Action operads, free algebras on invertible objects, and the classification of 3-groups

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Abstract

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Operads and their algebras

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The most important example of sub action operads are those of the symmetric operad, S. This is because Definition 1.3 itself makes explicit reference to the symmetric groups, and so every action operad will end up related to some sub-operad of S:

Definition 1.8. For an arbitrary action operad G the images of the underlying permutation maps $\pi_n^G : G(n) \to \mathrm{S}_n$ naturally form an action operad $\mathrm{im}(\pi^G)$, where

- the sets of operations are the images of G's sets of operations under the homomorphisms π^G:
$$\mathrm{im}(\pi^G)(n) \quad := \quad \mathrm{im}(\pi_n^G)$$

- the underlying permutation maps are the evident inclusions:

$$\pi_n^{\mathrm{im}(\pi^G)} \,:\, \mathrm{im}(\pi^G)(n) \hookrightarrow \mathrm{S}_n$$

- the operad multiplication is the appropriate restriction of the multiplication of S:

$$\mu^{\mathrm{im}(\pi^G)}(\,g\,;\,h_1,...,h_n\,) \quad := \quad \mu^{\mathrm{S}}(\,g\,;\,h_1,...,h_n\,)$$

Clearly this $\mathrm{im}(\pi^G)$ is a sub action operad of the symmetric operad S, and we will call the *underlying permutation operad* of G.

For example, consider the action operad B we just saw in Example 1.5. For a given n, the braid group B_n is generated by $n-1$ elementary braids. But the underlying permutations of these braids are just the $n-1$ adjacent transpositions which generate the symmetric group S_n, and so the underlying permutation maps $\pi_n^{\mathrm{B}} : \mathrm{B}_n \to \mathrm{S}_n$ are all surjective. Thus the underlying permutation operad of B is just the whole symmetric action operad S.

It is even easier to see that S itself will have underlying permutations S, as the maps $\pi_n^{\mathrm{S}} = \mathrm{id} : \mathrm{S}_n \to \mathrm{S}_n$ are obviously surjective. Similarly, the trivial operad T is also its own underlying permutation action operad, as the image of the homomorphims $\pi_n^{\mathrm{T}} : \{e\} \to \mathrm{S}_n$ are trivial. Faced with rather dull examples like these, it might be tempting to try and construct some action operads with more exotic underlying permutations, like maybe the alternating groups $\mathrm{A}_n \subset \mathrm{S}_n$. But it turns out that this is not possible; when it come to their underlying permutation operad, action operads come in exactly two flavours.

Definition 1.9. Let G be an action operad where $\mathrm{im}(\pi)(n)$ is the trivial group for each $n \in \mathbb{N}$. Then we say that G is *non-crossed*, since its operad multiplication will be a true group homomorphism:

$$\begin{aligned}\mu(\,gg'\,;\,h_1h_1',...,h_nh_n'\,) &= \mu(\,g\,;\,h_{\pi(g')^{-1}(1)},...,h_{\pi(g')^{-1}(n)}\,)\mu(\,g'\,;\,h_1',...,h_n'\,)\\ &= \mu(\,g\,;\,h_1,...,h_n\,)\mu(\,g'\,;\,h_1',...,h_n'\,)\end{aligned}$$

Likewise, a *crossed* action operad will refer to any that has a non-trivial underlying permutation operad.

Lemma 1.10. *An action operad G is crossed if and only if it has surjective underlying permutation maps $\pi_n : G(n) \to \mathrm{S}_n$. In other words, the underlying permutations operad of G must be either the trivial operad* T *or the symmetric operad* S*.*

Proof. Let $\mathrm{im}(\pi)$ be the underlying permutation operad of G, and let us assume that G is crossed, so that $\mathrm{im}(\pi)$ is not the trivial operad. This means that for some natural number n, the n-ary operations of $\mathrm{im}(\pi)$ include at least one permutation σ which is not the identity element of the relevant symmetric group S_n. Put another way, there must be some σ and some $1 \leq i \leq n$ for which $\sigma(i) \neq i$. But now consider evaluating the expression

$$\mu^{\mathrm{im}(\pi)}(\,\sigma\,;\,e_0,...,e_0,e_1,e_0,....,e_0,e_1,e_0,...,e_0\,)$$

where the e_1's above are appearing in the ith and $\sigma(i)$th coordinates, which we know are distinct. From the definitions of $\mathrm{im}(\pi)(n)$ and of operad multiplication in S, this permutation is really just

$$\mu^{\mathrm{S}}(\,\sigma\,;\,e_0,...,e_0,e_1,e_0,....,e_0,e_1,e_0,...,e_0\,) \quad = \quad (1\,2)$$

the only non-identity element of S_2. This proves that the map $\pi_2 : G(2) \to \mathrm{S}_2$ is indeed surjective, but more than that it shows that $\mathrm{im}(\pi)$ must contain every possible adjacent transposition, since for any $m \in \mathbb{N}$ we have

$$\begin{aligned} & \mu^{\mathrm{im}(\pi)}(\,e_n\,;\,e_1,...,e_1,(1\,2),e_1,....,e_1\,) \\ = \;& \mu^{\mathrm{S}}(\,e_n\,;\,e_1,...,e_1,(1\,2),e_1,....,e_1\,) \\ = \;& (m\;m+1) \in \mathrm{S}_n \end{aligned}$$

Then because adjacent transpositions generate the symmetric groups S_n, it follows that every permutation is actually an operation in $\mathrm{im}(\pi)$, so that it is really just the full symmetric operad S. Thus by only assuming that our action operad G was crossed, we have shown that all of the maps π_n must be surjective. □

Definition 1.11. G-operads

1.2 Operad algebras

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Definition 1.13. G-operad algebras

1.3 EG-algebras

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Definition 1.16. EG-algebras

Proposition 1.17. *G-operad algebras are monoidal categories with permutation-like structure*

Corollary 1.18. *Braided monoidal categories are G-operad algebras*

Definition 1.19. A strict monoidal category X is said to be *spacial* if, for any object $x \in \mathrm{Ob}(X)$ and any endomorphism of the unit object $f : I \to I$,

$$f \otimes \mathrm{id}_x = \mathrm{id}_x \otimes f$$

The motivation for the name 'spacial' comes from the context of string diagrams [4]. In a string diagram, the act of tensoring two strings together is represented by placing those strings side by side. Since the defining feature of the unit object is that tensoring it with other objects should have no effect, the unit object is therefore represented diagrammatically by the absense of a string. An endomorphism of the unit thus appears as an entity with no input or output strings, detached from the rest of the diagram. In a real-world version of these diagrams, made out of physical strings arranged in real space, we could use this detachedness to grab these endomorphisms and slide them over or under any strings we please, without affecting anything else in the diagram. This ability is embodied algebraically by the equation above, and hence categories which obey it are called 'spacial'.

Lemma 1.20. *If G is a crossed action operad, then all* EG*-algebras are spacial.*

Proof. Let G be a crossed action operad, let X be a EG-algebra, and fix $x \in \mathrm{Ob}(X)$ and $f : I \to I$. From Lemma 1.10 we know that $\pi : G(2) \to S_2$ is surjective, so that the set $\pi^{-1}(\,(1\,2)\,)$ is non-empty, and from the rules for composition of action morphisms we see that for any such $g \in \pi^{-1}(\,(1\,2)\,)$,

$$\begin{aligned} \alpha(\,g\,;\,\mathrm{id}_x,\,\mathrm{id}_I\,) \circ \alpha(\,e_2\,;\,\mathrm{id}_x,\,f\,) &= \alpha(\,g\,;\,\mathrm{id}_x,\,f\,) \\ &= \alpha(\,e_2\,;\,f,\,\mathrm{id}_x\,) \circ \alpha(\,g\,;\,\mathrm{id}_x,\,\mathrm{id}_I\,) \end{aligned}$$

Thus in order to obtain the result we're after, it will suffice to find a particular $g \in \pi^{-1}(\,(1\,2)\,)$ for which

$$\alpha(\,g\,;\,\mathrm{id}_x,\,\mathrm{id}_I\,) = \mathrm{id}_x$$

However, since

$$\begin{aligned}\alpha(\,g\,;\,\mathrm{id}_x,\,\mathrm{id}_I\,) &= \alpha(\,g\,;\,\mathrm{id}_x,\,\alpha(e_0;-)\,)\\ &= \alpha(\,\mu(g;e_1,e_0)\,;\,\mathrm{id}_x\,)\end{aligned}$$

all we really need is to find a $g \in \pi^{-1}(\,(1\,2)\,)$ for which

$$\mu(g;e_1,e_0) = e_1$$

To this end, choose an arbitrary element $h \in \pi^{-1}(\,(1\,2)\,)$. This h probably won't obey the above equation, but we can use it to construct a new element g which does. Specifically, define

$$k := \mu(\,h\,;\,e_1,\,e_0\,)$$

and then consider

$$g := h \cdot \mu(e_2;k^{-1},e_1)$$

To see that this is the correct choice of g, first note that we must have $\pi(k) = e_1$, since this is the only element of S_1. Following from that, we have

$$\begin{aligned}\pi\Big(\,\mu(e_2;k^{-1},e_1)\Big) &= \mu\Big(\,\pi(e_2)\,;\,\pi(k^{-1}),\,\pi(e_1)\Big)\\ &= \mu\Big(\,e_2\,;\,e_1,\,e_1\Big)\\ &= e_2\end{aligned}$$

and hence

$$\begin{aligned}\pi(g) &= \pi\Big(h\cdot\mu(e_2;k^{-1},e_1)\Big)\\ &= \pi(h)\cdot\pi\Big(\mu(e_2;k^{-1},e_1)\Big)\\ &= (1\,2)\cdot e_2\\ &= (1\,2).\end{aligned}$$

So g is indeed in $\pi^{-1}(\,(1\,2)\,)$, and furthermore

$$\begin{aligned}\mu(g;e_1,e_0) &= \mu\Big(\,h\cdot\mu(e_2;k^{-1},e_1)\,;\,e_1,\,e_0\,\Big)\\ &= \mu(\,h\,;\,e_1,\,e_0\,)\cdot\mu\Big(\,\mu(e_2;k^{-1},e_1)\,;\,e_1,\,e_0\,\Big)\\ &= \mu(\,h\,;\,e_1,\,e_0\,)\cdot\mu\Big(\,e_2\,;\,\mu(k^{-1};e_1),\,\mu(e_1;e_0)\,\Big)\\ &= \mu(\,h\,;\,e_1,\,e_0\,)\cdot\mu(\,e_2\,;\,k^{-1},e_0\,)\\ &= k\cdot k^{-1}\\ &= e_1\end{aligned}$$

Therefore, $h \cdot \mu(e_2; k^{-1}, e_1)$ is exactly the g we were looking for, and so working backwards through the proof we obtain the required result:

$$\begin{array}{rrcl} & \mu(g; e_1, e_0) & = & e_1 \\ \Longrightarrow & \alpha(\, g\, ;\, \mathrm{id}_x,\, \mathrm{id}_I\,) & = & \mathrm{id}_x \end{array}$$

$$\begin{array}{rcl} \alpha(\, g\, ;\, \mathrm{id}_x,\, \mathrm{id}_I\,) \circ \alpha(\, e_2\, ;\, \mathrm{id}_x,\, f\,) & = & \alpha(\, e_2\, ;\, f,\, \mathrm{id}_x\,) \circ \alpha(\, g\, ;\, \mathrm{id}_x,\, \mathrm{id}_I\,) \\ \Longrightarrow \qquad \alpha(\, e_2\, ;\, \mathrm{id}_x,\, f\,) & = & \alpha(\, e_2\, ;\, f,\, \mathrm{id}_I\,) \end{array}$$

□

1.4 The free $\mathrm{E}G$-algebra on n objects

Our goal for the next few chapters will be to understand the free braided monoidal category on an finite number of invertible objects. Thus, now that we have a firm grasp on action operads and their algebras, we should begin to think about the simpler free constructions they can form. We will use this extensively when calculating the invertible case later on.

In the paper [7], Gurski establishes how to contruct free G-operad algebras through the use of the monad $\mathrm{E}G$. What follows in this section is a quick summary of the results which will be useful for our purposes. For a more detailed treatment please refer to [7].

Proposition 1.21. *There exists a free* $\mathrm{E}G$*-algebra on* n *objects. That is, there is an* $\mathrm{E}G$*-algebra* Y *such that for any other* $\mathrm{E}G$*-algebra* X*, we have an isomorphism of categories*

$$\mathrm{E}G\mathrm{Alg}_S(Y, X) \cong X^n$$

Proof. There is an obvious forgetful 2-functor $U : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{Cat}$ sending $\mathrm{E}G$-algebras to their underlying categories. U has a left adjoint, which we call the free 2-functor $F : \mathrm{Cat} \to \mathrm{E}G\mathrm{Alg}_S$ adjoint to it. It follows immediately that

$$\begin{array}{rcl} U(X)^n & = & \mathrm{Cat}(\{z_1, ..., z_n\}, U(X)) \\ & \cong & \mathrm{E}G\mathrm{Alg}_S(F(\{z_1, ..., z_n\}), X) \end{array}$$

where $\{z_1, ..., z_n\}$ is any set with n distinct elements. Since X and $U(X)$ are obviously isomorphic as categories, this shows that $F(\{z_1, ..., z_n\})$ is the free algebra on n objects as required. □

Definition 1.22. Let $\{z_1, ..., z_n\}$ be an n-object set, which we will also consider as a discrete category. Then we will denote by $\mathbb{G}_n$ the EG-algebra whose underlying category is $\mathrm{E}G(\{z_1, ..., z_n\})$ and whose action

$$\alpha : \mathrm{E}G\Big(\mathrm{E}G(\{z_1, ..., z_n\})\Big) \to \mathrm{E}G(\{z_1, ..., z_n\})$$

is the appropriate component of the multiplication natural transformation $\mu : \mathrm{E}G \circ \mathrm{E}G \to \mathrm{E}G$ of the 2-monad EG.

Theorem 1.23. *$\mathbb{G}_n$ is the free* EG*-algebra on n objects. That is,*

$$F(\{z_1, ..., z_n\}) = \mathbb{G}_n$$

Proof. □

Definition 1.22 is a fairly opaque definition, so we'll spend a little time upacking it. Recall from Definition 1.15 that $\mathrm{E}G(\{z_1, ..., z_n\})$ is the coequalizer of the maps

$$\textstyle\coprod_{m\geq 0} \mathrm{E}G(m) \times G(m) \times \{z_1, ..., z_n\}^m \rightrightarrows \coprod_{m\geq 0} \mathrm{E}G(m) \times \{z_1, ..., z_n\}^m$$

that comes from the action of $G(m)$ on $\mathrm{E}G(m)$ by multiplication on the right,

$$\begin{array}{rcl} \mathrm{E}G(m) \times G(m) & \to & \mathrm{E}G(m) \\ (g, h) & \mapsto & gh \\ (\,! : g \to g', \mathrm{id}_h\,) & \mapsto & ! : gh \to g'h \end{array}$$

and the action of $G(m)$ on $\{z_1, ..., z_n\}^m$ by permutation,

$$\begin{array}{rcl} G(m) \times \{z_1, ..., z_n\}^m & \to & \{z_1, ..., z_n\}^m \\ (\,h\,;\, x_1, ..., x_m\,) & \mapsto & \big(x_{\pi(h^{-1})(1)}, ..., x_{\pi(h^{-1})(m)}\big) \\ (\mathrm{id}_h\,;\, \mathrm{id}_{(x_1,...,x_m)}\,) & \mapsto & \mathrm{id}_{(x_{\pi(h^{-1})(1)},...,x_{\pi(h^{-1})(m)})} \end{array}$$

First, objects in this algebra are equivalence classes of tuples $(g; x_1, ..., x_m)$, for $g \in G(m)$ and $x_i \in \{z_1, ..., z_n\}$, under the relation

$$(\,gh\,;\, x_1,\, ...,\, x_m\,) \sim (\,g\,;\, x_{\pi(h)^{-1}(1)},\, ...,\, x_{\pi(h)^{-1}(m)}\,)$$

Notice that using this relation we can rewrite any object uniquely in the form $[e; x_1, ..., x_m]$ for some $m \in \mathbb{N}$ and $x_i \in \{z_1, ..., z_n\}$. This means that each equivalence class is just the tensor product $x_1 \otimes ... \otimes x_m$ in the underlying monoidal category of $\mathbb{G}_n$, for some unique sequence of generators. That is, we can view the objects of $\mathbb{G}_n$ as elements of the monoid freely generated by each of the z_i, or in other words:

Lemma 1.24. $\mathrm{Ob}(\mathbb{G}_n)$ *is the free monoid on* n *generators,* $\mathbb{N}^{*n}$*, the free product of* n *copies of* $\mathbb{N}$*.*

Similarly, the morphisms of $\mathbb{G}_n$ are the maps

$$(!; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) : (g; x_1, ..., x_m) \to (g'; x_1, ..., x_m)$$

with $g, g' \in G(m)$ and $x_i \in \{z_1, ..., z_n\}$. Using the relation $\sim$ on objects we can rewrite each of these morphisms in the form

$$[h; \mathrm{id}_{y_1}, ..., \mathrm{id}_{y_m}] \,:\, y_1 \otimes ... \otimes y_m \;\to\; y_{\pi(h^{-1})(1)} \otimes ... \otimes y_{\pi(h^{-1})(m)}$$

where

$$h = g'g^{-1}, \qquad y_i = x_{\pi(g^{-1})(i)}$$

The EG-action of $\mathbb{G}_n$ is permutation and tensor product, and the action on morphisms is given by

$$\alpha(\, g\,;\, [h_1; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_{m_1}}],\, ...,\, [h_k; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_{m_k}}]\,) = [\,\mu(g; h_1, .., h_k)\,;\, \mathrm{id}_{x_1},\, ...,\, \mathrm{id}_{x_{m_k}}\,]$$

Notice that using tensor product notation the object $[e; x]$ is simply x, and so $[e; \mathrm{id}_x] = \mathrm{id}_{[e;x]}$ should be written as id_x. Hence by the above $[g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}]$ is really just $\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$, and so we have the following:

Lemma 1.25. *Every morphism of* $\mathbb{G}_n$ *can be expressed uniquely as an action morphism*

$$\alpha(\, g\,;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,) \,:\, x_1 \otimes ... \otimes x_m \;\to\; x_{\pi(g)^{-1}(1)} \otimes ... \otimes x_{\pi(g)^{-1}(m)}$$

for some $g, g' \in G(m)$ *and* $x_i \in \{z_1, ..., z_n\}$*.*

As an immediate consequence of this, the source and target of a given morphism in $\mathbb{G}_n$ must be related to one another by some permutation of the form $\pi(g)$. In toher words, the connected components of $\mathbb{G}_n$ will depend upon the underlying permutation operad of G, in the following way:

Proposition 1.26. *Considered as a monoid under tensor product,*

$$\pi_0(\mathbb{G}_n) \quad = \quad \begin{cases} \mathbb{N}^n & \textit{if } G \textit{ is crossed} \\ \mathbb{N}^{*n} & \textit{otherwise} \end{cases}$$

Also, the canonical homomorphism sending objects in $\mathbb{G}_n$ to their connected component,

$$[\_\,] \,:\, \mathrm{Ob}(\mathbb{G}_n) \to \pi_0(\mathbb{G}_n)$$

is the quotient map of abelianisation

$$\mathrm{ab} \,:\, \mathbb{N}^{*n} \to (\mathbb{N}^{*n})^{\mathrm{ab}} \,=\, \mathbb{N}^n$$

when G is crossed, and the identity map $\mathrm{id}_{\mathbb{N}^{*n}}$ otherwise.

Proof. By Lemma 1.25, all morphisms in $\mathbb{G}_n$ can be written uniquely as $\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$, for some $g \in G(m)$ and $x_i \in \{z_1, ..., z_n\}$, the set of generators of $\mathbb{N}^{*n}$. Since maps of this form have source $x_1 \otimes ... \otimes x_m$ and target $x_{\pi(g^{-1})(1)} \otimes ... \otimes x_{\pi(g^{-1})(m)}$, we see that the only pairs of object which might have a morphism between them are those that can be expanded as tensor products that differ by some permutation.

If our action operad G is crossed, then for any two objects like this — say source $x_1 \otimes ... \otimes x_m$ and target $x_{\sigma^{-1}(1)} \otimes ... \otimes x_{\sigma^{-1}(m)}$ for an arbitrary $\sigma \in \mathrm{S}_m$ — we can always find a map $\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$ between them, because by Lemma 1.10 the underlying permutations maps $\pi_m : G(m) \to S_m$ are all surjective and so there must exist at least one g with $\pi(g) = \sigma$. In particular, for any two generating objects z_i and $z_j$of $\mathbb{G}_n$ there must exist at least morphism between $z_i \otimes z_j$ and $z_j \otimes z_i$, and therefore

$$[z_i] \otimes [z_j] \quad = \quad [z_i \otimes z_j] \quad = \quad [z_j \otimes z_i] \quad = \quad [z_j] \otimes [z_i]$$

Thus the canonical map $[\_\,] : \mathrm{Ob}(\mathbb{G}_n) \to \pi_0(\mathbb{G}_n)$ is the one that makes the free product of $\mathbb{N}^{*n}$ commutative, that is, the quotient map for the abelianisation $\mathrm{ab} : \mathbb{N}^{*n} \to (\mathbb{N}^{*n})^{\mathrm{ab}}$, and so $\pi_0(\mathbb{G}_n) = \mathbb{N}^n$.

Conversely, if G is non-crossed then its underlying permutation operad $\mathrm{im}(\pi)$ is trivial, and so the only morphisms we have in $\mathbb{G}_n$ will be those of the form

$$\alpha(\, e_m \,;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m} \,) \quad = \quad \mathrm{id}_{x_1} \otimes ... \otimes \mathrm{id}_{x_m} \quad = \quad \mathrm{id}_{x_1 \otimes ... \otimes x_m}$$

Therefore the map $[\_]$ just sends each object to its identity morphism, and since that function is one-to-one and onto it follows that

$$\pi_0(\mathbb{G}_n) \quad = \quad \mathrm{Ob}(\mathbb{G}_n) \quad = \quad \mathbb{N}^{*n}, \qquad\qquad [\_] \quad = \quad \mathrm{id}_{\mathbb{N}^{*n}}$$

by Lemma 1.24. □

Finally, Lemma 1.25 also gives us a complete description of how the morphisms of $\mathbb{G}_n$ interact under tensor product, though we need a little new terminology in order to express it properly.

Definition 1.27. Let G be an action operad. Then we will also the notation G to denote the *underlying monoid* of this action operad. This is the natural way to consider G as a monoid, with its element set being all of its elements together, $\coprod_m G(m)$, and with tensor product as its binary operation, $g \otimes h = \mu(e_2; g, h)$.

Also, note that this monoid comes equipped with a homomorphism $|\_| : G \to \mathbb{N}$, sending each $g \in G$ to the natural number m if and only if g is an element of the group $G(m)$. We'll call this number $|g|$ the *length* of g.

Definition 1.28. Let S be a set and $F(S)$ the free monoid on S, the monoid whose elements are strings of elements of S and whose binary operation is concatenation. Then we will denote by

$$|\_| : F(S) \to \mathbb{N}$$

the monoid homomorphism defined by sending each element of $S \subseteq F(S)$ to 1, and therefore also each concatenation of n elements of S to the natural number n. Again, we will call $|x|$ the *length* of $x \in F(S)$.

Lemma 1.29. *The monoid of morphisms of the algebra $\mathbb{G}_n$ is*

$$\mathrm{Mor}(\mathbb{G}_n) \cong G \times_{\mathbb{N}} \mathbb{N}^{*n}$$

where this pullback is taken over the respective length homomorphisms,

$$\begin{tikzcd}
G \times_{\mathbb{N}} \mathbb{N}^{*n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{N}^{*n} \arrow[d, "|_|"] \\
G \arrow[r, "|_|"] & \mathbb{N}
\end{tikzcd}$$

using the fact that $\mathbb{N}^{*n}$ is the free monoid $F\Big(\{z_1, ..., z_n\}\Big)$.

Proof. An element of $G \times_{\mathbb{N}} F\Big(\{z_1, ..., z_n\}\Big)$ is just an element $g \in G(m)$ for some m, together with an m-tuple of objects $(x_1, ..., x_m)$ from the set of generators $\{z_1, ..., z_n\}$. Thus the action on $\mathbb{G}_n$ defines an obvious function

$$\begin{array}{rcrcl} \alpha & : & G \times_{\mathbb{N}} F\Big(\{z_1, ..., z_n\}\Big) & \to & \mathrm{Mor}(\mathbb{G}_n) \\ & : & (g; x_1, ..., x_m) & \mapsto & \alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) \end{array}$$

But by Lemma 1.25, each element of $\mathrm{Mor}(\mathbb{G}_n)$ can be expressed in the form $\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$ for a unique collection $(g; x_1, ..., x_m)$, and so this function α is actually a bijection of sets. Furthermore, this function preserves tensor product, since

$$\begin{array}{rcl} \alpha\Big((g; f_1, ..., f_m) \otimes (g'; f'_1, ..., f'_m)\Big) & = & \alpha(\, g \otimes g'\,;\, f_1, ..., f_m, f'_1, ..., f'_m\,) \\ & = & \alpha(\, g\,;\, f_1, ..., f_m\,) \otimes \alpha(\, g'\,;\, f'_1, ..., f'_m\,) \end{array}$$

and hence it is a monoid isomorphism, as required. □

Chapter 2

Free invertible algebras as initial objects

In this chapter we will start to consider how to construct free EG-algebras on some number of invertible objects. Specifically, we will begin by showing that such algebras are the initial objects of a particular comma category, in accordance with some well known properties of adjunctions and their units. Using this initial object prespective will allow us to recover all of the data associated with the objects of a given free invertible algebra — what those objects are, how they act under tensor product, and which pairs of objects form the source and target of at least one morphism. Unfortunately, a concrete decription of the morphisms themselves will ultimately remain elusive. We can get tantalisingly closer though, and an examiniation of the exact way that this method fails will provide the neccessary insight to motivate a more successful approach in **??**.

2.1 The free algebra on n invertible objects

We saw in Proposition 1.21 that the existence of a free EG-algebra on n objects can be proven by taking the left adjoint of a 2-functor which forgets about the algebra structure. Now we want to extend this idea into the realm of algebras on invertible objects. For the analogous approach, we will need to find a new 2-functor that lets us forget about non-invertible objects, and then hopefully we can find its left adjoint too, and use it to freely add inverses to $\mathbb{G}_n$. First though, we need to make this concept of 'forgetting non-invertible objects' a little more precise.

Definition 2.1. Given an EG-algebra X, we denote by X_{inv} the sub-EG-algebra containing all invertible objects in X and the isomorphisms between them.

Note that this is indeed a well-defined EG-algebra. If $x_1, ..., x_m$ are invertible objects with inverses $x_1^*, ..., x_m^*$, then $\alpha(g; x_1, ..., x_m)$ is an invertible object with inverse $\alpha(g; x_m^*, ..., x_1^*)$, since

$$\begin{aligned} & \alpha(g; x_1, ..., x_m) \otimes \alpha(g; x_m^*, ..., x_1^*) \\ = & \Big(x_{\pi(g)^{-1}(1)} \otimes ... \otimes x_{\pi(g)^{-1}(m)} \Big) \otimes \Big(x^*_{\pi(g)^{-1}(m)} \otimes ... \otimes x^*_{\pi(g)^{-1}(1)} \Big) \\ = & I \end{aligned}$$

$$\begin{aligned} & \alpha(g; x_m^*, ..., x_1^*) \otimes \alpha(g; x_1, ..., x_m) \\ = & \Big(x^*_{\pi(g)^{-1}(m)} \otimes ... \otimes x^*_{\pi(g)^{-1}(1)} \Big) \otimes \Big(x_{\pi(g)^{-1}(1)} \otimes ... \otimes x_{\pi(g)^{-1}(m)} \Big) \\ = & I \end{aligned}$$

Likewise, if $f_1, ..., f_m$ are isomorphisms from invertible objects $x_1, ..., x_m$ to invertible objects $y_1, ..., y_m$, then $\alpha(g; f_1, ..., f_m)$ is a map from the invertible object $\alpha(g; x_1, ..., x_m)$ to the invertible object $\alpha(g; y_1, ..., y_m)$, and it has an inverse $\alpha(g^{-1}; f^{-1}_{\pi(g)(1)}, ..., f^{-1}_{\pi(g)(m)})$, since

$$\begin{aligned} & \alpha\Big(g^{-1} \,;\, f^{-1}_{\pi(g)(1)},\, ...,\, f^{-1}_{\pi(g)(m)} \Big) \circ \alpha(\, g \,;\, f_1, ..., f_m \,) \\ = & \alpha\Big(g^{-1}g \,;\, f_1^{-1}f_1,\, ...,\, f_m^{-1}f_m \Big) \\ = & \mathrm{id}_{x_1 \otimes ... \otimes x_m} \end{aligned}$$

$$\begin{aligned} & \alpha(\, g \,;\, f_1, ..., f_m \,) \circ \alpha\Big(g^{-1} \,;\, f^{-1}_{\pi(g)(1)},\, ...,\, f^{-1}_{\pi(g)(m)} \Big) \\ = & \alpha\Big(gg^{-1} \,;\, f_{\pi(g)(1)} f^{-1}_{\pi(g)(1)},\, ...,\, f_{\pi(g)(m)} f^{-1}_{\pi(g)(m)} \Big) \\ = & \mathrm{id}_{y_{\pi(g)(1)} \otimes ... \otimes y_{\pi(g)(m)}} \end{aligned}$$

Clearly then, X_{inv} is the correct algebra for our new forgetful 2-functor to send X to. Knowing this, we can contruct the rest of the functor fairly easily.

