

18-799: Applied Computer Vision

Spring 2026

Image Formation

Objectives

In today's class we will learn

- **Homogeneous Coordinates and transformation**
- **Geometric primitive**
- Geometric Image formation
- Photometric image formation

Homogeneous Coordinates

- In computer vision, image formation is a chain of transformations:
- World → Camera → Image
- If each step is linear:

$$\mathbf{x}' = A\mathbf{x},$$

$$\mathbf{x}'' = B\mathbf{x}'$$

– then the whole pipeline collapses into:

$$\mathbf{x}'' = (BA)\mathbf{x}.$$

Homogeneous Coordinates

- Let's consider the function f , where:

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} x + 1 \\ y \end{bmatrix}$$

- Can we represent f with a matrix?
 - That's, is there a matrix A such that:

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 1 \\ y \end{bmatrix}$$

Homogeneous Coordinates

- Points in Euclidean Coordinates
 - In standard 2D space, a point is represented as a vector

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

- Points in Homogeneous Coordinates
 - The same point is represented by the augmented vector

$$\bar{\mathbf{x}} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

- More generally, the Euclidean point \mathbf{x} corresponds to the entire equivalence class

$$\tilde{\mathbf{x}} \sim \lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix}, \lambda \neq 0$$

- All such homogeneous vectors represent the same geometric point after normalization.

Homogeneous Coordinates

- Euclidean point $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ corresponds to the entire equivalence class in homogeneous coordinates:

$$\tilde{\mathbf{x}} \sim \lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix}, \lambda \neq 0$$

- Since λ can be anything ($\lambda \neq 0$), then $\tilde{\mathbf{x}}$ could be re-written $\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix}$
- To convert a homogeneous vector $\tilde{\mathbf{x}}$ back to inhomogeneous form, we divide by the last coordinate \tilde{w} :

$$\bar{\mathbf{x}} = \frac{1}{\tilde{w}} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} \tilde{x}/\tilde{w} \\ \tilde{y}/\tilde{w} \\ 1 \end{pmatrix}, \tilde{w} \neq 0.$$

- The corresponding inhomogeneous point is obtained by dropping the last coordinate:

$$\mathbf{x} = \begin{pmatrix} \tilde{x}/\tilde{w} \\ \tilde{y}/\tilde{w} \end{pmatrix}.$$

Homogeneous Coordinates

- Let's re-consider the function f , where:

$$f \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + 1 \\ y \\ 1 \end{pmatrix}$$

- Can we represent f with a matrix?
 - That's, is there a matrix A such that:

$$A \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + 1 \\ y \\ 1 \end{bmatrix}$$
$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Geometric primitive

- Geometric primitives are the basic building blocks used to describe 3D shapes
- Examples: Points, lines, planes

2D Points

- 2D points can be written in inhomogeneous coordinates as $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$
- or in homogeneous coordinates as $\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{pmatrix} \in \mathbb{P}^2$ where $\mathbb{P}^2 = \mathbb{R}^3 \setminus \{(0,0,0)\}$ is called projective space.
- Homogeneous vectors that differ only by scale are considered equivalent and define an equivalence class.

2D Points

- An inhomogeneous vector \mathbf{x} is converted to a homogeneous vector $\tilde{\mathbf{x}}$ as follows

$$\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \bar{\mathbf{x}}$$

with augmented vector $\bar{\mathbf{x}}$.

- To convert in the opposite direction we divide by \tilde{w} :

$$\bar{\mathbf{x}} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \frac{1}{\tilde{w}} \tilde{\mathbf{x}} = \frac{1}{\tilde{w}} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} \tilde{x}/\tilde{w} \\ \tilde{y}/\tilde{w} \\ 1 \end{pmatrix}$$

- Homogeneous points whose last element is $\tilde{w} = 0$ are called ideal points or points at infinity. These points can't be represented with inhomogeneous coordinates

2D Lines

- 2D lines can also be expressed using homogeneous coordinates

$$\tilde{\mathbf{l}} = (a, b, c)^\top : \{ \bar{\mathbf{x}} \mid \tilde{\mathbf{l}}^\top \bar{\mathbf{x}} = 0 \} \Leftrightarrow \{ x, y \mid ax + by + c = 0 \}$$

- An exception is the line at infinity $\tilde{\mathbf{l}}_\infty = (0, 0, 1)^\top$ which passes through all ideal points.

Cross Product

- Cross product of two vector:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- Cross product expressed as the product of a skew-symmetric matrix and a vector

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_{\times} \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

- See Levi-Civita symbol for more info.

Cross Product

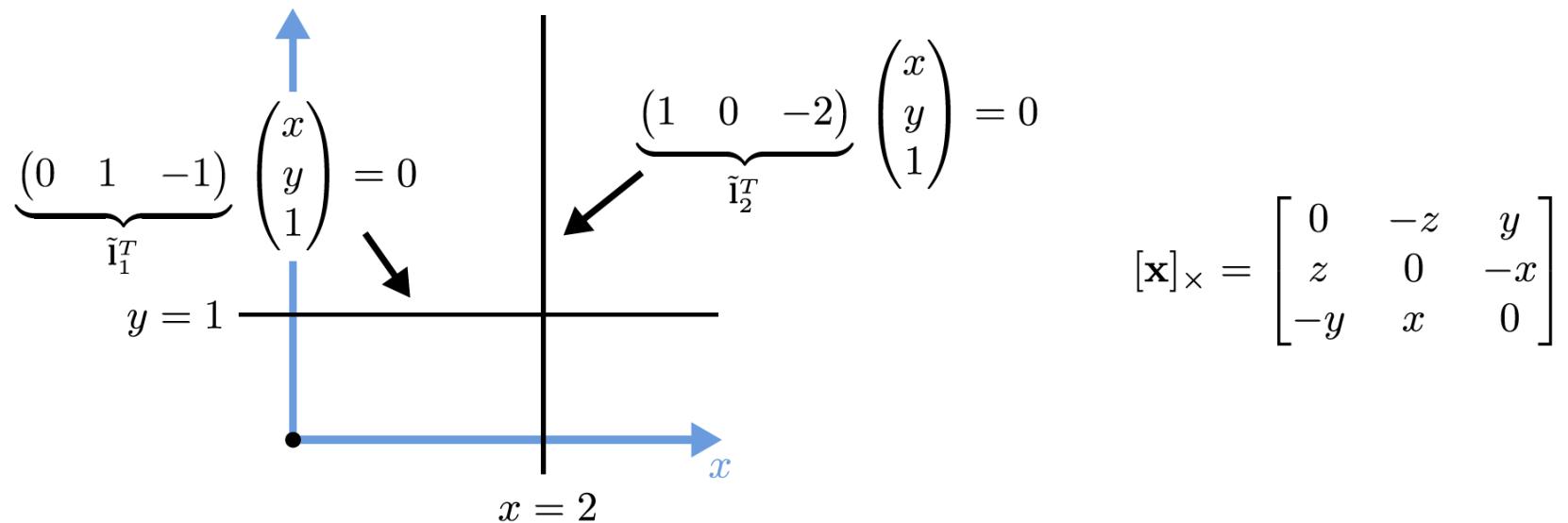
- In homogeneous coordinates, the intersection of two lines is given by:

$$\tilde{\mathbf{x}} = \tilde{\mathbf{l}}_1 \times \tilde{\mathbf{l}}_2$$

- Similarly, the line joining two points can be compactly written as:

$$\tilde{\mathbf{l}} = \bar{\mathbf{x}}_1 \times \bar{\mathbf{x}}_2$$

2D Line: Example



$$\tilde{\mathbf{l}}_1 \times \tilde{\mathbf{l}}_2 = [\tilde{\mathbf{l}}_1]_\times \tilde{\mathbf{l}}_2 = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

2D Line: Example

$$\underbrace{(1 \quad 0 \quad -1)}_{\tilde{\mathbf{l}}_1^T} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

$$\underbrace{(1 \quad 0 \quad -2)}_{\tilde{\mathbf{l}}_2^T} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

$$[\mathbf{x}]_\times = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

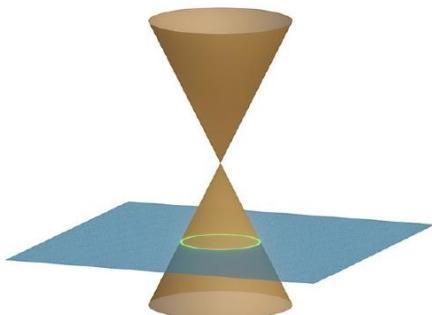
$$\tilde{\mathbf{l}}_1 \times \tilde{\mathbf{l}}_2 = [\tilde{\mathbf{l}}_1]_\times \tilde{\mathbf{l}}_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\underbrace{(0 \quad 0 \quad 1)}_{\tilde{\mathbf{l}}_\infty^T}^\top \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

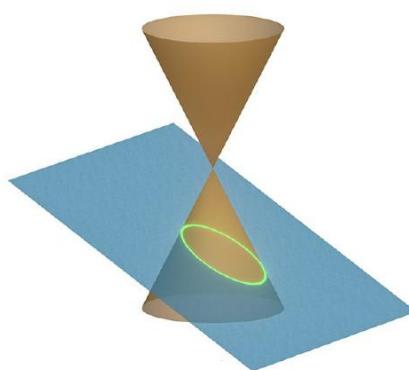
2D Conics

- More complex algebraic objects can be represented using polynomial homogeneous equations.
- For example, conic sections (arising as the intersection of a plane and a 3D cone) can be written using quadric equations:

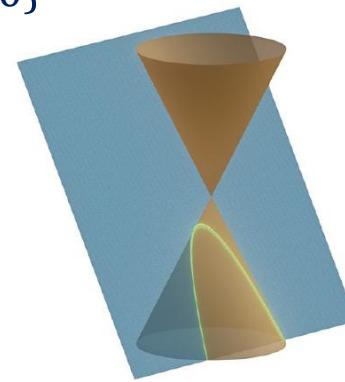
$$\{\bar{x} \mid \bar{x}^T Q \bar{x} = 0\}$$



