# ACTION OPERADS, FREE ALGEBRAS ON INVERTIBLE OBJECTS, AND THE CLASSIFICATION OF 3-GROUPS

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## Contents

## 1. Operads and their algebras

### 1.1. **Operads.**

**Definition 1.1.** Operads $O$
**Definition 1.2.** Action operads $G$
**Example 1.3.** The symmetric operad $S$
**Example 1.4.** The braid operad $B$
**Definition 1.5.** $G$-operads

### 1.2. **Operad algebras.**

**Definition 1.6.** Operad algbras
**Definition 1.7.** $G$-operad algebras

### 1.3. **E$G$-algebras.**

**Definition 1.8.** The $G$-operad E$G$
**Definition 1.9.** The monad E$G$
**Definition 1.10.** E$G$-algebras

**Proposition 1.11.** *$G$-operad algebras are monoidal categories with permutation-like structure*

**Corollary 1.12.** *Braided monoidal categories are $G$-operad algebras*

**Definition 1.13.** A strict monoidal category $X$ is said to be *spacial* if, for any object $x \in \mathrm{Ob}(X)$ and any endomorphism of the unit object $f : I \to I$,

$$f \otimes \mathrm{id}_x = \mathrm{id}_x \otimes f$$

The motivation for the name 'spacial' comes from the context of string diagrams [4]. In a string diagram, the act of tensoring two strings together is represented by placing those strings side by side. Since the defining feature of the unit object is that tensoring it with other objects should have no effect, the unit object is therefore represented diagrammatically by the absense of a string. An endomorphism of the unit thus appears as an entity with no input or output strings, detached from the rest of the diagram. In a real-world version of these diagrams, made out of physical strings arranged in real space, we could use this detachedness to grab these endomorphisms and slide them over or under any strings we please, without affecting anything else in the diagram. This ability is embodied algebraically by the equation above, and hence categories which obey it are called 'spacial'.

**Lemma 1.14.** *All* E$G$*-algebras are spacial.*

*Proof.* Let $X$ be an E$G$-algebra, and fix $x \in \mathrm{Ob}(X)$ and $f : I \to I$. From the surjectivity of $\pi : G(2) \to S_2$ we know that the set $\pi^{-1}(\,(1\,2)\,)$ is non-empty, and from the rules for composition of action morphisms we see that for any such $g \in \pi^{-1}(\,(1\,2)\,)$,

$$\begin{aligned} \alpha(\,g\,;\,\mathrm{id}_x,\,\mathrm{id}_I\,) \circ \alpha(\,e_2\,;\,\mathrm{id}_x,\,f\,) &= \alpha(\,g\,;\,\mathrm{id}_x,\,f\,) \\ &= \alpha(\,e_2\,;\,f,\,\mathrm{id}_x\,) \circ \alpha(\,g\,;\,\mathrm{id}_x,\,\mathrm{id}_I\,) \end{aligned}$$

Thus in order to obtain the result we're after, it will suffice to find a particular $g \in \pi^{-1}(\,(1\,2)\,)$ for which

$$\alpha(\,g\,;\,\mathrm{id}_x,\,\mathrm{id}_I\,) = \mathrm{id}_x$$

However, since

$$\begin{array}{rcl} \alpha(\,g\,;\,\mathrm{id}_x,\,\mathrm{id}_I\,) & = & \alpha(\,g\,;\,\mathrm{id}_x,\,\alpha(e_0;-)\,) \\ & = & \alpha(\,\mu(g;e_1,e_0)\,;\,\mathrm{id}_x\,) \end{array}$$

all we really need is to find a $g \in \pi^{-1}(\,(1\,2)\,)$ for which

$$\mu(g;e_1,e_0) = e_1$$

To this end, choose an arbtrary element $h \in \pi^{-1}(\,(1\,2)\,)$. This $h$ probably won't obey the above equation, but we can use it to construct a new element $g$ which does. Specifically, define

$$k \,:=\, \mu(\,h\,;\,e_1,\,e_0\,)$$

and then consider

$$g \,:=\, h \cdot \mu(e_2;k^{-1},e_1)$$

To see that this is the correct choice of $g$, first note that we must have $\pi(k) = e_1$, since this is the only element of $S_1$. Following from that, we have

$$\begin{array}{rcl} \pi\big(\,\mu(e_2;k^{-1},e_1)\,\big) & = & \mu\big(\,\pi(e_2)\,;\,\pi(k^{-1}),\,\pi(e_1)\,\big) \\ & = & \mu\big(\,e_2\,;\,e_1,\,e_1\,\big) \\ & = & e_2 \end{array}$$

and hence

$$\begin{array}{rcl} \pi(g) & = & \pi\big(h \cdot \mu(e_2;k^{-1},e_1)\big) \\ & = & \pi(h) \cdot \pi\big(\mu(e_2;k^{-1},e_1)\big) \\ & = & (1\,2) \cdot e_2 \\ & = & (1\,2). \end{array}$$

So $g$ is indeed in $\pi^{-1}(\,(1\,2)\,)$, and furthermore

$$\begin{array}{rcl} \mu(g;e_1,e_0) & = & \mu\big(\,h \cdot \mu(e_2;k^{-1},e_1)\,;\,e_1,\,e_0\,\big) \\ & = & \mu(\,h\,;\,e_1,\,e_0\,) \cdot \mu\big(\,\mu(e_2;k^{-1},e_1)\,;\,e_1,\,e_0\,\big) \\ & = & \mu(\,h\,;\,e_1,\,e_0\,) \cdot \mu\big(\,e_2\,;\,\mu(k^{-1};e_1),\,\mu(e_1;e_0)\,\big) \\ & = & \mu(\,h\,;\,e_1,\,e_0\,) \cdot \mu(\,e_2\,;\,k^{-1},e_0\,) \\ & = & k \cdot k^{-1} \\ & = & e_1 \end{array}$$

Therefore, $h \cdot \mu(e_2;k^{-1},e_1)$ is exactly the $g$ we were looking for, and so working backwards through the proof we obtain the required result:

$$\begin{array}{rrcl} & \mu(g;e_1,e_0) & = & e_1 \\ \Longrightarrow & \alpha(\,g\,;\,\mathrm{id}_x,\,\mathrm{id}_I\,) & = & \mathrm{id}_x \end{array}$$

$$\begin{array}{rcl} \alpha(\,g\,;\,\mathrm{id}_x,\,\mathrm{id}_I\,) \circ \alpha(\,e_2\,;\,\mathrm{id}_x,\,f\,) & = & \alpha(\,e_2\,;\,f,\,\mathrm{id}_x\,) \circ \alpha(\,g\,;\,\mathrm{id}_x,\,\mathrm{id}_I\,) \\ \Longrightarrow \quad \alpha(\,e_2\,;\,\mathrm{id}_x,\,f\,) & = & \alpha(\,e_2\,;\,f,\,\mathrm{id}_I\,) \end{array}$$

□

1.4. **The free E$G$-algebra on $n$ objects.** Our goal for the next few chapters will be to understand the free braided monoidal category on an finite number of invertible objects. Thus, now that we have a firm grasp on action operads and their algebras, we should begin to think about the simpler free constructions they can form. We will use this extensively when calculating the invertible case later on.

In the paper [5], Gurski establishes how to contruct free $G$-operad algebras through the use of the monad E$G$. What follows in this section is a quick summary of the results which will be useful for our purposes. For a more detailed treatment please refer to [5].

