\!\!\perp e^{itY_n}$, $\forall n$

$\because X_n \xrightarrow{D} X$, $Y_n \xrightarrow{D} Y$

$$\therefore \begin{cases} \phi_{X_n}(t) = E(e^{itX_n}) \to E(e^{itX}) = \phi_X(t) \\ \phi_{Y_n}(t) = E(e^{itY_n}) \to E(e^{itY}) = \phi_Y(t) \end{cases}$$

as $n \to \infty$

Then

$$\phi_{X_n+Y_n}(t) = E\left(e^{it(X_n+Y_n)}\right) = E\left(e^{itX_n}\right)E\left(e^{itY_n}\right)$$

$$\xrightarrow{n\to\infty} E\left(e^{itX}\right)E\left(e^{itY}\right) = E\left(e^{it(X+Y)}\right) = \phi_{X+Y}(t)$$

$\therefore$ $X_n + Y_n \longrightarrow X+Y$ in distribution

② Counter example for dependent cases

Let $\{Z_i\}_1^n$ iid $E(Z_i) = 0$, $Var(Z_i) = 1$

And $X_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_i$, $Y_n = -X_n$

By CLT : $\begin{cases} X_n \longrightarrow N(0,1) \\ Y_n \longrightarrow N(0,1) \end{cases}$ in distribution

However $X_n + Y_n$ is always zero $\square$

12. $\forall a, b \in \mathbb{R}$ $a^2+2ab+b^2 \geq 0$ and $a^2-2ab+b^2 \geq 0$

$\Rightarrow |ab| \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$

$\forall \varepsilon > 0$,

① $P\left(|(X_n - X)(Y_n - Y)| > \frac{\varepsilon}{3}\right)$

$\leq P\left(\frac{1}{2}(X_n - X)^2 + \frac{1}{2}(Y_n - Y)^2 > \frac{\varepsilon}{3}\right)$

$\leq P\left(\frac{1}{2}(X_n - X)^2 > \frac{\varepsilon}{6}\right) + P\left(\frac{1}{2}(Y_n - Y)^2 > \frac{\varepsilon}{6}\right)$

$= P\left(|X_n - X| > \sqrt{\frac{\varepsilon}{3}}\right) + P\left(|Y_n - Y| > \sqrt{\frac{\varepsilon}{3}}\right)$

converges to 0 as $n \to \infty$

② for any $k > 0$

$\because \left\{|(X_n - X)Y| > \frac{\varepsilon}{3}\right\} \subset \{|Y| > k\} \cup \left\{|X_n - X| > \frac{\varepsilon}{3k}\right\}$

$\therefore P\left(|(X_n - X)Y| > \frac{\varepsilon}{3}\right)$

$\leq P(|Y| > k) + P\left(|X_n - X| > \frac{\varepsilon}{3k}\right)$

$\Rightarrow P(|(X_n - X)Y| > \frac{\varepsilon}{3}) \leq P(|Y| > k)$ for any k

Let $k \to \infty$, then $P(|Y| > k) \to 0$

$\therefore P(|(X_n - X)Y| > \frac{\varepsilon}{3})$ converges to 0 as $n \to \infty$

③ Similarly as above,

$P(|(Y_n - Y)X| > \frac{\varepsilon}{3})$ converges to 0 as $n \to \infty$

By ① ② ③:

$P(|X_n Y_n - XY| > \varepsilon)$

$= P(|(X_n - X)(Y_n - Y) + (X_n - X)Y + (Y_n - Y)X| > \varepsilon)$

$\leq P(|(X_n - X)(Y_n - Y)| > \frac{\varepsilon}{3}) + P(|(X_n - X)Y| > \frac{\varepsilon}{3}) + P(|(Y_n - Y)X| > \frac{\varepsilon}{3})$

$\Rightarrow P(|X_n Y_n - XY| > \varepsilon)$ converges to 0 as $n \to \infty$

$\therefore X_n Y_n \xrightarrow{P} XY$ $\square$

13.

(a) $\because P(X_i > 0) = 1 \quad \forall i$

$\therefore P(X_i \leq 0) = 0 \quad \forall i$

Define $h_1(x) = \sqrt{x}$, $h_2(x) = \frac{a}{x}$ are both continuous functions for $x > 0$

By continuous mapping theorem:

$$h_1(X_i) = \sqrt{X_i} = Y_i \xrightarrow{p} \sqrt{a}$$

$$h_2(X_i) = \frac{a}{X_i} = Y_i' \xrightarrow{p} 1$$

$\square$

(b) We have known that $S_n^2 \xrightarrow{P} \sigma^2$

and $P(S_n^2 > 0) = 1$

Define $h_2(X) = \frac{\sigma}{X}$ and $h_1(x)$ same as above

By continuous mapping theorem:

$$h_1(S_n^2) = \sqrt{S_n^2} = S_n \xrightarrow{P} \sigma$$

and $h_2(S_n) = \frac{\sigma}{S_n} \xrightarrow{P} 1$ $\square$

14. (a) $h(\cdot)$ is a continuous function

$\forall \varepsilon > 0, \exists \delta > 0$, s.t.

$|h(X_n) - h(X)| < \varepsilon$ whenever $|X_n - X| < \delta$

$\therefore \{(X_n, X) : |h(X_n) - h(X)| < \varepsilon\} \supseteq \{(X_n, X) : |X_n - X| < \delta\}$

$\Rightarrow P(|h(X_n) - h(X)| < \varepsilon) \geq P(|X_n - X| < \delta)$

$\Rightarrow \lim_{n\to\infty} P(|h(X_n) - h(X)| < \varepsilon) \geq \lim_{n\to\infty} P(|X_n - X| < \delta) = 1$

$\Rightarrow \lim_{n\to\infty} P(|h(X_n) - h(X)| < \varepsilon) = 1$

$\therefore h(X_n) \xrightarrow{P} h(X)$ $\square$

15. $\because \{X_i\}_1^n$ are pairwise un-correlated

$$\therefore E[(X_i-\mu)(X_j-\mu)] = 0, \quad \forall\, i \neq j$$

Define $L_n = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)$

$$E(L_n) = 0$$

$$E(L_n^2)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{\sigma^2}{n} \to 0, \text{ as } n \to \infty$$

$$\therefore L_n \xrightarrow{\text{2nd mean}} 0 \Rightarrow L_n \xrightarrow{P} 0$$

By continuous mapping theorem

$$\frac{X_1 + \cdots + X_n}{n} \longrightarrow \mu \text{ in probability } \square$$

16. $\{X_i\}_1^n \overset{iid}{\sim} E(X_i^k) = m_k$

Define S is a sample space

and has elements donoted by $\omega \in S$

Then we need to show, $\forall \varepsilon > 0$

$$P\left(\omega : \lim_{n \to \infty} \left| \frac{1}{n}\left(X_1^k(\omega) + \cdots + X_n^k(\omega)\right) - m_k \right| < \varepsilon\right) = 1$$

to ensure that

$$\frac{1}{n}(X_1^k + \cdots + X_n^k) \longrightarrow m_k \text{ almost surely } \square$$

17. $S \sim U(-1, 1) \Rightarrow |S| \sim U(0, 1)$

$\forall \varepsilon > 0$, $P(|X_n(S) - X(S)| < \varepsilon)$

$= P(|S((-1)^n - 1)| < \varepsilon)$

$$= \begin{cases} P(0 < \varepsilon) = 1 & , n \text{ is even} \\ P(|S| < \frac{\varepsilon}{2}) = \frac{\varepsilon}{2} & , n \text{ is odd} \end{cases}$$

$\therefore X_n$ doesn't converge in probability to X $\square$