HW2 110024516 邱繼賢

1. $E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \underline{\mu}$

$\Sigma = E(X_i-\mu)(X_i-\mu)' = E(X_iX_i') - E(X_i)\mu' - \mu E(X_i)' + \mu\mu'$

$= E(X_iX_i') - \mu\mu' \Rightarrow E(X_iX_i') = \Sigma + \mu\mu'$

$Cov(\bar{X}) = E[(\bar{X}-\mu)(\bar{X}-\mu)'] = \frac{1}{n^2} E[(\sum_{i=1}^{n}X_i - n\mu)(\sum_{j=1}^{n}X_j - n\mu)']$

$= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} E[(X_i-\mu)(X_j-\mu)']$ $\left(\because X_i \perp X_j \Rightarrow E(X_i-\mu)(X_j-\mu)' = Cov(X_i, X_j) = 0 \quad \text{for } i \neq j\right)$

$= \frac{1}{n^2}\sum_{i=1}^{n} E[(X_i-\mu)(X_i-\mu)'] = \frac{1}{n^2}\sum_{i=1}^{n} Cov(X_i) = \underline{\frac{1}{n}\Sigma}$

$\sum_{i=1}^{n}(X_i-\bar{X})(X_i-\bar{X})' = \sum_{i=1}^{n}(X_i-\bar{X})X_i' - \sum_{i=1}^{n}(X_i-\bar{X})\bar{X}'$ $\left(\because \sum_{i=1}^{n}(X_i-\bar{X}) = 0\right)$

$= \sum_{i=1}^{n}X_iX_i' - \bar{X}\sum_{i=1}^{n}X_i' = \sum_{i=1}^{n}X_iX_i' - n\bar{X}\bar{X}'$

$\therefore E(S) = \frac{1}{n-1}E\left[\sum_{i=1}^{n}(X_i-\bar{X})(X_i-\bar{X})'\right] = \frac{1}{n-1}\left[\sum_{i=1}^{n}E(X_iX_i') - nE(\bar{X}\bar{X}')\right]$

$= \frac{1}{n-1}\left[\sum_{i=1}^{n}(\Sigma+\mu\mu') - n\left(\frac{1}{n}\Sigma + \mu\mu'\right)\right] = \frac{1}{n-1}\left[n\Sigma + n\mu\mu' - \Sigma - n\mu\mu'\right]$

$= \underline{\Sigma}$ $\square$

2. (a) Cauchy-Schwarz Inequality:

Let b and d be any two $p \times 1$ vectors. Then

$$(b'd)^2 \leq (b'b)(d'd)$$

with equality if and only if $b = cd$ for some constant c.

<pf>:

Consider the vector $(b - xd)$ where x is an arbitrary scalar.

$$\|b - xd\|^2 = (b - xd)'(b - xd) = b'b - xb'd - xd'b + x^2 d'd$$

$$= b'b - 2xb'd + x^2 d'd \geq 0$$

the equality holds only when $b = xd$

For $b \neq xd$ cases:

$$0 < b'b - 2xb'd + x^2 d'd = b'b - \frac{(b'd)^2}{d'd} + \frac{(b'd)^2}{d'd} - 2x(b'd) + x^2(d'd)$$

$$= b'b - \frac{(b'd)^2}{d'd} + d'd\left[x - \frac{b'd}{d'd}\right]^2$$

If we choose $x = \frac{b'd}{d'd}$, then $\left[x - \frac{b'd}{d'd}\right]^2 = 0$

$$\Rightarrow b'b - \frac{(b'd)^2}{d'd} > 0$$

$\therefore (b'd)^2 < (b'b)(d'd)$ if $b \neq xd$ for some x

And if $b = cd$, then $(b'd)^2 = (b'b)(d'd)$ $\square$

(b) Extended Cauchy - Schwarz Inequality :

Let $B_{p \times p}$ be a positive definite matrix, Then

$$(b'd) \le (b'Bb)(d'B^{-1}d)$$

with equality if and only if $b = cB^{-1}d$ for some constants c

<pf> :