Proposition 2.2. *The assignment $X \mapsto X_{\mathrm{inv}}$ can be extended to a 2-functor $(\_)_{\mathrm{inv}}$: $\mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$.*

Proof. Let $F : X \to Y$ be a (strict) map of EG-algebras. If x is an invertible object in X with inverse x^*, then $F(x)$ is an invertible object in Y with inverse $F(x^*)$, by

$$F(x) \otimes F(x^*) = F(x \otimes x^*) = F(I) = I$$

$$F(x^*) \otimes F(x) = F(x^* \otimes x) = F(I) = I$$

Since F sends invertible objects to invertible objects, it will also send isomorphisms of invertible objects to isomorphisms of invertible objects. In other words, the map $F : X \to Y$ can be restricted to a map $F_{\text{inv}} : X_{\text{inv}} \to Y_{\text{inv}}$. Moreover, we have that

$$(F \circ G)_{\text{inv}}(x) = F \circ G(x) = F_{\text{inv}} \circ G_{\text{inv}}(x)$$

$$(F \circ G)_{\text{inv}}(f) = F \circ G(f) = F_{\text{inv}} \circ G_{\text{inv}}(f)$$

and so the assignment $F \mapsto F_{\text{inv}}$ is clearly functorial. Next, let $\theta : F \Rightarrow G$ be an EG-monoidal natural transformation. Choose an invertible object x from X, and consider the component map of its inverse, $\theta_{x^*} : F(x^*) \to G(x^*)$. Since θ is monoidal, we have $\theta_{x^*} \otimes \theta_x = \theta_I = I$ and $\theta_x \otimes \theta_{x^*} = I$, or in other words that θ_{x^*} is the monoidal inverse of θ_x. We can use this fact to construct a compositional inverse as well, namely $\mathrm{id}_{F(x)} \otimes \theta_{x^*} \otimes \mathrm{id}_{G(x)}$, which can be seen as follows:

$$\begin{aligned} \left(\mathrm{id}_{F(x)} \otimes \theta_{x^*} \otimes \mathrm{id}_{G(x)}\right) \circ \theta_x &= \theta_x \otimes \theta_{x^*} \otimes \mathrm{id}_{G(x)} \\ &= \mathrm{id}_{G(x)} \end{aligned}$$

$$\begin{aligned} \theta_x \circ \left(\mathrm{id}_{F(x)} \otimes \theta_{x^*} \otimes \mathrm{id}_{G(x)}\right) &= \mathrm{id}_{F(x)} \otimes \theta_{x^*} \otimes \theta_x \\ &= \mathrm{id}_{F(x)} \end{aligned}$$

Therefore, we see that all the components of our transformation on invertible objects are isomorphisms, and hence we can define a new transformation $\theta_{\text{inv}} : F_{\text{inv}} \Rightarrow G_{\text{inv}}$ whose components are just $(\theta_{\text{inv}})_x = \theta_x$. The assignment $\theta \mapsto \theta_{\text{inv}}$ is also clearly functorial, and thus we have a complete 2-functor $(\_)_{\text{inv}} : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$. □

Proposition 2.3. *The 2-functor* $(\_)_{\text{inv}} : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$ *has a left adjoint,* $L : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$.

Proof. To begin, consider the 2-monad $\mathrm{E}G(\_)$. This is a finitary monad, that is it preserves all filtered colimits, and it is a 2-monad over Cat, which is locally finitely presentable. It follows from this that $\mathrm{E}G\mathrm{Alg}_S$ is itself locally finitely presentable. Thus if we want to prove $(\_)_{\text{inv}}$ has a left adjoint, we can use the Adjoint Functor Theorem for locally finitely presentable categories, which amounts to showing that $(\_)_{\text{inv}}$ preserves both limits and filtered colimits.

- Given an indexed collection of EG-algebras X_i, the EG-action of their product $\prod X_i$ is defined componentwise. In particular, this means that the tensor product of two objects in $\prod X_i$ is just the collection of the tensor products of their components in each of the X_i. An invertible object in $\prod X_i$ is thus simply a family of invertible objects from the X_i — in other words, $(\prod X_i)_{\text{inv}} = \prod (X_i)_{\text{inv}}$.

- Given maps of EG-algebras $F : X \to Z$, $G : Y \to Z$, the EG-action of their pullback $X \times_Z Y$ is also defined componentwise. It follows that an invertible object in $X \times_Z Y$ is just a pair of invertible objects (x, y) from X and Y, such that $F(x) = G(y)$. But this is the same as asking for a pair of objects (x, y) from X_{inv} and Y_{inv} such that $F_{\text{inv}}(x) = G_{\text{inv}}(y)$, and hence $(X \times_Z Y)_{\text{inv}} = X_{\text{inv}} \times_{Z_{\text{inv}}} Y_{\text{inv}}$.

- Given a filtered diagram D of EG-algebras, the EG-action of their colimit $\text{colim}(D_n)$ is defined in the following way: use filteredness to find an algebra which contains (representatives of the classes of) all the things you want to act on, then apply the action of that algebra. In the case of tensor products this means that $[x] \otimes [y] = [x \otimes y]$, and thus an invertible object in $\text{colim}(D_n)$ is just (the class of) an invertible object in one of the algebras of D. In other words, $\text{colim}(D_n)_{\text{inv}} = \text{colim}(D_{\text{inv}})$.

Preservation of products and pullbacks gives preservation of limits, and preservation of limits and filtered colimits gives the result. □

With this new 2-functor $L : \text{E}G\text{Alg}_S \to \text{E}G\text{Alg}_S$, we now have the ability to 'freely add inverses to objects' in any EG-algebra we want. The algebra $L\mathbb{G}_n$ is then a clear candidate for our free algebra on n invertible objects, and indeed the proof of this is very simple.

Theorem 2.4. *There exists a free* EG*-algebra on n invertible objects. Specifically, the algebra $L\mathbb{G}_n$ is such that for any other* EG*-algebra X, we have an isomorphism of categories*

$$\text{E}G\text{Alg}_S(L\mathbb{G}_n, X) \quad \cong \quad (X_{\text{inv}})^n$$

Proof. Using the adjunction from Proposition 2.3 along with the one from Proposition 1.21, we see that

$$\begin{aligned} U(X_{\text{inv}})^n &= \text{Cat}(\{z_1, ..., z_n\}, U(X_{\text{inv}})) \\ &\cong \text{E}G\text{Alg}_S(F(\{z_1, ..., z_n\}), X_{\text{inv}}) \\ &\cong \text{E}G\text{Alg}_S(LF(\{z_1, ..., z_n\}), X) \end{aligned}$$

As before, X_{inv} and $U(X_{\text{inv}})$ are obviously isomorphic as categories, and so $LF(\{z_1, ..., z_n\}) = L\mathbb{G}_n$ satisfies the requirements for the free algebra on n invertible objects. □

2.2 $L\mathbb{G}_n$ as an initial algebra

We have now proven that a free EG-algebra on n invertible objects indeed exists. But this fact on its own is not very helpful. To be able to actually use the free algebra $L\mathbb{G}_n$, we need to know how to contruct it explicitly, in terms of its objects and morphisms. We could do this by finding a detailed characterisation of the 2-functor L, and then applying this to our explicit description of $\mathbb{G}_n$ from Definition 1.22. However, this would probably take far more effort than is required, since it would involve determining the behaviour of L in many situtations that we aren't interested in. Also, we wouldn't be leveraging $\mathbb{G}_n$'s status as a free algebra to make the calculations any easier. We will try a different strategy instead, one that begins by noticing a special property of the functor L.

Proposition 2.5. *For any* EG*-algebra* X*, we have* $L(X)_{\mathrm{inv}} = L(X)$*.*

Proof. From the definition of adjunctions, the isomorphisms

$$\mathrm{E}G\mathrm{Alg}_S(LX, Y) \quad \cong \quad \mathrm{E}G\mathrm{Alg}_S(X, Y_{\mathrm{inv}})$$

are subject to certain naturality conditions. Specifically, given $F : X' \to X$ and $G : Y \to Y'$ we get a commutative diagram

$$\begin{tikzcd}
\mathrm{E}G\mathrm{Alg}_S(LX, Y) \arrow[r, "\sim"] \arrow[d, "G\circ_\circ LF"'] & \mathrm{E}G\mathrm{Alg}_S(X, Y_{\mathrm{inv}}) \arrow[d, "G_{\mathrm{inv}}\circ_\circ F"] \\
\mathrm{E}G\mathrm{Alg}_S(LX', Y') \arrow[r, "\sim"] & \mathrm{E}G\mathrm{Alg}_S(X', Y'_{\mathrm{inv}})
\end{tikzcd}$$

Consider the case where F is the identity map $\mathrm{id}_X : X \to X$ and G is the inclusion $j : L(X)_{\mathrm{inv}} \to L(X)$. Note that because j is an inclusion, the restriction $j_{\mathrm{inv}} : (L(X)_{\mathrm{inv}})_{\mathrm{inv}} \to L(X)_{\mathrm{inv}}$ is also an inclusion, but since $((\_)_{\mathrm{inv}})_{\mathrm{inv}} = (\_)_{\mathrm{inv}}$, we have that $j_{\mathrm{inv}} = \mathrm{id}$. It follows that

$$\begin{tikzcd}
\mathrm{E}G\mathrm{Alg}_S(LX, LX_{\mathrm{inv}}) \arrow[r, "\sim"] \arrow[d, "j\circ_"'] & \mathrm{E}G\mathrm{Alg}_S(X, LX_{\mathrm{inv}}) \arrow[d, equal] \\
\mathrm{E}G\mathrm{Alg}_S(LX, LX) \arrow[r, "\sim"] & \mathrm{E}G\mathrm{Alg}_S(X, LX_{\mathrm{inv}})
\end{tikzcd}$$

Therefore, for any map $f : LX \to LX$ there exists a unique $g : LX \to LX_{\text{inv}}$ such that $j \circ g = f$. But this means that for any such f, we must have $\text{im}(f) \subseteq L(X)_{\text{inv}}$, and so in particular $L(X) = \text{im}(\text{id}_{LX}) \subseteq L(X)_{\text{inv}}$. Since $L(X)_{\text{inv}} \subseteq L(X)$ by definition, we obtain the result. □

This result is not especially surprising. Intuitively, it just says that when you freely add inverses to an algebra, every object ends up with an inverse. But the upshot of this is that we now have another way of thinking about $L(X)$: as the target object of the unit of our adjunction, $\eta_X : X \to L(X)_{\text{inv}}$. This means that we don't really need to know the entirety of L in order to determine the free algebra $L\mathbb{G}_n$, just its unit. To find this unit directly, we can turn to the following fact about adjunctions, for which a proof can be found in Lemma 2.3.5 of Leinster's *Basic Category Theory* [8].

Proposition 2.6. *Let $F \dashv G : A \to B$ be an adjunction with unit η. For any object a in A, let $(a \downarrow G)$ denote the comma category whose objects are pairs (b, f) consisting of an object* B *from* B *and a morphism $f : a \to G(b)$ from A, and whose morphisms $h : (b, f) \to (b', f')$ are morphisms $f : b \to b'$ from* B *such that $G(f) \circ f = f'$. Then the pair $\big(F(a), \eta_a : a \to GF(a)\big)$ is an initial object of $(a \downarrow G)$.*

Corollary 2.7. *$\eta_{\mathbb{G}_n} : \mathbb{G}_n \to (L\mathbb{G}_n)_{\text{inv}} = L\mathbb{G}_n$ is an initial object of $(\mathbb{G}_n \downarrow \text{inv})$.*

Being able to view $L\mathbb{G}_n$ as the initial object in the comma category $(\mathbb{G}_n \downarrow \text{inv})$ will prove immensely useful in the coming sections. This is because it lets us think about the properties of $L\mathbb{G}_n$ in terms of maps $\psi : \mathbb{G}_n \to X_{\text{inv}}$, and this is exactly the context where we can exploit $\mathbb{G}_n$'s status as a free algebra. As a result, its worth taking some time to think about what exactly this map $\eta_{\mathbb{G}_n}$ is.

Lemma 2.8. *The initial object $\eta_{\mathbb{G}_n} : \mathbb{G}_n \to L\mathbb{G}_n$ is the obvious inclusion of the free* EG*-algebra on n objects into the free* EG*-algebra on n invertible objects. That is, $\eta_{\mathbb{G}_n}$ is the algebra map defined by*

$$\begin{array}{cccc} \eta_{\mathbb{G}_n} & : & \mathbb{G}_n & \to & L\mathbb{G}_n \\ & : & F(\{z_1, ..., z_n\}) & \to & LF(\{z_1, ..., z_n\}) \\ & : & z_i & \mapsto & z_i \end{array}$$

Proof. Consider the n-tuple $(z_1, ..., z_n)$ in $(\mathbb{G}_n)^n$. Clearly the image of $(z_1, ..., z_n)$ under the functor L is just the object $(z_1, ..., z_n)$ in the algebra

$$L\Big((\mathbb{G}_n)^n\Big) \quad = \quad (L\mathbb{G}_n)^n \quad = \quad LF(\{z_1, ..., z_n\})^n$$

But the image of $(z_1, ..., z_n) \in (\mathbb{G}_n)^n$ under the isomorphism

$$\mathrm{E}G\mathrm{Alg}_S(\,\mathbb{G}_n, \mathbb{G}_n\,) \quad \cong \quad (\mathbb{G}_n)^n$$

is just the identity map $\mathrm{id}_{\mathbb{G}_n}$. Thus by functoriality of L, the map $L(\mathrm{id}_{\mathbb{G}_n}) = \mathrm{id}_{L\mathbb{G}_n}$ must be the one which corresponds to the n-tuple $(z_1, ..., z_n) \in (L\mathbb{G}_n)^n$ image via the isomorphism

$$\mathrm{E}G\mathrm{Alg}_S(\,L\mathbb{G}_n, L\mathbb{G}_n\,) \quad \cong \quad (L\mathbb{G}_n)^n$$

Furthermore, the $\mathbb{G}_n$ component of the unit η is by definition the image of the identity map $\mathrm{id}_{L\mathbb{G}_n}$ under the isomorphism

$$\mathrm{E}G\mathrm{Alg}_S(\,L\mathbb{G}_n, L\mathbb{G}_n\,) \quad \cong \quad \mathrm{E}G\mathrm{Alg}_S(\,\mathbb{G}_n, L\mathbb{G}_n\,)$$

Hence it follows that $\eta_{\mathbb{G}_n}$ is the map that corresponds to $(z_1, ..., z_n)$ via

$$\mathrm{E}G\mathrm{Alg}_S(\,\mathbb{G}_n, L\mathbb{G}_n\,) \quad \cong \quad (L\mathbb{G}_n)^n$$

which is exactly the definition given in the statement of the lemma. $\square$

Before moving on, we'll make a small change in notation. From now on, rather than writing objects in $(\mathbb{G}_n \downarrow \mathrm{inv})$ as maps $\psi : \mathbb{G}_n \to Y_{\mathrm{inv}}$, we will instead just let $X = Y_{\mathrm{inv}}$ and speak of maps $\psi : \mathbb{G}_n \to X$. This is purely to prevent the notation from becoming cluttered, and shouldn't be a problem so long as we always remember that the targets of these maps only ever contain invertible objects and morphisms. We'll also drop the subscript from $\eta_{\mathbb{G}_n}$, since it is the only component of the unit we'll ever use.

2.3 The objects of $L\mathbb{G}_n$

So now we know that $L\mathbb{G}_n$ is an initial object in the category $(\mathbb{G}_n \downarrow \mathrm{inv})$. But what does this actually tell us? After all, we do not currently have a method for finding initial objects in an arbitrary collection of EG-algebra maps. Because of this, we'll have to approach the problem step-by-step, using the initiality of η to extract different pieces of information about the algebra $L\mathbb{G}_n$ as we go. We'll begin by tring to find its objects.

Definition 2.9. Denote by $\mathrm{Ob} : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{Mon}$ be the functor that sends EG-algebras X to their monoid of objects $\mathrm{Ob}(X)$, and algebra maps $F : X \to Y$ to their underlying monoid homomorphism $\mathrm{Ob}(F) : \mathrm{Ob}(X) \to \mathrm{Ob}(Y)$.

In order to find $\mathrm{Ob}(L\mathbb{G}_n)$, we'll need to make use of an important result about the nature of Ob.

Definition 2.10. Recall that given a monoid M, the monoidal category $\mathrm{E}M$ is the one whose monoid of objects is M and which has a unique isomorphism between any two objects. We can view $\mathrm{E}M$ as not just a category but an $\mathrm{E}G$-algebra, by letting the action on morphisms take the only possible values it can, given the required source and target. Similarly, for any monoid homomorphisms $h : M \to M'$ we can define a map of $\mathrm{E}G$-algebras

$$\begin{array}{rcrcl} \mathrm{E}h & : & \mathrm{E}M & \to & \mathrm{E}M' \\ & : & m & \mapsto & h(m) \\ & : & m \to m' & \mapsto & h(m) \to h(m') \end{array}$$

This definition of $\mathrm{E}h$ respects composition and identities, and so together with $\mathrm{E}M$ it describes a functor $\mathrm{E} : \mathrm{Mon} \to \mathrm{E}G\mathrm{Alg}_S$.

Proposition 2.11. E *is a right adjoint to the functor* Ob.

Proof. For any $\mathrm{E}G$-algebra X, a map $F : X \to \mathrm{E}M$ is determined entirely by its restriction to objects, the monoid homomorphism $\mathrm{Ob}(F) : \mathrm{Ob}(X) \to M$. This is because functorality of F ensures that any map $x \to x'$ in X must be sent to a map $F(x) \to F(x')$ in $\mathrm{E}M$, and by the definition of E there is always exactly one of these to choose from. In other words, we have an isomorphism between the homsets

$$\mathrm{E}G\mathrm{Alg}_S(\,X,\,\mathrm{E}M\,) \quad \cong \quad \mathrm{Mon}(\,\mathrm{Ob}(X),\,M\,)$$

Additionally, this isomorphism is natural in both coordinates. That is, for any $G : X \to X'$ in $\mathrm{E}G\mathrm{Alg}_S$ and $h : M \to M'$ in Mon, the diagram

$$\begin{tikzcd} \mathrm{E}G\mathrm{Alg}_S(X, \mathrm{E}M) \arrow[r, "\sim"] \arrow[d, "\mathrm{E}h\circ\_\circ G"'] & \mathrm{Mon}(\mathrm{Ob}(X), M) \arrow[d, "h\circ\_\circ\mathrm{Ob}(G)"] \\ \mathrm{E}G\mathrm{Alg}_S(X', \mathrm{E}M') \arrow[r, "\sim"] & \mathrm{Mon}(\mathrm{Ob}(X'), M') \end{tikzcd}$$

commutes, because

$$\mathrm{Ob}(\,\mathrm{E}h \circ F \circ G\,) \quad = \quad \mathrm{Ob}(Eh) \circ \mathrm{Ob}(F) \circ \mathrm{Ob}(G) \quad = \quad h \circ \mathrm{Ob}(F) \circ \mathrm{Ob}(G)$$

Therefore, $\mathrm{Ob} \dashv \mathrm{E}$. $\square$

What Proposition 2.11 is essentially saying is that the functor Ob provides a way for us to move back and forth between the categories $\mathrm{E}G\mathrm{Alg}_S$ and Mon. By applying this reasoning to the universal property of the initial object η, we can then determine the value of $\mathrm{Ob}(L\mathbb{G}_n)$ in terms of a new universal property of $\mathrm{Ob}(\eta)$ in the category Mon. In particular, the algebras in $(\mathbb{G}_n \downarrow \mathrm{inv})$ are those whose objects are all invertible, and so the induced property of $\mathrm{Ob}(\eta)$ will end up saying something about the relationship between $\mathrm{Ob}(\mathbb{G}_n)$ and groups — those monoids whose elements are all invertible.

Definition 2.12. Let M be a monoid, M^{gp} a group, and $i : M \to M^{\mathrm{gp}}$ a monoid homomorphism between them. Then we say that M^{gp} is the *group completion* of M if for any other group H and homomorphism $h : M \to H$, there exists a unique homomorphism $u : M^{\mathrm{gp}} \to H$ such that $u \circ i = h$.

There are several different ways to actually calculate the group completion of a monoid. One is to use that fact that M^{gp} is the group whose group presentation is the same as the monoid presentation of M. That is, if M is the quotient of the free monoid on generators $\mathcal{G}$ by the relations $\mathcal{R}$, then M^{gp} is the quotient of the free *group* on generators $\mathcal{G}$ by relations $\mathcal{R}$. This makes finding the completion of free monoids particularly simple.

Proposition 2.13. *The object monoid of $L\mathbb{G}_n$ is $\mathbb{Z}^{*n}$, the group completion of the object monoid of $\mathbb{G}_n$. The restriction of η on objects,* $\mathrm{Ob}(\eta)$*, is then the obvious inclusion $\mathbb{N}^{*n} \hookrightarrow \mathbb{Z}^{*n}$.*

Proof. Let H be a group, and $h : \mathrm{Ob}(\mathbb{G}_n) \to H$ a monoid homomorphism. By Proposition 2.11 we have an isomorphism of homsets

$$\mathrm{E}G\mathrm{Alg}_S(\,\mathbb{G}_n,\, \mathrm{E}H\,) \quad \cong \quad \mathrm{Mon}(\,\mathrm{Ob}(\mathbb{G}_n),\, H\,)$$

Denote by $h' : \mathbb{G}_n \to \mathrm{E}H$ the map of $\mathrm{E}G$-algebras corresponding to h under this isomorphism. Since H is a group, every object in $\mathrm{E}H$ is invertible, and so h' is an object of $(\mathbb{G}_n \downarrow \mathrm{inv})$. Thus, by initiality of η, there must exist a unique map $u : L\mathbb{G}_n \to \mathrm{E}G$ making the lefthand traingle below commute:

$$\begin{tikzcd}
\mathbb{G}_n \arrow[d, "\eta"'] \arrow[dr, "h'"] & \\
L\mathbb{G}_n \arrow[r, "u"'] & \mathrm{E}H
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathrm{Ob}(\mathbb{G}_n) \arrow[d, "\mathrm{Ob}(\eta)"'] \arrow[dr, "h"] & \\
\mathrm{Ob}(L\mathbb{G}_n) \arrow[r, "\mathrm{Ob}(u)"'] & H
\end{tikzcd}$$

It follows that the righthand triangle — which is the image of the first under Ob — also commutes. Hence for any group H and homomorphism $h : \mathrm{Ob}(\mathbb{G}_n) \to H$, there is at least one map which factors h through $\mathrm{Ob}(\eta)$.

But now let $v : \mathrm{Ob}(L\mathbb{G}_n) \to H$ be any homomorphism such that $v \circ \mathrm{Ob}(\eta) = h$. If $v' : L\mathbb{G}_n \to \mathrm{E}H$ is the image of v under the adjunction isomorphism, then by naturality $v' \circ \eta = h'$, a property that was supposed to be unique to u. Thus $v = \mathrm{Ob}(u)$, and so there is actually only one possible map which factors h through $\mathrm{Ob}(\eta)$.

Therefore every homomorphism from $\mathrm{Ob}(\mathbb{G}_n)$ onto a group factors uniquely through the $\mathrm{Ob}(L\mathbb{G}_n)$, or in other words $\mathrm{Ob}(L\mathbb{G}_n)$ is the group completion of the monoid $\mathrm{Ob}(\mathbb{G}_n)$. Since by Lemma 1.24 the object monoid of $\mathbb{G}_n$ is $\mathbb{N}^{*n}$, the free monoid on n generators, we can conclude that

$$\mathrm{Ob}(L\mathbb{G}_n) \;=\; \mathrm{Ob}(\mathbb{G}_n)^{\mathrm{gp}} \;=\; (\mathbb{N}^{*n})^{\mathrm{gp}} \;=\; \mathbb{Z}^{*n}$$

the free group on n generators. Moreover, the map $\mathrm{Ob}(\eta)$ is then the inclusion of $\mathrm{Ob}(\mathbb{G}_n)$ into its completion, which is just $\mathbb{N}^{*n} \hookrightarrow \mathbb{Z}^{*n}$. □

2.4 The connected components of $L\mathbb{G}_n$

The core result of Proposition 2.13 — that $\mathrm{Ob}(L\mathbb{G}_n)$ is the group completion of $\mathrm{Ob}(\mathbb{G}_n)$ — makes concrete the sense in which the functor L represents 'freely adding inverses' to objects. Extending this same logic to connected components as well, it would seem reasonable to expect that $\pi_0(L\mathbb{G}_n)$ is the group completion of $\pi_0(\mathbb{G}_n)$ as well. This is indeed the case, and the proof proceeds in a way completely analagous to Proposition 2.13.

First, we want to show that the process of taking connected components forms part of an adjunction. To do this we are going to need a category from which we can draw the kind of structures that can act as the components of an EG-algebra. Exactly which category this should be will depend on our choice of action operad G, or more precisely its underlying permutations.

Definition 2.14. For a given action operad G, denote by $\mathrm{im}(\pi)-\mathrm{Mon}$ the full subcategory of Mon on those monoids whose multiplication is invariant under the permutations in $\mathrm{im}(\pi)$. That is, a monoid M is in $\mathrm{im}(\pi)-\mathrm{Mon}$ if and only if

$$m_1, ..., m_n \in M,\ g \in G(n) \quad \Longrightarrow \quad m_1...m_n \;=\; m_{\pi(g)^{-1}(1)}...m_{\pi(g)^{-1}(n)}$$

Of course, by Lemma 1.10 there are really only two examples of such a $\mathrm{im}(\pi)-\mathrm{Mon}$. If the underlying permutations of G are trivial, then $\mathrm{im}(\pi)-\mathrm{Mon}$ is just the whole of the category Mon; if instead G is crossed then we are asking for monoids whose multiplication is invariant under arbitrary permutations from S, and so $\mathrm{im}(\pi)-\mathrm{Mon}$ is just the category of *commutative* monoids, CMon. Regardless, when we are working with an arbitrary action operad G, the category $\mathrm{im}(\pi)-\mathrm{Mon}$ is exactly the collection of possible connected components that we were looking for.

Lemma 2.15. *Let G be an action operad and* $\mathrm{im}(\pi)$ *its underlying permutation action operad. Then there is a functor*

$$\pi_0 : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{im}(\pi)-\mathrm{Mon}$$

which sends each algebra X to its monoid of connected components $\pi_0(X)$, and sends each map of algebras $F : X \to Y$ to its restriction to connected components $\pi_0(F) : \pi_0(X) \to \pi_0(Y)$.