**Proposition 1.15.** *There exists a free* E$G$*-algebra on* $n$ *objects. That is, there is an* E$G$*-algebra* $Y$ *such that for any other* E$G$*-algebra* $X$*, we have an isomorphism of categories*

$$\mathrm{E}G\mathrm{Alg}_S(Y, X) \cong X^n$$

*Proof.* There is an obvious forgetful 2-functor $U : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{Cat}$ sending E$G$-algebras to their underlying categories. $U$ has a left adjoint, which we call the free 2-functor $F : \mathrm{Cat} \to \mathrm{E}G\mathrm{Alg}_S$ adjoint to it. It follows immediately that

$$\begin{array}{rcl} U(X)^n & = & \mathrm{Cat}(\{z_1, ..., z_n\}, U(X)) \\ & \cong & \mathrm{E}G\mathrm{Alg}_S(F(\{z_1, ..., z_n\}), X) \end{array}$$

where $\{z_1, ..., z_n\}$ is any set with $n$ distinct elements. Since $X$ and $U(X)$ are obviously isomorphic as categories, this shows that $F(\{z_1, ..., z_n\})$ is the free algebra on $n$ objects as required. □

**Definition 1.16.** Let $\{z_1, ..., z_n\}$ be an $n$-object set, which we will also consider as a discrete category. Then we will denote by $\mathbb{G}_n$ the E$G$-algebra whose underlying category is $\mathrm{E}G(\{z_1, ..., z_n\})$ and whose action

$$\alpha : \mathrm{E}G\big(\,\mathrm{E}G(\{z_1, ..., z_n\})\,\big) \to \mathrm{E}G(\{z_1, ..., z_n\})$$

is the appropriate component of the multiplication natural transformation $\mu : \mathrm{E}G \circ \mathrm{E}G \to \mathrm{E}G$ of the 2-monad E$G$.

**Theorem 1.17.** $\mathbb{G}_n$ *is the free* E$G$*-algebra on* $n$ *objects. That is,*

$$F(\{z_1, ..., z_n\}) = \mathbb{G}_n$$

*Proof.* □

Definition 1.16 is a fairly opaque definition, so we'll spend a little time upacking it. Recall from Definition 1.9 that $\mathrm{E}G(\{z_1, ..., z_n\})$ is the coequalizer of the maps

$$\textstyle\coprod_{m\geq 0} \mathrm{E}G(m) \times G(m) \times \{z_1, ..., z_n\}^m \rightrightarrows \coprod_{m\geq 0} \mathrm{E}G(m) \times \{z_1, ..., z_n\}^m$$

that comes from the action of $G(m)$ on $\mathrm{E}G(m)$ by multiplication on the right,

$$\begin{array}{rcl} \mathrm{E}G(m) \times G(m) & \to & \mathrm{E}G(m) \\ (g, h) & \mapsto & gh \\ (\,! : g \to g', \mathrm{id}_h\,) & \mapsto & ! : gh \to g'h \end{array}$$

and the action of $G(m)$ on $\{z_1, ..., z_n\}^m$ by permutation,

$$\begin{array}{rcl} G(m) \times \{z_1, ..., z_n\}^m & \to & \{z_1, ..., z_n\}^m \\ (\,h\,;\, x_1, ..., x_m\,) & \mapsto & (x_{\pi(h^{-1})(1)}, ..., x_{\pi(h^{-1})(m)}) \\ (\mathrm{id}_h\,;\, \mathrm{id}_{(x_1,...,x_m)}\,) & \mapsto & \mathrm{id}_{(x_{\pi(h^{-1})(1)},...,x_{\pi(h^{-1})(m)})} \end{array}$$

First, objects in this algebra are equivalence classes of tuples $(g; x_1, ..., x_m)$, for $g \in G(m)$ and $x_i \in \{z_1, ..., z_n\}$, under the relation

$$(\,gh\,;\, x_1,\, ...,\, x_m\,) \sim (\,g\,;\, x_{\pi(h)^{-1}(1)},\, ...,\, x_{\pi(h)^{-1}(m)}\,)$$

Notice that using this relation we can rewrite any object uniquely in the form $[e; x_1, ..., x_m]$ for some $m \in \mathbb{N}$ and $x_i \in \{z_1, ..., z_n\}$. This means that each equivalence class is just the tensor product $x_1 \otimes ... \otimes x_m$ in the underlying monoidal category of $\mathbb{G}_n$, for some unique sequence of generators. That is, we can view the objects of $\mathbb{G}_n$ as elements of the monoid freely generated by each of the $z_i$, or in other words:

**Lemma 1.18.** $\mathrm{Ob}(\mathbb{G}_n)$ *is the free monoid on $n$ generators, $\mathbb{N}^{*n}$, the free product of $n$ copies of $\mathbb{N}$.*

Similarly, the morphisms of $\mathbb{G}_n$ are the maps

$$(!; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) : (g; x_1, ..., x_m) \to (g'; x_1, ..., x_m)$$

with $g, g' \in G(m)$ and $x_i \in \{z_1, ..., z_n\}$. Using the relation $\sim$ on objects we can rewrite each of these morphisms in the form

$$[h; \mathrm{id}_{y_1}, ..., \mathrm{id}_{y_m}] \,:\, y_1 \otimes ... \otimes y_m \;\to\; y_{\pi(h^{-1})(1)} \otimes ... \otimes y_{\pi(h^{-1})(m)}$$

where

$$h = g'g^{-1}, \qquad y_i = x_{\pi(g^{-1})(i)}$$

The E$G$-action of $\mathbb{G}_n$ is permutation and tensor product, and the action on morphisms is given by

$$\alpha(\, g\,;\, [h_1; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_{m_1}}],\, ...,\, [h_k; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_{m_k}}]\,) = [\, \mu(g; h_1, .., h_k)\,;\, \mathrm{id}_{x_1},\, ...,\, \mathrm{id}_{x_{m_k}}\,]$$

Notice that using tensor product notation the object $[e; x]$ is simply $x$, and so $[e; \mathrm{id}_x] = \mathrm{id}_{[e;x]}$ should be written as $\mathrm{id}_x$. Hence by the above $[g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}]$ is really just $\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$, and so we have the following:

**Lemma 1.19.** *Each morphism of $\mathbb{G}_n$ can be expressed uniquely as an action morphism $\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$, for some $g, g' \in G(m)$ and $x_i \in \{z_1, ..., z_n\}$.*

From this, we can also determine $\mathbb{G}_n$'s connected components.

**Proposition 1.20.** *The connected components of $\mathbb{G}_n$ are $\mathbb{N}^n$, with the assignment $[\;] : \mathbb{N}^{*n} \to \mathbb{N}^n$ of objects to their component being the quotient map of abelianisation.*

*Proof.* By Lemma 1.19, all morphisms in $\mathbb{G}_n$ can be written uniquely as $\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$, for some $g \in G(m)$ and $x_i \in \{z_1, ..., z_n\}$, the set of generators of $\mathbb{N}^{*n}$. Since maps of this form have source $x_1 \otimes ... \otimes x_m$ and target $x_{\pi(g^{-1})(1)} \otimes ... \otimes x_{\pi(g^{-1})(m)}$, we see that there can only exist a morphism between two objects if they can expanded as tensor products which are permutations of one another. Moreover, for any two objects where this is true — say source $x_1 \otimes ... \otimes x_m$ and target $x_{\sigma^{-1}(1)} \otimes ... \otimes x_{\sigma^{-1}(m)}$ — we can always find a map $\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$ between them, because $\pi : G(m) \to S_m$ is surjective and so there must exist at least one $g$ with $\pi(g) = \sigma$. Thus two objects of $\mathbb{G}_n$ share a connected component if and only if they are tensor products that differ by a permutation, and therefore the canonical map $[\;] : \mathrm{Ob}(\mathbb{G}_n) \to \pi_0(\mathbb{G}_n)$ sending each object to its connected component is just the map which forgets about these permutations, making the free product on $\mathbb{N}^{*n}$ commutative. That is, $[\;]$ is the quotient map for the abelianisation $\mathrm{ab} : \mathbb{N}^{*n} \to (\mathbb{N}^{*n})^{\mathrm{ab}}$, and so $\pi_0(\mathbb{G}_n) = \mathbb{N}^n$. $\square$

## 2. Free invertible algebras as initial objects

2.1. **The free algebra on $n$ invertible objects.** We saw in Proposition 1.15 that the existence of a free E$G$-algebra on $n$ objects can be proven by taking the left adjoint of a 2-functor which forgets about the algebra structure. Now we want to extend this idea into the realm of algebras on invertible objects. For the analogous approach, we will need to find a new 2-functor that lets us forget about non-invertible objects, and then hopefully we can find its left adjoint too, and use it to freely add inverses to $\mathbb{G}_n$. First though, we need to make this concept of 'forgetting non-invertible objects' a little more precise.

