38. Problem 7.7 in Casella and Berger (2001).

39. Problem 7.11 in Casella and Berger (2001).

40. Problem 7.14 in Casella and Berger (2001) (you could use results from Exercise 4.26 directly).

41. Problem 7.19 in Casella and Berger (2001).

42. Problem 1 of Keener (2010) Section 7.4.

43. Problem 8 of Keener (2010) Section 7.4.

44. Let X_1, X_2, X_3 be iid observations from a Cauchy distribution $f(x;\theta) = \frac{1}{\pi(1+(x-\theta)^2)}$, $-\infty < \theta < \infty$.
    (a) Could you derive a closed form expression for the roots of the likelihood equation (the first derivative = 0)?
    (b) Suppose that $X_1 = 0, X_2 = 2, X_3 = 10$. Using any graphical software, plot the log likelihood for $0 < \theta < 10$. Are there multiple local maximum? Does an MLE exist?

45. Show code and summary output. Each student should use different random seeds to generate data.
    Use R to perform the multinomial EM example in Dempster, Laird, and Rubin (1977). (1) Set a true value of $0 < \theta < 1$. (2) Generate a random sample of (Y_1, Y_2, Y_3, Y_4) of $n = 300$ independent trials with multinomial probabilities $(1/2 - \theta/2, \theta/4, \theta/4, 1/2)$. Write the likelihood of complete data. (3) Assume that you observe Y_1, Y_2, Y_3+Y_4, but not Y_3. Give the expressions of $E(Y_3|Y_3+Y_4)$ and $E(Y_4|Y_3+Y_4)$. (4) With an initial value of $\theta^{(0)} = 0.5$, perform the EM algorithm to find MLE of θ. Report the values of $\theta^{(j)}$ in each iterations.

    Practice
    Problems 7.6, 7.10, 7.22, and 7.24 in Casella and Berger (2001).
    Examples 7.2.16 and 7.3.4 in Casella and Berger (2001).
    Examples 7.3, 7.4 and Problem 7.3 of Chapter 7, Keener (2010).

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38. $L(\theta=0 \mid \underset{\sim}{X}) = \prod_{i=1}^{n} I(0<X_i<1)$

$L(\theta=1 \mid \underset{\sim}{X}) = \prod_{i=1}^{n} \frac{1}{2\sqrt{X_i}} I(0<X_i<1)$

$\therefore \hat{\theta}_{MLE} = \begin{cases} 0, & \text{if } 1 \geq \prod_{i=1}^{n} \frac{1}{2\sqrt{X_i}} \\ 1, & \text{if } 1 < \prod_{i=1}^{n} \frac{1}{2\sqrt{X_i}} \end{cases}$ $\square$

39.(a) $L(\theta \mid \underset{\sim}{X}) = \theta^n \prod_{i=1}^{n} X_i^{\theta-1} I(0 \leq X_i \leq 1)$

$\ell(\theta \mid \underset{\sim}{X}) = n \log\theta + (\theta-1)\sum_{i=1}^{n} \log X_i \quad , \quad 0 \leq X_i \leq 1$

$\frac{d}{d\theta}\ell(\theta \mid \underset{\sim}{X}) = \frac{n}{\theta} + \sum_{i=1}^{n} \log X_i \overset{set}{=\!=} 0 \Rightarrow \hat{\theta} = \left(\frac{-1}{n}\sum_{i=1}^{n}\log X_i\right)^{-1}$

check $\frac{d^2}{d\theta^2}\ell(\theta \mid \underset{\sim}{X}) = \frac{-n}{\theta^2} < 0$ for $0 < \theta < \infty$

$\therefore \hat{\theta}_{MLE} = \left(\frac{-1}{n}\sum_{i=1}^{n}\log X_i\right)^{-1}$