Consider square $B^{\frac{1}{2}} = \sum_{i=1}^{p} \sqrt{\lambda_i}\, e_i e_i'$ and $B^{-\frac{1}{2}} = \sum_{i=1}^{p} \frac{1}{\sqrt{\lambda_i}}\, e_i e_i'$

where λ_i's are the eigenvalues of B and

e_i's are the normalized eigenvectors

By Cauchy - Schwarz inequality :

$$b'd = (B^{\frac{1}{2}}b)'(B^{-\frac{1}{2}}d) \le \left[(B^{\frac{1}{2}}b)'(B^{\frac{1}{2}}b)\right]\left[(B^{-\frac{1}{2}}d)'(B^{-\frac{1}{2}}d)\right]$$

$$= (b'Bb)(d'B^{-1}d)$$

the equation holds when $B^{\frac{1}{2}}b = cB^{-\frac{1}{2}}d$

$\Rightarrow b = cB^{-1}d$ for some constants c $\square$

(c) Maximization Lemma :

Let $B_{p\times p}$ and $d_{p\times 1}$ be defined as above

for an arbitrary nonzero vector $X_{p\times 1}$,

$$\max_{X\neq 0} \frac{(X'd)^2}{X'BX} = d'B^{-1}d$$

with the maximum attained when $X = cB^{-1}d$ for any constant $c \neq 0$

<pf> :

$\because$ $X \neq 0$ and B is positive definite

$\therefore$ $X'BX > 0$

By the extended Cauchy - Schwarz inequality

$$(X'd)^2 \leq (X'BX)(d'B^{-1}d)$$

$$\Rightarrow \frac{(X'd)^2}{X'BX} \leq d'B^{-1}d$$

The equation holds , namely $\frac{(X'd)^2}{X'BX}$ attains its maximum

when $X = cB'd$ for any constant $c \neq 0$ $\square$

(d) Maximization of Quadratic Forms for Points on the Unit Sphere:

Let $B_{p\times p}$ be a positive definite matrix with eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_p \ge 0$ and associated normalized eigenvectors $e_1, e_2, \cdots, e_p$. Then

① $\max\limits_{x\neq 0} \dfrac{x'Bx}{x'x} = \lambda_1$ (attained when $x = e_1$)

② $\min\limits_{x\neq 0} \dfrac{x'Bx}{x'x} = \lambda_p$ (attained when $x = e_p$)

Moreover. ③ $\max\limits_{x \perp e_1, \cdots, e_k} \dfrac{x'Bx}{x'x} = \lambda_{k+1}$ (attained when $x = e_{k+1}$, $k = 1, \cdots, p-1$)

<pf>: Let $P = [e_1, \cdots, e_p]$ and $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_p)$

$$B^{\frac{1}{2}} = P\Lambda^{\frac{1}{2}}P' \text{ and } y = P'x$$

Then $\dfrac{x'Bx}{x'x} = \dfrac{x'B^{\frac{1}{2}}B^{\frac{1}{2}}x}{x'PP'x} = \dfrac{x'P\Lambda^{\frac{1}{2}}P'P\Lambda^{\frac{1}{2}}P'x}{y'y} = \dfrac{y'\Lambda y}{y'y} = \dfrac{\sum_1^p \lambda_i y_i^2}{\sum_1^p y_i^2}$

$$\Rightarrow \lambda_p \frac{\sum_1^p y_i^2}{\sum_1^p y_i^2} = \lambda_p \le \frac{x'Bx}{x'x} \le \lambda_1 = \lambda_1 \frac{\sum_1^p y_i^2}{\sum_1^p y_i^2}$$

①
Setting $x = e_1 \Rightarrow y = P'e_1 = (1, 0, \cdots, 0)'$

$$\frac{e_1'Be_1}{e_1'e_1} = e_1'B^{\frac{1}{2}}B^{\frac{1}{2}}e_1 = e_1'P\Lambda^{\frac{1}{2}}P'P\Lambda^{\frac{1}{2}}P'e_1 = y'\Lambda y = \lambda_1$$

(attained maximization)

②
Setting $x = e_p \Rightarrow y = P'e_p = (0, \cdots, 0, 1)'$

$$\frac{e_p'Be_p}{e_p'e_p} = e_p'B^{\frac{1}{2}}B^{\frac{1}{2}}e_p = e_p'P\Lambda^{\frac{1}{2}}P'P\Lambda^{\frac{1}{2}}P'e_p = y'\Lambda y = \lambda_p \text{ (attained minimization)}$$

