55. **(Poisson one-sided UMP)** Problem 8.31 in Casella and Berger (2001).

56. **(two-sample t-test with equal variance is LRT)** Problem 8.41(a)(b) in Casella and Berger (2001).

57. **(Clopper and Pearson CI for p)** Problem 9.21 in Casella and Berger (2001).

58. **(distribution of p-values)** Problem 17(b)(c) of Keener (2010) Section 12.8.

59. For Example 10.4.6 on class notes on CI for a proportion of $B(1, p)$,
(a) we discussed in class that the roots are real. Verify that the roots are ≥ 0 and ≤ 1.
(b) Construct another $100(1-\alpha)\%$ confidence interval for p based on the following steps. (1) Find a variance-stablizing transformation of $\hat{p}$, $h(\hat{p})$, so that the variance of $h(\hat{p})$ is free of p. (2) Construct a confidence interval of p based on the asymptotic distribution of $h(\hat{p})$.

60. Let $Y_i = (\theta/2)t_i^2 + \varepsilon_i$, $i = 1\ldots, n$, where ε_i are independent normal random variables with mean 0 and variance σ^2.
(a) Derive the MLE for θ.
(b) Using a pivot based on the MLE of θ, find a level $1-\alpha$ confidence interval for θ.

61. Let $X_1, \ldots, X_n$ be an i.i.d. sample from the uniform distribution on $[0, \theta]$ for $\theta > 0$.

    (a) Consider the problem of testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$. Show that any test ϕ for which $\phi(x) = 1$ when $x_{(n)} = \max\{x_1, \ldots, x_n\} > \theta_0$ is UMP at level $\alpha = \mathbb{E}_{\theta_0}[\phi(X)]$.

    (b) Now consider the problem of testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$. For a level-$\alpha$ test given by
$$\phi(x) = 1\{x_{(n)} > \theta_0 \text{ or } x_{(n)} < \theta_0\alpha^{1/n}\},$$
is it UMP?

Practice
8.31, 8.33, 8.34, 9.2, 9.6, 9.12, 9.16 in Casella and Berger (2001).
Problem 16 of Keener (2010) Section 12.8.

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55. (a)

for $\lambda_1 < \lambda_2$, $\frac{f(\underline{x};\lambda_2)}{f(\underline{x};\lambda_1)} = e^{-n(\lambda_2-\lambda_1)}\left(\frac{\lambda_2}{\lambda_1}\right)^{\Sigma X_i}$ is a non-decreasing function of ΣX_i

$\therefore \Sigma X_i \sim Poi(n\lambda)$ has MLR

By the Karlin - Rubin Theorem, the UMP test is to reject H_0

if $\Sigma X_i > k$ where k satisfies $P(\Sigma X_i > k \mid \lambda = \lambda_0) = \alpha$ □

(b) $\Sigma X_i \sim Poi(n\lambda) \xrightarrow{D} N(n\lambda, n\lambda)$

$$\begin{cases} P(\Sigma X_i > k \mid \lambda = 1) \approx P\left(Z > \frac{k-n}{\sqrt{n}}\right) = 0.05 \\ P(\Sigma X_i > k \mid \lambda = 2) \approx P\left(Z > \frac{k-2n}{\sqrt{2n}}\right) = 0.9 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{k-n}{\sqrt{n}} = 1.645 \\ \frac{k-2n}{\sqrt{2n}} = -1.28 \end{cases} \Rightarrow n = 12$$

□

56. (a).

Note that $\Omega_0 = \{\mu \mid \mu = \mu_X = \mu_Y\}$, $\Omega = \{(\mu_X, \mu_Y)\}$

$$L(\mu, \sigma^2 \mid \underline{X}, \underline{Y}) = (2\pi\sigma^2)^{-\frac{1}{2}(n+m)} \exp\left\{ \frac{-1}{2\sigma^2}\left[\sum_1^n (X_i-\mu)^2 + \sum_1^m (Y_i-\mu)^2\right]\right\}$$

$$l(\mu, \sigma^2 \mid \underline{X}, \underline{Y}) = \log L = -\frac{1}{2}(n+m)\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\left[\sum_1^n (X_i-\mu)^2 + \sum_1^m (Y_i-\mu)^2\right]$$

$$\Rightarrow \text{Let} \begin{cases} \frac{\partial l}{\partial \mu} = 0 \\ \frac{\partial l}{\partial \sigma^2} = 0 \end{cases} \Rightarrow \begin{cases} \hat{\mu} = \frac{n\bar{X} + m\bar{Y}}{n+m} \\ \hat{\sigma}_0^2 = \frac{\sum_1^n (X_i-\hat{\mu})^2 + \sum_1^m (Y_i-\hat{\mu})^2}{n+m} \end{cases}$$

$$L(\mu_X, \mu_Y, \sigma^2 \mid \underline{X}, \underline{Y}) = (2\pi\sigma^2)^{-\frac{1}{2}(n+m)} \exp\left\{ \frac{-1}{2\sigma^2}\left[\sum_1^n (X_i-\mu_X)^2 + \sum_1^m (Y_i-\mu_Y)^2\right]\right\}$$

$$l(\mu_X, \mu_Y, \sigma^2 \mid \underline{X}, \underline{Y}) = \log L = -\frac{1}{2}(n+m)\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\left[\sum_1^n (X_i-\mu_X)^2 + \sum_1^m (Y_i-\mu_Y)^2\right]$$

$$\Rightarrow \text{Let} \begin{cases} \frac{\partial l}{\partial \mu_X} = 0 \\ \frac{\partial l}{\partial \mu_Y} = 0 \\ \frac{\partial l}{\partial \sigma^2} = 0 \end{cases} \Rightarrow \begin{cases} \hat{\mu}_X = \bar{X} \\ \hat{\mu}_Y = \bar{Y} \\ \hat{\sigma}^2 = \frac{\sum_1^n (X_i-\bar{X})^2 + \sum_1^m (Y_i-\bar{Y})^2}{n+m} \end{cases}$$

$$\lambda(\underline{X}, \underline{Y}) = \frac{\sup_{\Omega_0} L(\mu, \sigma^2)}{\sup_{\Omega} L(\mu_X, \mu_Y, \sigma^2)} = \frac{L(\hat{\mu}, \hat{\sigma}_0^2)}{L(\bar{X}, \bar{Y}, \hat{\sigma}^2)} = \frac{(2\pi\hat{\sigma}_0^2)^{-\frac{1}{2}(n+m)} \exp\left\{\frac{-1}{2\hat{\sigma}_0^2}\left[\sum_1^n (X_i-\hat{\mu})^2 + \sum_1^m (Y_i-\hat{\mu})^2\right]\right\}}{(2\pi\hat{\sigma}^2)^{-\frac{1}{2}(n+m)} \exp\left\{\frac{-1}{2\hat{\sigma}^2}\left[\sum_1^n (X_i-\bar{X})^2 + \sum_1^m (Y_i-\bar{Y})^2\right]\right\}} = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2}\right)^{-\frac{(n+m)}{2}}$$

$\therefore$ The LRT rejects H_0 when $\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2} > k$

$$\Rightarrow \frac{\sum_1^n \left(X_i - \frac{n\bar{X}+m\bar{Y}}{n+m}\right)^2 + \sum_1^m \left(Y_i - \frac{n\bar{X}+m\bar{Y}}{n+m}\right)^2}{\sum_1^n (X_i-\bar{X})^2 + \sum_1^m (Y_i-\bar{Y})^2} = \frac{\sum_1^n (X_i-\bar{X})^2 + \sum_1^m (Y_i-\bar{Y})^2 + \frac{nm}{n+m}(\bar{X}-\bar{Y})^2}{\sum_1^n (X_i-\bar{X})^2 + \sum_1^m (Y_i-\bar{Y})^2} > k$$

$$\Rightarrow \frac{(\bar{X}-\bar{Y})^2}{\left(\frac{1}{n}+\frac{1}{m}\right)\left[\frac{\sum_1^n (X_i-\bar{X})^2 + \sum_1^m (Y_i-\bar{Y})^2}{n+m-2}\right]} > k' \Rightarrow |T| = \frac{|\bar{X}-\bar{Y}|}{\sqrt{S_p^2\left(\frac{1}{n}+\frac{1}{m}\right)}} > k'' \quad \square$$

(b) Under H_0, $\bar{X}-\bar{Y} \sim N\left(0, \left(\frac{1}{n}+\frac{1}{m}\right)\sigma^2\right)$

$$\frac{(n+m-2)S_p^2}{\sigma^2} \sim \chi^2_{n+m-2}$$

and $\dfrac{\bar{X}-\bar{Y}}{\sqrt{\frac{1}{n}+\frac{1}{m}}\,\sigma} \perp\!\!\!\perp \dfrac{(n+m-2)S_p^2}{\sigma^2}$

$$\therefore\ T = \frac{\bar{X}-\bar{Y}}{\sqrt{S_p^2\left(\frac{1}{n}+\frac{1}{m}\right)}} = \frac{\dfrac{\bar{X}-\bar{Y}}{\sqrt{\frac{1}{n}+\frac{1}{m}}\,\sigma}}{\sqrt{\dfrac{(n+m-2)S_p^2}{\sigma^2}\Big/(n+m-2)}} \sim t_{n+m-2} \ \square$$

