Reliability Analysis
Homework 2

Problem 1: Exercise 3.1

Use the ball-bearing life-test data in Table 1.1 and file LZBearing.csv to do the following:

(a) Compute a nonparametric estimate of the population fraction failing by 75 million cycles.

(b) Use the conservative method in Section 3.4.1 to compute a conservative 90% confidence interval for the population fraction failing by 75 million cycles.

(c) Repeat part (b) using the Jeffreys method in Section 3.4.2.

(d) Repeat part (b) using the Wald method in Section 3.4.3.

(e) Comparing the intervals from parts (b), (c), and (d), what do you conclude about the adequacy of the Wald method for these data?

Table 1.1 Ball-bearing failure times in millions of revolutions

17.88	28.92	33.00	41.52	42.12	45.60	48.40	51.84
51.96	54.12	55.56	67.80	68.64	68.64	68.88	84.12
93.12	98.64	105.12	105.84	127.92	128.04	173.40	

Problem 2: Exercise 3.6

Parida (1991) gives data from a load-controlled high-cycle fatigue test conducted on 130 chain links. The 130 links were selected randomly from a population of different heats used to manufacture the links. Each link was tested until failure or until it had run for 80 thousand cycles, whichever came first. There were 10 failures reported-one each at $33, 46, 50, 59, 62, 71,$ 74, and 75 thousand cycles and 2 at 78 thousand cycles. The other 120 links had not failed by 80 thousand cycles. The data are in file ChainLink.csv.

(a) Use (3.1)

$$\widehat{F}\left(t_i\right) = \frac{\#\text{ of failures up to time } t_i}{n} = \frac{\sum_{j=1}^{i} d_j}{n}$$

to compute the nonparametric estimate of $F(t)$ and corresponding standard errors.

(b) Compute pointwise approximate 90% confidence intervals for $F(t)$. Give a proper interpretation of these intervals.

(c) For the first three failures, compare the numerical estimates from (3.8)

$$\widehat{F}\left(t_i\right) = 1 - \widehat{S}\left(t_i\right), \quad i = 1, \ldots, m$$

with the numerical estimates from (3.1)

$$\widehat{F}\left(t_i\right) = \frac{\#\text{ of failures up to time } t_i}{n} = \frac{\sum_{j=1}^{i} d_j}{n}.$$

(d) The original paper reported the number of cycles to failure, as given above. Suggest reasons why the numbers of cycles to failures were not given with more precision and describe the effect that this has on the results of the analysis.

(e) The original paper reported that the tested units had been selected from a random sample of heats. What might have happened in the experiment if all of the sample links had been selected from just one or two heats?

(f) The original paper did not report the order in which the tests were run. Typically, fatigue tests require the use of one or a few expensive test stands and tests are done in sequence. The order in which the failures occurred was not described in the original paper. Is it possible that there was some useful information in knowing the order in which the 130 units had been tested? Discuss.

Problem 3: Exercise 3.7

The supplier of an electromechanical control for a household appliance ran an accelerated life test on sample controls. In the test, 25 controls were put on test and run until failure or until 30 thousand cycles had been accumulated. Failures occurred at 5, 21, and 28 thousand cycles. The other 22 controls did not fail by the end of the test.

(a) Compute and plot a nonparametric estimate for $F(t)$.

(b) Compute an approximate 95% confidence interval for the probability that an electromechanical device from the same production process, tested in the same way, would fail before 30 thousand cycles. Use the conservative binomial distribution method.

(c) Repeat part (b) using the Jeffrey method.

(d) Repeat part (b) using the Wald method, based on $Z_{\widehat{F}(30)} \sim \text{NORM}(0, 1)$.

(e) Explain why, in this situation, the method in part (b) or (c) would be preferred to the method in part (d).

(f) The appliance manufacturer is really interested in the probability of the number of days to failure for its product. Use rate differs from household to household, but the average rate is 2.3 cycles per day. What can the manufacturer say about the fraction of devices that would fail in 10 years of operation (the expected technological life of the product)?

Problem 4: Exercise 3.8

Over the past 18 months, ten separate copies of an electronic system have been deployed in Earth orbit, where repair is impossible. Continuous remote monitoring, however, provides information on the state of the system and each of its main subsystems. Each system contains three nominally identical devices and it was learned, after deployment, that these devices are, in the system's environment, failing unexpectedly. The failures cause degradation to the overall system operation. For future systems that are to be deployed, the problem will be fixed, but owners of the systems have asked for information on the amount of degradation that can be expected in future years of operation among those currently deployed. To date, 5 of the 30 devices have failed. Due to the staggered entry of the systems into service, the available data are multiply censored. The following table summarizes the available information with times given in hours. Times of unfailed units are marked with a "+". File ElectronicSystem.csv contains the data.

System	Device 1	Device 2	Device 3
1	564+	564+	564+
2	1321+	1104	1321+
3	1933+	1933+	1933+
5	1965+	1965+	1965+
4	2578+	2345	2578+
6	3122+	3122+	3122+
7	5918+	5918+	4467
8	7912+	7912+	6623
9	8156+	8156+	8156+
10	7885	12229+	12229+

(a) Compute a nonparametric estimate of $F(t)$, the failure-time cdf of the devices, assuming that the devices are operating and failing independently.

(b) Plot the nonparametric estimate of $F(t)$.

