46. Problem 7.38 in Casella and Berger (2001).
47. Problem 7.42 in Casella and Berger (2001).
48. Problem 8.6 in Casella and Berger (2001).
49. Problem 8.13(a)(b)(c) in Casella and Berger (2001).
50. Problem 8.29 in Casella and Berger (2001).
51. Problem 4 of Keener (2010) Section 4.7.
52. Problem 28 of Keener (2010) Section 4.7.
53. Prove Theorem 12.9(c) of Keener (2010).
54. Assuming $N(\theta, 1)$ and testing $H_0 : \theta = 0$ vs. $\theta \neq 0$, with α =0.05, use some software to plot the 3 power functions in Example 8.3.20 in Casella and Berger (2001). (a) Are the one-sided tests unbiased? (b) (bonus +10 points) Prove by mathematical arguments that the two-sided test is unbiased.

    Practice
    7.45, 7.48(b), 8.22, and 8.28(a) in Casella and Berger (2001).
    Problems 2, 7, 11, 21 of Chapter 4, Keener (2010).

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46. (a) $L(\theta;\underline{X}) = \prod_{i=1}^{n} f(x_i;\theta) = \theta^n \left(\prod_{i=1}^{n} X_i\right)^{\theta-1} \Rightarrow l(\theta;\underline{X}) = \log L(\theta;\underline{X}) = n\log\theta + (\theta-1)\sum_{i=1}^{n}\log X_i$

$$\frac{\partial}{\partial\theta} l(\theta) = \frac{n}{\theta} + \sum_{i=1}^{n}\log X_i = -n\left[\frac{-\sum_{i=1}^{n}\log X_i}{n} - \frac{1}{\theta}\right]$$

$\therefore \dfrac{-\sum_{i=1}^{n}\log X_i}{n}$ is the UMVUE of $\frac{1}{\theta}$ and attains the CRLB □

(b) $L(\theta;\underline{X}) = \left[\dfrac{\log\theta}{\theta-1}\right]^n \theta^{\sum_{i=1}^{n} X_i} \Rightarrow l(\theta;\underline{X}) = \log L(\theta;\underline{X}) = n\left[\log\log\theta - \log(\theta-1)\right] + \sum_{i=1}^{n} X_i\log\theta$

$$\frac{\partial}{\partial\theta} l(\theta) = n\left[\frac{1}{\theta\log\theta} - \frac{1}{\theta-1}\right] + \frac{\sum X_i}{\theta} = \frac{n}{\theta}\left[\bar{X} - \left(\frac{\theta}{\theta-1} - \frac{1}{\log\theta}\right)\right]$$

$\therefore \bar{X}$ is the UMVUE of $\left(\dfrac{\theta}{\theta-1} - \dfrac{1}{\log\theta}\right)$ and attains the CRLB □.

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47. $E(W_i) = \theta$, $Var(W_i) = \sigma_i^2$, $Cov(W_i, W_j) = 0$ if $i \neq j$

(a). $E(\sum a_i W_i) = \sum a_i E(W_i) = \sum a_i\theta = \theta \Rightarrow \sum a_i = 1$

$Var(\sum a_i W_i) = \sum a_i^2 Var(W_i) = \sum a_i^2\sigma_i^2$

By Cauchy Schwarz inequality: $\left(\sum a_i^2\sigma_i^2\right)\left(\sum \frac{1}{\sigma_i^2}\right) \geq \left(\sum a_i\right)^2 = 1$

The equality holds $\Leftrightarrow a_i\sigma_i = \dfrac{\lambda}{\sigma_i}$ for some fixed λ

$\Rightarrow \sum a_i = \sum \dfrac{\lambda}{\sigma_i^2} = 1 \Rightarrow \lambda = \dfrac{1}{\sum \frac{1}{\sigma_i^2}}$

$\therefore a_i = \dfrac{1/\sigma_i^2}{\sum(1/\sigma_i^2)}$ s.t. $W^* = \dfrac{\sum W_i/\sigma_i^2}{\sum(1/\sigma_i^2)}$ has minimum variance □