Proof. Let $x_1, ..., x_n$ be an arbitrary collection of objects from the algebra X, and g an element of the group $G(n)$. Then the action of G guarantees the existence of a morphism

$$\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_n}) \ : \ x_1 \otimes ... \otimes x_n \to x_{\pi(g^{-1})(1)} \otimes ... \otimes x_{\pi(g^{-1})(n)}$$

By definition the source and target of this morphism belong to the same connected component, and hence

$$\begin{array}{rrcl} & [\, x_1 \otimes ... \otimes x_n \,] & = & [\, x_{\pi(g^{-1})(1)} \otimes ... \otimes x_{\pi(g^{-1})(n)} \,] \\ \implies & [x_1] \otimes ... \otimes [x_n] & = & [x_{\pi(g^{-1})(1)}] \otimes ... \otimes [x_{\pi(g^{-1})(n)}] \end{array}$$

But since the x_i are just arbitrary objects of X, the components $[x_i]$ are an arbitrary collection of elements from $\pi_0(X)$, and likewise for the group element g and the permutation $\pi(g)$. Therefore multiplication in the monoid $\pi_0(X)$ is invariant under all permutations in the images of the homomorphisms $\pi_n : G(n) \to S_n$, and thus $\pi_0(X)$ is an object of $\mathrm{im}(\pi)-\mathrm{Mon}$, as required. Well-definedness of the functor π_0 on morphisms then follows immediately from the fullness of $\mathrm{im}(\pi)-\mathrm{Mon}$. □

Now that we have a functor which represents the act of finding the connected component monoid of an algebra, we need another functor heading in the opposite direction, so that we can construct an adjunction between them.

Definition 2.16. There exists an inclusion of 2-categories $\mathrm{D} : \mathrm{Set} \hookrightarrow \mathrm{Cat}$ which allows us to view any set S as a *discrete category*, one whose objects are just the elements of S and whose morphisms are all identities. If the given set also happens to be a monoid M, then there is an obvious way to see the discete category $\mathrm{D}M$ as a monoidal category, and so we have a similar inclusion $\mathrm{D} : \mathrm{Mon} \hookrightarrow \mathrm{MonCat}$. Finally, for any action operad G and object M of the category $\mathrm{im}(\pi){-}\mathrm{Mon}$, there is a unique way to assign an $\mathrm{E}G$-action to the discete category $\mathrm{D}M$. This works because for any elements $m_1, ..., m_n \in M$ and $g \in G(n)$, the morphism $\alpha(g; \mathrm{id}_{m_1}, ..., \mathrm{id}_{m_n})$ must have source and target

$$m_1 \otimes ... \otimes m_n \quad = \quad m_{\pi(g^{-1})(1)} \otimes ... \otimes m_{\pi(g^{-1})(m)}$$

and therefore it can only be the morphism $\mathrm{id}_{m_1 \otimes ... \otimes m_n}$. This choice of action yields one last inclusion $\mathrm{CMon} \hookrightarrow \mathrm{E}G\mathrm{Alg}_S$, which we shall also call D.

Proposition 2.17. D *is a right adjoint to the functor* π_0.

Proof. Consider a map of $F : X \to \mathrm{D}C$ from some $\mathrm{E}G$-algebra X onto the discrete $\mathrm{E}G$-algebra for a monoid M in $\mathrm{im}(\pi){-}\mathrm{Mon}$. For any $f : x \to x'$ in X, the morphism $F(f)$ must be an identity map in $\mathrm{D}M$, since these are the only morphisms that $\mathrm{D}M$ has. It follows that x and x' being in the same connected component will imply $F(x) = F(x')$, and so F is determined entirely by its restriction to connected components, the monoid homomorphism $\pi_0(F) : \pi_0(X) \to M$. In other words, we have an isomorphism between the homsets

$$\mathrm{E}G\mathrm{Alg}_S(\, X, \mathrm{D}M\,) \quad \cong \quad \mathrm{im}(\pi){-}\mathrm{Mon}(\, \pi_0(X), M\,)$$

This isomorphism is natural in both coordinates, since for any $G : X \to X'$ in $\mathrm{E}G\mathrm{Alg}_S$ and $h : M \to M'$ in $\mathrm{im}(\pi){-}\mathrm{Mon}$,

$$\pi_0(\,\mathrm{D}h \circ F \circ G\,) \quad = \quad \pi_0(\mathrm{D}h) \circ \pi_0(F) \circ \pi_0(G) \quad = \quad h \circ \pi_0(F) \circ \pi_0(G)$$

and so the diagram

$$\begin{tikzcd}
\mathrm{E}G\mathrm{Alg}_S(X, \mathrm{D}M) \arrow[r, "\sim"] \arrow[d, "\mathrm{D}h \circ _ \circ G"'] & \mathrm{im}(\pi){-}\mathrm{Mon}\Big(\pi_0(X), M\Big) \arrow[d, "h \circ _ \circ \pi_0(G)"] \\
\mathrm{E}G\mathrm{Alg}_S(X', \mathrm{D}M') \arrow[r, "\sim"] & \mathrm{im}(\pi){-}\mathrm{Mon}\Big(\pi_0(X'), M'\Big)
\end{tikzcd}$$

commutes. Therefore, $\pi_0 \dashv \mathrm{D}$. $\square$

Now we can utilise Proposition 2.17 to draw out a universal property of $\pi_0(L\mathbb{G}_n)$, just as we did with $\mathrm{Ob}(L\mathbb{G}_n)$ in Proposition 2.11.

Proposition 2.18. *The connected components of $L\mathbb{G}_n$ are the group completion of the connected components of $\mathbb{G}_n$. Also, the restriction of η onto connected components, $\pi_0(\eta)$, is the canonical map $\pi_0(\mathbb{G}_n) \to \pi_0(\mathbb{G}_n)^{\mathrm{gp}}$ associated with that group completion.*

Proof. Let H be a group which is also an object of $\mathrm{im}(\pi)-\mathrm{Mon}$, and let $h : \pi_0(\mathbb{G}_n) \to H$ be a monoid homomorphism. By Proposition 2.17 there is a homset isomorphism

$$\mathrm{E}G\mathrm{Alg}_S(\, \mathbb{G}_n,\, \mathrm{D}H\,) \quad \cong \quad \mathrm{im}(\pi)-\mathrm{Mon}(\, \pi_0(\mathbb{G}_n),\, H\,)$$

and thus some EG-algebra map $h' : \mathbb{G}_n \to \mathrm{D}H$ corresponding to h. As H is a group, every object of DH is invertible, and so h' is an object of $(\mathbb{G}_n \downarrow \mathrm{inv})$. It follows that there exists a unique map $u : L\mathbb{G}_n \to \mathrm{D}M$ which factors h' through the initial object η:

$$\begin{tikzcd}
\mathbb{G}_n \arrow[d, "\eta"'] \arrow[dr, "h'"] & \\
L\mathbb{G}_n \arrow[r, "u"'] & \mathrm{D}H
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\pi_0(\mathbb{G}_n) \arrow[d, "\pi_0(\eta)"'] \arrow[dr, "h"] & \\
\pi_0(L\mathbb{G}_n) \arrow[r, "\pi_0(u)"'] & H
\end{tikzcd}$$

Applying the functor π_0 everywhere, we see that $\pi_0(u)$ must also factor h through the homomorphism $\pi_0(\eta)$. Moreover, $\pi_0(u)$ is the only map with this property, since for any other map $v : \pi_0(L\mathbb{G}_n) \to H$ with $v \circ \pi_0(\eta) = h$, its image under the adjunction isomorphism $v' : L\mathbb{G}_n \to \mathrm{D}H$ would have $v' \circ \eta = h'$ by naturality, which would mean that it was actually u. Therefore, any monoid homomorphism $\pi_0(\mathbb{G}_n) \to H$ will factor uniquely through $\pi_0(L\mathbb{G}_n)$, so long as H is a group.

Now consider another monoid homomorphism $k : \pi_0(\mathbb{G}_n) \to K$, where this time K is still a group but not neccessarily in $\mathrm{im}(\pi)-\mathrm{Mon}$. From Lemma 2.15, we know that $\pi_0(\mathbb{G}_n)$ is still an object of $\mathrm{im}(\pi)-\mathrm{Mon}$, and from this we can conclude that the image $\mathrm{im}(k)$ will be too:

$$\begin{array}{rcrcl}
x_1, ..., x_m \in \pi_0(\mathbb{G}_n),\, g \in G(n) & \Longrightarrow & x_1 \otimes ... \otimes x_m & = & x_{g(1)} \otimes ... \otimes x_{g(m)} \\
 & \Longrightarrow & k(\, x_1 \otimes ... \otimes x_m\,) & = & k(\, x_{g(1)} \otimes ... \otimes x_{g(m)}\,) \\
 & \Longrightarrow & k(x_1) \otimes ... \otimes k(x_m) & = & k(x_{g(1)}) \otimes ... \otimes k(x_{g(m)})
\end{array}$$

Also, since $\mathrm{im}(k)$ is a submonoid of the group K, it is a group as well. Thus if we denote by $k_{\mathrm{im}} : \mathrm{Ob}(\mathbb{G}_n) \to \mathrm{im}(k)$ the restriction of k to it image, then k_{im} is a map in $\mathrm{im}(\pi)-\mathrm{Mon}$ out of $\mathrm{Ob}(\mathbb{G}_n)$ and onto a group, and therefore by what we showed earlier there exists a unique homomorphism $v : \mathrm{Ob}(L\mathbb{G}_n) \to \mathrm{im}(k)$ with the property $v \circ \pi_0(\eta) = k_{\mathrm{im}}$. Composing this v with the inclusion $i : \mathrm{im}(k) \hookrightarrow K$, we see that

$$i \circ v \circ \pi_0(\eta) \;=\; i \circ k_{\mathrm{im}} \;=\; k$$

and $i \circ v$ must be the only map for which this is true, for restricting this equation back on $\mathrm{im}(k)$ yields the unique property of v again. Thus $\pi_0(\eta)$ will actually take any homomorphism from $\mathrm{Ob}(\mathbb{G}_n)$ onto a group and factor it through $\pi_0(L\mathbb{G}_n)$ in a unique way, not just those homomorphisms in $\mathrm{im}(\pi)-\mathrm{Mon}$. In other words,

$$\pi_0(L\mathbb{G}_n) \quad = \quad \pi_0(\mathbb{G}_n)^{\mathrm{gp}}$$

and $\pi_0(\eta)$ is the canonical map of this group completion. □

As we've said before, this result is a reflection of the fact that the functor L is trying to add inverses the objects of $\mathbb{G}_n$ freely, that is, with as little effect on the rest of the algebra as possible. Indeed, if we happen to know whether or not our action operad G is crossed then we can now calculate exactly what the effect on the components will be.

Corollary 2.19. *If G is a crossed action algebra then*

- *the connected components of $L\mathbb{G}_n$ are the monoid $\mathbb{Z}^n$*
- *the restriction of η to components is the obvious inclusion $\mathbb{N}^n \hookrightarrow \mathbb{Z}^n$*
- *the assignment of objects to their component is given by the quotient map of abelianisation $\mathrm{ab} : \mathbb{Z}^{*n} \to \mathbb{Z}^n$*

If instead G is non-crossed, then

- *the connected components of $L\mathbb{G}_n$ are the monoid $\mathbb{Z}^{*n}$*
- *the restriction of η to components is the obvious inclusion $\mathbb{N}^{*n} \hookrightarrow \mathbb{Z}^{*n}$*
- *the assignment of objects to their component is $\mathrm{id}_{\mathbb{Z}^{*n}}$*

Proof. Combining Propositions 1.26 and 2.18, we see that

$$\pi_0(L\mathbb{G}_n) \quad = \quad \pi_0(\mathbb{G}_n)^{\mathrm{gp}} \quad = \quad \begin{cases} (\mathbb{N}^n)^{\mathrm{gp}} \;=\; \mathbb{Z}^n & \text{if } G \text{ is crossed} \\ (\mathbb{N}^{*n})^{\mathrm{gp}} \;=\; \mathbb{Z}^{*n} & \text{otherwise} \end{cases}$$

Moreover, Proposition 2.18 says that restriction of η to connected components, $\pi_0(\eta)$, will be the homomorphism associated with these group completion, which means the inclusion $\mathbb{N}^n \hookrightarrow \mathbb{Z}^n$ when G is crossed and $\mathbb{N}^{*n} \hookrightarrow \mathbb{Z}^{*n}$ when it is not.

Next, by Proposition 1.26 we know that the map $[\_] : \mathrm{Ob}(\mathbb{G}_n) \to \pi_0(\mathbb{G}_n)$ sending objects of $\mathbb{G}_n$ to their connected component is either the quotient map of abelianisation $\mathbb{N}^{*n} \to \mathbb{N}^n$ or the identity on $\mathbb{N}^{*n}$, depending on whether or not it is crossed. If we also use $[\_]$ to denote the map sending objects of $L\mathbb{G}_n$ to their components, it then follows from functoriality of η that the corresponding choice of the followings two diagrams will commute:

$$\begin{tikzcd}
\mathbb{N}^{*n} \arrow[r, hook, "\mathrm{Ob}(\eta)"] \arrow[d, "{[_]}"'] & \mathbb{Z}^{*n} \arrow[d, "{[_]}"] \\
\mathbb{N}^{n} \arrow[r, hook, "\pi_0(\eta)"'] & \mathbb{Z}^{n}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{N}^{*n} \arrow[r, hook, "\mathrm{Ob}(\eta)"] \arrow[d, equal, "{[_]}"'] & \mathbb{Z}^{*n} \arrow[d, "{[_]}"] \\
\mathbb{N}^{*n} \arrow[r, hook, "\pi_0(\eta)"'] & \mathbb{Z}^{*n}
\end{tikzcd}$$

Using the values of $[\_]$ from Proposition 1.26, $\mathrm{Ob}(\eta)$ from Proposition 2.13, and $\pi_0(\eta)$ from earlier in this proof, it follows that for any generator z_i of $\mathbb{Z}^{*n}$,

$$[z_i] \quad = \quad [\mathrm{Ob}(\eta)(z_i)] \quad = \quad \pi_0(\eta)([z_i]) \quad = \quad \pi_0(\eta)(z_i) \quad = \quad z_i$$

But this description of $[\_] : \mathrm{Ob}(L\mathbb{G}_n) \to \pi_0(L\mathbb{G}_n)$ on generators is either the definition of the quotient map $\mathrm{ab} : \mathbb{Z}^{*n} \to (\mathbb{Z}^{*n})^{\mathrm{ab}}$ or the identity $\mathrm{id} : \mathbb{Z}^{*n} \to \mathbb{Z}^{*n}$, depending on the value of target monoid, as required. $\square$

2.5 The morphisms of $L\mathbb{G}_n$

Now that we understand the objects and connected components of the algebra $L\mathbb{G}_n$, the next most obvious thing to look for are its morphisms, $\mathrm{Mor}(L\mathbb{G}_n)$. It would be nice to construct this collection in the same way we constructed $\mathrm{Ob}(L\mathbb{G}_n)$ and $\pi_0(L\mathbb{G}_n)$, by applying the left adjoint of some adjunction to the initial map η. Before we can do this however, we need to ask ourselves a question. What sort of mathematical object is $\mathrm{Mor}(L\mathbb{G}_n)$, exactly?

Given a pair of morphisms $f : x \to y, f' : y' \to z$ in an EG-algebra X, there are two basic binary operations we can perform. First, we can take their tensor product $f \otimes f'$, and this together with the unit map id_I imbues $\mathrm{Mor}(X)$ with the structure of a monoid. Second, if we have $y = y'$ then we can form the composite morphism $f' \circ f$. However, these two operations are not as different as they first appear.

Lemma 2.20. *Let $f : x \to y$ and $f' : y \to z$ be morphisms in some monoidal category, and y is an invertible object of that category. Then*

$$f' \circ f \quad = \quad f' \otimes \mathrm{id}_{y*} \otimes f$$

Proof. By the interchange law for monoidal categories,

$$\begin{aligned} f' \circ f &= (f' \otimes \mathrm{id}_I) \circ (\mathrm{id}_I \otimes f) \\ &= (f' \otimes \mathrm{id}_{y*} \otimes \mathrm{id}_y) \circ (\mathrm{id}_y \otimes \mathrm{id}_{y*} \otimes f) \\ &= (f' \circ \mathrm{id}_y) \otimes (\mathrm{id}_{y*} \circ \mathrm{id}_{y*}) \otimes (\mathrm{id}_y \circ f) \\ &= f' \otimes \mathrm{id}_{y*} \otimes f \end{aligned}$$

□

In other words, composition along invertible objects in X is completely determined by the monoidal structure of X. In the case of $L\mathbb{G}_n$, where every object is invertible, this means that if we understand $\mathrm{Mor}(L\mathbb{G}_n)$ as a monoid then we will be able to recover the operation $\circ$ in its entirety. For that reason, we will choose to ignore composition of elements of $\mathrm{Mor}(X)$ for the time being, and focus on its status as a monoid.

Now we try to proceed as we did before, by showing that $\mathrm{Mor}(X)$ is part of an adjunction.

Definition 2.21. Let $\mathrm{Mor} : \mathrm{MonCat} \to \mathrm{Mon}$ be the functor which sends algebras X to their monoid of morphisms $\mathrm{Mor}(X)$, and sends algebra maps $F : X \to Y$ to the monoid homomorphism

$$\begin{aligned} \mathrm{Mor}(F) \quad : &\quad \mathrm{Mor}(X) \;\to\; \mathrm{Mor}(Y) \\ : &\quad f : x \to x' \;\mapsto\; F(f) : F(x) \to F(x') \end{aligned}$$

Definition 2.22. For a given abelian group A, let $C(A)$ represent the monoidal category defines as follows:

- The objects of $C(A)$ are the monoid A, with the monoid multiplication as the tensor product and the identity element e as the monoidal unit.
- For any two objects $a, a' \in A$, the homset $C(A)(a, a')$ is isomorphic to the underlying set of A.

- From the above, the morphisms of $C(A)$ will clearly be

$$\mathrm{Mor}\Big(C(A)\Big) \;=\; A \times A^2 \;=\; A^3$$

  when viewed as a set, but this equality also holds as monoids, so that the tensor product is defined componentwise using the monoid multiplication of A.

- For any two composable morphisms (a, b, b'), (a', b', b'') of $C(A)$, their composite is the morphism

$$(a', b', b'') \circ (a, b, b') \;=\; \Big(a(b')^* a',\; b,\; b''\Big)$$

Likewise, for any group homomorphism $h : A \to A'$ between abelian groups, denote by $C(h) : C(A) \to C(A')$ the obvious monoidal functor which acts like h on objects and h^3 on morphisms. This defines a functor $C : \mathrm{Ab} \to \mathrm{MonCat}$ from the category of abelian groups onto the category of monoidal categories.

Intuitively, $C(A)$ is the the monoidal category that we can build out of A by using the trick we discussed before for extracting composition from the tensor product, $f' \circ f = f' \otimes \mathrm{id}_{y*} \otimes f$. This is why we had to choose A to be a group, as this can only work when all of the objects of $C(A)$ are invertible. Notice also that commutativity is required in order for $C(A)$ to be a well-defined monoidal category, since we need its operations to obey an interchange law, and thus

$$\begin{aligned}
(aa', e, e) &= (a, e, e) \otimes (a', e, e) \\
&= \Big(\mathrm{id}_e \circ (a, e, e)\Big) \otimes \Big((a', e, e) \circ \mathrm{id}_e\Big) \\
&= \Big((a', e, e) \otimes \mathrm{id}_e\Big) \circ \Big(\mathrm{id}_e \otimes (a, e, e)\Big) \\
&= (a', e, e) \circ (a, e, e) \\
&= (a'a, e, e)
\end{aligned}$$

This is the classic Eckmann-Hilton argument.

Proposition 2.23. *C is a right adjoint to the functor* $\mathrm{Mor}(\_)^{\mathrm{gp,ab}} : \mathrm{MonCat} \to \mathrm{Ab}$.

Proof. Let X be a monoidal category, A an abelian group, and $F : X \to C(A)$ a monoidal functor. For any morphism $f : x \to y$ in X, by functoriality F will send it onto some morphism $F(f)$ in the homset $C(A)(F(x), F(y))$. However, every homset of $C(A)$ is isomorphic to a copy of A, and so clearly there is some sense in which F

contains a map $\mathrm{Mor}(X) \to A$. Specifically, if we define ϵ_A to be the projection

$$\begin{aligned} \epsilon_A \ : \ \mathrm{Mor}\Big(C(A)\Big) &\to A \\ : \ A^3 &\to A \\ : \ (a, b, b') &\mapsto a \end{aligned}$$

then we can use the functor Mor and the map ϵ_A to form the following composite map:

$$\mathrm{Mor}(X) \xrightarrow{\mathrm{Mor}(F)} \mathrm{Mor}\Big(C(A)\Big) \xrightarrow{\epsilon_A} A$$

Then, since A is not just a monoid but an abelian group, we can factor the homomorphism $\mathrm{Mor}(F)$ through the group completion of $\mathrm{Mor}(X)$, and then through the abelianisation of that group, at last yielding a map

$$F' := \epsilon_A \circ \Big(\mathrm{Mor}(F)^{\mathrm{gp}}\Big)^{\mathrm{ab}} : \Big(\mathrm{Mor}(X)^{\mathrm{gp}}\Big)^{\mathrm{ab}} \to A$$

This ϵ will be the counit of our adjunction, with the assignment $F \mapsto F'$ being one direction of the eventual homset adjunction.

Conversely, let η_X be the monoidal functor defined by

$$\begin{aligned} \eta_X \ : \ X &\to C\Big(\mathrm{Mor}(X)^{\mathrm{gp,ab}}\Big) \\ : \ x &\mapsto \mathrm{ag}(\mathrm{id}_x) \\ : \ f : x \to y &\mapsto \Big(\mathrm{ag}(f), \mathrm{ag}(\mathrm{id}_x), \mathrm{ag}(\mathrm{id}_y)\Big) \end{aligned}$$

where for brevity's sake we're using ag to refer to $\mathrm{ab} \circ \mathrm{gp}$, the composite of the group completion map $\mathrm{Mor}(X) \to \mathrm{Mor}(X)^{\mathrm{gp}}$ with the quotient of abelianisation $\mathrm{Mor}(X)^{\mathrm{gp}} \to \mathrm{Mor}(X)^{\mathrm{gp,ab}}$. Then any homomorphism $h : \mathrm{Mor}(X)^{\mathrm{gp,ab}} \to A$ can be used to construct a monoidal functor $h' : X \to C(A)$ as follows:

$$X \xrightarrow{\eta_X} C\Big(\mathrm{Mor}(X)^{\mathrm{gp,ab}}\Big) \xrightarrow{C(h)} C(A)$$

Moreover, if we compare this η with ϵ then we see that the composites

$$\begin{tikzcd}
\mathrm{Mor}(X)^{\mathrm{gp,ab}} \arrow[d, "\mathrm{Mor}(\eta_X)^{\mathrm{gp,ab}}"] \\
\mathrm{Mor}\Big(C\Big(\mathrm{Mor}(X)^{\mathrm{gp,ab}} \Big) \Big)^{\mathrm{gp,ab}} \arrow[d, equal] \\
\mathrm{Mor}\Big(C\Big(\mathrm{Mor}(X)^{\mathrm{gp,ab}} \Big) \Big) \arrow[d, "\epsilon_{\mathrm{Mor}(X)^{\mathrm{gp,ab}}}"] \\
\mathrm{Mor}(X)^{\mathrm{gp,ab}}
\end{tikzcd}
\qquad
\begin{tikzcd}
C(A) \arrow[d, "\eta_{C(A)}"] \\
C\Big(\mathrm{Mor}\Big(C(A) \Big)^{\mathrm{gp,ab}} \Big) \arrow[d, equal] \\
C\Big(\mathrm{Mor}\Big(C(A) \Big) \Big) \arrow[d, "C(\epsilon_A)"] \\
C(A)
\end{tikzcd}$$

must be the respective identity maps:

$$\begin{aligned}
\epsilon_{\mathrm{Mor}(X)^{\mathrm{gp,ab}}} \circ \mathrm{Mor}(\eta_X)^{\mathrm{gp,ab}}\Big(\mathrm{ag}(f : x \to y) \Big) &= \epsilon_{\mathrm{Mor}(X)^{\mathrm{gp,ab}}}\Big(\mathrm{ag}(f), \mathrm{ag}(\mathrm{id}_x), \mathrm{ag}(\mathrm{id}_y) \Big) \\
&= \mathrm{ag}(f)
\end{aligned}$$

$$\begin{aligned}
C(\epsilon_A) \circ \epsilon_{\mathrm{Mor}(X)^{\mathrm{gp,ab}}}(a) &= C(\epsilon_A)(a, a, a) \\
&= a
\end{aligned}$$

$$\begin{aligned}
C(\epsilon_A) \circ \epsilon_{\mathrm{Mor}(X)^{\mathrm{gp,ab}}}(a, b, b') &= C(\epsilon_A)\Big((a, b, b'),\ (b, b, b),\ (b', b', b') \Big) \\
&= (a, b, b')
\end{aligned}$$

In other words, η_X and ϵ_A really are the unit and counit of an adjunction $\mathrm{Mor}(\_)^{\mathrm{gp,ab}} \dashv C$, whose isomorphism of homsets

$$\mathrm{MonCat}(\, X, C(A)\,) \quad \cong \quad \mathrm{Ab}(\, \mathrm{Mor}(X)^{\mathrm{gp,ab}}, A\,)$$

is given by the assigments $F \mapsto F'$ and $h \mapsto h'$. $\square$

Proposition 2.23 seems at first glance very similar to Propositions 2.11 and 2.17. However, our goal was to discover the relationship between the morphisms of $\mathbb{G}_n$ and $L\mathbb{G}_n$, paralleling what we did in Propositions 2.13 and 2.18, and in that regard the adjunction $\mathrm{Mor}(\_)^{\mathrm{gp,ab}} \dashv C$ falls short in two very important ways.

1. What we really wanted was an adjunction involving EGAlg$_S$, not MonCat. This is because η is an initial object in $(\mathbb{G}_n \downarrow \mathrm{inv})$, and so we only know how to use it to factor algebra maps $\mathbb{G}_n \to X_{\mathrm{inv}}$, and not general monoidal functors.

2. We would rather have had the other side of the adjunction be the category Mon instead of Ab. After all, we wanted to use this adjunction to calculate the monoid $\mathrm{Mor}(L\mathbb{G}_n)$, and not the group-completed, abelianised version.

Unfortunately, this adjunction seems to be the best we can do. We already saw that we need A to be an abelian group for $C(A)$ to have composition and interchange, and further for an arbitrary abelian group that we want to be the morphisms of an algebra there does not seem to be a general method for assigning it an EG-action. But all is not lost. It turns out that this approach is fixable, though first we will need to reframe the problem somewhat.

Chapter 3

Free invertible algebras as colimits

In the previous chapter, we made progress towards understanding the structure of $L\mathbb{G}_n$ by showing that the algebra was an initial object in a certain comma category. Specifically, we saw that the map $\eta : \mathbb{G}_n \to L\mathbb{G}_n$ is initial among all EG-algebra maps $\mathbb{G}_n \to X_{\text{inv}}$. This fact is the rigourous way of expressing a fairly obvious intuition about $L\mathbb{G}_n$ — that we should expect the free algebra on n invertible objects to be like the free algebra on n objects, except that its objects are invertible.

However, this not the only way of thinking about $L\mathbb{G}_n$. Consider for a moment the free EG-algebra on $2n$ objects, $\mathbb{G}_{2n}$. Intuitively, if we were to take this algebra and then enforce upon it the extra relations $z_{n+1} = z_1^*, ..., z_{2n} = z_n^*$, then we would be changing it from a structure with $2n$ independent generators into one with n indepedent generators and their inverses. That is, there seems to be a natural way to think about $L\mathbb{G}_n$ as a quotient of the larger algebra $\mathbb{G}_{2n}$. In this chapter we will work towards making this idea precise, and then examine some of its consequences. Together with the information we have already gleaned from the initial object persepctive, this will then provide us with a complete description of the algebra $L\mathbb{G}_n$.

3.1 $L\mathbb{G}_n$ as a cokernel in $\mathrm{E}G\mathrm{Alg}_S$

We'll begin with some definitions.

Definition 3.1. Let δ be the map of EG-algebras defined on generators by

$$\begin{array}{rcrcl} \delta & : & \mathbb{G}_{2n} & \to & \mathbb{G}_{2n} \\ & : & z_i & \mapsto & z_i \otimes z_{n+i} \\ & : & z_{n+i} & \mapsto & z_{n+i} \otimes z_i \end{array}$$

for $1 \leq i \leq n$. We will also denote by $q : \mathbb{G}_{2n} \to Q$ the cokernel this map.

Note that the above definition does actually make sense. The given descriptions of δ is enough to specify it uniquely because $\mathbb{G}_{2n}$ is the free EG-algebra on $2n$ objects, and hence algebra maps $\mathbb{G}_{2n} \to \mathbb{G}_{2n}$ are canonically isomorphic to functions $\{z_1, ..., z_{2n}\} \to \mathrm{ob}(\mathbb{G}_{2n})$. Also we can be sure that the map q exists, because E$G\mathrm{Alg}_S$ is a locally finitely presentable category and thus has all finite colimits.

