CONVEX OPTIMIZATION

1. ONE-DIMENSIONAL

2. MULTI-DIMENSIONAL CASE

Consider a smooth function $f : \Omega \to \mathbb{R}$, where Ω is a nonempty open subset of $\mathbb{R}^n$. We focus on the problem of finding $\mathbf{x}_0 \in \Omega$ so that $f(\mathbf{x}_0) = \max_{\mathbf{x} \in \Omega} f(\mathbf{x})$ or $f(\mathbf{x}_0) = \min_{\mathbf{x} \in \Omega} f(\mathbf{x})$, denoted by

$$\mathbf{x}_0 = \arg\max_{\mathbf{x} \in \Omega} f(\mathbf{x}) \text{ or } \mathbf{x}_0 = \arg\min_{\mathbf{x} \in \Omega} f(\mathbf{x})$$

Such an $\mathbf{x}_0$ is called a **global maximum** or **global minimum** of f on Ω. When f is a differentiable function, we only have the necessary condition for $\mathbf{x}_0 : \nabla f(\mathbf{x}_0) = \mathbf{0}$. This condition is also sufficient in convex optimization.

Outlines

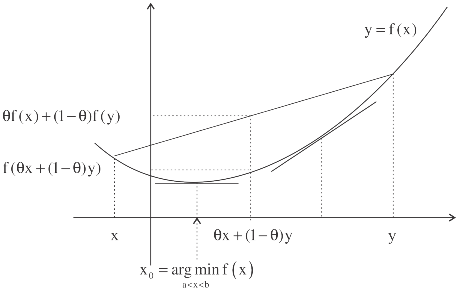
Let $f : (a, b) \to \mathbb{R}$, $-\infty \leq a < b \leq \infty$.

Definition 1. The function $f : (a, b) \to \mathbb{R}$ is **convex** if for all $x, y \in (a, b)$ and $0 \leq \theta \leq 1$, we have

$$f\left(\theta x + (1-\theta)\, y\right) \leq \theta f(x) + (1-\theta)\, f(y)\,. \tag{1}$$

The function f is **concave** if $-f$ is convex.

Geometrically, the inequality (1) means that the line segment between $(x, f(x))$ and $(y, f(y))$, which is a **chord** from x to y, lies above the graph of f.



Theorem 2. Let $f : (a, b) \to \mathbb{R}$, where $-\infty \leq a < b \leq \infty$, be of class C^2, i.e. the second derivative f'' exists and is a continuous function. The following statements are equivalent
(i) f is a convex function.
(ii) $f(y) \geq f(x) + f'(x)(y - x)$, for all $x, y \in (a, b)$.
(iii) $f''(x) \geq 0$, for all $x \in (a, b)$.
In order to prove it, we need the following result, which is a particular case of Taylor's formula.
Proposition 3. Let $f : (a, b) \to \mathbb{R}$ be of class C^2. For each $x, y \in (a, b)$, $x < y$, there exists $x \leq z \leq y$ such that

$$f(y) = f(x) + f'(x)(y - x) + f''(z)\frac{(y - x)^2}{2!}.$$

Proof of Theorem 2. We shall prove $(i) \Leftrightarrow (ii)$ and $(ii) \Leftrightarrow (iii)$. When (i) satisfies, with $x, y \in (a, b)$ and $0 \leq \theta \leq 1$, (1) gives

$$f(y + \theta(x - y)) \leq f(y) + \theta(f(x) - f(y))$$

so with $0 < \theta \leq 1$, since

$$f(x) - f(y) \geq \frac{f(y + \theta(x-y)) - f(y)}{\theta} \to f'(y)(x-y)$$

whenever $\theta \downarrow 0$, (ii) is satisfied. Conversely, when (ii) holds, then with $z = \theta x + (1-\theta) y$, we have

$$f(x) \geq f(z) + f'(z)(x-z), \tag{2}$$

and

$$f(y) \geq f(z) + f'(z)(y-z). \tag{3}$$

Multiply both sides of (2) by θ, (3) by $1-\theta$, and add, side by side, we get

$$\theta f(x) + (1-\theta) f(y) \geq f(z) = f(\theta x + (1-\theta) y),$$

and (i) is proved.

