# COMP-2310 Formula Sheet
# Midterm Test 2

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# Chapter 3

**Definition** (Principle of Extension)**.** *Two sets $A$ and $B$ are **equal**, denoted by $A = B$, iff $(\forall x)(x \in A \Leftrightarrow x \in B)$*

**Lemma** (3.2.1)**.** *Let $A$, $B$, $C$ be sets.*

*(i)* $A = A$

*(ii)* *If $(A = B)$, then $B = A$*

*(iii)* *If $(A = B)$ and $(B = C)$, then $A = C$*

**Lemma** (3.2.2)**.** $A \neq B$ *iff* $(\exists x)(x \in A \wedge x \notin B) \vee (\exists x)(x \notin A \wedge x \in B)$

**Corollary** (3.2.2.1)**.** *Let $A$ and $B$ be sets.*
$(\exists x)(x \in A \wedge x \notin B) \Rightarrow A \neq B$

**Definition** (Subset)**.** *Let $A$, $B$ be two sets. $A$ is a **subset** of $B$ or $B$ **includes** $A$, denoted by $A \subseteq B$, iff $(\forall x)(x \in A \Rightarrow x \in B)$.*
*$A$ is a **proper subset** of $B$, denoted by $A \subset B$, iff $A \subseteq B$ and $A \neq B$.*

**Lemma** (3.2.3)**.** *Let $A$, $B$, $C$ be sets.*

*(i)* $A \subseteq A$

*(ii)* *If $A \subseteq B$ and $B \subseteq A$, then $A = B$*

*(iii)* *If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$*

**Lemma** (3.2.4)**.** *Let $A$, $B$ be sets. If $A = B$, then $(A \subseteq B) \wedge (B \subseteq A)$*

**Lemma** (3.2.5)**.** $A \nsubseteq B \Leftrightarrow (\exists x)(x \in A \wedge x \notin B)$

**Lemma** (3.2.6)**.** *If $A \subset B$, then $(\exists x)(x \in B \wedge x \notin A)$*

**Definition** (Principle of Specification)**.** *For every set $A$ and every formula $S(x)$, there exists a set $B$ whose elements are exactly those elements of $A$ for which $S(x)$ is true. The set $B$ is denoted by: $\{x | x \in A \wedge S(x)\}$ or $\{x \in A | S(x)\}$*

**Definition** (Power Set)**.** *Let $A$ be a set. The **power set** of $A$, denoted by $\mathcal{P}(A)$, is the set $\{X | X \subseteq A\}$*

# Chapter 4

**Definition** (Union). *Let $A$, $B$ be two sets. The **union** of $A$ and $B$ is the set $A \cup B = \{x | x \in A \vee x \in B\}$*

**Theorem** (4.1.1). *Let $A$, $B$ be sets.*

*(i)* $A \cup \emptyset = A$

*(ii)* $A \cup A = A$

*(iii)* $A \cup B = B \cup A$

*(iv)* $(A \cup B) \cup C = A \cup (B \cup C)$

*(v)* $A \subseteq B$ *iff* $A \cup B = B$

**Definition** (Intersection). *Let $A$, $B$ be two sets. The **intersection** of $A$ and $B$ is the set $A \cap B = \{x | x \in A \wedge x \in B\}$*

**Theorem** (4.2.2). *Let $A$, $B$ be sets.*

*(i)* $A \cap \emptyset = \emptyset$

*(ii)* $A \cap A = A$

*(iii)* $A \cap B = B \cap A$

*(iv)* $(A \cap B) \cap C = A \cap (B \cap C)$

*(v)* $A \subseteq B$ *iff* $A \cap B = A$

**Corollary** (4.2.2.1). *Let $A$, $B$ be any two sets. Then, $A \cap B \subseteq A \subseteq A \cup B$*

**Theorem** (4.2.3). *Let $A$, $B$, $C$ be sets.*

*(i)* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

*(ii)* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

**Definition** (Disjoint). *Two sets $A$ and $B$ are **disjoint** if $A \cap B = \emptyset$*

**Definition** (Relative Complement). *Let $A$, $B$ be two sets. The **relative complement** of $B$ in $A$ is the set $A - B = \{x | x \in A \wedge x \notin B\}$*

