62. **(ratio of normal variances)** Problem 9.4 (b)(c) in Casella and Berger (2001) (based on class discussion on 9.4(a)).

63. Problem 9.25 in Casella and Berger (2001).

64. Problem 9.29 in Casella and Berger (2001).

65. Problem 9.41 in Casella and Berger (2001).

66. **(exponential)** Problem 9.45(a)(b)(c) in Casella and Berger (2001).

67. Problem 6(a)(c) of Keener (2010) Section 13.4, p:266.

68. For $X_1, \ldots, X_n$ i.i.d. $\sim Unif(0, \theta)$, construct a shortest-length $(1-\alpha)$ confidence interval based on a pivot $\frac{\bar{X}}{\theta}$. Compare its length with the shortest-length confidence interval in Exercise 9.37 discussed in class.

    Practice
    9.34, 9.50, 9.52 in Casella and Berger (2001).

HW10 110024516 統研碩一 邱繼賢

62. (b)

Note that under $\Omega = \{H_0 \cup H_1\}$: $\hat{\sigma}_X^2 = \frac{1}{n}\sum_1^n X_i^2$, $\hat{\sigma}_Y^2 = \frac{1}{m}\sum_1^m Y_i^2$

$\omega = \{H_0 : \lambda = \lambda_0\}$: $\hat{\sigma}_0^2 = \dfrac{\lambda_0 \sum_1^n X_i^2 + \sum_1^m Y_i^2}{\lambda_0 (m+n)}$

The test rejects H_0 if $\lambda(\underline{X}, \underline{Y}) = \dfrac{(\hat{\sigma}_X^2)^{\frac{n}{2}} (\hat{\sigma}_Y^2)^{\frac{m}{2}}}{\lambda_0^{\frac{m}{2}} (\hat{\sigma}_0^2)^{\frac{n+m}{2}}} < k$,

where k is chosen to give that the test size $= \alpha$

$$\lambda(\underline{X}, \underline{Y}) = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_X^2}\right)^{\frac{-n}{2}} \left(\frac{\lambda_0 \hat{\sigma}_0^2}{\hat{\sigma}_Y^2}\right)^{\frac{-m}{2}}$$

$$= \left[\frac{n}{m+n} + \frac{\left(\frac{\Sigma Y_i^2}{\lambda_0 \sigma_X^2}\right)/m}{\left(\frac{\Sigma X_i^2}{\sigma_X^2}\right)/n} \frac{m}{m+n}\right]^{\frac{-n}{2}} \left[\frac{m}{m+n} + \frac{\left(\frac{\Sigma X_i^2}{\sigma_X^2}\right)/n}{\left(\frac{\Sigma Y_i^2}{\lambda_0 \sigma_X^2}\right)/m} \frac{n}{m+n}\right]^{\frac{-m}{2}}$$

Under H_0 : $\dfrac{\Sigma Y_i^2}{\lambda_0 \sigma_X^2} \sim \chi^2_m$, $\dfrac{\Sigma X_i^2}{\sigma_X^2} \sim \chi^2_n \Rightarrow F = \dfrac{\left(\frac{\Sigma Y_i^2}{\lambda_0 \sigma_X^2}\right)/m}{\left(\frac{\Sigma X_i^2}{\sigma_X^2}\right)/n} \sim F_{m,n}$

$\therefore$ The rejection region is

$$\left\{(x,y) \,\middle|\, \left[\frac{n}{m+n} + \frac{m}{m+n} F\right]^{\frac{n}{2}} \left[\frac{m}{m+n} + \frac{n}{m+n} F^{-1}\right]^{\frac{m}{2}} > k_\alpha \right\}$$

where k_α satisfies

$$P\left\{\left[\frac{n}{m+n} + \frac{m}{m+n} F\right]^{\frac{n}{2}} \left[\frac{m}{m+n} + \frac{n}{m+n} F^{-1}\right]^{\frac{m}{2}} > k_\alpha \right\} = \alpha \quad \square$$

(c). $\left\{ \lambda : \left[\frac{n}{m+n} \left(1 + \frac{\Sigma Y_i^2}{\lambda \Sigma X_i^2} \right) \right]^{\frac{n}{2}} \left[\frac{m}{m+n} \left(1 + \frac{\lambda \Sigma X_i^2}{\Sigma Y_i^2} \right) \right]^{\frac{m}{2}} < k_\alpha \right\}$

is a $(1-\alpha)$ C. I. for λ □

63. ① The LRT method :

$$\lambda(y) = \frac{\sup_{\mu=\mu_0} L(\mu|y)}{\sup_{\mu\in\mathbb{R}} L(\mu|y)} = \frac{n e^{-n(y-\mu_0)} I_{[\mu_0,\infty)}(y)}{n e^{-n(y-y)} I_{[y,\infty)}(y)}$$

$$= n e^{-n(y-\mu_0)} I_{[\mu_0,\infty)}(y) = \begin{cases} 0 & , \text{ if } y < \mu_0 \\ e^{-n(y-\mu_0)} & , \text{ if } y \geq \mu_0 \end{cases}$$