(b) $Var(W^*) = \dfrac{\sum \frac{1}{\sigma_i^4} Var(W_i)}{\left[\sum(1/\sigma_i^2)\right]^2} = \dfrac{\sum(1/\sigma_i^2)}{\left[\sum(1/\sigma_i^2)\right]^2} = \dfrac{1}{\sum(1/\sigma_i^2)}$ □

48. $\{X_i\}_1^n \overset{iid}{\sim} Exp(\theta)$, $\{Y_j\}_1^m \overset{iid}{\sim} Exp(\mu)$

(a) $L(\theta, \mu; \underline{x}, \underline{y}) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x_i}{\theta}} \prod_{j=1}^{m} \frac{1}{\mu} e^{-\frac{y_j}{\mu}} = \theta^{-n} \mu^{-m} \exp\{-\frac{1}{\theta}\sum_1^n x_i - \frac{1}{\mu}\sum_1^m y_i\}$

$$\lambda(x,y) = \frac{\sup_{\Theta} L(\theta,\mu)}{\sup_{\Theta_0} L(\theta, \mu=\theta)} = \frac{\sup_{\theta,\mu}\{\theta^{-n}\mu^{-m}\exp[\frac{-\sum_1^n x_i}{\theta} + \frac{-\sum_1^m y_i}{\mu}]\}}{\sup_\theta\{\theta^{-n-m}\exp[\frac{-1}{\theta}(\sum_1^n x_i + \sum_1^m y_i)]\}}$$

(For $(\theta,\mu) \in \Theta_0$: $\hat{\theta}_0 = \frac{\sum_1^n x_i + \sum_1^m y_j}{n+m}$, $(\theta,\mu) \in \Theta$: $\hat{\theta} = \bar{x}$, $\hat{\mu} = \bar{y}$)

$$= \frac{\bar{x}^{-n}\,\bar{y}^{-m}\,\exp(-n-m)}{\left(\frac{n+m}{\sum_1^n x_i + \sum_1^m y_i}\right)^{n+m}\exp(-n-m)} = \frac{n^n m^m (\sum_1^n x_i + \sum_1^m y_i)^{n+m}}{(n+m)^{n+m}(\sum_1^n x_i)^n(\sum_1^m y_i)^m}$$

$\Rightarrow$ The LRT is to reject H_0 if $\lambda(x,y) > k$ □

(b) $\lambda(x,y) = \frac{n^n m^m}{(n+m)^{n+m}}\left(\frac{\sum_1^n x_i + \sum_1^m y_i}{\sum_1^n x_i}\right)^n\left(\frac{\sum_1^n x_i + \sum_1^m y_i}{\sum_1^m y_i}\right)^m = \frac{n^n m^m}{(n+m)^{n+m}} T^{-n}(1-T)^{-m}$

$\lambda(x,y) = \frac{n^n m^m}{(n+m)^{n+m}} T^{-n}(1-T)^{-m} > k \Rightarrow k_1' < T < k_2'$

$\therefore$ The test can be based on T □

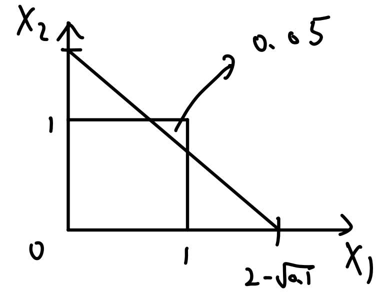
(c) When $H_0 : \theta = \mu$ is true, $\sum_1^n X_i \sim Gamma(n, \theta)$, $\sum_1^m Y_i \sim Gamma(m, \theta)$

$\therefore T \sim Beta(n, m)$ □

49. $X_1, X_2 \overset{iid}{\sim} U(\theta, \theta+1)$, $H_0: \theta = 0$ v.s. $H_1: \theta > 0$

