1. Problem 5 in Keener (2010) Section 1.11.

2. Problem 10 in Keener (2010) Section 1.11.

3. We outline the steps for proving Theorem 1.8 of Keener (2010). Provide the details to prove Theorem 1.8.

4. Problem 2 of Keener (2010) Section 2.5.

5. The gamma family is a two-parameter family of distributions on $\mathbb{R}_+ = [0, \infty)$ and its density is
$$p_{k,\theta}(x) = \frac{x^{k-1}e^{-x/\theta}}{\Gamma(k)\theta^k}$$
with respect to the Lebesgue measure on $\mathbb{R}_+$, where $k > 0$ and $\theta > 0$ are respectively called the shape and scale parameters, and $\Gamma(k)$ is the gamma function, defined as
$$\Gamma(k) = \int_0^\infty x^{k-1}e^{-x}\,dx.$$
Show that the gamma is a 2-parameter exponential family by putting it into its canonical form.

6. Let $\mathcal{P} = \{p_\eta : \eta \in \Xi_1\}$ denote an s-parameter exponential family in canonical form
$$p_\eta(x) = e^{\eta' T(x) - A(\eta)}h(x), \qquad A(\eta) = \log \int_{\mathcal{X}} e^{\eta' T(x)}h(x)\,d\mu(x),$$
where $\Xi_1 = \{\eta : A(\eta) < \infty\}$ is the natural (canonical) parameter space.

   Recall the Hölder's inequality: if $q_1, q_2 \geq 1$ with $q_1^{-1} + q_2^{-1} = 1$, and f_1 and f_2 are (μ-measurable) functions from $\mathcal{X}$ to $\mathbb{R}$, then
$$\|f_1 f_2\|_{L^1(\mu)} \leq \|f_1\|_{L^{q_1}(\mu)}\|f_2\|_{L^{q_2}(\mu)}, \quad \text{where } \|f\|_{L^q(\mu)} = \left(\int_{\mathcal{X}} |f(x)|^q\,d\mu(x)\right)^{1/q}.$$
($q_1 = q_2 = 2$ reduces to Cauchy-Schwarz).

   Show that $A(\eta)$ is a convex function: that is, for *any* $\eta_1, \eta_2 \in \mathbb{R}^s$, and $c \in [0, 1]$ then
$$A(c\eta_1 + (1-c)\eta_2) \leq cA(\eta_1) + (1-c)A(\eta_2).$$

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1、 $\because A, B \in \mathcal{S}$, $\therefore (A \cap B) \in \mathcal{S}$

and μ is a measure on $(\mathcal{X}, \mathcal{S})$

$$\Rightarrow \quad 0 \leq \nu(B) = \mu(A \cap B) < \infty \quad \text{——} \ ①$$

Let $B_1, B_2, \ldots$ be disjoint elements of $\mathcal{S}$

$\therefore A \cap B_1, A \cap B_2, \ldots$ are also disjoint elements of $\mathcal{S}$

$$\Rightarrow \nu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left\{A \cap \left(\bigcup_{i=1}^{\infty} B_i\right)\right\} = \mu\left\{\bigcup_{i=1}^{\infty} (A \cap B_i)\right\}$$

$$= \sum_{i=1}^{\infty} \mu(A \cap B_i) = \sum_{i=1}^{\infty} \nu(B_i) \quad \text{——} \ ②$$

By ① and ②, ν is a measure on $(\mathcal{X}, \mathcal{S})$

$\square$

2.

a). $\because$ μ and ν are measures on $(\Sigma, \mathcal{B})$

$$\therefore \begin{cases} 0 \leq \mu(B) < \infty \\ 0 \leq \nu(B) < \infty \end{cases}, \quad \text{for } B \in \mathcal{B}$$

$$\Rightarrow \quad 0 \leq \eta(B) = \mu(B) + \nu(B) < \infty \quad \text{— ①}$$

Let $B_1, B_2, \ldots$ be disjoint elements of $\mathcal{B}$

$$\eta\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) + \nu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i) + \sum_{i=1}^{\infty} \nu(B_i)$$

$$= \sum_{i=1}^{\infty} \left\{ \mu(B_i) + \nu(B_i) \right\} = \sum_{i=1}^{\infty} \eta(B_i) \quad \text{— ②}$$