The goal of this approach will be show that Q is in fact that same algebra as $L\mathbb{G}_n$. In order to do this, it would help if we could easily compare $q : \mathbb{G}_{2n} \to Q$ to our initial object $\eta : \mathbb{G}_{2n} \to L\mathbb{G}_n$. In other words, we really want to show that q is an object of $(\mathbb{G}_n \downarrow \mathrm{inv})$ — that Q has only invertible objects. This can be done using the adjunction we found in Proposition 2.11.

Proposition 3.2. *The object monoid of Q is $\mathbb{Z}^{*n}$, and the restriction of q to objects $\mathrm{Ob}(q) : \mathrm{Ob}(\mathbb{G}_{2n}) \to \mathrm{Ob}(Q)$ is the monoid homomorphism defined on generators as*

$$\begin{array}{rcrcl} \mathrm{Ob}(q) & : & \mathbb{N}^{*2n} & \to & \mathbb{Z}^{*n} \\ & : & z_i & \mapsto & z_i \\ & : & z_{n+i} & \mapsto & z_i^* \end{array}$$

Proof. Consider $\mathrm{Ob}(\delta)$, the restrictions on objects of the algebra maps $\delta : \mathbb{G}_{2n} \to \mathbb{G}_{2n}$. By Lemma 1.24, this is a monoid homomorphism $\mathbb{N}^{*2n} \to \mathbb{N}^{*2n}$, and since Mon is cocomplete it too must have a cokernel. This will be a new homomorphism whose source is $\mathbb{N}^{*2n}$ and whose target is the quotient of $\mathbb{N}^{*2n}$ by the relations $\mathrm{Ob}(\delta)(x) = I$. Remembering Definition 3.1, and that $\mathbb{N}^{*2n}$ is the free monoid on $2n$ generators, this quotient monoid will have the following presentation:

$$\begin{array}{ll} \text{Generators:} & z_1, \, ..., \, z_{2n} \\ \text{Relations:} & z_i \otimes z_{n+i} = I, \\ & z_{n+i} \otimes z_i = I \end{array}$$

This is just the same as

$$\begin{array}{ll} \text{Generators:} & z_1, \, ..., \, z_{2n} \\ \text{Relations:} & z_{n+i} = z_i^*, \end{array}$$

which is the presentation of $\mathbb{Z}^{*n}$.

But by Proposition 2.11, Ob is a left adjoint and hence preserves all colimits. Thus the cokernel of $\mathrm{Ob}(\delta)$ is just the underlying homomorphism of the cokernel of δ. Therefore $\mathrm{Ob}(Q) = \mathbb{Z}^{*n}$, and $\mathrm{Ob}(q)$ is the quotient map $\mathbb{N}^{*2n} \to \mathbb{Z}^{*n}$ sending $z_i \mapsto z_i$ and $z_{n+i} \mapsto z_i^*$ for $1 \leq i \leq n$. $\square$

An immediate corollary of Proposition 3.2 is that every object of the cokernel algebra Q is invertible. Thus $q : \mathbb{G}_{2n} \to Q$ is an object of the category $(\mathbb{G}_n \downarrow \mathrm{inv})$, and hence we can use the initiality of η to determine the following result:

Proposition 3.3. *Let $i : \mathbb{G}_n \to \mathbb{G}_{2n}$ be the inclusion of* EG*-algebras defined on generators by $i(z_i) = z_i$. Then $i \circ q$ is an initial object of* $(\mathbb{G}_n \downarrow \mathrm{inv})$. *In particular, this means that*

$$Q \quad \cong \quad L\mathbb{G}_n$$

Proof. Let $\psi : \mathbb{G}_n \to X$ be an arbitrary object of $(\mathbb{G}_n \downarrow \mathrm{inv})$. Since $\mathbb{G}_n$ is the free EG-algebra on n objects, we can use it and ψ to define a new map, $\psi^* : \mathbb{G}_n \to X$, which takes the values

$$\psi^*(z_i) \quad := \quad \psi(z_i)^*$$

on generators. Clearly psi^* is also an object of $(\mathbb{G}_n \downarrow \mathrm{inv})$, and so using these two we can define a new map, $\psi + \psi^*$, via the universal property of the colimit:

$$\begin{tikzcd}
 & \mathbb{G}_n + \mathbb{G}_n \arrow[dd, dashed, "\psi+\psi^*"] & \\
\mathbb{G}_n \arrow[ur, hook, "i"] \arrow[dr, "\psi"'] & & \mathbb{G}_n \arrow[ul, hook', "i'"'] \arrow[dl, "\psi^*"] \\
 & X &
\end{tikzcd}$$

But because $\mathbb{G}_n$ is the free algebra on n objects, and the free functor $F : \mathrm{Cat} \to \mathrm{E}G\mathrm{Alg}_S$ is a left adjoint and thus preserves colimits, we must have

$$\begin{aligned}
\mathbb{G}_n + \mathbb{G}_n &= F(\{z_1, ..., z_n\}) + F(\{z'_1, ..., z'_n\}) \\
&= F(\ \{z_1, ..., z_n\} + \{z'_1, ..., z'_n\}\ ) \\
&= F(\{z_1, ..., z_{2n}\}) \\
&= \mathbb{G}_{2n}
\end{aligned}$$

This means that we can compose $\psi + \psi^* : \mathbb{G}_{2n} \to X$ with the map $\delta : \mathbb{G}_{2n} \to \mathbb{G}_{2n}$, though we need to be careful to specify exactly which inclusions we used in the definition of $\psi + \psi^*$. Suppose that the lefthand inclusion is i, the one given in the statement of the proposition, and the other is defined by the assignment $z_i \mapsto z_{i+n}$. Then for

$1 \leq i \leq n$,

$$\begin{aligned}(\psi + \psi^*)\delta(z_i) &= (\psi + \psi^*)(z_i \otimes z_{n+i}) \\ &= \psi(z_i) \otimes \psi(z_i)^* \\ &= I\end{aligned}$$

$$\begin{aligned}(\psi + \psi^*)\delta(z_{n+i}) &= (\psi + \psi^*)(z_{n+i} \otimes z_i) \\ &= \psi(z_i)^* \otimes \psi(z_i) \\ &= I\end{aligned}$$

That is, $(\psi + \psi^*) \circ \delta = I$. But we've already defined $q : \mathbb{G}_{2n} \to Q$ to be the cokernel of δ, the universal map with this property, and so there must exist a unique EG-algebra map $u : Q \to X$ making the righthand triangle below diagram commute:

$$\begin{tikzcd} \mathbb{G}_n \arrow[r, hook, "i"] \arrow[dr, "\psi"'] & \mathbb{G}_{2n} \arrow[r, "q"] \arrow[d, "\psi+\psi^*"] & Q \arrow[dl, "u"] \\ & X & \end{tikzcd}$$

The other triangle commutes by the definition of $\psi + \psi^*$, and so together the diagram tells us that for any object ψ of $(\mathbb{G}_n \downarrow \text{inv})$, there exists at least one morphism u in $(\mathbb{G}_n \downarrow \text{inv})$ going from $q \circ i$ to ψ.

Next, let $v : Q \to X$ be an arbitrary morphism $q \circ i \to \psi$ in $(\mathbb{G}_n \downarrow \text{inv})$. By definition, this means that

$$\begin{aligned} \psi &= vqi \\ \Longrightarrow \quad \psi + \psi^* &= vqi + (vqi)^* \end{aligned}$$

Also, for $1 \leq i \leq n$ we have

$$\begin{aligned} q(z_i) \otimes q(z_{n+i}) &= q(z_{i-n} \otimes z_i) = q\delta(z_i) = I \\ q(z_{n+i}) \otimes q(z_i) &= q(z_i \otimes z_{n+i}) = q\delta(z_{n+i}) = I \\ \Longrightarrow \quad & q(z_{n+i}) = q(z_i)^* \end{aligned}$$

Therefore,

$$\begin{aligned}(\psi + \psi^*)(z_i) &= \big(vqi + (vqi)^*\big)(z_i) \\ &= vqi(z_i) \\ &= vq(z_i)\end{aligned}$$

$$
\begin{aligned}
(\psi + \psi^*)(z_{n+i}) &= \Big(vqi + (vqi)^*\Big)(z_{n+i}) \\
&= vqi(z_i)^* \\
&= v\Big(q(z_i)^*\Big) \\
&= vq(z_{n+i})
\end{aligned}
$$

or in other words $\psi + \psi^* = v \circ q$ for any morphism $v : q \circ i \to \psi$ in $(\mathbb{G}_n \downarrow \mathrm{inv})$. But this is the property that the map u was supposed to satisfy uniquely, and thus it must be the only morphism $q \circ i \to \psi$ in $(\mathbb{G}_n \downarrow \mathrm{inv})$. Therefore $q \circ i$ is an initial object, and hence it is isomorphic in $(\mathbb{G}_n \downarrow \mathrm{inv})$ to any other initial object, such as η. It follows that the targets of these two maps, Q and $L\mathbb{G}_n$ respectively, are isomorphic as EG-algebras. □

It's worth noting that we have not given a method for actually taking cokernels in EGAlg$_S$, and so Proposition 3.3 doesn't immediately provide an explicit description for the whole of $L\mathbb{G}_n$. However, it does offer us another way to extract partial information, like what we were doing in Chapter 2. Consider Proposition 3.2; now that we know that Q is actually $L\mathbb{G}_n$, the statement of this proposition is just the same as that of Proposition 2.13. But the proof of the former uses the ability of cokernels to preserve left adjoint functors, rather than any of the initial algebra and group completion properties that appear in the latter.

Of course, by Proposition 3.3 the fact that q is a cokernel is equivalent to it being initial, and so while they may not look it at first glance, these two approaches are secretly the same. Thus from now on whenever we are trying to determine some aspect of $L\mathbb{G}_n$, we will make sure to take a look at both methods, just in case there are some properties of our free algebra which are more readily apparent from one description than another.

3.2 $L\mathbb{G}_n$ as a surjective coequaliser

One immediate consequence our new cokernel perspective of $L\mathbb{G}_n$ is that, since left adjoint functor all preserve colimits, Propositions 2.11 and 2.17 now both imply results about the partial surjectivity of this new map q. The former says that since $\mathrm{Ob}(q)$ is a cokernel map of monoids, and hence that every object of $L\mathbb{G}_n$ is the image under q of some object of $\mathbb{G}_{2n}$; the latter says a similar thing for connected components. From this one might guess that q is will just turn out to be a surjective map of EG-algebras, and indeed this is the case. Moreover, much as Proposition 2.13 is an analogue of

Lemma 1.24, the fact that q is surjective on morphisms means that there will be a result analagous to Lemma 1.25 as well. That is, since every morphism of $\mathbb{G}_{2n}$ is an action morphism, and since EG-algebra maps always send action morphisms to action morphisms, if q is surjective then every morphism of $L\mathbb{G}_n$ is also an action morphism.

Unfortunately, we can not go about proving that q is surjective on morphisms by a similar adjunction technique, since this best we have is the one from Proposition 2.23 and it will only tell us about the map $\mathrm{Mor}(q)^{\mathrm{gp,ab}}$. However, there is a general result about the coequalisers of EG-algebras that we can prove to get us around this.

Proposition 3.4. *Let $\phi, \phi' : X \to Y$ be a pair of parallel* EG*-algebra maps, and $k : Y \to Z$ their coequalizer in* $\mathrm{E}G\mathrm{Alg}_S$*. If the monoid* $\mathrm{Ob}(Z)$ *is also a group, then the functor k is surjective.*

Proof. We begin by mirroring the proof of Proposition 3.2. We know that the functor $\mathrm{Ob} : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{Mon}$ is a left adjoint, by Proposition 2.11, and thus preserves all colimits. It follows that the monoid homomorphism $\mathrm{Ob}(k) : \mathrm{Ob}(Y) \to \mathrm{Ob}(Z)$ is the coequaliser of the parallel pair $\mathrm{Ob}(\phi), \mathrm{Ob}(\phi') : \mathrm{Ob}(X) \to \mathrm{Ob}(Y)$ in Mon, or in other words

$$\mathrm{Ob}(Z) \quad = \quad {\mathrm{Ob}(Y)}\big/{\sim}$$

where $\sim$ is the relation defined by

$$\mathrm{Ob}(\phi)(y) \sim \mathrm{Ob}(\phi')(y), \qquad\qquad a \sim a', b \sim b' \implies ab \sim a'b'$$

The map $\mathrm{Ob}(k) : \mathrm{Ob}(Y) \to \mathrm{Ob}(Y)/\sim$ is then clearly surjective.

Next, let $f : v \to w$ and $f' : w' \to v'$ be any two morphisms of the algebra Y for which $k(f)$ and $k(f')$ are composable in Z. Since these maps are composable we know that $k(w)$ and $k(w')$ must be the same object of Z, and since Z is a group we know this object has an inverse $k(w)^* = k(w')^*$. So by the surjectivity of k we can find another object y of Y for which $k(y) = k(w)^*$. Using this, define the morphism $h : x \to x'$ to be the tensor product $f' \otimes \mathrm{id}_y \otimes f$. Then

$$\begin{aligned} k(h) &= k(f' \otimes \mathrm{id}_y \otimes f) \\ &= k(f') \otimes \mathrm{id}_{k(y)} \otimes k(f) \\ &= k(f') \otimes \mathrm{id}_{k(w)^*} \otimes k(f) \end{aligned}$$

But by Lemma 2.20, this is really just the composite $k(f') \circ k(f)$. Thus the set of morphisms of Z which are images of morphisms of Y is closed under composition.

So now consider $k(Y)$, the subcategory of Z that contains every object x' for which there exists x in Y with $k(x) = x'$, and every morphism f' for which there exists f in Y with $q(f) = f'$. We know that the morphisms of $k(Y)$ are closed under composition, and so this is indeed a well-defined category. Moreover, for any collection of morphisms $f'_1, ..., f'_m$ of $k(Y)$ we'll have

$$\begin{aligned} \alpha_Z(g; f'_1, ..., f'_m) &= \alpha_Z\Big(g\,;\, k(f_1), ..., k(f_m) \Big) \\ &= k\Big(\alpha_Y(g; f_1, ..., f_m) \Big) \\ &\in k(Y) \end{aligned}$$

for some $f_1, ..., f_m$, since k is a map of EG-algebras. Thus $k(Y)$ is also a well-defined sub-EG-algebra of Z. There is also clearly a canonical map $k' : Y \to k(Y)$, the unique surjective map of EG-algebras with the property that $k'(x) = k(x)$ for any object x and $k'(f) = k(f)$ for any morphism f. If we denote by i the evident inclusion of algebras $i : k(Y) \hookrightarrow Z$, then these maps are related by the fact that $i \circ k' = k$.

$$\begin{tikzcd} & X \arrow[d, bend right, "\phi"'] \arrow[d, bend left, "\phi'"] & \\ & Y \arrow[dl, "k'"'] \arrow[d, "k"] \arrow[dr, "j"] & \\ k(Y) \arrow[r, hook, "i"] & Z \arrow[r, "u"] & U \end{tikzcd}$$

Given all of this, let $j : Y \to U$ be any map of EG-algebras with the property that $j \circ \phi = j \circ \phi'$. Since h is the coequaliser of ϕ and ϕ', it follows that there exists a unique map $u : Y \to U$ such that $j = u \circ k$. This means that $j = u \circ i \circ k'$, and hence there is obviously at least one map, $u \circ i$, which lets us factors j through k'. But for any other map $v : k(Y) \to U$ that factors j like this, we'll have

$$\begin{aligned} v \circ k' &= j \\ &= u \circ i \circ k' \\ \implies \qquad v &= u \circ i \end{aligned}$$

because k' is surjective, and thus $u \circ i$ is the unique map with this property. That is, k' is also a coequaliser of ϕ and ϕ'. But colimits are always unique up to a unique isomorphism, and so there should be a unique invertible map $k(Y) \to Z$ factoring k

through k'. This is clearly just the inclusion i, and as a result $k(Y) = Z$ and $k' = k$. In other words, the map coequaliser mapm k is surjective. $\square$

Because a cokernel of a morphism is just its coequaliser with the zero map, and since we know that the objects of $L\mathbb{G}_n$ form a group, we can immediately apply this result to the functor q.

Corollary 3.5. *The cokernel map $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$ is surjective.*

As we noted earlier, knowing that q is surjective will now allow use to take Lemma 1.25, a statement about the morphisms $\mathbb{G}_{2n}$, and extend the result onto $L\mathbb{G}_n$:

Lemma 3.6. *Every morphism in $L\mathbb{G}_n$ can be expressed as $\alpha_{L\mathbb{G}_n}(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$, for some $g \in G(m)$ and $x_i \in \{z_1, ..., z_n, z_1^*, ..., z_n^*\}$.*

Proof. Let f be an arbitrary morphism in $L\mathbb{G}_n$. By surjectivity of q, there must exist at least one morphism f' in $\mathbb{G}_{2n}$ such that $q(f') = f$, and from Lemma 1.25 we know that this f' can be expressed uniquely as $\alpha(g; \mathrm{id}_{x_1'}, ..., \mathrm{id}_{x_m'})$ for some $g \in G(m)$ and $x_i' \in \{z_1, ..., z_{2n}\}$. Thus, because q is a map of EG-algebras, we will have

$$\begin{aligned} f &= q(f') \\ &= q\Big(\alpha_{\mathbb{G}_{2n}}(\, g \,;\, \mathrm{id}_{x_1'}, ..., \mathrm{id}_{x_m'} \,) \Big) \\ &= \alpha_{L\mathbb{G}_n}(\, g \,;\, \mathrm{id}_{q(x_1')}, ..., \mathrm{id}_{q(x_m')} \,) \end{aligned}$$

Therefore there is at least one collection of $x_i = q(x_i')$ for which the statement of the proposition holds. $\square$

Lemma 3.6 formalises a certain intuition about how the functor L should act on algebras, the idea that a 'free' structure really shouldn't have any 'superfluous' components, only whatever data is absolutely required for it to be well-defined. In the case of $L\mathbb{G}_n$, we have proven that the only morphisms contained in the free EG-algebra on invertible objects are EG-action morphisms. However, while this is very similar to what we have in the non-invertible case it should be stressed that Lemma 3.6 does *not* prove that the morphisms of $L\mathbb{G}_n$ have *unique* representations $\alpha(g; \mathrm{id}_{w_1}, ..., \mathrm{id}_{w_m})$, as morphisms of $\mathbb{G}_n$ do.

3.3 $L\mathbb{G}_n$ as a colimit in MonCat

Looking back at the proof of Proposition 3.4, notice that we never needed to use the fact that ϕ, ϕ' and k were maps of EG-algebras, only that they were monoidal functors.

Because we had assumed from the beginning that we were working in $\mathrm{E}G\mathrm{Alg}_S$, we did at one point have to show that the category $k(Y)$ was an algebra, so that we could then use the universal property of k in $\mathrm{E}G\mathrm{Alg}_S$, but if k had just been a coequaliser in MonCat from the start then this part would not have been necessary. We also had to invoke Proposition 2.11 — which says that $\mathrm{Ob} : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{Mon}$ is a left adjoint — so that we could exploit preservation of colimits. But since Ob clearly doesn't care about the morphisms of an algebra, it doesn't really matter whether we are applying it to an algebra in the first place. The actions of X, Y and Z just never came into play.

With that in mind, we can co-opt all of these previous proofs about EG-algebra maps to prove the analagous statements about monoidal functors.

Proposition 3.7. *Let the functors*

$$\mathrm{Ob} \;:\; \mathrm{MonCat} \to \mathrm{Mon}, \qquad\qquad \mathrm{E} \;:\; \mathrm{Mon} \to \mathrm{MonCat}$$

be defined exactly as those from Definitions 2.9 and 2.10, except without the requirement that the monoidal categories be EG*-algebras. Then* E *is a right adjoint to the functor* Ob*.*

Proof. The same as the proof of Proposition 2.11. □

Proposition 3.8. *Let $\phi, \phi' : X \to Y$ be a pair of parallel monoidal functors, and $k : Y \to Z$ their coequalizer in* Mon*. If the monoid* $\mathrm{Ob}(Z)$ *is also a group, then the functor k is surjective.*

Proof. The same as the proof of Proposition 3.4, but with Proposition 3.7 in place of Proposition 2.11, and no reference to $k(Y)$ being a sub-EG-algebra. □

Further, these new propositions prove a surjectivity statement just like Corollary 3.5.

Definition 3.9. Let the monoidal functor $c : \mathbb{G}_{2n} \to C$ onto some monoidal category C be the cokernel of the underlying monoidal functor of δ in MonCat. This map definitely exists because MonCat is cocomplete, and like with q we can show that its target has a group of objects.

Proposition 3.10. *The object monoid of C is $\mathbb{Z}^{*n}$, and the restriction of c to objects* $\mathrm{Ob}(c) : \mathrm{Ob}(\mathbb{G}_{2n}) \to \mathrm{Ob}(C)$ *is the monoid homomorphism defined on generators as*

$$\begin{array}{rcrcl} \mathrm{Ob}(c) & : & \mathbb{N}^{*2n} & \to & \mathbb{Z}^{*n} \\ & : & z_i & \mapsto & z_i \\ & : & z_{n+i} & \mapsto & z_i^* \end{array}$$

Proof. The same as the proof of Proposition 3.2, but with $c : \mathbb{G}_{2n} \to C$ in place of $q : \mathbb{G}_{2n} \to Q$ and Proposition 3.7 in place of Proposition 2.11. $\square$

Propositions 3.8 and 3.10 then immediately combine to give:

Corollary 3.11. *The cokernel map $c : \mathbb{G}_{2n} \to C$ is surjective.*

This statement is actually pretty unusual. In Corollary 3.5 it made sense that q would be surjective, but that was because its source and target were special. $\mathbb{G}_{2n}$ is the free EG-algebra on $2n$ objects, and $L\mathbb{G}_n$ is the free EG-algebra on n objects and their n inverses, and so intuitively the map identifying those sets generators would tell us everything we need to know about the algebra structure of $L\mathbb{G}_n$. And since by freeness we expect algebra maps to be all there really is to $L\mathbb{G}_n$, it was a safe bet that q was going to be surjective.

But none of that is true for c. The underlying monoidal category of $\mathbb{G}_{2n}$ is not anything special in MonCat, and neither is C. So what is going on here? The answer is that category C is *almost* the algebra $L\mathbb{G}_n$, and likewise the functor c is *almost* the map q. To see this, consider the following niave method for assigning an EG-action α^C to C:

$$\alpha^C(\, g\, ;\, c(f_1), ..., c(f_m)\,) \quad := \quad c\Big(\, \alpha^{\mathbb{G}_{2n}}(\, g\, ;\, f_1, ..., f_m\,)\Big)$$

Any action on C that made c into a map of EG-algebras would have to satify this condition, of course. But because c is surjective, every collection of morphisms in C can be written as $c(f_1), ..., c(f_m)$, and this forces α^C to take a unique value everywhere, assuming it is well-defined. Then, since the the cokernel of δ in MonCat would be an EG-algebra map, we could conclude that it was also the cokernel of δ in E$G\mathrm{Alg}_S$ too.

However, 'assuming it is well-defined' is where the problems lie. In particular, since c is not injective on objects we can find $w_1, ..., w_m$ and $w'_1, ..., w'_m$ in $\mathbb{G}_{2n}$ for which $c(w_i) = c(w'_i)$, and so α^C would only be well-defined if

$$c\Big(\, \alpha^{\mathbb{G}_{2n}}(\, g\, ;\, \mathrm{id}_{w_1}, ..., \mathrm{id}_{w_m}\,)\Big) \quad = \quad c\Big(\, \alpha^{\mathbb{G}_{2n}}(\, g\, ;\, \mathrm{id}_{w'_1}, ..., \mathrm{id}_{w'_m}\,)\Big)$$

which we have no reason to believe is true. Fixing this issue is not to difficult though, it just means that we have to employ yet more colimits.

Definition 3.12. Denote by P following the pushout in MonCat:

$$\begin{tikzcd}
\mathrm{E}G \times_{\mathbb{N}} F(\mathbb{G}_{2n}) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}\times F(c)"'] & \mathbb{G}_{2n} \arrow[r, "c"] & C \arrow[d, "i_C"] \\
\mathrm{E}G \times_{\mathbb{N}} F(C) \arrow[rr, "i_{\mathrm{E}G\times F(C)}"] & & P \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

We'll also denote the composite $i_C \circ c$ by the name p.

The form of this pushout has been chosen precisely so that the category P and the functor p will retain all of the important features that we liked about C and c, whilst eventually allowing us to show that they belong in the category $\mathrm{E}G\mathrm{Alg}_S$. In particular:

Lemma 3.13. *The functor $p : \mathbb{G}_{2n} \to P$ is surjective.*

Proof. We know from Corollary 3.11 that c is surjective, and hence so is the map $\coprod_m(\mathrm{id} \times c^m) : \coprod_m \mathrm{E}G(m) \times (\mathbb{G}_{2n})^m \to \coprod_m \mathrm{E}G(m) \times C^m$. But surjective functors are preserved by pullback, and so the map i_C must also be surjective, from which it follows that $p = i_C \circ c$ is too. □

Proposition 3.14. *The object monoid of P is $\mathbb{Z}^{*n}$, and the restriction of $p : \mathbb{G}_{2n} \to P$ to objects is the same as the underlying monoid homomorphism of c.*

Proof. To start off with, take the pushout diagram in Definition 3.12 and apply the functor $\mathrm{Ob} : \mathrm{MonCat} \to \mathrm{Mon}$ to the whole thing. Since this is a left adjoint functor, the resulting diagram,

$$\begin{tikzcd}
G \times_{\mathbb{N}} F(\mathbb{N}^{*2n}) \arrow[r, "\mathrm{Ob}(\alpha)"] \arrow[d, "\mathrm{id}\times \mathrm{Ob}(Fc)"'] & \mathbb{N}^{*2n} \arrow[r, "\mathrm{Ob}(c)"] & \mathbb{Z}^{*n} \arrow[d, "\mathrm{Ob}(i_C)"] \\
G \times F(\mathbb{Z}^{*n}) \arrow[rr, "\mathrm{Ob}(i_{\mathrm{E}G\times F(C)})"] & & \mathrm{Ob}(P) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

must be a pullback diagram of monoids. But recall that $\alpha(g; x_1, ..., x_m) = x_1 \otimes ... \otimes x_m$ for any objects x_i and element $g \in G(m)$. This means that the homomorphism $\mathrm{Ob}(\alpha)$ is just the map turning concatenation in $F(\mathbb{N}^{*2n})$ into multiplication and ignoring the G component. It is also possible to form a similar concatenation-to-multiplication map $f : G \times F(\mathbb{Z}^{*n}) \to \mathbb{Z}^{*n}$, and since $\mathrm{Ob}(c)$ as a monoid homomorphism respects multiplication, the top-left triangle created by this new map below will commute.

$$\begin{tikzcd}
G \times_{\mathbb{N}} F(\mathbb{N}^{*2n}) \arrow[r, "\mathrm{Ob}(\alpha)"] \arrow[d, "\mathrm{id}\times\mathrm{Ob}(Fc)"'] & \mathbb{N}^{*2n} \arrow[r, "\mathrm{Ob}(c)"] & \mathbb{Z}^{*n} \arrow[d, "\mathrm{Ob}(i_C)"] \\
G \times F(\mathbb{Z}^{*n}) \arrow[rru, "f"] \arrow[rr, "\mathrm{Ob}(i_{\mathrm{E}G\times F(C)})"] & & \mathrm{Ob}(P) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Because c is surjective, $\mathrm{id} \times \mathrm{Ob}(Fc)$ will be to, and this lets us conclude that the bottom-right triangle will also commute. In other words, the pushout map $\mathrm{Ob}(i_{\mathrm{E}G\times F(C)})$ is just the composite $\mathrm{Ob}(i_C) \circ f$.

Indeed, the same reasoning tells us that for any monoid M which is a cocone of the maps $\mathrm{id} \times \mathrm{Ob}(Fc)$ and $\mathrm{Ob}(c) \circ \mathrm{Ob}(\alpha)$, the canonical map $G \times F(\mathbb{Z}^{*n}) \to M$ will just be the composite of the map $\mathbb{Z}^{*n} \to M$ with f. If we denote by $u : \mathrm{Ob}(P) \to M$ the unique map we get from the universal property of $\mathrm{Ob}(P)$, the condition that u factorises both of the maps into M will then just depend on its ability to uniquely factorise the one corresponding to $\mathrm{Ob}(i_C)$. But if we choose $\mathrm{Ob}(i_C)$ to be the identity homomorphism, then this property holds automatically — the unique value of u would just be the same as the canonical $\mathbb{Z}^{*n} \to M$. Thus the monoid $\mathbb{Z}^{*n}$ equipped with the maps $\mathrm{id}_{\mathbb{Z}^{*n}}$ and f is pushout of our diagram in Mon, and hence

$$\mathrm{Ob}(P) \;\cong\; \mathbb{Z}^{*n}, \qquad\qquad \begin{aligned} \mathrm{Ob}(p) &= \mathrm{Ob}(i_C) \circ \mathrm{Ob}(c) \\ &= \mathrm{id}_{\mathbb{Z}^{*n}} \circ \mathrm{Ob}(c) \\ &= \mathrm{Ob}(c) \end{aligned}$$

□

With these results in hand, a simple calculation causes the algebra nature of P to drop straight out of its definition.

Proposition 3.15. *There is a unique action α^P making the category P into* EG*-algebra and the functor $p : \mathbb{G}_{2n} \to P$ into a map of* EG*-algebras.*

Proof. We will try to affix an action to P in the same way we thought about doing with the category C. In order for the functor $p : \mathbb{G}_{2n} \to P$ to be an EG-algebra map with respect to some α^P, it must satisfy

$$\alpha^P(\, g\, ;\, p(f_1), ..., p(f_m)\,) \quad = \quad p\Big(\, \alpha^{\mathbb{G}_{2n}}(\, g\, ;\, f_1, ..., f_m\,)\Big)$$

for all morphisms $f_1, ..., f_m$ in $\mathbb{G}_{2n}$. But from Lemma 3.13 we know that p is surjective, and so this condition will actually suffice as a definition for α^P, provided that we can prove it to be well-defined. To that end, let $f_i : v_i \to w_i$ and $f_i' : v_i' \to w_i'$, for $1 \leq i \leq m$, be any two sequences of morphisms in $\mathbb{G}_{2n}$ such that $p(f_i) = p(f_i')$. Then by Proposition 3.14,

$$\begin{array}{rrcl} & p(f_i : v_i \to w_i) & = & p(f_i' : v_i' \to w_i') \\ \Longrightarrow & p(w_i) & = & p(w_i') \\ \Longrightarrow & c(w_i) & = & c(w_i') \end{array}$$

and hence

$$\begin{array}{rcl} p\alpha^{\mathbb{G}_{2n}}(\, g\, ;\, f_1, ..., f_m\,) & = & p\Big(\, \alpha^{\mathbb{G}_{2n}}(\, g\, ;\, \mathrm{id}_{w_1}, ..., \mathrm{id}_{w_m}\,) \circ (f_1 \otimes ... \otimes f_m)\,\Big) \\ & = & p\alpha^{\mathbb{G}_{2n}}(\, g\, ;\, \mathrm{id}_{w_1}, ..., \mathrm{id}_{w_m}\,) \circ \Big(\, p(f_1) \otimes ... \otimes p(f_m)\,\Big) \\ & = & i_C c\alpha^{\mathbb{G}_{2n}}(\, g\, ;\, \mathrm{id}_{w_1}, ..., \mathrm{id}_{w_m}\,) \circ \Big(\, p(f_1) \otimes ... \otimes p(f_m)\,\Big) \\ & = & i_{\mathrm{E}G\times F(C)}(\, g\, ;\, \mathrm{id}_{c(w_1)}, ..., \mathrm{id}_{c(w_m)}\,) \circ \Big(\, p(f_1) \otimes ... \otimes p(f_m)\,\Big) \\ & = & i_{\mathrm{E}G\times F(C)}(\, g\, ;\, \mathrm{id}_{c(w_1')}, ..., \mathrm{id}_{c(w_m')}\,) \circ \Big(\, p(f_1') \otimes ... \otimes p(f_m')\,\Big) \\ & = & p\alpha^{\mathbb{G}_{2n}}(\, g\, ;\, f_1', ..., f_m'\,) \end{array}$$

Therefore the value of $\alpha^P(\, g\, ;\, p(f_1), ..., p(f_m)\,)$ does not depend on our particular choice of f_i, and so α^P is indeed well-defined. Thus the pushout category P is an EG-algebra, and p an EG-algebra map. □

Now we're finally ready to address problem 1 from the end of the previous chapter: how can we deal with the fact that our adjunction $\mathrm{Mor}(\_)^{\mathrm{gp,ab}} \dashv C$ just involves monoidal categories and not full EG-algebras? The answer is that this is all we really need, as despite us originally conceiving of $L\mathbb{G}_n$ as a colimit in E$G\mathrm{Alg}_S$ it can equally be viewed as a slightly more complicated colimit in MonCat.

Proposition 3.16. *The colimit functor $p : \mathbb{G}_{2n} \to P$ defined in Definition 3.12 is isomorphic as an map of* EG*-algebras to $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$, the cokernel of δ in* E$G\mathrm{Alg}_S$.

Proof. First, consider what we know of the functor p. By definition it is equal to the composite $i_C \circ c$, where $c : \mathbb{G}_{2n} \to C$ is the cokernel of the map $\delta : \mathbb{G}_{2n} \to \mathbb{G}_{2n}$ in MonCat, and hence p has the property

$$p \circ \delta \quad = \quad i_C \circ c \circ \delta \quad = \quad i_C \circ I \quad = \quad I$$

Moreover, given what we saw in Proposition 3.15 we know that p is map of EG-algebras with this property. But the cokernel map q is universal amongst maps like these, and so it follows that there must exist a unique map of EG-algebras $u : L\mathbb{G}_n \to P$ factoring p through q.

Conversely, now consider the algebra map q. We know that it is a monoidal functor for which $q \circ \delta = I$, and that c is the universal map in MonCat with this property. Thus there exists a unique monoidal functor $v : C \to L\mathbb{G}_n$ with $q = v \circ c$, or put another way, the two triangles in the diagram below both commute.

$$\begin{tikzcd}
\mathrm{E}G \times_{\mathbb{N}} F(\mathbb{G}_{2n}) \arrow[r, "\alpha^{\mathbb{G}_{2n}}"] \arrow[d, "\mathrm{id}\times F(c)"'] \arrow[dr, "\mathrm{id}\times F(q)"] & \mathbb{G}_{2n} \arrow[r, "c"] \arrow[dr, "q"] & C \arrow[d, "v"] \\
\mathrm{E}G \times_{\mathbb{N}} F(C) \arrow[r, "\mathrm{id}\times F(v)"'] & \mathrm{E}G \times_{\mathbb{N}} F(L\mathbb{G}_n) \arrow[r, "\alpha^{L\mathbb{G}_n}"'] & L\mathbb{G}_n
\end{tikzcd}$$

The middle square here also commutes — due to the fact that q is a map of EG-algebras — and so we can conclude that the entire diagram does as well. But this just says that $L\mathbb{G}_n$ is a cocone of the same diagram we used to define the pushout category P, and hence that there exists some monoidal functor $u' : P \to L\mathbb{G}_n$ factoring the diagram above through the one in Definition 3.12. In particular, we get that

$$u' \circ i_C \quad = \quad v \qquad \Longrightarrow \qquad u' \circ p \quad = \quad u' \circ i_C \circ c \quad = \quad v \circ c \quad = \quad q$$

Putting this fact together with $u \circ q = p$ that we saw earlier, and also the surjectivity of both q and p (from Corollary 3.5 and Lemma 3.13 respectively), we can conclude that the maps u and u' form a isomorphism of monoidal categories:

$$\begin{array}{rclclcrcl}
u \circ u' \circ p & = & u \circ q & = & p & \Longrightarrow & u \circ u' & = & \mathrm{id}_P \\
u' \circ u \circ q & = & u' \circ p & = & q & \Longrightarrow & u' \circ u & = & \mathrm{id}_{L\mathbb{G}_n}
\end{array}$$

Furthermore, not only is u an algebra map but u' is too, as by surjectivity of p we have

$$\begin{aligned}
u'\Big(\alpha^P(\, g \,;\, f_1, ..., f_m \,) \Big) &= u'\Big(\alpha^P(\, g \,;\, p(f_1'), ..., p(f_m') \,) \Big) \\
&= u'p\Big(\alpha^{\mathbb{G}_{2n}}(\, g \,;\, f_1', ..., f_m' \,) \Big) \\
&= q\Big(\alpha^{\mathbb{G}_{2n}}(\, g \,;\, f_1', ..., f_m' \,) \Big) \\
&= \alpha^{L\mathbb{G}_n}(\, g \,;\, q(f_1'), ..., q(f_m') \,) \Big) \\
&= \alpha^{L\mathbb{G}_n}(\, g \,;\, u'p(f_1'), ..., u'p(f_m') \,) \Big) \\
&= \alpha^{L\mathbb{G}_n}(\, g \,;\, u'(f_1), ..., u(f_m) \,) \Big)
\end{aligned}$$

Therefore u and u' are also an isomorphism of EG-algebras. In other words $P \cong L\mathbb{G}_n$, and up to this isomorphism the maps $p : \mathbb{G}_{2n} \to P$ and $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$ are the same. □

3.4 The abelianised morphisms of $L\mathbb{G}_n$

With our newfound ability to express the map $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$ as a colimit of monoidal categories, we can now set about using the adjunction from Proposition 2.23 to calculate $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$. The most obvious way to do this is to mimic what we did in Proposition 3.2 — apply the left adjoint functor to q and then commute it with the colimit to get a formula in terms of the known monoid $\mathrm{Mor}(\mathbb{G}_{2n})$.

Proposition 3.17. *The abelianisation of the group completion of the morphism monoid of $L\mathbb{G}_n$ is*

$$\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \quad = \quad \mathrm{Mor}(C)^{\mathrm{gp,ab}} \quad = \quad \mathrm{Mor}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \Big/ \mathrm{im}\Big(\mathrm{Mor}(\delta)^{\mathrm{gp,ab}}\Big)$$

Proof. From Proposition 2.23, we know that $\mathrm{Mor}(\_)^{\mathrm{gp,ab}} : \mathrm{MonCat} \to \mathrm{Ab}$ is a left adjoint functor. This means that it preserves all colimits in MonCat, including the cokernel category C and the pushout category P, which from Proposition 3.16 we now know to be $L\mathbb{G}_n$.

For the first of these, we have

$$\mathrm{coker}\Big(\mathrm{Mor}(\delta)^{\mathrm{gp,ab}}\Big) \quad = \quad \mathrm{Mor}\Big(\mathrm{coker}(\delta)\Big)^{\mathrm{gp,ab}} \quad = \quad \mathrm{Mor}(c)^{\mathrm{gp,ab}}$$

or in other words, the following is a cokernel diagram in the category of abelian groups:

$$\mathrm{Mor}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \xrightarrow{\mathrm{Mor}(\delta)^{\mathrm{gp,ab}}} \mathrm{Mor}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \xrightarrow{\mathrm{Mor}(q)^{\mathrm{gp,ab}}} \mathrm{Mor}(C)^{\mathrm{gp,ab}}$$

But the cokernel of an abelian group homomorphism is just the quotient of its target group by its image. Hence in this case we have

$$\mathrm{Mor}(C)^{\mathrm{gp,ab}} \quad = \quad \mathrm{Mor}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \Big/ \mathrm{im}\Big(\mathrm{Mor}(\delta)^{\mathrm{gp,ab}}\Big)$$

Likewise, using the definition of P we can construct the pushout diagram of abelian groups below:

$$\begin{tikzcd}
\mathrm{Mor}\Big(\mathrm{E}G \times_{\mathbb{N}} F(\mathbb{G}_{2n})\Big)^{\mathrm{gp,ab}} \arrow[r, "\mathrm{Mor}(\alpha)^{\mathrm{gp,ab}}"] \arrow[d, "\mathrm{Mor}\big(\mathrm{id}\times F(c)\big)^{\mathrm{gp,ab}}"] & \mathrm{Mor}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \arrow[r, "\mathrm{Mor}(c)^{\mathrm{gp,ab}}"] & \mathrm{Mor}(C)^{\mathrm{gp,ab}} \arrow[d, "\mathrm{Mor}(i_C)^{\mathrm{gp,ab}}"] \\
\mathrm{Mor}\Big(\mathrm{E}G \times_{\mathbb{N}} F(C)\Big)^{\mathrm{gp,ab}} \arrow[rr, "\mathrm{Mor}(i_{\mathrm{E}G\times F(C)})^{\mathrm{gp,ab}}"] & & \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Now, for two arbitrary abelian group homomorphisms $h : A \to B$, $h' : A \to B'$, their pushout is the quotient of $B \times B'$ by a subgroup H which contains all elements of the form $(h_1(a), h_2(a)^{-1})$. Moreover, if the map h' is surjective then the image in $(B \times B')/H$ of any element $(b, b') \in B \times B'$ is the same as one whose second coordinate is the identity, since

$$\begin{aligned}
[\,(b, b')\,] &= \Big[\Big(b, h'(a)\Big)\Big] \\
&= \Big[\Big(b, h'(a)\Big) \cdot \Big(h(a), h'(a)^{-1}\Big)\Big] \\
&= \Big[\Big(b \cdot h(a), e\Big)\Big]
\end{aligned}$$

and elements of this form are equivalent if and only if their first coordinates differ by an element of $h(\ker(h'))$:

$$\begin{array}{lcccl}
[\,(b_1, e)\,] = [\,(b_2, e)\,] & \Longrightarrow & \exists\, a \in A & : & (b_1, e) = (b_2, e)\Big(h(a), h'(a)^{-1}\Big) \\
& \Longrightarrow & \exists\, a \in A & : & (b_1, e) = \Big(b_2 \cdot h(a), h'(a)^{-1}\Big) \\
& \Longrightarrow & \exists\, a \in \ker(h') & : & b_1 = b_2 \cdot h(a)
\end{array}$$

Therefore the pushout $(B \times B')/H$ is really just the same as the quotient $B/h(\ker(h'))$.

Returning to our specific homomorphisms, we know from Corollary 3.11 that c is surjective, and hence so is $\mathrm{id} \times F(c)$. In particular it is surjective on morphisms, and so it follows that the homomorphism $\mathrm{Mor}(\mathrm{id}_{\mathrm{E}G} \times F(c))^{\mathrm{gp,ab}}$ will also be surjective. This means that for our pushout we'll have

$$\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}} = \mathrm{Mor}(C)^{\mathrm{gp,ab}} \Big/ \mathrm{Mor}(c \circ \alpha)^{\mathrm{gp,ab}}\Big(\ker\Big(\mathrm{Mor}\Big(\mathrm{id}_{\mathrm{E}G} \times F(c)\Big)^{\mathrm{gp,ab}}\Big)\Big)$$

Futhermore,

$$\begin{aligned}
\ker\Big(\mathrm{Mor}\Big(\mathrm{id}_{\mathrm{E}G} \times F(c) \Big)^{\mathrm{gp,ab}} \Big) &= \ker\Big(\mathrm{Mor}(\mathrm{id}_{\mathrm{E}G})^{\mathrm{gp,ab}} \times \mathrm{Mor}\Big(F(c) \Big)^{\mathrm{gp,ab}} \Big) \\
&= \{e\} \times \ker\Big(F\Big(\mathrm{Mor}(c)^{\mathrm{gp,ab}} \Big) \Big) \\
&= \{e\} \times F\Big(\ker\Big(\mathrm{Mor}(c)^{\mathrm{gp,ab}} \Big) \Big) \\
&= \{e\} \times F\Big(\ker \operatorname{coker}\Big(\mathrm{Mor}(\delta)^{\mathrm{gp,ab}} \Big) \Big) \\
&= \{e\} \times F\Big(\mathrm{im}\Big(\mathrm{Mor}(\delta)^{\mathrm{gp,ab}} \Big) \Big)
\end{aligned}$$

where this e is the identity element of the group

$$\mathrm{Mor}(\mathrm{E}G)^{\mathrm{gp,ab}} \quad = \quad G^{\mathrm{gp,ab}}$$

Recalling **??**, we see that elements of $\ker(\mathrm{Mor}(\mathrm{id}_{\mathrm{E}G} \times F(c))^{\mathrm{gp,ab}})$ can therefore all be expressed uniquely in the form $[(e_m; \delta(f_1), ..., \delta(f_m))]$, for some $m \in \mathbb{N}$ and $f_1, ..., f_m \in \mathrm{Mor}(\mathbb{G}_{2n})$. But then

$$\begin{aligned}
\mathrm{Mor}(c \circ \alpha)^{\mathrm{gp,ab}}\Big[\Big(e_m \,;\, \delta(f_1), ..., \delta(f_m) \Big) \Big] &= \Big[c\Big(\alpha\Big(e_m \,;\, \delta(f_1), ..., \delta(f_m) \Big) \Big) \Big] \\
&= \Big[c\delta\Big(\alpha(\, e_m \,;\, f_1, ..., f_m \,) \Big) \Big] \\
&= [\mathrm{id}_I]
\end{aligned}$$

and so we have

$$\mathrm{Mor}(c \circ \alpha)^{\mathrm{gp,ab}}\Big(\ker\Big(\mathrm{Mor}\Big(\mathrm{id}_{\mathrm{E}G} \times F(c) \Big)^{\mathrm{gp,ab}} \Big) \Big) \quad = \quad \{\mathrm{id}_{[I]}\}$$

$$\begin{aligned}
\implies \quad \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}} &= \mathrm{Mor}(C)^{\mathrm{gp,ab}} \big/ \{\mathrm{id}_{[I]}\} \\
&= \mathrm{Mor}(C)^{\mathrm{gp,ab}} \\
&= \mathrm{Mor}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \big/ \mathrm{im}\Big(\mathrm{Mor}(\delta)^{\mathrm{gp,ab}} \Big)
\end{aligned}$$

as required. □

Proposition 3.17 has at last given us a complete, usable description of the abelian group $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$. However, its worth noting that the resulting quotient group is slightly odd, almost inefficient in some sense. The numerator involves the free

EG-algebra on $2n$ objects, twice as many as we're really interested in, and then we essentially fix this by having the denominator be the image of δ, a map defined by sending each generator to a *pair* of generators. We might ask if there is some way for these two doublings to cancel out, yielding a simpler description of $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$.

But of course there is, we've seen it already. We know that the algebra map q isn't just a cokernel, but also an initial object in the comma category $(\mathbb{G}_n \downarrow \mathrm{inv})$ — indeed, we saw in **??** that these are really equivalent properties of q. Now that we have this way of expanding the view of q from being cokernel in $\mathrm{E}G\mathrm{Alg}_S$ to a larger colimit in the context of MonCat, we can pass this back through the equivalence, and produce a corresponding extension of q's initial object description.

We need to be careful here though. Up until now, we've been rather loose about distinguishing algebras from their underlying monoidal categories. In what follows it will help to be more precise.

Definition 3.18. Denote by $U : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{MonCat}$ the functor sending each EG to its underlying monoidal category, and each EG-algebra map to its underlying monoidal functor.

Proposition 3.19. *Let $U(i) : U(\mathbb{G}_n) \to U(\mathbb{G}_{2n})$ be the obvious inclusion of monoidal categories, which is also the underlying monoidal functor of the obvious inclusion of algebras defined on generators by $i(z_i) = z_i$. Then $U(i) \circ c$ is an initial object of the comma category $(U(\mathbb{G}_n) \downarrow U(\mathrm{inv}))$.*

Proof. This proposition is clearly a direct parallel of Proposition 3.3, and it proceeds as such.

Let $\psi : U(\mathbb{G}_n) \to X$ be an arbitrary object of $(U(\mathbb{G}_n) \downarrow U(\mathrm{inv}))$. Since $\mathbb{G}_n$ is the free EG-algebra on n objects, we can use it and ψ to define a new map, $\psi^* : \mathbb{G}_n \to X$, which takes the values

$$\psi^*(z_i) \quad := \quad \psi(z_i)^*$$

on generators. Clearly *psi** is also an object of $(\mathbb{G}_n \downarrow \mathrm{inv})$, and so using these two we can define a new map, $\psi + \psi^*$, via the universal property of the colimit:

$$\begin{tikzcd}
 & \mathbb{G}_n + \mathbb{G}_n \arrow[d, dashed, "\psi+\psi^*"] & \\
\mathbb{G}_n \arrow[ur, hook, "i"] \arrow[dr, "\psi"'] & & \mathbb{G}_n \arrow[ul, hook', "i'"'] \arrow[dl, "\psi^*"] \\
 & X &
\end{tikzcd}$$

But because $\mathbb{G}_n$ is the free algebra on n objects, and the free functor $F : \mathrm{Cat} \to \mathrm{E}G\mathrm{Alg}_S$ is a left adjoint and thus preserves colimits, we must have

$$\begin{aligned} \mathbb{G}_n + \mathbb{G}_n &= F(\{z_1, ..., z_n\}) + F(\{z'_1, ..., z'_n\}) \\ &= F(\ \{z_1, ..., z_n\} + \{z'_1, ..., z'_n\}\ ) \\ &= F(\{z_1, ..., z_{2n}\}) \\ &= \mathbb{G}_{2n} \end{aligned}$$

This means that we can compose $\psi + \psi^* : \mathbb{G}_{2n} \to X$ with the maps $\delta : \mathbb{G}_{2n} \to \mathbb{G}_{2n}$, though we need to be careful to specify exactly which inclusions we used in the definition of $\psi + \psi^*$. Suppose that the lefthand inclusion is i, the one given in the statement of the proposition, and the other is defined by the assignment $z_i \mapsto z_{i+n}$. Then for $1 \leq i \leq n$,

$$\begin{aligned} (\psi + \psi^*)\delta(z_i) &= (\psi + \psi^*)(z_i \otimes z_{n+i}) \\ &= \psi(z_i) \otimes \psi(z_i)^* \\ &= I \end{aligned}$$

$$\begin{aligned} (\psi + \psi^*)\delta(z_{n+i}) &= (\psi + \psi^*)(z_{n+i} \otimes z_i) \\ &= \psi(z_i)^* \otimes \psi(z_i) \\ &= I \end{aligned}$$

That is, $(\psi + \psi^*) \circ \delta = I$. But we've already defined $q : \mathbb{G}_{2n} \to Q$ to be the cokernel of δ, the universal map with this property, and so there must exist a unique $\mathrm{E}G$-algebra map $u : Q \to X$ making the righthand triangle below diagram commute:

$$\begin{tikzcd} \mathbb{G}_n \arrow[r, hook, "i"] \arrow[dr, "\psi"'] & \mathbb{G}_{2n} \arrow[r, "q"] \arrow[d, "\psi+\psi^*"] & Q \arrow[dl, "u"] \\ & X & \end{tikzcd}$$

The other triangle commutes by the definition of $\psi + \psi^*$, and so together the diagram tells us that for any object ψ of $(\mathbb{G}_n \downarrow \mathrm{inv})$, there exists at least one morphism u in $(\mathbb{G}_n \downarrow \mathrm{inv})$ going from $q \circ i$ to ψ.

Next, let $v : Q \to X$ be an arbitrary morphism $q \circ i \to \psi$ in $(\mathbb{G}_n \downarrow \mathrm{inv})$. By definition, this means that

$$\begin{aligned} & \psi &= vqi \\ \Longrightarrow \quad & \psi + \psi^* &= vqi + (vqi)^* \end{aligned}$$

Also, for $1 \leq i \leq n$ we have

$$\begin{array}{rclcl} q(z_i) \otimes q(z_{n+i}) & = & q(z_{i-n} \otimes z_i) & = & q\delta(z_i) & = & I \\ q(z_{n+i}) \otimes q(z_i) & = & q(z_i \otimes z_{n+i}) & = & q\delta(z_{n+i}) & = & I \\ & \Longrightarrow & q(z_{n+i}) & = & q(z_i)^* & & \end{array}$$

Therefore,

$$\begin{aligned} (\psi + \psi^*)(z_i) &= \Big(vqi + (vqi)^*\Big)(z_i) \\ &= vqi(z_i) \\ &= vq(z_i) \end{aligned}$$

$$\begin{aligned} (\psi + \psi^*)(z_{n+i}) &= \Big(vqi + (vqi)^*\Big)(z_{n+i}) \\ &= vqi(z_i)^* \\ &= v\Big(q(z_i)^*\Big) \\ &= vq(z_{n+i}) \end{aligned}$$

or in other words $\psi + \psi^* = v \circ q$ for any morphism $v : q \circ i \to \psi$ in $(\mathbb{G}_n \downarrow \mathrm{inv})$. But this is the property that the map u was supposed to satisfy uniquely, and thus it must be the only morphism $q \circ i \to \psi$ in $(\mathbb{G}_n \downarrow \mathrm{inv})$. Therefore $q \circ i$ is an initial object, and hence it is isomorphic in $(\mathbb{G}_n \downarrow \mathrm{inv})$ to any other initial object, such as η. It follows that the targets of these two maps, Q and $L\mathbb{G}_n$ respectively, are isomorphic as EG-algebras. $\square$

Chapter 4

Complete descriptions of free invertible algebras

The goal of these next couple of sections will be to show that we can reconstruct the all of morphisms of $L\mathbb{G}_n$ from just the abelian group $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$, and therefore that we can actually use the adjunction from **??** to help find a description of $L\mathbb{G}_n$. The way we will do this is by splitting $\mathrm{Mor}(L\mathbb{G}_n)$ up as the product of two other monoids. The first of these will encode all of the possible combinations of source and target data for morphisms in $L\mathbb{G}_n$, while the second will just be the endomorphisms of the unit object, $L\mathbb{G}_n(I, I)$. In other words, we will see that the monoid $\mathrm{Mor}(L\mathbb{G}_n)$ can be broken down into a context where source and target are the only thing that matters, and another where they are irrelevant. Once we have done this, we can then use the fact that $L\mathbb{G}_n(I, I)$ is always an abelian group to rewrite $\mathrm{Mor}(L\mathbb{G}_n)$ in terms of $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$.

4.1 Sources and targets in $L\mathbb{G}_n$

To get things started, we will spend this section considering the source and target information of morphisms in $L\mathbb{G}_n$.

Definition 4.1. For any EG-algebra X, denote by $s : \mathrm{Mor}(X) \to \mathrm{Ob}(X)$ and $t : \mathrm{Mor}(X) \to \mathrm{Ob}X)$ the monoid homomorphisms which send each morphism of X to its source and target, respectively. That is,

$$s(\,f : x \to y) \,=\, x, \qquad t(\,f : x \to y) \,=\, y$$

If we use the universal property of products, we can combine these source and target homomorphisms into a single map, $s \times t : \mathrm{Mor}(X) \to \mathrm{Ob}(X) \times \mathrm{Ob}(X)$. The monoid we are interested in finding is the image $L\mathbb{G}_n$ under its instance of this map.

Lemma 4.2. *Let X be an $\mathrm{E}G$-algebra, and $s \times t : \mathrm{Mor}(X) \to \mathrm{Ob}(X)^2$ the map built from s and t using the universal property of products. Then the image of this map is*

$$(s \times t)(X) \;=\; \mathrm{Ob}(X) \times_{\pi_0(X)} \mathrm{Ob}(X)$$

where this pullback is taken over the canonical maps sending objects of X to their connected components:

$$\begin{tikzcd}
\mathrm{Ob}(X) \times_{\pi_0(X)} \mathrm{Ob}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ob}(X) \arrow[d, "{[_]}"] \\
\mathrm{Ob}(X) \arrow[r, "{[_]}"] & \pi_0(X)
\end{tikzcd}$$

Proof. By definition, there exists a morphism $f : x \to y$ between objects x, y of X if and only if they are in the same connected component, $[x] = [y]$. Thus

$$\begin{aligned}
(x, y) \in (s \times t)(X) \quad &\iff \quad \exists f : \quad s(f) = x, \quad t(f) = y \\
&\iff \quad [x] = [y] \\
&\iff \quad (x, y) \in \mathrm{Ob}(X) \times_{\pi_0(X)} \mathrm{Ob}(X)
\end{aligned}$$

as required. $\square$

Recalling Lemma 1.24, Propositions 1.26 and 2.13, and Corollary 2.19, we can immediately conclude the following:

Corollary 4.3.

$$(s \times t)(\mathbb{G}_n) \;=\; \begin{cases} \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} & \textit{if } G \textit{ is crossed} \\ \mathbb{N}^{*n} & \textit{otherwise} \end{cases}$$

$$(s \times t)(L\mathbb{G}_n) \;=\; \begin{cases} \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} & \textit{if } G \textit{ is crossed} \\ \mathbb{Z}^{*n} & \textit{otherwise} \end{cases}$$

where the pullbacks are taken over the quotients of abelianisation for $(\mathbb{N}^{*n})^{\mathrm{ab}} = \mathbb{N}^n$ and $(\mathbb{Z}^{*n})^{\mathrm{ab}} = \mathbb{Z}^n$ respectively.

Next, we want to show that this $(s \times t)(L\mathbb{G}_n)$ we have described is in fact a submonoid of $\mathrm{Mor}(L\mathbb{G}_n)$. This is a little tricky though, since we don't currently know what the morphisms of $L\mathbb{G}_n$ even are. We will sidestep this problem by first proving the analogous statement for all $\mathbb{G}_n$, and then recovering the $L\mathbb{G}_n$ version from it later.

Now, by Lemma 1.29 we know that wanting $(s\times t)(\mathbb{G}_n)$ to be a submonoid of $\mathrm{Mor}(\mathbb{G}_n)$ is the same as asking if we can find an injective homomorphism $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \to G \times_{\mathbb{N}} \mathbb{N}^{*n}$, assuming G is crossed, or $\mathbb{N}^{*n} \to G \times_{\mathbb{N}} \mathbb{N}^{*n}$ if it is not. The latter case is pretty obvious, so we'll focus on crossed G for the moment. Creating a injective *function* $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \to G \times_{\mathbb{N}} \mathbb{N}^{*n}$ is not especially hard. For any pair $(w, w') \in \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$, the image of w and w' in the abelian group $\mathbb{N}^n$ is the same, which is to say that the words $w, w' \in \mathbb{N}^{*n}$ are permuations of each other. Since the underlying permutation maps $\pi : G(m) \to \mathrm{S}_m$ of a crossed action operad G are all surjective, we can always find an element of $g \in G(|w|)$ for which $\pi(g)(w) = w'$. Thus in order to make our injective function all we need to do is make a choice $g_{(w,w')}$ like this for each (w, w'), and then set

$$\begin{array}{rcl} \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} & \to & G \times_{\mathbb{N}} \mathbb{N}^{*n} \\ (w, w') & \mapsto & (\, g_{(w,w')}, w\,) \end{array}$$

Injectivity follows from

$$(\, g_{(w,w')}, w\,) \;=\; (\, g_{(v,v')}, v\,) \quad \Longrightarrow \quad \begin{array}{rcl} g_{(w,w')} & = & g_{(v,v')} \\ w & = & v \\ w' & = & \pi(g_{(w,w')})(w) \\ & = & \pi(g_{(v,v')})(v) \\ & = & v' \end{array}$$

So how do we know if we can choose these $g_{(w,w')}$ in such a way that the resulting function is also a monoid homomorphism? If we could find a presentation of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ in terms of generators and relations then this would help a little, since we would only need to pick a $g_{(z,z')}$ for each generator (z, z'), and then define all other g by way of products.

$$g_{(vw,v'w')} \;=\; g_{(v,v')} g_{(w,w')}$$

But we would still need to know if our choice of $g_{(z,z')}$ obeyed the necessary relations on the generators of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$. Luckily for us though, this turns out to be no problem at all.

Proposition 4.4. $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ *is a free monoid.*

Proof. Given an element (w, w') of the monoid $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$, let $d(w, w')$ be the following set:

$$d(w,w') = \left\{ (u,u'),(v,v') \in \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \;:\; \begin{array}{rcl} (w,w') & = & (u,u') \otimes (v,v'), \\ (u,u') & \neq & (I,I), \\ (v,v') & \neq & (I,I) \end{array} \right\}$$

We can use these sets to recursively define a decomposition of any element (w, w') as a product of other elements of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$. Specifically, if $d(w, w')$ is empty then we say that the decomposition of (w, w') is just (w, w') itself, and otherwise we choose any $\Big((u,u'),(v,v') \Big) \in d(w,w')$ and say that the decomposition of (w, w') is the concatenation of the decomposition of (u, u') with the decomposition of (v, v'). Note that this process definitely terminates, since $|u|$ and $|v|$ are always strictly smaller that $|w|$, and any strictly decreasing sequence of natural numbers is finite.

Of course, we need to check that this decomposition of (w, w') is well-defined, which amounts to checking that the choice of $(u, u'), (v, v')$ we make at each stage won't change the eventual output. To that end, suppose for the sake of contradiction that $(u_1, u'_1), ..., (u_m, u'_m)$ and $(v_1, v'_1), ..., (v'_m, v'_{m'})$ are distinct decompositions of (w, w') we could arrive at using the above process. Notice that we can assume without loss of generality that $|u_1| < |v_1|$. If instead $|u_1| > |w_1|$, we can just swap the labels of the sequences, and if $|u_1| = |v_1|$ then we can just discard those elements and instead consider the decompositions $(u_2, u'_2), ..., (u_m, u'_m)$ and $(v_2, v'_2), ..., (v'_m, v'_{m'})$ of $(u_1, u'_1) \otimes ... \otimes (u_m, u'_m) = (v_1, v'_1) \otimes ... \otimes (v'_m, v'_{m'})$. Since $(u_1, u'_1), ..., (u_m, u'_m)$ and $(v_1, v'_1), ..., (v'_m, v'_{m'})$ were distinct decompositions of (w, w'), in this way we will eventually reach some subsequences whose first elements are different; once we have, we can relabel them so that $|u_1| < |v_1|$.

Then by definition,

$$u_1 \otimes \Big(\bigotimes_{i=2}^{m} u_i \Big) \;=\; w \;=\; v_1 \otimes \Big(\bigotimes_{i=2}^{m'} v_i \Big)$$

But $w, u_1, v_1, \bigotimes_{i=2}^{m} u_i, \bigotimes_{i=2}^{m'} v_i$ are all elements of $\mathbb{N}^{*n}$, which is a free monoid, and so they each have a unique decomposition as products of the generators $\{z_1, ..., z_n\}$, and these all respect tensor products. Therefore, since $|u_1| < |v_1|$, there must exist some element a of $\mathbb{N}^{*n}$ such that

$$w \;=\; u_1 \otimes a \otimes \Big(\bigotimes_{i=2}^{m'} v_i \Big) \qquad \Longrightarrow \qquad v_1 \;=\; u_1 \otimes a$$

Since

$$|u_1'| = |u_1| < |v_1| = |v_1'|$$

we can also use exactly the same reasoning to find an a' in $\mathbb{N}^{*n}$ with $v_1' = u_1' \otimes a'$, and hence $(v_1, v_1') = (u_1, u_1') \otimes (a, a')$. Moreover, this (a, a') is an element of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$, because

$$\begin{array}{rrcccl} & v_1 & = & u_1 \otimes a & & \\ \Longrightarrow & [v_1] & = & [u_1 \otimes a] & = & [u_1] + [a] \\ \\ & v_1' & = & u_1' \otimes a' & & \\ \Longrightarrow & [v_1'] & = & [u_1' \otimes a'] & = & [u_1'] + [a'] \\ \\ \Longrightarrow & [a] & = & [v_1] - [u_1] & & \\ & & & [v_1'] - [u_1'] & = & [a'] \end{array}$$

In other words, we have shown that the pair $\Big((u_1, u_1')(a, a') \Big)$ is an element of $d(v_1, v_1')$. But by assumption $(v_1, v_1'), ..., (v_m', v_{m'}')$ was a decomposition of (w, w'), and hence the $d(v_i, v_i')$ were supposed to be empty for each i, since that is when the decomposition finding process terminates. This is a contradiction, and hence our assumption that $(u_1, u_1'), ..., (u_m, u_m')$ and $(v_1, v_1'), ..., (v_m', v_{m'}')$ were distinct decompositions of (w, w') is false. Therefore, each (w, w') in $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ has a unique decomposition in terms of elements (v_i, v_i') for which $d(v_i, v_i')$ is empty, and so $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ is the free monoid whose generators are all such elements. □

It follows immediately from this that our earlier contruction of an injective function $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \to G \times_{\mathbb{N}} \mathbb{N}^{*n}$ can be extended to be an inclusion of monoids.

Proposition 4.5. *$(s \times t)(\mathbb{G}_n)$ is (isomorphic to) a submonoid of* $\mathrm{Mor}(\mathbb{G}_n)$

Proof. First, assume that the action operad G is non-crossed. Then there exists an obvious injective monoid homomorphism

$$\begin{array}{rcrcl} i & : & (s \times t)(\mathbb{G}_n) & \to & \mathrm{Mor}(\mathbb{G}_n) \\ & : & \mathbb{N}^{*n} & \to & G \times_{\mathbb{N}} \mathbb{N}^{*n} \\ & : & w & \mapsto & (\, e_{|w|}, w \,) \end{array}$$

The homomorphism property follows from the fact that the length $|w|$ defined in Definition 1.28 is itself a homomorphism, so $|w \otimes w'| = |w| + |w'|$. Thus $(s \times t)(\mathbb{G}_n) \subseteq \mathrm{Mor}(\mathbb{G}_n)$ for non-crossed G.

Now assume that G is crossed. For each generator (z, z') of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$, choose an element of $g_{(z,z')} \in G(|z|)$ with the property that $\pi(g_{(z,z')})(z) = z'$. This is always possible, since $(z, z') \in \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ implies that the words $z, z' \in \mathbb{N}^{*n}$ are permuations of each other, and the maps $\pi : G(m) \to \mathrm{S}_m$ are always surjective. Then we can define the homomorphism i to be

$$\begin{array}{rrcl} i : & (s \times t)(\mathbb{G}_n) & \to & \mathrm{Mor}(\mathbb{G}_n) \\ : & \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} & \to & G \times_{\mathbb{N}} \mathbb{N}^{*n} \\ : & (z, z') & \mapsto & (\, g_{(z,z')}, z\,) \end{array}$$

on generators. Since by Proposition 4.4 $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ is free, this i extends to a well-defined monoid homomorphism, as long as we choose $g_{(I,I)} = e_0$ so that it preserves the identity. Moreover, for any two generators $(z_1, z_1'), (z_2, z_2')$, we have

$$(\, g_{(z_1,z_1')}, z_1\,) \;=\; (\, g_{(z_2,z_2')}, z_2\,) \quad\Longrightarrow\quad \begin{array}{rcl} g_{(z_1,z_1')} & = & g_{(z_2,z_2')} \\ z_1 & = & z_2 \\ z_1' & = & \pi(g_{(z_1,z_1')})(z_1) \\ & = & \pi(g_{(z_2,z_2')})(z_2) \\ & = & z_2' \end{array}$$

and thus i is injective. Therefore the image of this i is a submonoid of $G \times_{\mathbb{N}} \mathbb{N}^{*n}$ which is isomorphic to $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$, so again $(s \times t)(\mathbb{G}_n) \subseteq \mathrm{Mor}(\mathbb{G}_n)$ as required. □

So, now we know that $(s \times t)(\mathbb{G}_n)$ is a submonoid of $\mathrm{Mor}(\mathbb{G}_n)$, but what we are really interested in is whether $(s \times t)(\mathbb{G}_n)$ is a submonoid of $\mathrm{Mor}(\mathbb{G}_n)$. To recover the latter result from the former, we will use our cokernel map $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$. In particular, the surjectivity of q combined with the case $(s \times t)(\mathbb{G}_{2n}) \subseteq \mathrm{Mor}(\mathbb{G}_{2n})$ from Proposition 4.5, immediately gives us what we need.

Corollary 4.6. *$(s \times t)(L\mathbb{G}_n)$ is (isomorphic to) a submonoid of $\mathrm{Mor}(L\mathbb{G}_n)$*

Proof. Let $i : (s \times t)(\mathbb{G}_{2n}) \hookrightarrow \mathrm{Mor}(\mathbb{G}_{2n})$ be an inclusion which allows us to view $(s \times t)(\mathbb{G}_{2n})$ as a submonoid of $\mathrm{Mor}(\mathbb{G}_{2n})$, as in Proposition 4.5. Also, let $\mathrm{Mor}(q) : \mathrm{Mor}(\mathbb{G}_{2n}) \to \mathrm{Mor}(L\mathbb{G}_n)$ the restriction of the cokernel map $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$ onto morphisms. Then the image of the composite of these two homomorphisms,

$$\mathrm{im}\Big(\,\mathrm{Mor}(q) \circ i\,\Big) \;=\; q\Big(\,\mathrm{im}(i)\,\Big) \;\cong\; q\Big(\,(s \times t)(\mathbb{G}_{2n})\,\Big)$$

is clearly a submonoid of $\mathrm{Mor}(L\mathbb{G}_n)$.

But by **??** q is a surjective functor. This means that there can exist a map $w \to v$ in $L\mathbb{G}_n$ if and only if there exists at least one map $w' \to v'$ in $\mathbb{G}_{2n}$, for some w', v' which have $q(w') = w$ and $q(v') = v$. In other words,

$$q\Big((s \times t)(\mathbb{G}_{2n}) \Big) \;=\; (s \times t)(L\mathbb{G}_n)$$

and therefore the monoid $\mathrm{im}\Big(\mathrm{Mor}(q) \circ i \Big)$ that we saw above is really a submonoid of $\mathrm{Mor}(L\mathbb{G}_n)$ isomorphic to $(s \times t)(L\mathbb{G}_n)$, as required. □

4.2 Unit endomorphisms of $L\mathbb{G}_n$

To help us understand $\mathrm{Mor}(L\mathbb{G}_n)$, we decided to break it down into two smaller pieces. The first of these was the source/target data $(s \times t)(L\mathbb{G}_n)$, which we explored in the previous section. The other piece that we now have to consider is the monoid of unit endomorphisms, $L\mathbb{G}_n(I, I)$.

This is a particularly important submonoid of the morphisms $\mathrm{Mor}(L\mathbb{G}_n)$, since it is the only submonoid which is also a homset of the category $L\mathbb{G}_n$. Moreover, because the maps in $L\mathbb{G}_n(I, I)$ all share the same source and target, what we have is not just a monoid under tensor product but also under composition as well. This fact leads to a series of special properties for $L\mathbb{G}_n(I, I)$, the first of which is just another instance of the classic Eckmann-Hilton argument.

Lemma 4.7. *$L\mathbb{G}_n(I, I)$ is a commutative monoid under both tensor product and composition, with $f \otimes f' = f \circ f'$.*

Proof. Let f, f' be arbitrary elements of the monoid $L\mathbb{G}_n(I, I)$. Since both of these are morphisms in the monoidal category $L\mathbb{G}_n$, we can use the law of interchange to show that

$$\begin{aligned}
f \otimes f' &= (f \circ \mathrm{id}_I) \otimes (\mathrm{id}_I \circ f') \\
&= (f \otimes \mathrm{id}_I) \circ (\mathrm{id}_I \otimes f') \\
&= f \circ f' \\
&= (\mathrm{id}_I \otimes f) \circ (f' \otimes \mathrm{id}_I) \\
&= (f' \circ \mathrm{id}_I) \otimes (\mathrm{id}_I \circ f) \\
&= f' \otimes f
\end{aligned}$$

□

In fact, since we already proved that the morphisms of $L\mathbb{G}_n$ are all actions morphisms, we can take this one step further.

Proposition 4.8. *$L\mathbb{G}_n(I,I)$ is an abelian group.*

Proof. From Lemma 3.6 we know that every morphism f in $L\mathbb{G}_n$ is of the form $\alpha(g;\mathrm{id}_{x_1},...,\mathrm{id}_{x_m})$, for some $g \in G(m)$ and $x_i \in \mathbb{Z}^{*n}$. It follows immediately that

$$\begin{aligned} & \alpha(\, g\,;\, \mathrm{id}_{x_1},...,\mathrm{id}_{x_m}\,) \circ \alpha(\, g^{-1}\,;\, \mathrm{id}_{x_{\pi(g^{-1})(1)}},...,\mathrm{id}_{x_{\pi(g^{-1})(m)}}\,) \\ = \; & \alpha(\, gg^{-1}\,;\, \mathrm{id}_{x_{\pi(g^{-1})(1)}},...,\mathrm{id}_{x_{\pi(g^{-1})(m)}}\,) \\ = \; & \alpha(\, e_m\,;\, \mathrm{id}_{x_{\pi(g^{-1})(1)}},...,\mathrm{id}_{x_{\pi(g^{-1})(m)}}\,) \\ = \; & \mathrm{id}_{x_{\pi(g^{-1})(1)}\otimes...\otimes x_{\pi(g^{-1})(m)}} \end{aligned}$$

$$\begin{aligned} & \alpha(\, g^{-1}\,;\, \mathrm{id}_{x_{\pi(g^{-1})(1)}},...,\mathrm{id}_{x_{\pi(g^{-1})(m)}}\,) \circ \alpha(\, g\,;\, \mathrm{id}_{x_1},...,\mathrm{id}_{x_m}\,) \\ = \; & \alpha(\, g^{-1}g\,;\, \mathrm{id}_{x_1},...,\mathrm{id}_{x_m}\,) \\ = \; & \alpha(\, e_m\,;\, \mathrm{id}_{x_1},...,\mathrm{id}_{x_m}\,) \\ = \; & \mathrm{id}_{x_1\otimes...\otimes x_m} \end{aligned}$$

In other words, every morphism $f : w \to v$ in $L\mathbb{G}_n$ has an inverse under composition,

$$f^{-1} \quad := \quad \alpha(g^{-1};\mathrm{id}_{x_{\pi(g^{-1})(1)}},...,\mathrm{id}_{x_{\pi(g^{-1})(m)}})$$

But we know from Lemma 4.7 that tensor product and composition are the same for endomorphisms of the unit object of $L\mathbb{G}_n$. In particular this means that if some morphism $f : I \to I$ has a compositional inverse f^{-1}, then it will also be its monoidal inverse f^*. Thus every element of the commutative monoid $L\mathbb{G}_n(I,I)$ is invertible, or in other words $L\mathbb{G}_n(I,I)$ is an abelian group. □

Indeed, by using a slightly broader argument we can extend this result to every morphism of $L\mathbb{G}_n$.

Proposition 4.9. *Every morphism $f : w \to v$ in $L\mathbb{G}_n$ has an inverse under tensor product, $f^* : w^* \to v^*$. That is, the monoid $\mathrm{Mor}(L\mathbb{G}_n)$ is actually a group.*

Proof. For any $f : w \to v$ in $L\mathbb{G}_n$, consider the map $\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*}$, where f^{-1} is the compositional inverse of f, as in the proof of Proposition 4.8. This morphism has source $w^* \otimes v \otimes v^* = w^*$ and target $w^* \otimes w \otimes v^* = v^*$, which allows us to apply the law of interchange to get

$$\begin{aligned}
f \otimes (\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*}) &= \Big(f \circ \mathrm{id}_w\Big) \otimes \Big(\mathrm{id}_{v^*} \circ (\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*})\Big) \\
&= \Big(f \otimes \mathrm{id}_{v^*}\Big) \circ \Big(\mathrm{id}_w \otimes (\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*})\Big) \\
&= (f \otimes \mathrm{id}_{v^*}) \circ (f^{-1} \otimes \mathrm{id}_{v^*}) \\
&= (f \circ f^{-1}) \otimes (\mathrm{id}_{v^*} \circ \mathrm{id}_{v^*}) \\
&= \mathrm{id}_v \otimes \mathrm{id}_{v^*} \\
&= \mathrm{id}_I
\end{aligned}$$

and likewise

$$\begin{aligned}
(\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*}) \otimes f &= \Big((\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*}) \circ \mathrm{id}_{w^*}\Big) \otimes \Big(\mathrm{id}_v \circ f\Big) \\
&= \Big((\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*}) \otimes \mathrm{id}_v\Big) \circ \Big(\mathrm{id}_{w^*} \otimes f\Big) \\
&= (\mathrm{id}_{w^*} \otimes f^{-1}) \circ (\mathrm{id}_{w^*} \otimes f) \\
&= (\mathrm{id}_{w^*} \circ \mathrm{id}_{w^*}) \otimes (f^{-1} \circ f) \\
&= \mathrm{id}_{w^*} \otimes \mathrm{id}_w \\
&= \mathrm{id}_I
\end{aligned}$$

In other words, $f^* := \mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*}$ is the inverse of f in the monoid $\mathrm{Mor}(L\mathbb{G}_n)$, as required. □

So $\mathrm{Mor}(L\mathbb{G}_n)$ and $L\mathbb{G}_n(I,I)$ both turn out to be groups under tensor product. Obviously it follows from this that $L\mathbb{G}_n(I,I)$ is a not just a submonoid of $\mathrm{Mor}(L\mathbb{G}_n)$ but a subgroup — in particular an abelian subgroup, going by Proposition 4.8. But $L\mathbb{G}_n(I,I)$ is actually an even more special subgroup than this.

Proposition 4.10. *$L\mathbb{G}_n(I,I)$ is a normal subgroup of* $\mathrm{Mor}(L\mathbb{G}_n)$.

Proof. From Propositions 4.8 and 4.9, we know that $L\mathbb{G}_n(I,I)$ is a subgroup of $\mathrm{Mor}(L\mathbb{G}_n)$. For normality, we need to again consider both crossed and non-crossed action operads separately.

If G is non-crossed, then by Corollary 2.19 we know that the map assigning objects of $L\mathbb{G}_n$ to their connected component is just the identity $\mathrm{id}_{\mathbb{Z}^{*n}}$. In other words, every objects belongs to its own unique component, so that every morphisms of $L\mathbb{G}_n$ is actually an endomorphism. It follows that the group $L\mathbb{G}_n(I,I)$ is the kernel of the source homomorphism s from Definition 4.1 — or equally the target homomorphism t.

$$L\mathbb{G}_n(I,I) \longrightarrow \mathrm{Mor}(L\mathbb{G}_n) \xrightarrow{\ s\ } \mathrm{Ob}(L\mathbb{G}_n)$$

The kernel of a group homomorphism is always a normal subgroup of that homomorphisms source, and so in our case we have $L\mathbb{G}_n(I,I) \leq \mathrm{Mor}(L\mathbb{G}_n)$.

For crossed G, recall from Lemma 1.20 that all crossed EG-algebras are spacial, and so in particular $L\mathbb{G}_n$ is. This means that for any $h \in L\mathbb{G}_n(I,I)$ and $w \in \mathrm{Ob}(L\mathbb{G}_n)$ we will always have $h \otimes \mathrm{id}_w = \mathrm{id}_w \otimes h$. Thus for any $f : w \to v$ in $\mathrm{Mor}(L\mathbb{G}_n)$, we get

$$\begin{aligned}
h \otimes f &= (\mathrm{id}_I \circ h) \otimes (f \circ \mathrm{id}_w) \\
&= (\mathrm{id}_I \otimes f) \circ (h \otimes \mathrm{id}_w) \\
&= (f \otimes \mathrm{id}_I) \circ (\mathrm{id}_w \otimes h) \\
&= (f \circ \mathrm{id}_w) \otimes (\mathrm{id}_I \circ h) \\
&= f \otimes h
\end{aligned}$$

That is, $L\mathbb{G}_n(I,I)$ is a subgroup of the centre of $\mathrm{Mor}(L\mathbb{G}_n)$. Then because

$$f \otimes h \otimes f^* \;=\; h \otimes f \otimes f^* \;=\; h \;\in L\mathbb{G}_n(I,I)$$

it follows that $L\mathbb{G}_n(I,I)$ is a normal subgroup of $\mathrm{Mor}(L\mathbb{G}_n)$. □

This is the last important property of $L\mathbb{G}_n(I,I)$ that we need. Now we finally have enough information to show that the morphism monoid of $L\mathbb{G}_n$ really does split apart into the smaller pieces that we claimed it did.

4.3 Recovering the morphisms of $L\mathbb{G}_n$

Proposition 4.11.

$$\mathrm{Mor}(L\mathbb{G}_n) \quad = \quad (s \times t)(L\mathbb{G}_n) \times L\mathbb{G}_n(I,I)$$

Proof. Because we know that $L\mathbb{G}_n(I,I)$ is a normal subgroup of $\mathrm{Mor}(L\mathbb{G}_n)$, we can consider the quotient group

$$L\mathbb{G}_n(I,I) \hookrightarrow \mathrm{Mor}(L\mathbb{G}_n) \longrightarrow \mathrm{Mor}(L\mathbb{G}_n) \big/ L\mathbb{G}_n(I,I)$$

Whenever they exist, quotient groups are an example of a cokernel in the category of groups and group homomorphisms. This means that the quotient map $\mathrm{Mor}(L\mathbb{G}_n) \to \mathrm{Mor}(L\mathbb{G}_n)/L\mathbb{G}_n(I,I)$ will factor any homomorphism whose composite with the inclusion $L\mathbb{G}_n(I,I) \to \mathrm{Mor}(L\mathbb{G}_n)$ is the zero map. But our source/target map

$s \times t : \mathrm{Mor}(L\mathbb{G}_n) \to (s \times t)(L\mathbb{G}_n)$ is one such homomorphism, since for any $h : I \to I$ clearly $(s \times t)(h) = (I, I)$, which is the identity element in $(s \times t)(L\mathbb{G}_n)$. Therefore there must exist a unique homomorphism u making the triangle below commute:

$$\begin{tikzcd}
\mathrm{Mor}(L\mathbb{G}_n) \arrow[d] \arrow[dr, "s\times t"] & \\
\mathrm{Mor}(L\mathbb{G}_n) / L\mathbb{G}_n(I,I) \arrow[r, "u"] & (s \times t)(L\mathbb{G}_n)
\end{tikzcd}$$

This map u will be surjective — because $s \times t$ is — but in fact it will also be injective. This is because if two morphisms f, f' of $L\mathbb{G}_n$ have the same source and target, then the map $h = f^* \otimes f'$ is an element of $L\mathbb{G}_n(I, I)$ for which $f \otimes h = f'$, and so f and f' are a part of the same equivalence classes in $\mathrm{Mor}(L\mathbb{G}_n)/L\mathbb{G}_n(I, I)$. More precisely,

$$\begin{aligned}
[f] \neq [f'] \implies [f]^* \otimes [f'] &\neq [I] \\
\implies [f^* \otimes f'] &\neq [I] \\
\implies f^* \otimes f' &\notin L\mathbb{G}_n(I, I)
\end{aligned}$$

$$\begin{aligned}
\implies (s \times t)(f^* \otimes f') &\neq (I, I) \\
\implies (s \times t)(f)^* \otimes (s \times t)(f') &\neq (I, I) \\
\implies (s \times t)(f) &\neq (s \times t)(f')
\end{aligned}$$

Thus u is bijective, or in other words

$$\mathrm{Mor}(L\mathbb{G}_n) / L\mathbb{G}_n(I, I) \cong (s \times t)(L\mathbb{G}_n)$$

Finally, by Corollary 4.6 $(s \times t)(L\mathbb{G}_n)$ is also a submonoid (hence subgroup) of $\mathrm{Mor}(L\mathbb{G}_n)$. Combined with the identity above, we see that what have here is a split exact sequence of groups

$$L\mathbb{G}_n(I, I) \longrightarrow \mathrm{Mor}(L\mathbb{G}_n) \longrightarrow (s \times t)(L\mathbb{G}_n)$$

That is, $\mathrm{Mor}(L\mathbb{G}_n)$ is a split group extension of $(s \times t)(L\mathbb{G}_n)$ by $L\mathbb{G}_n(I, I)$, or equivalently $\mathrm{Mor}(L\mathbb{G}_n)$ is a semi direct product $L\mathbb{G}_n(I, I) \rtimes (s \times t)(L\mathbb{G}_n)$. Moreover, we saw earlier that $L\mathbb{G}_n(I, I)$ is a subgroup of the centre of $\mathrm{Mor}(L\mathbb{G}_n)$, and so it follows that $\mathrm{Mor}(L\mathbb{G}_n)$ is also a central extension of $(s \times t)(L\mathbb{G}_n)$. However, the only extensions which are both central and split are the trivial extensions, and therefore $\mathrm{Mor}(L\mathbb{G}_n)$ is really just the direct product $L\mathbb{G}_n(I,I) \times (s \times t)(L\mathbb{G}_n)$, as required. □

Proposition 4.12. *The endomorphisms of the unit object of $L\mathbb{G}_n$ are*

$$L\mathbb{G}_n(I,I) \quad = \quad {\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}}}\Big/{\mathbb{Z}^n}$$

and therefore

$$\mathrm{Mor}(L\mathbb{G}_n) \quad = \quad \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \times {\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}}}\Big/{\mathbb{Z}^n}$$

Proof. From Proposition 4.11, we know that

$$\mathrm{Mor}(L\mathbb{G}_n) \quad = \quad (s \times t)(L\mathbb{G}_n) \times L\mathbb{G}_n(I,I)$$

Abelianising both sides of this equation, we get

$$\begin{aligned} \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}} &= \Big((s \times t)(L\mathbb{G}_n) \times L\mathbb{G}_n(I,I)\Big)^{\mathrm{ab}} \\ &= (s \times t)(L\mathbb{G}_n)^{\mathrm{ab}} \times L\mathbb{G}_n(I,I)^{\mathrm{ab}} \\ &= (s \times t)(L\mathbb{G}_n)^{\mathrm{ab}} \times L\mathbb{G}_n(I,I) \end{aligned}$$

since $L\mathbb{G}_n(I,I)$ is already abelian. Now, there is an obvious inclusion $(s \times t)(L\mathbb{G}_n)^{\mathrm{ab}} \hookrightarrow (s \times t)(L\mathbb{G}_n)^{\mathrm{ab}} \times L\mathbb{G}_n(I,I)$, and since everything here is abelain, all subgroups are normal subgroups. Thus we can take the quotient of the above equation by this map, to obtain

$$L\mathbb{G}_n(I,I) \quad = \quad {\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}}}\Big/{(s \times t)(L\mathbb{G}_n)^{\mathrm{ab}}}$$

We can also now substitute this expression back into our original equation, which yields

$$\mathrm{Mor}(L\mathbb{G}_n) \quad = \quad (s \times t)(L\mathbb{G}_n) \times {\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}}}\Big/{(s \times t)(L\mathbb{G}_n)^{\mathrm{ab}}}$$

But from Corollary 4.3 we already know that the value of $(s \times t)(L\mathbb{G}_n)$ is $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$. Moreover, the homomorphisms that this pullback is taken over are both the quotient map of abelianisation $\mathbb{Z}^{*n} \to \mathbb{Z}^n$, and as a result,

$$(\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}} \quad = \quad \mathbb{Z}^n$$

Putting this all together, we get the two equations in the statement of the proposition. $\square$

Note that its not entirely clear here exactly which $\mathbb{Z}^n$ subgroup of $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$ is being referenced in the statement of Proposition 4.12. This is because the existence of such a quotient relied on our assumption that the algebra map $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$ exists, and so we will not be able to actually perform this quotient until we understand where q comes from.