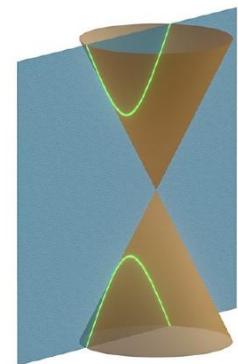
Circle



Ellipse



Parabola



Hyperbola

3D Quadrics

The 3D analog of 2D conics is a quadric surface:

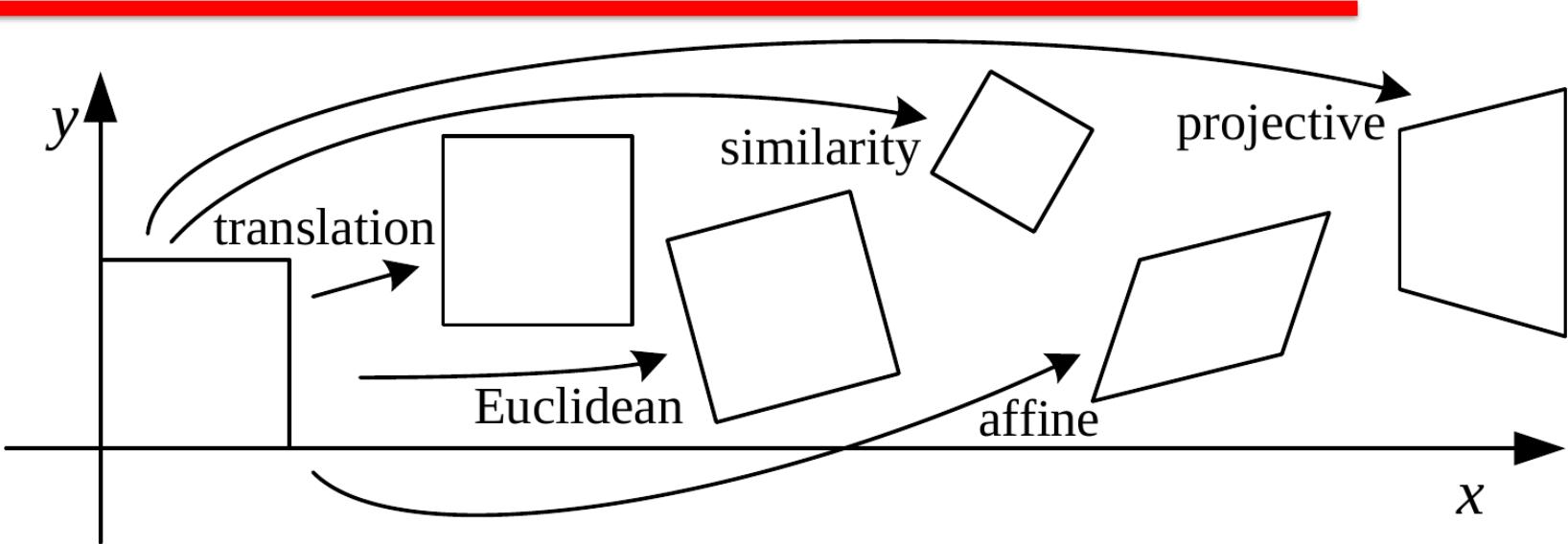
$$\{\bar{\mathbf{x}} \mid \bar{\mathbf{x}}^T \mathbf{Q} \bar{\mathbf{x}} = 0\}$$



Superquadrics (generalization of quadrics) for shape abstraction and compression.

Paschalidou, Ulusoy and Geiger: Superquadrics Revisited: Learning 3D Shape Parsing beyond Cuboids. CVPR, 2019

Translation

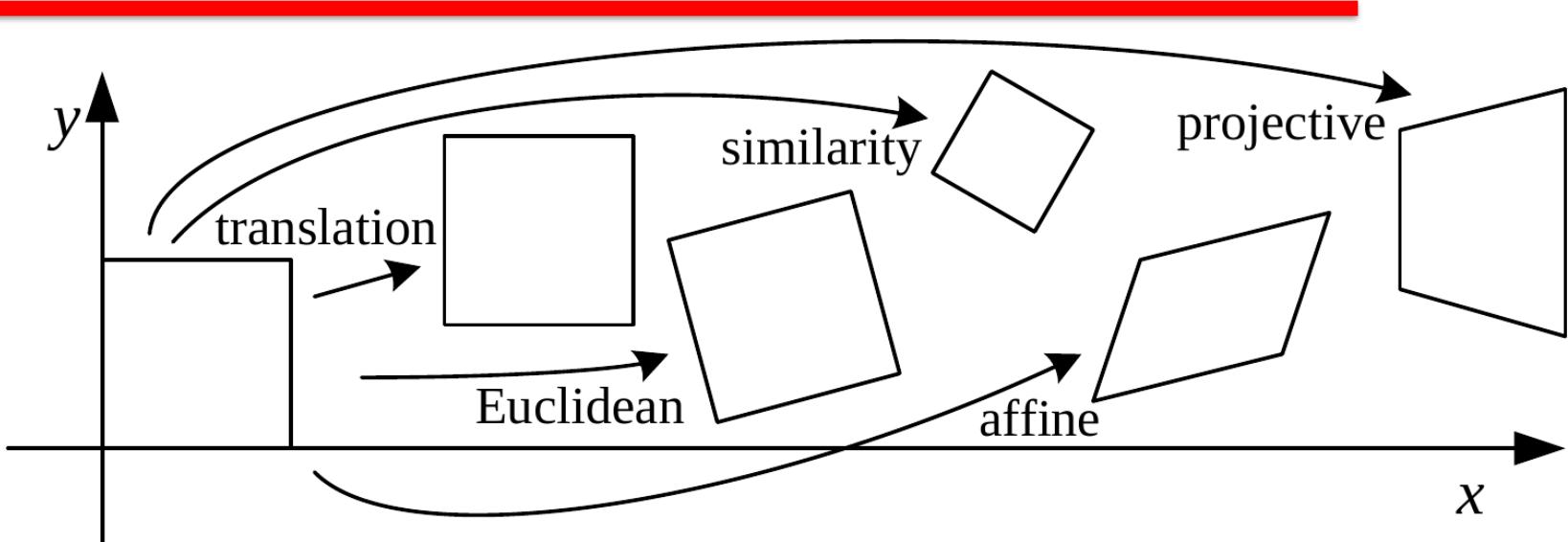


- [2D Translation of the Input, 2 DoF]

$$\mathbf{x}' = \mathbf{x} + \mathbf{t} \Leftrightarrow \bar{\mathbf{x}}' = \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \bar{\mathbf{x}} \Rightarrow \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- Using homogeneous representations allows to chain/invert transformations
- Augmented vectors $\bar{\mathbf{x}}$ can always be replaced by general homogeneous ones $\hat{\mathbf{x}}$

Affine

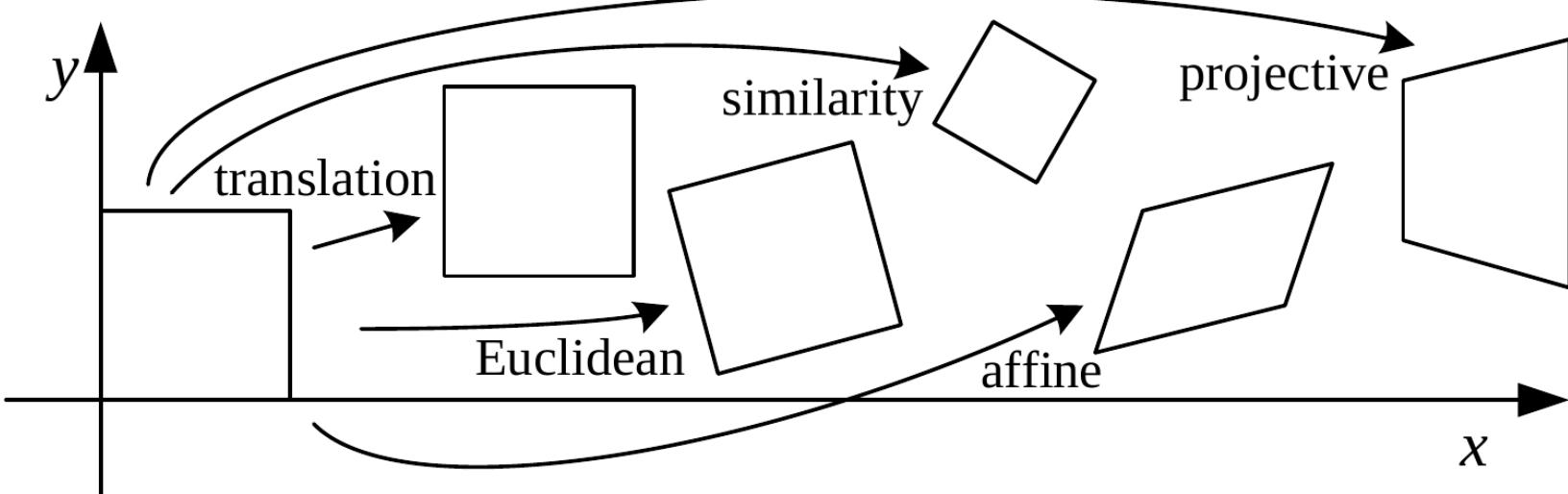


- Equivalent to 2D Linear Transformation, 6 DoF

$$\mathbf{x}' = \mathbf{Ax} + \mathbf{t} \Leftrightarrow \bar{\mathbf{x}}' = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \bar{\mathbf{x}}$$