$\Rightarrow$ Reject H_0 if $\lambda(y) < k_\alpha$

and $\alpha = P\{ y < \mu_0 \text{ or } e^{-n(y-\mu_0)} < k_\alpha \mid \mu = \mu_0 \}$

$$= P\left\{ y > \mu_0 - \frac{\log k_\alpha}{n} \,\middle|\, \mu = \mu_0 \right\} = \int_{\mu_0 - \frac{\log k_\alpha}{n}}^{\infty} n e^{-n(y-\mu_0)} \, dy$$

$$= -e^{-n(y-\mu_0)} \Big|_{y=\mu_0 - \frac{\log k_\alpha}{n}}^{\infty} = k_\alpha$$

$\therefore$ The $(1-\alpha)$ C.I. $\Rightarrow C_1(y) = \left\{ \mu : \mu \leq y \leq \mu - \frac{\log \alpha}{n} \right\}$

$$= \left\{ \mu : y + \frac{1}{n} \log \alpha \leq \mu \leq y \right\}$$

② The pivotal method:

Note that $Z = Y - \mu \sim Exp(\lambda = n)$ and $P\{a \le Z \le b\} = 1-\alpha$

s.t. $\begin{cases} \frac{\alpha}{2} = \int_0^a n e^{-nz} dz = -e^{-nz}\Big|_{z=0}^{a} = 1 - e^{-na} \\ \frac{\alpha}{2} = \int_b^\infty n e^{-nz} dz = -e^{-nz}\Big|_{z=b}^{\infty} = e^{-nb} \end{cases}$

$\Rightarrow \begin{cases} a = \frac{-1}{n}\log(1-\frac{\alpha}{2}) \\ b = \frac{-1}{n}\log(\frac{\alpha}{2}) \end{cases}$

$\therefore$ The $(1-\alpha)$ C.I. $\Rightarrow C_2(y) = \{\mu : \frac{-1}{n}\log(1-\frac{\alpha}{2}) \le y - \mu \le \frac{-1}{n}\log(\frac{\alpha}{2})\}$

$= \{\mu : y + \frac{1}{n}\log(\frac{\alpha}{2}) \le \mu \le y + \frac{1}{n}\log(1-\frac{\alpha}{2})\}$

length of $C_1(y) = y - (y + \frac{1}{n}\log\alpha) = \frac{1}{n}\log\frac{1}{\alpha}$

length of $C_2(y) = [y + \frac{1}{n}\log(1-\frac{\alpha}{2})] - [y + \frac{1}{n}\log(\frac{\alpha}{2})] = \frac{1}{n}\log(\frac{2-\alpha}{\alpha})$

$\because 0 < \alpha < 1 \Rightarrow \frac{1}{\alpha} < \frac{2-\alpha}{\alpha} \Rightarrow \log(\frac{1}{\alpha}) < \log(\frac{2-\alpha}{\alpha})$

$\therefore$ length of $C_1(y)$ < length of $C_2(y)$ $\square$

64. (a) $\{X_i | p\}_{i=1}^n \sim Ber(p)$, $\pi(p) \sim Beta(a,b)$

$$\pi(p|\underline{X}) \propto_p p^{a-1+\Sigma X_i} (1-p)^{b-1+n-\Sigma X_i}$$

$$\Rightarrow \pi(p|\underline{X}) \sim Beta(a+\Sigma X_i, b+n-\Sigma X_i)$$

$$\therefore \{p : Beta(a+\Sigma X_i, b+n-\Sigma X_i)_{1-\frac{\alpha}{2}} < p < Beta(a+\Sigma X_i, b+n-\Sigma X_i)_{\frac{\alpha}{2}}\} \quad \square$$

(b) $$Beta(a+\Sigma X_i, b+n-\Sigma X_i) = \frac{\frac{a+\Sigma X_i}{b+n-\Sigma X_i} F_{2(a+\Sigma X_i), 2(b+n-\Sigma X_i)}}{1+\frac{a+\Sigma X_i}{b+n-\Sigma X_i} F_{2(a+\Sigma X_i), 2(b+n-\Sigma X_i)}}$$