Now, when (ii) holds, we have, with $x, y \in (a, b)$, $x < y$,
$f(y) \geq f(x) + f'(x)(y - x)$ and $f(x) \geq f(y) + f'(y)(x - y)$.
Therefore

$$f'(x)(y - x) \leq f(y) - f(x) \leq f'(y)(y - x).$$

Divide both sides by $(y - x)^2$, we get

$$\frac{f'(y) - f'(x)}{y - x} \geq 0,$$

for all $x, y \in (a, b)$, $x < y$. Letting $y \to x$, we get (iii). Conversely, by proposition 3, there exists $z \in [x, y]$ so that

$$f(y) = f(x) + f'(x)(y - x) + \frac{1}{2} f''(z)(y - x)^2.$$

Since $f''(z) \geq 0$, we must have $f(y) \geq f(x) + f'(x)(y - x)$, i.e., (ii) holds.
Using Theorem 2 with $-f$ when f is a concave function, we have

Corollary 4. Let $f : (a, b) \to \mathbb{R}$, where $-\infty \leq a < b \leq \infty$, be of class C^2. The following statements are equivalent
(i) f is a concave function.
(ii) $f(y) \leq f(x) + f'(x)(y - x)$, for all $x, y \in (a, b)$.
(iii) $f''(x) \leq 0$, for all $x \in (a, b)$.

Example 1. The functions $y = x^n$, for even $n \in \mathbb{N}$, and $y = e^x$ are convex. The function $y = \ln x$ is concave.
Let f be a convex/concave function on (a, b) and let $x_0 \in (a, b)$ that satisfies $f'(x_0) = 0$. The inequality (ii) of Theorem 2 or Corollary 3, with $x = x_0$, yields $f(x) \geq f(x_0)$ or $f(x) \leq f(x_0)$, for all $x \in (a, b)$. Hence, x_0 is a global minimum or a global maximum of f on (a, b) which shows that $f'(x_0) = 0$ is also a sufficient condition for global extremum.

Corollary 5. Let $f : (a, b) \to \mathbb{R}$, where $-\infty \le a < b \le \infty$, be of class C^1, i.e., the first derivative f' exists and is continuous and let $x_0 \in (a, b)$ such that $f'(x_0) = 0$.

(i) If f is convex then $x_0 = \underset{x \in (a,b)}{\arg\min} f(x)$.

(ii) If f is concave then $x_0 = \underset{x \in (a,b)}{\arg\max} f(x)$.

Example 2. (i) The function $y = x^4 + 6x^2$, defined on $\mathbb{R}$, has $y' = 4x^3 + 12x$, $y'' = 12x^2$. Since $y'' \geq 0$, for all $x \in \mathbb{R}$, $y = x^4 + 6x^2$ is convex on $\mathbb{R}$. Now, $y' = 0 \Leftrightarrow x = 3$ implies that $\underset{x \in \mathbb{R}}{\arg\min} \left(x^4 + 6x^2\right) = 3$.

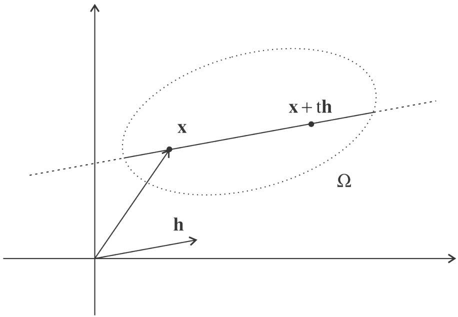
(ii) The function $y = -x \ln x$, defined on $(0, +\infty)$, has $y' = -\ln x + 1$, $y'' = -\frac{1}{x}$. Since $y'' < 0$, for all $x \in (0, +\infty)$, $y = -x \ln x$ is concave on $(0, +\infty)$. Now, $y' = 0 \Leftrightarrow x = e$ implies that $\underset{x \in (0,+\infty)}{\arg\max} \left(-x \ln x\right) = e$.

Outlines

1. ONE-DIMENSIONAL
2. MULTI-DIMENSIONAL CASE

MULTI-DIMENSIONAL CASE

Let $f : \Omega \to \mathbb{R}$, where Ω is a nonempty subset of $\mathbb{R}^n$.
First, we set some notations and terminologies.
Definition 6. A nonempty subset $\Omega \subset \mathbb{R}^n$ is said to be an **open convex** set in $\mathbb{R}^n$ if for all $\mathbf{x} \in \Omega$ and $\mathbf{h} \in \mathbb{R}^n$, $\{t \in \mathbb{R} | \mathbf{x} + t\mathbf{h} \in \Omega\}$ is an open interval (a, b) in $\mathbb{R}$, for some $-\infty \leq a < b \leq \infty$.



Example 3. Any open convex set Ω in $\mathbb{R}$ must be an open interval. $\mathbb{R}^n$ is itself an open convex subset.
$\mathbb{R} \times (0, \infty) = \{(x, y) \in \mathbb{R}^2 | \, y > 0\}$ and
$(0, \infty) \times (0, \infty) = \{(x, y) \in \mathbb{R}^2 | \, x, y > 0\}$ are two convex open sets in $\mathbb{R}^2$.