- $\mathbf{A} \in \mathbb{R}^{2 \times 2}$ is an arbitrary 2×2 matrix
- Parallel lines remain parallel under affine transformations

Projective

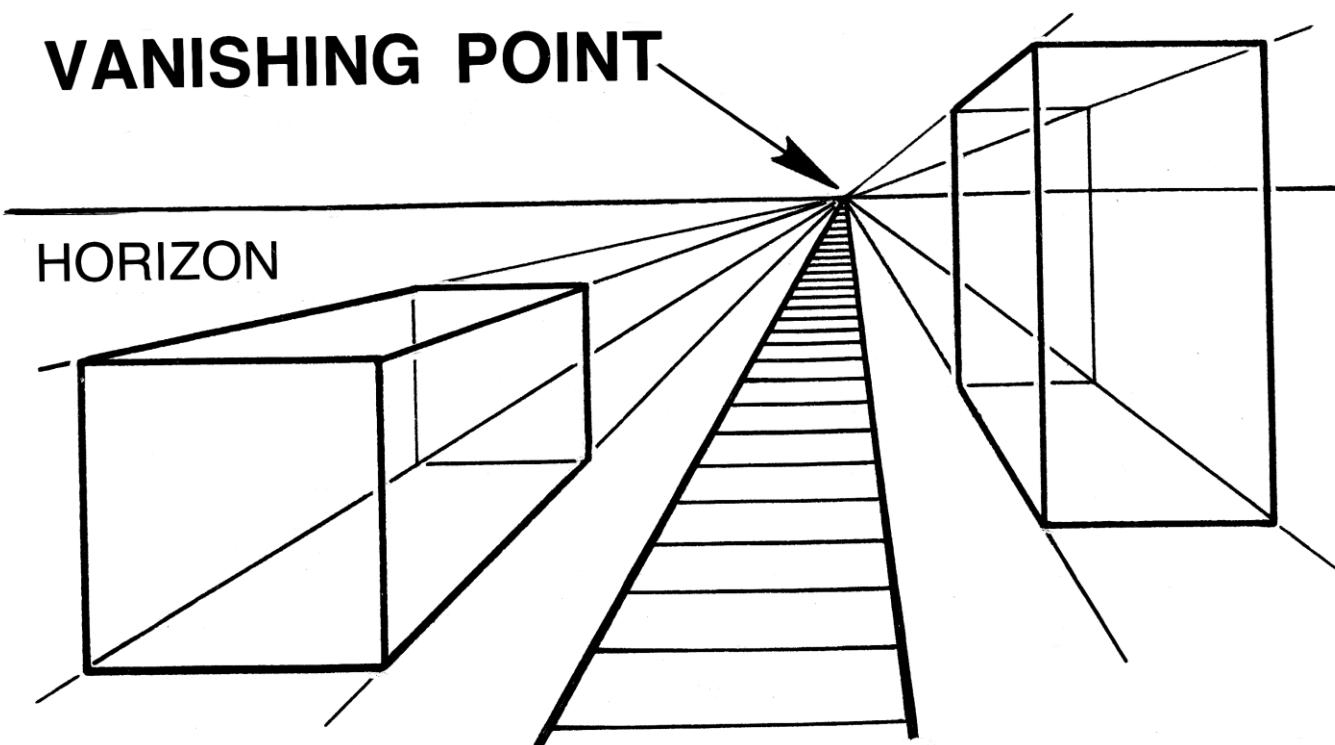


- Homography, 8 DoF

$$\tilde{\mathbf{x}}' = \tilde{\mathbf{H}}\tilde{\mathbf{x}} \left(\bar{\mathbf{x}} = \frac{1}{\tilde{w}}\tilde{\mathbf{x}} \right)$$

- $\tilde{\mathbf{H}} \in \mathbb{R}^{3 \times 3}$ is an arbitrary homogeneous 3×3 matrix (specified up to scale)
- Projective transformations preserve straight lines

Projective



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Projective

- A homography is defined in homogeneous coordinates as

$$\tilde{\mathbf{x}}' = H\tilde{\mathbf{x}}, H \in \mathbb{R}^{3 \times 3}.$$

- Write the point (x, y) in homogeneous form:

$$\tilde{\mathbf{x}} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Let

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}.$$

- We now have:

$$\begin{bmatrix} \tilde{x}' \\ \tilde{y}' \\ \tilde{w}' \end{bmatrix} = \begin{bmatrix} h_{11}x + h_{12}y + h_{13} \\ h_{21}x + h_{22}y + h_{23} \\ h_{31}x + h_{32}y + h_{33} \end{bmatrix}.$$

Projective

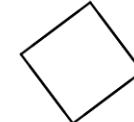
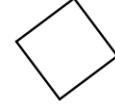
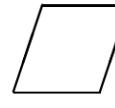
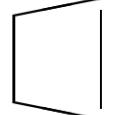
- Homogeneous coordinates represent points up to scale.
To recover Euclidean coordinates, we divide by the last component:

$$x' = \frac{\tilde{x}'}{\tilde{w}'}, y' = \frac{\tilde{y}'}{\tilde{w}'}$$

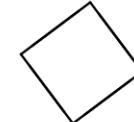
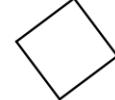
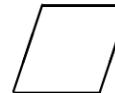
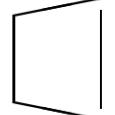
- Substituting gives exactly:

$$\begin{aligned}x' &= \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \\y' &= \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}\end{aligned}$$

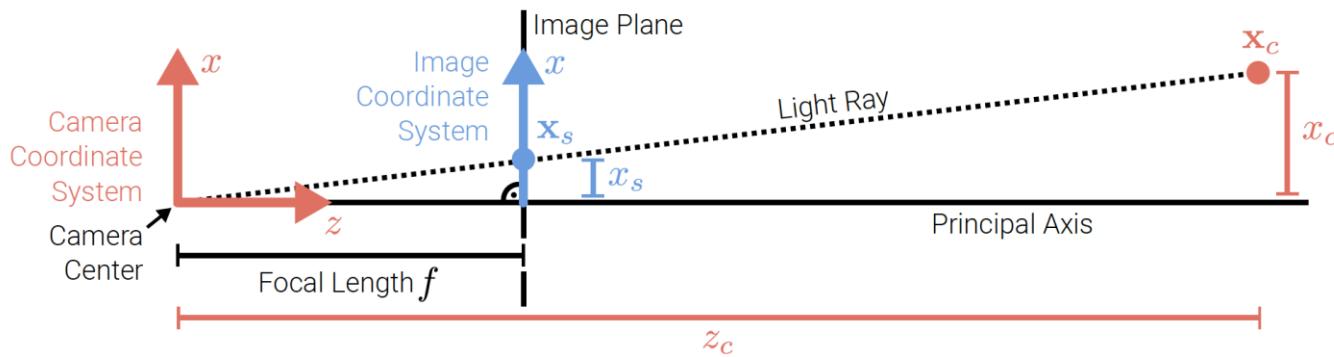
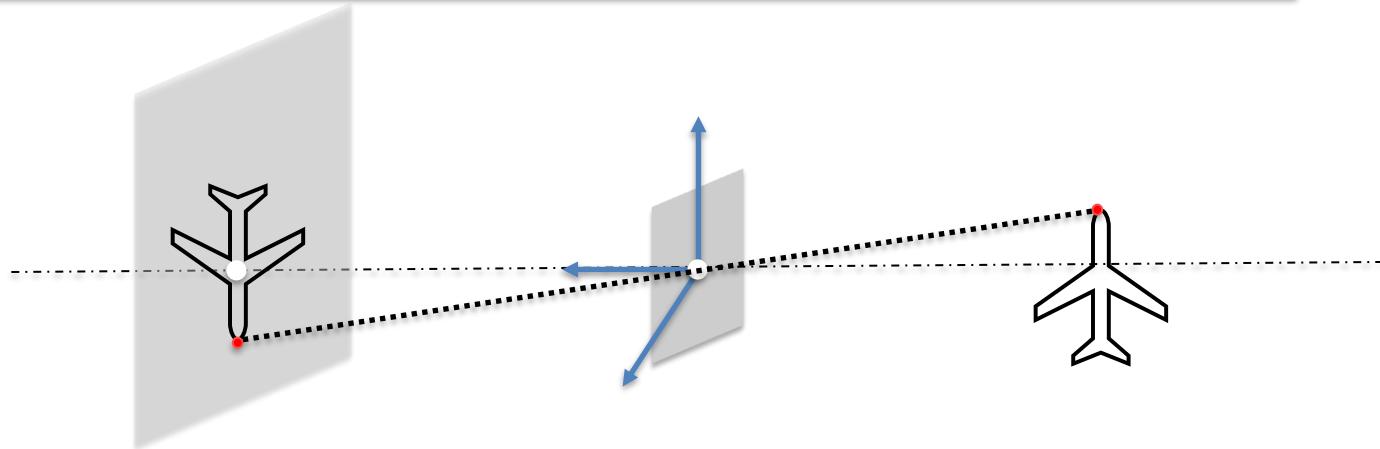
2D Transformation

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix}_{2 \times 3}$	2	orientation	
rigid (Euclidean)	$\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}_{2 \times 3}$	3	lengths	
similarity	$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \end{bmatrix}_{2 \times 3}$	4	angles	
affine	$\begin{bmatrix} \mathbf{A} \end{bmatrix}_{2 \times 3}$	6	parallelism	
projective	$\begin{bmatrix} \tilde{\mathbf{H}} \end{bmatrix}_{3 \times 3}$	8	straight lines	