**Definition 2.1.** Given an E$G$-algebra $X$, we denote by $X_{\text{inv}}$ the sub-E$G$-algebra containing all invertible objects in $X$ and the isomorphisms between them.

Note that this is indeed a well-defined E$G$-algebra. If $x_1, ..., x_m$ are invertible objects with inverses $x_1^*, ..., x_m^*$, then $\alpha(g; x_1, ..., x_m)$ is an invertible object with inverse $\alpha(g; x_m^*, ..., x_1^*)$, since

$$\begin{array}{rl} & \alpha(g; x_1, ..., x_m) \otimes \alpha(g; x_m^*, ..., x_1^*) \\ = & \big(x_{\pi(g)^{-1}(1)} \otimes ... \otimes x_{\pi(g)^{-1}(m)}\big) \otimes \big(x^*_{\pi(g)^{-1}(m)} \otimes ... \otimes x^*_{\pi(g)^{-1}(1)}\big) \\ = & I \end{array}$$

$$\begin{array}{rl} & \alpha(g; x_m^*, ..., x_1^*) \otimes \alpha(g; x_1, ..., x_m) \\ = & \big(x^*_{\pi(g)^{-1}(m)} \otimes ... \otimes x^*_{\pi(g)^{-1}(1)}\big) \otimes \big(x_{\pi(g)^{-1}(1)} \otimes ... \otimes x_{\pi(g)^{-1}(m)}\big) \\ = & I \end{array}$$

Likewise, if $f_1, ..., f_m$ are isomorphisms from invertible objects $x_1, ..., x_m$ to invertible objects $y_1, ..., y_m$, then $\alpha(g; f_1, ..., f_m)$ is a map from the invertible object $\alpha(g; x_1, ..., x_m)$ to the invertible object $\alpha(g; y_1, ..., y_m)$, and it has an inverse $\alpha(g^{-1}; f^{-1}_{\pi(g)(1)}, ..., f^{-1}_{\pi(g)(m)})$, since

$$\begin{array}{rl} & \alpha\big(\, g^{-1}\, ;\, f^{-1}_{\pi(g)(1)},\, ...,\, f^{-1}_{\pi(g)(m)}\,\big) \circ \alpha(\, g\, ;\, f_1, ..., f_m\,) \\ = & \alpha\big(\, g^{-1}g\, ;\, f_1^{-1}f_1,\, ...,\, f_m^{-1}f_m\,\big) \\ = & \mathrm{id}_{x_1 \otimes ... \otimes x_m} \end{array}$$

$$\begin{array}{rl} & \alpha(\, g\, ;\, f_1, ..., f_m\,) \circ \alpha\big(\, g^{-1}\, ;\, f^{-1}_{\pi(g)(1)},\, ...,\, f^{-1}_{\pi(g)(m)}\,\big) \\ = & \alpha\big(\, gg^{-1}\, ;\, f_{\pi(g)(1)}f^{-1}_{\pi(g)(1)},\, ...,\, f_{\pi(g)(m)}f^{-1}_{\pi(g)(m)}\,\big) \\ = & \mathrm{id}_{y_{\pi(g)(1)} \otimes ... \otimes y_{\pi(g)(m)}} \end{array}$$

Clearly then, $X_{\text{inv}}$ is the correct algebra for our new forgetful 2-functor to send $X$ to. Knowing this, we can contruct the rest of the functor fairly easily.

**Proposition 2.2.** *The assignment $X \mapsto X_{\text{inv}}$ can be extended to a 2-functor $(\_)_{\text{inv}}$ : $\mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$.*

*Proof.* Let $F : X \to Y$ be a (strict) map of E$G$-algebras. If $x$ is an invertible object in $X$ with inverse $x^*$, then $F(x)$ is an invertible object in $Y$ with inverse $F(x^*)$, by

$$F(x) \otimes F(x^*) = F(x \otimes x^*) = F(I) = I$$

$$F(x^*) \otimes F(x) = F(x^* \otimes x) = F(I) = I$$

Since $F$ sends invertible objects to invertible objects, it will also send isomorphisms of invertible objects to isomorphisms of invertible objects. In other words, the map $F : X \to Y$ can be restricted to a map $F_{\text{inv}} : X_{\text{inv}} \to Y_{\text{inv}}$. Moreover, we have that

$$(F \circ G)_{\text{inv}}(x) = F \circ G(x) = F_{\text{inv}} \circ G_{\text{inv}}(x)$$

$$(F \circ G)_{\text{inv}}(f) = F \circ G(f) = F_{\text{inv}} \circ G_{\text{inv}}(f)$$

and so the assignment $F \mapsto F_{\mathrm{inv}}$ is clearly functorial. Next, let $\theta : F \Rightarrow G$ be an E$G$-monoidal natural transformation. Choose an invertible object $x$ from $X$, and consider the component map of its inverse, $\theta_{x^*} : F(x^*) \to G(x^*)$. Since $\theta$ is monoidal, we have $\theta_{x^*} \otimes \theta_x = \theta_I = I$ and $\theta_x \otimes \theta_{x^*} = I$, or in other words that $\theta_{x^*}$ is the monoidal inverse of $\theta_x$. We can use this fact to construct a compositional inverse as well, namely $\mathrm{id}_{F(x)} \otimes \theta_{x^*} \otimes \mathrm{id}_{G(x)}$, which can be seen as follows:

$$\begin{aligned}
\big(\mathrm{id}_{F(x)} \otimes \theta_{x^*} \otimes \mathrm{id}_{G(x)}\big) \circ \theta_x &= \theta_x \otimes \theta_{x^*} \otimes \mathrm{id}_{G(x)} \\
&= \mathrm{id}_{G(x)} \\[1em]
\theta_x \circ \big(\mathrm{id}_{F(x)} \otimes \theta_{x^*} \otimes \mathrm{id}_{G(x)}\big) &= \mathrm{id}_{F(x)} \otimes \theta_{x^*} \otimes \theta_x \\
&= \mathrm{id}_{F(x)}
\end{aligned}$$

Therefore, we see that all the components of our transformation on invertible objects are isomorphisms, and hence we can define a new transformation $\theta_{\mathrm{inv}} : F_{\mathrm{inv}} \Rightarrow G_{\mathrm{inv}}$ whose components are just $(\theta_{\mathrm{inv}})_x = \theta_x$. The assignment $\theta \mapsto \theta_{\mathrm{inv}}$ is also clearly functorial, and thus we have a complete 2-functor $(\_)_{\mathrm{inv}} : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$. □

**Proposition 2.3.** *The 2-functor* $(\_)_{\mathrm{inv}} : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$ *has a left adjoint,* $L : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$.

*Proof.* To begin, consider the 2-monad $\mathrm{E}G(\_)$. This is a finitary monad, that is it preserves all filtered colimits, and it is a 2-monad over Cat, which is locally finitely presentable. It follows from this that $\mathrm{E}G\mathrm{Alg}_S$ is itself locally finitely presentable. Thus if we want to prove $(\_)_{\mathrm{inv}}$ has a left adjoint, we can use the Adjoint Functor Theorem for locally finitely presentable categories, which amounts to showing that $(\_)_{\mathrm{inv}}$ preserves both limits and filtered colimits.