Let $Y = -\log X \Rightarrow X = e^{-Y} \Rightarrow J = -e^{-Y}$

$$f_Y(y;\theta) = f_X(e^{-y};\theta)|J| = \theta e^{-\theta y}, \quad y > 0$$

$$\Rightarrow \{Y_i\}_1^n \overset{iid}{\sim} Exp(\theta) \Rightarrow T = \sum_{i=1}^n Y_i \sim Gamma(\alpha = n, \lambda = \theta)$$

$$E\left(\frac{1}{T}\right) = \int_0^\infty \frac{\theta^n}{\Gamma(n)} t^{n-2} e^{-\theta t}\, dt = \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-1)}{\theta^{n-1}} = \frac{\theta}{n-1}$$

$$E\left(\frac{1}{T^2}\right) = \int_0^\infty \frac{\theta^n}{\Gamma(n)} t^{n-3} e^{-\theta t}\, dt = \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-2)}{\theta^{n-2}} = \frac{\theta^2}{(n-1)(n-2)}$$

$$\Rightarrow Var\left(\frac{1}{T^2}\right) = \frac{\theta^2}{(n-1)(n-2)} - \frac{\theta^2}{(n-1)^2} = \frac{\theta^2}{(n-1)^2(n-2)}$$

$$\therefore Var(\hat{\theta}_{MLE}) = Var\left(\frac{n}{T^2}\right) = \frac{n^2\theta^2}{(n-1)^2(n-2)} \xrightarrow{n\to\infty} 0$$

$\square$

(b) $X \sim Beta(\theta, 1) \Rightarrow E(X) = \dfrac{\theta}{\theta+1} \overset{set}{=\!=} \bar{X}$

$$\therefore \hat{\theta}_{MOM} = \frac{\bar{X}}{1-\bar{X}}$$ $\square$

40. $P(Z \le z, W=0) = P(Y \le z, Y \le X)$

$$= \int_0^z \int_y^\infty \frac{1}{\lambda\mu} \exp\left\{ \frac{-x}{\lambda} - \frac{y}{\mu} \right\} dx\, dy$$

$$= \frac{\lambda}{\lambda+\mu} \left\{ 1 - \exp\left[-\left(\frac{1}{\mu} + \frac{1}{\lambda}\right) z \right] \right\}$$

$P(Z \le z, W=1) = P(X \le z, X \le Y)$

$$= \int_0^z \int_x^\infty \frac{1}{\lambda\mu} \exp\left\{ \frac{-x}{\lambda} - \frac{y}{\mu} \right\} dy\, dx$$

$$= \frac{\mu}{\lambda+\mu} \left\{ 1 - \exp\left[-\left(\frac{1}{\mu} + \frac{1}{\lambda}\right) z \right] \right\}$$

$$\Rightarrow f(z \mid W=0) = \frac{d}{dz} P(Z \le z, W=0) = \frac{1}{\mu} \exp\left[-\left(\frac{1}{\mu} + \frac{1}{\lambda}\right) z \right]$$

$$f(z \mid W=1) = \frac{d}{dz} P(Z \le z, W=1) = \frac{1}{\lambda} \exp\left[-\left(\frac{1}{\mu} + \frac{1}{\lambda}\right) z \right]$$

$$\therefore f(z, w) = \left(\frac{1}{\lambda}\right)^w \left(\frac{1}{\mu}\right)^{1-w} \exp\left[-\left(\frac{1}{\mu} + \frac{1}{\lambda}\right) z \right], \quad z \ge 0,\ w = 0, 1$$

$$L(\mu,\lambda \mid \underline{(z,w)}) = \left(\frac{1}{\lambda}\right)^{\Sigma w_i} \left(\frac{1}{\mu}\right)^{n-\Sigma w} \exp\left[-\left(\frac{1}{\mu}+\frac{1}{\lambda}\right)\Sigma z_i\right]$$

$$l(\mu,\lambda \mid \underline{(z,w)}) = -\Sigma w_i \log\lambda - (n-\Sigma w_i)\log\mu - \left(\frac{1}{\mu}+\frac{1}{\lambda}\right)\Sigma z_i$$

