VECTORS IN EUCLIDEAN SPACES

Outlines

Consider the Eclidean space $\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n) | \, x_1, x_2, \cdots, x_n \in \mathbb{R}\}$ where its element is called a vector, $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. We have The Operations on Vectors in $\mathbb{R}^\ltimes$

Let $\mathbf{u} = (u_1, u_2, \cdots, u_n), \mathbf{v} = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$. We define

- The relation "equality" : $\mathbf{u} = v$ if $u_i = v_i$, for all $i = 1, 2, \cdots, n$.
- The algebraic operations :
  $u + v = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$, and
  $\alpha\mathbf{u} = (\alpha u_1, \alpha u_2, \cdots, \alpha u_n)$, for all $\alpha \in \mathbb{R}$.

Combining these two operations, with $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in \mathbb{R}^n$ and $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{R}$, the vector is called a linear combination of with coefficients $\alpha_1, \alpha_2, \cdots, \alpha_k$.

Vector

- The dot (scalar) product :

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_m v_m = \sum_{i=1}^{m} u_i v_i \ .$$

- The norm (length)

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{v_1^2 + v_2^2 + \cdots + v_m^2} = \sqrt{\sum_{i=1}^{m} v_i^2}$$

A vector of norm 1 is called a unit vector. Recall that, for any nonzero vector $v \in \mathbb{R}^n$, the vector $\mathbf{u} = \dfrac{1}{\|\mathbf{v}\|}\mathbf{v} \equiv \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector.

Outlines

Basis and Orthogonal Basis

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ be a set of vectors in $\mathbb{R}^n$. Consider the linear combination

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k = 0 \tag{1}$$

in k unknowns $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{R}$. With $\mathbf{u}_i = (a_{1i}, a_{2i}, \cdots, a_{ni})$, for $i = 1, 2, \cdots, k$, (1) gives a homogeneous system of linear equations

$$\left\{\begin{array}{ccccccccc} a_{11}\alpha_1 & + & a_{12}\alpha_2 & + & \cdots & + & a_{1k}\alpha_k & = & 0 \\ a_{21}\alpha_1 & + & a_{22}\alpha_2 & + & \cdots & + & a_{2k}\alpha_k & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{n1}\alpha_1 & + & a_{n2}\alpha_2 & + & \cdots & + & a_{nk}\alpha_k & = & 0 \end{array}\right. \tag{2}$$

If the system (2) has only the trivial solution, $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, we say that S is linearly independent. Otherwise, S is called linearly dependent.

A linearly independent set $S = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ in $\mathbb{R}^n$ is called a basis for $\mathbb{R}^n$.
A set of nonzero vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ is called orthogonal if $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$, for all $i, j = 1, 2, \cdots, k$, $i \neq j$. It is clear that an orthogonal set of vectors is linearly independent. An orthogonal set of vectors is called orthonormal is it consists of unit vector, i.e., $\|\mathbf{u}_i\| = 1$, for all $i = 1, 2, \cdots, k$.
An orthogonal set (orthonormal set, resp.) $S = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ in $\mathbb{R}^n$ is called an orthogonal basis (orthonormal basis, resp) for $\mathbb{R}^n$.

Outlines

GRAM-SCHMIDT PROCESS

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ be an orthogonal set in $\mathbb{R}^n$. We have
Theorem 1. For each vector $\mathbf{v} \in \mathbb{R}^n$, let

$$\mathbf{u}_{k+1} = \mathbf{v} - \frac{\langle \mathbf{v}, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 - \frac{\langle \mathbf{v}, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 - \cdots - \frac{\langle \mathbf{v}, \mathbf{u}_k \rangle}{\|\mathbf{u}_k\|^2}\mathbf{u}_k \qquad (3)$$

Then $\langle \mathbf{u}_{k+1}, \mathbf{u}_i \rangle = 0$, for all $i = 1, 2, \cdots, n$.
Proof.

$$\begin{gathered}\langle u_{k+1}, u_1 \rangle = \left\langle v - \tfrac{\langle v, u_1 \rangle}{\|u_1\|^2} u_1 - \tfrac{\langle v, u_2 \rangle}{\|u_2\|^2} u_2 - \cdots - \tfrac{\langle v, u_k \rangle}{\|u_k\|^2} u_k, u_1 \right\rangle \\ = \langle v, u_1 \rangle - \tfrac{\langle v, u_1 \rangle}{\|u_1\|^2} \langle u_1, u_1 \rangle - \tfrac{\langle v, u_2 \rangle}{\|u_2\|^2} \langle u_2, u_1 \rangle - \cdots - \tfrac{\langle v, u_k \rangle}{\|u_k\|^2} \langle u_k, u_1 \rangle \\ = \langle v, u_1 \rangle - \langle v, u_1 \rangle = 0\end{gathered}$$

since $\langle u_1, u_1 \rangle = \|u_1\|^2$ and $\langle u_i, u_1 \rangle = 0$, for $i = 2, \cdots, k$. The equalities $\langle u_{k+1}, u_j \rangle = 0$, $j = 2, \cdots, k$, are proved similarly.