3D Transformation

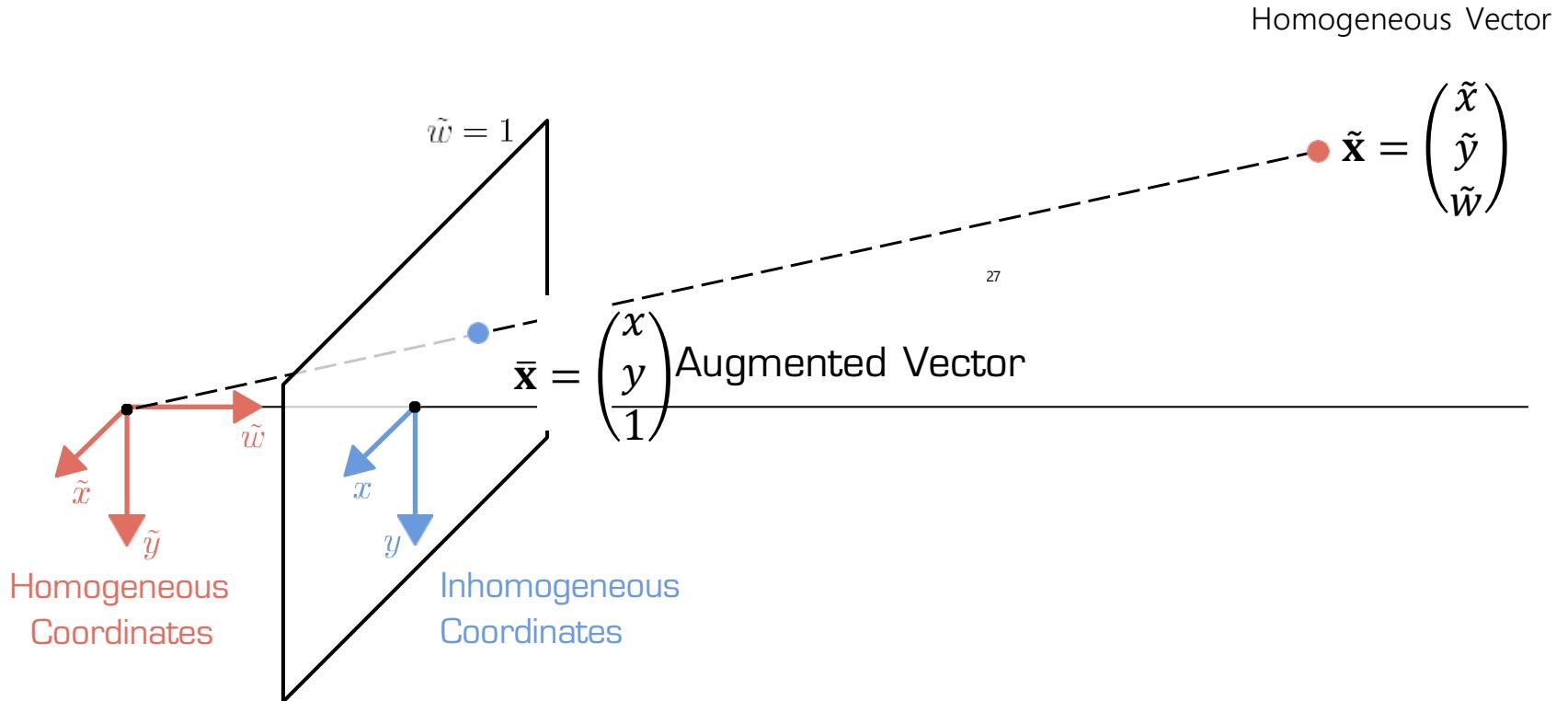
Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix}_{3 \times 4}$	3	orientation	
rigid (Euclidean)	$\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}_{3 \times 4}$	6	lengths	
similarity	$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \end{bmatrix}_{3 \times 4}$	7	angles	
affine	$\begin{bmatrix} \mathbf{A} \end{bmatrix}_{3 \times 4}$	12	parallelism	
projective	$\begin{bmatrix} \tilde{\mathbf{H}} \end{bmatrix}_{4 \times 4}$	15	straight lines	

Geometric Image Formation



$$\frac{x_s}{f} = \frac{x_c}{z_c}$$

2D Points



Perspective Projection

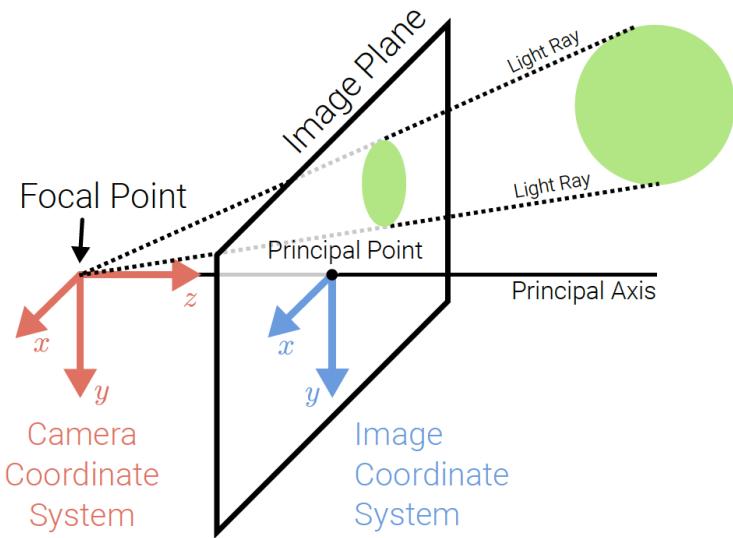
- In perspective projection, we have:

$$\begin{pmatrix} x_s \\ y_s \end{pmatrix} = \begin{pmatrix} fx_c/z_c \\ fy_c/z_c \end{pmatrix} \Leftrightarrow \tilde{\mathbf{x}}_s = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \bar{\mathbf{x}}_c, \text{ where } \bar{\mathbf{x}}_c = \begin{pmatrix} x_c \\ y_c \\ z_c \\ 1 \end{pmatrix}$$

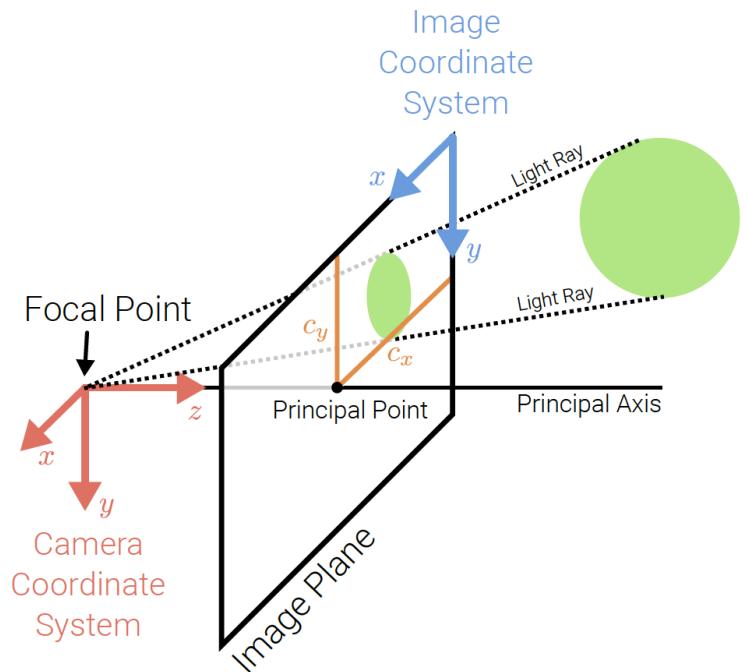
- Note that this projection is linear when using homogeneous coordinates.
- After the projection it is not possible to recover the distance of the 3D point from the image.
- Calibration matrix

Perspective Projection

Without Principal Point Offset



With Principal Point Offset



- To ensure positive pixel coordinates, a **principal point offset c** is usually added
- This moves the image coordinate system to the corner of the image plane

Perspective Projection

- The complete perspective projection model is given by:

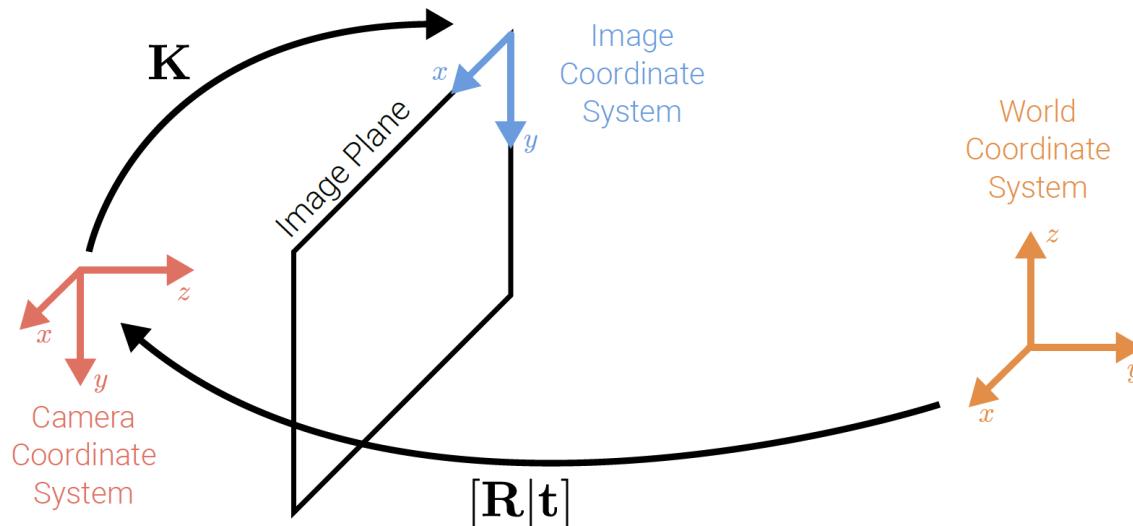
$$\begin{pmatrix} x_s \\ y_s \end{pmatrix} = \begin{pmatrix} f_x x_c / z_c + s y_c / z_c + c_x \\ f_y y_c / z_c + c_y \end{pmatrix} \Leftrightarrow \tilde{\mathbf{x}}_s = \begin{bmatrix} f_x & s & c_x & 0 \\ 0 & f_y & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \bar{\mathbf{x}}_c$$

