

# 18-799: Applied Computer Vision

Spring 2026

Image Formation

# Objectives

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In today's class we will learn

- **Homogeneous Coordinates and transformation**
- **Geometric primitive**
- Geometric Image formation
- Photometric image formation

# Homogeneous Coordinates

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- In computer vision, image formation is a chain of transformations:
- World  $\rightarrow$  Camera  $\rightarrow$  Image
- If each step is linear:

$$\mathbf{x}' = A\mathbf{x},$$

$$\mathbf{x}'' = B\mathbf{x}'$$

– then the whole pipeline collapses into:

$$\mathbf{x}'' = (BA)\mathbf{x}.$$

# Homogeneous Coordinates

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- Let's consider the function  $f$ , where:

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 1 \\ y \end{bmatrix}$$

- Can we represent  $f$  with a matrix?
  - That's, is there a matrix  $A$  such that:

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 1 \\ y \end{bmatrix}$$

# Homogeneous Coordinates

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- Points in Euclidean Coordinates
  - In standard 2D space, a point is represented as a vector

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

- Points in Homogeneous Coordinates
  - The same point is represented by the augmented vector

$$\bar{\mathbf{x}} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

- More generally, the Euclidean point  $\mathbf{x}$  corresponds to the entire equivalence class

$$\tilde{\mathbf{x}} \sim \lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix}, \lambda \neq 0$$

- All such homogeneous vectors represent the same geometric point after normalization.

# Homogeneous Coordinates

- Euclidean point  $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$  corresponds to the entire equivalence class in homogeneous coordinates:

$$\tilde{\mathbf{x}} \sim \lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix}, \lambda \neq 0$$

- Since  $\lambda$  can be anything ( $\lambda \neq 0$ ), then  $\tilde{\mathbf{x}}$  could be re-written  $\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix}$
- To convert a homogeneous vector  $\tilde{\mathbf{x}}$  back to inhomogeneous form, we divide by the last coordinate  $\tilde{w}$ :

$$\bar{\mathbf{x}} = \frac{1}{\tilde{w}} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} \tilde{x}/\tilde{w} \\ \tilde{y}/\tilde{w} \\ 1 \end{pmatrix}, \tilde{w} \neq 0.$$

- The corresponding inhomogeneous point is obtained by dropping the last coordinate:

$$\mathbf{x} = \begin{pmatrix} \tilde{x}/\tilde{w} \\ \tilde{y}/\tilde{w} \end{pmatrix}.$$

# Homogeneous Coordinates

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- Let's re-consider the function  $f$ , where:

$$f\left(\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}\right) = \begin{bmatrix} x + 1 \\ y \\ 1 \end{bmatrix}$$

- Can we represent  $f$  with a matrix?
  - That's, is there a matrix  $A$  such that:

$$A \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + 1 \\ y \\ 1 \end{bmatrix}$$
$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Geometric primitive

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- Geometric primitives are the basic building blocks used to describe 3D shapes
- Examples: Points, lines, planes



# 2D Points

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- 2D points can be written in inhomogeneous coordinates as  $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$
- or in homogeneous coordinates as  $\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{pmatrix} \in \mathbb{P}^2$  where  $\mathbb{P}^2 = \mathbb{R}^3 \setminus \{(0,0,0)\}$  is called projective space.
- Homogeneous vectors that differ only by scale are considered equivalent and define an equivalence class.

# 2D Points

- An inhomogeneous vector  $\mathbf{x}$  is converted to a homogeneous vector  $\tilde{\mathbf{x}}$  as follows

$$\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \bar{\mathbf{x}}$$

with augmented vector  $\bar{\mathbf{x}}$ .

- To convert in the opposite direction we divide by  $\tilde{w}$  :

$$\bar{\mathbf{x}} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \frac{1}{\tilde{w}} \tilde{\mathbf{x}} = \frac{1}{\tilde{w}} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} \tilde{x}/\tilde{w} \\ \tilde{y}/\tilde{w} \\ 1 \end{pmatrix}$$

- Homogeneous points whose last element is  $\tilde{w} = 0$  are called ideal points or points at infinity. These points can't be represented with inhomogeneous coordinates

# 2D Lines

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- 2D lines can also be expressed using homogeneous coordinates

$$\tilde{\mathbf{l}} = (a, b, c)^T : \{\bar{\mathbf{x}} \mid \tilde{\mathbf{l}}^T \bar{\mathbf{x}} = 0\} \Leftrightarrow \{x, y \mid ax + by + c = 0\}$$

- An exception is the line at infinity  $\tilde{\mathbf{l}}_\infty = (0, 0, 1)^T$  which passes through all ideal points.

# Cross Product

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- Cross product of two vector:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- Cross product expressed as the product of a skew-symmetric matrix and a vector

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_{\times} \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

- **See** Levi-Civita symbol for more info.

# Cross Product

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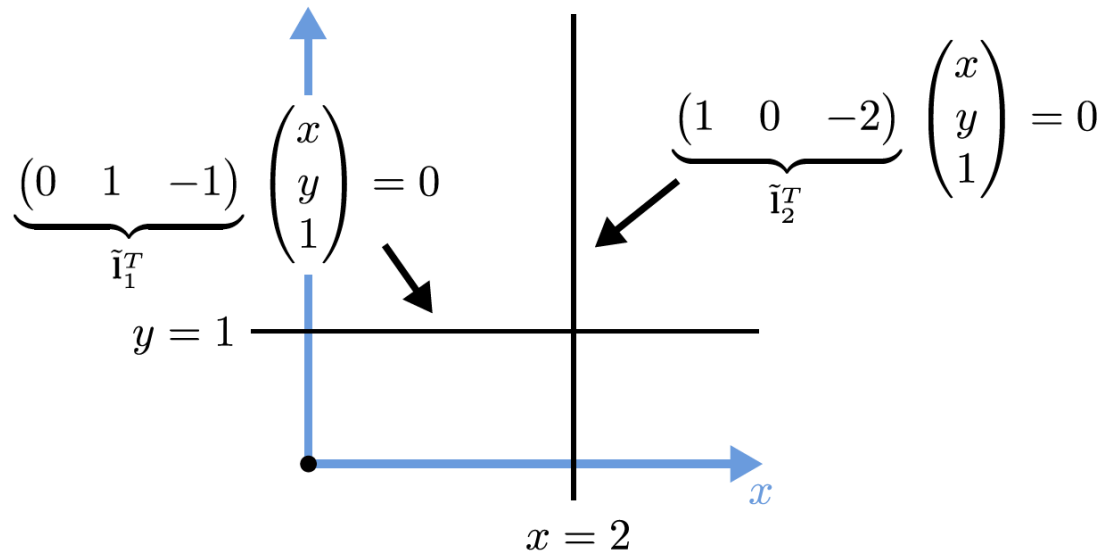
- In homogeneous coordinates, the intersection of two lines is given by:

$$\tilde{\mathbf{x}} = \tilde{\mathbf{l}}_1 \times \tilde{\mathbf{l}}_2$$

- Similarly, the line joining two points can be compactly written as:

$$\tilde{\mathbf{l}} = \bar{\mathbf{x}}_1 \times \bar{\mathbf{x}}_2$$

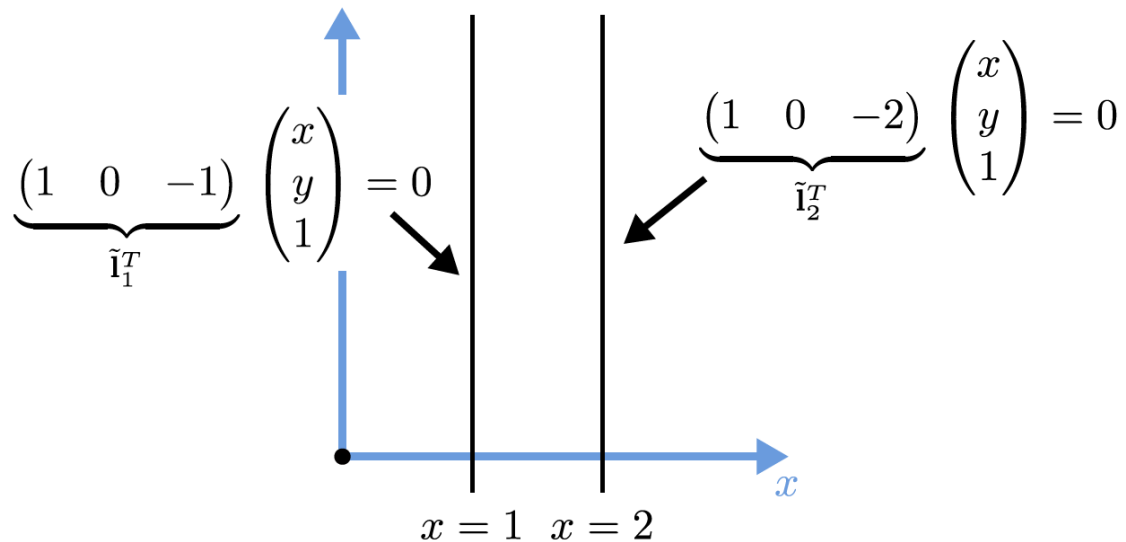
# 2D Line: Example



$$[\mathbf{x}]_{\times} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

$$\tilde{\mathbf{l}}_1 \times \tilde{\mathbf{l}}_2 = [\tilde{\mathbf{l}}_1]_{\times} \tilde{\mathbf{l}}_2 = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

# 2D Line: Example



$$[\mathbf{x}]_{\times} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

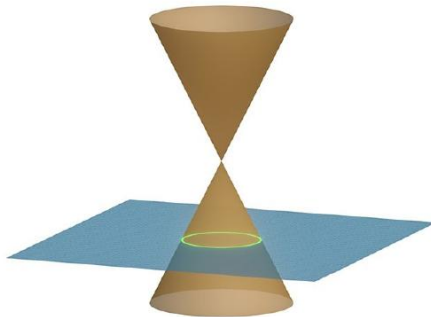
$$\tilde{\mathbf{l}}_1 \times \tilde{\mathbf{l}}_2 = [\tilde{\mathbf{l}}_1]_{\times} \tilde{\mathbf{l}}_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\underbrace{(0 \ 0 \ 1)}_{\tilde{\mathbf{l}}_{\infty}}^T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

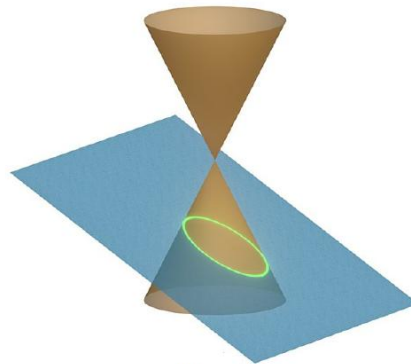
# 2D Conics

- More complex algebraic objects can be represented using polynomial homogeneous equations.
- For example, conic sections (arising as the intersection of a plane and a 3D cone) can be written using quadric equations:

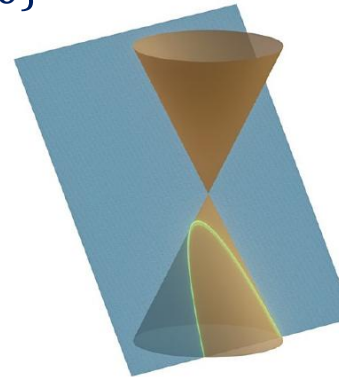
$$\{\bar{\mathbf{x}} \mid \bar{\mathbf{x}}^T \mathbf{Q} \bar{\mathbf{x}} = 0\}$$



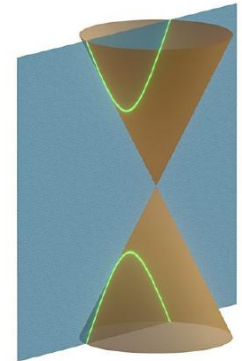
Circle



Ellipse



Parabola



Hyperbola



# 3D Quadrics

The 3D analog of 2D conics is a quadric surface:

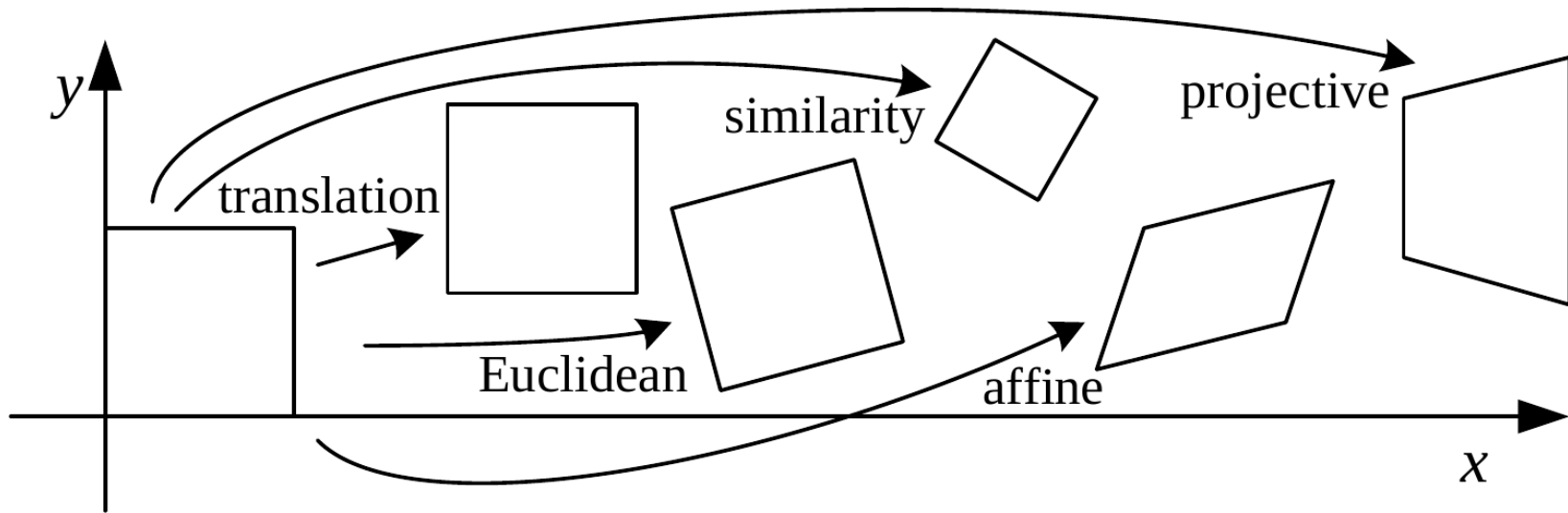
$$\{\bar{\mathbf{x}} \mid \bar{\mathbf{x}}^T \mathbf{Q} \bar{\mathbf{x}} = 0\}$$



**Superquadrics** (generalization of quadrics) for shape abstraction and compression.

Paschalidou, Ulusoy and Geiger: Superquadrics Revisited: Learning 3D Shape Parsing beyond Cuboids. CVPR, 2019

# Translation

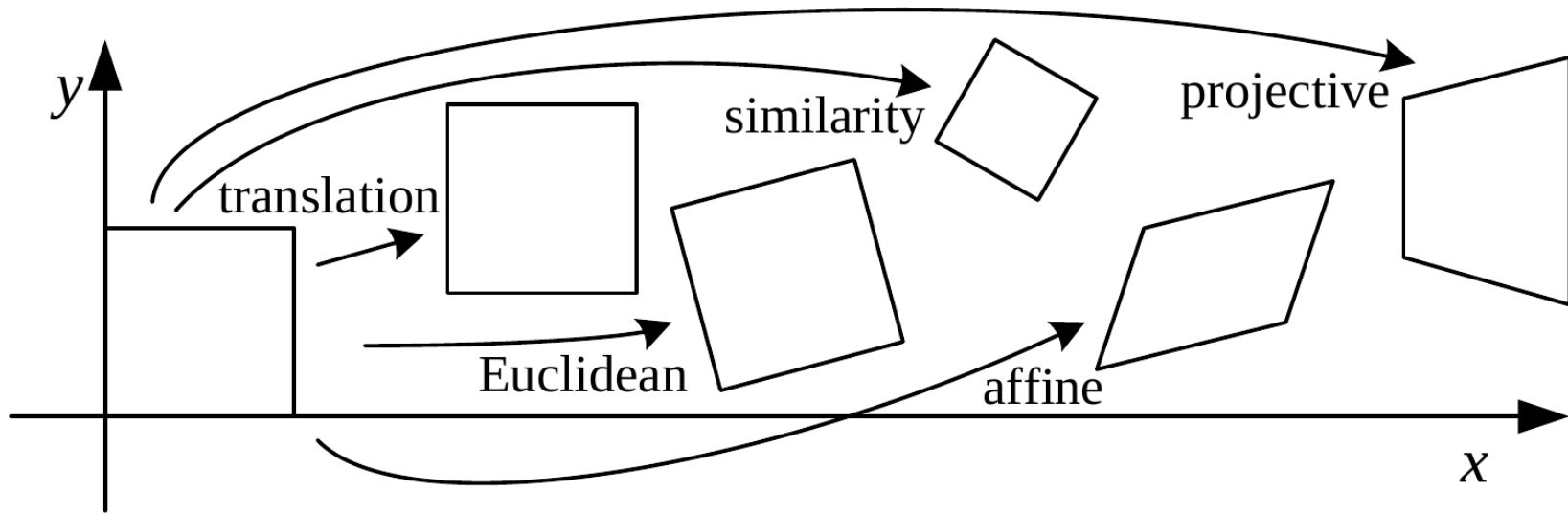


- [2D Translation of the Input, 2 DoF]

$$\mathbf{x}' = \mathbf{x} + \mathbf{t} \Leftrightarrow \bar{\mathbf{x}}' = \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \bar{\mathbf{x}} \Rightarrow \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- Using homogeneous representations allows to chain/invert transformations
- Augmented vectors  $\bar{\mathbf{x}}$  can always be replaced by general homogeneous ones  $\tilde{\mathbf{x}}$