(c) Compute pointwise approximate 95% confidence intervals for $F(t)$ and add these to your plot.

Problem 5: Exercise 3.12

Weis et al. (1986) report on the results of a life test on silicon photodiode detectors in which 28 detectors were tested at 85°C and 40 volts reverse bias. These conditions, which were more stressful than normal use conditions, were used in order to get failures quickly. Specified electrical tests were made at $0, 10, 25, 75, 100, 500, 750, 1000, 1500, 2000, 2500, 3000, 3500,$ 3600, 3700, and 3800 hours to determine if the detectors were still performing properly. Failures were found after the inspections at 2500 (1 failure), 3000 (1 failure), 3500 (2 failures), 3600 (1 failure), 3700 (1 failure), and 3800 (1 failure). The other 21 detectors had not failed after 3800 hours of operation. These data are also available in file PhotoDetector.csv. Use these data to estimate the failure-time cdf of such photodiode detectors running at the test conditions.

(b) Compute and plot a nonparametric estimate of the cdf for time to failure at the test conditions.

(c) Compute standard errors for the nonparametric estimate in part (b).

(d) Compute pointwise approximate 95% confidence intervals for $F(t)$ and add these to your plot.

(e) Compute nonparametric simultaneous approximate 95% confidence bands for $F(t)$ over the complete range of observation.

(f) Provide a careful explanation of the differences in interpretation and application of the nonparametric pointwise confidence intervals and the nonparametric simultaneous confidence bands.

Problem 6: Exercise 4.5

Let $T \sim \mathrm{WEIB}(\mu, \sigma), \eta = \exp(\mu)$, and $\beta = 1/\sigma$.

(a) For $m > 0$, show that $\mathrm{E}\left(T^m\right) = \eta^m \Gamma(1 + m/\beta)$, where $\Gamma(x)$ is the gamma function. Hint: Change integration variable using $w = (t/\eta)^\beta$ as the new variable.

(b) Use the result in (a) to show that

$$\mathrm{E}(T) = \eta\Gamma(1 + 1/\beta), \quad \mathrm{Var}(T) = \eta^2\left[\Gamma(1 + 2/\beta) - \Gamma^2(1 + 1/\beta)\right]$$

Problem 7: Exercise 4.6

Consider the Weibull distributions with parameters $\eta = 10$ years and $\beta = 0.5, 1,\ 2,$ and 4.

(a) Plot the Weibull distribution hf for t ranging between 0 and 10 .

(b) Explain the practical interpretation of the hf at $t = 1$ and $t = 10$ years.

(c) Plot the Weibull cdfs over the same range of t. For which shape parameter value is the probability of failing the largest at 1 year? At 10 years? Explain.

(note: probability of falling at time $t = F(t)$)

Problem 8: Exercise 4.15

Assume that T is $\mathrm{LNORM}(\mu, \sigma)$ and m is an arbitrary real number.

(a) Show that $\mathrm{E}\left(T^m\right) = \exp\left(\mu m + \sigma^2 m^2/2\right)$.
Hint: Because $T \sim \mathrm{LNORM}(\mu, \sigma)$, $T = \exp(Y)$ where $Y \sim \mathrm{NORM}(\mu, \sigma)$. Thus, $\mathrm{E}\left(T^m\right) = \mathrm{E}[\exp(mY)]$, where the later expectation is the moment-generating function of a normal distribution.

(b) Use the result in (a) to show that

$$\mathrm{E}(T) = \exp\left(\mu + \sigma^2/2\right), \quad \mathrm{Var}(T) = \exp\left(2\mu + \sigma^2\right)\left[\exp\left(\sigma^2\right) - 1\right]$$

Problem 9: Exercise 4.24

Consider a random variable T which has a log-location-scale distribution given by $F(t; \mu, \sigma) = \Phi([\log(t) - \mu]/\sigma)$, where $t > 0$ and $\sigma > 0$.

(a) For any arbitrary positive constant m, show that the variable T^m also has a loglocation-scale distribution with the same standard cdf $\Phi(z)$.
Hint: Use the relationship $\log(T) = \mu + \sigma Z$, where $Z \sim \Phi(z)$, to obtain a similar relationship for $\log\left(T^m\right)$.

(b) Using the result in (a), find the distribution of the square root of a Weibull random variable.

Problem 10: Exercise 4.25

If failure times in a population are adequately described by a distribution with a decreasing hf, one might think that the surviving units in the population are getting better with time. In fact, decreasing hfs are common for certain solid-state electronic components and electronic systems. Weaker units fail early, after which the hf decreases. For a mixture of two exponential distributions with $\gamma = 0$ but different values of θ (say, $\theta_1 = 1$ and $\theta_2 = 5$), and equal proportions from the two populations (i.e., the mixture distribution has a CDF $F(t) = 0.5[1 - \exp(-t)] + 0.5[1 - \exp(-t/5)], t > 0$), do the following:

(a) Obtain an expression for the cdf of the mixture.

(b) Obtain an expression for the pdf of the mixture.

(c) Use parts (a) and (b) to derive an expression for the hf of the mixture.

(d) Graph the mixture hf from $t = 0$ to $t = 10$.

(e) What is the shape of the mixture hf? What is the intuition for this result?

(f) In what sense is the exponential-mixture population "improving" with time (as suggested by the decreasing hf ?