77. **(parametric bootstrap)** Let $X_1, \ldots, X_n$ i.i.d. from Bernoulli(p) and $\hat{p} = \sum_i X_i/n$. Denote a bootstrap sample $X_1^*, \ldots, X_n^*$ and let $\hat{p}^* = \sum X_i^*/n$. What is the exact distribution of $n\hat{p}^*$, conditional on $X_1, \ldots, X_n$?

78. **(McNemar test)** Problem 10.32(a) in Casella and Berger (2001) p:512.

79. **(Gamma(α, β), α known)** Problem 10.38 in Casella and Berger (2001) p:513.

80. **(testing equal probabilities for multinomial)** In Example 10.3.4 of Casella and Berger (2001) (class notes pp:9-10), we discussed testing under a multinomial distribution with $\theta = (p_1, p_2, \ldots, p_5), \sum p_j = 1, p_j \geq 0$. Derive the likelihood ratio test for testing $H_0 : p_1 = p_2 = \cdots = p_5 = 0.2$ and state its asymptotic distribution.

81. **(likelihood ratio test at boundary)** For $X_1, \ldots, X_n$ i.i.d. $N(\mu, I_{2\times2})$ with $\mu = (\mu_1, \mu_2)^T$, $\mu_1 \geq 0$ and μ_2 unrestricted. Derive the likelihood ratio test for $H_0 : \mu_1 = 0$ and discuss its distribution.

82. **(two-sample tests for identical normal distribution)** Let $X_1, \ldots, X_n$ be a sample from $N(\mu_x, \sigma_x^2)$ and $Y_1, \ldots, Y_n$ be an independent sample from $N(\mu_y, \sigma_y^2)$. For testing $H_0 : \mu_x = \mu_y$ and $\sigma_x^2 = \sigma_y^2$, derive the likelihood ratio, Wald, and score test statistics and state their asymptotic distributions.

83. **(exponential)** Problem 4(a) of Keener (2010) Section 17.5, p:363. .

Practice
10.33, 10.34(a), 10.35, 10.37 of Casella and Berger (2001)
Problem 1, 10(a) of Keener (2010) Section 17.5.

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77. $\{X_1^*, \dots, X_n^*\}$ are resampled from $\{X_1, \dots, X_n\}$ with replacement

$\Rightarrow n\hat{p}^* = \sum X_i^* \sim Bin(n, \hat{p})$ $\square$

78. under $H_0: p_1 = p_2 = p$

$$L(p, p_3, \dots, p_n \mid \underline{X}) = p^{X_1+X_2} p_3^{X_3} \cdots p_{n-1}^{X_{n-1}} \left(1-2p-\sum_3^{n-1} p_i\right)^{m-X_1-X_2-\cdots-X_{n-1}}$$

$$\ell(p, p_3, \dots, p_n \mid \underline{X}) = (X_1+X_2)\log p + X_3 \log p_3 + \cdots + X_{n-1}\log p_{n-1} + \left(m - \sum_1^{n-1} X_i\right)\log\left(1-2p-\sum_3^{n-1} p_i\right)$$

$$\overset{set}{\Longrightarrow} \begin{cases} \dfrac{\partial \ell}{\partial p} = \dfrac{X_1+X_2}{p} - \dfrac{2\left(m-\sum_1^{n-1} X_i\right)}{1-2p-\sum_3^{n-1} p_i} = \dfrac{X_1+X_2}{p} - \dfrac{2X_n}{p_n} = 0 \\ \dfrac{\partial \ell}{\partial p_i} = \dfrac{X_i}{p_i} - \dfrac{m-\sum_1^{n-1} X_i}{1-2p-\sum_3^{n-1} p_j} = \dfrac{X_i}{p_i} - \dfrac{X_n}{p_n} = 0 \end{cases} \Rightarrow \begin{cases} \hat{p} = \dfrac{X_1+X_2}{2m} \\ \hat{p}_j = \dfrac{X_j}{m} \\ \hat{p}_n = \dfrac{X_n}{m} \end{cases}$$

$$\begin{cases} \text{observed} = O_i = X_i \\ \text{expected} = E_i = m\hat{p}_i = \begin{cases} \dfrac{X_1+X_2}{2}, & i = 1, 2 \\ X_i, & i = 3, \dots, n \end{cases} \end{cases}$$

$$\therefore \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} = \frac{\left(X_1 - \frac{X_1+X_2}{2}\right)^2}{\frac{X_1+X_2}{2}} + \frac{\left(X_2 - \frac{X_1+X_2}{2}\right)^2}{\frac{X_1+X_2}{2}} = \frac{(X_1 - X_2)^2}{X_1 + X_2} \quad \square$$

