A Brief-ish Introduction to Vector Math

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October 25, 2012

In mathematics, physics, and engineering, the term ***vector*** (more precisely, ***Euclidean vector***) is often used to refer to quantities that have both a ***magnitude*** and a ***direction***, such as the acceleration of or net force acting upon a physical object. Geometrically speaking, a vector may be represented as a directed line segment in a Euclidean space; in this case, the direction of this line segment represents the direction of the corresponding vector, while its length represents the vector's magnitude.

The vector's ***dimension***, meanwhile, is the dimension of the Euclidean space in which the corresponding line segment appears. If the line segment appears in one-dimensional Euclidean space (that is to say, $\mathbb{R}$, the real number line), its vector is a vector in one dimension; if the line segment appears in two-dimensional Euclidean space ($\mathbb{R}^2$, the real coordinate plane), its vector is a vector in two dimensions; and, generally, if the line segment appears in n-dimensional Euclidean space ($\mathbb{R}^n$), its vector is a vector in n dimensions.

Some vectors, known as ***bound vectors***, are associated with particular initial and terminal points (those of their corresponding line segments in Euclidean space). We will only be concerned with ***free vectors***, for which only the magnitude and direction of a vector are regarded as important; all bound vectors with the same magnitude and direction are represented by the same free vector, regardless of their initial and terminal points.

By restricting our notion of vectors to free vectors, we gain the ability to uniquely represent n-dimensional vectors in a convenient (and mathematically conventional) algebraic manner: as ordered collections of n real numbers (such as $[3.6, 10, 0]$ if $n = 3$). These collections of numbers constitute the ***coordinates*** of the terminal point of a particular bound vector: the bound vector whose initial point is the ***origin*** of $\mathbb{R}^n$ and whose magnitude and direction are equal to that of the free vector we wish to represent. In general, then, if $v \in \mathbb{R}^n$, $v = [v_1, v_2, ..., v_n]$ where $v_1, v_2, ..., v_n \in \mathbb{R}$ are the terminal coordinates of the aforementioned bound vector. Using this notation and given that it provides a unique representation of each free vector, v and another vector $w \in \mathbb{R}^n$ are ***equivalent*** if and only if all of the equations $v_1 = w_1, v_2 = w_2, ..., v_n = w_n$ hold true.

Numerous operations are defined for vectors according to the rules of vector algebra. The simplest of these are ***scalar multiplication*** of a vector and ***vector addition*** of two vectors.

Scalar multiplication is defined as follows: let $a \in \mathbb{R}$ be a ***scalar*** (a quantity with a magnitude but no direction) and let $v \in \mathbb{R}^n$ be a vector. Then $av = [av_1, av_2, ..., av_n]$; that is, scalar multiplication of a and v produces an n-dimensional vector whose entries are the products of a and each of the corresponding entries of v. Geometrically, this means that av's direction is the same as v's, but that its magnitude is a times as much as v's. (Note that the first part of this statement is only true when a is positive. If a is zero, then av produces the ***zero vector***, which has no particular direction; if a is negative, then av's direction is the opposite of v's.) Scalar multiplication is commutative, associative, and distributive over vector addition (the next subject we will discuss).

Vector addition is defined as follows: let $v, w \in \mathbb{R}^n$ be two n-dimensional vectors. Then $v + w = [v_1 + w_1, v_2 + w_2, ..., v_n + w_n]$; that is, vector addition of v and w produces an n-dimensional vector whose entries are the sums of each of the corresponding entries of v and w. Geometrically, this means that $v + w$ is the vector represented as the coordinates of the terminal point of w if w is treated as a bound vector and translated such that its initial point is the terminal point of v. If v and w have the same direction, $v + w$ has the same direction as both; if w is the zero vector, $v + w = v$ (making the zero vector the ***additive identity*** element for vectors); and if $w = -v$ (that is, if w is the result of multiplying v by the scalar -1), $w + v$ is the zero vector (making $-w$ the ***additive inverse*** of v). Vector addition is commutative and associative.

The rules governing multiplication of vectors by other vectors are more complicated, mainly because there is more than one way to multiply a vector with another vector. In particular, we will be concerned with the ***dot product*** of two vectors (also known as their ***scalar product*** because its result is a scalar) and the ***cross product*** of two vectors (also known as their ***vector product*** because its result is a vector).

The dot product is defined as follows: let $v, w \in \mathbb{R}^n$ be two n-dimensional vectors. Then $v \cdot w = v_1 w_1 + v_2 w_2 + ... + v_n w_n$; that is, the dot product of v and w is the sum of the products of each of the corresponding entries of v and w. The resulting scalar value might be thought of (loosely) as representing a measure of the extent to which v and w have the same direction; if v and w are ***orthogonal*** (perpendicular) to each other, $v \cdot w = 0$, whereas if the angle between v and w is acute, $v \cdot w > 0$, and if it is obtuse, $v \cdot w < 0$, with the absolute value of $v \cdot w$ increasing with the acuteness or obtuseness of the angle between v and w. The dot product is commutative, associative, and distributive over vector addition.

The cross product, meanwhile, is only defined for vectors in $\mathbb{R}^3$ and $\mathbb{R}^7$ (not for vectors generally); we will only concern ourselves with the cross product of three-dimensional vectors. Let $v = [v_1, v_2, v_3]$ and $w = [w_1, w_2, w_3]$ be vectors in $\mathbb{R}^3$. Then $v \times w = [v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1]$. Equivalently, $v \times w$ may be represented as the ***determinant*** of a formal matrix (where i, j, and k represent the vectors $[1, 0, 0], [0, 1, 0]$, and $[0, 0, 1]$ respectively):

$$\begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

The vector resulting from $v \times w$ has a useful property: it is orthogonal to both v and w (with the consequence that taking the dot product of either v or w and $v \times w$ will always produce 0, regardless of the actual values of v and w). It is also notable that the cross product is distributive over vector addition; however, unlike the dot product, it is not associative and it is not commutative. Instead, it is ***anticommutative***; that is, $v \times w = -w \times v$.

Finally, the Euclidean ***norm*** of a vector $v \in \mathbb{R}^n$ is a scalar value representing its magnitude, i.e., the length of the corresponding line segment in $\mathbb{R}^n$. We may calculate the norm $||v||$ using the formula $||v|| = \sqrt{v_1^2 + v_2^2 + ... + v_n^2}$, which follows readily from an n-dimensional generalization of the Pythagorean theorem. We may then use this norm to ***normalize*** v by dividing v by $||v||$ (or, more precisely, multiplying v by the scalar value $\frac{1}{||v||}$). As long as v is not the zero vector, this will result in a ***unit vector*** for v: a vector with the same direction as v, but with a magnitude of 1.

Note that while we have restricted our discussion to Euclidean vectors, an alternative definition of vector identifies vectors as elements of what linear algebra calls a ***vector space***. The set of Euclidean vectors constitutes a vector space; thus, the definition of vectors as elements of a vector space represents a generalization of the definition of vectors we have used. Many (though not all) of the concepts we have discussed likewise generalize readily when vectors are redefined thus.