# Affine

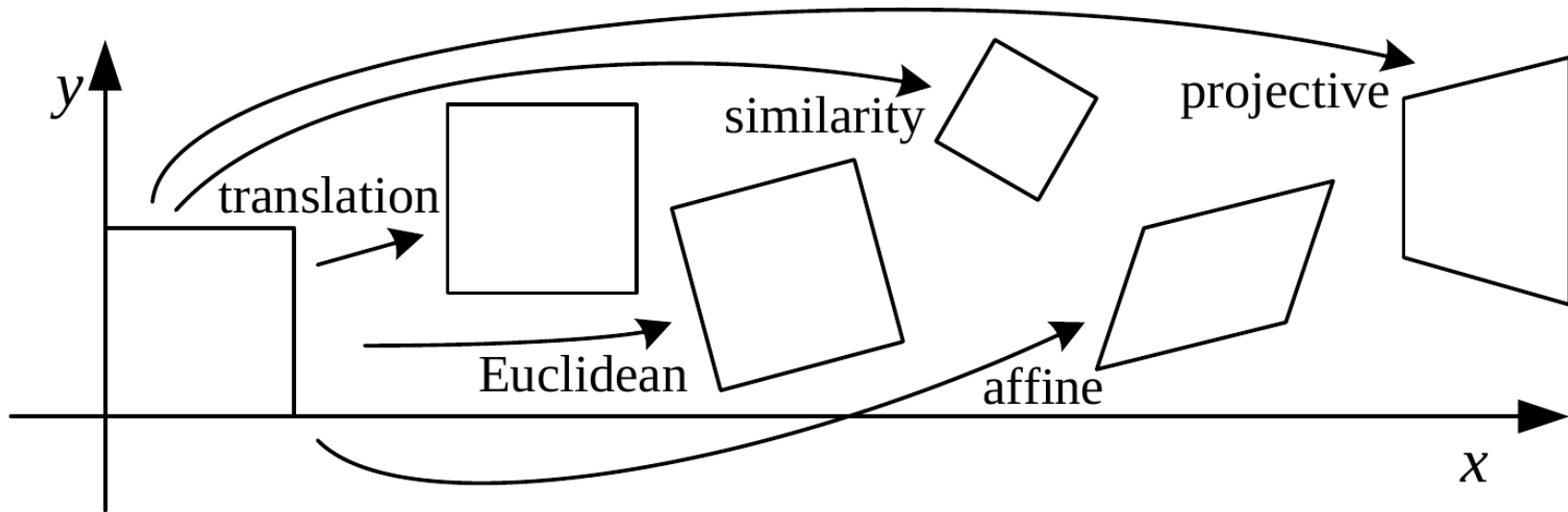


- Equivalent to 2D Linear Transformation, 6 DoF

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{t} \Leftrightarrow \bar{\mathbf{x}}' = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \bar{\mathbf{x}}$$

- $\mathbf{A} \in \mathbb{R}^{2 \times 2}$  is an arbitrary  $2 \times 2$  matrix
- Parallel lines remain parallel under affine transformations

# Projective

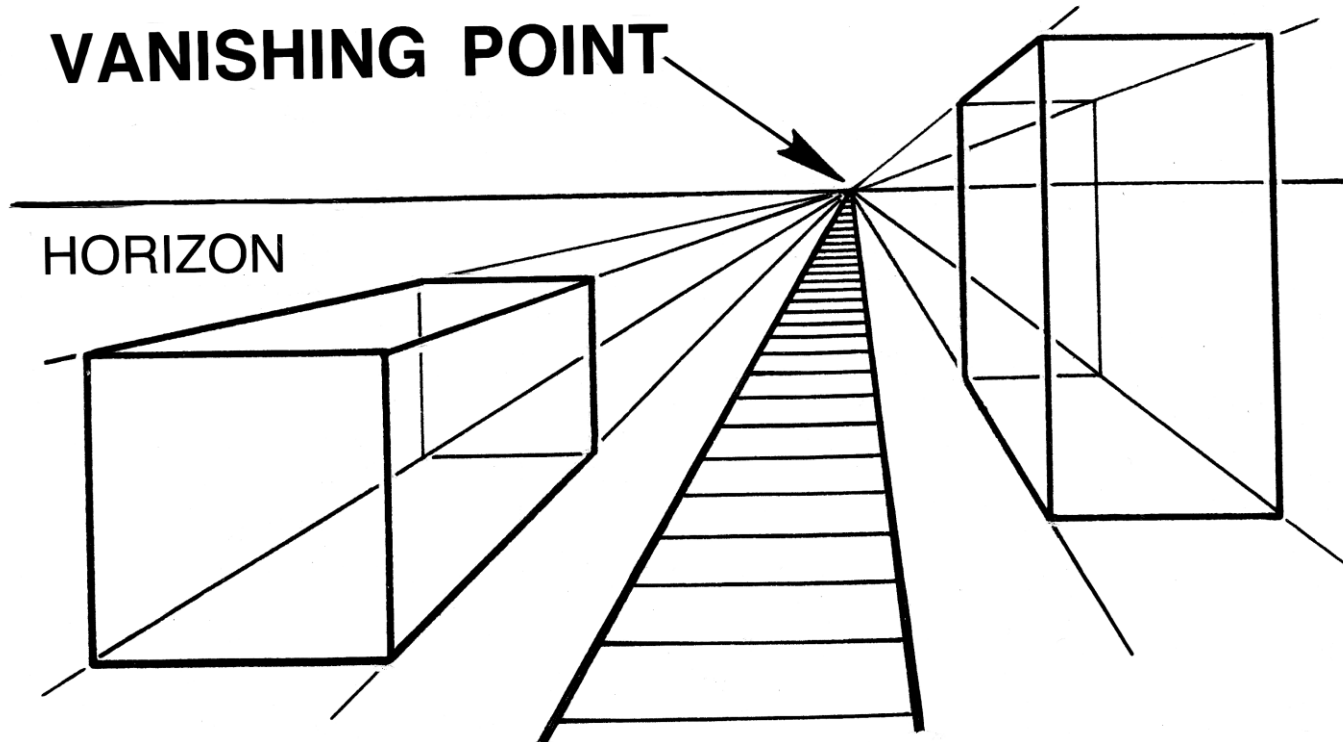


- Homography, 8 DoF

$$\tilde{\mathbf{x}}' = \tilde{\mathbf{H}}\tilde{\mathbf{x}} \quad \left( \bar{\mathbf{x}} = \frac{1}{\tilde{w}}\tilde{\mathbf{x}} \right)$$

- $\tilde{\mathbf{H}} \in \mathbb{R}^{3 \times 3}$  is an arbitrary homogeneous  $3 \times 3$  matrix (specified up to scale)
- Projective transformations preserve straight lines

# Projective



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# Projective

- A homography is defined in homogeneous coordinates as

$$\tilde{\mathbf{x}}' = H\tilde{\mathbf{x}}, H \in \mathbb{R}^{3 \times 3}.$$

- Write the point  $(x, y)$  in homogeneous form:

$$\tilde{\mathbf{x}} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Let

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}.$$

- We now have:

$$\begin{bmatrix} \tilde{x}' \\ \tilde{y}' \\ \tilde{w}' \end{bmatrix} = \begin{bmatrix} h_{11}x + h_{12}y + h_{13} \\ h_{21}x + h_{22}y + h_{23} \\ h_{31}x + h_{32}y + h_{33} \end{bmatrix}.$$

# Projective

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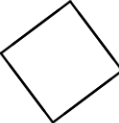
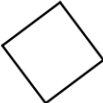

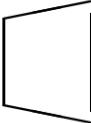
- Homogeneous coordinates represent points up to scale.  
To recover Euclidean coordinates, we divide by the last component:

$$x' = \frac{\tilde{x}'}{\tilde{w}'}, y' = \frac{\tilde{y}'}{\tilde{w}'}$$

- Substituting gives exactly:


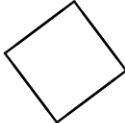
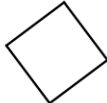

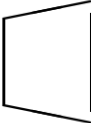
$$\begin{aligned} x' &= \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \\ y' &= \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}} \end{aligned}$$

# 2D Transformation

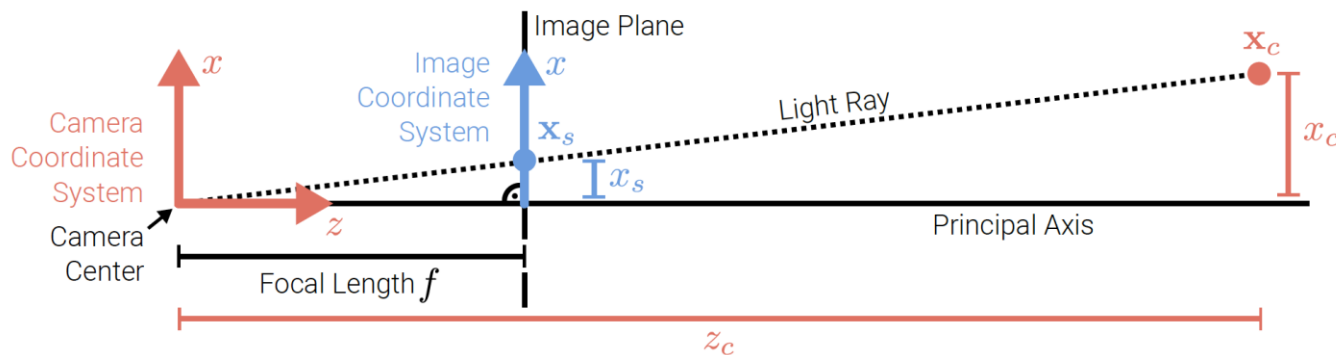
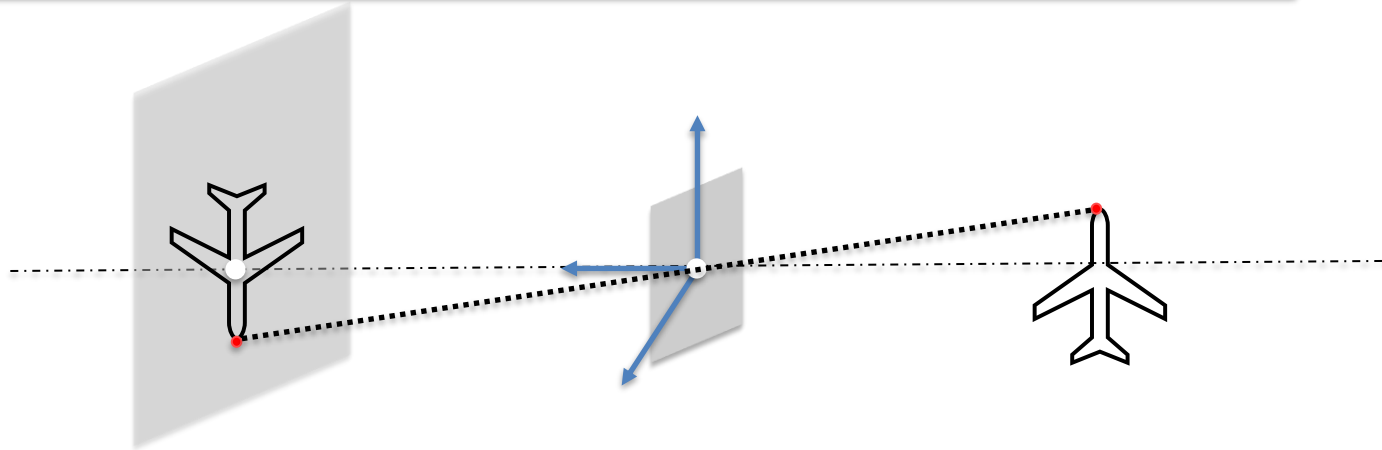
Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix}_{2 \times 3}$	2	orientation	
rigid (Euclidean)	$\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}_{2 \times 3}$	3	lengths	
similarity	$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \end{bmatrix}_{2 \times 3}$	4	angles	
affine	$\begin{bmatrix} \mathbf{A} \end{bmatrix}_{2 \times 3}$	6	parallelism	
projective	$\begin{bmatrix} \tilde{\mathbf{H}} \end{bmatrix}_{3 \times 3}$	8	straight lines	



# 3D Transformation

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix}_{3 \times 4}$	3	orientation	
rigid (Euclidean)	$\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}_{3 \times 4}$	6	lengths	
similarity	$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \end{bmatrix}_{3 \times 4}$	7	angles	
affine	$\begin{bmatrix} \mathbf{A} \end{bmatrix}_{3 \times 4}$	12	parallelism	
projective	$\begin{bmatrix} \tilde{\mathbf{H}} \end{bmatrix}_{4 \times 4}$	15	straight lines	

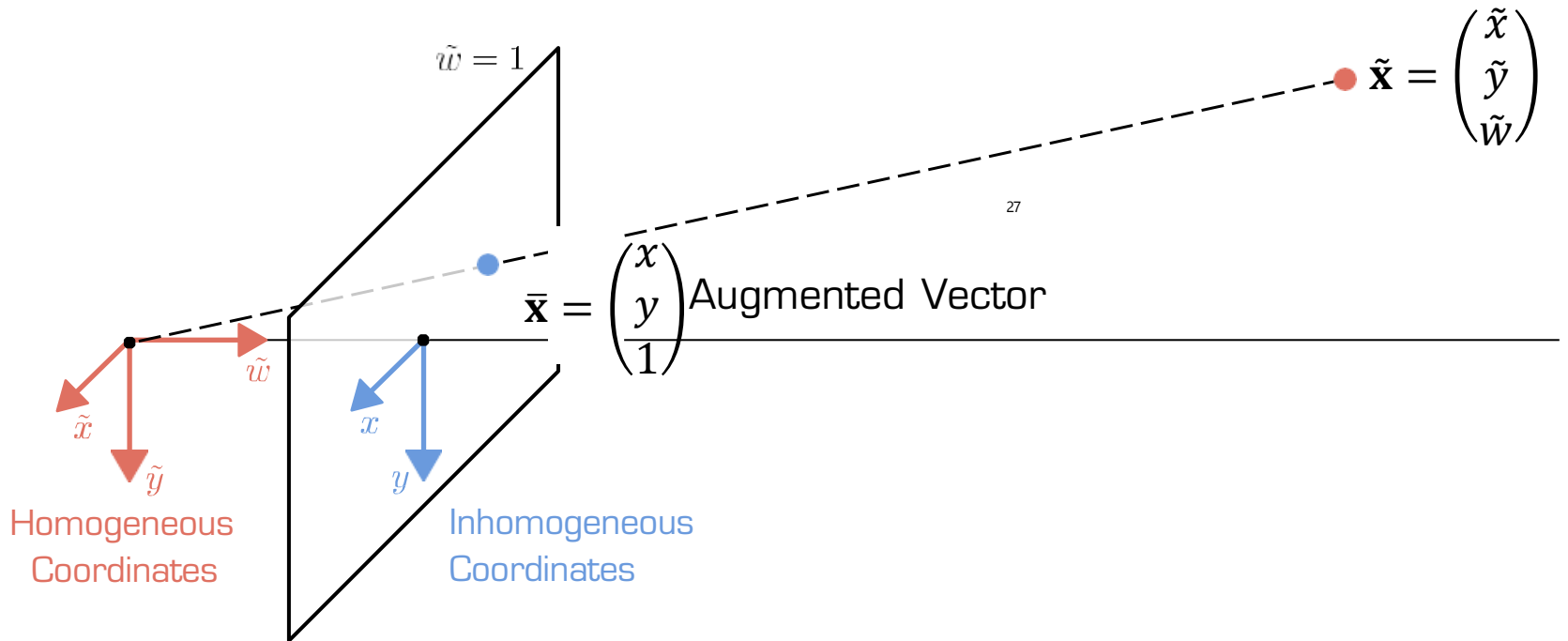
# Geometric Image Formation



$$\frac{x_s}{f} = \frac{x_c}{z_c}$$

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## Homogeneous Vector



# Perspective Projection

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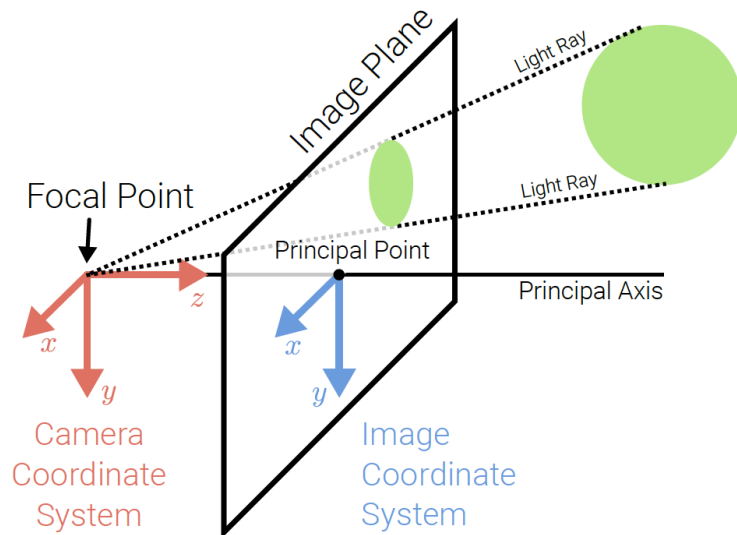
- In perspective projection, we have:

$$\begin{pmatrix} x_s \\ y_s \end{pmatrix} = \begin{pmatrix} f x_c / z_c \\ f y_c / z_c \end{pmatrix} \Leftrightarrow \tilde{\mathbf{x}}_s = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \bar{\mathbf{x}}_c, \text{ where } \bar{\mathbf{x}}_c = \begin{pmatrix} x_c \\ y_c \\ z_c \\ 1 \end{pmatrix}$$

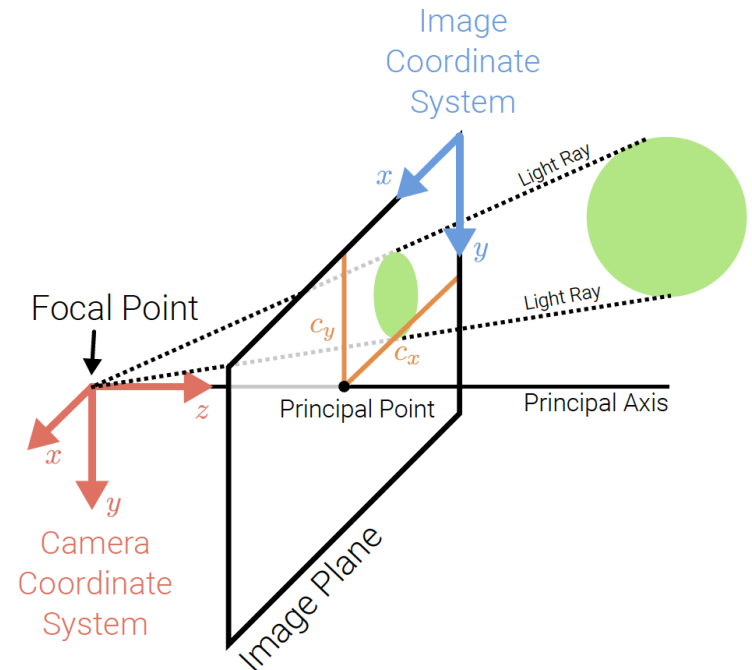
- Note that this projection is linear when using homogeneous coordinates.
- After the projection it is not possible to recover the distance of the 3D point from the image.
- Calibration matrix