By ① and ②, η is also a measure $\square$

b). Suppose f is a nonnegative simple function, $f = \sum_{i=1}^{m} a_i I_{A_i}$

$$\int f \, d\eta = \int \sum_{i=1}^{m} a_i I_{A_i} \, d\eta = \sum_{i=1}^{m} a_i \eta(A_i)$$

$$= \sum_{i=1}^{m} a_i \left[\mu(A_i) + \nu(A_i) \right] = \sum_{i=1}^{m} a_i \mu(A_i) + \sum_{i=1}^{m} a_i \nu(A_i)$$

$$= \int f \, d\mu + \int f \, d\nu$$

Then suppose f is a nonnegative nonsimple function

$$f = \lim_{m \to \infty} f_m = \lim_{m \to \infty} \sum_{i=1}^{m} a_i I_{A_i}$$

where f and f_m are both measurable functions

$$\int f \, d\eta = \int \lim_{m \to \infty} f_m \, d\eta = \lim_{m \to \infty} \int f_m \, d\eta$$

$$= \lim_{m \to \infty} \int \sum_{i=1}^{m} a_i I_{A_i} \, d\eta = \lim_{m \to \infty} \sum_{i=1}^{m} a_i \eta(A_i)$$

$$= \lim_{m \to \infty} \sum_{i=1}^{m} a_i \mu(A_i) + \lim_{m \to \infty} \sum_{i=1}^{m} a_i \nu(A_i)$$

$$= \int f \, d\mu + \int f \, d\nu \quad \square$$

3. Define: $f_n(x) = \begin{cases} \frac{i-1}{2^n}, & \text{if } \frac{i-1}{2^n} \le f(x) < \frac{i}{2^n}, i=1,2,\dots,n2^n \\ n, & \text{if } f(x) \ge n \end{cases}$

$\because \begin{cases} f^{-1}([\frac{i-1}{2^n}, \frac{i}{2^n})) \in \mathcal{S} \\ f^{-1}([n, \infty)) \in \mathcal{S} \end{cases}$