- The left 3×3 submatrix of the projection matrix is called calibration matrix \mathbf{K}
- The parameters of \mathbf{K} are called camera intrinsics (as opposed to extrinsic pose)
- Here, f_x and f_y are independent, allowing for different pixel aspect ratios
- The skew s arises due to the sensor not mounted perpendicular to the optical axis
- In practice, we often set $f_x = f_y$ and $s = 0$, but model $\mathbf{c} = (c_x, c_y)^\top$

Chaining Transformation

- Let \mathbf{K} be the calibration matrix (intrinsics) and $[\mathbf{R} \mid \mathbf{t}]$ the camera pose (extrinsics).
- We chain both transformations to project a point in world coordinates to the image:

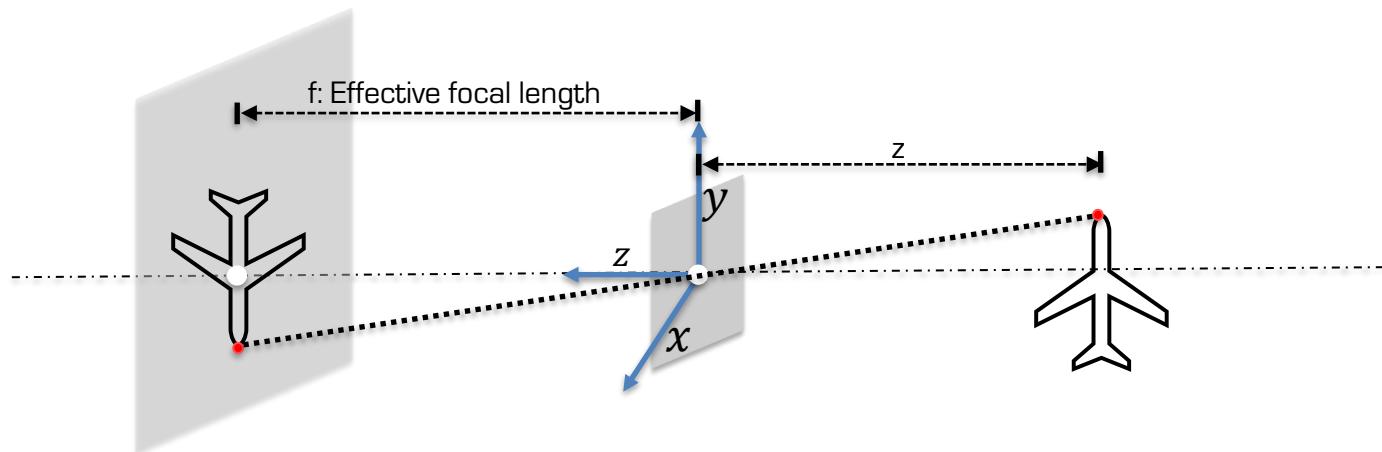
$$\tilde{\mathbf{x}}_s = [\mathbf{K} \quad \mathbf{0}] \bar{\mathbf{x}}_c = [\mathbf{K} \quad \mathbf{0}] \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \bar{\mathbf{x}}_w = \mathbf{K} [\mathbf{R} \quad \mathbf{t}] \bar{\mathbf{x}}_w = \mathbf{P} \bar{\mathbf{x}}_w$$