# Perspective Projection

Without Principal Point Offset



With Principal Point Offset



- ▶ To ensure positive pixel coordinates, a **principal point offset  $\mathbf{c}$**  is usually added
- ▶ This moves the image coordinate system to the corner of the image plane

# Perspective Projection

- The complete perspective projection model is given by:

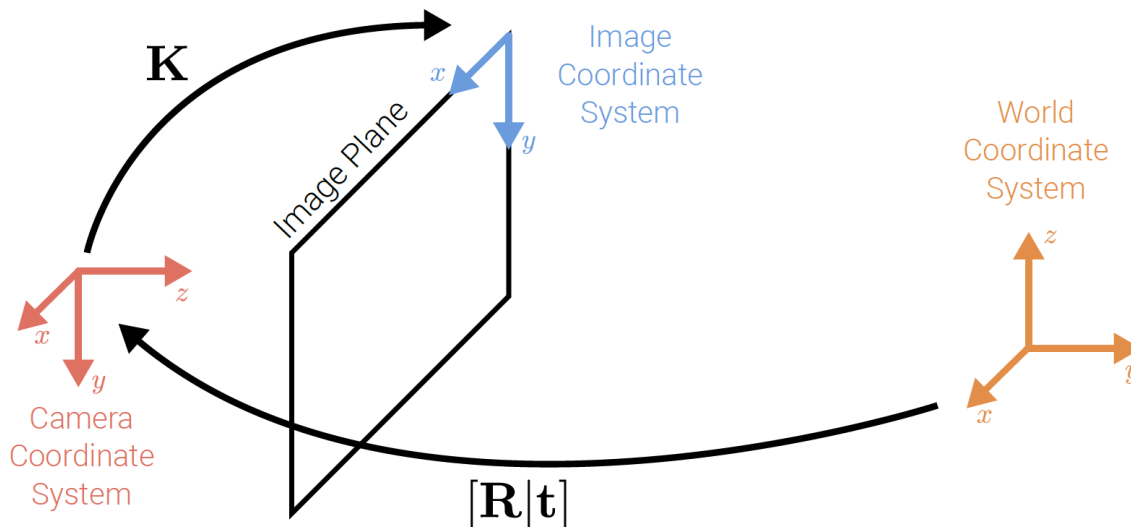
$$\begin{pmatrix} x_s \\ y_s \end{pmatrix} = \begin{pmatrix} f_x x_c / z_c + s y_c / z_c + c_x \\ f_y y_c / z_c + c_y \end{pmatrix} \Leftrightarrow \tilde{\mathbf{x}}_s = \begin{bmatrix} f_x & s & c_x & 0 \\ \mathbf{0} & f_y & c_y & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 \end{bmatrix} \bar{\mathbf{x}}_c$$

- The left  $3 \times 3$  submatrix of the projection matrix is called calibration matrix  $\mathbf{K}$
- The parameters of  $\mathbf{K}$  are called camera intrinsics (as opposed to extrinsic pose)
- Here,  $f_x$  and  $f_y$  are independent, allowing for different pixel aspect ratios
- The skew  $s$  arises due to the sensor not mounted perpendicular to the optical axis
- In practice, we often set  $f_x = f_y$  and  $s = 0$ , but model  $\mathbf{c} = (c_x, c_y)^\top$

# Chaining Transformation

- Let  $\mathbf{K}$  be the calibration matrix (intrinsics) and  $[\mathbf{R} \mid \mathbf{t}]$  the camera pose (extrinsics).
- We chain both transformations to project a point in world coordinates to the image:

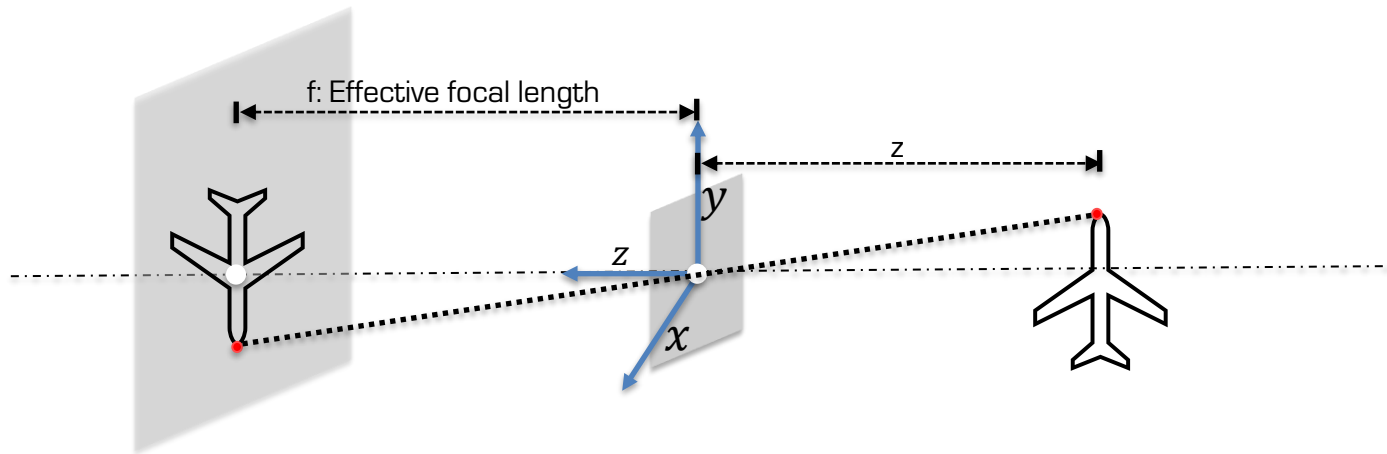
$$\tilde{\mathbf{x}}_s = [\mathbf{K} \quad \mathbf{0}] \bar{\mathbf{x}}_c = [\mathbf{K} \quad \mathbf{0}] \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \bar{\mathbf{x}}_w = \mathbf{K}[\mathbf{R} \quad \mathbf{t}] \bar{\mathbf{x}}_w = \mathbf{P} \bar{\mathbf{x}}_w$$



# Perspective Projection

Image Magnification:  $m = \frac{f}{z}$

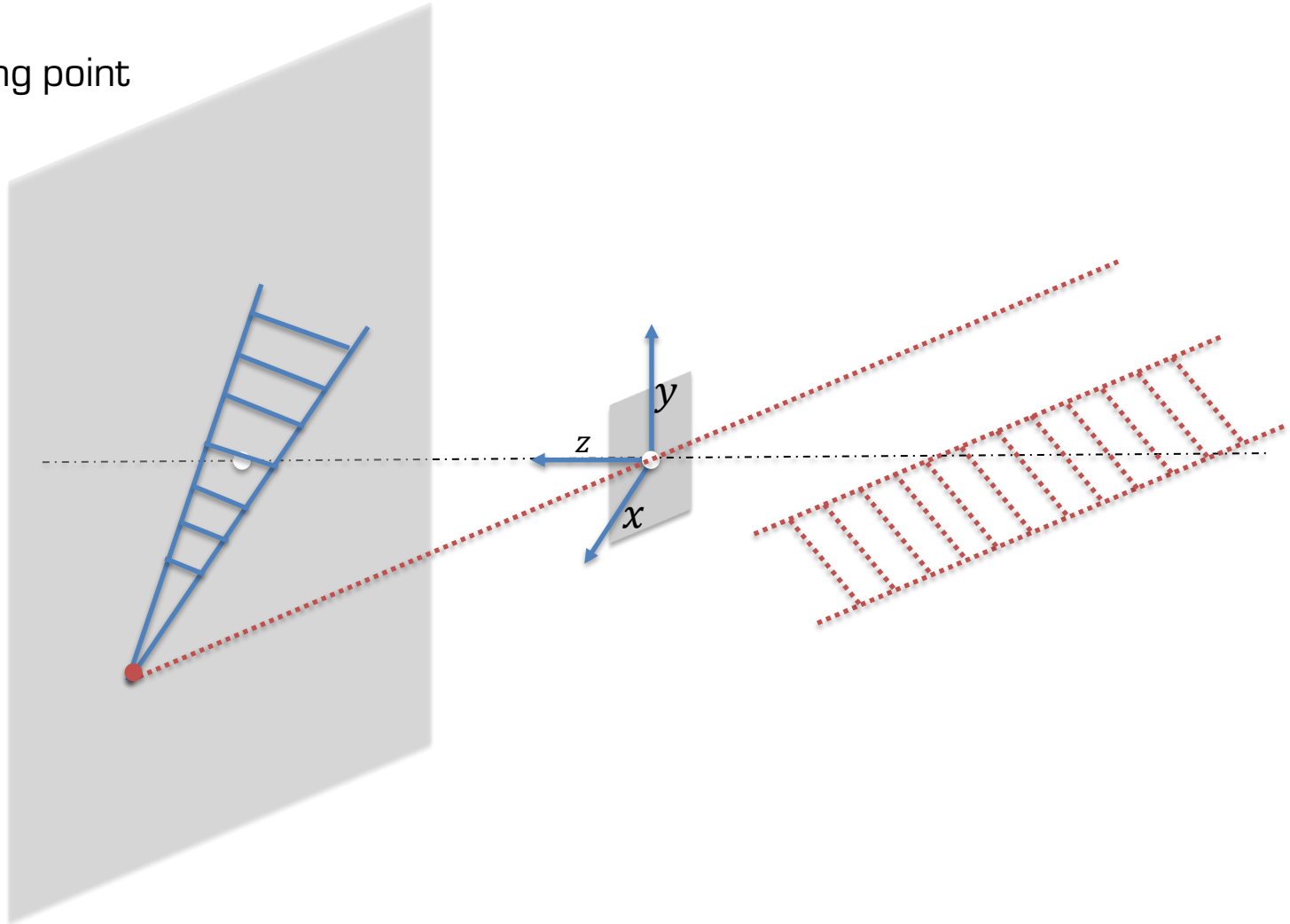
- Image size inversely proportional to depth