$$\Rightarrow \begin{cases} \dfrac{dl}{d\mu} = \dfrac{\Sigma w_i - n}{\mu} + \dfrac{\Sigma z_i}{\mu^2} \overset{set}{=\!=} 0 \\[2ex] \dfrac{dl}{d\lambda} = \dfrac{-\Sigma w_i}{\lambda} + \dfrac{\Sigma z_i}{\lambda^2} \overset{set}{=\!=} 0 \end{cases}$$

$$\therefore \begin{cases} \hat{\mu}_{MLE} = \dfrac{\bar{z}}{1-\bar{w}} \\[2ex] \hat{\lambda}_{MLE} = \dfrac{\bar{z}}{\bar{w}} \end{cases} \quad \square$$

41. (a) $\{Y_i\}_1^n \overset{\text{indep}}{\sim} N(\beta X_i, \sigma^2)$

$$L(\beta, \sigma^2 | \underline{Y}) = (2\pi\sigma^2)^{\frac{-n}{2}} \exp\left\{\frac{-1}{2\sigma^2}\left[\sum_1^n Y_j^2 - 2\beta\sum_1^n X_j Y_j + \beta^2 \sum_1^n X_i^2\right]\right\}$$

$$= (2\pi\sigma^2)^{\frac{-n}{2}} \exp\left\{\frac{-\beta^2 \sum_1^n X_i^2}{2\sigma^2}\right\} \exp\left\{\frac{-1}{2\sigma^2}\left[\sum_1^n Y_j^2 - 2\beta\sum_1^n X_i Y_i\right]\right\}$$

By Factorization Theorem:

$\left(\sum_{i=1}^n Y_i^2, \sum_{j=1}^n X_i Y_i\right)$ is a sufficient statistic for (β, σ^2) $\square$

(b) $\ell(\beta, \sigma^2 | \underline{Y}) = \frac{-n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\left[\sum_1^n Y_i^2 - 2\beta\sum_1^n X_i Y_i + \beta^2 \sum_1^n X_i^2\right]$

$$\frac{d\ell}{d\beta} = \frac{1}{\sigma^2}\sum_1^n X_i Y_i - \frac{1}{\sigma^2}\beta\sum_1^n X_i^2 \overset{\text{set}}{=\!=} 0 \Rightarrow \hat{\beta}_{MLE} = \frac{\sum_1^n X_j Y_j}{\sum_1^n X_i^2}$$

$$E(\hat{\beta}_{MLE}) = \frac{\sum_1^n X_i E(Y_i)}{\sum_1^n X_j^2} = \frac{\beta \sum_1^n X_i^2}{\sum_1^n X_i^2} = \beta$$

$\therefore \hat{\beta}_{MLE}$ is an unbiased estimator of β $\square$

(c) $\because \hat{\beta}_{MLE}$ is a linear combination of Y_i which follows a normal distribution

$\therefore \hat{\beta}_{MLE}$ also follows a normal distribution

$$Var(\hat{\beta}_{MLE}) = \frac{\sum_1^n x_i^2 Var(Y_i)}{(\sum_1^n x_i^2)^2} = \frac{\sigma^2}{\sum_1^n x_i^2}$$

$$\therefore \hat{\beta}_{MLE} \sim N\left(\beta, \frac{\sigma^2}{\sum_1^n x_i^2}\right) \quad \square$$

---

42. $\Theta \sim Exp(\eta)$, $\{X_i \mid \Theta = \theta\}_1^n \overset{iid}{\sim} Poi(\theta)$

$$k(\theta \mid \underline{X}) \propto_\theta \lambda(\theta) f(\underline{X} \mid \theta) \propto_\theta e^{-\eta\theta} e^{-n\theta} \theta^{\sum_1^n X_i}$$

$$= \theta^{\sum_1^n X_i} e^{-(\eta+n)\theta}$$

$$\Rightarrow \Theta \mid \underline{X} \sim Gamma\left(\alpha = \sum_1^n X_i + 1, \lambda = n + \eta\right)$$