79. $\{X_i\}_1^n \overset{iid}{\sim} \text{Gamma}(\alpha, \beta)$ where α is known and β is unknown

$$\begin{cases} H_0: \beta = \beta_0 \\ H_1: \beta \neq \beta_0 \end{cases}$$

$$L(\beta; \underline{X}) = \left[\Gamma(\alpha)\, \beta^{\alpha}\right]^{-n} \left(\prod_{i=1}^{n} X_i\right)^{\alpha-1} \exp\left[\frac{-1}{\beta}\sum_1^n X_i\right]$$

$$l(\beta; \underline{X}) = \log L = -n\log\Gamma(\alpha) - n\alpha\log\beta + (\alpha-1)\sum_1^n \log X_i - \frac{1}{\beta}\sum_1^n X_i$$

$$\Rightarrow \begin{cases} S(\beta) = \dfrac{dl}{d\beta} = \dfrac{-n\alpha}{\beta} + \dfrac{1}{\beta^2}\sum_1^n X_i \\ I_1(\beta) = -E\left[\dfrac{d^2 l_1(\beta)}{d\beta^2}\right] = \dfrac{2}{\beta^3}E(X_i) - \dfrac{\alpha}{\beta^2} = \dfrac{\alpha}{\beta^2} \end{cases}$$

$$\therefore S_n = \left[\frac{S(\beta_0)}{\sqrt{nI_1(\beta_0)}}\right]^2 = \frac{n(\bar{X}-\alpha\beta_0)^2}{\alpha\beta_0^2} \xrightarrow{D} \chi_1^2 \text{ under } H_0 \; \square$$

80. $L(p_1, \cdots, p_5) = p_1^{y_1} p_2^{y_2} p_3^{y_3} p_4^{y_4} p_5^{y_5}$, where $y_j = \#\{X_i = j,\ i=1, \cdots, n,\ j=1, \cdots, 5\}$

MLE: $\hat{p}_j = \frac{y_j}{n}$, $j = 1, \cdots, 5$

$$\lambda = \frac{\sup_{H_0} L}{\sup_{H_0 \cup H_1} L} = \frac{(0.2)^n}{\prod_1^5 \left(\frac{y_j}{n}\right)^{y_j}} < k$$

$$\Rightarrow -2\log\lambda = 2n\log 5 + 2\sum_1^5 y_i \log y_i - (2\log n)\sum_1^5 y_i \xrightarrow{D} \chi_4^2$$

Reject H_0 when $-2\log\lambda > k$ $\square$

81. $\{X_i\}_1^n = \left\{\begin{pmatrix} X_{i1} \\ X_{i2} \end{pmatrix}\right\}_1^n \overset{iid}{\sim} N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, I\right)$

$$\begin{cases} H_0: \mu_1 = 0 \\ H_1: \mu_1 \neq 0 \end{cases}$$

under H_0: $\hat{\mu}_{02} = \frac{1}{n}\sum X_{i2} = \bar{X}_2$

under H_1: $\hat{\mu}_{11} = \max\{0, \bar{X}_1\}$, $\hat{\mu}_{12} = \bar{X}_2$

$$\lambda = \frac{\sup_{H_0} L(\mu_2)}{\sup_{H_1} L(\mu_1, \mu_2)} = \frac{\exp\left\{\frac{-1}{2}\left[\sum X_{i1}^2 + \sum (X_{i2} - \bar{X}_2)^2\right]\right\}}{\exp\left\{\frac{-1}{2}\left[\sum (X_{i1} - \hat{\mu}_{11})^2 + \sum (X_{i2} - \bar{X}_2)^2\right]\right\}} = \exp\left\{\frac{-1}{2}\left[\sum X_{i1}^2 - \sum (X_{i1} - \hat{\mu}_{11})^2\right]\right\}$$