③ Consider $X \perp e_1, e_2, \cdots, e_k$, s.t. $e_i' X = 0 \quad \forall i = 1, \cdots, k$

$$0 = e_i' X = e_i' P y = e_i' [e_1 \ e_2 \ \cdots \ e_p] \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}$$

$$= e_i' e_1 y_1 + e_i' e_2 y_2 + \cdots + e_i' e_p y_p = y_i \ , \ \forall i = 1, \cdots, k$$

$$\Rightarrow \sum_{i=1}^{k} y_i^2 = \sum_{i=1}^{k} \lambda_i y_i^2 = 0$$

$$\Rightarrow \frac{X' B X}{X' X} = \frac{\sum_{i=1}^{p} \lambda_i y_i^2}{\sum_{i=1}^{p} y_i^2} = \frac{\sum_{i=k+1}^{p} \lambda_i y_i^2}{\sum_{i=k+1}^{p} y_i^2} \leq \lambda_{k+1} \frac{\sum_{i=k+1}^{p} y_i^2}{\sum_{i=k+1}^{p} y_i^2} = \lambda_{k+1}$$

$\therefore \max\limits_{X \perp e_1, \cdots, e_k} \dfrac{X' B X}{X' X} = \lambda_{k+1}$ with maximum attained

when $X = e_{k+1}$ s.t. $BX = \lambda_{k+1} X$ $\square$

3. $S = \frac{1}{n-1} \tilde{X}' (I_n - \frac{1}{n} J_n) \tilde{X}$

$\because$ S , $\tilde{X}$, $(I_n - \frac{1}{n} J_n)$ are totally $n \times n$ square matrixs

$\therefore |S| = \frac{1}{n-1} |\tilde{X}'| \, |I_n - \frac{1}{n} J_n| \, |\tilde{X}|$

$$\det(I_n - \frac{1}{n} J_n) = \det\left(\begin{bmatrix} 1-\frac{1}{n} & \frac{-1}{n} & \cdots & \frac{-1}{n} \\ \frac{-1}{n} & 1-\frac{1}{n} & & \vdots \\ \vdots & & \ddots & \frac{-1}{n} \\ \frac{-1}{n} & \cdots & \frac{-1}{n} & 1-\frac{1}{n} \end{bmatrix}_{n \times n} \right) = \det\left(\begin{bmatrix} 0 & \frac{-1}{n} & \cdots & \frac{-1}{n} \\ 0 & 1-\frac{1}{n} & & \vdots \\ \vdots & \vdots & \ddots & \frac{-1}{n} \\ 0 & \frac{-1}{n} \cdots & \frac{-1}{n} & 1-\frac{1}{n} \end{bmatrix} \right) = 0$$

$\therefore |S| = 0$ ◻

4. Ex 2.32

(a) $E(X^{(1)}) = E\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

(b) $E(AX^{(1)}) = A\,E(X^{(1)}) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

(c) $Cov(X^{(1)}) = E\left[\begin{pmatrix} X_1 - 2 \\ X_2 - 4 \end{pmatrix}\begin{pmatrix} X_1 - 2 \\ X_2 - 4 \end{pmatrix}'\right] = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$

(d) $Cov(AX^{(1)}) = A\,Cov(X^{(1)})\,A' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ 1 & 5 \end{bmatrix}$

(e) $E(X^{(2)}) = E\left(\begin{bmatrix} X_3 \\ X_4 \\ X_5 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$

(f) $E(BX^{(2)}) = B\,E(X^{(2)}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(g) $Cov(X^{(2)}) = E\left[\begin{pmatrix} X_3 + 1 \\ X_4 - 3 \\ X_5 \end{pmatrix}\begin{pmatrix} X_3 + 1 \\ X_4 - 3 \\ X_5 \end{pmatrix}'\right] = \begin{bmatrix} 6 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

(h) $Cov(BX^{(2)}) = B\,Cov(X^{(2)})\,B' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} 6 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 12 & 9 \\ 9 & 24 \end{bmatrix}$

