Three dimensional coordinates into two dimensional coordinates conversion

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Version 0.1.4 (very drafty paper)
(will definitivly be rewritten again)
June 24, 2015

Lemma On a piece of paper you see three coordinate axis pointing into three directions in space. In reality these vectors are two dimensional. Because they point into three directions on the paper, and not into the real space.

[Picture of a 3-D coordinate system with ijk-vectors on the axis pointing into three Directions]

In this document we will design a basis for the coordinate transformation. A basis is multiplied with the value of the coordinate to move to the correct new point.

Definition Let φ_n be the set of axis angles, one for each axis. The angles start at the same place, at the number zero. You have to arrange the x, y, and z axes like on a piece of paper around the unit circle by giving them the appropriate angles. All three angles start at the default at zero.

$$\varphi_n := \{\varphi_x, \varphi_y, \varphi_z\}$$

Definition Let e_n be the set of three two dimensional unit base vectors, namely $\vec{e}_x$,$\vec{e}_y$ and e_z. Other names are i, j or k for example, like on the picture of the coordinate system mentioned. The three vectors point into the three directions of the three axis. On a piece of paper they are two dimensional, because they point into three directions on the paper. The space being shown is what our brain completes, seeing three correct axis. The three base vectors represent exactly one unit of each axis. That is, why these vectors are called the unit vectors. They denote the unit of the local coordinate system.

$$e_n := \{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$$

This is the set of three unit vectors in set notation. To guess no numbers, its easier for us, for each vector, to go around the unit circle by the angles we already defined and to use cosine and sine for the correct x-distance and y-distance. For help, you should remember this parametrization of x and y from the unit circle.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$$

$$(x, y) = (r\cos\varphi, r\sin\varphi)$$

Definition

Modeling the three two dimensional base vectors with this information, we get the following three two dimensional base vectors.

$$\vec{e}_x := (r_x \cos(\varphi_x), r_x \sin(\varphi_x))^T = \begin{pmatrix} r_x \cos(\varphi_x) \\ r_x \sin(\varphi_x) \end{pmatrix}$$
$$\vec{e}_y := (r_y \cos(\varphi_y), r_y \sin(\varphi_y))^T = \begin{pmatrix} r_y \cos(\varphi_y) \\ r_y \sin(\varphi_y) \end{pmatrix}$$
$$\vec{e}_z := (r_z \cos(\varphi_z), r_z \sin(\varphi_z))^T = \begin{pmatrix} r_z \cos(\varphi_z) \\ r_z \sin(\varphi_z) \end{pmatrix}$$

One for each component of (x, y, z) By multiplying with, we move the points into the right pieces of direction. On the plane we use to point into three directions.

Remark.The values of r_x, r_y and r_z decide, how long one unit into each direction is. To preserve affine graphical transformations all three axes should have the same unit length, which can generally be enlarged or made smaller than unit length. By default the resulting vector of the cos and sin Terms has unit length, if you dont multiply with r_x, r_y and r_z.

The other help we take is from the orthogonal base formula . The sum of the basis multiplied with the coordinates is nothing new. Literature and scripts dont show. I will tell you. But before, here is the orthogonal basis application first. [multiplying and elonginating a vector, little information is planned]

$$\vec{v} = \sum_n \vec{e}_n \vec{x}_n$$

Each (x, y, z) coordinate has to be multiplied for the new (x', y') with its corresponding term of the unit vectors in the matrix. That means, to sum the products with (x, y, z) and the cos terms up for x' and to sum the products of (x, y, z) and the sin terms up for y'. This is the same as imagining walking left and right with $\cos\varphi$ and up and down with $\sin\varphi$. Or mathematically adding positive or negative values.

$$\begin{pmatrix} \text{x'} \\ \text{y'} \end{pmatrix} = \begin{pmatrix} x r_x \cos(\varphi_x) + y r_y \cos(\varphi_y) + z r_z \cos(\varphi_z) \\ x r_x \sin(\varphi_x) + y r_y \sin(\varphi_y) + z r_z \sin(\varphi_z) \end{pmatrix}$$

Definition Let A be the matrix containing the three two dimensional unit vectors in order, one each column. You get a 2x3 matrix. With the bases in the colums. A 2x3 Matrix, which i call the Gerhold Matrix to distinguish it from other matrices, making sure it has three times $\cos\varphi_n$ and $\sin\varphi_n$ inside.

$$A := \begin{pmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \end{pmatrix} = \begin{pmatrix} r_x \cos(\varphi_x) & r_y \cos(\varphi_y) & r_z \cos(\varphi_z) \\ r_x \sin(\varphi_x) & r_y \sin(\varphi_y) & r_z \sin(\varphi_z)) \end{pmatrix}$$

Theorem *(Fundamental Theorem of converting 3-D Points into 2-D Points)*:

If you multiply the Gerhold matrix of the three two-dimensional unit vectors with the three-coordinate points (x, y, z), the result is a two coordinate point, (x', y'). This point (x', y') is the correct point on the two dimensional plane, representing the point from the three dimensional coordinate system we would like to display.

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Applying the operator performs the following operation

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$$

$$= \begin{pmatrix} xr_x\cos(\varphi_x) + yr_y\cos(\varphi_y) + zr_z\cos(\varphi_z) \\ xr_x\sin(\varphi_x) + yr_y\sin(\varphi_y) + zr_z\sin(\varphi_z) \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Proof

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} xr_x\cos(\varphi_x) + yr_y\cos(\varphi_y) + zr_z\cos(\varphi_z) \\ xr_x\sin(\varphi_x) + yr_y\sin(\varphi_y) + zr_z\sin(\varphi_z) \end{pmatrix}$$

Corollary *(Converting any Dimensions down to less dimensions)*:

The theorem can be extended to more dimensions, for example can four two-dimensional vectors represent a 4-D space on the 2-D plane. They get converted into the correct 2-D points. For Example, if you use a 2x4 matrix and convert all points at each instance of t you have a moving object into the direction of the fourth base vector.

$$A := \begin{pmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z & \vec{e}_t \end{pmatrix}$$

$$= \begin{pmatrix} r_x\cos(\varphi_x) & r_y\cos(\varphi_y) & r_z\cos(\varphi_z) & r_t\cos(\varphi_t) \\ r_x\sin(\varphi_x) & r_y\sin(\varphi_y) & r_z\sin(\varphi_z) & r_t\sin(\varphi_t) \end{pmatrix}$$

Here the basis is four times two dimensions.

$$A\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \sum_n \vec{e}_n\vec{x}_n = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Proof:

$$A\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} xr_x\cos(\varphi_x) + yr_y\cos(\varphi_y) + zr_z\cos(\varphi_z) + zr_t\cos(\varphi_t) \\ xr_x\sin(\varphi_x) + yr_y\sin(\varphi_y) + zr_z\sin(\varphi_z) + zr_t\sin(\varphi_t) \end{pmatrix}$$

$$= x\vec{e}_x + y\vec{e}_y + z\vec{e}_z + t\vec{e}_t = \sum_n \vec{e}_n\vec{x}_n = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

The same method can be used to convert any other number of dimensions to the xy-plane. If you know the base vectors for the other dimensions you can convert them down as well.

References: