Lecture Notes – Maximum Flow

Objectives

- Be familiar with max flow problems and applications.
- Be able to solve optimization problems using max flow algorithms

Key Ideas

- Flow networks
- Flow conservation
- Ford-Fulkerson method
- Residual networks
- Augmenting paths
- Cuts of flow networks
- Min-cut and max flow (dual problems)

Lecture Notes

Reference: Chapter 26 on textbook

1. Basic concepts:

   Given a flow network $G = (V, E)$, $u \in V$ and $v \in V$ and each edge $(u, v) \in E$ has a nonnegative **capacity** $c(u, v) \geq 0$.

   (a) The capacity constraint principle of a flow between nodes says the flow from node u to node v must satisfy

   $$0 \leq f(u, v) \leq c(u, v)$$

   .

   (b) The flow conservation principle states that the total flow entering a node should be equal to the total flow leaving that node. In other words, there is no accumulation of flow in a node anytime. Mathematically, for all $u \in V - \{s, t\}$

   $$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v)$$

(c) The residual capacity of an edge is difference between its capacity and the flow passing through the edge.

$$c_f(v, u) = c(u, v) - f(u, v)$$

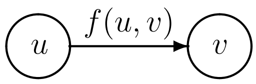
2. Finding max flow of a flow network

   (a) Ford-Fulkerson Method

   (b) Find augmenting paths

   (c) Build residual graphs

   (d) Suppose there is a flow $f(u, v)$ from node u to v. The capacity of the edge (u, v) is $c(u, v)$. Also, there is no edge from v to u, that is, $c(v, u) = 0$.



   The residual capacity of from node v to u is

$$c_f(v, u) = c(v, u) - f(v, u) = 0 - (-f(u, v)) = f(u, v)$$

   This means that we can push a flow backward from node v to u with the same amount as $f(u, v)$, even though there is no edge (v, u). This is because those two flows are in opposite direction and they cancel each other. The capacity of edge (v, u) is still zero.

3. Cuts:

   (a) A cut of a flow network divides V of a graph $G = (V, E)$ into two parts (sets) S and $T = V - S$ such that the source node $s \in S$ and the sink node $t \in T$. Any flow f in a network from the source to the sink must pass across such cuts (S, T). The capacity of a cut (S, T) is the maximum total flow across the cut.

   (b) The **min-cut** of a flow network is a cut (S, T) such that its capacity is minimum.

   (c) The capacity of the min-cut of a flow network is equal to the max flow of the flow network.