We have to minimum $E(L(\theta, d) \mid \underline{X}) = E\left[\theta^{9}(d-\theta)^2 \mid \underline{X}\right)$

$$\Rightarrow \frac{d}{dd}\left[d^2 E(\theta^p|\underline{X}) - 2d E(\theta^{p+1}|\underline{X}) + E(\theta^{p+2}|\underline{X}) \right]$$

$$= 2d E(\theta^p|\underline{X}) - 2E(\theta^{p+1}|\underline{X}) \overset{set}{=\!=\!=} 0$$

$$\Rightarrow \text{The Bayes estimator } \delta(X) = \frac{E(\theta^{p+1}|\underline{X})}{E(\theta^p|\underline{X})}$$

$$= \frac{\int_0^\infty \theta^{p+1}\, \theta^{\Sigma X_i} (n+\eta)^{\Sigma X_i+1} e^{-(n+\eta)\theta}\, d\theta}{\int_0^\infty \theta^{p}\, \theta^{\Sigma X_i} (n+\eta)^{\Sigma X_i+1} e^{-(n+\eta)\theta}\, d\theta} \quad \left(\begin{array}{l} \text{Let } t = (n+\eta)\theta \\ dt = (n+\eta)\, d\theta \end{array} \right)$$

$$= \frac{\int_0^\infty t^{p+1+\Sigma X_i} e^{-t}\, dt}{\int_0^\infty t^{p+\Sigma X_i} e^{-t}\, dt\, (n+\eta)} = \frac{\Gamma(p+2+\Sigma X_i)}{\Gamma(p+1+\Sigma X_i)(n+\eta)}$$

$$= \frac{p+1+\sum_{i=1}^{n} X_i}{n+\eta} \qquad \square$$

43. $\Theta \sim Exp(1)$, $X \mid \Theta = \theta \sim P_\theta(x)$

(a) $f_{\Theta,X}(\theta, x) = e^{-\theta} e^{\theta - x} = e^{-x}$, $0 < \theta < x$

$f_X(x) = \int_0^x e^{-x}\, d\theta = \theta e^{-x} \Big|_0^x = x e^{-x}$, $x > 0$

$\Rightarrow X \sim \text{Gamma}(\alpha = 2, \lambda = 1)$

$\therefore E(X) = \frac{\alpha}{\lambda} = 2$ □

(b) $\pi(\theta \mid x) = \frac{f(x \mid \theta) f(\theta)}{f(x)} = \frac{e^{\theta - x}\, e^{-\theta}}{x e^{-x}} = \frac{1}{x}$, $x > \theta > 0$

$\Rightarrow \Theta \mid X \sim U(0, x)$

$\therefore$ The Bayes estimator for Θ under squared error loss

$\delta(X) = E(\Theta \mid X) = \frac{X}{2}$ □

44. (a) $L(\theta; X_1, X_2, X_3) = \prod_{i=1}^{3} f(X_i; \theta)$

$$= \pi^{-3} \prod_{i=1}^{3} \left(1 + (X_i - \theta)^2\right)^{-1}$$

$$\Rightarrow \ell(\theta; X_1, X_2, X_3) = -3\log \pi - \sum_{i=1}^{3} \log\left(1 + (X_i - \theta)^2\right)$$

$$\frac{d\ell}{d\theta} = \sum_{i=1}^{3} \frac{2(X_i - \theta)}{1+(X_i - \theta)^2}$$