Definition 7. A square matrix $A \in \mathbb{R}^{n \times n}$ is said to be **positive semidefinite** (**negative semidefinite**, resp), noted $A \geq 0$ ($A \leq 0$, resp.), if $\mathbf{h}^T A\mathbf{h} \geq 0$ ($\mathbf{h}^T A\mathbf{h} \leq 0$, resp.) for all $\mathbf{h} \in \mathbb{R}^n$.

Example 4. Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}.$$

For all $\mathbf{h} = (h, k) \in \mathbb{R}^2$,

$$\mathbf{h}^T A\mathbf{h} = \begin{bmatrix} h & k \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} h & k \end{bmatrix} \begin{bmatrix} h + 2k \\ 2h + 5k \end{bmatrix}$$

$$= h\,(h + 2k) + k\,(2h + 5k) = h^2 + 4hk + 5k^2$$

By writing $h^2 + 4hk = (h + 2k)^2 - 4k^2$, we have

$$h^2 + 4hk + 5k^2 = (h + 2k)^2 - 4k^2 + 5k^2 = (h + 2k)^2 + k^2 \geq 0,$$

for all $\mathbf{h} = (h, k) \in \mathbb{R}^2$. Therefore, A is positive semidefinite.
Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix with (repeated) eigenvalues λ_i, $i = 1, \cdots, n$. We have

Theorem 8. The symmetric matrix A is positive semidefinite if $\lambda_i \geq 0$, for all $i = 1, \cdots, n$, and is negative semidefinite if $\lambda_i \leq 0$, for all $i = 1, \cdots, n$.

Proof. Let Q be an orthogonal matrix that orthogonally diagonalizes A, $Q^T A Q = D$ with $D = diag\left(\lambda_1, \lambda_2, \cdots, \lambda_n\right)$. Since and $QDQ^T = A$, it follows that $Q^T = Q^{-1}$ and for each $\mathbf{h} \in \mathbb{R}^n$, we have

$$\mathbf{h}^T A \mathbf{h} = \mathbf{h}^T \left(QDQ^T\right) \mathbf{h} = \left(Q^T \mathbf{h}\right)^T D \left(Q^T \mathbf{h}\right).$$

Let $Q^T \mathbf{h} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$, the right hand side becomes

$$\left(Q^T\mathbf{h}\right)^T D\left(Q^T\mathbf{h}\right) = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \sum_{i=1}^{n} \lambda_i y_i^2$$

which shows that $\mathbf{h}^T A\mathbf{h} \geq 0$ when $\lambda_i \geq 0$, for all $i = 1, \cdots, n$, and it is ≤ 0 when $\lambda_i \leq 0$, for all $i = 1, \cdots, n$.

Example 5. Consider again the matrix A of example 4. The characteristic equation

$$\det\left(A - \lambda I_2\right) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda) - 4 = \lambda^2 - 6\lambda + 5 = ($$

shows that A has two positive eigenvalues, 1 and 5. Therefore, A is positive semidefinite.

Definition 9. Let $f : \Omega \to \mathbb{R}$, where Ω is a nonempty open convex set in $\mathbb{R}^n$.
(i) f is said to be a **convex function** if for all $\mathbf{x}, \mathbf{y} \in \Omega$ and $0 \leq \theta \leq 1$, we have

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y}).$$

(ii) f is said to be a **concave function** if $-f$ is a convex function.

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y}).$$

Given a function $f : \Omega \to \mathbb{R}$, where Ω is a nonempty open convex set in $\mathbb{R}^n$, for each $\mathbf{x} \in \Omega$ and $\mathbf{h} \in \mathbb{R}^n$, let $I = \{ t \in \mathbb{R} | \, \mathbf{x} + t\mathbf{h} \in \Omega \}$ which is an open interval in $\mathbb{R}$, and let $g_{\mathbf{x},\mathbf{h}} : I \to \mathbb{R}$ be defined by $g_{\mathbf{x},\mathbf{h}}(t) = f(\mathbf{x} + t\mathbf{h})$, for each $t \in I$.
It is easy to have the following important connection between the multivariate function f and the one-dimensional function $g_{\mathbf{x},\mathbf{h}}$ as follows.