57. Suppose the $1-\alpha$ C.I. for p is $\{p : l \le p \le u\}$

where l and u satisfy
$$\begin{cases} \dfrac{\alpha}{2} = \displaystyle\sum_{k=0}^{x} \binom{n}{k} u^k (1-u)^{n-k} & \text{———— (1)} \\ \dfrac{\alpha}{2} = \displaystyle\sum_{k=x}^{n} \binom{n}{k} l^k (1-l)^{n-k} & \text{———— (2)} \end{cases}$$

(1) : $\dfrac{\alpha}{2} = P(K_1 \le x) = P(Y_1 \le 1-u)$ where $\begin{cases} K_1 \sim Bin(n, u) \\ Y_1 \sim Beta(n-x, x+1) \end{cases}$

Let $Z_1 \sim F_{(2(n-x),\, 2(x+1))}$, and $c = \dfrac{n-x}{x+1} \Rightarrow \dfrac{cZ_1}{1+cZ_1} \sim Beta(n-x, x+1)$

$\therefore \dfrac{\alpha}{2} = P\left(\dfrac{cZ_1}{1+cZ_1} \le 1-u\right) = P\left(\dfrac{1}{Z_1} \ge \dfrac{cu}{1-u}\right)$, where $\dfrac{1}{Z_1} \sim F_{(2(x+1),\, 2(n-x))}$

$\Rightarrow \dfrac{cu}{1-u} = F_{(2(x+1),\, 2(n-x)),\, \frac{\alpha}{2}} \Rightarrow u = \dfrac{\frac{x+1}{n-x} F_{(2(x+1),\, 2(n-x)),\, \frac{\alpha}{2}}}{1 + \frac{x+1}{n-x} F_{(2(x+1),\, 2(n-x)),\, \frac{\alpha}{2}}}$

(2) : $\dfrac{\alpha}{2} = 1 - P(K_2 \le x-1) = 1 - P(Y_2 \le 1-l)$ where $\begin{cases} K_2 \sim Bin(n, l) \\ Y_2 \sim Beta(n-x+1, x) \end{cases}$

Let $Z_2 \sim F_{(2(n-x+1),\, 2x)}$ and $d = \dfrac{n-x+1}{x} \Rightarrow \dfrac{dZ_2}{1+dZ_2} \sim Beta(n-x+1, x)$

$\therefore \dfrac{\alpha}{2} = 1 - P\left(\dfrac{dZ_2}{1+dZ_2} \le 1-l\right) = P\left(\dfrac{dZ_2}{1+dZ_2} > 1-l\right) = P\left(Z_2 > \dfrac{1-l}{dl}\right)$

$\Rightarrow \dfrac{1-l}{dl} = F_{(2(n-x+1),\, 2x),\, \frac{\alpha}{2}} \Rightarrow l = \dfrac{1}{1 + \frac{n-x+1}{x} F_{(2(n-x+1),\, 2x),\, \frac{\alpha}{2}}}$

$\therefore$ The $1-\alpha$ C.I. of $p \Rightarrow \left[\dfrac{1}{1 + \frac{n-x+1}{x} F_{(2(n-x+1),\, 2x),\, \frac{\alpha}{2}}},\ \dfrac{\frac{x+1}{n-x} F_{(2(x+1),\, 2(n-x)),\, \frac{\alpha}{2}}}{1 + \frac{x+1}{n-x} F_{(2(x+1),\, 2(n-x)),\, \frac{\alpha}{2}}} \right]$

58. (b)

Define $F(t) = P_{\theta_0}(T \le t)$.