- Given an indexed collection of E$G$-algebras $X_i$, the E$G$-action of their product $\prod X_i$ is defined componentwise. In particular, this means that the tensor product of two objects in $\prod X_i$ is just the collection of the tensor products of their components in each of the $X_i$. An invertible object in $\prod X_i$ is thus simply a family of invertible objects from the $X_i$ — in other words, $(\prod X_i)_{\mathrm{inv}} = \prod (X_i)_{\mathrm{inv}}$.
- Given maps of E$G$-algebras $F : X \to Z$, $G : Y \to Z$, the E$G$-action of their pullback $X \times_Z Y$ is also defined componentwise. It follows that an invertible object in $X \times_Z Y$ is just a pair of invertible objects $(x, y)$ from $X$ and $Y$, such that $F(x) = G(y)$. But this is the same as asking for a pair of objects $(x, y)$ from $X_{\mathrm{inv}}$ and $Y_{\mathrm{inv}}$ such that $F_{\mathrm{inv}}(x) = G_{\mathrm{inv}}(y)$, and hence $(X \times_Z Y)_{\mathrm{inv}} = X_{\mathrm{inv}} \times_{Z_{\mathrm{inv}}} Y_{\mathrm{inv}}$.
- Given a filtered diagram $D$ of E$G$-algebras, the E$G$-action of their colimit $\mathrm{colim}(D_n)$ is defined in the following way: use filteredness to find an algebra which contains (representatives of the classes of) all the things you want to act on, then apply the action of that algebra. In the case of tensor products this means that $[x] \otimes [y] = [x \otimes y]$, and thus an invertible object in $\mathrm{colim}(D_n)$ is just (the class of) an invertible object in one of the algebras of $D$. In other words, $\mathrm{colim}(D_n)_{\mathrm{inv}} = \mathrm{colim}(D_{\mathrm{inv}})$.

Preservation of products and pullbacks gives preservation of limits, and preservation of limits and filtered colimits gives the result. □

With this new 2-functor $L : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$, we now have the ability to 'freely add inverses to objects' in any E$G$-algebra we want. The algebra $L\mathbb{G}_n$ is then a clear candidate for our free algebra on $n$ invertible objects, and indeed the proof of this is very simple.

**Theorem 2.4.** *There exists a free* E$G$*-algebra on $n$ invertible objects. Specifically, the algebra $L\mathbb{G}_n$ is such that for any other* E$G$*-algebra $X$, we have an isomorphism of categories*

$$\mathrm{E}G\mathrm{Alg}_S(L\mathbb{G}_n, X) \cong (X_{\mathrm{inv}})^n$$

*Proof.* Using the adjunction from the previous Proposition along with the one from Proposition 1.15, we see that

$$\begin{array}{rcl} U(X_{\mathrm{inv}})^n & = & \mathrm{Cat}(\{z_1, ..., z_n\}, U(X_{\mathrm{inv}})) \\ & \cong & \mathrm{E}G\mathrm{Alg}_S(F(\{z_1, ..., z_n\}), X_{\mathrm{inv}}) \\ & \cong & \mathrm{E}G\mathrm{Alg}_S(LF(\{z_1, ..., z_n\}), X) \end{array}$$

As before, $X_{\mathrm{inv}}$ and $U(X_{\mathrm{inv}})$ are obviously isomorphic as categories, and so $LF(\{z_1, ..., z_n\}) = L\mathbb{G}_n$ satisfies the requirements for the free algebra on $n$ invertible objects. □

2.2. **$L\mathbb{G}_n$ as an initial algebra.** We have now proven that a free E$G$-algebra on $n$ invertible objects does indeed exist. But this fact on its own is not very helpful. To be able to actually use the free algebra $L\mathbb{G}_n$, we need to know how to contruct it explicitly, in terms of its objects and morphisms. We could do this by finding a detailed characterisation of the 2-functor $L$, and then applying this to our explicit description of $\mathbb{G}_n$ from Definition 1.16. However, this would probably be much more effort than is required, since it would involve determining the behaviour of $L$ in many situtations we aren't interested in, and we also wouldn't be leveraging $\mathbb{G}_n$'s status as a free algebra to make the calculations any easier. We will try a different strategy instead. We begin by noticing a special property of the functor $L$.

**Proposition 2.5.** *For any* E$G$*-algebra $X$, we have $L(X)_{\mathrm{inv}} = L(X)$.*

*Proof.* From the definition of adjunctions, the isomorphisms

$$\mathrm{E}G\mathrm{Alg}_S(LX, Y) \cong \mathrm{E}G\mathrm{Alg}_S(X, Y_{\mathrm{inv}})$$

are subject to certain naturality conditions. Specifically, given $F : X' \to X$ and $G : Y \to Y'$ we get a commutative diagram

$$\begin{tikzcd} \mathrm{E}G\mathrm{Alg}_S(LX, Y) \arrow[r, "\sim"] \arrow[d, "G\circ\_\circ LF"'] & \mathrm{E}G\mathrm{Alg}_S(X, Y_{\mathrm{inv}}) \arrow[d, "G_{\mathrm{inv}}\circ\_\circ F"] \\ \mathrm{E}G\mathrm{Alg}_S(LX', Y') \arrow[r, "\sim"] & \mathrm{E}G\mathrm{Alg}_S(X', Y'_{\mathrm{inv}}) \end{tikzcd}$$

Consider the case where $F$ is the identity map $\mathrm{id}_X : X \to X$ and $G$ is the inclusion $j : L(X)_{\mathrm{inv}} \to L(X)$. Note that because $j$ is an inclusion, the restriction $j_{\mathrm{inv}} : (L(X)_{\mathrm{inv}})_{\mathrm{inv}} \to L(X)_{\mathrm{inv}}$ is also an inclusion, but since $((\_)_{\mathrm{inv}})_{\mathrm{inv}} = (\_)_{\mathrm{inv}}$, we have that $j_{\mathrm{inv}} = \mathrm{id}$. It follows that

$$\begin{tikzcd} \mathrm{E}G\mathrm{Alg}_S(LX, LX_{\mathrm{inv}}) \arrow[r, "\sim"] \arrow[d, "j\circ\_"'] & \mathrm{E}G\mathrm{Alg}_S(X, LX_{\mathrm{inv}}) \arrow[d, equal] \\ \mathrm{E}G\mathrm{Alg}_S(LX, LX) \arrow[r, "\sim"] & \mathrm{E}G\mathrm{Alg}_S(X, LX_{\mathrm{inv}}) \end{tikzcd}$$

Therefore, for any map $f : LX \to LX$ there exists a unique $g : LX \to LX_{\mathrm{inv}}$ such that $j \circ g = f$. But this means that for any such $f$, we must have $\mathrm{im}(f) \subseteq L(X)_{\mathrm{inv}}$, and so in particular $L(X) = \mathrm{im}(\mathrm{id}_{LX}) \subseteq L(X)_{\mathrm{inv}}$. Since $L(X)_{\mathrm{inv}} \subseteq L(X)$ by definition, we obtain the result. $\square$

This result is not especially surprising. Intuitively, it just says that when you freely add inverses to an algebra, every object ends up with an inverse. But the upshot of this is that we now have another way of thinking about $L(X)$: as the target object of the unit of our adjunction, $\eta_X : X \to L(X)_{\mathrm{inv}}$. This means that we don't really need to know the entirety of $L$ in order to determine the free algebra $L\mathbb{G}_n$, just its unit. To find this unit directly, we can turn to the following fact about adjunctions, for which a proof can be found in Lemma 2.3.5 of Leinster's *Basic Category Theory* [6].

**Proposition 2.6.** *Let $F \dashv G : A \to B$ be an adjunction with unit $\eta$. For any object $a$ in $A$, let $(a \downarrow G)$ denote the comma category whose objects are pairs $(b, f)$ consisting of an object $b$ from $B$ and a morphism $f : a \to G(b)$ from $A$, and whose morphisms $h : (b, f) \to (b', f')$ are morphisms $f : b \to b'$ from $B$ such that $G(f) \circ f = f'$. Then the pair $\big(F(a), \eta_a : a \to GF(a)\big)$ is an initial object of $(a \downarrow G)$.*

**Corollary 2.7.** *If $\phi : \mathbb{G}_n \to Z$ is an initial object of $(\mathbb{G}_n \downarrow \mathrm{inv})$, then*

$$Z \,\cong\, (L\mathbb{G}_n)_{\mathrm{inv}} \,=\, L\mathbb{G}_n$$

Being able to view $L\mathbb{G}_n$ as the initial object in the comma category $(\mathbb{G}_n \downarrow \mathrm{inv})$ will prove immensely useful in the coming sections. This is because it lets us think about the properties of $L\mathbb{G}_n$ in terms of maps $\psi : \mathbb{G}_n \to X_{\mathrm{inv}}$, and this is exactly the context where we can exploit $\mathbb{G}_n$'s status as a free algebra.