$\Rightarrow f_n(x)$ is nonnegative simple and measurable

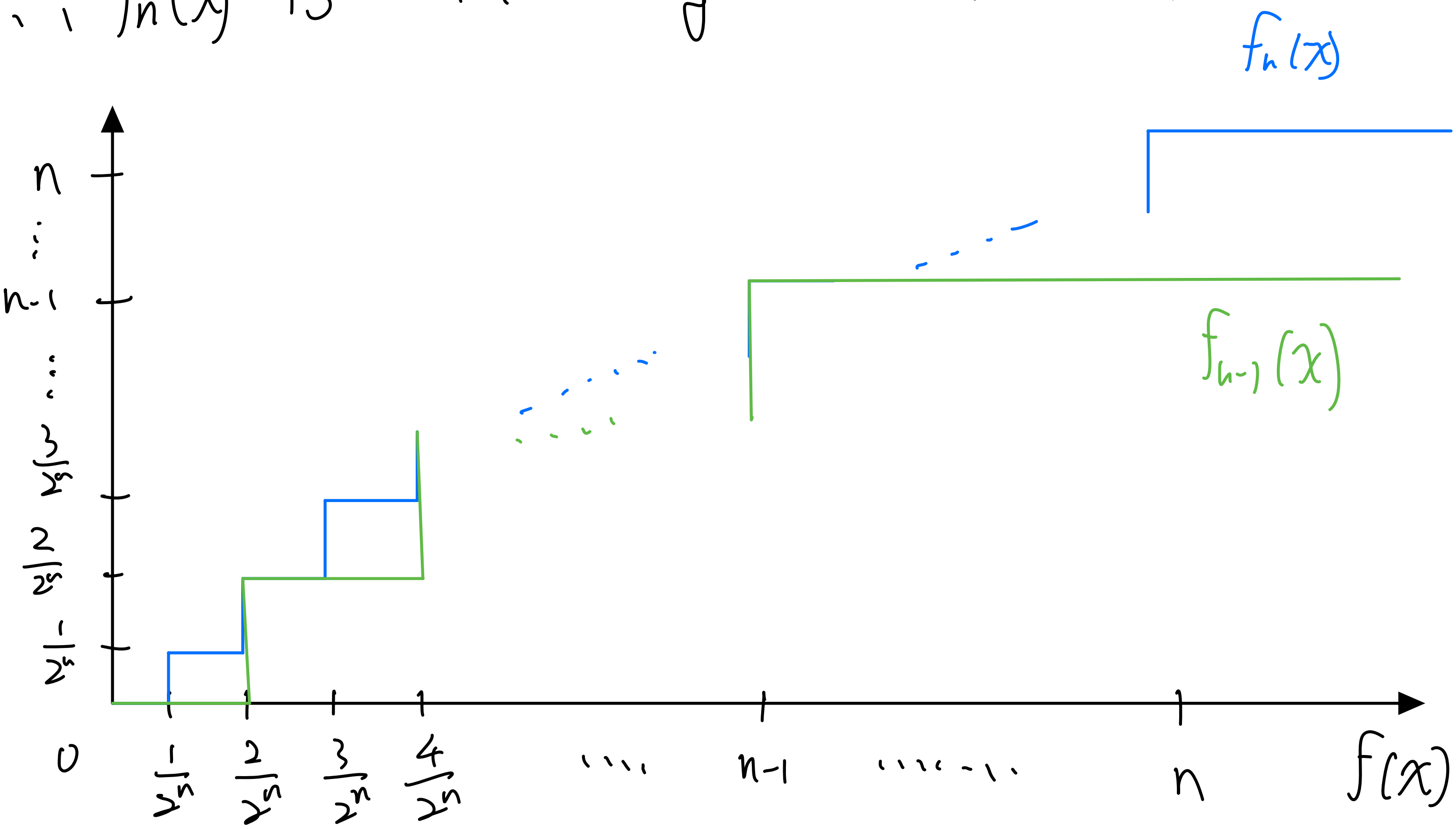
$f_{n-1}(x) = \begin{cases} \frac{2(j-1)}{2^n}, & \text{if } \frac{2(j-1)}{2^n} \le f(x) < \frac{2j}{2^n}, j=1,2,\dots,(n-1)2^{n-1} \\ n-1, & \text{if } f(x) \ge n-1 \end{cases}$

$f_n(x) - f_{n-1}(x)$

$= \begin{cases} 0, & \text{if } \frac{i-1}{2^n} \le f(x) < \frac{i}{2^n}, i \text{ is odd and } i \le (n-1)2^{n-1} \\ \frac{1}{2^n}, & \text{if } \frac{i-1}{2^n} \le f(x) < \frac{i}{2^n}, i \text{ is even and } i \le (n-1)2^{n-1} \\ \frac{i-1}{2^n} - (n-1), & \text{if } \frac{i-1}{2^n} \le f(x) < \frac{i}{2^n}, i = (n-1)2^{n-1}+1, \dots, n2^n \\ 1, & \text{if } f(x) \ge n \end{cases}$

$\Rightarrow f_n(x) - f_{n-1}(x) \geq 0$ for every x

$\therefore f_n(x)$ is nondecreasing in n for each x



If $x \in f^{-1}\left(\frac{i-1}{2^n}, \frac{i}{2^n}\right)$, then for some $1 \leq i \leq n2^n$

$\Rightarrow 0 \leq f(x) - f_n(x) \leq \frac{i}{2^n} - \frac{i-1}{2^n} = \frac{1}{2^n}$

If $x \in f^{-1}(n, \infty)$, $f_n(x) = n$, $\forall n$

$\therefore \forall x \in \mathcal{S}$, $\lim_{n\to\infty} f_n(x) = f(x)$ $\square$

