Summary of Wavelets

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Continuous Wavelet Transform

Def: Suppose $f \in L^2(\mathbb{R})$. Then, its *continuous wavelet transform* (CWT) is given by

$$\begin{aligned} F_w(a,b) &= \frac{1}{|a|^{1/2}} \int dx\ f(x)\ \bar{\psi}\left(\frac{x-b}{a}\right) \\ &= \langle f, \psi^{a,b} \rangle, \end{aligned}$$

where $a \in \mathbb{R}^{+*}$, $b \in \mathbb{R}$ denotes the *dilation parameter* and the *shift parameter.* The function $\bar{\psi} \in L^2(\mathbb{R})$ is referred to as a *mother wavelet*.

Remark: Note this is a convolution on f.

Thm: Suppose the mother wavelet satisfies the *admissibility condition*

$$C_\psi = 2\pi \int d\xi\ |\xi|^{-1}\ |\tilde{\psi}(\xi)|^2 < \infty,$$

then f can be expressed as an *inverse CWT* (ICWT):

$$f(x) = C_\psi^{-1} \int_0^\infty \int_{-\infty}^\infty \frac{da\ db}{a^2}\ F_w(a,b)\ \psi\left(\frac{x-b}{a}\right).$$

Remark: This convergence is defined in the *weak* sense in $L^2(\mathbb{R})$.

— References: Wikipedia, Daubechies ch. 2

Discrete Wavelet Transform

Like applying the Fourier transform on a periodic function, the CWT is highly redundant and computationally intractable. Therefore, it is desirable to discretize the CWT so that functions may be expanded in terms of wavelet series (like Fourier series in the periodic setting).

Orthonormal Wavelet Basis

Defs: For simplicity, suppose $\psi \in L^2(\mathbb{R})$ such that the following set of functions is an orthonormal basis of $L^2(\mathbb{R})$.

$$\{\psi_{j,k}(x) = \psi\left(2^j x - k\right)\ :\ j \in \mathbb{Z},\ k \in \mathbb{Z},\ x \in \mathbb{R}\}$$

ψ is again referred to as a *mother wavelet* and the basis above is the *wavelet family* generated by ψ. From this, any function $f \in L^2(\mathbb{R})$ can be expressed as a *wavelet expansion*

$$f(x) = \sum_{j,k} d_{j,k}\ \psi_{j,k}(x),$$

where the coefficients $d_{j,k} = \langle f, \psi_{j,k} \rangle$ are the *wavelet coefficients* of f. They are also referred to as the *detail coefficients* or *discrete wavelet transforms* of f. In this way, f is the *inverse discrete wavelet transform* of of the detail coefficients $d_{j,k}$.

Remarks: The dilation parameter need not take the form 2^j. Its base can take on any $a_0 \in \mathbb{Q}$. Also, the shift parameters can be integer multiples of a constant b_0. This *dyadic* sampling is made for convenience.

We only consider mother wavelets with compact support in time and space. Admissibility also requires them to have zero mean. We do not consider further the notion of *vanishing moments*.

Lastly, symmetric wavelets are not orthogonal, but rather they are biorthogonal and have pseudoinverses (with respect to a space of wavelet duals). This approach breaks down in that case, but the same ideas apply.

Functional Multiresolution Analysis

We have discretized the wavelet transform. However, it still requires an infinite number of scales to characterize. The tiling in frequency space appears in the figure below:

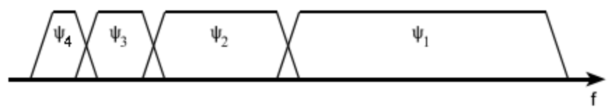


Figure 2

Touching wavelet spectra resulting from scaling of the mother wavelet in the time domain.

For our purposes, we want to reconstruct a function without having to specify its details at coarse scales. That is, we want a single function to take care of the coarse scales as shown below:

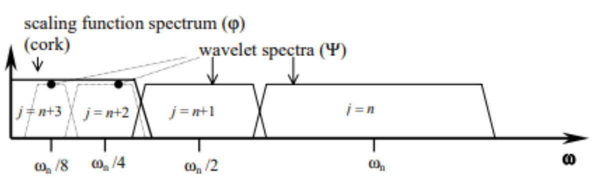


Figure 3

How an infinite set of wavelets is replaced by one scaling function.

Note, we're not taking a shortcut. Finding this scaling function is an art because it would be a linear combination of an infinite-number of coarse-scale wavelets.

Def: The wavelets $\psi_{j,k}$ span a space $\mathcal{W}_j \subset L^2$ and $\mathcal{W}_j \ \cap \ \mathcal{W}_k = \mathcal{W}_j \ \delta_{jk}$ since $\mathcal{W}_j$ would live in the orthogonal complement of $\mathcal{W}_k$ if they do not share the same wavelets.