Proposition 10. Let $f : \Omega \to \mathbb{R}$ be a function defined on an open convex set Ω in $\mathbb{R}^n$ and let $\{g_{\mathbf{x},\mathbf{h}}\}$ be the family of functions of one variable, defined by (**??**). Then
f is a convex function if and only if $g_{\mathbf{x},\mathbf{h}}$ are convex functions, for all $\mathbf{x} \in \Omega$ and $\mathbf{h} \in \mathbb{R}^n$.
and
f is a concave function if and only if $g_{\mathbf{x},\mathbf{h}}$ are concave functions, for all $\mathbf{x} \in \Omega$ and $\mathbf{h} \in \mathbb{R}^n$.
Let $f : \mathbb{R}^n \to \mathbb{R}$ be of class C^2. The gradient and the Hessian of f at $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, noted $\nabla f(\mathbf{x})$ and $\nabla^2 f(\mathbf{x})$, respectively,

are defined by $\nabla f(\mathbf{x}) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \frac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix}$ and

$$\nabla^2 f(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{x}) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_2^2}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2}(\mathbf{x}) \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(\mathbf{x}) \end{pmatrix}.$$ When f is of class C^2, $\frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}) = \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x})$, for all $i, j = 1, \cdots, n$. It follows that $\nabla^2 f(\mathbf{x})$ is a symmetric matrix, $\nabla^2 f(\mathbf{x})^T = \nabla^2 f(\mathbf{x})$.

Example 6. (a) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x_1, x_2, x_3) = 2x_1 - x_2 + 3x_3$. We have

$$\nabla f(x_1, x_2, x_3) = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$ and $\nabla^2 f(x_1, x_2, x_3) = \mathbf{0}$, the zero matrix in $\mathbb{R}^{3\times 3}$.

(b) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 + x_1x_3$. We have

$$\nabla f(x_1, x_2, x_3) = \begin{pmatrix} 2x_1 - x_2 + x_3 \\ 2x_2 - x_1 \\ 2x_3 + x_1 \end{pmatrix} \text{ and}$$

$$\nabla^2 f(x_1, x_2, x_3) = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Proposition 11. Let $f : \Omega \to \mathbb{R}$ be of class C^2, where Ω is a nonempty open convex set in $\mathbb{R}^n$. For each $\mathbf{x} \in \Omega$ and $\mathbf{h} = (h_1, h_2, \cdots, h_n) \in \mathbb{R}^n$, let $I = \{t \in \mathbb{R} | \mathbf{x} + t\mathbf{h} \in \Omega\}$ and let $g_{\mathbf{x},\mathbf{h}} : I \to \mathbb{R}$ be defined. We have
$g'_{\mathbf{x},\mathbf{h}}(t) = \nabla f(\mathbf{x} + t\mathbf{h})\mathbf{h}$ và $g''_{\mathbf{x},\mathbf{h}}(t) = \mathbf{h}^T \nabla^2 f(\mathbf{x} + t\mathbf{h})\mathbf{h}$.

Proof. The differentability of f gives the existence of a function $\varepsilon(\mathbf{h})$, defined on $\|\mathbf{h}\| < r$, such that $\varepsilon(\mathbf{h}) \to 0$ as $\mathbf{h} \to 0$, and

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \nabla f(\mathbf{x})\mathbf{h} + \|\mathbf{h}\| \varepsilon(\mathbf{h}).$$

Now, let $t \in I$ be fixed. For all $k \in \mathbb{R}$ so that $t + k \in I$, we have

$$g_{\mathbf{x},\mathbf{h}}(t+k) - g_{\mathbf{x},\mathbf{h}}(t) = f(\mathbf{x}+(t+k)\mathbf{h}) - f(\mathbf{x}+t\mathbf{h}) = \nabla f(\mathbf{x}+t\mathbf{h}$$
$$= k\nabla f(\mathbf{x}+t\mathbf{h})\,\mathbf{h} + |k|\,\|\mathbf{h}\|\,\varepsilon(k\mathbf{h})$$

and then

$$\frac{g_{\mathbf{x},\mathbf{h}}(t+k) - g_{\mathbf{x},\mathbf{h}}(t)}{k} = \nabla f(\mathbf{x}+t\mathbf{h})\,\mathbf{h} + \frac{|k|}{k}\|\mathbf{h}\|\,\varepsilon(k\mathbf{h}).$$

Letting $k \to 0$, we have $\|k\mathbf{h}\| \to 0$ which gives

$$\left|\frac{|k|}{k}\|\mathbf{h}\|\,\varepsilon(k\mathbf{h})\right| = \|\mathbf{h}\|\,|\varepsilon(k\mathbf{h})| \to 0.$$

Therefore

$$g'_{\mathbf{x},\mathbf{h}}(t) = \lim_{k\to 0}\frac{g_{\mathbf{x},\mathbf{h}}(t+k) - g_{\mathbf{x},\mathbf{h}}(t)}{k}$$
$$= \nabla f(\mathbf{x}+t\mathbf{h})\,\mathbf{h} = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(\mathbf{x}+t\mathbf{h})\,h_i$$

Now, for each $i = 1, 2, \cdots, n$, let $g_i(t) = \frac{\partial f}{\partial x_i}(\mathbf{x} + t\mathbf{h})$. Apply the above result for $\frac{\partial f}{\partial x_i}$ in place of f, we get

$$g_i'(t) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(\mathbf{x} + t\mathbf{h})\, h_j = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j \partial x_i}(\mathbf{x} + t\mathbf{h})\, h_j$$

and then

$$\begin{aligned} g''_{\mathbf{x},\mathbf{h}}(t) &= \textstyle\sum_{i=1}^{n} h_i g_i'(t) = \sum_{i=1}^{n} h_i \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j \partial x_i}(\mathbf{x} + t\mathbf{h})\, h_j \\ &= \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix} \begin{bmatrix} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j \partial x_1}(\mathbf{x} + t\mathbf{h})\, h_j \\ \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j \partial x_2}(\mathbf{x} + t\mathbf{h})\, h_j \\ \vdots \\ \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j \partial x_n}(\mathbf{x} + t\mathbf{h})\, h_j \end{bmatrix} \end{aligned}$$

as desired.

$$= \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix} \begin{bmatrix} \frac{\partial^2}{\partial x_1^2}(\mathbf{x}+t\mathbf{h}) & \frac{\partial^2}{\partial x_2 \partial x_1}(\mathbf{x}+t\mathbf{h}) & \cdots & \frac{\partial^2}{\partial x_n \partial x_1} \\ \frac{\partial^2}{\partial x_1 \partial x_2}(\mathbf{x}+t\mathbf{h}) & \frac{\partial^2}{\partial x_2^2}(\mathbf{x}+t\mathbf{h}) & \cdots & \frac{\partial^2}{\partial x_n \partial x_2} \\ \vdots & \vdots & & \\ \frac{\partial^2}{\partial x_1 \partial x_n}(\mathbf{x}+t\mathbf{h}) & \frac{\partial^2}{\partial x_2 \partial x_n}(\mathbf{x}+t\mathbf{h}) & \cdots & \frac{\partial^2}{\partial x_n^2}(\end{bmatrix}$$

$$= \mathbf{h}^T \nabla^2 f(\mathbf{x}+t\mathbf{h})\,\mathbf{h}.$$

as desired.

We have the following multidimensional version of Theorem 2.

Theorem 12. Let $f : \Omega \to \mathbb{R}$ be a function of class C^2 on an open convex set Ω in $\mathbb{R}^n$. The following statements are equivalent.

(i) f is a convex function.

(ii) $f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T (\mathbf{y}-\mathbf{x})$, for all $\mathbf{x}, \mathbf{y} \in \Omega$.

(iii) $\nabla^2 f(\mathbf{x}) \geq 0$, for all $\mathbf{x} \in \Omega$.

Proof. For each $\mathbf{z} \in \Omega$ and $\mathbf{h} \in \mathbb{R}^n$, $I = \{ t \in \mathbb{R} | \mathbf{z} + t\mathbf{h} \in \Omega \}$, and let $g_{\mathbf{z},\mathbf{h}} : I \to \mathbb{R}$ be defined by $g_{\mathbf{z},\mathbf{h}}(t) = f(\mathbf{z} + t\mathbf{h})$, for all $t \in I$. By proposition 10, (i) is satisfied if and only if all functions $g_{\mathbf{z},\mathbf{h}}$ are convex. Using Theorem 2, the statement that $g_{\mathbf{z},\mathbf{h}}$ is convex is equivalent with the two following statements

(a) $g_{\mathbf{z},\mathbf{h}}(s) \geq g_{\mathbf{z},\mathbf{h}}(t) + g'_{\mathbf{z},\mathbf{h}}(t)(s-t)$, for all $s, t \in I$.

(b) $g''_{\mathbf{z},\mathbf{h}}(t) \geq 0$, for all $t \in I$.

By proposition 11,

$g'_{\mathbf{z},\mathbf{h}}(t) = \nabla f(\mathbf{z} + t\mathbf{h})\,\mathbf{h}$ và $g''_{\mathbf{z},\mathbf{h}}(t) = \mathbf{h}^T \nabla^2 f(\mathbf{z} + t\mathbf{h})\,\mathbf{h}$.