① If $\bar{X}_1 \leq 0$, then $\hat{\mu}_{11} = 0 \Rightarrow \lambda = 1$ and $-2\log\lambda = 0$

② If $\bar{X}_1 > 0$, then $\hat{\mu}_{11} = \bar{X}_1 \Rightarrow -2\log\lambda = n\bar{X}_1^2 \longrightarrow \chi_1^2$ □

82. $\{X_i\}_1^n \overset{iid}{\sim} N(\mu_X, \sigma_X^2)$, $\{Y_i\}_1^n \overset{iid}{\sim} N(\mu_Y, \sigma_Y^2)$

$H_0: \mu_X = \mu_Y = \mu$ and $\sigma_X^2 = \sigma_Y^2 = \sigma^2$

① LRT:

$$\lambda = \frac{\sup_{H_0} L(\mu, \sigma^2)}{\sup_{H_0 \cup H_1} L(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2)} = \frac{(\hat{\sigma}^2)^{-n}}{(\hat{\sigma}_X^2)^{\frac{-n}{2}} (\hat{\sigma}_X^2)^{\frac{-n}{2}}} = \frac{\left[\frac{1}{2n}\sum\left((X_i - \frac{\bar{X}+\bar{Y}}{2})^2 + (Y_i - \frac{\bar{X}+\bar{Y}}{2})^2\right)\right]^{-n}}{\left[\frac{1}{n^2}\sum(X_i - \bar{X})^2 \sum(Y_i - \bar{Y})^2\right]^{\frac{-n}{2}}}$$

$$\Rightarrow -2\log\lambda = n\left\{2\log\left[\sum\left(X_i - \frac{\bar{X}+\bar{Y}}{2}\right)^2 + \sum\left(Y_i - \frac{\bar{X}+\bar{Y}}{2}\right)^2\right] - \log\sum(X_i - \bar{X})^2 - \log\sum(Y_i - \bar{Y})^2 - 2\log 2\right\}$$

$\xrightarrow{D} \chi_2^2$ under H_0.

② Wald test:

$$H_0: g_1(\theta) = \mu_X - \mu_Y = 0,\ g_2(\theta) = \sigma_X^2 - \sigma_Y^2 = 0,\quad H_\theta = \left(\frac{\partial g_i}{\partial \theta_j}\right)_{4\times 2} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$l_1(\theta) = \frac{-1}{2}\log(2\pi\sigma_X^2) - \frac{1}{2\sigma_X^2}(x-\mu_X)^2 - \frac{1}{2}\log(2\pi\sigma_Y^2) - \frac{1}{2\sigma_Y^2}(y-\mu_Y)^2$$

$$I_1(\theta) = -E\begin{bmatrix} \frac{\partial^2 l_1}{\partial \mu_X^2} & \frac{\partial^2 l_1}{\partial \mu_X \partial \mu_Y} & \frac{\partial^2 l_1}{\partial \mu_X \partial \sigma_X^2} & \frac{\partial^2 l_1}{\partial \mu_X \partial \sigma_Y^2} \\ \frac{\partial^2 l_1}{\partial \mu_X \partial \mu_Y} & \frac{\partial^2 l_1}{\partial \mu_Y^2} & \frac{\partial^2 l_1}{\partial \mu_Y \partial \sigma_X^2} & \frac{\partial^2 l_1}{\partial \mu_Y \partial \sigma_Y^2} \\ \frac{\partial^2 l_1}{\partial \mu_X \partial \sigma_X^2} & \frac{\partial^2 l_1}{\partial \mu_Y \partial \sigma_X^2} & \frac{\partial^2 l_1}{\partial (\sigma_X^2)^2} & \frac{\partial^2 l_1}{\partial \sigma_X^2 \partial \sigma_Y^2} \\ \frac{\partial^2 l_1}{\partial \mu_X \partial \sigma_Y^2} & \frac{\partial^2 l_1}{\partial \mu_Y \partial \sigma_Y^2} & \frac{\partial^2 l_1}{\partial \sigma_X^2 \partial \sigma_Y^2} & \frac{\partial^2 l_1}{\partial (\sigma_Y^2)^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma_X^2} & & & 0 \\ & \frac{1}{\sigma_Y^2} & & \\ & & \frac{1}{2(\sigma_X^2)^2} & \\ 0 & & & \frac{1}{2(\sigma_Y^2)^2} \end{bmatrix}$$