(a) $P(X_1 > 0.95 \mid \theta = 0) = P(X_1 + X_2 > c \mid \theta = 0)$

$\Rightarrow \quad 0.05 = P(X_1 + X_2 > c \mid \theta = 0)$



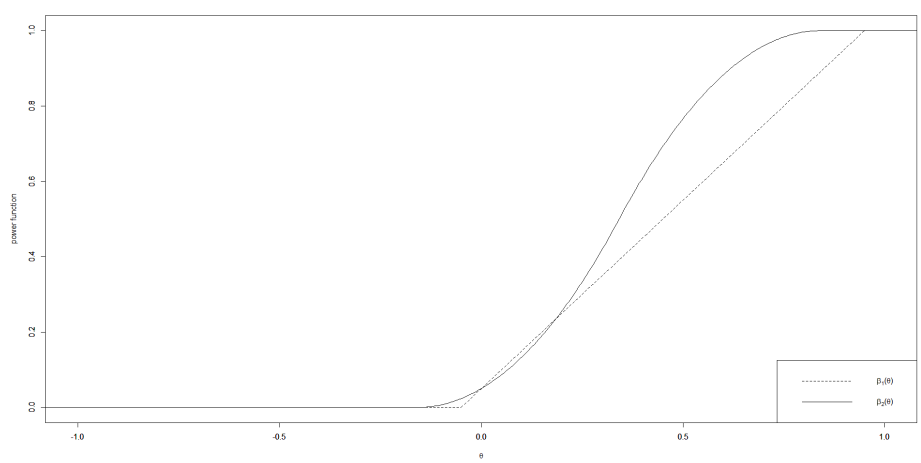
$\Rightarrow c = 2 - \sqrt{0.1}$ $\square$

(b) $\beta_1(\theta) = P(X_1 > 0.95 \mid \theta) = \begin{cases} 0, & \theta < -0.05 \\ \theta + 0.05, & -0.05 \le \theta < 0.95 \\ 1, & \theta \ge 0.95 \end{cases}$

$\beta_2(\theta) = P(X_1 + X_2 > c \mid \theta) = \begin{cases} 0, & \theta < \frac{c}{2} - 1 \\ \frac{1}{2}(2\theta + 2 - c)^2, & \frac{c}{2} - 1 \le \theta < \frac{c-1}{2} \\ 1 - \frac{1}{2}(c - 2\theta)^2, & \frac{c-1}{2} \le \theta < \frac{c}{2} \\ 1, & \theta \ge \frac{c}{2} \end{cases}$

(c) ϕ_2 is not a more powerful test than ϕ_1.

$\because$ about θ near 0, there is a region where $\beta_1(\theta) > \beta_2(\theta)$ $\square$



50. (a) Let $\theta_2 > \theta_1$, then

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{1+(x-\theta_1)^2}{1+(x-\theta_2)^2} = \frac{x^2-2\theta_1 x+\theta_1^2+1}{x^2-2\theta_2 x+\theta_2^2+1} \longrightarrow 1, \text{ as } x \longrightarrow \infty \text{ or } -\infty$$

$\therefore$ The ratio would not be monotone increasing (or decreasing) function as $x \in (-\infty, \infty)$ $\Rightarrow$ The family does not have an MLR $\square$

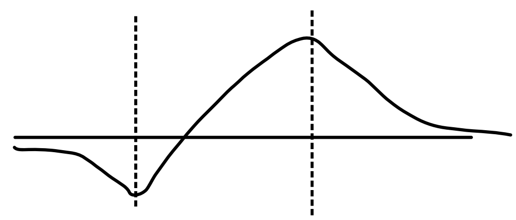
(b) By the Neyman - Pearson Lemma, a test will be UMP if it rejects when

$$\frac{f(x|\theta=1)}{f(x|\theta=0)} = \frac{1+x^2}{1+(x-1)^2} = \frac{x^2+1}{x^2-2x+2} > k$$