**Theorem** (4.3.4)**.** *Let $A$, $B$, $C$ be sets.*

*(i)* $A \subseteq B$ *iff* $A - B = \emptyset$

*(ii)* $A - (A - B) = A \cap B$

*(iii)* $A \cap (B - C) = (A \cap B) - (A \cap C)$

**Definition** (Absolute Complement)**.** *The* ***(relative) Universal set****, denoted by* $\mathbf{U}$*, is a set which contains every object (thereby includes every set) in our discussion. Let* $A$ *be any set such that* $A \subseteq \mathbf{U}$*. The* ***absolute complement*** *of* $A$ *is the set* $\overline{A} = \mathbf{U} - A$

**Lemma** (4.3.5)**.** *Let* $A \subseteq \mathbf{U}$*. Then,* $x \in \overline{A} \Leftrightarrow x \notin A$

**Theorem** (4.3.6 DeMorgan's Theorem)**.** *Let $A$, $B$ be any two subsets of* $\mathbf{U}$

*(i)* $\overline{A \cup B} = \overline{A} \cap \overline{B}$

*(ii)* $\overline{A \cap B} = \overline{A} \cup \overline{B}$

**Theorem** (4.3.7)**.** *Let $A$, $B$ be any two subsets of* $\mathbf{U}$*.*

*(i)* $\overline{\overline{A}} = A$

*(ii)* $\overline{\emptyset} = \mathbf{U}$ *and* $\overline{\mathbf{U}} = \emptyset$

*(iii)* $A \cap \overline{A} = \emptyset$ *and* $A \cup \overline{A} = \mathbf{U}$

*(iv)* $A \subseteq B$ *iff* $\overline{B} \subseteq \overline{A}$

*(v)* $A - B = A \cap \overline{B}$

**Definition** (Ordered Pair)**.** *The* ***ordered pair*** *of $a$ and $b$ is the set*
$(a, b) = \{\{a\}, \{a, b\}\} = \{x | x = \{a\} \vee x = \{a, b\}\}$

**Lemma** (4.4.8)**.** $(a, b) = (x, y)$ *implies that* $a = x$ *and* $b = y$

**Lemma** (4.4.9)**.** $a = c \wedge b = d \Rightarrow (a, b) = (c, d)$

**Definition** (Cartesian Product)**.** *Let $A$, $B$ be two sets. The* ***Cartesian Product*** *of $A$ and $B$ is the set* $A \times B = \{x | (\exists a)(\exists b)(a \in A \wedge b \in B \wedge x = (a, b))\} = \{(a, b) | a \in A \wedge b \in B\}$

**Lemma** (4.5.10)**.** *Let $A$, $B$, $X$ and $Y$ be sets.*

*(i)* $(A \cup B) \times X = (A \times X) \cup (B \times X)$

*(ii)* $(A \cap B) \times X = (A \times X) \cap (B \times X)$

*(iii)* $(A - B) \times X = (A \times X) - (B \times X)$

*(iv)* $(A \cap B) \times (X \cap Y) = (A \times X) \cap (B \times Y)$

*(v)* $(A = \emptyset \vee B = \emptyset) \Leftrightarrow A \times B = \emptyset$

**Definition** (Union of a Collection of Sets)**.** *Let $\mathcal{C}$ be a collection of sets. The* ***union of*** *$\mathcal{C}$ is the set:*

$$\bigcup_{X \in \mathcal{C}} X = \{x | (\exists X)(X \in \mathcal{C} \wedge x \in X)\} = \{x | x \in X \text{ for some } X \in \mathcal{C}\}$$

**Definition** (Intersection of a Collection of Sets)**.** *Let $\mathcal{C}$ be a collection of sets such that $\mathcal{C} \neq \emptyset$. The* ***intersection of*** *$\mathcal{C}$ is the set:*

$$\bigcap_{X \in \mathcal{C}} X = \{x | (\forall X)(X \in \mathcal{C} \Rightarrow x \in X)\} = \{x | x \in X \text{ for all } X \in \mathcal{C}\}$$

**Lemma.** *A:* $(X \cap Y) \cup X = X$

# Chapter 5

**Definition** (Relation). *A **relation** is a set of ordered pairs. Specifically, a set $R$ is a relation if* $(\forall x)(x \in R \Rightarrow (\exists a)(\exists b)x = (a, b))$

**Definition** (Domain & Range of Relations). *Let $R$ be a relation. The **domain** of $R$ is the set $Dom(R) = \{x|(\exists y)(x, y) \in R\}$ and the **range** of $R$ is the set* $Ran(R) = \{y|(\exists x)(x, y) \in R\}$