However, it does let us answer a lingering question about Proposition 4.12. Recall that we found that

$$L\mathbb{G}_n(I, I) \quad = \quad {\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}}} \big/ {\mathbb{Z}^n}$$

but at the time it was not obvious which $\mathbb{Z}^n$ subgroup of $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$ this equation was refering to. But now we have Proposition 3.2 and **??** to tell us how q acts on objects, which allows us conclude the following:

Proposition 4.13.

$$L\mathbb{G}_n(I, I) \quad = \quad {\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}}} \big/ {\langle\, [\mathrm{id}_x] \;:\; x \in \mathrm{Ob}(L\mathbb{G}_n)\,\rangle}$$

Proof. In the proof of Proposition 4.12, the $\mathbb{Z}^n$ term first appears when we form the quotient group for the inclusion

$$\mathbb{Z}^n \quad = \quad (s \times t)(L\mathbb{G}_n)^{\mathrm{ab}} \hookrightarrow (s \times t)(L\mathbb{G}_n)^{\mathrm{ab}} \times L\mathbb{G}_n(I, I) \quad = \quad \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}}$$

This is just the image under the abelianisation functor $\mathrm{ab} : \mathrm{Grp} \to \mathrm{Ab}$ of the inclusion

$$(s \times t)(L\mathbb{G}_n) \hookrightarrow \mathrm{Mor}(L\mathbb{G}_n)$$

given in **??**, which is in turn just the image under the algebra map $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$ of whichever inclusion

$$i \;:\; (s \times t)(\mathbb{G}_{2n}) \hookrightarrow \mathrm{Mor}(\mathbb{G}_{2n})$$

we decided to use in **??**.

Now, rememeber that making a choice for this inclusion amounted to choosing for each generator (z, z') of $\mathbb{N}^{*2n} \times_{\mathbb{N}^{2n}} \mathbb{N}^{*2n}$ an element $g_{z,z'}$ of $G(|z|)$ for which $\pi(g_{z,z'})(z) = z'$. Moreover, for each of the generators $z_1, ..., z_{2n}$ of the monoid $\mathbb{N}^{*2n}$, the pair (z_i, z_i)

is definitely a generator of $\mathbb{N}^{*2n} \times_{\mathbb{N}^{2n}} \mathbb{N}^{*2n}$. This is because there are no non-unit elements $a, b \in \mathbb{N}^{*2n}$ with the property that $a \otimes b = z_i$, and hence no non-unit elements $(a, a'), (b, b') \in \mathbb{N}^{*2n} \times_{\mathbb{N}^{2n}} \mathbb{N}^{*2n}$ for which $(a, a') \otimes (b, b') = (a \otimes b, a' \otimes b') = (z_i, z_i)$. Therefore, when we are making a choice for the inclusion i we must at some point pick a sequence $g_{z_1,z_1}, ..., g_{z_{2n},z_{2n}}$ of independent elements of $G(1)$. We would need to ask that their underlying permutations satisfy $\pi(g_{z_i,z_i})(z_i) = z_i$ as well, but this will always be true, since in this case π is a map $\pi_1 : G(1) \to \mathrm{S}_1$, and $\mathrm{S}_1 = \{e\}$.

However, notice that regardless of which G we are using we are always free to choose each of the g_{z_i,z_i} to be the identity element $e_1 \in G(1)$. If we do this then our inclusion i will end up sending the (z_i, z_i) to the elements (e_1, z_i) of $G \times_{\mathbb{N}} \mathbb{N}^{*2n} \cong \mathrm{Mor}(\mathbb{G}_{2n})$, which correspond to the identity morphisms id_{z_i} of $\mathbb{G}_{2n}$.

$$\begin{array}{rcrcl} i & : & (s \times t)(\mathbb{G}_{2n}) & \hookrightarrow & \mathrm{Mor}(\mathbb{G}_{2n}) \\ & : & (z_i, z_i) & \mapsto & \mathrm{id}_{z_i} \end{array}$$

Working our way back towards the statement of Proposition 4.12 again, we then have

$$\begin{array}{rrcrcl} & q(i) & : & (s \times t)(L\mathbb{G}_n) & \hookrightarrow & \mathrm{Mor}(L\mathbb{G}_n) \\ & & : & \Big(q(z_i), q(z_i) \Big) & \mapsto & q(\mathrm{id}_{z_i}) \\ \\ \Longrightarrow & q(i) & : & (z_i, z_i) & \mapsto & \mathrm{id}_{z_i} \\ & & & (z_i^*, z_i^*) & \mapsto & \mathrm{id}_{z_i^*} \\ \\ & q(i)^{\mathrm{ab}} & : & (s \times t)(L\mathbb{G}_n)^{\mathrm{ab}} & \hookrightarrow & \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}} \\ & & : & [\,(z_i, z_i)\,] & \mapsto & [\mathrm{id}_{z_i}] \\ & & : & [\,(z_i^*, z_i^*)\,] & \mapsto & [\mathrm{id}_{z_i^*}] \\ \\ \Longrightarrow & q(i)^{\mathrm{ab}} & : & \mathbb{Z}^n & \hookrightarrow & \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}} \\ & & : & z_i & \mapsto & [\mathrm{id}_{z_i}] \\ & & : & z_i^* & \mapsto & [\mathrm{id}_{z_i^*}] \end{array}$$

Therefore the particular $\mathbb{Z}^n$ that we are quotienting out of $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}}$ is the one generated by the equivalence classes of the identity maps $\mathrm{id}_{z_i}, \mathrm{id}_{z_i^*}$ under abelianisation. But since $\mathrm{Ob}(L\mathbb{G}_n) = \mathrm{Z}^{*n}$ is generated by the objects z_i, z_i^*, and

$$[\mathrm{id}_x] \otimes [\mathrm{id}_y] \;=\; [\mathrm{id}_x \otimes \mathrm{id}_y] \;=\; [\mathrm{id}_{x \otimes y}]$$

the group generated by the $[\mathrm{id}_{z_i}], [\mathrm{id}_{z_i^*}]$ will just contain the equivalence classes for every identity morphism of $L\mathbb{G}_n$. That is, we will have

$$L\mathbb{G}_n(I, I) \quad = \quad \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}} \Big/ \langle\, [\mathrm{id}_x] \,:\, x \in \mathrm{Ob}(L\mathbb{G}_n)\,\rangle$$

as required. □

4.4 The action of $L\mathbb{G}_n$

Proposition 4.14. *The action of $L\mathbb{G}_n$ is given by the following map:*

Proof. □

4.5 A full description of $L\mathbb{G}_n$

With this last proposition proven, the results in this chapter now collectively describe how to construct free EG-algebras on n invertible objects. However, since this characterization was discovered by us in such a piecemeal fashion, it would be best to restate the complete conclusion all in one place.

Theorem 4.15. *Let $\mathbb{G}_n$ be the free* EG*-algebra on n objects. Then the free* EG*-algebra on n invertible objects, $L\mathbb{G}_n$, is the algebra described by*

Proof. □

With Theorem 4.15 proven we can now finally achieve the first main goal of this paper — to describe the free braided monoidal category on n invertible objects. In addition, this section will provide a few other simple applications of the theorem, in an effort to build up to the main result more gently. The definition of $L\mathbb{G}_n$ given in 4.15 is after all a little difficult to parse on first reading, because of the fairly abstract way it is presented, and hopefully the following concrete examples should allow the braided case to be properly understood.

4.6 Freely generated action operads

At this stage, the obvious next question to ask is can we simplify the expression $(G \times_{\mathbb{N}} \mathbb{N}^{*2n})^{\mathrm{gp,ab}}$?

Definition 4.16. Let G be an action operad and $\mathcal{G} \subseteq G$ a subset. If

- the monoid G is freely generated by the set $\mathcal{G}$ under tensor product

- the group $G(0) \subseteq G$ is the trivial group $\{I\}$

the we say that $(G, \mathcal{G})$ is a

Lemma 4.17. *If $(G, \mathcal{G})$ is , then $e_1 \in \mathcal{G}$, where e_1 is the identity element of the group $G(1) \subseteq G$.*

Proof. Consider the identity element $e_1 \in G(1)$. By definition, $|e_1| = 1$, which means that

$$\begin{array}{rcl} g_1 \otimes ... \otimes g_k = e_1 \implies |g_1| + ... + |g_k| & = & |g_1 \otimes ... \otimes g_k| \\ & = & |e_1| \\ & = & 1 \\ & & \\ \implies \exists i \in \mathbb{N} \quad : \qquad |g_i| & = & 1 \\ |g_j| & = & 0, \quad i \neq j \end{array}$$

But since $G(0) = \{I\}$, the only element of G of length 0 is I, and thus the only ways to express e_1 as a tensor product of other elements are the trivial ones,

$$e_1 = I \otimes ... \otimes I \otimes e_1 \otimes I \otimes ... \otimes I$$

Therefore e_1 cannot be generated by the set $\mathcal{G}$ unless e_1 itself is in $\mathcal{G}$, and so since $\mathcal{G}$ does generate all of the elements of G, this will in fact be the case. □

Proposition 4.18. *Let $(G, \mathcal{G})$ be a . Then*

$$(G \times_{\mathbb{N}} \mathbb{N}^{*2n})^{\mathrm{gp}} = G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{2n}$$

Proof. First, consider the fact that the monoid G is freely generated by the set $\mathcal{G}$, and that $\mathbb{N}^{*2n}$ is not only a free monoid but one whose generators $z_1, ..., z_{2n}$ all have length $|z_i| = 1$. This means that we can factorise any (g, w) in the pullback monoid $G \times_{\mathbb{N}} \mathbb{N}^{*2n}$

as a tensor product of elements of the pullback set $\mathcal{G} \times_{\mathbb{N}} \mathbb{N}^{*2n}$:

$$\begin{array}{rcll} g & = & g_1 \otimes ... \otimes g_k, & g_i \in \mathcal{G} \\ w & = & x_1 \otimes ... \otimes x_{|w|}, & x_i \in \{z_1, ..., z_{2n}\} \\ |g| & = & |w| & \end{array}$$

$$\begin{array}{rrcl} \Longrightarrow & (g, w) & = & (\, g_1,\, x_1 \otimes ... \otimes x_{|g_1|}\,) \otimes ... \otimes (\, g_k,\, x_{|w|-|g_k|} \otimes ... \otimes x_{|w|}\,) \\ & & =: & (g_1, w_1) \otimes ... \otimes (g_k, w_k), \\ & & & (g_i, w_i) \in \mathcal{G} \times_{\mathbb{N}} \mathbb{N}^{*2n} \end{array}$$

This expansion will also always be unique, since

$$\begin{array}{rcl} (g_1, w_1) \otimes ... \otimes (g_k, w_k) & = & (g'_1, w'_1) \otimes ... \otimes (g'_{k'}, w_{k'}), \\ & & (g_i, w_i), (g'_i, w'_i) \in \mathcal{G} \times_{\mathbb{N}} \mathbb{N}^{*2n} \end{array}$$

$$\Longrightarrow \quad (\, g_1 \otimes ... \otimes g_k,\, w_1 \otimes ... \otimes w_k\,) = (\, g'_1 \otimes ... \otimes g'_{k'},\, w'_1 \otimes ... \otimes w_{k'}\,)$$

$$\begin{array}{rrcl} \Longrightarrow & g_1 \otimes ... \otimes g_k & = & g'_1 \otimes ... \otimes g'_{k'} \\ & w_1 \otimes ... \otimes w_k & = & w'_1 \otimes ... \otimes w_{k'} \end{array}$$

$$\begin{array}{rrcl} \Longrightarrow & g_i & = & g'_i \\ & k & = & k' \\ \Longrightarrow & |g_i| & = & |g'_i| \\ \Longrightarrow & |w_i| & = & |w'_i| \\ \Longrightarrow & w_i & = & w'_i \end{array}$$

$$\Longrightarrow \quad (g_i, w_i) = (g'_i, w'_i)$$

In other words, the monoid $G \times_{\mathbb{N}} \mathbb{N}^{*2n}$ is freely generated by its subset $\mathcal{G} \times_{\mathbb{N}} \mathbb{N}^{*2n}$.

Now, recall that if M is a monoid which is presented by some generators $\mathcal{M}$ subject to relations $\mathcal{R}$, then the group completion M^{gp} will be the group given by the *group* presentation $(\mathcal{M}, \mathcal{R})$. In particular, if M is a free monoid, then M^{gp} is the free group on the set $\mathcal{M}$, and so it follows that G^{gp} is the free group on the set $\mathcal{G}$, $(\mathbb{N}^{*2n})^{\mathrm{gp}}$ is the free group on $\{z_1, ..., z_{2n}\}$ — that is, $\mathbb{Z}^{*2n}$ — and $(G \times_{\mathbb{N}} \mathbb{N}^{*2n})^{\mathrm{gp}}$ is the free group on $\mathcal{G} \times_{\mathbb{N}} \mathbb{N}^{*2n}$. Moreover, the canonical maps $G \to G^{\mathrm{gp}}$ and $\mathbb{N}^{*2n} \to (\mathbb{N}^{*2n})^{\mathrm{gp}} = \mathbb{Z}^{*2n}$ act as the identity on the shared generating sets $\mathcal{G}$ and $\{z_1, ..., z_{2n}\}$ respectively, and so these homomorphisms are actually inclusions of monoids.

Finally, define the pullback

$$\begin{tikzcd}[column sep=small]
 & G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & \\
G^{\mathrm{gp}} \arrow[dr, "|_|^{\mathrm{gp}}"'] & & \mathbb{Z}^{*2n} \arrow[dl, "|_|^{\mathrm{gp}}"] \\
 & \mathbb{Z} &
\end{tikzcd}$$

where the $|\_|^{\mathrm{gp}}$ are the obvious extensions of the length homomorphisms $|\_|$ defined on their generators by

$$\begin{array}{rcrcl}
|_|^{\mathrm{gp}} & : & G^{\mathrm{gp}} & \to & \mathbb{Z} \\
 & : & g & \mapsto & |g|, \quad g \in \mathcal{G} \\
 & : & g^* & \mapsto & -|g|, \quad g \in \mathcal{G}
\end{array}$$

$$\begin{array}{rcrcl}
|_|^{\mathrm{gp}} & : & \mathbb{Z}^{*2n} & \to & \mathbb{Z} \\
 & : & z_i & \mapsto & 1 \\
 & : & z_i^* & \mapsto & -1
\end{array}$$

Notice that we can use the inclusions $\mathcal{G} \hookrightarrow G \hookrightarrow G^{\mathrm{gp}}$ and $\mathbb{N}^{*2n} \hookrightarrow \mathbb{Z}^{*2n}$ to see any element (g, w) of our generator pullback $\mathcal{G} \times_{\mathbb{N}} \mathbb{N}^{*2n}$ as an element $(\, i(g), i(w)\,)$ of this new pullback $G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n}$, because

$$|i(g)|^{\mathrm{gp}} \;=\; |g| \;=\; |w| \;=\; |i(w)|^{\mathrm{gp}}$$

Furthermore, for any $(g, w) \in G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n}$ we must have some

$$g_1, ..., g_k \in \{\, h \in G^{\mathrm{gp}} \;:\; h \in \mathcal{G} \text{ or } h^* \in \mathcal{G} \,\}, \qquad g \;=\; g_1 \otimes ... \otimes g_k$$

and hence

$$\begin{array}{rcl}
(g, w) & = & (\, g_1 \otimes ... \otimes g_k,\; w\,) \\
 & = & \Big(\, g_1 \otimes ... \otimes g_k \otimes {e_{|g|}}^* \otimes e_{|g|},\;\; z_1^{|g|} \otimes (z_1^{|g|})^* \otimes w\,\Big) \\
 & = & \Big(\, g_1 \otimes ... \otimes g_k \otimes {e_{|g|}}^* \otimes e_{|w|},\;\; z_1^{|g_1|+...+|g_k|} \otimes (z_1^{|g|})^* \otimes w\,\Big) \\
 & = & \Big(g_1, {z_1}^{|g_1|}\Big) \otimes ... \otimes \Big(g_k, {z_1}^{|g_k|}\Big) \otimes \Big({e_{|g|}}^*, (z_1^{|g|})^*\Big) \otimes \Big(e_{|w|}, w\Big) \\
 & = & (g_1, {z_1}^{|g_1|}) \otimes ... \otimes (g_k, {z_1}^{|g_k|}) \otimes (e_{|g|}, {z_1}^{|g|})^* \otimes (e_{|w|}, w)
\end{array}$$

$$(g_1, {z_1}^{|g_1|}),\; ...,\; (g_k, {z_1}^{|g_k|}),\; (e_{|g|}, {z_1}^{|g|}),\; (e_{|w|}, w) \;\in\; G \times_{\mathbb{N}} \mathbb{N}^{*2n}$$

Therefore the subset $\mathcal{G} \times_{\mathbb{N}} \mathbb{N}^{*2n}$ generates the whole of the free group $G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n}$, and thus since $(G \times_{\mathbb{N}} \mathbb{N}^{*2n})^{\mathrm{gp}}$ is also freely generated by the same set, we must have

$$(G \times_{\mathbb{N}} \mathbb{N}^{*2n})^{\mathrm{gp}} \quad = \quad G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{2n}$$

as required. $\square$

Proposition 4.19.

$$(G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n})^{\mathrm{ab}} \quad \cong \quad G^{\mathrm{gp,ab}} \times_{\mathbb{Z}} \mathbb{Z}^{2n}$$

Proof. Recall that the monoid $G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n}$ is a pullback over the length homomorphisms $|\_|^{\mathrm{gp}} : G \to \mathbb{N}$ and $|\_|^{\mathrm{gp}} : \mathbb{N}^{*2n} \to \mathbb{N}$, which from now on we will just write as $|\_|$ to avoid clutter. Their shared target, $\mathbb{Z}$, is a abelian group, and this means that the $|\_|$ will factor through the abelianisations $(G^{\mathrm{gp}})^{\mathrm{ab}}$ and $(\mathbb{Z}^{*2n})^{\mathrm{ab}} = \mathbb{Z}^{2n}$, respectively. Expanding them like this, the pullback square for $G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n}$ becomes the following commutative diagram:

$$\begin{tikzcd}
 & G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n} \arrow[dl] \arrow[dr] & \\
G^{\mathrm{gp}} \arrow[d, "\mathrm{ab}"'] \arrow[ddr, "|_|"] & & \mathbb{Z}^{*2n} \arrow[d, "\mathrm{ab}"] \arrow[ddl, "|_|"'] \\
G^{\mathrm{gp,ab}} \arrow[dr, "|_|^{\mathrm{ab}}"'] & & \mathbb{Z}^{2n} \arrow[dl, "|_|^{\mathrm{ab}}"] \\
 & \mathbb{Z} &
\end{tikzcd}$$

Now, if we take the pullback of the bottom two maps in this diagram, the $|\_|^{\mathrm{ab}}$, then we obtain a new monoid $G^{\mathrm{gp,ab}} \times_{\mathbb{Z}} \mathbb{Z}^{2n}$. Then because the diagram above also forms a commutative square over the maps $|\_|^{\mathrm{ab}}$, the universal property of the pullback $G^{\mathrm{gp,ab}} \times_{\mathbb{Z}} \mathbb{Z}^{2n}$ will give us a unique homomorphism $u : G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n} \to G^{\mathrm{gp,ab}} \times_{\mathbb{Z}} \mathbb{Z}^{2n}$, making the top-left and top-right regions in the following diagram commute:

$$\begin{tikzcd}
 & G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n} \arrow[dl] \arrow[dr] \arrow[dd, dashed, "u"] & \\
G^{\mathrm{gp}} \arrow[dd, "\mathrm{ab}"'] & & \mathbb{Z}^{*2n} \arrow[dd, "\mathrm{ab}"] \\
 & G^{\mathrm{gp,ab}} \times_{\mathbb{Z}} \mathbb{Z}^{2n} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & \\
G^{\mathrm{gp,ab}} \arrow[dr, "|_|^{\mathrm{ab}}"'] & & \mathbb{Z}^{2n} \arrow[dl, "|_|^{\mathrm{ab}}"] \\
 & \mathbb{Z} &
\end{tikzcd}$$

However, since the monoid $G^{\mathrm{gp,ab}} \times_{\mathbb{Z}} \mathbb{Z}^{2n}$ is a pullback of abelian groups, it must be abelian itself. It follows then that the map u will also factor through the abelianisation of its source monoid, via a new homomorphism that we shall call u^{ab}.

$$\begin{tikzcd}
 & G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n} \arrow[dl, "\mathrm{ab}"'] \arrow[dr, "u"] & \\
(G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n})^{\mathrm{ab}} \arrow[rr, dashed, "u^{\mathrm{ab}}"] & & G^{\mathrm{gp,ab}} \times_{\mathbb{Z}} \mathbb{Z}^{2n}
\end{tikzcd}$$

It is not too hard to find an explicit description of the map u^{ab}. For any group H, the abelianisation H^{ab} is just the quotient group $H/[H,H]$, where

$$[H,H] \;=\; \{\, h \in H \;:\; \exists a, b \in H,\; h \;=\; aba^{-1}b^{-1} \,\}$$

is the commutator subgroup of H. Thus elements of the monoid $G^{\mathrm{gp,ab}} \times_{\mathbb{Z}} \mathbb{Z}^{2n}$ are pairs of equivalence classes

$$\Big(\, [g], [w] \,\Big), \quad \text{for} \quad g \in G^{\mathrm{gp}},\; w \in \mathbb{Z}^{*2n}, \quad |\,[g]\,|^{\mathrm{ab}} \;=\; |\,[w]\,|^{\mathrm{ab}}$$

and the elements of $(G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n})^{\mathrm{ab}}$ are equivalence classes of pairs

$$\Big[\, (g, w) \,\Big], \quad \text{for} \quad g \in G^{\mathrm{gp}},\; w \in \mathbb{Z}^{*2n}, \quad |g| \;=\; |w|$$

By the universal property of pullbacks, the unique map u is then simply the monoid homomorphism defined by

$$\begin{array}{rcrcl} u & : & (G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n}) & \to & G^{\mathrm{gp,ab}} \times_{\mathbb{Z}} \mathbb{Z}^{2n} \\ & : & (g, w) & \mapsto & \big([g], [w] \big) \end{array}$$

and hence u^{ab} is

$$\begin{array}{rcrcl} u^{\mathrm{ab}} & : & (G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n})^{\mathrm{ab}} & \to & G^{\mathrm{gp,ab}} \times_{\mathbb{Z}} \mathbb{Z}^{2n} \\ & : & \big[(g, w) \big] & \mapsto & \big([g], [w] \big) \end{array}$$

To complete the proof, we just need to demonstrate that this map is actually an isomorphism of monoids. In other words, we must show that the obvious reverse assignment, $([g], [w]) \mapsto [(g, w)]$, is well-defined.