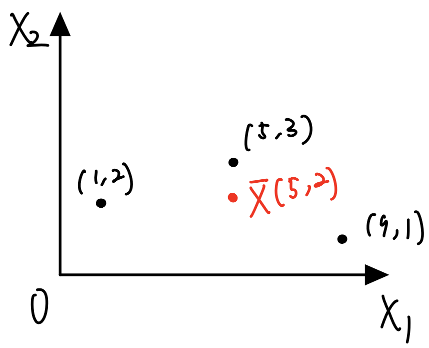
(i) $Cov(X^{(1)}, X^{(2)}) = E\left[\begin{pmatrix} X_1 - 2 \\ X_2 - 4 \end{pmatrix}\begin{pmatrix} X_3 + 1 \\ X_4 - 3 \\ X_5 \end{pmatrix}'\right] = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & 0 \\ 1 & -1 & 0 \end{bmatrix}$

(j) $Cov(AX^{(1)}, BX^{(2)}) = E\left[A(X^{(1)} - EX^{(1)})\right]\left[B(X^{(2)} - EX^{(2)})\right]' = A\,E\left[(X^{(1)} - EX^{(1)})(X^{(2)} - EX^{(2)})\right]B'$

$= A\,Cov(X^{(1)}, X^{(2)})\,B' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & 0 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ □

5. Ex 3.1

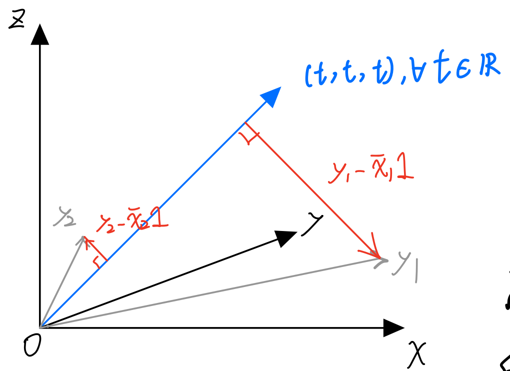
(a)



$$\bar{x}_1 = \frac{1}{3}(9+5+1) = 5$$

$$\bar{x}_2 = \frac{1}{3}(1+3+2) = 2$$

(b)

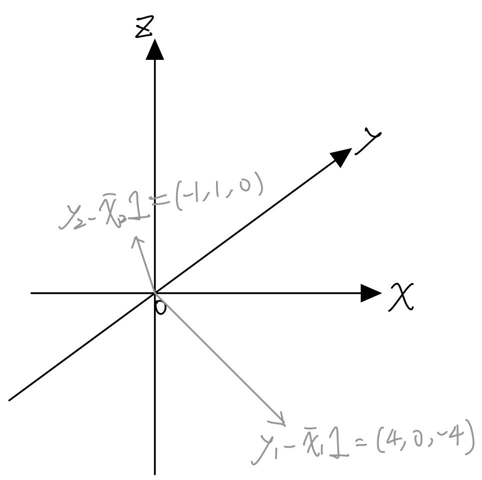


$$\begin{cases} y_1 - \bar{x}_1 1 = \begin{bmatrix} 9 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix} \\ y_2 - \bar{x}_2 1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \end{cases}$$

兩向量為 y_i 向量投影到 $(1,1,1)$ 向量後 $(\bar{x}_i 1)$ 與其相減後得到和 $(1,1,1)$ 直交之向量。

$\frac{32}{3}$

(c)



$$\begin{cases} |y_1 - \bar{x}_1 1| = \sqrt{(y_1-\bar{x}_1 1)'(y_1-\bar{x}_1 1)} = \sqrt{4^2+(-4)^2} = 4\sqrt{2} = \sqrt{n S_{11}} \\ |y_2 - \bar{x}_2 1| = \sqrt{(y_2-\bar{x}_2 1)'(y_2-\bar{x}_2 1)} = \sqrt{(-1)^2+1^2} = \sqrt{2} = \sqrt{n S_{22}} \\ \cos(\theta) = \frac{(y_1-\bar{x}_1 1)'(y_2-\bar{x}_2 1)}{|y_1-\bar{x}_1 1|\,|y_2-\bar{x}_2 1|} = \frac{-4}{8} = \frac{-1}{2} = \frac{S_{12}}{\sqrt{S_{11} S_{12}}} \end{cases}$$