4. $A(\eta) = \log\left\{\int \exp[\eta\, T(x,y)]\, h(x,y)\, d\mu\right.$

$$= \log\left\{\iint \exp[\eta\, xy]\, \exp\left[-\tfrac{1}{2}(x^2+y^2)\right] / (2\pi)\, dx\, dy\right\}$$

$$= \log\left\{\frac{1}{2\pi}\iint \exp\left[\eta\, xy - \tfrac{1}{2}(x^2+y^2)\right] dx\, dy\right\}$$

$$p_\eta(x,y) = \exp\{\eta\, T(x,y) - A(\eta)\}\, h(x,y)$$

$$= \exp\{\eta\, xy - A(\eta)\}\, \exp\left[-\tfrac{1}{2}(x^2+y^2)\right] / (2\pi)$$

$$= \frac{\exp\left[\eta\, xy - \frac{1}{2}(x^2+y^2)\right]}{\iint \exp\left[\eta\, xy - \frac{1}{2}(x^2+y^2)\right] dx\, dy}, \quad (x,y) \in \mathbb{R}^2 \quad \square$$

5. $$p_{k,\theta}(x) = \frac{x^{k-1} e^{-\frac{x}{\theta}}}{\Gamma(k)\ \theta^k}, \quad x \in \mathbb{R}_+ = [0, \infty)$$

$$= \exp\left\{ (k-1)\log x - \frac{x}{\theta} - \log\Gamma(k) - k\log\theta \right\}$$

$$= \exp\left\{ \begin{pmatrix} k-1 \\ \frac{-1}{\theta} \end{pmatrix}^T \begin{pmatrix} \log x \\ x \end{pmatrix} - \left[\log\Gamma(k) + k\log\theta\right] \right\} 1$$

Let: $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} k-1 \\ \frac{-1}{\theta} \end{pmatrix}$, $T(x) = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} \log x \\ x \end{pmatrix}$

$$A(\eta) = \log\Gamma(k) + k\log\theta = \log\Gamma(\eta_1+1) - (\eta_1+1)\log(-\eta_2)$$

$h(x) = 1$ $\square$

6.

Let $\begin{cases} f_1(x) = \left\{\exp[\eta_1^T T(x)]\, h(x)\right\}^{c} \\ f_2(x) = \left\{\exp[\eta_2^T T(x)]\, h(x)\right\}^{1-c} \end{cases}$ and $\begin{cases} q_1 = \frac{1}{c} \\ q_2 = \frac{1}{1-c} \end{cases}$

$$\Rightarrow \begin{cases} \underline{\|f_1\|_{L^{q_1}(\mu)}} = \left[\int_{\mathcal{X}} |f_1(x)|^{q_1}\, d\mu(x)\right]^{\frac{1}{q_1}} \\ \underline{\|f_2\|_{L^{q_2}(\mu)}} = \left[\int_{\mathcal{X}} |f_2(x)|^{q_2}\, d\mu(x)\right]^{\frac{1}{q_2}} \end{cases}$$

$$\Rightarrow \begin{cases} = \left[\int_{\mathcal{X}} \exp[\eta_1^T T(x)]\, h(x)\, d\mu(x)\right]^{c} = \underline{e^{cA(\eta_1)}} \\ = \left[\int_{\mathcal{X}} \exp[\eta_2^T T(x)]\, h(x)\, d\mu(x)\right]^{1-c} = \underline{e^{(1-c)A(\eta_2)}} \end{cases}$$

$$\underline{\|f_1 f_2\|_{L^1(\mu)}} = \int_{\mathcal{X}} \{\exp[\eta_1^T T(x)] h(x)\}^c \{\exp[\eta_2^T T(x)] h(x)\}^{1-c} d\mu(x)$$

$$= \int_{\mathcal{X}} \exp\{c\eta_1^T T(x) + (1-c)\eta_2^T T(x)\} h(x) d\mu(x)$$

$$= \underline{e^{A(c\eta_1 + c(1-c)\eta_2)}}$$

By Hölder's inequality:

$$\|f_1 f_2\|_{L^1(\mu)} \leq \|f_1\|_{L^{q_1}(\mu)} \|f_2\|_{L^{q_2}(\mu)}$$

$$\Rightarrow \exp\{A(c\eta_1 + (1-c)\eta_2)\} \leq \exp\{cA(\eta_1) + (1-c)A(\eta_2)\}$$

$$\Rightarrow A(c\eta_1 + (1-c)\eta_2) \leq cA(\eta_1) + (1-c)A(\eta_2) \quad \square$$