$$W_n = n\, g(\hat\theta)^T \left(H_{\hat\theta}^T I_1^{-1}(\hat\theta) H_{\hat\theta}\right)^{-1} g(\hat\theta)$$

$$= \frac{n^2(\bar X - \bar Y)^2}{\Sigma\left[(X_i-\bar X)^2 + (Y_i-\bar Y)^2\right]} + \frac{n\left\{\Sigma\left[(X_i-\bar X)^2 + (Y_i - \bar Y)^2\right]\right\}^2}{2\left\{\left[\Sigma(X_i-\bar X)^2\right]^2 + \left[\Sigma(Y_i-\bar Y)^2\right]^2\right\}} \xrightarrow{D} \chi_2^2 \text{ under } H_0$$

③ Score test: $\tilde\theta_0 = (\hat\mu, \hat\sigma^2)$ is the $\hat\theta_{MLE}$ under H_0

$$S(\theta) = \left(\frac{\partial l_n}{\partial \theta}\right)_{4\times 1} = \begin{bmatrix} \frac{1}{\sigma_X^2}\Sigma(X_i - \mu_X) \\ \frac{1}{\sigma_Y^2}\Sigma(Y_i - \mu_Y) \\ \frac{-n}{2\sigma_X^2} + \frac{1}{2(\sigma_X^2)^2}\Sigma(X_i-\mu_X)^2 \\ \frac{-n}{2\sigma_Y^2} + \frac{1}{2(\sigma_Y^2)^2}\Sigma(Y_i-\mu_Y)^2 \end{bmatrix}$$

$$\Rightarrow S_n = n^{-1} S(\tilde\theta_0)^T I^{-1}(\tilde\theta_0) S(\tilde\theta_0)$$

$$= -n + n^2 \frac{(\bar X - \bar Y)^2}{\Sigma(X_i - \frac{\bar X + \bar Y}{2})^2 + \Sigma(Y_i - \frac{\bar X+\bar Y}{2})^2} + 2n\frac{\left[\Sigma(X_i - \frac{\bar X+\bar Y}{2})^2\right]^2 + \left[\Sigma(Y_i - \frac{\bar X+\bar Y}{2})^2\right]^2}{\left[\Sigma(X_i - \frac{\bar X+\bar Y}{2})^2 + \Sigma(Y_i - \frac{\bar X+\bar Y}{2})^2\right]^2}$$

$\xrightarrow{D} \chi_2^2$ under H_0.

83. $\begin{cases} H_0: \theta_x = 2\theta_y \\ H_1: \theta_x \neq 2\theta_y \end{cases}$

$$L(\theta_x, \theta_y) = \theta_x \theta_y \exp\{-\theta_x X - \theta_y Y\}$$

$$\ell(\theta_x, \theta_y) = \log L = \log \theta_x + \log \theta_y - \theta_x X - \theta_y Y$$

$$\xRightarrow{\text{set}} \begin{cases} \frac{\partial \ell}{\partial \theta_x} = \frac{1}{\theta_x} - X = 0 \\ \frac{\partial \ell}{\partial \theta_y} = \frac{1}{\theta_y} - Y = 0 \end{cases} \Rightarrow \begin{cases} \hat{\theta}_x = \frac{1}{X} \\ \hat{\theta}_y = \frac{1}{Y} \end{cases}$$

Under H_0:

$$L(2\theta = \theta_x = 2\theta_y) = 2\theta^2 \exp\{-(2X+Y)\theta\}$$

$$\ell(\theta) = \log \theta = \log 2 + 2\log\theta - (2X+Y)\theta$$

$$\Rightarrow \frac{\partial \ell}{\partial \theta} = \frac{2}{\theta} - (2X+Y) \overset{\text{set}}{=\!=} 0 \Rightarrow \hat{\theta} = \frac{2}{2X+Y}$$

$$\therefore \lambda = \frac{\sup_{H_0} L(\theta)}{\sup_{H_1} L(\theta_x, \theta_y)} = \frac{\frac{8}{(2X+Y)^2}\exp(-2)}{\frac{1}{XY}\exp(-2)} = \frac{8XY}{(2X+Y)^2} < k \ \square$$