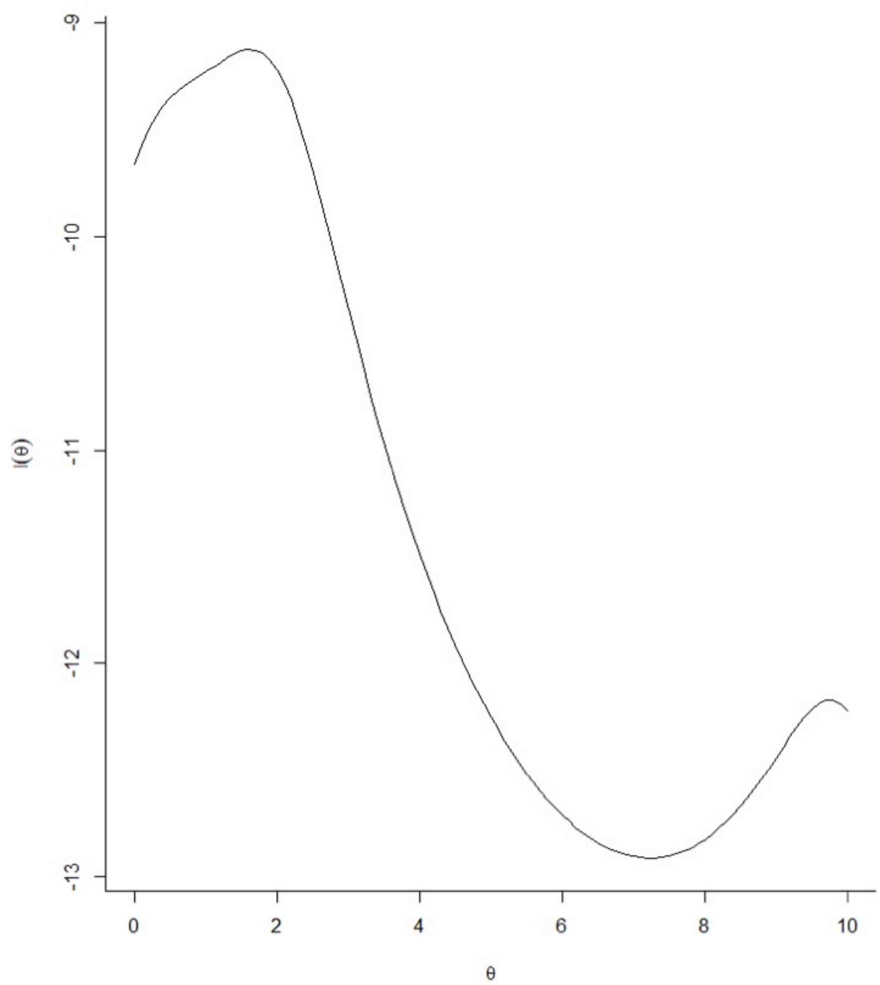
$$= \frac{2(X_1 - \theta)}{1+(X_1-\theta)^2} + \frac{2(X_2 - \theta)}{1+(X_2-\theta)^2} + \frac{2(X_3 - \theta)}{1+(X_3-\theta)^2} \overset{set}{=\!=} 0$$

$$\Rightarrow (X_1 - \theta)\left[1+(X_2-\theta)^2\right]\left[1+(X_3-\theta)^2\right] + (X_2-\theta)\left[1+(X_1-\theta)^2\right]\left[1+(X_3-\theta)^2\right]$$
$$+ (X_3-\theta)\left[1+(X_1-\theta)^2\right]\left[1+(X_2-\theta)^2\right] = 0$$

By Galois Theorem, the degree 5 polynomial do not have close form. Thus, we couldn't derive a closed form expression.

(b) $\ell(\theta ; X_1 = 0, X_2 = 2, X_3 = 10)$

$$= -3\log\pi - \left[\log(1+\theta^2) + \log(1+(2-\theta)^2) + \log(1+(10-\theta)^2)\right]$$



There are local maximum at about $\theta \doteq 2$ and $8 < \theta < 10$

At about $\theta \doteq 2$, that point is the global maximum of the log likelihood function, that is the point where MLE exists. □