84. Problem 10.23(a)(b)(c) in Casella and Berger (2001) p:509.

85. Problem 10.25 in Casella and Berger (2001) p:509.

86. Extending the discussion in Eg 10.2.3 on asymptotic normality of the median, derive asymptotic normality of the 3rd quartile (75%-tile).

87. Derive the A and B matrices in Example 10 of
Stefanski, L. A., and Boos, D. D. (2002). The calculus of M-estimation. The American Statistician, 56(1), 29-38.

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84. (a) $ARE = [2\sigma f(\mu)]^2 = 4\sigma^2 f(\mu)^2$

normal : $(\mu, \sigma) = (1, 0) \Rightarrow f(\mu) = \frac{1}{\sqrt{2\pi}} \exp(\frac{-1}{2} \cdot 0) \approx 0.3989$

$\therefore ARE \approx 0.64$

logistic : $(\mu, s) = (0, 1) \Rightarrow \sigma^2 = \frac{s^2\pi^2}{3} = \frac{\pi^2}{3}$ and $f(\mu) = \frac{e^0}{(1+e^0)^2} = \frac{1}{4}$

$\therefore ARE \approx 0.82$

double exp. : $(\mu, b) = (0, 1) \Rightarrow \sigma^2 = 2b^2 = 2$ and $f(\mu) = \frac{1}{2}\exp(0) = \frac{1}{2}$

$\therefore ARE = 2$

(b) $\{X_i\}_1^n \overset{iid}{\sim} E(X) = \mu,\ Var(X) = \sigma^2 \Rightarrow ARE_1 = 4\sigma^2 f_X(\mu)^2$

Suppose $Y = \frac{X}{\sigma}$ is a scale change of $X \Rightarrow X = \sigma Y \Rightarrow J = \sigma$

$f_Y(y) = f_X(\sigma y)\sigma \Rightarrow ARE_2 = 4\frac{\sigma^2}{\sigma^2} f_Y(\frac{\mu}{\sigma})^2 = 4\sigma^2 f_X(\mu) = ARE_1$ □

(c) If $X \sim t_\nu$, then $\mu = E(X) = 0$, $\sigma^2 = Var(X) = \frac{\nu}{\nu-2}$ $f(\mu) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})}$

ν	σ^2	$f(0)$	$ARE = 4\sigma^2 f(0)^2$
3	3	0.367	1.62
5	$\frac{5}{3}$	0.379	0.96
10	$\frac{5}{4}$	0.389	0.757
25	$\frac{25}{23}$	0.395	0.678
50	$\frac{25}{24}$	0.397	0.657
∞	1	0.399	0.637

□

85. Let $y = x - \theta$

and $\because$ ① f is symmetric around 0 $\Rightarrow f(-y) = f(y)$

② ψ is an odd function $\Rightarrow \psi(y) = -\psi(-y)$

$$\int_{-\infty}^{\infty} \psi(x-\theta) f(x-\theta)\, dx = \int_{-\infty}^{\infty} \psi(y) f(y)\, dy = \int_{-\infty}^{0} \psi(y) f(y)\, dy + \int_{0}^{\infty} \psi(y) f(y)\, dy$$

$$= \int_{-\infty}^{0} -\psi(-y) f(-y)\, dy + \int_{0}^{\infty} \psi(y) f(y)\, dy$$

$$= -\int_{0}^{\infty} \psi(y) f(y)\, dy + \int_{0}^{\infty} \psi(y) f(y)\, dy = 0$$

The integrals add to 0 by the symmetry of f.