$\therefore$ The interval becomes

$$\frac{\frac{a+\Sigma X_i}{b+n-\Sigma X_i} F_{2(a+\Sigma X_i), 2(b+n-\Sigma X_i), 1-\frac{\alpha}{2}}}{1+\frac{a+\Sigma X_i}{b+n-\Sigma X_i} F_{2(a+\Sigma X_i), 2(b+n-\Sigma X_i), 1-\frac{\alpha}{2}}} < p < \frac{\frac{a+\Sigma X_i}{b+n-\Sigma X_i} F_{2(a+\Sigma X_i), 2(b+n-\Sigma X_i), \frac{\alpha}{2}}}{1+\frac{a+\Sigma X_i}{b+n-\Sigma X_i} F_{2(a+\Sigma X_i), 2(b+n-\Sigma X_i), \frac{\alpha}{2}}}$$

$$\Rightarrow \frac{1}{1+\frac{b+n-\Sigma X_i}{a+\Sigma X_i} F_{2(b+n-\Sigma X_i), 2(a+\Sigma X_i), \frac{\alpha}{2}}} < p < \frac{\frac{a+\Sigma X_i}{b+n-\Sigma X_i} F_{2(a+\Sigma X_i), 2(b+n-\Sigma X_i), \frac{\alpha}{2}}}{1+\frac{a+\Sigma X_i}{b+n-\Sigma X_i} F_{2(a+\Sigma X_i), 2(b+n-\Sigma X_i), \frac{\alpha}{2}}}$$

Compares to Ex 9.21:

Lower bound:

$a = 0$

$b = 1$

Upper bound:

$a = 1$

$b = 0$

$\therefore$ No values of a and b will make the intervals match $\square$

65. (a) To show that $\forall$ interval $[a, b]$ and $\varepsilon > 0$, $\int_a^b f\,dx < \int_{a-\varepsilon}^{b-\varepsilon} f\,dx$

$$\int_a^b f(x)\,dx - \int_{a-\varepsilon}^{b-\varepsilon} f(x)\,dx = \int_{b-\varepsilon}^{b} f(x)\,dx - \int_{a-\varepsilon}^{a} f(x)\,dx$$

$$< f(b-\varepsilon)\,\varepsilon - f(a)\,\varepsilon = \varepsilon[f(b-\varepsilon) - f(a)] < 0$$

$\Rightarrow$ As moving the interval toward zero, then increasing the integration

$\therefore$ The shortest $(1-\alpha)$ C.I. is obtained by $a=0$ s.t. $\int_0^b f(x)\,dx = 1-\alpha$ $\square$

(b) $(Y-\mu) \sim Exp(\lambda = n)$ which has a strictly decreasing pdf. on $[0, \infty)$

and $\int_0^b n e^{-nt}\,dt = 1-\alpha \Rightarrow b = -\frac{1}{n}\log\alpha$

$\therefore$ The shortest $(1-\alpha)$ C.I. of $(Y-\mu)$ is $[0, \frac{-1}{n}\log\alpha]$

$\Rightarrow$ The best $(1-\alpha)$ C.I. of μ is $[Y + \frac{1}{n}\log\alpha, Y]$ $\square$

66. (a) For $\lambda_1 < \lambda_2$,

$\frac{L(\lambda_2; \underline{X})}{L(\lambda_1; \underline{X})} = \left(\frac{\lambda_1}{\lambda_2}\right)^n \exp\left\{\Sigma X_i \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)\right\}$ is a non-decreasing function of ΣX_i

$\therefore$ ΣX_i has MLR and by Karlin - Rubin Theorem

$\Rightarrow$ The UMP test is to reject H_0

if $\Sigma X_i < k$, where $P(\Sigma X_i < k \mid \lambda = \lambda_0) = \alpha$ □

(b) $\Sigma X_i \sim \text{Gamma}(n, \lambda) \Rightarrow \frac{2}{\lambda} \Sigma X_i \sim \chi^2_{2n}$

$\Rightarrow$ The rejection region : $\left\{ \frac{2}{\lambda} \Sigma X_i < \chi^2_{2n,\alpha} \right\}$

The acception region : $\left\{ \frac{2}{\lambda} \Sigma X_i \geq \chi^2_{2n,\alpha} \right\}$

$\therefore$ $\left\{ \lambda : \frac{2}{\lambda} \Sigma X_i \geq \chi^2_{2n,\alpha} \right\} = \left\{ \lambda : 0 \leq \lambda \leq \frac{2 \Sigma X_i}{\chi^2_{2n,\alpha}} \right\}$

is the UMA $(1-\alpha)$ C. I. for λ □

(c) $E\left\{ \frac{2 \Sigma X_i}{\chi^2_{2n,\alpha}} \right\} = \frac{2n\lambda}{\chi^2_{2n,\alpha}}$ □

