System of linear equations

Outlines

Let $x_1, x_2, \cdots, x_n$ be n unknown real number. A linear equation in these unknowns is of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \cdots, a_n$ and b are constants.
A finite set of linear equations is called **a system of linear equations**,

$$\left\{ \begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \right. \tag{1}$$

where $a_{ij}, b_i, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$ are constants.

A **solution** of (1) is an ordered n-tuple $(s_1, s_2, \cdots, s_n)$ so that the substitution $x_i = s_i, i = 1, 2, \cdots, n$ makes each equation in (1) a true statement. The set of all solutions of a system is called the **solution set**.

Each system of linear equation is represented by its augmented matrix which consists all informations of the system, the $a_{ij}'s$ and the $b_i's$. For example, the augmented matrix of the system (1) is given by

$$\bar{A} = \begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \in \mathbb{R}^{m\times(n+1)} \tag{2}$$

Since the **algebraic operations** on a system as

- Interchange two equations,
- Multiply an equation by a nonzero constant,
- Add a constant times one equation to another,

give an **equivalent system**, i.e., the two solution sets are equal, and the corresponding augmented matrices are affected by the so-called **elementary row operations**,

- Interchange two equations,
- Multiply an equation by a nonzero constant,
- Add a constant times one equation to another,

the **Gaussian elimination** method try to reduce a matrix into **row echelon form** which satisfies the following properties

1. If there are any rows that consist entirely of zeros, called **zeros row**, then they are grouped together at the bottom of the matrix,
2. In any two successive rows that do not consist entirely of zeros, called **non-zeros row**, the **leading entry** (the first non-zero number in the row) in the lower row occurs farther to the right than the leading entry in the higher row.

Moreover, if

3. The leading entry of each non-zero row is 1, called leading 1,
4. Each column that contains a leading 1 has zeros everywhere else in that column, then the matrix is called of **reduced row echelon form**.

Example 1

Let

$$A = \begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 2 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 3 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the symbol * stands for any numbers. The matrix A is in row echelon form whereas B is in reduced row echelon form.

Outlines

In order to get the row echelon form for a matrix $A \in \mathbb{R}^{m \times n}$

Step 1. Locate the leftmost column that does not consist entirely of zeros.

Step 2. Interchange the top row with another row, if necessary, to bring the non-zero entry to the top of the column found in step 1, which becomes the leading entry.

Step 3. (optional). Multiply the first row by an appropritate number to get a leading 1.

Step 4. Add suitable multiples of the top row to the rows below so that all entries below the leading entry become zeros.

Step 5. Cover the top row in the matrix come back to step 1 with the submatrix that remains.

Example 2

Transform the following matrix into the row echelon form

$$A = \begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step 1. The first column is non-zero. Step 2. Interchange two rows to get the leading entry "2" for the first row,

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step 4. Add the third row to a multiple of the first row,

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

Step 5. Cover the first row,

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

and come back to step 1 with this submatrix. The third column is non-zero with leading entry . Step 4. Add the second row to a multiple of the first row,

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

Step 5. Cover the first row,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \quad 1 \end{bmatrix}$$

which already consists of the leading entry . The algorithm finish and we get the row echelon form of A,

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \end{bmatrix}$$

Outlines

Gauss-Jordan Elimination Algorithm

In order to get the reduced row echelon form for a matrix $A \in \mathbb{R}^{m \times n}$,
Step 1. Use Gauss elimination algorithm to get a row echelon form of A.
Step 2. Multiply nonzero rows by suitable numbers to get a leading 1.
Step 3. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

Example 3

Transform the matrix A of example 2 into the reduced row echelon form.

Step 1. From example 2, we get the row echelon form of A,

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \end{bmatrix}$$

Step 2. Multiply nonzero rows by suitable numbers,

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Step 3. Add suitable multiplies of the third row to the rows above,

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

add suitable multiply of the second row to the row above,

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

which is of the reduced row echelon form.

Outlines

Solving System of Linear Equations by Gauss Elimination

Consider the system of linear equations (1).
Step 1. Form the augmented matrix, , and transform it into the (reduced) row echelon form.
Step 2. If the last leading entry is on the last column, conclude that the system has no solution. Otherwise, go to step 3.
Step 3. The variables which correspond to the leading entries are called leading variables. The remaining variables are called free variables. Solving for the leading variables in terms of the free variables.
Note. According to step 3, if there is no free variable, the system has an unique solution. Otherwise, it has an infinite number of solutions.

Example 4

Solve the following system

$$\begin{cases} & & & & -2x_3 & & & + & 7x_5 & = & 12 \\ 2x_1 & + & 4x_2 & - & 10x_3 & + & 6x_4 & + & 12x_5 & = & 28 \\ 2x_1 & + & 4x_2 & - & 5x_3 & + & 6x_4 & - & 5x_5 & = & -1 \end{cases}$$

(i) Use Gauss elimination algorithm

Step 1. Form the augmented marix

$$\bar{A} = \begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

and transform it into the row echelon form (of example 2),

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \end{bmatrix}$$

Step 2. The last leading entry is not on the last column, go to step 3.
Step 3. The leading variables are x_1, x_3, x_5 , the remaining x_2, x_4 are free variables.
Solving x_1, x_3, x_5 in terms of x_2, x_4 by the backward substitutions : the last equation, $\frac{1}{2}x_5 = 1$, gives $x_5 = 2$; the second equation, $-2x_3 + 7x_5 = 12$, gives $x_3 = 1$; the first equation, $2x_1 + 4x_2 - 10x_3 + 6x_4 + 12x_5 = 28$, gives $x_1 = 7 - 2x_2 - 3x_4$.
The solutions of the system are $(7 - 2x_2 - 3x_4, x_2, 1, x_4, 2)$, for arbitrary $x_2, x_4 \in \mathbb{R}$.

(ii) Use Gauss-Jordan elimination algorithm

Step 1. Form the augmented marix

$$\bar{A} = \begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

and transform it into the reduced row echelon form (of example 3),

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Step 2. The last leading entry is not on the last column, go to step 3.

Step 3. The leading variables are x_1, x_3, x_5 , the remaining x_2, x_4 are free variables.

Solving x_1, x_3, x_5 in terms of x_2, x_4 : the last equation gives $x_5 = 1$; the second equation gives $x_3 = 1$;

The first equation, $x_1 + 2x_2 + 3x_4 = 7$, gives $x_1 = 7 - 2x_2 - 3x_4$, and we obtain the solutions of the system, $(7 - 2x_2 - 3x_4, x_2, 1, x_4, 2)$, for arbitrary $x_2, x_4 \in \mathbb{R}$.