Def: Consider the nested family $\{\mathcal{V}_j\}_{j \in Z}$ starting from $\{0\}$ to L^2; i.e.,

$$\{0\} \subset ... \subset \mathcal{V}_{-2} \subset \mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset ... \subset L^2$$

Suppose there exists a *scaling function* $\varphi(x)$ such that $\varphi(2^j x - k)$ spans the space $\mathcal{V}_j$. These spaces definitely have a notion of *resolution*.

Furthermore, suppose a correponding wavelet family spans the *differences* between the various sets. For example, take $\mathcal{V}_0$ and $\mathcal{V}_1$. Then, by definition,

$$\mathcal{V}_1 = \mathcal{V}_0 \oplus \mathcal{W}_0$$

Moreover, by taking the finer set to $j \to \infty$,

$$L^2 = \mathcal{V}_0 \oplus \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus ...$$

Or, by also taking the coarser set to $j \to -\infty$

$$L^2 = \bigoplus_j \mathcal{W}_j,$$

as expected for the wavelet basis.

Remark: Noting that, for band-limited $\mathcal{V}_j$, $\mathcal{V}_{j+1}$ would be made up of a wider band and similarly for the wavelets. We see how the scaling function can take care of the coarser scales.

Thm: Choose the scaling function $\varphi_{j_0,k}$ and the corresponding wavelet family $\psi_{j,k}$. Then, any function $f \in L^2$ is given by

$$f(x) = \sum_{k\in Z} c_{j_0,k}\ \varphi_{j_0,,k}(x) + \sum_{j=j_0}^{\infty}\sum_{k\in Z} d_{j,k}\ \psi_{j,,k}(x),$$

where the scaling coefficients $c_{j_0,k}$ and the wavelet coefficients $d_{j,k}$ are obtained by the typical inner products.

Practical Multiresolution Analysis

In practice, we will need to set a cut-off for the scale j. Hopefully, our function f has a maximum frequency bounded above by the scale j_1. If so, then $f \in \mathcal{V}_{j_1}$, implying that

$$f(x) = \sum_{k\in Z} c_{j_1,k}\ \varphi_{j_0,,k}(x),$$

which would then imply that the multiresolution wavelet decomposition of f is given by

$$f(x) = \sum_{k\in Z} c_{j_0,k}\ \varphi_{j_0,,k}(x) + \sum_{j=j_0}^{j_1-1}\sum_{k\in Z} d_{j,k}\ \psi_{j,,k}(x),$$

for some convenient coarser scale $j_0 < j_1$.

Additionally, if our function is defined on a finite domain. Then, we can further simplify the above expansion. The simplest way is to periodize it over its region of support. Intuitively, one sees that a scaling function/wavelet with sufficiently-large $k > 0$ values will lead to the support of f and the support of the scaling function/wavelet being disjoint. Taking the support of f and the mother wavelet/scaling function to be $[0, 1]$ for simplicity. Therefore, we need only sum over the k-values where they have non-zero support.

$$f(x) = \sum_{k\subset[0,1]} c_{j_0,k}\ \varphi_{j_0,,k}(x) + \sum_{j=j_0}^{j_1-1}\sum_{k\subset[0,1]} d_{j,k}\ \psi_{j,,k}(x),$$

For arbitrary supports, we need only rescale f and the basis functions.

Filter Bank Multiresolution Analysis

Suppose we want to perform an MRA of $f \in L^2([0,1]) \cap \mathcal{V}_{j_1}$ down to the coarse scale j_0. To do this, we would need to perform $2^{j_0} + \sum_{j=j_0}^{j_1-1} 2^j = 2^{j_1}$ inner products for the scaling/wavelet coefficients. If we want to change our choice of coarse scale, we need not calculate all inner products again. The decomposition that we perform for a certain choice of a coarse scale j_0 contains most of the information we need to perform a decomposition or reconstruction at some other scale. In developing these ideas, the filter bank approach will follow.

Recall the nested structure of the subspaces $\{\mathcal{V}_j\}_{j\in Z}$ spanned by the scaling function. Starting with the scaling functions $\varphi(t-k)$ that spans $\mathcal{V}_0$, by noting that $\varphi(t) \in \mathcal{V}_1$, which is also spanned by $\varphi(2t-k)$, we can expand

$\varphi(t)$ in terms of the $\varphi(2t-k)$ functions:

$$\varphi(t) = \sum_k h(k)\, \sqrt{2}\, \varphi(2x-k).$$

Similarly, one obtains an expansion for the wavelet $\psi(t)$.

$$\psi(t) = \sum_k h_1(k)\, \sqrt{2}\, \varphi(2x-k),$$

where h_1 is closely related to h through the relation $\mathcal{V}_1 = \mathcal{V}_0 \oplus \mathcal{W}_0$ and the orthogonality of the scaling function/wavelets. In particular,

$$h_1(k) = (-1)^k h(1-k)$$

By plugging this into the scaling function expansion for f, one obtains the following multiresolution decomposition of the scaling and wavelet coefficients:

$$c_j(k) = \sum_m h(m-2k)\, c_{j+1}(m)$$
$$d_j(k) = \sum_m h_1(m-2k) c_{j+1}(m),$$

By thinking of the scaling function has being band-limited at low frequencies and the wavelets as having high frequencies. It is clear that $h(k)$ is acting as a low-pass filter and $h_1(k)$ is acting as a high-pass filter.