**Definition** (Cartesian Product of Relations). *Let $R \subseteq X \times Y$, we say that $R$ is a **relation from X to Y**. In particular, when $X = Y$, (ie. $R \subseteq X \times X$), we say that $R$ is a **relation in X**.*

**Lemma** (5.1.1). *Let $R$ be a relation from $X$ to $Y$. Then $Dom(R) \subseteq X$ and $Ran(R) \subseteq Y$*

**Definition** (Properties of Relation). *Let $R$ be a relation in a set $X$*

*$R$ is **reflexive** if* $(\forall x \in X)(x, x) \in R$;

*$R$ is **irreflexive** if* $(\forall x \in X)(x, x) \notin R$;

*$R$ is **symmetric** if* $(x, y) \in R \Rightarrow (y, x) \in R$;

*$R$ is **antisymmetric** if* $(x, y) \in R \wedge (y, x) \in R \Rightarrow x = y$;

*$R$ is **asymmetric** if* $\forall x, y \in X,\ ((x, y) \in R \wedge (y, x) \in R)$, *or equivalently,*
$\forall x, y \in X, (x, y) \in R \Rightarrow (y, x) \notin R$;

*$R$ is **transitive** if* $(x, y) \in R \wedge (y, z) \in R \Rightarrow (x, z) \in R$.

**Lemma** (5.2.1). *Let $R$ be a relation in $X$. If $(\exists x \in X)(x, x) \in R$, then $R$ is not asymmetric.*

**Definition** (Equivalence Relation). *A relation $R$ in a set $X$ is an **equivalence relation** if $R$ is reflexive, symmetric and transitive.*

**Definition** (Equivalence Class). *Let $R$ be an equivalence relation in a set $X$. For each $x \in X$, the **equivalence class of x w.r.t. R** is the set*
$[x]/R = \{y|y \in X \wedge (x, y) \in R\} = \{y \in X|(x, y) \in R\}$ *or equivalently,*
$[x]/R = \{y|(x, y) \in R\}$

**Definition** (Collection of Equivalence Classes). *Let $R$ be an equivalence relation in a set $X$. **The collection of equivalence classes w.r.t. R** is the set:* $[X]/R = \{S|(\exists x)(x \in X \wedge S = [x]/R)\} = \{[x]/R|x \in X\}$

**Definition** (Partition of a Collection of Sets). *Let $X$ be a set. A collection of sets $\mathcal{C}$ is a **partition** of $X$ if*

*(i)* $\bigcup_{S\in\mathcal{C}} S = X$, *and*

*(ii)* $\forall S \in \mathcal{C}, S \neq \emptyset$, *and*

*(iii)* $\forall S, S' \in \mathcal{C}, S \neq S' \Rightarrow S \cap S' = \emptyset$

**Lemma** (5.3.1). *If $R$ is an equivalence relation in $X$, then the collection of equivalence classes $[X]/R$ is a partition of $X$.*

**Definition** (Induced Relation). *Let $\mathcal{C}$ be a partition of $X$. The **relation induced by** $\mathcal{C}$, denoted by $X/\mathcal{C}$, is a relation in $X$, such that:*
$X/\mathcal{C} = \{(x, y) \in X \times X | (\exists S \in \mathcal{C})(x \in S \wedge y \in S)\} = \{(x, y) | (x \in X \wedge y \in X) \wedge (\exists S \in \mathcal{C})(x \in S \wedge y \in S)\}$

**Lemma** (5.3.2). *Let $R$ be an equivalence relation in $X$. The relation induced by the collection of equivalence classes $[X]/R$ is identical to $R$. In other words, $X/([X]/R) = R$*

**Lemma** (5.3.3). *Let $\mathcal{C}$ be a partition of $X$. The induced relation $X/\mathcal{C}$ is an equivalence relation in $X$.*

**Lemma** (5.3.4). *Let $\mathcal{C}$ be a partition of $X$. Then, $A \in \mathcal{C} \wedge a \in A \Rightarrow A = [a]/(X/\mathcal{C})$*

**Lemma** (5.3.5). *Let $\mathcal{C}$ be a partition of $X$. Then, $[X]/(X/\mathcal{C}) = \mathcal{C}$*

**Theorem** (5.3.6). *Properties of Equivalence Relations*

*(i) If $R$ is an equivalence relation in $X$, then the set of equivalence classes $[X]/R$ is a partition of $X$ that induces the relation $R$, and*