The UMP level α test is $\phi_\alpha(x) = I\{T(x) > k(\alpha)\}$

where $k(\alpha)$ is chosen s.t. $F(k(\alpha)) = 1-\alpha$

$\because$ F is non-decreasing and continuous function

$\therefore$ If $t > k(\alpha)$, then $F(t) \ge F(k(\alpha)) = 1-\alpha$

$\Rightarrow$ p-value $= \inf\{\alpha : t > k(\alpha)\} \ge \inf\{\alpha : F(t) \ge 1-\alpha\} = 1 - F(t) = P_{\theta_0}[T > t]$

But if $F(t) > F(k(\alpha)) = 1-\alpha$, then $t > k(\alpha)$

$\Rightarrow$ p-value $= \inf\{\alpha : t > k(\alpha)\} \le \inf\{\alpha : F(t) > 1-\alpha\} = P_{\theta_0}[T > t]$

$\therefore$ p-value $= P_{\theta_0}[T(X) > t]$ $\square$

(c) $P_{\theta_0}[\text{p-value} \le x] = P_{\theta_0}[1 - F(T) \le x] = P_{\theta_0}[F(T) \ge 1-x]$

$= P_{\theta_0}[T \ge F^{-1}(1-x)] = 1 - P_{\theta_0}[T < F^{-1}(1-x)] = 1 - F(F^{-1}(1-x)) = x$

$\therefore$ p-value $= 1 - F(T) \sim U(0,1)$ under $\theta = \theta_0$ $\square$

59. (a).

The quadratic equation : $(1+\frac{c}{n})p^2 - (\frac{c}{n}+2\hat{p})p + \hat{p}^2 = 0$

$$\Rightarrow \begin{cases} 2\text{根和} = \dfrac{\frac{c}{n}+2\hat{p}}{1+\frac{c}{n}} = \dfrac{2\Sigma X_i + c}{n+c} > 0 \\ 2\text{根積} = \dfrac{\hat{p}^2}{1+\frac{c}{n}} > 0 \end{cases} \Rightarrow 2 \text{ roots are } > 0$$

check $\dfrac{\frac{c}{n}+2\hat{p} \pm \sqrt{\frac{c}{n}(\frac{c}{n}+4(\hat{p}-\hat{p}^2))}}{2(1+\frac{c}{n})} - 1 = \dfrac{1}{2(1+\frac{c}{n})}\left[\frac{-c}{n}+2(\hat{p}-1) \pm \sqrt{\frac{c}{n}(\frac{c}{n}+4(\hat{p}-\hat{p}^2))}\right]$

It's easy to see that $\frac{-c}{n}+2(\hat{p}-1) - \sqrt{\frac{c}{n}(\frac{c}{n}+4(\hat{p}-\hat{p}^2))} < 0$

Just check $\left[\frac{-c}{n}+2(\hat{p}-1)\right]^2 - \sqrt{\frac{c}{n}(\frac{c}{n}+4(\hat{p}-\hat{p}^2))}^2$

$$= \left[\frac{c^2}{n^2} - \frac{4c}{n}(\hat{p}-1) + 4(\hat{p}-1)^2\right] - \left[\frac{c^2}{n^2} + \frac{4c}{n}\hat{p} - \frac{4c}{n}\hat{p}^2\right]$$

$$= 4\left[(1+\frac{c}{n})\hat{p}^2 - 2(1+\frac{c}{n})\hat{p} + (1+\frac{c}{n})\right] = 4(1+\frac{c}{n})(\hat{p}-1)^2 > 0$$

$\therefore \left[\frac{-c}{n}+2(\hat{p}-1)\right]^2 - \sqrt{\frac{c}{n}(\frac{c}{n}+4(\hat{p}-\hat{p}^2))}^2 > 0$

$\Rightarrow \frac{-c}{n}+2(\hat{p}-1) - \sqrt{\frac{c}{n}(\frac{c}{n}+4(\hat{p}-\hat{p}^2))} < 0$

$\therefore$ The two roots are ≥ 0 and ≤ 1 $\square$

(b)

(1) Let $h(p) = \arcsin\sqrt{p}$, then $h'(p) = \frac{1}{2\sqrt{p(1-p)}} \Rightarrow [h'(p)]^2 = \frac{1}{4p(1-p)}$

(2) We have known that $\sqrt{n}(\hat{p}-p) \xrightarrow{D} N(0, p(1-p))$

Then by Delta Method:

$$\sqrt{n}\left(h(\hat{p}) - h(p)\right) \xrightarrow{D} N\left(0, [h'(p)]^2 p(1-p)\right) = N\left(0, \tfrac{1}{4}\right)$$

$$\Rightarrow 2\sqrt{n}\left(\arcsin\sqrt{\hat{p}} - \arcsin\sqrt{p}\right) \xrightarrow{D} N(0,1)$$

$$1-\alpha = P\left[-Z_{\frac{\alpha}{2}} < 2\sqrt{n}\left(\arcsin\sqrt{\hat{p}} - \arcsin\sqrt{p}\right) < Z_{\frac{\alpha}{2}}\right]$$