Before moving on, we'll make a small change in notation. From now on, rather than writing objects in $(\mathbb{G}_n \downarrow \mathrm{inv})$ as maps $\psi : \mathbb{G}_n \to Y_{\mathrm{inv}}$, we will instead just let $X = Y_{\mathrm{inv}}$ and speak of maps $\psi : \mathbb{G}_n \to X$. This is purely to prevent the notation from becoming cluttered, and shouldn't be a problem so long as we always remember that the targets of these maps only ever contain invertible objects and morphisms.

2.3. **The objects of $L\mathbb{G}_n$.** So now we know that $L\mathbb{G}_n$ is an initial object in the category $(\mathbb{G}_n \downarrow \mathrm{inv})$. But what does this actually tell us? After all, we do not currently have a method for finding initial objects in an arbitrary collection of E$G$-algebra maps. Because of this, we'll have to approach the problem step-by-step, using the initiality of $\eta$ to extract different pieces of information about the algebra $L\mathbb{G}_n$ as we go. We'll begin by tring to find its objects.

**Definition 2.8.** Denote by $\mathrm{Ob} : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{Mon}$ be the functor that sends E$G$-algebras $X$ to their monoid of objects $\mathrm{Ob}(X)$, and algebra maps $F : X \to Y$ to their underlying monoid homomorphism $\mathrm{Ob}(F) : \mathrm{Ob}(X) \to \mathrm{Ob}(Y)$.

In order to find $\mathrm{Ob}(L\mathbb{G}_n)$, we'll need to make use of an important result about the nature of Ob.

**Definition 2.9.** Recall that given a monoid $M$, the monoidal category $\mathrm{E}M$ is the one whose monoid of objects is $M$ and which has a unique isomorphism between any two objects. We can view $\mathrm{E}M$ as not just a category but an $\mathrm{E}G$-algebra, by letting the action on morphisms take the only possible values it can, given the required source and target. Similarly, for any monoid homomorphisms $h : M \to M'$ we can define a map of $\mathrm{E}G$-algebras

$$\begin{array}{rcrcl} \mathrm{E}h & : & \mathrm{E}M & \to & \mathrm{E}M' \\ & : & m & \mapsto & h(m) \\ & : & m \to m' & \mapsto & h(m) \to h(m') \end{array}$$

This definition of $\mathrm{E}h$ respects composition and identities, and so together with $\mathrm{E}M$ it describes a functor $\mathrm{E} : \mathrm{Mon} \to \mathrm{E}G\mathrm{Alg}_S$.

**Proposition 2.10.** *$E$ is a right adjoint to the functor* Ob.

*Proof.* For any $\mathrm{E}G$-algebra $X$, a map $F : X \to \mathrm{E}M$ is determined entirely by its restriction to objects, the monoid homomorphism $\mathrm{Ob}(F) : \mathrm{Ob}(X) \to M$. This is because functorality of $F$ ensures that any map $x \to x'$ in $X$ must be sent to a map $F(x) \to F(x')$ in $\mathrm{E}M$, and by the definition of E there is always exactly one of these to choose from. In other words, we have an isomorphism between the homsets

$$\mathrm{E}G\mathrm{Alg}_S(\, X, \mathrm{E}M\,) \quad \cong \quad \mathrm{Mon}(\, \mathrm{Ob}(X), M\,)$$

Additionally, this isomorphism is natural in both coordinates. That is, for any $G : X \to X'$ in $\mathrm{E}G\mathrm{Alg}_S$ and $h : M \to M'$ in Mon, the diagram

$$\begin{tikzcd} \mathrm{E}G\mathrm{Alg}_S(X, \mathrm{E}M) \arrow[r, "\sim"] \arrow[d, "\mathrm{E}h \circ \_ \circ G"'] & \mathrm{Mon}(\mathrm{Ob}(X), M) \arrow[d, "h \circ \_ \circ \mathrm{Ob}(G)"] \\ \mathrm{E}G\mathrm{Alg}_S(X', \mathrm{E}M') \arrow[r, "\sim"] & \mathrm{Mon}(\mathrm{Ob}(X'), M') \end{tikzcd}$$

commutes, because

$$\mathrm{Ob}(\, \mathrm{E}h \circ F \circ G\,) \quad = \quad \mathrm{Ob}(Eh) \circ \mathrm{Ob}(F) \circ \mathrm{Ob}(G) \quad = \quad h \circ \mathrm{Ob}(F) \circ \mathrm{Ob}(G)$$

Therefore, $\mathrm{Ob} \dashv \mathrm{E}$. ∎

What Proposition 2.10 is essentially saying is that the functor Ob provides a way for us to move back and forth between the categories $\mathrm{E}G\mathrm{Alg}_S$ and Mon. By applying this reasoning to the universal property of the initial object $\eta$, we can then determine the value of $\mathrm{Ob}(L\mathbb{G}_n)$ in terms of a new universal property of $\mathrm{Ob}(\eta)$ in the category Mon. In particular, the algebras in $(\mathbb{G}_n \downarrow \mathrm{inv})$ are those whose objects are all invertible, and so the induced property of $\mathrm{Ob}(\eta)$ will end up saying something about the relationship between $\mathrm{Ob}(\mathbb{G}_n)$ and groups — those monoids whose elements are all invertible.

**Definition 2.11.** Let $M$ be a monoid, $M^{\mathrm{gp}}$ a group, and $i : M \to M^{\mathrm{gp}}$ a monoid homomorphism between them. Then we say that $M^{\mathrm{gp}}$ is the *group completion* of $M$ if for any other group $H$ and homomorphism $h : M \to H$, there exists a unique homomorphism $u : M^{\mathrm{gp}} \to H$ such that $u \circ i = h$.

In practice, $M^{\mathrm{gp}}$ is the group whose group presentation is the same as the monoid presentation of $M$. That is, if $M$ is the quotient of the free monoid on generators $\mathcal{G}$ by the relations $\mathcal{R}$, then $M^{\mathrm{gp}}$ is the quotient of the free *group* on generators $\mathcal{G}$ by relations $\mathcal{R}$.

**Proposition 2.12.** *The object monoid of $L\mathbb{G}_n$ is $\mathbb{Z}^{*n}$, and the restriction of $\eta$ to objects* $\mathrm{Ob}(\eta)$ *is the obvious inclusion $\mathbb{N}^{*n} \hookrightarrow \mathbb{Z}^{*n}$.*

*Proof.* Let $H$ be a group, and $h : \mathrm{Ob}(\mathbb{G}_n) \to H$ a monoid homomorphism. By Proposition 2.10 we have an isomorphism of homsets

$$\mathrm{E}G\mathrm{Alg}_S(\, \mathbb{G}_n, \mathrm{E}H\,) \quad \cong \quad \mathrm{Mon}(\,\mathrm{Ob}(\mathbb{G}_n),\, H\,)$$

Denote by $h' : \mathbb{G}_n \to \mathrm{E}H$ the map of $\mathrm{E}G$-algebras corresponding to $h$ under this isomorphism. Since $H$ is a group, every object in $\mathrm{E}H$ is invertible, and so $h'$ is an object of $(\mathbb{G}_n \downarrow \mathrm{inv})$. Thus, by initiality of $\eta$, there must exist a unique map $u : L\mathbb{G}_n \to \mathrm{E}G$ making the lefthand traingle below commute:

$$\begin{tikzcd}
\mathbb{G}_n \arrow[d, "\eta"'] \arrow[dr, "h'"] & \\
L\mathbb{G}_n \arrow[r, "u"'] & \mathrm{E}H
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathrm{Ob}(\mathbb{G}_n) \arrow[d, "\mathrm{Ob}(\eta)"'] \arrow[dr, "h"] & \\
\mathrm{Ob}(L\mathbb{G}_n) \arrow[r, "\mathrm{Ob}(u)"'] & H
\end{tikzcd}$$

It follows that the righthand triangle — which is the image of the first under Ob — also commutes. Hence for any group $H$ and homomorphism $h : \mathrm{Ob}(\mathbb{G}_n) \to H$, there is at least one map which factors $h$ through $\mathrm{Ob}(\eta)$.