Letting $\mathbf{z} = \mathbf{x}$, $\mathbf{h} = \mathbf{y} - \mathbf{x}$, $s = 1$, and $t = 0$ in (a), we have

$$f(\mathbf{y}) = g_{\mathbf{x},\mathbf{h}}(1) \geq g_{\mathbf{z},\mathbf{h}}(0) + g'_{\mathbf{z},\mathbf{h}}(0) = f(\mathbf{x}) + \nabla f(\mathbf{x})\,\mathbf{h}$$

which is (ii). Letting $\mathbf{z} = \mathbf{x}$, we have

$\mathbf{h}^T \nabla^2 f(\mathbf{x})\,\mathbf{h} = g''_{\mathbf{x},\mathbf{h}}(0) \geq 0$, for all $\mathbf{h} \in \mathbb{R}^n$, which implies that $\nabla^2 f(\mathbf{x}) \geq 0$, i.e., (ii) is proved. Conversely, consider the function $g_{\mathbf{z},\mathbf{h}}(t) = f(\mathbf{z} + t\mathbf{h})$, $\mathbf{z} \in \Omega$ and $\mathbf{h} \in \mathbb{R}^n$. For $s, t \in I$ by letting $\mathbf{x} = \mathbf{z} + t\mathbf{h}$ and $\mathbf{y} = \mathbf{z} + s\mathbf{h}$, we get

$$g_{\mathbf{z},\mathbf{h}}(s) = f(\mathbf{z} + s\mathbf{h}) \geq f(\mathbf{z} + t\mathbf{h}) + \nabla f(\mathbf{z} + t\mathbf{h})^T((s-t)\mathbf{h}) = g_{\mathbf{z},\mathbf{h}}(t)$$

whenever (ii) holds, and $g''_{\mathbf{z},\mathbf{h}}(t) = \mathbf{h}^T \nabla^2 f(\mathbf{z} + t\mathbf{h})\mathbf{h} \geq 0$ whenever (iii) holds. Therefore, if (ii) or (iii) hold, then, by Theorem 2, $g_{\mathbf{z},\mathbf{h}}$ is a convex function and the Theorem is proved.

Conditions (iii) is used to check if f is a convex function and condition (ii) shows that a solution of the equation $\nabla f(\mathbf{x}) = \mathbf{0}$ is a global minimum of f on Ω.

As in the one-dimensional case, letting $-f$ in Theorem 12, we get the following result for concave functions.

Corollary 13. Let $f : \Omega \to \mathbb{R}$ be a function of class C^2 on an open convex set Ω in $\mathbb{R}^n$. The following statements are equivalent.

(i) f is a concave function.
(ii) $f(\mathbf{y}) \leq f(\mathbf{x}) + \nabla f(\mathbf{x})^T (\mathbf{y} - \mathbf{x})$, for all $\mathbf{x}, \mathbf{y} \in \Omega$.
(iii) $\nabla^2 f(\mathbf{x}) \leq 0$, for all $\mathbf{x} \in \Omega$.
By combining these results, we have the following important result for applications.
Corollary 14. Let Ω be an open convex set of $\mathbb{R}^n$, let $f : \Omega \to \mathbb{R}$ be of class C^1, i.e., all partial derivatives of f exist and are continuous continuous functions, and let $\mathbf{x}_0 \in \Omega$ such that $\nabla f(\mathbf{x}_0) = \mathbf{0}$.
(i) If f is a convex function then $\mathbf{x}_0 = \arg\min_{\mathbf{x} \in \Omega} f(\mathbf{x})$.
(ii) If f is a concave function then $\mathbf{x}_0 = \arg\max_{\mathbf{x} \in \Omega} f(\mathbf{x})$.

Example 7. Consider the quadratic form of example 6 (b), with

$$f\left(x_1, x_2, x_3\right) = x_1^2 + x_2^2 + x_3^2 - x_1 x_2 + x_1 x_3, \left(x_1, x_2, x_3\right) \in \mathbb{R}^3,$$

$$\nabla f\left(x_1, x_2, x_3\right) = \begin{pmatrix} 2x_1 - x_2 + x_3 \\ 2x_2 - x_1 \\ 2x_3 + x_1 \end{pmatrix} \text{ and}$$

$$\nabla^2 f\left(x_1, x_2, x_3\right) = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

The matrix $\nabla^2 f\left(x_1, x_2, x_3\right)$ has positive eigenvalues 2, $2 + \sqrt{2}$, and $2 - \sqrt{2}$ which shows that $\nabla^2 f\left(x_1, x_2, x_3\right) \geq 0$ and then f is a convex function. Therefore, the unique solution of the equation $\nabla f\left(x_1, x_2, x_3\right) = (0, 0, 0)$ is the global minimum of f on $\mathbb{R}^3$.

Definition 15. Let $\mathbf{q} = (q_1, q_2, \cdots, q_n) \in \mathbb{R}^n$ and let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$.
(i) The function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f_{\mathbf{q}}(\mathbf{x}) = \langle \mathbf{q}, \mathbf{x} \rangle = \sum_{i=1}^{n} q_i x_i \equiv \mathbf{q}^T \mathbf{x},$$

for $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, is called a **linear map**.
(ii) The function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f_A(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = \mathbf{x}^T A \mathbf{x},$$

for $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, is called a **quadratic form**.