$$\xrightarrow{\frac{d}{dx}} \frac{2x(x^2-2x+2)-(x^2+1)(2x-2)}{(x^2-2x+2)^2} = \frac{-2(x^2-x-1)}{(x^2-2x+2)^2}$$

$$\therefore \frac{f_1(x)}{f_0(x)} \Rightarrow \begin{cases} \text{decreasing}, & x < \frac{1-\sqrt{5}}{2} \text{ or } x \geq \frac{1+\sqrt{5}}{2} \\ \text{increasing}, & \frac{1-\sqrt{5}}{2} \leq x < \frac{1+\sqrt{5}}{2} \end{cases} \quad \text{and } \frac{f_1(1)}{f_0(1)} = \frac{f_1(3)}{f_0(3)} = 2$$

The rejection region $\left\{x \,\middle|\, \frac{f_1(x)}{f_0(x)} > 2\right\} = \{x \mid 1 < x < 3\}$



Therefore, the given test is UMP of its size.

$$\text{Type I error} = P(1 < X < 3 \mid \theta = 0) = \int_1^3 \frac{1}{\pi} \frac{1}{1+x^2}\,dx = \frac{1}{\pi}\arctan 3 - \frac{1}{4}$$

$$\text{Type II error} = 1 - P(1 < X < 3 \mid \theta = 1) = 1 - \int_1^3 \frac{1}{\pi} \frac{1}{1+(x-1)^2}\,dx$$

$$= 1 - \frac{1}{\pi}(\arctan 2 - \arctan 0) = 1 - \frac{1}{\pi}\arctan 2 \ \square$$

(c) disprove : For $\theta=2$: $\frac{f(x|\theta=2)}{f(x|\theta=0)} = \frac{1+x^2}{1+(x-2)^2} = \frac{x^2+1}{x^2-4x+5}$

Then $\frac{f(1|\theta=2)}{f(1|\theta=0)} = 1 \neq 5 = \frac{f(3|\theta=2)}{f(3|\theta=0)}$ $\Rightarrow$ The $RR = \{1<X<3\}$ will not be MP at this case

$\therefore$ The test is not UMP for $H_0: \theta \leq 0$ v.s. $H_1: \theta > 0$ $\square$

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51. $\{X_i\}_1^n \overset{\text{indep.}}{\sim} N(\alpha t_i + \beta t_i^2, 1)$

$f(\underline{X}; \alpha, \beta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\{\frac{-1}{2}(x_i - \alpha t_i - \beta t_i^2)^2\}$

$= (2\pi)^{\frac{-n}{2}} \exp\{\frac{-1}{2}(\Sigma x_i^2 + \alpha^2 \Sigma t_i^2 + \beta \Sigma t_i^4 - 2\alpha \Sigma t_i x_i - 2\beta \Sigma t_i^2 x_i + 2\alpha\beta \Sigma t_i^3)\}$

which is a full rank pdf of exp family

$\Rightarrow (T_1(X), T_2(X)) = (\Sigma t_i x_i, \Sigma t_i^2 x_i)$ is a 2-dim complete suff. stat. for (α, β)

$E(T_1) = \alpha \Sigma t_i^2 + \beta \Sigma t_i^3$, $E(T_2) = \alpha \Sigma t_i^3 + \beta \Sigma t_i^4$

Then $E\left[\frac{T_1 \Sigma t_i^4 - T_2 \Sigma t_i^3}{\Sigma t_i^2 \Sigma t_i^4 - (\Sigma t_i^3)^2}\right] = \alpha$ and $E\left[\frac{T_1 \Sigma t_i^3 - T_2 \Sigma t_i^2}{(\Sigma t_i^3)^2 - \Sigma t_i^2 \Sigma t_i^4}\right] = \beta$

are unbiased estimator for (α, β) and both function of (T_1, T_2)