# Perspective Projection

Vanishing point



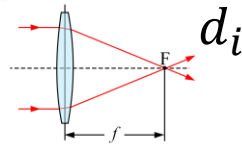
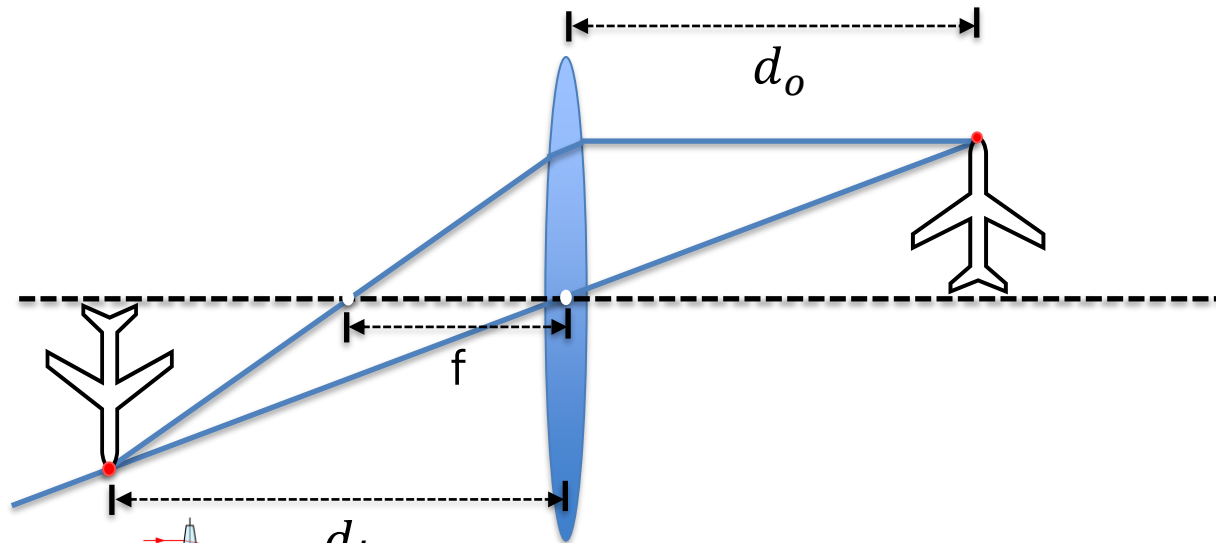
# Perspective Projection

Vanishing point:  $(x_{vp}, y_{vp}) = \left(f \frac{l_x}{l_z}, f \frac{l_y}{l_z}\right)$

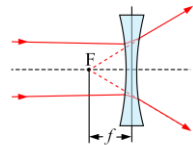


[https://commons.wikimedia.org/wiki/File:1\\_point\\_perspective.svg](https://commons.wikimedia.org/wiki/File:1_point_perspective.svg)

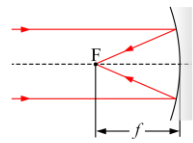
# Lens



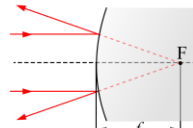
Focal length ( $f$ ): It determines the lens' bending power



This Lens' law:  $\frac{1}{d_i} + \frac{1}{d_o} = \frac{1}{f}$



Magnification ( $m$ ):  $= \frac{h_i}{h_o} = \frac{d_i}{d_o}$



# Lens

**Aperture:** Area of the lens receiving light

$f/1.4$



$f/2.0$



$f/2.8$



$f/4.0$



$f/5.6$



$f/8.0$



# Lens

**Aperture:** Area of the lens receiving light

**f-number:** a measure of the light-gathering ability of a camera lens  
it equals focal length divided by the aperture.

- Aperture:  $= \frac{f}{f\text{-Number}}$
- where  $N$  is called the f-Number of lens.