Example 8. (a) The function $f(x_1, x_2, x_3) = 2x_1 - x_2 + 3x_3$ of example 6 (a) is a linear map, $f = f_{\mathbf{q}}$, with $\mathbf{q} = (2, -1, 3)$.
(b) The function $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 + x_1x_3$ of example 6 (b) is a quadratic form, $f = f_A$, with

$$A = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}.$$

Remark. Let $f_A(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ be a quadratic form on $\mathbb{R}^n$ and let $B = \frac{1}{2}\left(A + A^T\right)$. Then B is symmetric and

$$\mathbf{x}^T B\mathbf{x} = \frac{1}{2}\left(\mathbf{x}^T A\mathbf{x} + \mathbf{x}^T A^T\mathbf{x}\right) = \mathbf{x}^T A\mathbf{x},$$

since $\mathbf{x}^T A^T\mathbf{x} = \left(\mathbf{x}^T A^T\mathbf{x}\right)^T = \mathbf{x}^T\left(A^T\right)^T\mathbf{x} = \mathbf{x}^T A\mathbf{x} \in \mathbb{R}^{1\times 1}$. So we can assume that the matrix A that defines the quadratic form $f_A(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is symmetric.

Proposition 16. Let $f_{\mathbf{q}}(\mathbf{x})$ be a linear map with $\mathbf{q} = (q_1, q_2, \cdots, q_n) \in \mathbb{R}^n$ and let $f_A(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ be a quadratic form with symmetric matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$. We have
(i) $\nabla f_{\mathbf{q}}(\mathbf{x}) = \mathbf{q}$ and $\nabla^2 f_{\mathbf{q}}(\mathbf{x}) = \mathbf{0}$, for all $\mathbf{x} \in \mathbb{R}^n$.
(ii) $\nabla f_A(\mathbf{x}) = 2A\mathbf{x}$ and $\nabla^2 f_A(\mathbf{x}) = 2A$, for all $\mathbf{x} \in \mathbb{R}^n$.
Proof. (i) The identity $f_{\mathbf{q}}(\mathbf{x}) = q_1x_1 + \cdots + q_ix_i + \cdots + q_nx_n$ gives $\frac{\partial f_{\mathbf{q}}}{\partial x_i}(\mathbf{x}) = q_i$, for all $i = 1, 2, \cdots, n$, and $\frac{\partial^2 f_{\mathbf{q}}}{\partial x_j \partial x_i}(\mathbf{x}) = 0$, for all $i, j = 1, 2, \cdots, n$.
(ii) Since $\frac{\partial}{\partial x_k}(x_ix_j) = 0$, when $k \neq i, j$, $\frac{\partial}{\partial x_k}(x_kx_j) = x_j$, when $i = k \neq j$, and $\frac{\partial}{\partial x_k}(x_kx_k) = 2x_k$, it follows from the identity $f_A(\mathbf{x}) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}x_ix_j$ that, for each $k = 1, 2, \cdots, n$,

$$
\begin{aligned}
\frac{\partial f_A}{\partial x_k}(\mathbf{x}) &= \sum_{i=1}^n \sum_{j=1}^n a_{ij} \frac{\partial}{\partial x_k}(x_i x_j) = \sum_{i=1}^n \left(a_{ik} \frac{\partial}{\partial x_k}(x_i x_k) + \sum_{j \neq k} a_{ij} \frac{\partial}{\partial x_k} \right. \\
&= \sum_{i=1}^n a_{ik} \frac{\partial}{\partial x_k}(x_i x_k) + \sum_{i=1}^n \left(\sum_{j \neq k} a_{ij} \frac{\partial}{\partial x_k}(x_i x_j) \right) = 2a_{kk}x_k + \sum_{i \neq k} a_{ik} \\
&= 2a_{kk}x_k + \sum_{i \neq k} a_{ik}x_i + \sum_{j \neq k} a_{kj}x_j = 2\sum_{i=1}^n a_{ki}x_i
\end{aligned}
\tag{4}
$$

which is the k-th row of $2A\mathbf{x}$. Hence, $\nabla f_A(\mathbf{x}) = 2A\mathbf{x}$. Now, by (4), $\frac{\partial f_A}{\partial x_k}(\mathbf{x})$ is a linear function with $\mathbf{q} = (2a_{k1}, 2a_{k2}, \cdots, 2a_{kn})$, the k-th row of $2A$. By (i), $\frac{\partial^2 f_A}{\partial x_l \partial x_k}(\mathbf{x}) = a_{lk}$, for each $l, k = 1, 2, \cdots, n$, which gives $\nabla^2 f_A(\mathbf{x}) = 2A$as desired.