$\therefore$ They are UMVU estimators of α, β $\square$

52. $\{X_i\}_1^n \overset{iid}{\sim} U(0, \theta)$

(a) For any unbiased estimator δ

$$Var(\delta) \geq \frac{[g(\theta+\Delta) - g(\theta)]^2}{E_\theta\left[\frac{P_{\theta+\Delta}(x)}{P_\theta(x)} - 1\right]^2}$$, and Δ satisfies (i) $\theta+\Delta \in \Omega \Rightarrow \theta+\Delta > 0$

(ii) $E_{\theta+\Delta}(\delta) - E_\theta(\delta) = g(\theta+\Delta) - g(\theta) = \Delta$

(iii) $P_{\theta+\Delta}(X) = 0$ when $P_\theta(x) = 0 \Rightarrow \Delta < 0$ $\quad \therefore \Delta \in (-\theta, 0)$

$$\because \frac{P_{\theta+\Delta}(X)}{P_\theta(X)} = \begin{cases} \left(\frac{\theta}{\theta+\Delta}\right)^n, & X_{(n)} < \theta+\Delta \\ 0, & \text{o.w.} \end{cases}$$

$$\therefore E_\theta\left(\frac{P_{\theta+\Delta}(x)}{P_\theta(x)} - 1\right)^2 = E_\theta\left(\left(\frac{P_{\theta+\Delta}(x)}{P_\theta(x)}\right)^2 - 2\left(\frac{P_{\theta+\Delta}(x)}{P_\theta(x)}\right) + 1\right) = \left(\frac{\theta}{\theta+\Delta}\right)^{2n}\left(\frac{\theta+\Delta}{\theta}\right)^n - 2\left(\frac{\theta}{\theta+\Delta}\right)^n\left(\frac{\theta+\Delta}{\theta}\right)^n + 1$$

$$= \left(\frac{\theta}{\theta+\Delta}\right)^n - 1$$

$$\Rightarrow Var(\delta) \geq \frac{\Delta^2}{\left(\frac{\theta}{\theta+\Delta}\right)^n - 1} \quad \square$$

(b)

$$\frac{\Delta^2}{\left(\frac{\theta}{\theta+\Delta}\right)^n - 1} = \frac{\frac{c^2\theta^2}{n^2}}{\left(\frac{1}{1-\frac{c}{n}}\right)^n - 1} \Rightarrow g_n(c) = \frac{c^2}{\left(1-\frac{c}{n}\right)^{-n} - 1}$$

$$g(c) = \lim_{n\to\infty} g_n(c) = \lim_{n\to\infty} \frac{c^2}{\left(1-\frac{c}{n}\right)^{-n} - 1} = \frac{c^2}{e^c - 1} \quad \square$$

(c) $g'(c) = \dfrac{2c(e^c-1) - c^2 e^c}{(e^c-1)^2} > 0$ for $c \in (0,1)$

$\therefore$ $g(c)$ is an increasing function for $c \in (0,1)$

$\Rightarrow C_0 = 1$ maximize $g(c) = \dfrac{1}{e-1} \approx 0.582$

$\therefore$ The approximate lower bound $= \dfrac{\theta^2 g(c_0)}{n^2} \approx \dfrac{0.582\theta^2}{n^2}$ $\square$

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53. Suppose $\theta_1 < \theta_2$ and $L(x) = \dfrac{p_{\theta_2}(x)}{p_{\theta_1}(x)}$

$\because$ The family has MLR, L is a non-decreasing function of T.

If k is the value of L when $T = c$,

then $\phi^*(x) = \begin{cases} 1, & L > k \\ 0, & L < k \end{cases}$, and ϕ^* has level $\alpha = E_{\theta_0}\phi^*$

To show that ϕ^* is UMP, suppose $\tilde{\phi}$ has level at most α and $\theta_1 > \theta_0$

Then $E_{\theta_0}\tilde{\phi} \le \alpha$, and $\because$ ϕ^* is LRT of $\theta = \theta_0$ v.s. $\theta = \theta_1$ maximizing $E_{\theta_1}\phi$.

among all tests with $E_{\theta_0}\phi \le E_{\theta_0}\phi^* = \alpha$, $E_{\theta_1}\phi^* \ge E_{\theta_1}\tilde{\phi}$.

