31. Problem 6.8 in Casella and Berger (2001).

32. Problem 6.10 in Casella and Berger (2001) (this is an example of minimal sufficient statistics but not complete.)

33. Problem 6.15 in Casella and Berger (2001).

34. Problem 18 of Keener (2010) Section 3.7.

35. Problem 25 of Keener (2010) Section 3.7.

36. Consider r independent normal populations $N(\mu_i, \sigma_i^2)$. Let $\bar{Y}_i$ and S_i^2 be the sample mean and sample variance from the i-th population. The i-th sample size is n_i. We are interested in estimating a linear combination of the means $L = \sum_{i=1}^r c_i\mu_i$. An estimator of L is

$$\hat{L} = \sum_{i=1}^r c_i\bar{Y}_i.$$

Now consider the following statistic for estimating the variance of $\hat{L}$:

$$s^2\{\hat{L}\} = \sum_{i=1}^r c_i^2 S_i^2/n_i.$$

Explain how to use the Satterthwaite approximation to approximate the distribution of $s^2\{\hat{L}\}$.

37. Suppose that i.i.d. $X_1, \dots, X_n$ have a Beta(α_1, α_2) didtribution. Find the method of moments estimates of (α_1, α_2) based on the first two moments.

Practice
Problems 6.16, 6.21(a)(b), 6.22, and Example 6.2.18 in Casella and Berger (2001).
Example 3.13 and 3.16 of Keener (2010).

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31. $\{X_i\}_{i=1}^{n} \overset{iid}{\sim} f(x-\theta)$

$$f(\underline{x}|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} f(x_i - \theta) = \prod_{i=1}^{n} f(x_{(i)} - \theta)$$

$$= h(\underline{x})\, g(T(\underline{x})|\theta)$$

where $h(\underline{x}) = 1$, $T(\underline{x}) = (X_{(1)}, \dots, X_{(n)})$

$$g(T(\underline{x})|\theta) = \prod_{i=1}^{n} f(x_{(i)} - \theta)$$

By Factorization Theorem: $T(\underline{x})$ is sufficient for θ

And $$\frac{f(\underline{x}|\theta)}{f(\underline{y}|\theta)} = \frac{\prod_{i=1}^{n} f(x_{(i)} - \theta)}{\prod_{i=1}^{n} f(y_{(i)} - \theta)} = c(\underline{x}, \underline{y}) \text{ when } T(\underline{x}) = T(\underline{y})$$

$\therefore T(\underline{X}) = (X_{(1)}, X_{(2)}, \dots, X_{(n)})$ is a

minimal sufficient statistic for θ □

32. By Eg 6.2.15,

$T(\underline{X}) = (X_{(1)}, X_{(n)})$ is a MSS for θ

$$f_{X_{(1)}, X_{(n)}}(r, s) = n(n-1) f(r) f(s) \left[F(s) - F(r)\right]^{n-2}$$

$$= n(n-1)(s-r)^{n-2}, \quad \theta < r \leq s < \theta + 1$$

Define $g(T) = \dfrac{1}{n(n-1)(X_{(n)} - X_{(1)})^{n-2}} - \dfrac{1}{2}$

$$E[g(T)] = \int_{\theta}^{\theta+1} \int_{\theta}^{s} 1 - \tfrac{1}{2} n(n-1)(s-r)^{n-2} \, dr \, ds$$

$$= \int_{\theta}^{\theta+1} \int_{\theta}^{s} 1 \, dr \, ds - \tfrac{1}{2} \int_{\theta}^{\theta+1} \int_{\theta}^{s} n(n-1)(s-r)^{n-2} \, dr \, ds$$

$$= \int_{\theta}^{\theta+1} (s-\theta) \, ds - \frac{1}{2} = \left[\frac{1}{2} s^2 - \theta s\right]_{s=\theta}^{\theta+1} - \frac{1}{2} = 0$$

But $g(T) \neq 0 \Rightarrow T$ is not complete $\square$

33. (a) The parameter space $\Omega = \{(\theta, a\theta^2) : \theta > 0\}$

, where a is a known positive constant, contains only the points of the right part of a parabola.

$\therefore$ Ω does not contain a two-dimensional open set.