$$S_n = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} \frac{|y_1-\bar{x}_1 1|^2}{n} & \frac{(y_1-\bar{x}_1 1)'(y_2-\bar{x}_2 1)}{n} \\ \frac{(y_1-\bar{x}_1 1)'(y_2-\bar{x}_2 1)}{n} & \frac{|y_2-\bar{x}_2 1|^2}{n} \end{bmatrix} = \begin{bmatrix} \frac{32}{3} & \frac{-4}{3} \\ \frac{-4}{3} & \frac{2}{3} \end{bmatrix}$$

$$r_{12} = \frac{S_{12}}{\sqrt{S_{11} S_{12}}} = \frac{-1}{2}, \quad R = \begin{bmatrix} 1 & r_{12} \\ r_{21} & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{-1}{2} \\ \frac{-1}{2} & 1 \end{bmatrix} \square$$

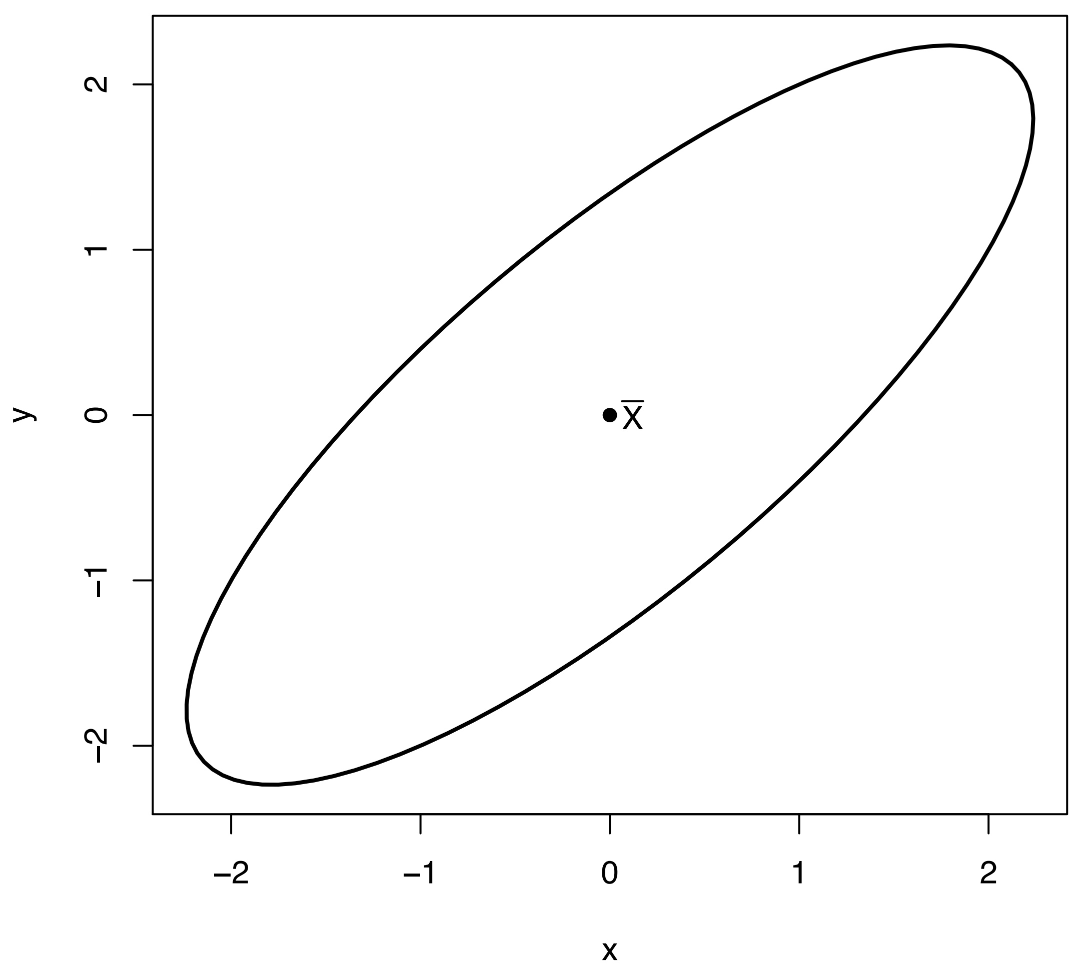
6. Exercise 3.7

I take $\overline{X} = (0,0)'$ for example in the following graphics.