Let $g, g' \in G^{\mathrm{gp}}$ and $w, w' \in \mathbb{Z}^{*2n}$ with $|g| = |g'| = |w| = |w'|$, so that (g, w) and (g', w') are valid elements of $G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n}$ and $([g], [w]), ([g'], [w'])$ are valid elements of $G^{\mathrm{gp,ab}} \times_{\mathbb{Z}} \mathbb{Z}^{2n}$, and furthermore let $([g], [w]) = ([g'], [w'])$. It follows immediately that $[g] = [g']$ and $[w] = [w']$, or equivalently

$$\begin{array}{rcrcl} \exists\, h, h' \in [G^{\mathrm{gp}}, G^{\mathrm{gp}}] & : & gh & = & g'h' \\ \exists\, v, v' \in [\mathbb{Z}^{*2n}, \mathbb{Z}^{*2n}] & : & wv & = & w'v' \end{array}$$

But notice that

$$\begin{array}{rcrcll} h \in [G^{\mathrm{gp}}, G^{\mathrm{gp}}] & \implies & h & = & a \otimes b \otimes a^* \otimes b^*, & a, b \in G^{\mathrm{gp}} \\ & & & & & \\ & \implies & |h| & = & |a| + |b| + |a^*| + |b^*| & \\ & & & = & |a| + |b| - |a| - |b| & \\ & & & = & 0 & \end{array}$$

$$\begin{array}{rcl} h' \in [G^{\mathrm{gp}}, G^{\mathrm{gp}}] & \implies & |h'| \;=\; 0 \\ v \in [\mathbb{Z}^{*2n}, \mathbb{Z}^{*2n}] & \implies & |v| \;=\; 0 \\ v' \in [\mathbb{Z}^{*2n}, \mathbb{Z}^{*2n}] & \implies & |v'| \;=\; 0 \end{array}$$

and so in particular

$$|h| = |v|, \quad |h'| = |v'| \quad \implies \quad (h, v), (h', v') \in G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n}$$

Moreover, if e_1 is the identity element of the group $G(1) \subseteq G \subseteq G^{\mathrm{gp}}$ and z_1 is the first generator of Z^{*2n} then $|e_1| = |z_1| = 1$, and thus

$$\begin{array}{rcccl} \exists\, a, b \in G^{\mathrm{gp}} & : & h & = & a \otimes b \otimes a^* \otimes b^* \\ \exists\, x, y \in \mathbb{Z}^{*2n} & : & v & = & x \otimes y \otimes x^* \otimes y^* \end{array}$$

$$\begin{array}{rrcl} \Longrightarrow & h & = & a \otimes b \otimes a^* \otimes b^* \\ & & = & a \otimes b \otimes a^* \otimes b^* \otimes e_1^{|x|+|y|} \otimes (e_1^{|x|+|y|})^* \\ & & = & a \otimes b \otimes a^* \otimes b^* \otimes e_1^{|x|} \otimes e_1^{|y|} \otimes (e_1^{|x|})^* \otimes (e_1^{|y|})^* \\ & & & \\ & v & = & x \otimes y \otimes x^* \otimes y^* \\ & & = & z_1^{|a|+|b|} \otimes (z_1^{|a|+|b|})^* \otimes x \otimes y \otimes x^* \otimes y^* \\ & & = & z_1^{|a|} \otimes z_1^{|b|} \otimes (z_1^{|a|})^* \otimes (z_1^{|b|})^* \otimes x \otimes y \otimes x^* \otimes y^* \end{array}$$

$$\begin{array}{rrcl} \Longrightarrow & (h, v) & = & \Big(a,\, z_1^{|a|}\Big) \otimes \Big(b,\, z_1^{|b|}\Big) \otimes \Big(a^*,\, (z_1^{|a|})^*\Big) \otimes \Big(b^*,\, (z_1^{|b|})^*\Big) \\ & & & \otimes \Big(e_1^{|x|},\, x\Big) \otimes \Big(e_1^{|y|},\, y\Big) \otimes \Big((e_1^{|x|})^*,\, x^*\Big) \otimes \Big((e_1^{|y|})^*,\, y^*\Big) \\ & & \in & \big[\, G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n}, G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n} \,\big] \end{array}$$

and also

$$(h', v') \in [\, G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n}, G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n} \,]$$

for similar reasons. Therefore

$$\exists\, (h, v), (h', v') \in [\, G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n}, G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n} \,] \quad \text{such that}$$

$$\begin{array}{rcl} (g, w) \otimes (h, v) & = & (g \otimes h, w \otimes v) \\ & = & (g' \otimes h', w' \otimes v') \\ & = & (g', w') \otimes (h, v) \\ & & \\ \Longrightarrow \Big[(g, w) \Big] & = & \Big[(g', w') \Big] \end{array}$$

That is, we have shown that

$$\Big([g], [w] \Big) = \Big([g'], [w'] \Big) \quad \Longrightarrow \quad \Big[(g, w) \Big] = \Big[(g', w') \Big]$$

and so the mapping $([g], [w]) \mapsto [(g, w)]$ is indeed well-defined. From this we can conclude that the homomorphism u^{ab} has an inverse, and hence we have an isomorphism

$$(G^{\mathrm{gp}} \times_{\mathbb{Z}} \mathbb{Z}^{*2n})^{\mathrm{ab}} \quad \cong \quad G^{\mathrm{gp,ab}} \times_{\mathbb{Z}} \mathbb{Z}^{2n}$$

as required ☐

Definition 4.20. We say that a monoid M is *left-cancellative* if for any $x, y, z \in M$, we have

$$x \otimes y = x \otimes z \quad \Longrightarrow \quad y = z$$

That is, common factors in tensor products may be cancelled out on the left. Similarly, we call M *right-cancellative* if common factorscan be cancelled on the left:

$$x \otimes z = y \otimes z \quad \Longrightarrow \quad x = y$$

A monoid that is both left- and right-cancellative is simply referred to as *cancellative*

Proposition 4.21. *Let G be an action operad. Then $(G, \otimes)$ is a cancellative monoid.*

Proof. Let g, g', and h be elements of G with the property that $g \otimes h = g' \otimes h$. Since the length map $|\_| : G \to \mathbb{N}$ is a monoid homomorphism, applying it to both sides of this equation yields

$$\begin{array}{rcl} |g \otimes h| & = & |g' \otimes h| \\ \Longrightarrow \quad |g| \otimes |h| & = & |g'| \otimes |h| \end{array}$$

Then because $\mathbb{N}$ is a definitely a cancellative monoid, it follows from this that $|g| = |g'|$. In other words, g and g' are both elements of the same group of operations, $G(\,|g|\,)$, and so in particular it makes sense to multiply them. Thus

$$\begin{array}{rcl} g \otimes h = g' \otimes h \quad \Longrightarrow \quad e_{|g|+|h|} & = & (g \otimes h)^{-1}(g' \otimes h) \\ & = & (g^{-1} \otimes h^{-1})(g' \otimes h) \\ & = & (g^{-1}g') \otimes (h^{-1}h) \\ & = & (g^{-1}g') \otimes e_{|h|} \end{array}$$

But then

$$e_{|g|} \otimes e_{|h|} = e_{|g|+|h|} = (g^{-1}g') \otimes e_{|h|}$$

☐

Lemma 4.22. *Let be a subset of whose elements generate G by tensor product and group multiplication, subject to some relations $\mathcal{R}$. Then $G^{\mathrm{gp,ab}}$ is just the group with generators $\mathcal{G}$, subject to relations*

$$\mathcal{R}' = \mathcal{R} \cup \{\, ab = ba \;:\; \forall a, b \in \mathcal{G} \,\}$$

Proof. □

Proposition 4.23. *One object case*

Proposition 4.24. *Symmetric case*

Proposition 4.25. *Braided case*

Proposition 4.26. *Cactus group case*

. . . . .

4.7 The free algebra on n weakly invertible objects

Up until now, we've been working under the convention that by 'invertible' objects we mean stictly invertible — $x \otimes x^* = I$. As an additional exercise, we can ask ourselves how all of this would change if we permitted our objects to be only weakly invertible, that is $x \otimes x^* \cong I$. The situation is actually quite elegant, in that the effect of weakening in our objects can be offset completely by the effect of also weakening our algebra homomorphisms, such that we won't need to calculate any new free algebras other than those given by Theorem 4.15. Before proving this though, we first to need to set out some definitions.

Definition 4.27. Given an EG-algebra X, we denote by X_{wkinv} the category whose

- objects are tuples (x, x^*, η, ϵ), where x and x^* are objects of X and $\eta : I \to x^* \otimes x$ and $\epsilon : x \otimes x^* \to I$ are morphisms such that the composites

$$x \xrightarrow{\mathrm{id}\otimes\eta} x \otimes x^* \otimes x \xrightarrow{\epsilon\otimes\mathrm{id}} x \qquad x^* \xrightarrow{\eta\otimes\mathrm{id}} x^* \otimes x \otimes x^* \xrightarrow{\mathrm{id}\otimes\epsilon} x^*$$

  are identity morphisms.

- maps $(f, f^*) : (x, x^*, \eta_x, \epsilon_x) \to (y, y^*, \eta_y, \epsilon_y)$ are pairs $f : x \to y$, $f^* : x^* \to y^*$ of morphisms such that the diagrams

$$\begin{tikzcd} & I \arrow[dl, "\eta_x"'] \arrow[dr, "\eta_y"] & \\ x^* \otimes x \arrow[rr, "f^*\otimes f"] & & y \otimes y^* \end{tikzcd} \qquad \begin{tikzcd} x \otimes x^* \arrow[rr, "f\otimes f^*"] \arrow[dr, "\epsilon_x"'] & & y \otimes y^* \arrow[dl, "\epsilon_y"] \\ & I & \end{tikzcd}$$

  commute.

Definition 4.28. Let (X, α) and (Y, β) be EG-algebras. A *weak* EG*-algebra homorphism* between them is a weak monoidal functor $\psi : X \to Y$ such that all diagrams of the form

$$\begin{tikzcd}
\psi(x_1 \otimes ... \otimes x_m) \arrow[rr, "\sim"] \arrow[d, "\psi(\alpha(g;h_1,...,h_m))"'] & & \psi(x_1) \otimes ... \otimes \psi(x_m) \arrow[d, "\beta(g;\psi(h_1),...,\psi(h_m))"] \\
\psi(y_{\pi(g)^{-1}(1)} \otimes ... \otimes y_{\pi(g)^{-1}(m)}) \arrow[rr, "\sim"] & & \psi(y_{\pi(g)^{-1}(1)}) \otimes ... \otimes \psi(y_{\pi(g)^{-1}(m)})
\end{tikzcd}$$

commute.

Definition 4.29. We denote by EGAlg$_W$ the 2-category of EG-algebras, weak EG-algebra homomorphisms, and weak monoidal transformations.

Now we can properly express what we mean by the free algebras on weakly invertible objects being the same as those in the strict case.

Theorem 4.30. *The algebra $L\mathbb{G}_n$ is also the free* EG*-algebra on n weakly invertible objects. Specifically, for any other* EG*-algebra X there is an equivalence of categories*

$$\mathrm{E}G\mathrm{Alg}_W(L\mathbb{G}_n, X) \simeq (X_{\mathrm{wkinv}})^n$$

Proof. We begin by defining a functor $F : \mathrm{E}G\mathrm{Alg}_W(L\mathbb{G}_n, X) \to (X_{\mathrm{wkinv}})^n$. On weak maps, F acts as

$$F(\,\psi : L\mathbb{G}_n \to X\,) = \Big\{\,(\,\psi(z_i),\, \psi(z_i^*),\, I \xrightarrow{\sim} \psi(I) \xrightarrow{\sim} \psi(z_i^*)\psi(z_i),\, \psi(z_i)\psi(z_i^*) \xrightarrow{\sim} \psi(I) \xrightarrow{\sim} I\,)\,\Big\}_{i\in\{z_1,...,z_n\}}$$

where the z_i are the generators of $\mathbb{Z}^{*n}$ and the isomorphisms are those given by ψ being a weak moniodal functor. On weak monoidal transformations, F acts as

$$F(\,\theta : \psi \to \chi\,) = \Big\{\,(\,\theta_{z_i},\, \theta_{z_i^*}\,)\,\Big\}_{i\in\{z_1,...,z_n\}}$$

This choice does satisfy the condition on morphisms of $(X_{\mathrm{wkinv}})^n$, since we can build the required commuting diagrams out of smaller ones given by θ being a weak monoidal transfomation:

$$\begin{tikzcd} & I \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ \psi(I) \arrow[rr, "\theta_I"] \arrow[d, "\sim"'] & & \chi(I) \arrow[d, "\sim"] \\ \psi(z_i^*) \otimes \psi(z_i) \arrow[rr, "\theta_{z_i^*} \otimes \theta_{z_i}"] & & \chi(z_i^*) \otimes \chi(z_i) \end{tikzcd} \qquad \begin{tikzcd} \psi(z_i) \otimes \psi(z_i^*) \arrow[rr, "\theta_{z_i} \otimes \theta_{z_i^*}"] \arrow[d, "\sim"'] & & \chi(z_i) \otimes \chi(z_i^*) \arrow[d, "\sim"] \\ \psi(I) \arrow[rr, "\theta_I"] \arrow[dr, "\sim"'] & & \chi(I) \arrow[dl, "\sim"] \\ & I & \end{tikzcd}$$

Now we need to check if F is an equivalence of categories. First, let $\Big\{(x_i, x_i^*, \eta_i, \epsilon_i)\Big\}_{i \in \{z_1, \ldots, z_n\}}$ be an arbitrary object of $(X_{\text{wkinv}})^n$. We can construct a weak algebra map $\psi : L\mathbb{G}_n \to X$ from it as follows. Define

$$\psi(I) = I, \quad \psi(z_i) = x_i, \quad \psi(z_i^*) = x_i^*$$

and choose the isomorphisms

$$\begin{array}{rclcrcl} \psi_I & : & I \to \psi(I) & = & \mathrm{id}_I & : & I \to I \\ \psi_{z_i, z_i^*} & : & \psi(z_i) \otimes \psi(z_i^*) \to \psi(I) & = & \epsilon_i & : & x_i \otimes x_i^* \to I \\ \psi_{z_i^*, z_i} & : & \psi(z_i^*) \otimes \psi(z_i) \to \psi(I) & = & \eta_i^{-1} & : & x_i^* \otimes x_i \to I \end{array} .$$

Then for any $w, w' \in \mathrm{Ob}(L\mathbb{G}_n)$ such that $d(w \otimes w') = d(w) \otimes d(w')$, where $d(-)$ is the minimal generator decomposition from **??**, set

$$\psi(w \otimes w') = \psi(w) \otimes \psi(w'), \qquad \psi_{w,w'} = \mathrm{id}_{\psi(w) \otimes \psi(w')}$$

This is enough to determine the value of ψ on all of the remaining objects, via successive decompositions. For the isomorphisms, first note that the ones we have already defined satisfy the associativity and unitality required of weak monoidal functors. Now consider some w, w' with $d(w \otimes w') \neq d(w) \otimes d(w')$. The fact that they differ implies that tensoring w with w' causes some cancellation of inverses to occur where the end of one sequence meets the beginning of another. In particular, if we let b be the last term in the minimal generator decomposition of w, and let $c = w'$, then we conclude that the length $d(b \otimes c)$ is smaller than the length of $d(c)$. Let a be the product of the rest of

$d(w)$, so that $a \otimes b = w$. Then we can use requirement for associativity,

$$\begin{tikzcd} \psi(a) \otimes \psi(b) \otimes \psi(c) \arrow[r, "\mathrm{id}\otimes\psi_{b,c}"] \arrow[d, "\psi_{a,b}\otimes\mathrm{id}"'] & \psi(a) \otimes \psi(b \otimes c) \arrow[d, "\psi_{a,b\otimes c}"] \\ \psi(a \otimes b) \otimes \psi(c) \arrow[r, "\psi_{a\otimes b,c}"] & \psi(a \otimes b \otimes c) \end{tikzcd}$$

to define $\psi_{w,w'} = \psi a \otimes b, c$ in terms of three other isomorphisms that each have strictly smaller decompositions. Repeating this process will therefore eventually yield a definition in terms of our previous isomorphisms.

By Lemma 3.6, every morphism in $L\mathbb{G}_n$ can be written as $\alpha(g; \mathrm{id}_{w_1}, ..., \mathrm{id}_{w_m})$ for some $g \in G(m)$, $w_i \in \mathbb{Z}^{*n}$. The action of ψ on morphisms is thus determined by the diagram in Definition 4.28, that is

$$\psi(\alpha(g; w_1, ...w_m)) \;=\; \psi_{\mathbf{w}_{\pi(g)^{-1}}} \circ \beta(\, g\,;\, \mathrm{id}_{\psi(w_1)},\; ...,\; \mathrm{id}_{\psi(w_m)}\,) \circ \psi_{\mathbf{w}}^{-1}$$

However, morphisms do not have a unique representation of this form, so we must check that whenever we have different representations of the same morphism

$$\alpha(g; \mathrm{id}_{w_1}, ..., \mathrm{id}_{w_m}) = \alpha(g'; \mathrm{id}_{w'_1}, ..., \mathrm{id}_{w'_{m'}})$$

their diagrams give the same image under ψ. There are two cases to consider here;

$$\alpha(g; \mathrm{id}_{w_1}, ..., \mathrm{id}_{w_m}) = \alpha(\, g \otimes e_k\,;\, \mathrm{id}_{w_1},\; ...,\; \mathrm{id}_{w_m},\; \mathrm{id}_{v_1},\; ...,\; \mathrm{id}_{v_k}\,)$$

when $v_1 \otimes ... \otimes v_k = 0$, which comes from the edges of the colimit diagram D_n in **??**; and

$$\begin{aligned} \alpha(g; \mathrm{id}_{w_1}, ..., \mathrm{id}_{w_m}) \;&=\; \alpha(\, h\,;\, \mathrm{id}_{w'_1},\; ...,\; \mathrm{id}_{w_{m'}}\,) \\ &\quad \circ\, \alpha(\, j\,;\, \mathrm{id}_{w''_1},\; ...,\; \mathrm{id}_{w''_{m''}}\,) \\ &\quad \circ\, \alpha(\, h^{-1}\,;\, \mathrm{id}_{w'_1},\; ...,\; \mathrm{id}_{w'_{m'}}\,) \\ &\quad \circ\, \alpha(\, j^{-1}\,;\, \mathrm{id}_{w''_1},\; ...,\; \mathrm{id}_{w''_{m''}}\,) \\ &=\; \mathrm{id}_{w_1\otimes ...\otimes w_m} \end{aligned}$$

for $\alpha(\, h\,;\, \mathrm{id}_{w'_1},\; ...,\; \mathrm{id}_{w_{m'}}\,), \alpha(\, j\,;\, \mathrm{id}_{w''_1},\; ...,\; \mathrm{id}_{w''_{m''}}\,) \in \mathbb{G}_n(w_1 \otimes ... \otimes w_m, w_1 \otimes ... \otimes w_m)$, which comes from the abelianisation of the vertices of D_n. All other ways for a morphism to have different representations must be generated by successive examples of these cases, since otherwise they wouldn't be coequalised by the colimit in **??**. In the first case we just have

$$\begin{aligned}
& \psi(\,\alpha(\,g \otimes e_k\,;\,\mathrm{id}_{w_1},\,...,\,\mathrm{id}_{w_m},\,\mathrm{id}_{v_1},\,...,\,\mathrm{id}_{v_k}\,)\,) \\
= \;& \psi_{\mathbf{w}_{\pi(g)^{-1}},\mathbf{v}} \circ \beta(\,g \otimes e_k\,;\,\mathrm{id}_{\psi(w_1)},\,...,\,\mathrm{id}_{\psi(w_m)},\,\mathrm{id}_{\psi(v_1)},\,...,\,\mathrm{id}_{\psi(v_k)}\,) \circ \psi^{-1}_{\mathbf{w},\mathbf{v}} \\
= \;& \left(\psi_{\mathbf{w}_{\pi(g)^{-1}}} \otimes \psi_{\mathbf{v}}\right) \circ \left(\beta(g;\mathrm{id}_{\psi(w_1)},...,\mathrm{id}_{\psi(w_m)}) \otimes \mathrm{id}_{\psi(\mathbf{v})}\right) \circ \left(\psi^{-1}_{\mathbf{w}} \otimes \psi^{-1}_{\mathbf{v}}\right) \\
= \;& \left(\psi_{\mathbf{w}_{\pi(g)^{-1}}} \circ \beta(g;\mathrm{id}_{\psi(w_1)},...,\mathrm{id}_{\psi(w_m)}) \circ \psi^{-1}_{\mathbf{w}}\right) \otimes \left(\psi_{\mathbf{v}} \circ \mathrm{id}_{\psi(\mathbf{v})} \circ \psi^{-1}_{\mathbf{v}}\right) \\
= \;& \psi_{\mathbf{w}_{\pi(g)^{-1}}} \circ \beta(g;\mathrm{id}_{\psi(w_1)},...,\mathrm{id}_{\psi(w_m)}) \circ \psi^{-1}_{\mathbf{w}} \\
= \;& \psi(\,\alpha(g;\mathrm{id}_{w_1},...,\mathrm{id}_{w_m})\,)
\end{aligned}$$

as required. The second case is more subtle. We begin by expanding

$$\begin{aligned}
& \psi(\,\alpha(\,g\,;\,\mathrm{id}_{w_1},\,...,\,\mathrm{id}_{w_m}\,) \\
= \;& \psi(\,\alpha(\,h\,;\,\mathrm{id}_{w'_1},\,...,\,\mathrm{id}_{w_{m'}}\,)\,) \\
& \circ \psi(\,\alpha(\,j\,;\,\mathrm{id}_{w''_1},\,...,\,\mathrm{id}_{w''_{m''}}\,)\,) \\
& \circ \psi(\,\alpha(\,h^{-1}\,;\,\mathrm{id}_{w'_1},\,...,\,\mathrm{id}_{w'_{m'}}\,)\,) \\
& \circ \psi(\,\alpha(\,j^{-1}\,;\,\mathrm{id}_{w''_1},\,...,\,\mathrm{id}_{w''_{m''}}\,)\,) \\
= \;& \psi_{\mathbf{w}'} \circ \beta(\,h\,;\,\mathrm{id}_{\psi(w'_1)},\,...,\,\mathrm{id}_{\psi(w_{m'})}\,) \circ \psi^{-1}_{\mathbf{w}'} \\
& \circ \psi_{\mathbf{w}''} \circ \beta(\,j\,;\,\mathrm{id}_{\psi(w''_1)},\,...,\,\mathrm{id}_{\psi(w''_{m''})}\,) \circ \psi^{-1}_{\mathbf{w}''} \\
& \circ \psi_{\mathbf{w}'} \circ \beta(\,h^{-1}\,;\,\mathrm{id}_{\psi(w'_1)},\,...,\,\mathrm{id}_{\psi(w'_{m'})}\,) \circ \psi^{-1}_{\mathbf{w}'} \\
& \circ \psi_{\mathbf{w}''} \circ \beta(\,j^{-1}\,;\,\mathrm{id}_{\psi(w''_1)},\,...,\,\mathrm{id}_{\psi(w''_{m''})}\,) \circ \psi^{-1}_{\mathbf{w}''}
\end{aligned}$$

Here the objects w_i, w'_i, w''_i are all in $\mathbb{G}_n \subseteq L\mathbb{G}_n$, and so we know their minimal generator decompositions are also in $\mathbb{G}_n$. It follows that $d(w_i \otimes w_j) = d(w_i) \otimes d(w_j)$ for all i, j, and hence by our definition of ψ we have $\psi(w_i \otimes w_j) = \psi(w_i) \otimes \psi(w_j)$ and also $\psi_{\mathbf{w}_\sigma} = id$ for any permuation σ — and the same for $\mathbf{w}'$ and $\mathbf{w}''$. Also, note that since we are working in $\mathbb{G}_n(w_1 \otimes ... \otimes w_m, w_1 \otimes ... \otimes w_m)$, all of the action morphisms in the above composite have the same source and target, $\psi(w_1 \otimes ... \otimes w_m)$. This object is weakly invertible, because each of the w_i are invertible. However, the automorphisms of any weakly invertible object are isomorphic to the automorphisms of the unit object, as in the proof of **??**, and hence form an abelian group, by an Eckmann-Hilton argument like in the proof of **??**. Therefore we may permute these action morphisms freely, and so

$$\begin{aligned}
& \psi(\, \alpha(\, g\, ;\, \mathrm{id}_{w_1},\, \ldots,\, \mathrm{id}_{w_m}\,) \\
= \;& \beta(\, h\, ;\, \mathrm{id}_{\psi(w'_1)},\, \ldots,\, \mathrm{id}_{\psi(w_{m'})}\,) \\
& \circ\, \beta(\, h^{-1}\, ;\, \mathrm{id}_{\psi(w'_1)},\, \ldots,\, \mathrm{id}_{\psi(w'_{m'})}\,) \\
& \circ\, \beta(\, j\, ;\, \mathrm{id}_{\psi(w''_1)},\, \ldots,\, \mathrm{id}_{\psi(w''_{m''})}\,) \\
& \circ\, \beta(\, j^{-1}\, ;\, \mathrm{id}_{\psi(w''_1)},\, \ldots,\, \mathrm{id}_{\psi(w''_{m''})}\,) \\
= \;& \mathrm{id}_{\psi(w_1)\otimes \ldots \otimes \psi(w_m)} \\
= \;& \psi_{\mathbf{w}} \circ \beta(\, e_m\, ;\, \mathrm{id}_{\psi(w_1)},\, \ldots,\, \mathrm{id}_{\psi(w_m)}\,) \circ \psi_{\mathbf{w}}^{-1}
\end{aligned}$$

as required.

With ψ now fully defined, notice that

$$\begin{aligned}
F(\psi) \;&=\; \Big\{\, (\, \psi(z_i),\, \psi(z_i^*),\, I \xrightarrow{\sim} \psi(I) \xrightarrow{\sim} \psi(z_i^*)\psi(z_i),\, \psi(z_i)\psi(z_i^*) \xrightarrow{\sim} \psi(I) \xrightarrow{\sim} I\,)\, \Big\}_{i\in\{z_1,\ldots,z_n\}} \\
&=\; \Big\{\, (\, x_i,\, x_i^*,\, \eta_i,\, \epsilon_i\,)\, \Big\}_{i\in\{z_1,\ldots,z_n\}}
\end{aligned}$$

which was our arbitrary object in $(X_{\mathrm{wkinv}})^n$. Therefore, F is surjective on objects.

Next, choose an arbitrary monoidal transformation $\theta : \psi \to \chi$ from $\mathrm{E}G\mathrm{Alg}_W(L\mathbb{G}_n, X)$. By naturality, for any $w, w' \in \mathrm{Ob}(L\mathbb{G}_n)$ we have that

$$\begin{array}{ccc}
\psi(w)\otimes\psi(w') & \xrightarrow{\;\sim\;} & \psi(w\otimes w') \\
{\scriptstyle \theta_w\otimes\theta_{w'}}\big\downarrow & & \big\downarrow{\scriptstyle \theta_{w\otimes w'}} \\
\chi(w)\otimes\chi(w') & \xrightarrow{\;\sim\;} & \chi(w\otimes w')
\end{array}$$

or equivalently, $\theta_{w\otimes w'} = \chi_{w,w'} \circ (\theta_w \otimes \theta_{w'}) \circ \psi_{w,w'}^{-1}$. It follows from this that the components of θ are generated by the components on the generators of $\mathrm{Ob}(L\mathbb{G}_n)$, namely $\{\, (\, \theta_{z_i},\, \theta_{z_i^*}\,)\, \}_{i\in\{z_1,\ldots,z_n\}}$. But this is just $F(\theta)$, and thus any monoidal transformation θ is determined uniquely by its image under F, or in other words F is faithful.

Finally, let ψ, χ be objects of $\mathrm{E}G\mathrm{Alg}_W(L\mathbb{G}_n, X)$, and choose an arbitrary map $\{\, (\, f_i,\, f_i^*\,)\, \}_{i\in\{z_1,\ldots,z_n\}} : F(\psi) \to F(\chi)$ from $(X_{\mathrm{wkinv}})^n$. We can use this to construct a monoidal transformation $\theta : \psi \to \chi$ via the reverse of process we just used. Specifically, if we define

$$\theta_I = \chi_I \circ \psi_I^{-1}, \qquad \theta_{z_i} = f_i, \qquad \theta_{z_i^*} = f_i^*$$

then these will automatically form the naturality squares

$$\begin{tikzcd}
\psi(z_i) \otimes \psi(z_i^*) \arrow[r, "\psi_{z_i, z_i^*}"] \arrow[dd, "f_i \otimes f_i^*"'] & \psi(I) \arrow[d, "\psi_I^{-1}"] \\
& I \arrow[d, "\chi_I"] \\
\chi(z_i) \otimes \chi(z_i^*) \arrow[r, "\chi_{z_i, z_i^*}"] & \chi(I)
\end{tikzcd}
\qquad
\begin{tikzcd}
\psi(z_i^*) \otimes \psi(z_i) \arrow[r, "\psi_{z_i^*, z_i}"] \arrow[dd, "f_i^* \otimes f_i"'] & \psi(I) \arrow[d, "\psi_I^{-1}"] \\
& I \arrow[d, "\chi_I"] \\
\chi(z_i^*) \otimes \chi(z_i) \arrow[r, "\chi_{z_i^*, z_i}"] & \chi(I)
\end{tikzcd}$$

since these are just the conditions for $\{ \, (\, f_i, \, f_i^* \,) \, \}_{i \in \{z_1, \ldots, z_n\}}$ to be a map $F(\psi) \to F(\chi)$ in $(X_{\text{wkinv}})^n$. Repeatedly applying the naturality condition $\theta_{w \otimes w'} = \chi_{w,w'} \circ (\theta_w \otimes \theta_{w'}) \circ \psi_{w,w'}^{-1}$ will then generate all of the other components of θ, in a way that clearly satisfies naturality. Thus we have a well-defined monoidal transformation $\theta : \psi \to \chi$, and applying F to it gives

$$\begin{aligned}
F(\theta) &= \left\{ \, (\, \theta_{z_i}, \, \theta_{z_i^*} \,) \, \right\}_{i \in \{z_1, \ldots, z_n\}} \\
&= \left\{ \, (\, f_i, \, f_i^* \,) \, \right\}_{i \in \{z_1, \ldots, z_n\}},
\end{aligned}$$

our arbitrary map. Therefore F is full and, putting this together with the previous results, is an equivalence of categories. $\square$

Chapter 5

The classification of 2-groups

See [1] for more detail.

5.1 n-groups

Definition 5.1. Weak n-categories

Definition 5.2. n-groupoids

Definition 5.3. n-groups

Example 5.4. Examples

5.2 Classifying 2-groups

Definition 5.5. 2-groups

Definition 5.6. Coherent 2-groups

Theorem 5.7. *The classification of 2-groups*

5.3 Group cohomology

Definition 5.8. Cochain complex of groups

Definition 5.9. Cocycles, coboundaries, and cohomology classes

Definition 5.10. Group cohomology

5.4 Proof of the classification theorem

Proof of Theorem 5.7. □

5.5 Generalizing to 3-groups

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Chapter 6

2-group cohomology

See [2] and [3] for more details.

6.1 Cohomology of categories

Definition 6.1. Cochain complex of categories

Definition 6.2. Cocycles, coboundaries, and cohomology classes

6.2 Symmetric 2-group cohomology

Definition 6.3. Symmetric 2-group cohomology

Proposition 6.4. *Coherence of symmetric 2-group cohomology*

6.3 Braided 2-group cohomology

Definition 6.5. Braided 2-group cohomology

Proposition 6.6. *Coherence of braided 2-group cohomology*

Chapter 7

Classifying 3-groups

Proposition 7.1. *3-group coherence data as elements of cohomology classes*

Theorem 7.2. *The classification of 3-groups*

Example 7.3. Examples

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