$f/1.4$



$f/2.0$



$f/2.8$



$f/4.0$



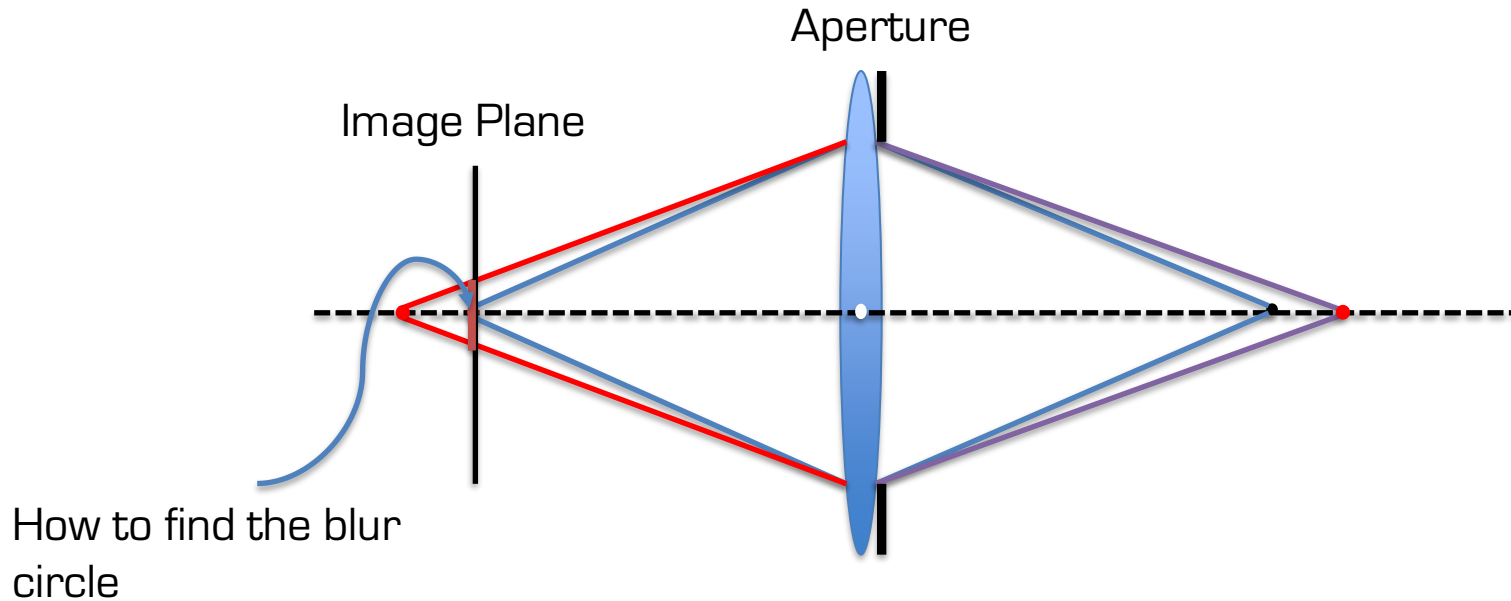
$f/5.6$



$f/8.0$

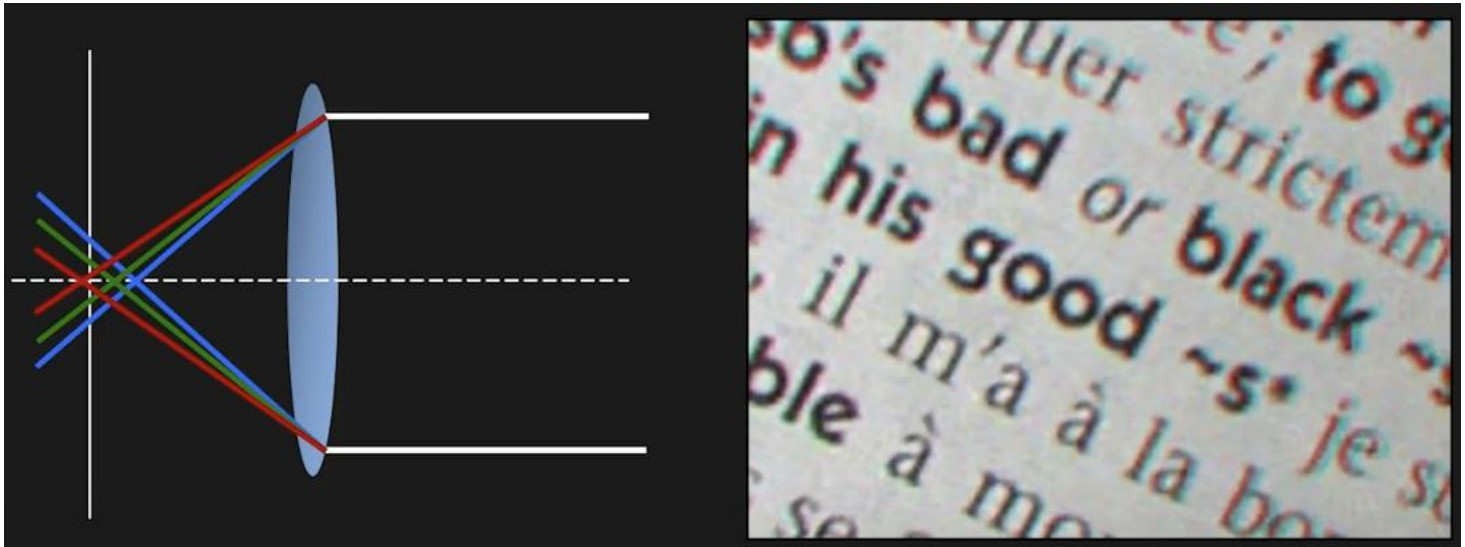


# Lens Defocus



# Lens Distortion

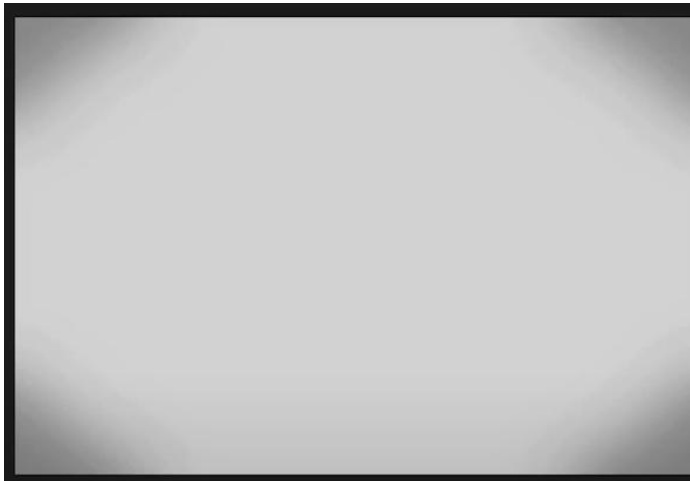
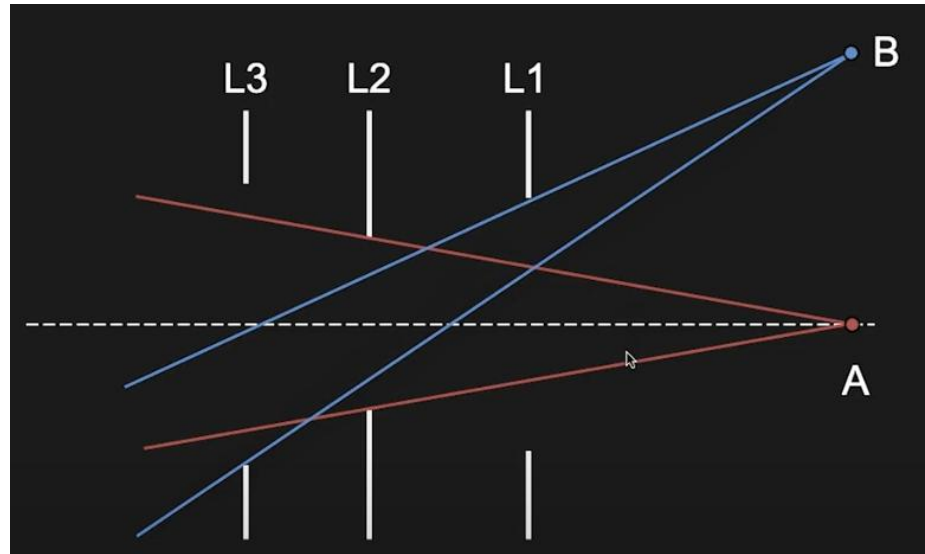
- Chromatic Aberration



Refractive index (and hence focal length) of lens is different for different wavelengths.

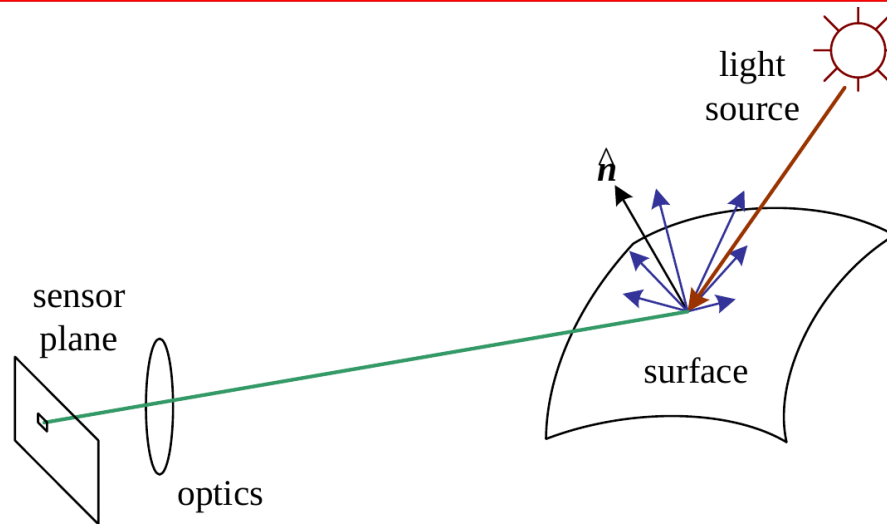
# Lens Distortion

- Vignetting





# Photometric Image Formation



- How an image is formed in terms of pixel intensities and colors.
- Light is emitted by one or more light sources and reflected or refracted (once or multiple times) at surfaces of objects (or media) in the scene

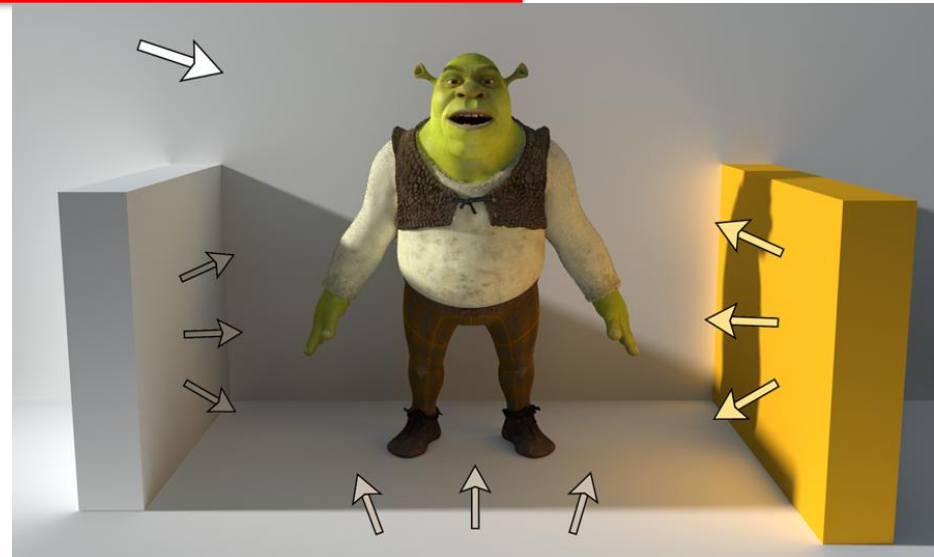
# Rendering Equation

- Let  $\mathbf{p} \in \mathbb{R}^3$  denote a 3D surface point,  $\mathbf{v} \in \mathbb{R}^3$  the viewing direction and  $\mathbf{s} \in \mathbb{R}^3$  the incoming light direction.
- The rendering equation describes how much of the light  $L_{\text{in}}$  with wavelength  $\lambda$  arriving at  $\mathbf{p}$  is reflected into the viewing direction  $\mathbf{v}$  :

$$L_{\text{out}}(\mathbf{p}, \mathbf{v}, \lambda) = L_{\text{emit}}(\mathbf{p}, \mathbf{v}, \lambda) + \int_{\Omega} \text{BRDF}(\mathbf{p}, \mathbf{s}, \mathbf{v}, \lambda) \cdot L_{\text{in}}(\mathbf{p}, \mathbf{s}, \lambda) \cdot (-\mathbf{n}^T \mathbf{s}) d\mathbf{s}$$

- $\Omega$  is the unit hemisphere at normal  $\mathbf{n}$
- The bidirectional reflectance distribution function  $\text{BRDF}(\mathbf{p}, \mathbf{s}, \mathbf{v}, \lambda)$  defines how light is reflected at an opaque surface.
- $L_{\text{emit}} > 0$  only for light emitting surfaces

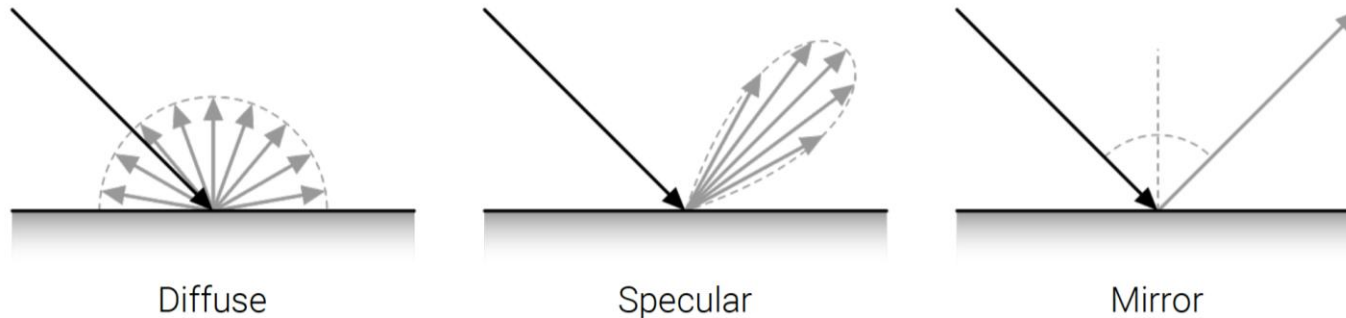
# Global Illumination



- Modeling one light bounce is insufficient for rendering complex scenes
- Light sources can be shadowed by occluders and rays can bounce multiple times
- Global illumination techniques also take indirect illumination into account