But now let $v : \mathrm{Ob}(L\mathbb{G}_n) \to H$ be any homomorphism such that $v \circ \mathrm{Ob}(\eta) = h$. If $v' : L\mathbb{G}_n \to \mathrm{E}H$ is the image of $v$ under the adjunction isomorphism, then by naturality $v' \circ \eta = h'$, a property that was supposed to be unique to $u$. Thus $v = \mathrm{Ob}(u)$, and so there is actually only one possible map which factors $h$ through $\mathrm{Ob}(\eta)$.

Therefore every homomorphism from $\mathrm{Ob}(\mathbb{G}_n)$ onto a group factors uniquely through the $\mathrm{Ob}(L\mathbb{G}_n)$, or in other words $\mathrm{Ob}(L\mathbb{G}_n)$ is the group completion of the monoid $\mathrm{Ob}(\mathbb{G}_n)$. Since by Lemma 1.18 the object monoid of $\mathbb{G}_n$ is $\mathbb{N}^{*n}$, the free monoid on $n$ generators, we can conclude that

$$\mathrm{Ob}(L\mathbb{G}_n) \;=\; \mathrm{Ob}(\mathbb{G}_n)^{\mathrm{gp}} \;=\; (\mathbb{N}^{*n})^{\mathrm{gp}} \;=\; \mathbb{Z}^{*n}$$

the free group on $n$ generators. Moreover, the map $\mathrm{Ob}(\eta)$ is then the inclusion of $\mathrm{Ob}(\mathbb{G}_n)$ into its completion, which is just $\mathbb{N}^{*n} \hookrightarrow \mathbb{Z}^{*n}$. □

This result makes concrete the sense in which the functor $L$ represents 'freely adding inverses' to objects. Extending this same logic to connected components as well, it seems reasonable to expect that $\pi_0(L\mathbb{G}_n)$ should be $\mathbb{Z}^n$, the group completion of $\pi_0(\mathbb{G}_n) = \mathbb{N}^n$. This is indeed the case, and the proof proceeds in a way completely analagous to Proposition 2.12. First, we show that the process of taking connected components is part of an adjunction.

**Definition 2.13.** Denote by $\pi_0 : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{CMon}$ be the functor that sends each algebra $X$ to its commutaive monoid of connected components, $\pi_0(X)$, and sends each map of algebras $F : X \to Y$ to its restriction to connected components $\pi_0(F) : \pi_0(X) \to \pi_0(Y)$.

**Definition 2.14.** There exists an inclusion of 2-categories Set $\hookrightarrow$ Cat which allows us to view any set $S$ as a discrete category, one whose objects are just the elements of $S$ and whose morphisms are all identities. If the given set also happens to be a monoid then there is an obvious way to see this discete category as a monoidal category, and so we have a similar inclusion Mon $\hookrightarrow$ MonCat. Finally, if a given monoid happens to be commutative then there is a unique way to assign an E$G$-action to its discete category. This works because for any elements $c_i$ of a commutive monoid $C$, the morphism $\alpha(g; \mathrm{id}_{c_1}, ..., \mathrm{id}_{c_m})$ must have source and target $c_1 \otimes ... \otimes c_m = c_{\pi(g^{-1})(1)} \otimes ... \otimes c_{\pi(g^{-1})(m)}$, and therefore it can only be $\mathrm{id}_{c_1 \otimes ... \otimes c_m}$. This choice of action yields one last inclusion CMon $\hookrightarrow$ E$G$Alg$_S$, which we shall call D.

**Proposition 2.15.** *D is a right adjoint to the functor* $\pi_0$.

*Proof.* Consider a map $F : X \to \mathrm{D}C$ from some E$G$-algebra $X$ to the discrete E$G$-algebra on a commutative monoid $C$. For any $f : x \to x'$ in $X$, the morphism $F(f)$ must be an identity map in D$C$, since these are the only morphisms that D$C$ has. It follows that $x$ and $x'$ being in the same connected component implies $F(x) = F(x')$, and so $F$ is determined entirely by its restriction to connected components, the monoid homomorphism $\pi_0(F) : \pi_0(X) \to C$. In other words, we have an isomorphism between the homsets

$$\mathrm{E}G\mathrm{Alg}_S(\, X, \mathrm{D}C\,) \quad \cong \quad \mathrm{CMon}(\, \pi_0(X), C\,)$$

This isomorphism is natural in both coordinates, since for any $G : X \to X'$ in E$G$Alg$_S$ and $h : C \to C'$ in CMon,

$$\pi_0(\, \mathrm{D}h \circ F \circ G\,) \quad = \quad \pi_0(\mathrm{D}h) \circ \pi_0(F) \circ \pi_0(G) \quad = \quad h \circ \pi_0(F) \circ \pi_0(G)$$

and so the diagram

$$\begin{tikzcd}
\mathrm{E}G\mathrm{Alg}_S(X, \mathrm{D}C) \arrow[r, "\sim"] \arrow[d, "\mathrm{D}h\circ_\circ G"'] & \mathrm{CMon}(\pi_0(X), C) \arrow[d, "h\circ_\circ\pi_0(G)"] \\
\mathrm{E}G\mathrm{Alg}_S(X', \mathrm{D}C') \arrow[r, "\sim"] & \mathrm{CMon}(\pi_0(X'), C')
\end{tikzcd}$$

commutes. Therefore, $\pi_0 \dashv \mathrm{D}$. □

Now we can utilise Proposition 2.15 to draw out a universal property of $\pi_0(L\mathbb{G}_n)$, just as we did with Proposition 2.10 and $\mathrm{Ob}(L\mathbb{G}_n)$. This time, since we are dealing with commutative monoids, the requirement that everything be invertible will lead us to consider abelian groups.

**Proposition 2.16.** *The connected components of* $L\mathbb{G}_n$ *are* $\mathbb{Z}^n$*, with the restriction of* $\eta$ *to components* $\pi_0(\eta)$ *being the obvious inclusion* $\mathbb{N}^n \hookrightarrow \mathbb{Z}^n$*, and the assignment of objects to their component* $[\,\_\,] : \mathrm{Ob}(L\mathbb{G}_n) \to \pi_0(L\mathbb{G}_n)$ *being the quotient map of abelianisation* $\mathbb{Z}^{*n} \to \mathbb{Z}^n$.

*Proof.* Let $A$ be an abelian group and $h : \pi_0(\mathbb{G}_n) \to A$ a monoid homomorphism. By Proposition 2.15 there is a homset isomorphism

$$\mathrm{E}G\mathrm{Alg}_S(\, \mathbb{G}_n, \, \mathrm{D}A\,) \quad \cong \quad \mathrm{Mon}(\, \pi_0(\mathbb{G}_n), \, A\,)$$

and thus some E$G$-algebra map $h' : \mathbb{G}_n \to \mathrm{D}A$ corresponding to $h$. As every object of D$A$ is invertible, $h'$ is in $(\mathbb{G}_n \downarrow \mathrm{inv})$, and hence there exists a unique map

$u : L\mathbb{G}_n \to \mathrm{D}A$ factoring $h'$ through the initial object $\eta$:

$$\begin{tikzcd} \mathbb{G}_n \arrow[d, "\eta"'] \arrow[dr, "h'"] & \\ L\mathbb{G}_n \arrow[r, "u"'] & \mathrm{D}A \end{tikzcd} \qquad \begin{tikzcd} \pi_0(\mathbb{G}_n) \arrow[d, "\pi_0(\eta)"'] \arrow[dr, "h"] & \\ \pi_0(L\mathbb{G}_n) \arrow[r, "\pi_0(u)"'] & A \end{tikzcd}$$

Applying the functor $\pi_0$ everywhere, we see that $\pi_0(u)$ must also factor $h$ through the homomorphism $\pi_0(\eta)$. Moreover, $\pi_0(u)$ is the only map with this property, since for any other map $v : \pi_0(L\mathbb{G}_n) \to A$ with $v \circ \pi_0(\eta) = h$, its image under the adjunction isomorphism $v' : L\mathbb{G}_n \to \mathrm{D}A$ would have $v' \circ \eta = h'$ by naturality, which would mean that it was actually $u$. Therefore, any homomorphism $\mathrm{Ob}(\mathbb{G}_n) \to A$ will factor uniquely through $\mathrm{Ob}(L\mathbb{G}_n)$, so long as $A$ is an abelian group.