Similarly, if $\theta_1 < \theta_0$, $\because$ ϕ^* is LRT of $\theta = \theta_1$ v.s. $\theta = \theta_0$ with some

critical value k, it must maximize $E_{\theta_0}\phi - kE_{\theta_1}\phi$.

$\therefore$ If $\tilde{\phi}$ is a competing test with $E_{\theta_0}\tilde{\phi} = \alpha = E_{\theta_0}\phi^*$

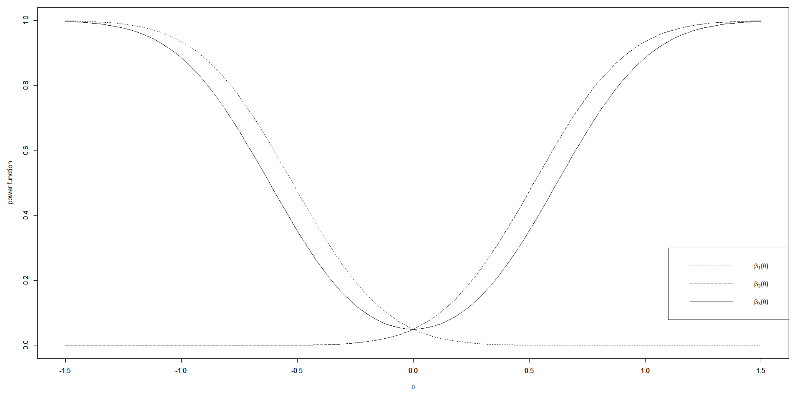
$\Rightarrow E_{\theta_1}\tilde{\phi} \ge E_{\theta_1}\phi^*$ $\square$

54. $\beta_1(\theta) = P_\theta(\bar{X} < -Z_\alpha \sqrt{n}) = P(\sqrt{n}(\bar{X}-\theta) < -Z_\alpha - \sqrt{n}\theta)$

$\beta_2(\theta) = P_\theta(\bar{X} > Z_\alpha \sqrt{n}) = P(\sqrt{n}(\bar{X}-\theta) > Z_\alpha - \sqrt{n}\theta)$

$\beta_3(\theta) = P_\theta(\bar{X} < -Z_{\frac{\alpha}{2}}\sqrt{n} \text{ or } \bar{X} > Z_{\frac{\alpha}{2}}\sqrt{n}) = P(\sqrt{n}(\bar{X}-\theta) < -Z_{\frac{\alpha}{2}} - \sqrt{n}\theta) + P(\sqrt{n}(\bar{X}-\theta) > Z_{\frac{\alpha}{2}} - \sqrt{n}\theta)$

Take $n = 10$ for example:



(a) From the above plot, both one-sided tests are not unbiased.

(b) To show $\forall \theta_1 \neq 0$, $\beta_3(\theta_1) \geq \beta_3(0)$

$\beta_3(0) = \alpha = 0.05$

$\beta_3(\theta_1) = P(\sqrt{n}(\bar{X}-\theta_1) < -Z_{0.025} - \sqrt{n}\theta_1) + P(\sqrt{n}(\bar{X}-\theta_1) > Z_{0.025} - \sqrt{n}\theta_1)$

$\Rightarrow \beta_3(\theta_1) - \beta_3(0) = P(Z_{0.025} - \sqrt{n}\theta_1 < Z < Z_{0.025}) - P(-Z_{0.025} - \sqrt{n}\theta_1 < Z < -Z_{0.025})$

($\because$ The pdf value of $N(0,1)$ becomes larger as X going closer to 0)

$\therefore P(Z_{0.025} - \sqrt{n}\theta_1 < Z < Z_{0.025}) > P(-Z_{0.025} - \sqrt{n}\theta_1 < Z < -Z_{0.025})$

$\Rightarrow \beta_3(\theta_1) > \beta_3(0)$ $\square$