Perspective Projection

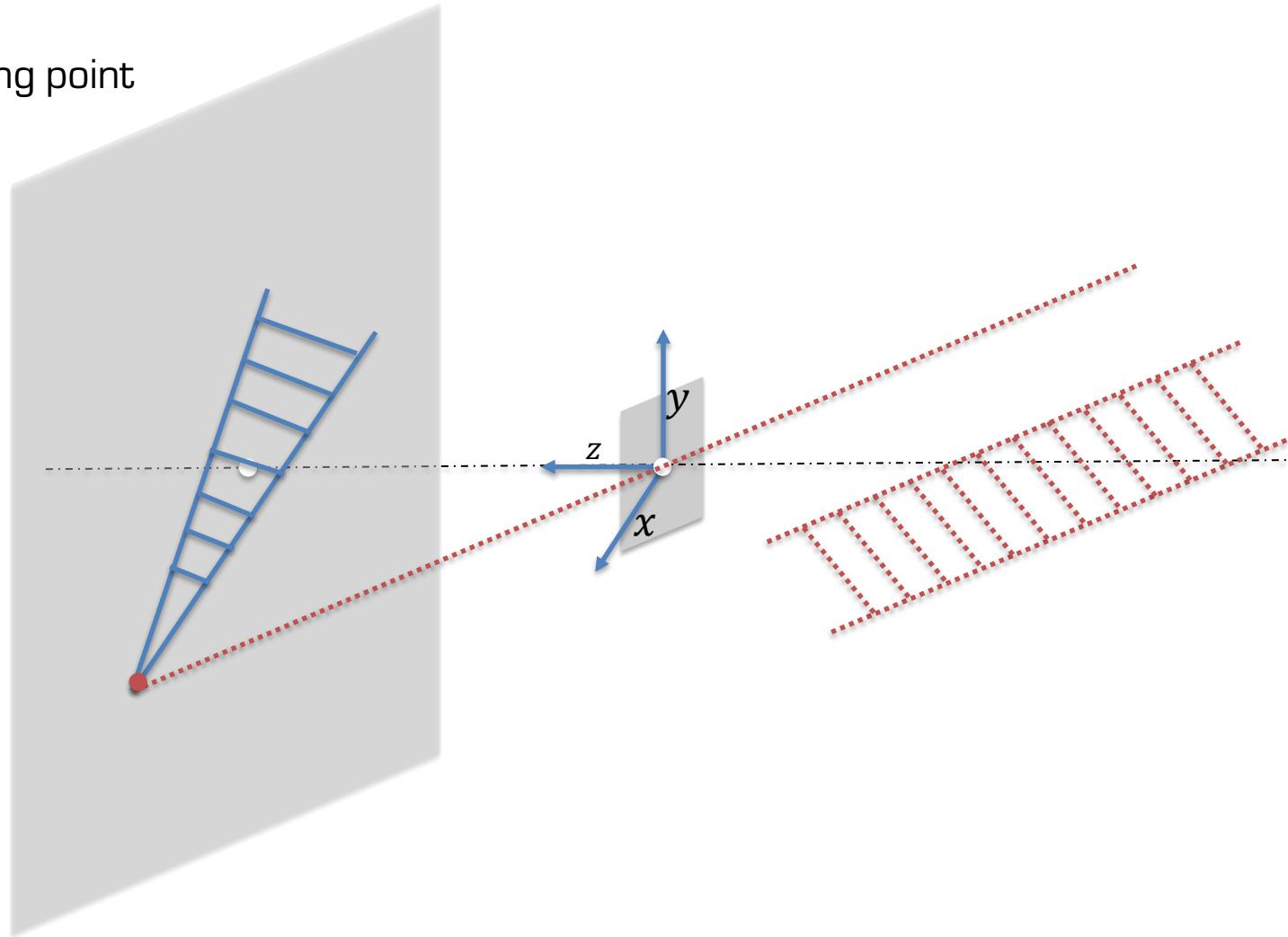
Image Magnification: $m = \frac{f}{z}$

- Image size inversely proportional to depth



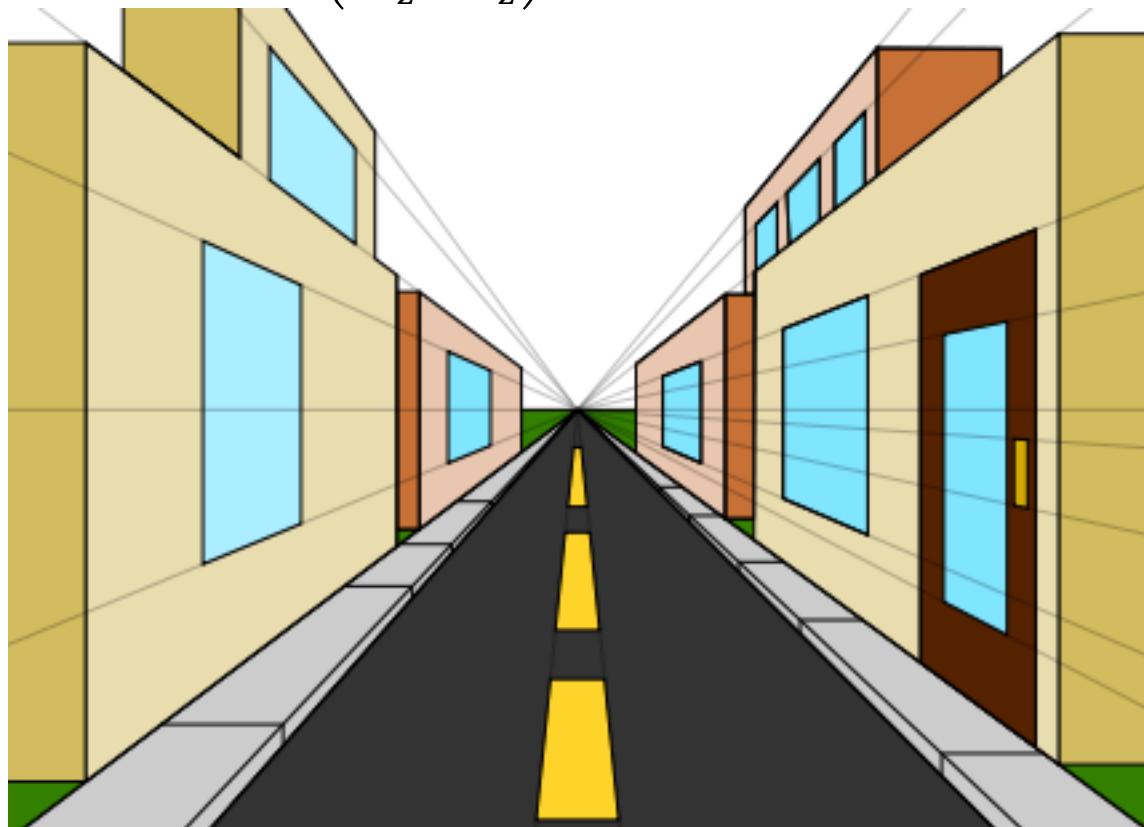
Perspective Projection

Vanishing point



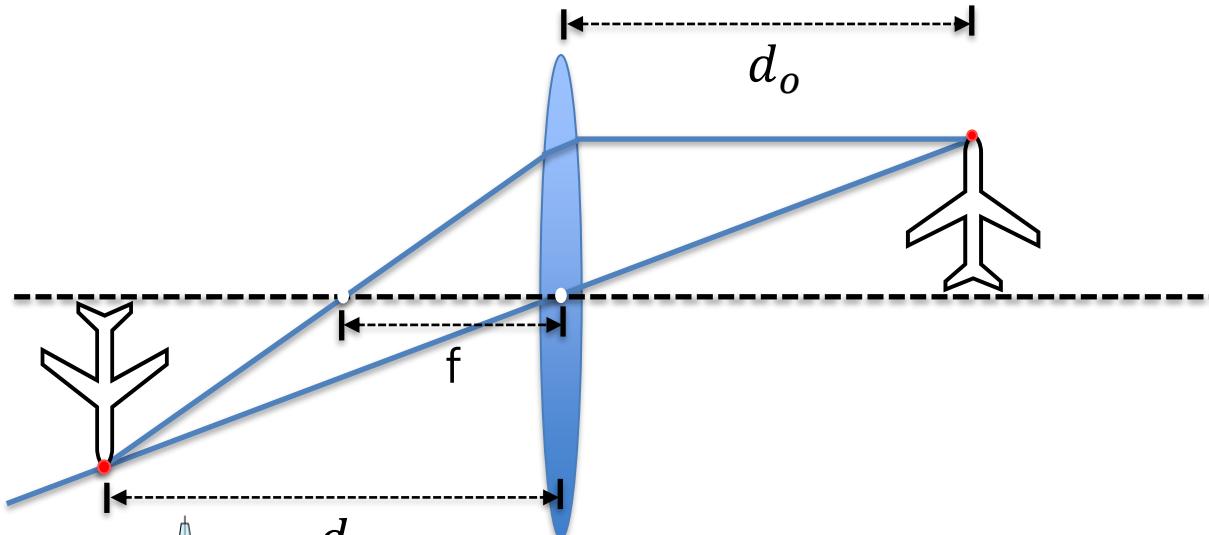
Perspective Projection

Vanishing point: $(x_{vp}, y_{vp}) = \left(f \frac{l_x}{l_z}, f \frac{l_y}{l_z} \right)$

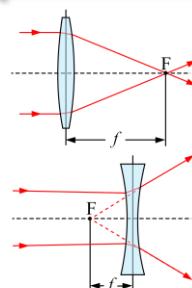


https://commons.wikimedia.org/wiki/File:1_point_perspective.svg

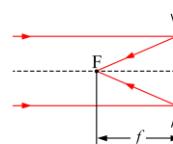
Lens



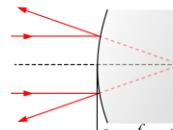
Focal length (f): It determines the lens' bending power



$$\text{This Lens' law: } \frac{1}{d_i} + \frac{1}{d_o} = \frac{1}{f}$$



$$\text{Magnification (m): } m = \frac{h_i}{h_o} = \frac{d_i}{d_o}$$



Lens

Aperture: Area of the lens receiving light

$f/1.4$



$f/2.0$



$f/2.8$



$f/4.0$



$f/5.6$



$f/8.0$



Lens

Aperture: Area of the lens receiving light

f-number: a measure of the light-gathering ability of a camera lens

it equals focal length divided by the aperture.

$f/1.4$



$f/2.0$



$f/2.8$



- Aperture:
$$= \frac{f}{f-Number}$$
- where N is called the f-Number of lens.