Finally, consider a matrix $A \in \mathbb{R}^{N \times n}$, with $n \leq N$, *rank* $A = n$, and a vector $\mathbf{b} \in \mathbb{R}^N$. Let the function $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(\mathbf{x}) = \|A\mathbf{x} - \mathbf{b}\|^2$. We have

$$
\begin{aligned}
f(\mathbf{x}) &= \langle A\mathbf{x} - \mathbf{b}, A\mathbf{x} - \mathbf{b} \rangle = (A\mathbf{x} - \mathbf{b})^T (A\mathbf{x} - \mathbf{b}) \\
&= \left(\mathbf{x}^T A^T - \mathbf{b}^T\right)(A\mathbf{x} - \mathbf{b}) \\
&= \mathbf{x}^T A^T A\mathbf{x} - \mathbf{b}^T A\mathbf{x} - \mathbf{x}^T A^T \mathbf{b} + \mathbf{b}^T \mathbf{b} \\
&= \mathbf{x}^T \left(A^T A\right)\mathbf{x} - \left(2\mathbf{b}^T A\right)\mathbf{x} + \mathbf{b}^T \mathbf{b},
\end{aligned}
$$

since $\mathbf{b}^T A\mathbf{x} = \mathbf{x}^T A^T \mathbf{b}$. Therefore
$\nabla f(\mathbf{x}) = 2\left(A^T A\right)\mathbf{x} - 2A^T \mathbf{b}$ and $\nabla^2 f(\mathbf{x}) = 2\left(A^T A\right)$.
Since
$\mathbf{h}^T \nabla^2 f(\mathbf{x})\,\mathbf{h} = 2\left[\mathbf{h}^T \left(A^T A\right)\mathbf{h}\right] = 2\left[\left(\mathbf{h}^T A^T\right)(A\mathbf{h})\right] = 2\|A\mathbf{h}\|^2 \geq 0$,
for all $\mathbf{h} \in \mathbb{R}^N$, the Hessian $\nabla^2 f(\mathbf{x})$ is a positive semidefinite matrix.
Moreover, *rank* $\left(A^T A\right) = n$ implies that the matrix $A^T A$ is invertible.

Hence, the solution $\mathbf{x} = \left(A^T A\right)^{-1} A^T \mathbf{b}$ of the equation $\nabla f(\mathbf{x}) = \mathbf{0}$ is the global minimum of f.

We state this result in the following theorem for future applications.

Theorem 17. Let a matrix $A \in \mathbb{R}^{N \times n}$, with $n \leq N$, so that *rank* $A = n = \min(N, n)$, and let $\mathbf{b} \in \mathbb{R}^N$. We have

$$\left(A^T A\right)^{-1} A^T \mathbf{b} = \arg\min_{\mathbf{x} \in \mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|^2 .$$

Example 9. Let $\mathbf{b} = (2, 2, 5, 8) \in \mathbb{R}^4$ and let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \in \mathbb{R}^{4 \times 2}.$$

It is easy to see that $\dim(col(A)) = 2 = rank(A) = \min(4, 2)$, $A^T A = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$ is an invertible matrix, and

$$\left(A^T A\right)^{-1} A^T \mathbf{b} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}^{-1} \begin{bmatrix} 17 \\ 53 \end{bmatrix} = \begin{bmatrix} -1 \\ 2.1 \end{bmatrix}.$$ Hence

$$(-1, 2.1) = \underset{\mathbf{x} \in \mathbb{R}^2}{\arg\min} \left\| A\mathbf{x} - \mathbf{b} \right\|^2.$$

Now, with

$$A = \begin{bmatrix} 1 & 1 & 1^2 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ 1 & 4 & 4^2 \end{bmatrix} \in \mathbb{R}^{4\times 3}.$$

We have $\dim\left(col\left(A\right)\right) = 3 = rank\left(A\right) = \min\left(4, 3\right)$,
$A^T A = \begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix}$ is an invertible matrix, and

$$\left(A^TA\right)^{-1}A^T\mathbf{b} = \begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix}^{-1} \begin{bmatrix} 17 \\ 53 \\ 183 \end{bmatrix} = \begin{bmatrix} 2.75 \\ -1.65 \\ 0.75 \end{bmatrix}.$$

Hence

$$(2.75, -1.65, 0.75) = \arg\min_{\mathbf{x}\in\mathbb{R}^3} \|A\mathbf{x} - \mathbf{b}\|^2 .$$