18. Problem 5.44 in Casella and Berger (2001).

19. Consider the following linear model, $Y_j = \alpha + j\beta + \varepsilon_j$, $j = 1, \ldots, n$, where ε_j, j= $1, \ldots, n$ are i.i.d. with mean 0, finite variance σ^2, and $E(\varepsilon_j^4) < \infty$. Then the least squares estimate of β is given by

$$\hat{\beta}_n = \sum_{i=1}^{n}(j - a_n)Y_j / \sum_{j=1}^{n}(j - a_n)^2, \qquad a_n = (n+1)/2.$$

(Do not need to derive the expression of $\hat{\beta}_n$.) Find the asymptotic distribution of $\hat{\beta}_n$. You must provide all the important logical steps to show work.

20. Prove that if $X_n = O_P(r_n)$ where $0 < r_n < \infty$ and $r_n \to 0$ as $n \to \infty$, then $|X_n|^r = O_P(r_n^r)$, $r > 0$.

21. Prove the following statements: if $X_n = o_p(a_n)$ and $Y_n = o_p(b_n)$, where $a_n > 0$, $b_n > 0$, $n = 1, 2, \ldots$, then (a) $X_n + Y_n = o_p(\max(a_n, b_n))$; (b) $X_n Y_n = o_p(a_n b_n)$. You may refer to the proof of Proposition 6.1.1 in Brockwell and Davis (1991).

22. Prove the probabilistic Taylor's expansion in the univariate case. You may refer to the proof of Proposition 6.1.5 in Brockwell and Davis (1991).

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18. $\{X_i\}_{i=1}^{n} \overset{iid}{\sim} Ber(p)$, $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

(a) $E(X_i) = p$, $Var(X_i) = p(1-p)$

$Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}_n$

By CLT: $\sqrt{n}\left(\overline{X}_n - E(X_i)\right) \xrightarrow{D} N(0, Var(X_i))$

$\therefore \sqrt{n}(Y_n - p) \xrightarrow{D} N(0, p(1-p))$ □

(b) Define $g(p) = p(1-p)$

Then $g'(p) = 1-2p$ exists and $\neq 0$ ($\because p \neq \frac{1}{2}$)

By Delta Method:

$\sqrt{n}\left(g(Y_n) - g(p)\right) \xrightarrow{D} N(0, p(1-p)[g'(p)]^2)$

$\therefore \sqrt{n}\left(Y_n(1-Y_n) - p(1-p)\right) \xrightarrow{D} N(0, (1-2p)^2 p(1-p))$ □

(c) By CLT:

$$\frac{\sqrt{n}(Y_n - p)}{\sqrt{p(1-p)}} \xrightarrow{D} N(0,1)$$

By countinuous mapping theorem:

$$\left[\frac{\sqrt{n}(Y_n - p)}{\sqrt{p(1-p)}}\right]^2 = \frac{n(Y_n - p)^2}{p(1-p)} \xrightarrow{D} \chi_1^2$$

For $p = \frac{1}{2}$: $4n\left[\frac{1}{4} - Y_n(1-Y_n)\right] \xrightarrow{D} \chi_1^2$

$$\therefore n\left[Y_n(1-Y_n) - \frac{1}{4}\right] \xrightarrow{D} -\frac{1}{4}\chi_1^2$$

19.

$\{\varepsilon_j\}_1^n \overset{iid}{\sim} E(\varepsilon_j) = 0, \ Var(\varepsilon_j) = \sigma^2 < \infty$

Define $S^2 = \sum_1^n (j - a_n)^2$

$$\hat{\beta}_n = \frac{\sum_1^n (j - a_n) Y_j}{\sum_1^n (j - a_n)^2} = \beta + \frac{\sum_1^n (j - a_n)\varepsilon_j}{S^2}$$

$$= \beta + \frac{\sigma}{S}\left[\frac{\sum_1^n \left(\frac{\varepsilon_j}{\sigma}\right)(j - a_n)}{S}\right]$$

Let $C_{nj} = j - a_n$, $\sum_{j=1}^n C_{nj}^2 = \sum_{j=1}^n (j - a_n)^2 = S^2$

$$X_j = \frac{\varepsilon_j}{\sigma} \overset{iid}{\sim} E(X_j) = 0, \ Var(X_j) = 1$$

$$Z_n = \sum_{j=1}^n \frac{X_j C_{nj}}{S} = \frac{\sum_1^n \left(\frac{\varepsilon_j}{\sigma}\right)(j - a_n)}{S}$$

and $E(Z_n) = 0$, $Var(Z_n) = 1$

$$\sum_{j=1}^{n} E\left|\frac{X_j C_{nj}}{S} - 0\right|^2 I\left\{\left|\frac{X_j C_{nj}}{S} - 0\right| > \varepsilon\right\}$$

$$= \frac{1}{S^2}\sum_{j=1}^{n} C_{nj}^2 \, E|X_j|^2 \, I\left\{|X_j| > \frac{\varepsilon S}{|C_{nj}|}\right\} \longrightarrow 0$$

if $\frac{S}{|C_{nj}|} \longrightarrow \infty$, for all, and $E(X_j^4) = \frac{E(\varepsilon_j^4)}{\sigma^4} < \infty$

Want to show $\frac{|C_{nj}|}{S} \longrightarrow 0$, for all j

$$\Rightarrow \frac{\max_j |C_{nj}|^2}{S^2} = \frac{\max_j (j - a_n)^2}{\sum_1^n (j - a_n)^2} = \frac{(n^2 - 2n + 1)/2}{(n^3 - n)/12}$$

$\longrightarrow 0$, as $n \longrightarrow \infty$

By Lindeberg - Feller CLT:

$$Z_n = \frac{\sum_1^n \left(\frac{\varepsilon_j}{\sigma}\right)(j - a_n)}{S} \xrightarrow{\mathcal{D}} N(0, 1)$$

By Slutsky's Theorem:

$$\hat{\beta} \xrightarrow{\mathcal{D}} N\left(\beta, \frac{\sigma^2}{\sum_1^n (j - a_n)^2}\right) = N\left(\beta, \frac{12\sigma^2}{n^3 - n}\right)_{\square}$$

20. $\because X_n = O_p(r_n) \Rightarrow \dfrac{X_n}{r_n} = O_p(1)$

$\therefore \forall \varepsilon > 0, \exists \sqrt[r]{M_\varepsilon} > 0$

s.t. $P\left(\dfrac{|X_n|}{r_n} > \sqrt[r]{M_\varepsilon}\right) < \varepsilon, \forall n$

$\Rightarrow P\left(\dfrac{|X_n|^r}{r_n^{\,r}} > M_\varepsilon\right) < \varepsilon, \forall n$

$\therefore |X_n|^r = O_p(r_n^{\,r}), \quad r > 0 \quad \square$

21. $X_n = o_p(a_n)$, $Y_n = o_p(b_n)$

(a) $\frac{X_n}{a_n} \xrightarrow{p} 0$, $\frac{Y_n}{b_n} \xrightarrow{p} 0$

$$\forall \varepsilon > 0, \begin{cases} P\left(\left|\frac{X_n}{a_n}\right| > \frac{\varepsilon}{2}\right) \longrightarrow 0 \\ P\left(\left|\frac{Y_n}{b_n}\right| > \frac{\varepsilon}{2}\right) \longrightarrow 0 \end{cases}, \text{ as } n \to \infty$$

Note that $M = \max(a_n, b_n)$

$$\Rightarrow \begin{cases} P\left(\left|\frac{X_n}{M}\right| > \frac{\varepsilon}{2}\right) \le P\left(\left|\frac{X_n}{a_n}\right| > \frac{\varepsilon}{2}\right) \to 0 \\ P\left(\left|\frac{Y_n}{M}\right| > \frac{\varepsilon}{2}\right) \le P\left(\left|\frac{Y_n}{b_n}\right| > \frac{\varepsilon}{2}\right) \to 0 \end{cases}, \text{ as } n \to \infty$$

$$P\left(\left|\frac{X_n + Y_n}{M}\right| > \varepsilon\right) \le P\left(\left|\frac{X_n}{M}\right| + \left|\frac{Y_n}{M}\right| > \varepsilon\right)$$

$$\le P\left(\left|\frac{X_n}{M}\right| > \frac{\varepsilon}{2}\right) + P\left(\left|\frac{Y_n}{M}\right| > \frac{\varepsilon}{2}\right) \to 0, \text{ as } n \to \infty$$

$$\Rightarrow \frac{X_n + Y_n}{\max(a_n, b_n)} \xrightarrow{p} 0$$

$\therefore X_n + Y_n = o_p(\max(a_n, b_n))$ $\square$

(b) By homework ex 12:

If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, then $X_n Y_n \xrightarrow{P} XY$

(We have proved it last week)

$\because \frac{X_n}{a_n} \xrightarrow{P} 0$, $\frac{Y_n}{b_n} \xrightarrow{P} 0$

$\therefore \left(\frac{X_n}{a_n}\right)\left(\frac{Y_n}{b_n}\right) = \frac{X_n Y_n}{a_n b_n} \xrightarrow{P} 0$

$\Rightarrow X_n Y_n = o_p(a_n b_n)$ $\square$

22. We have known that

$$X_n = a + O_p(r_n), \quad 0 < r_n \to 0, \text{ as } n \to \infty$$

$$\Rightarrow (X_n - a)^s = O_p(r_n^s)$$

Define

$$h(x) = \begin{cases} \left[g(x) - \sum_{j=0}^{s} \frac{g^{(j)}(a)}{j!}(x-a)^j \right] \Big/ \left[\frac{(x-a)^s}{s!} \right], & x \neq a \\ 0, & x = a \end{cases}$$

Then $h(x)$ is a continuous function at a

$$\therefore h(X_n) = h(a) + o_p(1) \Rightarrow h(X_n) = o_p(1)$$

$$\Rightarrow (X_n - a)^s h(X_n) = o_p(r_n^s)$$

$$\Rightarrow g(X_n) - \sum_{j=0}^{s} \frac{g^{(j)}(a)}{j!}(X_n - a)^j = o_p(r_n^s)$$

$$\therefore g(X_n) = \sum_{j=0}^{s} \frac{g^{(j)}(a)}{j!}(X_n - a)^j + o_p(r_n^s) \quad \square$$