$f/4.0$



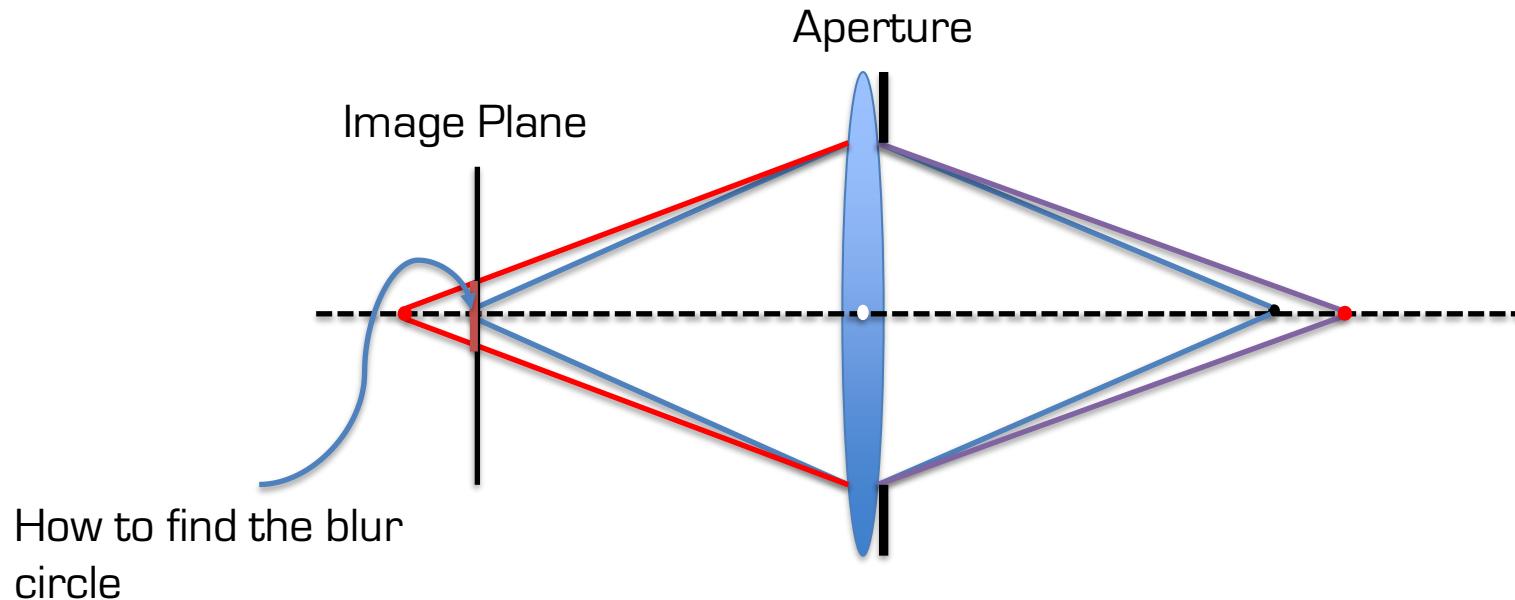
$f/5.6$



$f/8.0$

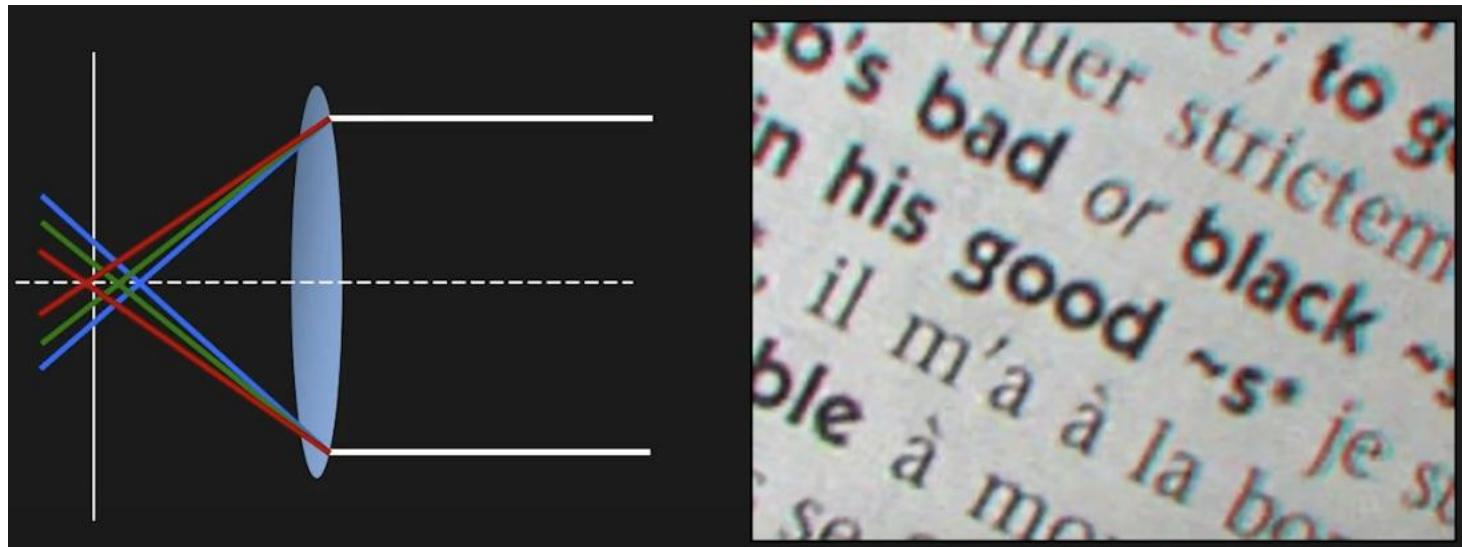


Lens Defocus



Lens Distortion

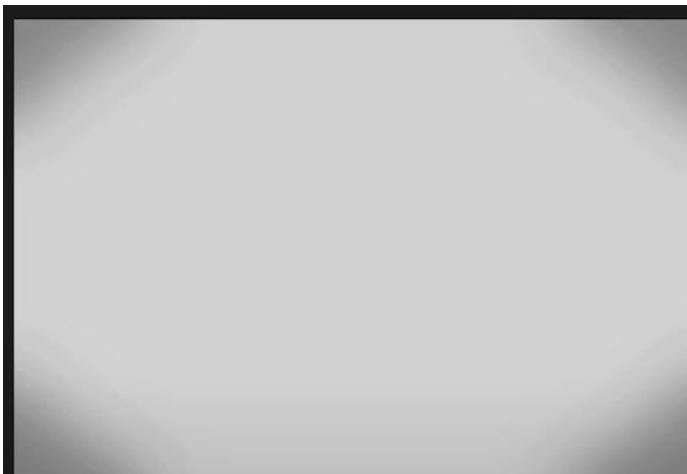
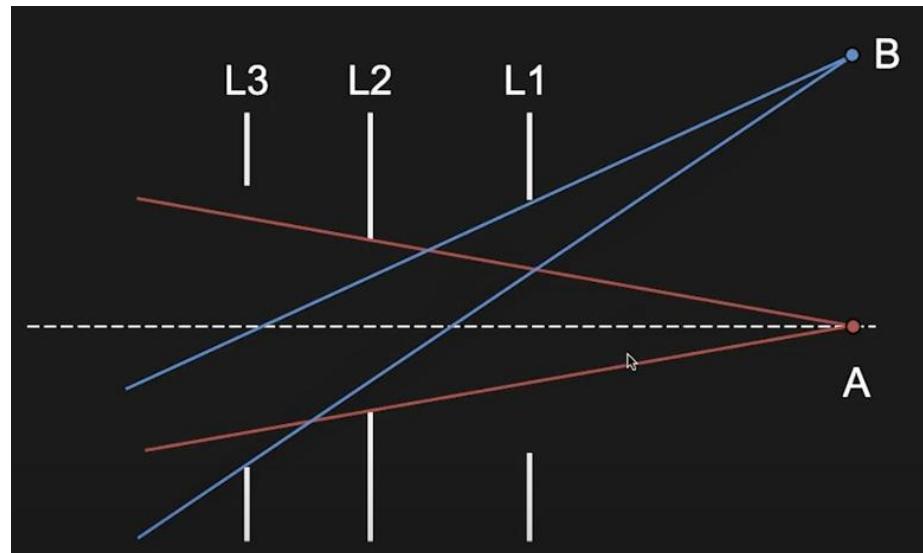
- Chromatic Aberration



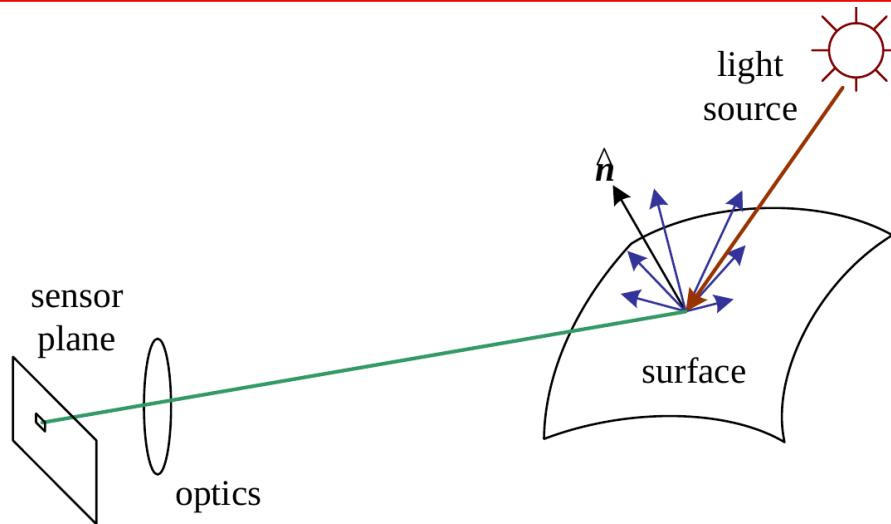
Refractive index (and hence focal length) of lens is different for different wavelengths.

Lens Distortion

- Vignetting



Photometric Image Formation



- How an image is formed in terms of pixel intensities and colors.
- Light is emitted by one or more light sources and reflected or refracted (once or multiple times) at surfaces of objects (or media) in the scene

Rendering Equation

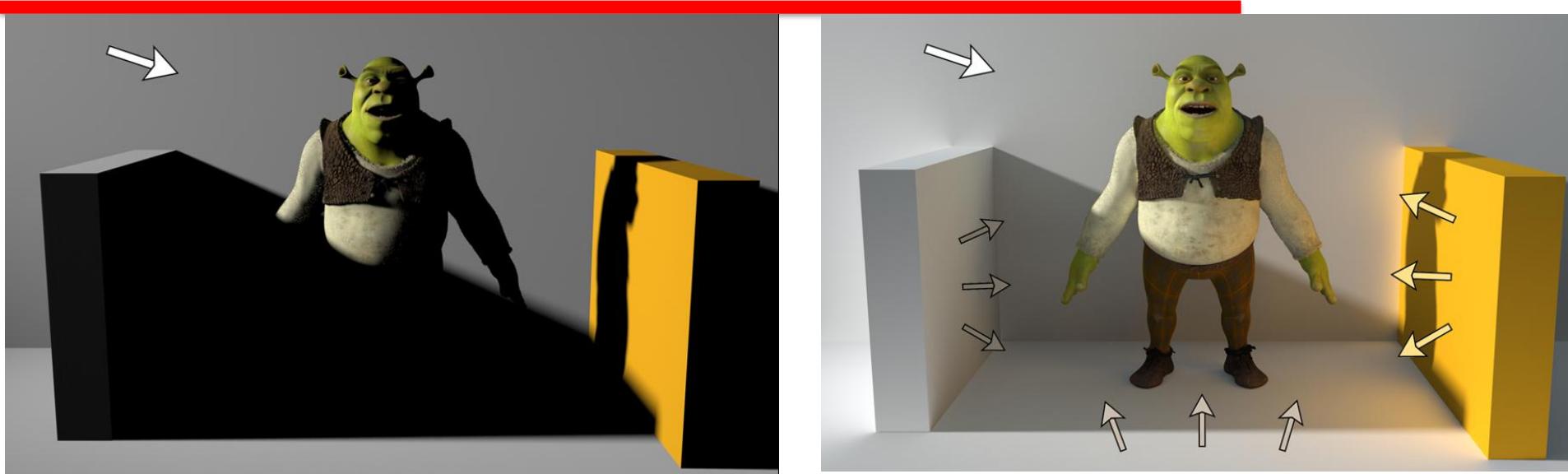
- Let $\mathbf{p} \in \mathbb{R}^3$ denote a 3D surface point, $\mathbf{v} \in \mathbb{R}^3$ the viewing direction and $\mathbf{s} \in \mathbb{R}^3$ the incoming light direction.
- The rendering equation describes how much of the light L_{in} with wavelength λ arriving at \mathbf{p} is reflected into the viewing direction \mathbf{v} :

$$L_{\text{out}}(\mathbf{p}, \mathbf{v}, \lambda)$$

$$= L_{\text{emit}}(\mathbf{p}, \mathbf{v}, \lambda) + \int_{\Omega} \text{BRDF}(\mathbf{p}, \mathbf{s}, \mathbf{v}, \lambda) \cdot L_{\text{in}}(\mathbf{p}, \mathbf{s}, \lambda) \cdot (-\mathbf{n}^T \mathbf{s}) d\mathbf{s}$$

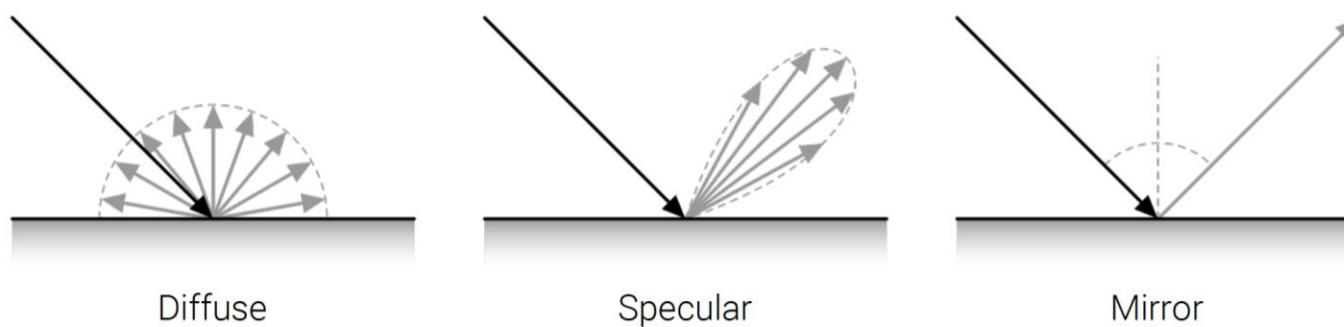
- Ω is the unit hemisphere at normal \mathbf{n}
- The bidirectional reflectance distribution function $\text{BRDF}(\mathbf{p}, \mathbf{s}, \mathbf{v}, \lambda)$ defines how light is reflected at an opaque surface.
- $L_{\text{emit}} > 0$ only for light emitting surfaces