$$= P\left[\arcsin\sqrt{\hat{p}} - \frac{Z_{\frac{\alpha}{2}}}{2\sqrt{n}} < \arcsin\sqrt{p} < \arcsin\sqrt{\hat{p}} + \frac{Z_{\frac{\alpha}{2}}}{2\sqrt{n}}\right]$$

$$= P\left\{\max\left[0, \sin^2\left(\arcsin\sqrt{\hat{p}} - \frac{Z_{\frac{\alpha}{2}}}{2\sqrt{n}}\right)\right] < p < \sin^2\left(\arcsin\sqrt{\hat{p}} + \frac{Z_{\frac{\alpha}{2}}}{2\sqrt{n}}\right)\right\} \square$$

60. (a)

$$\{Y_i\}_{i=1}^n \overset{indep.}{\sim} N\left(\frac{\theta t_i^2}{2}, \sigma^2\right)$$

$$L(\theta, \sigma^2 | \underline{Y}) = \prod_{i=1}^n f(Y_i; \theta, \sigma^2) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{\frac{-1}{2\sigma^2}\sum_1^n \left(Y_i - \frac{\theta t_i^2}{2}\right)^2\right\}$$

$$l(\theta, \sigma^2 | \underline{Y}) = \log L = \frac{-n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_1^n\left[Y_i^2 - \theta t_i^2 Y_i + \frac{1}{4}\theta^2 t_i^4\right]$$

$$\Rightarrow \frac{\partial l}{\partial \theta} = \frac{-1}{2\sigma^2}\left[-\sum_1^n t_i^2 Y_i + \frac{\theta}{2}\sum_1^n t_i^4\right] \overset{set}{=\!=} 0 \Rightarrow \hat{\theta}_{MLE} = \frac{2\sum_1^n t_i^2 Y_i}{\sum_1^n t_i^4} \square$$

(b) $\frac{2t_i^2 Y_i}{\Sigma t_i^4} \sim N\left(\frac{\theta t_i^4}{\Sigma t_i^4}, \frac{4t_i^4 \sigma^2}{(\Sigma t_i^4)^2}\right)$

$$\Rightarrow \hat{\theta}_{MLE} = \frac{2\Sigma t_i^2 Y_i}{\Sigma t_i^4} \sim N\left(\theta, \frac{4\sigma^2}{\Sigma t_i^4}\right) \Rightarrow \frac{\hat{\theta} - \theta}{2\sigma / \sqrt{\Sigma t_i^4}} \sim N(0, 1)$$

$$1-\alpha = P_\theta\left[-Z_{\frac{\alpha}{2}} < \frac{\hat{\theta} - \theta}{2\sigma / \sqrt{\Sigma t_i^4}} < Z_{\frac{\alpha}{2}}\right]$$

$$= P_\theta\left[\hat{\theta} - Z_{\frac{\alpha}{2}} \frac{2\sigma}{\sqrt{\Sigma t_i^4}} < \theta < \hat{\theta} + Z_{\frac{\alpha}{2}} \frac{2\sigma}{\sqrt{\Sigma t_i^4}}\right]$$

$\therefore$ the $1-\alpha$ C.I. for θ is $\left(\hat{\theta} - Z_{\frac{\alpha}{2}} \frac{2\sigma}{\sqrt{\Sigma t_i^4}}, \hat{\theta} + Z_{\frac{\alpha}{2}} \frac{2\sigma}{\sqrt{\Sigma t_i^4}}\right)$ $\square$

61. (a).

For $\theta_2 > \theta_1$, $\frac{L(\underline{X}; \theta_2)}{L(\underline{X}; \theta_1)} = \left(\frac{\theta_1}{\theta_2}\right)^n \frac{I\{0 < X_{(1)} \le X_{(n)} < \theta_2\}}{I\{0 < X_{(1)} \le X_{(n)} < \theta_1\}} = \begin{cases} \left(\frac{\theta_1}{\theta_2}\right)^n, & 0 < X_{(n)} < \theta_1 \\ \infty, & \theta_1 \le X_{(n)} < \theta_2 \end{cases}$

is a non-decreasing function of $X_{(n)} \Rightarrow X_{(n)}$ has MLR

$\therefore \phi(x) = \begin{cases} 1, & X_{(n)} > \theta_0 \\ r, & X_{(n)} = \theta_0 \\ 0, & X_{(n)} < \theta_0 \end{cases}$ is UMP test with $\alpha = E_{\theta_0}[\phi(X)]$ $\square$