GRAM-SCHMIDT PROCESS

Remark.

If $\mathbf{u}_{n+1} \neq 0$ in (3), then $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k, \mathbf{v}\}$ is linearly dependent.
Using **Theorem 1**, we arrive at the following algorithm to convert a linearly independent set $S = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ into an orthogonal set $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$.

Gram-Schmidt Process

Continue the following k steps:

Step 1. Let $v_1 = u_1$.

Step 2. Let $v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1$.

Step 3. Let $v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_2, v_2\rangle}{\|v_2\|^2} v_2$

Step 4. Let $v_4 = u_4 - \frac{\langle u_4, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_4, v_2\rangle}{\|v_2\|^2} v_2 - \frac{\langle u_4, v_3\rangle}{\|v_3\|^2} v_3$.

until Step k.

Moreover, we can get an orthonormal set $\{q_1, q_2, \cdots, q_n\}$ from S by letting

$$q_i = \frac{v_i}{\|v_i\|}, i = 1, 2, \cdots, n$$

Remark

If $\{u_1, u_2, \cdots, u_n\}$ is linearly dependent, there exists $i = 1, 2, \cdots, n$ such that $v_i = 0$.

In this case, the Gram-Schmidt process finishes after Step i.

Examples

Using the Gram-Schmidt process, construct an orthonormal set from the following linearly independent sets $\{u_1 = (1, 0, 1), u_2 = (1, 1, 0)\}$

Step 1. $v_1 = u_1 = (1, 0, 1)$.

Step 2.

$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1 = (1, 1, 0) - \frac{(1)(1)+(1)(0)+(0)(1)}{1^2+0^2+1^2} (1, 0, 1) = \left(\frac{1}{2}, 1, -\frac{1}{2}\right)$

We get the orthogonal set $\left\{v_1 = (1,0,1), v_2 = \left(\frac{1}{2}, 1, -\frac{1}{2}\right)\right\}$. Now,
let $q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{1^2+0^2+1^2}}(1,0,1) = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$,
$q_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2+1^2+\left(-\frac{1}{2}\right)^2}}\left(\frac{1}{2}, 1, -\frac{1}{2}\right) = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$
We have the orthonormal set
$q_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2+1^2+\left(-\frac{1}{2}\right)^2}}\left(\frac{1}{2}, 1, -\frac{1}{2}\right) = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$.

Example

Using the Gram-Schmidt process, construct an orthonormal set from the following linearly independent sets
$\{u_1 = (1,0,1), u_2 = (1,1,0), u_3 = (0,1,1)\}$

Step 1. $v_1 = u_1 = (1, 0, 1)$.
Step 2.
$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1 = (1, 1, 0) - \frac{(1)(1)+(1)(0)+(0)(1)}{1^2+0^2+1^2} (1, 0, 1) = \left(\frac{1}{2}, 1, -\frac{1}{2}\right)$
Step 3. $v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$ gives

$$v_3 = (0, 1, 1) - \frac{(0)(1) + (1)(0) + (1)(1)}{1^2 + 0^2 + 1^2} (1, 0, 1) - \frac{(0)\left(\frac{1}{2}\right) + (1)(1) +}{\left(\frac{1}{2}\right)^2 + 1^2 + (}$$

$$= \left(-\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

We get the orthogonal set
$\left\{ v_1 = (1, 0, 1), v_2 = \left(\frac{1}{2}, 1, -\frac{1}{2}\right), v_3 = \left(-\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) \right\}$.

Now, let $q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{1^2+0^2+1^2}}(1,0,1) = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$,
$q_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2+1^2+\left(-\frac{1}{2}\right)^2}}\left(\frac{1}{2}, 1, -\frac{1}{2}\right) = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$.
$q_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{\left(-\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^2}}\left(-\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.
We have the orthonormal set
$\left\{q_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), q_2 = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right), q_3 = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right\}$
which is an orthonormal basis for $\mathbb{R}^3$.