Global Illumination



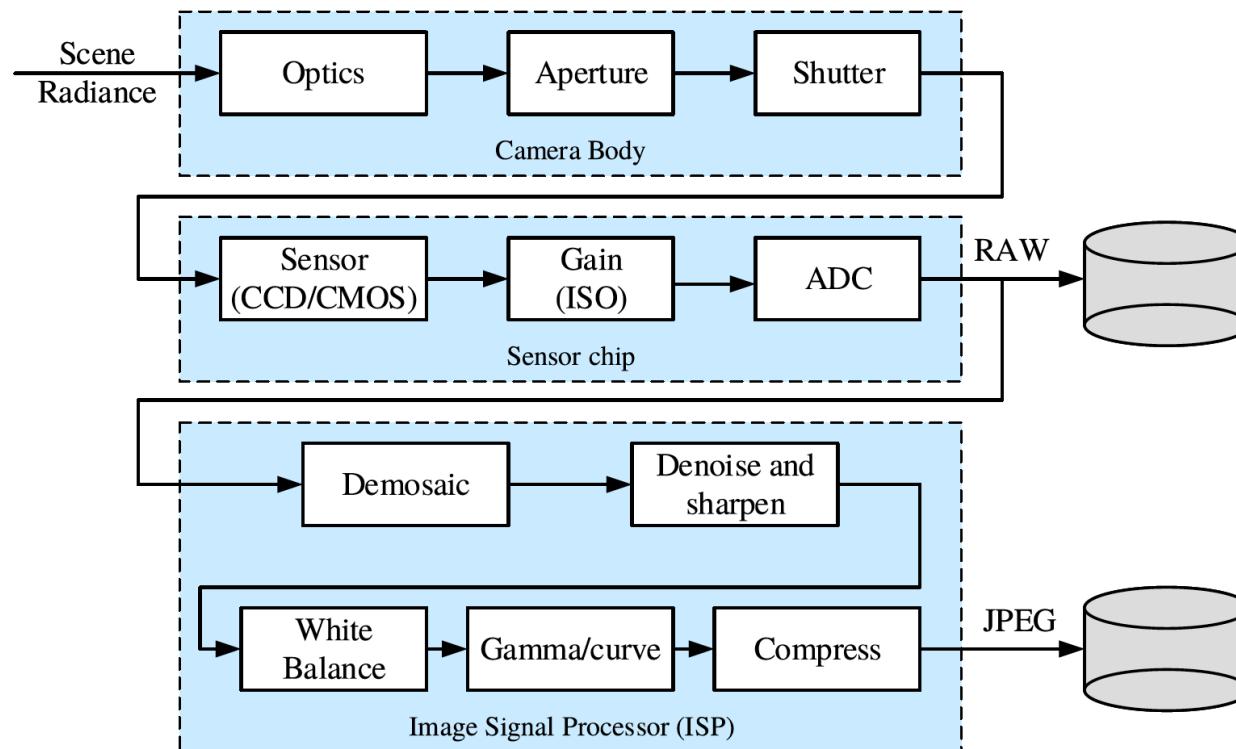
- Modeling one light bounce is insufficient for rendering complex scenes
- | Light sources can be shadowed by occluders and rays can bounce multiple times
- | Global illumination techniques also take indirect illumination into account

Diffuse and Specular Reflection



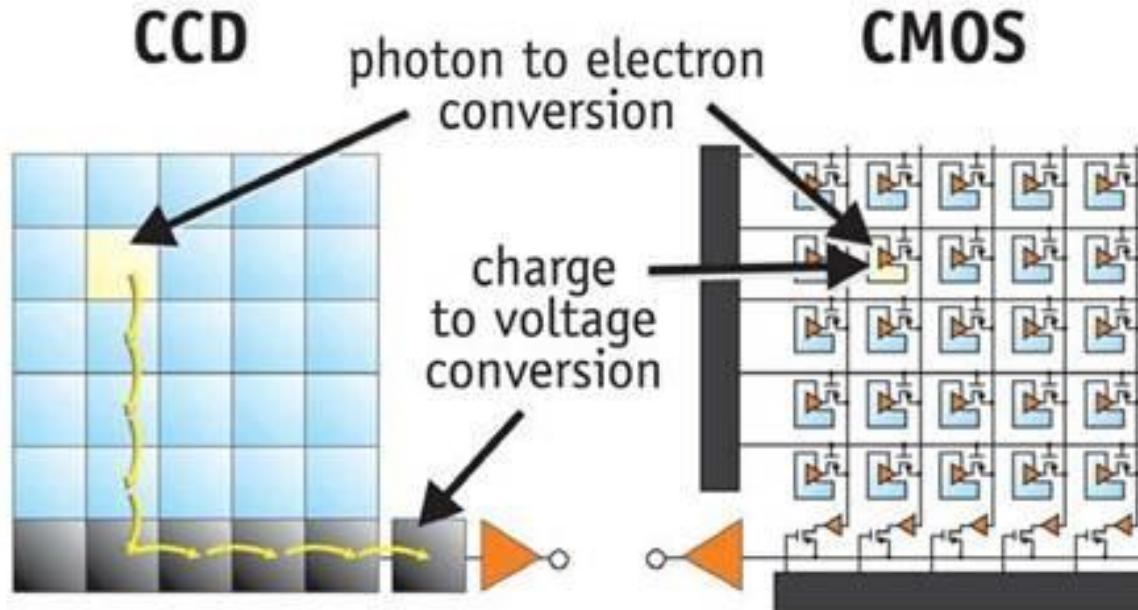
- ▶ Typical BRDFs have a **diffuse** and a **specular** component
- ▶ The diffuse (=constant) component scatters light uniformly in all directions
- ▶ This leads to shading, i.e., smooth variation of intensity wrt. surface normal
- ▶ The specular component depends strongly on the outgoing light direction

Image Sensor



- Physical light transport in the camera lens/body
- Photon measurement and conversion on the sensor chip
- Image signal processing (ISP) and image compression

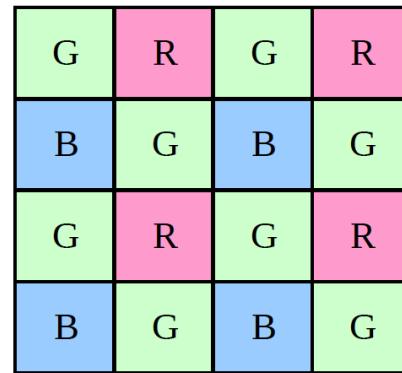
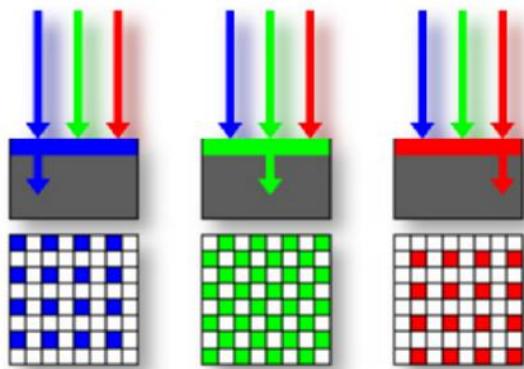
Image Sensor



- I CCDs move charge from pixel to pixel and convert it to voltage at the output node
- I CMOS images convert charge to voltage inside each pixel and are standard

Image Sensor

- **Color Filter:**



Bayer RGB Pattern

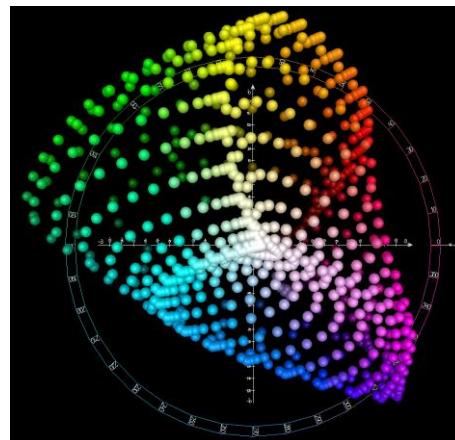
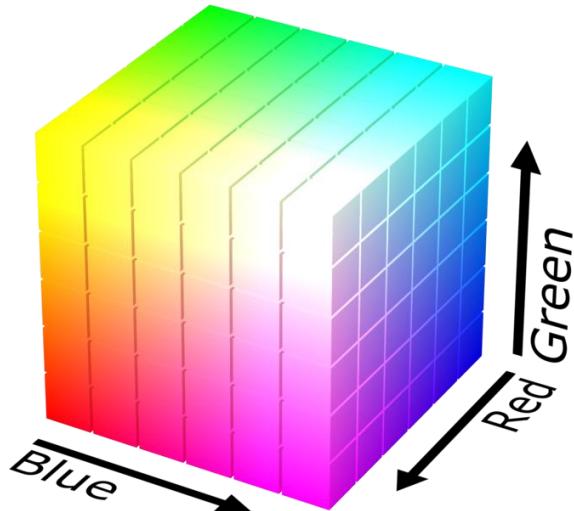
rGb	Rgb	rGb	Rgb
rgB	rGb	rgB	rGb
rGb	Rgb	rGb	Rgb
rgB	rGb	rgB	rGb

Interpolated Pixels

- To measure color, pixels are arranged in a color array, e.g.: Bayer RGB pattern
- I Missing colors at each pixel are interpolated from the neighbors (demosaicing)

Image Sensor

- **Color Filter:**



- To measure color, pixels are arranged in a color array, e.g.: Bayer RGB pattern
- I Missing colors at each pixel are interpolated from the neighbors (demosaicing)

Reference

- <https://uni-tuebingen.de/fakultaeten/mathematisch-naturwissenschaftliche-fakultaet/fachbereiche/informatik/lehrstuhle/autonomous-vision/lectures/computer-vision/>