(b) $$f(\underline{X} ; \theta) = (2\pi a)^{-\frac{n}{2}} \theta^{-n} \exp\left\{ \frac{1}{2a\theta^2} \sum_{i=1}^{n} (X_i - \theta)^2 \right\}$$

$$= (2\pi a)^{-\frac{n}{2}} \theta^{-n} \exp\left\{ \frac{1}{2a\theta^2} \left[\sum_{i=1}^{n} X_i^2 - 2\theta \sum_{i=1}^{n} X_i + n\theta^2 \right] \right\}$$

By Factorization Theorem:

$\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^2\right)$ is a sufficient statistic for θ

And $\exists$ a function h

s.t. $h\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^2\right) = T = (\bar{X}, S^2)$

$\therefore$ T is also a sufficient statistic for θ.

$$E(\bar{X}^2) = Var(\bar{X}) + E(\bar{X})^2 = \frac{a\theta^2}{n} + \theta^2 = \frac{a+1}{n}\theta^2$$

$$E(S^2) = a\theta^2$$

Define $g(\bar{X}, S^2) = \frac{n}{a+1}\bar{X}^2 - \frac{1}{a}S^2$

And $E_\theta[g(\bar{X}, S^2)] = \frac{n}{a+1}E_\theta[\bar{X}^2] - \frac{1}{a}E_\theta[S^2]$

$= \theta^2 - \theta^2 = 0$

However, $g(\bar{X}, S^2) \neq 0$

$\therefore T = (\bar{X}, S^2)$ is not complete $\square$

34. (a) $\{X_i\}_{i=1}^{n} \overset{\text{indep.}}{\sim} N(t_i\theta, 1)$

$$f(\underline{x};\theta) = (2\pi)^{-\frac{n}{2}} \exp\left\{ \frac{1}{2} \sum_{j=1}^{n} (x_j - t_j\theta)^2 \right\}$$

$$= (2\pi)^{-\frac{n}{2}} \exp\left\{ \frac{1}{2} \sum_{i=1}^{n} x_i^2 - \theta \sum_{j=1}^{n} t_i x_i + \frac{\theta}{2} \sum_{i=1}^{n} t_i^2 \right\}$$

$\because$ the parameter space $\Theta = \{(t_i\theta, 1) \mid t_1, \dots, t_n \text{ are known}\}$

contains an open set in $\mathbb{R}$

$\therefore \sum_{i=1}^{n} t_i X_i$ is complete sufficient for θ

And $\exists$ a function $h(\cdot)$,

s.t. $h\left(\sum_{i=1}^{n} t_i X_i\right) = \dfrac{\sum_{i=1}^{n} t_i X_i}{\sum_{i=1}^{n} t_i^2} = \hat{\theta}$

$\therefore \hat{\theta}$ is complete sufficient

for the family of joint distributions $\square$

(b) $\hat{\theta} \sim N\left(\theta, \frac{1}{\sum t_j^2}\right) \Rightarrow t_i \hat{\theta} \sim N\left(t_i\theta, \frac{t_i^2}{\sum t_j^2}\right)$

$$Cov(X_i, t_i\hat{\theta}) = \frac{t_i}{\sum t_i^2} \sum_{i=1}^{n} Cov(X_i, X_i t_i) = \frac{t_i \sum t_i}{\sum t_i^2}$$

$\Rightarrow (X_i - t_i\hat{\theta}) \sim N\left(0, 1 + \frac{t_i^2}{\sum t_i^2} - 2\frac{t_i \sum t_i}{\sum t_i^2}\right)$ is free for θ

$\therefore (X_i - t_i\hat{\theta})^2$ is also free for θ

$\Rightarrow \sum_{i=1}^{n} (X_i - t_i\hat{\theta})^2$ is free for θ

$\therefore \sum_{i=1}^{n} (X_i - t_i\hat{\theta})^2$ is an ancillary statistic of θ

By Basu's Theorem,

$\hat{\theta}$ and $\sum_{i=1}^{n} (X_i - t_i\hat{\theta})^2$ are independent $\square$

35. (a)

Define $T = \theta X_i \Rightarrow X_i = \frac{1}{\theta} T \Rightarrow J = \frac{dX_i}{dT} = \frac{1}{\theta}$

Then $f_T(t) = f_{X_i}(\frac{1}{\theta}t)\,|J| = \theta e^{-t} \cdot \frac{1}{\theta} = e^{-t}$

$$\therefore f_{\theta X_i}(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases} \sim Exp(\lambda = 1)$$

(b) $f(\underline{x}) = \theta^n e^{-\theta \Sigma X_i} = \theta^n \exp\{-n\theta \bar{X}\}$