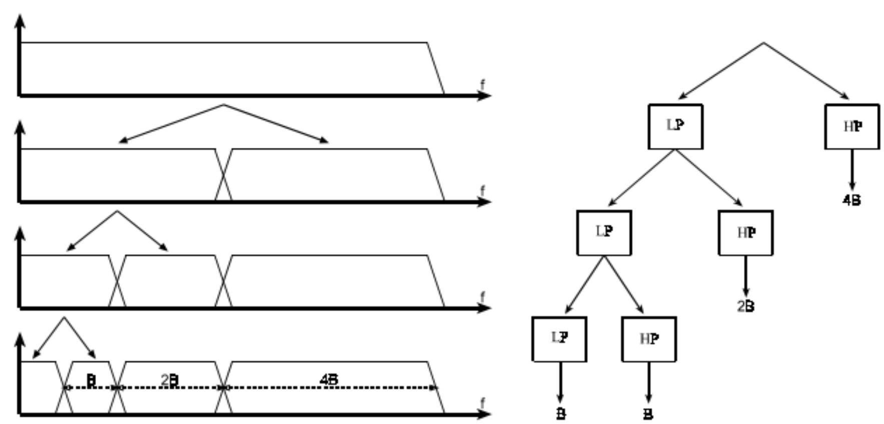
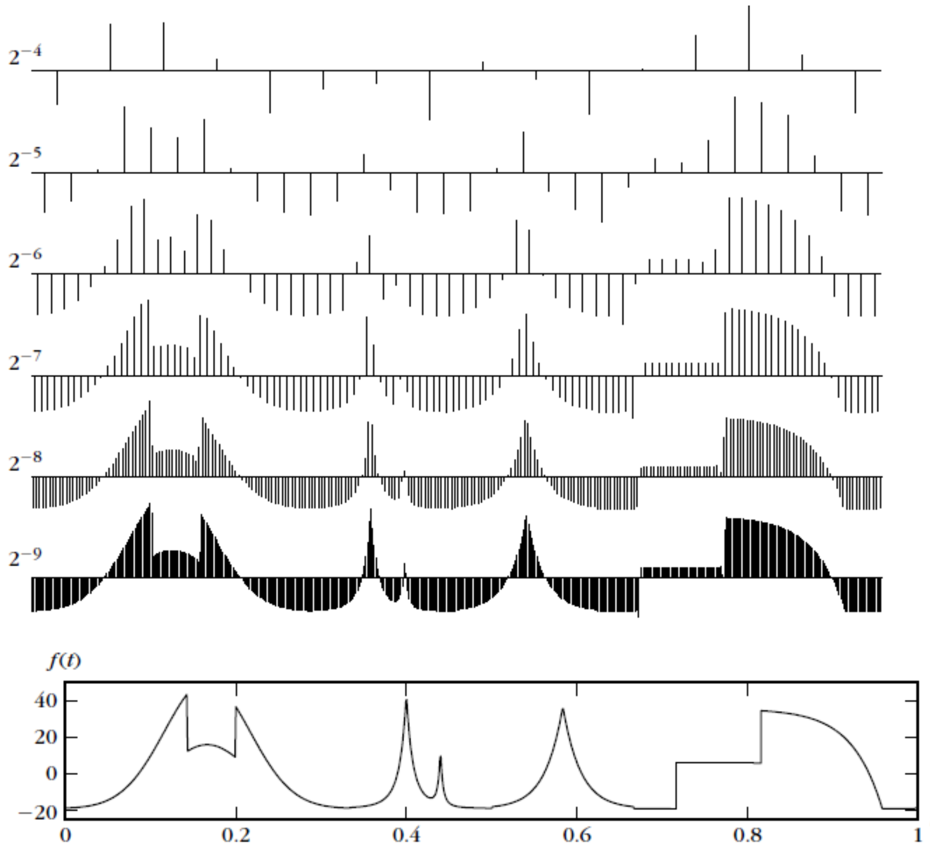


Figure 4
Splitting the signal spectrum with an iterated filter bank.

Note: As $j \to \infty$, $c_j(k) \sim \langle f, \delta_k \rangle = f(k)$, so we can, in principle, decompose all the information of a function into a coarse approximation and corrections at different scales.



— References: Daubechies ch. 3; Burrus ch. 2, 3, 9 ; "A Really Friendly Guide to Wavelets"