Now assume that $h : \mathrm{Ob}(\mathbb{G}_n) \to H$ is more general sort of homomorphism, where $H$ is a group but not neccessarily an abelian one. Since $\pi_0(L\mathbb{G}_n)$ is a commutative monoid, it follows that its image $\langle \mathrm{im}(h) \rangle$ will be commutative too:

$$h(x)h(y) \;=\; h(xy) \;=\; h(yx) \;=\; h(y)h(x)$$

But then $\langle \mathrm{im}(h) \rangle$ is a commutative submonoid of the group $H$, and so it is really an abelian group. If we denote by $h_{\mathrm{im}} : \mathrm{Ob}(\mathbb{G}_n) \to \langle \mathrm{im}(h) \rangle$ the restriction of $h$ to it image, then from what we have already shown we can conclude that there will be a unique homomorphism $v : \mathrm{Ob}(L\mathbb{G}_n) \to \langle \mathrm{im}(h) \rangle$ with the property $v \circ \pi_0(\eta) = h_{\mathrm{im}}$. Composing this $v$ with the inclusion $i : \langle \mathrm{im}(h) \rangle \hookrightarrow A$, we see that

$$i \circ v \circ \pi_0(\eta) \;=\; i \circ h_{\mathrm{im}} \;=\; h$$

and $i \circ v$ must be the only map for which this is true, for restricting this equation back on $\langle \mathrm{im}(h) \rangle$ yields the unique property of $v$ again. Thus $\pi_0(L\mathbb{G}_n)$ will actually factor any homomorphism $\mathrm{Ob}(\mathbb{G}_n) \to H$ in a unique way, and hence by Proposition 1.20 it is the group completion

$$\pi_0(L\mathbb{G}_n) \;=\; \pi_0(\mathbb{G}_n)^{\mathrm{gp}} \;=\; (\mathbb{N}^n)^{\mathrm{gp}} \;=\; \mathbb{Z}^n$$

The map $\mathrm{Ob}(\eta)$ is then the inclusion of $\mathrm{Ob}(\mathbb{G}_n)$ into its completion, $\mathbb{N}^{*n} \hookrightarrow \mathbb{Z}^{*n}$.

Something else we previously saw in Proposition 1.20 was that that the map $[\,\_\,] : \mathrm{Ob}(\mathbb{G}_n) \to \pi_0(\mathbb{G}_n)$ sending objects of $\mathbb{G}_n$ to their connected component is the quotient map of abelianisation, $\mathbb{N}^{*n} \to (\mathbb{N}^{*n})^{\mathrm{ab}} = \mathbb{N}^n$. If we also use $[\,\_\,]$ to denote the map sending objects of $L\mathbb{G}_n$ to their components, then functoriality of $\eta$ tells us that

$$\begin{tikzcd} \mathbb{N}^{*n} \arrow[r, "\mathrm{Ob}(\eta)"] \arrow[d, "{[\,\_\,]}"'] & \mathbb{Z}^{*n} \arrow[d, "{[\,\_\,]}"] \\ \mathbb{N}^n \arrow[r, "\pi_0(\eta)"] & \mathbb{Z}^n \end{tikzcd}$$

commutes. Using the values of $[\,\_\,]$ from Proposition 1.20, $\mathrm{Ob}(\eta)$ from Proposition 2.12, and $\pi_0(\eta)$ from the earlier parts of this proposition, it follows that for any generator $z_i$ of $\mathbb{Z}^{*n}$,

$$[z_i] \;=\; [\mathrm{Ob}(\eta)(z_i)] \;=\; \pi_0(\eta)([z_i]) \;=\; \pi_0(\eta)(z_i) \;=\; z_i$$

But this description of $[\,\_\,] : \mathrm{Ob}(L\mathbb{G}_n) \to \pi_0(L\mathbb{G}_n)$ on generators is then just the definition of the quotient map of abelianisation $\mathbb{Z}^{*n} \to (\mathbb{Z}^{*n})^{\mathrm{ab}}$, as required $\square$

2.4. **The morphisms of $L\mathbb{G}_n$.** Now that we understand the objects of the algebra $L\mathbb{G}_n$, the next most obvious thing to look for are its morphisms, $\mathrm{Mor}(L\mathbb{G}_n)$. It would be nice to construct this collection in the same way we constructed $\mathrm{Ob}(L\mathbb{G}_n)$ and $\pi_0(L\mathbb{G}_n)$, by applying the left adjoint of some adjunction to the initial map $\eta$. Before we can do this however, we need to ask ourselves a question. What sort of mathematical object is $\mathrm{Mor}(L\mathbb{G}_n)$, exactly?

Given a pair of morphisms $f : x \to y, f' : x' \to y'$ in an E$G$-algebra $X$, there are two basic binary operations we can perform. First, we can take their tensor product $f \otimes f'$, and this together with the unit map $\mathrm{id}_I$ imbues $\mathrm{Mor}(X)$ with the structure of a monoid. Second, if we have $y = x'$ then we can form the composite morphism $f' \circ f$. However, consider the following fact: if $y$ is an invertible object, then

$$\begin{aligned} f' \circ f &= (f' \otimes \mathrm{id}_I) \circ (\mathrm{id}_I \otimes f) \\ &= (f' \otimes \mathrm{id}_{y*} \otimes \mathrm{id}_y) \circ (\mathrm{id}_y \otimes \mathrm{id}_{y*} \otimes f) \\ &= (f' \circ \mathrm{id}_y) \otimes (\mathrm{id}_{y*} \circ \mathrm{id}_{y*}) \otimes (\mathrm{id}_y \circ f) \\ &= f' \otimes \mathrm{id}_{y*} \otimes f \end{aligned}$$

In other words, composition along invertible objects in $X$ is determined completely by the monoidal structure of $X$. In the case of $L\mathbb{G}_n$, where every object is invertible, this means that if we understand $\mathrm{Mor}(L\mathbb{G}_n)$ as a monoid then we will be able to recover the operation $\circ$ in its entirety. For that reason, we will choose to ignore composition of elements of $\mathrm{Mor}(X)$ for the time being, and focus on its status as a monoid.

Now we try to proceed as we did before, by showing that $\mathrm{Mor}(X)$ is part of an adjunction.

**Definition 2.17.** Let $\mathrm{Mor} : \mathrm{MonCat} \to \mathrm{Mon}$ be the functor which sends algebras $X$ to their monoid of morphisms $\mathrm{Mor}(X)$, and sends algebra maps $F : X \to Y$ to the monoid homomorphism

$$\begin{array}{rccl} \mathrm{Mor}(F) & : & \mathrm{Mor}(X) & \to & \mathrm{Mor}(Y) \\ & : & f : x \to x' & \mapsto & F(f) : F(x) \to F(x') \end{array}$$

**Definition 2.18.** For a given abelian group $A$, let $C(A)$ represent the monoidal category defines as follows:

- The objects of $C(A)$ are the monoid $A$, with the monoid multiplication as the tensor product and the identity element $e$ as the monoidal unit.
- For any two objects $a, a' \in A$, the homset $C(A)(a, a')$ is isomorphic to the underlying set of $A$.
- From the above, the morphisms of $C(A)$ will clearly be
$$\mathrm{Mor}\big(C(A)\big) \;=\; A \times A^2 \;=\; A^3$$
when viewed as a set, but this equality also holds as monoids, so that the tensor product is defined componentwise using the monoid multiplication of $A$.
- For any two composable morphisms $(a, b, b')$, $(a', b', b'')$ of $C(A)$, their composite is the morphism
$$(a', b', b'') \circ (a, b, b') \;=\; \big(a(b')^* a',\, b,\, b''\big)$$

Likewise, for any group homomorphism $h : A \to A'$ between abelian groups, denote by $C(h) : C(A) \to C(A')$ the obvious monoidal functor which acts like $h$ on objects and $h^3$ on morphisms. This defines a functor $C : \mathrm{Ab} \to \mathrm{MonCat}$ from the category of abelian groups onto the category of monoidal categories.