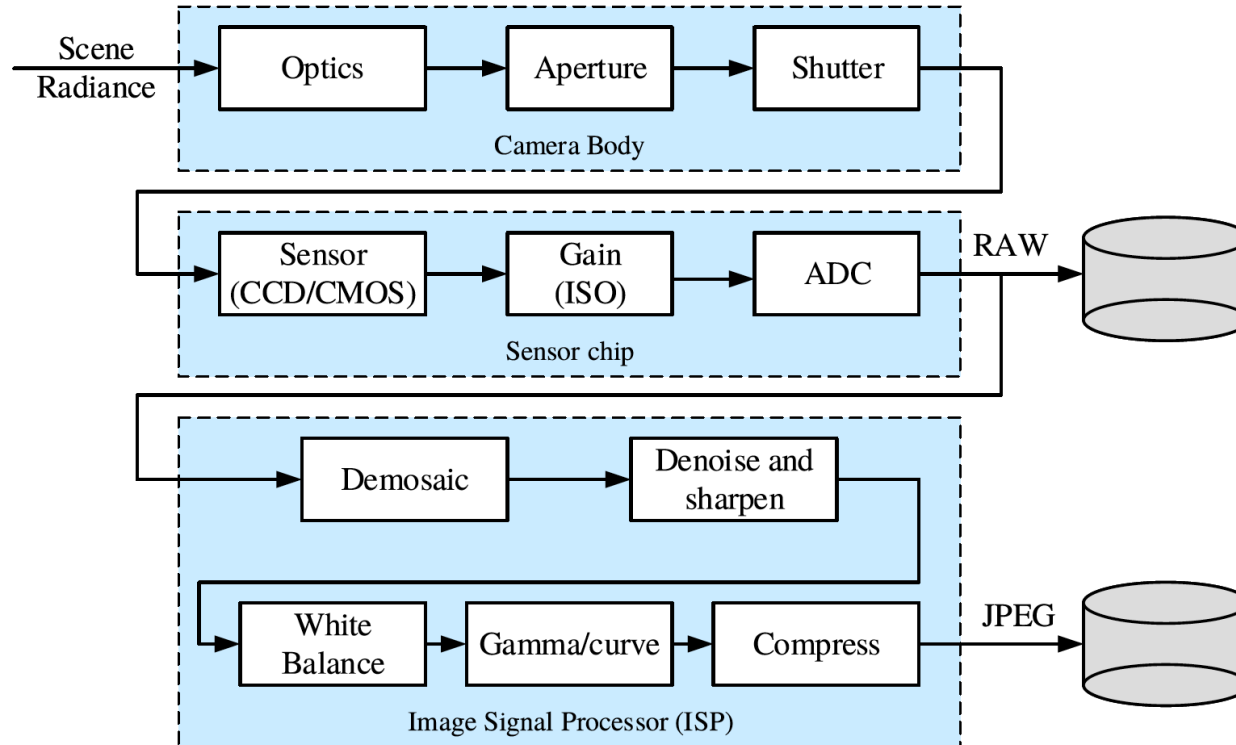
# Diffuse and Specular Reflection

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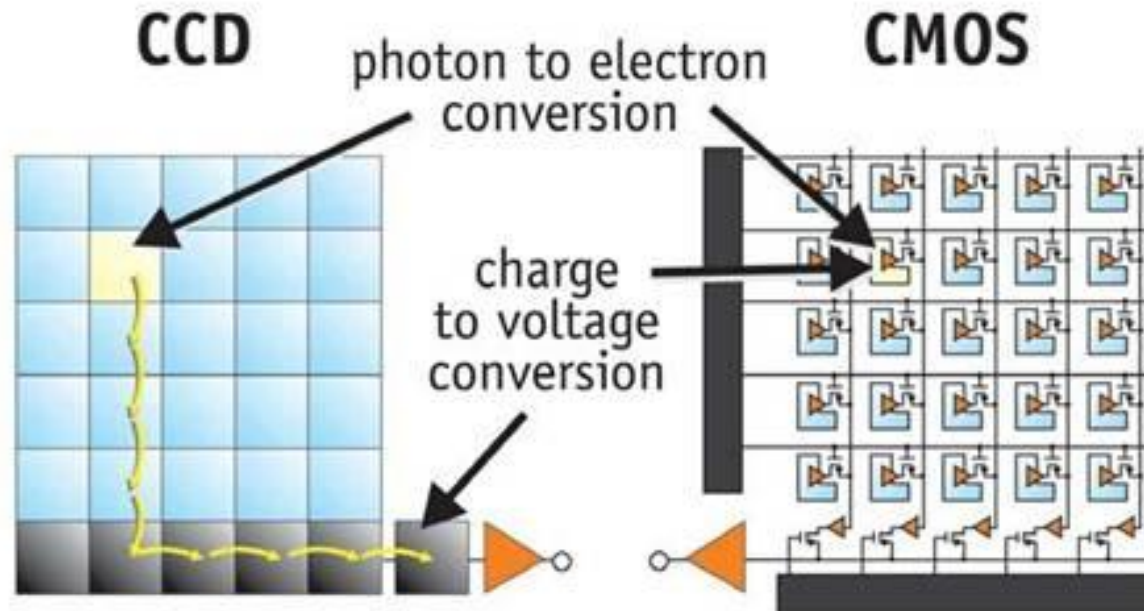
- ▶ Typical BRDFs have a **diffuse** and a **specular** component
- ▶ The diffuse (=constant) component scatters light uniformly in all directions
- ▶ This leads to shading, i.e., smooth variation of intensity wrt. surface normal
- ▶ The specular component depends strongly on the outgoing light direction

# Image Sensor



- Physical light transport in the camera lens/body
- Photon measurement and conversion on the sensor chip
- Image signal processing (ISP) and image compression

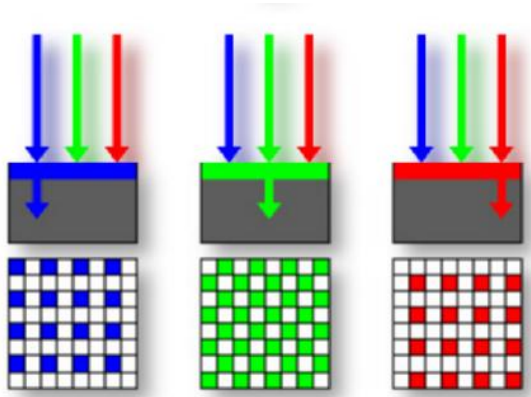
# Image Sensor



- | CCDs move charge from pixel to pixel and convert it to voltage at the output node
- | CMOS images convert charge to voltage inside each pixel and are standard

# Image Sensor

- **Color Filter:**



G	R	G	R
B	G	B	G
G	R	G	R
B	G	B	G

Bayer RGB Pattern

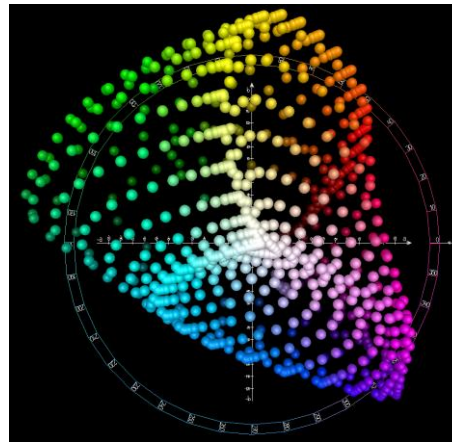
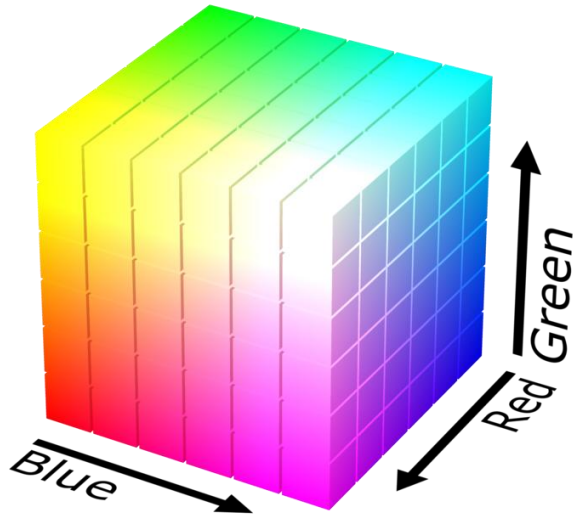
rGb	Rgb	rGb	Rgb
rgB	rGb	rgB	rGb
rGb	Rgb	rGb	Rgb
rgB	rGb	rgB	rGb

Interpolated Pixels

- To measure color, pixels are arranged in a color array, e.g.: Bayer RGB pattern
- Missing colors at each pixel are interpolated from the neighbors (demosaicing)

# Image Sensor

- **Color Filter:**



- To measure color, pixels are arranged in a color array, e.g.: Bayer RGB pattern
- Missing colors at each pixel are interpolated from the neighbors (demosaicing)



# Reference

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- <https://uni-tuebingen.de/fakultaeten/mathematisch-naturwissenschaftliche-fakultaet/fachbereiche/informatik/lehrstuehle/autonomous-vision/lectures/computer-vision/>