(i)

$$S = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

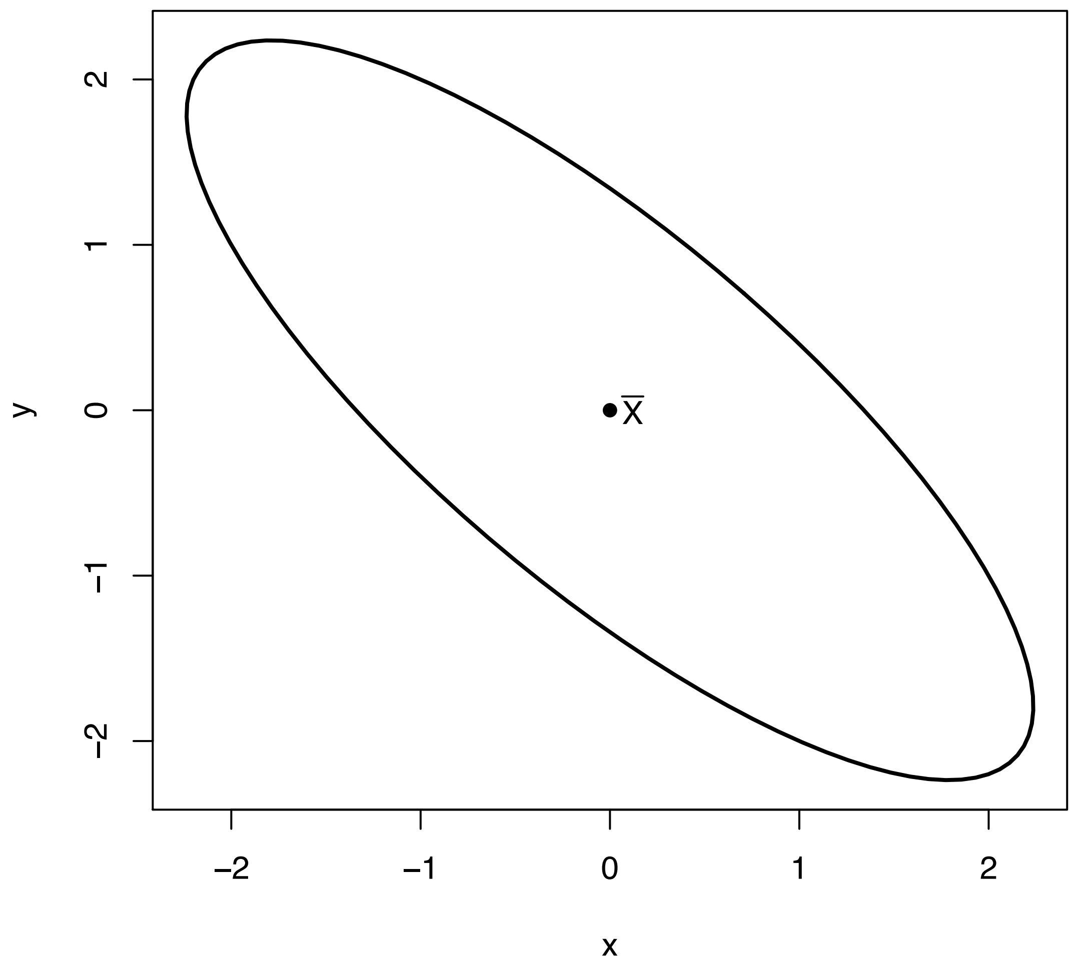
```
library(ellipse)
library(latex2exp)
plot.ellipse = function(cov, c) {
    plot(ellipse(cov, centre = c(0,0), level = pchisq(c,2)), type = "l", lwd = 2)
    points(0,0, pch = 16)
    text(0.12,0,TeX("$\\bar{X}$"))}
s1 = matrix(c(5,4,4,5),2,2)
plot.ellipse(s1, 1)
```



(ii)

$$S = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

```
s2 = matrix(c(5,-4,-4,5),2,2)
plot.ellipse(s2, 1)
```



(iii)

$$S = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

```
s3 = matrix(c(3,0,0,3),2,2)
plot.ellipse(s3, 1)
```