$\because$ the parameter space $\Theta = \{\theta : \theta > 0\}$ contains an open set in $\mathbb{R}$

$\therefore \bar{X}$ is a complete sufficient statistic for θ

$\frac{T_{(1)}}{T_{(n)}} = \frac{\theta X_{(1)}}{\theta X_{(n)}} = \frac{X_{(1)}}{X_{(n)}}$, $\because f_T(t)$ is free for θ

$\therefore \frac{T_{(1)}}{T_{(n)}} = \frac{X_{(1)}}{X_{(n)}}$ is an ancillary statistic for θ

By Basu's Theorem $\Rightarrow \bar{X}$ and $\frac{X_{(1)}}{X_{(n)}}$ are independent $\square$

36. Define $U_i = \frac{(n_i-1)S_i^2}{\sigma_i^2} \sim \chi^2_{(\nu_i)}$ where $\nu_i = n_i - 1$

Then $S^2\{\hat{L}\} = \sum_{i=1}^{r} \frac{c_i^2 S_i^2}{n_i} = \sum_{i=1}^{r} a_i U_i$, where $a_i = \frac{c_i^2 \sigma_i^2}{(n_i-1)n_i}$

By Satterthwaite approximation, we have to find $\hat{\nu}$

s.t. $S^2\{\hat{L}\} \sim \frac{E(S^2\{\hat{L}\})}{\hat{\nu}} \chi^2_{(\hat{\nu})}$

$\therefore Var(S^2\{\hat{L}\}) = \frac{E(S^2\{\hat{L}\})^2}{\hat{\nu}^2}(2\hat{\nu}) = \frac{2E(S^2\{\hat{L}\})^2}{\hat{\nu}}$

And $S^2\{\hat{L}\} = \sum_{i=1}^{r} a_i U_i$

$\therefore Var(S^2\{\hat{L}\}) = \sum_{i=1}^{r} a_i^2 (2\nu_i) = 2\sum_{i=1}^{r} \frac{(a_i\nu_i)^2}{\nu_i} = 2\sum_{i=1}^{r} \frac{(a_i E(U_i))^2}{\nu_i}$

By method of moment estimator:

$$E(U_i) = U_j \ , \ E(s^2\{\hat{L}\}) = \sum_{i=1}^{r} a_i E(U_j) = s^2\{L\}$$

Then $$Var(s^2\{\hat{L}\}) = \frac{2(s^2\{\hat{L}\})^2}{\hat{\nu}} = 2\sum_{i=1}^{r} \frac{(a_i U_j)^2}{n_j - 1}$$

$$\Rightarrow \hat{\nu} = \frac{(s^2\{\hat{L}\})^2}{\sum_{i=1}^{r}(a_i U_i)^2/(n_j-1)}$$

$$= \frac{\left(\sum_{i=1}^{r} c_j^2 s_j^2/n_j\right)^2}{\sum_{j=1}^{r}(c_j^2 s_i^2/n_i)^2/(n_i-1)} \quad \square$$

37. $\{X_i\}_{i=1}^{n} \overset{iid}{\sim} Beta(\alpha_1, \alpha_2)$

$$\begin{cases} E(X_i) = \dfrac{\alpha_1}{\alpha_1+\alpha_2} \overset{set}{=\!=} \bar{X} \ni \alpha_2 = \dfrac{\alpha_1(1-\bar{X})}{\bar{X}} \\[2em] Var(X_i) = \dfrac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2(\alpha_1+\alpha_2+1)} \overset{set}{=\!=} \dfrac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2 = S^2 \end{cases}$$

$$\ni S^2 = \frac{\alpha_1^2(1-\bar{X})/\bar{X}}{(\alpha_1/\bar{X})^2(\alpha_1/\bar{X}+1)} = \frac{\alpha_1^2\bar{X}^2(1-\bar{X})}{\alpha_1^3+\alpha_1^2\bar{X}} = \frac{\bar{X}^2(1-\bar{X})}{\alpha_1+\bar{X}}$$

$\therefore$ the method of moments estimates:

$$\begin{cases} \hat{\alpha}_1 = \bar{X}\left[\dfrac{\bar{X}(1-\bar{X})}{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2} - 1\right] \\[2em] \hat{\alpha}_2 = \dfrac{\hat{\alpha}_1(1-\bar{X})}{\bar{X}} = (1-\bar{X})\left[\dfrac{\bar{X}(1-\bar{X})}{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2} - 1\right] \end{cases} \square$$