67. $L(\lambda_x, \lambda_y; \underline{X}, \underline{Y}) = \frac{1}{\prod X_i \prod Y_i} \exp\left\{-m\lambda_x - n\lambda_y + \sum X_i \log \lambda_x + \sum Y_i \log \lambda_y\right\}$

$$= \frac{e^{-m\lambda_x - n\lambda_y}}{\prod X_i! \prod Y_i!} \exp\left\{\sum X_i(\log \lambda_x - \log \lambda_y) + (\sum X_i + \sum Y_i)\log \lambda_y\right\}$$

$\because$ Poisson distribution is exp family with full rank

$\therefore$ The UMPU test is

$$\phi = \begin{cases} 1 & , \sum X_i > c(\sum X_i + \sum Y_i) \\ \gamma & , \sum X_i = c(\sum X_i + \sum Y_i) \\ 0 & , \sum X_i < c(\sum X_i + \sum Y_i) \end{cases} \quad \text{where } \gamma \text{ and } c \text{ satisfy}$$

$$P\left(\sum X_i > c(\sum X_i + \sum Y_i) \,\middle|\, \lambda_x = \lambda_y\right) + \gamma P\left(\sum X_i = c(\sum X_i + \sum Y_i) \,\middle|\, \lambda_x = \lambda_y\right) = \alpha \quad \square$$

(c) Note that when $\lambda_x = \lambda_y = \lambda$

$$\frac{P(\sum X_i = t_1, \sum Y_i = t_2 - t_1)}{P(\sum X_i + \sum Y_i = t_2)} = \frac{(m\lambda)^{t_1}(n\lambda)^{t_2 - t_1} \exp[-(m+n)\lambda] / [t_1!(t_2-t_1)!]}{\exp[-(m+n)\lambda][(m+n)\lambda]^{t_2} / t_2!} = \binom{t_2}{t_1}\left(\frac{m}{m+n}\right)^{t_1}\left(\frac{n}{m+n}\right)^{t_2 - t_1}$$

$\therefore$ when $\lambda_x = \lambda_y$, $\sum X_i \,\middle|\, (\sum X_i + \sum Y_i) \sim Bin\left(\sum X_i + \sum Y_i, \frac{m}{m+n}\right)$

And using normal approximation :

$$\sum X_i \,\middle|\, (\sum X_i + \sum Y_i) \xrightarrow{D} N\left(\frac{m(\sum X_i + \sum Y_i)}{m+n}, \frac{mn(\sum X_i + \sum Y_i)}{(m+n)^2}\right)$$

$\therefore$ The approximate test will reject H_0 if

$$\sum X_i > \frac{m(\sum X_i + \sum Y_i) + Z_\alpha \sqrt{mn(\sum X_i + \sum Y_i)}}{m+n} \quad \square$$

68. $\frac{X_1}{\theta}, \cdots, \frac{X_n}{\theta} \overset{iid}{\sim} U(0,1)$

By CLT, $\frac{\bar{X}}{\theta} \xrightarrow{D} N\left(\frac{1}{2}, \frac{1}{12n}\right)$

Let $h(t) = t^2 f_{\frac{\bar{X}}{\theta}}(t) = t^2 \sqrt{\frac{6n}{\pi}} \exp\left\{-6n\left(t-\frac{1}{2}\right)^2\right\}$

$h'(t) = \sqrt{\frac{6n}{\pi}} \left\{2t \exp\left[-6n\left(t-\frac{1}{2}\right)^2\right] - 12n\left(t-\frac{1}{2}\right)t^2 \exp\left[-6\left(t-\frac{1}{2}\right)^2\right]\right\} \overset{set}{=} 0$

$\Rightarrow t = 0 \vee \frac{3n \pm \sqrt{3n(3n+8)}}{12n}$

check $h''(0) = 2\sqrt{\frac{6n}{\pi}} \exp\left\{\frac{-3n}{2}\right\} > 0$ where $t = 0$ is a minimum point

$\Rightarrow t = \frac{3n + \sqrt{3n(3n+8)}}{12n}$ is a maximum point

$\Rightarrow t^2 f(t)$ is unimodal at $t > 0$

$\therefore C = \left\{\theta : a \leq \frac{\bar{X}}{\theta} \leq b\right\} = \left\{\theta : \frac{\bar{X}}{b} \leq \theta \leq \frac{\bar{X}}{a}\right\}$

is the shortest length $(1-\alpha)$ C. I. where

① $\int_a^b f(t)\,dt = 1-\alpha$

② $a^2 f(a) = b^2 f(b) > 0$

③ $a \leq \frac{3n+\sqrt{3n(3n+8)}}{12n} \leq b$ and solve a, b $\square$