*(ii) If $\mathcal{C}$ is a partition of $X$, then the induced relation $X/\mathcal{C}$ is an equivalence relation in $X$ whose set of equivalence classes is identical to $\mathcal{C}$.*

**Definition** (Partial Order). *A relation $R$ in a set $X$ is a **partial order** if $R$ is reflexive, antisymmetric and transitive.*

**Definition** (Partially Ordered Set). *A **partially ordered set** is an ordered pair $(X, \precsim)$ in which $X$ is a set and $\precsim$ is a partial order in $X$. If $\forall x, y \in X, (x \precsim y) \vee (y \precsim x)$, then $\precsim$ is called a **total order** and $(X, \precsim)$ is a **totally ordered set** or a **chain**.*

**Definition** (Strict Order). *A relation $R$ in a set $X$ is a **strict order** if $R$ is asymmetric and transitive.*

**Lemma** (5.4.1). *Let $\precsim$ be a partial order in $X$, and $X_=$ be the relation of equality in $X$. The relation $\precsim - X_=$ is a strict order in $X$.*

**Definition** (Strict Order Corresponding to a Partial Order)**.** *Let $\precsim$ be a partial order and $X_=$ is the relation of equality in $X$. Then $\precsim - X_=$ is called the* ***strict order corresponding to*** *$\precsim$.*

**Lemma** (5.4.2)**.** *Let $\prec$ be a strict order in $X$, and $X_=$ be the relation of equality in $X$. The relation $\prec \cup X_=$ is a partial order in $X$.*

**Definition** (Smallest, Minimal, Largest, Maximal Elemnt of a Partially Ordered Set)**.** *Let $(X, \precsim)$ be a partially ordered set. An element $a \in X$ is called the* ***smallest element*** *or* ***least element*** *of $X$ if $\forall x \in X, a \precsim x$. An element $a \in X$ is called the* ***largest element*** *or* ***greatest element*** *of $X$ if $\forall x, \in X, x \precsim a$. An element $a \in X$ is called the* ***minimal element*** *of $X$ if $\;(\exists x \in X) x \prec a$ where $\prec$ is the strict order corresponding to $\precsim$. An element $a \in X$ is called a* ***maximal element*** *of $X$ if $\;(\exists x \in X) a \prec x$.*

**Lemma** (5.4.3)**.** *The smallest (largest, respectively) element is unique, if exists.*

**Lemma** (5.4.4)**.** *Let $(X, \precsim)$ be a partially ordered set. An element $a$ is a minimal element in $X$ iff $(\forall x \in X)(x \precsim a \Rightarrow x = a)$*

**Definition** (Well Ordering Property)**.** *A partially ordered set $(X, \precsim)$ is a* ***well-ordered set*** *and $\precsim$ is a* ***well ordering*** *if every non-empty subset of $X$ has a smallest element. Let $\mathcal{N}_0$ be the set of non-negative integers. ie. $\mathcal{N}_0 = \mathcal{N} \cup \{0\}$, where $\mathcal{N}$ is the set of positive integers.*

*Every non-empty set of nonnegative integers has a smallest element. Formally,*

$$(\forall X)((X \subseteq \mathcal{N}_0 \land X \neq \emptyset) \Rightarrow (\exists x \in X)(\forall y)(y \in X \Rightarrow x \leq y))$$

**Definition** (Inverse of Relation)**.** *Let $R$ be a relation in a set $X$. The* ***inverse*** *of $R$ is the relation $R^{-1} = \{(y, x) | (x, y) \in R\}$*

**Definition** (Composition of Relation)**.** *Let $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$. The* ***composition*** *of $R_1$ and $R_2$ is the relation of $R_1 \circ R_2 = \{(x, z) \in A \times C | (\exists y \in B)((x, y) \in R_1 \land (y, z) \in R_2)\}$*

**Theorem** (5.6.5)**.** *Let $R$, $S$ and $T$ be relations.*

*(i)* $(R^{-1})^{-1} = R$

*(ii)* $\circ$ *is not commutative*

*(iii)* $(R \circ S) \circ T = R \circ (S \circ T)$ *where* $R \subseteq X \times Y, S \subseteq Y \times Z, T \subseteq Z \times W$

*(iv)* $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$

*(v)* $R \subseteq S \Rightarrow R^{-1} \subseteq S^{-1}$