Note that θ_0 is the true value of θ

and Huber estimator $\hat{\theta}_M$ minimizes $\sum_{1}^{n} \rho(x_i - \theta)$

$\because \int_{-\infty}^{\infty} \psi(x-\theta_0) f(x-\theta_0)\, dx = 0$ and ρ is symmetric

$\therefore$ By 1-dim asy. normality of M-estimator

$$\sqrt{n}\,(\hat{\theta}_M - \theta_0) \xrightarrow{D} N\left(0, \frac{E_{\theta_0}[\psi^2(x_i - \theta_0)]}{E_{\theta_0}[\psi'(x_i - \theta_0)]^2}\right) \square$$

86. Let $\{X_i\}_{i=1}^n \overset{iid}{\sim}$ cdf F with $F(q_3) = \frac{3}{4}$

and note that Q_3 is the 3rd quartile of sample

Let $A_n = \frac{1}{n}\sum_1^n I\{X_i \le q_3\} = \frac{1}{n}\sum_1^n Y_i$ where $\{Y_i\}_1^n \overset{iid}{\sim} Ber(\frac{3}{4})$

By CLT: $\sqrt{n}\left(A_n - \frac{3}{4}\right) \xrightarrow{D} N\left(0, \frac{3}{16}\right)$

Let $g(t) = F^{-1}(t) \Rightarrow g'(t) = \dfrac{1}{f(F^{-1}(t))}$

By Delta-method:

$$\sqrt{n}\left(F^{-1}(A_n) - F^{-1}\left(\frac{3}{4}\right)\right) = \sqrt{n}\left(Q_3 - q_3\right)$$

$$\xrightarrow{D} N\left(0, \frac{3}{16}\left(\frac{1}{f(F^{-1}(\frac{3}{4}))}\right)^2\right) = N\left(0, \frac{3}{16 f(q_3)^2}\right) \quad \square$$

87.

Note that
$$\begin{cases} \psi_1(Y_i, n_i, \theta_1, p) = \dfrac{(Y_i - n_i p)^2}{n_i p(1-p)} - \theta_1 \\ \psi_2(Y_i, n_i, \theta_1, p) = Y_i - n_i p \end{cases}$$

$$\Rightarrow A = E\begin{bmatrix} \frac{-\partial\psi_1}{\partial\theta_1} & \frac{-\partial\psi_1}{\partial p} \\ \frac{-\partial\psi_2}{\partial\theta_1} & \frac{-\partial\psi_2}{\partial p} \end{bmatrix} = E\begin{bmatrix} 1 & \frac{2n_i(Y_i-n_ip)p(1-p) + (Y_i-n_ip)^2(1-2p)}{n_i p^2(1-p)^2} \\ 0 & n_i \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{2(E(Y_i)-n_ip)}{p(1-p)} + E\left[E\left(\frac{(Y_i-n_ip)^2}{n_i}\middle| n_i\right)\right]\frac{(1-2p)}{p^2(1-p)^2} \\ 0 & E(n_i) \end{bmatrix} = \begin{bmatrix} 1 & \frac{1-2p}{p(1-p)} \\ 0 & \mu_n \end{bmatrix}$$

$$B = E\begin{bmatrix} \psi_1^2 & \psi_1\psi_2 \\ \psi_1\psi_2 & \psi_2^2 \end{bmatrix} = E\begin{bmatrix} \frac{(Y_i-n_ip)^4}{n_i^2p^2(1-p)^2} - \frac{2\theta_1(Y_i-n_ip)^2}{n_ip(1-p)} + \theta_1^2 & \frac{(Y_i-n_ip)^3}{n_ip(1-p)} - (Y_i-n_ip)\theta_1 \\ \frac{(Y_i-n_ip)^3}{n_ip(1-p)} - (Y_i-n_ip)\theta_1 & (Y_i-n_ip)^2 \end{bmatrix}$$

$$= \begin{bmatrix} E\left[E\left(\frac{(Y_i-n_ip)^4}{n_i^2}\middle| n_i\right)\right]\frac{1}{p^2(1-p)^2} - 2 + 1 & E\left[E\left(\frac{(Y_i-n_ip)^3}{n_i}\middle| n_i\right)\right]\frac{1}{p(1-p)} \\ E\left[E\left(\frac{(Y_i-n_ip)^3}{n_i}\middle| n_i\right)\right]\frac{1}{p(1-p)} & E(n_i)p(1-p) \end{bmatrix}$$

$$= \begin{bmatrix} 2 + \frac{1-6p+6p^2}{p(1-p)}E\left(\frac{1}{n_i}\right) & 1-2p \\ 1-2p & \mu_n p(1-p) \end{bmatrix} \qquad \square$$