Intuitively, $C(A)$ is the the monoidal category that we can build out of $A$ by using the trick we discussed before for extracting composition from the tensor product, $f' \circ f = f' \otimes \mathrm{id}_{y*} \otimes f$. This is why we had to choose $A$ to be a group, as this can only work when all of the objects of $C(A)$ are invertible. Notice also that commutativity is required in order for $C(A)$ to be a well-defined monoidal category, since we need its operations to obey an interchange law, and thus

$$\begin{array}{rcl} (aa', e, e) & = & (a, e, e) \otimes (a', e, e) \\ & = & \big(\, \mathrm{id}_e \circ (a, e, e) \,\big) \otimes \big(\, (a', e, e) \circ \mathrm{id}_e \,\big) \\ & = & \big(\, (a', e, e) \otimes \mathrm{id}_e \,\big) \circ \big(\, \mathrm{id}_e \otimes (a, e, e) \,\big) \\ & = & (a', e, e) \circ (a, e, e) \\ & = & (a'a, e, e) \end{array}$$

This is the classic Eckmann-Hilton argument.

**Proposition 2.19.** *The functor $C$ is a right adjoint to the functor* $\mathrm{Mor}(\_)^{\mathrm{gp,ab}} : \mathrm{MonCat} \to \mathrm{Ab}$.

*Proof.* Let $X$ be a monoidal category, $A$ an abelian group, and $F : X \to C(A)$ a monoidal functor. For any morphism $f : x \to y$ in $X$, by functoriality $F$ will send it onto some morphism $F(f)$ in the homset $C(A)\big(\, F(x), F(y)\,\big)$. However, every homset of $C(A)$ is isomorphic to a copy of $A$, and so clearly there is a way to extract from $F$ a map $\mathrm{Mor}(X) \to A$. Specifically, if we define the map $\epsilon_A$ to be the projection

$$\begin{array}{rrcl} \epsilon_A & : \quad \mathrm{Mor}\big(\, C(A)\,\big) & \to & A \\ & : \quad A^3 & \to & A \\ & : \quad (a, b, b') & \mapsto & a \end{array}$$

then we can use the functor Mor and the map $\epsilon_A$ to form the following composite map:

$$\mathrm{Mor}(X) \xrightarrow{\ \mathrm{Mor}(F)\ } \mathrm{Mor}\big(\, C(A)\,\big) \xrightarrow{\ \ \epsilon_A\ \ } A$$

Then, since $A$ is not just a monoid but an abelian group, we can factor the homomorphism $\mathrm{Mor}(F)$ through the group completion of $\mathrm{Mor}(X)$, and then through the abelianisation of that group, at last yielding a map

$$F' := \epsilon_A \circ \big(\, \mathrm{Mor}(F)^{\mathrm{gp}}\,\big)^{\mathrm{ab}} : \big(\, \mathrm{Mor}(X)^{\mathrm{gp}}\,\big)^{\mathrm{ab}} \to A$$

This $\epsilon$ will be the counit of our adjunction, with the assignment $F \mapsto F'$ being one direction of the eventual homset adjunction.

Conversely, let $\eta_X$ be the monoidal functor defined by

$$\begin{array}{rrcl} \eta_X & : \quad X & \to & C\big(\, \mathrm{Mor}(X)^{\mathrm{gp,ab}}\,\big) \\ & : \quad x & \mapsto & [\mathrm{id}_x] \\ & : \quad f : x \to y & \mapsto & (\, [f], [\mathrm{id}_x], [\mathrm{id}_y]\,) \end{array}$$

where $[\_]$ is the quotient of abelianisation $\mathrm{Mor}(X) \to \mathrm{Mor}(X)^{\mathrm{gp,ab}}$. Then any homomorphism $h : \mathrm{Mor}(X)^{\mathrm{gp,ab}} \to A$ can be used to construct a monoidal functor

$h' : X \to C(A)$ as follows:

$$X \xrightarrow{\eta_X} C\big(\,\mathrm{Mor}(X)\,\big) \xrightarrow{C(\,[\,\_\,]\,)} C\big(\,\mathrm{Mor}(X)^{\mathrm{gp,ab}}\,\big) \xrightarrow{C(h)} C(A)$$

Moreover, if we compare this $\eta$ with $\epsilon$ then we see that the composites

$$\begin{tikzcd}
\mathrm{Mor}(X)^{\mathrm{gp,ab}} \arrow[d, "\mathrm{Mor}(\eta_X)^{\mathrm{gp,ab}}"] & & C(A) \arrow[d, "\eta_{C(A)}"] \\
\mathrm{Mor}\Big(\,C\big(\,\mathrm{Mor}(X)^{\mathrm{gp,ab}}\,\big)\,\Big)^{\mathrm{gp,ab}} \arrow[d, equal] & & C\Big(\,\mathrm{Mor}\big(\,C(A)\,\big)^{\mathrm{gp,ab}}\,\Big) \arrow[d, equal] \\
\mathrm{Mor}\Big(\,C\big(\,\mathrm{Mor}(X)^{\mathrm{gp,ab}}\,\big)\,\Big) \arrow[d, "\epsilon_{\mathrm{Mor}(X)^{\mathrm{gp,ab}}}"] & & C\Big(\,\mathrm{Mor}\big(\,C(A)\,\big)\,\Big) \arrow[d, "C(\epsilon_A)"] \\
\mathrm{Mor}(X)^{\mathrm{gp,ab}} & & C(A)
\end{tikzcd}$$

must be the respective identity maps:

$$\begin{aligned}
\epsilon_{\mathrm{Mor}(X)^{\mathrm{gp,ab}}} \circ \mathrm{Mor}(\eta_X)^{\mathrm{gp,ab}}\big(\,[f : x \to y]\,\big) &= \epsilon_{\mathrm{Mor}(X)^{\mathrm{gp,ab}}}\big(\,[f], [\mathrm{id}_x], [\mathrm{id}_y]\,\big) \\
&= [f] \\
C(\epsilon_A) \circ \epsilon_{\mathrm{Mor}(X)^{\mathrm{gp,ab}}}\big(\,a\,\big) &= C(\epsilon_A)\big(\,a, a, a\,\big) \\
&= a \\
C(\epsilon_A) \circ \epsilon_{\mathrm{Mor}(X)^{\mathrm{gp,ab}}}\big(\,a, b, b'\,\big) &= C(\epsilon_A)\big(\,(a, b, b'),\ (b, b, b),\ (b', b', b')\,\big) \\
&= (a, b, b')
\end{aligned}$$

In other words, $\eta_X$ and $\epsilon_A$ really are the unit and counit of an adjunction $\mathrm{Mor}(\,\_\,)^{\mathrm{gp,ab}} \dashv C$, whose isomorphism of homsets

$$\mathrm{MonCat}(\,X, C(A)\,) \quad \cong \quad \mathrm{Ab}(\,\mathrm{Mor}(X)^{\mathrm{gp,ab}}, A\,)$$

is given by the assigments $F \mapsto F'$ and $h \mapsto h'$. □

Proposition 2.19 is very similar to Propositions 2.10 and 2.15, but it falls short in a few very important ways. First, if we want to use an adjunction to find a relationship between the morphisms of $\mathbb{G}_n$ and $L\mathbb{G}_n$, like what we did in Propositions 2.12 and 2.16, then what we need is an adjunction involving $\mathrm{E}G\mathrm{Alg}_S$, not MonCat. This is because $\eta$ can only be used to factor algebra maps $\mathbb{G}_n \to X_{\mathrm{inv}}$ through $L\mathbb{G}_n$, and not arbitrary monoidal functors. Likewise, we would rather have the other side of the adjunction be Mon instead of Ab so that we could work with the functor Mor directly, and not its group completed, abelianised version.

Unfortunately this adjunction seems to be the best we can do. We already saw that we need $A$ to be an abelian group for $C(A)$ to have composition and interchange, and given an arbitrary abelian group that we want to be the morphisms of an algebra, there is no general method for assigning it an $\mathrm{E}G$-action. As this is what we are stuck with, we will not be able to use the $\eta$ method to extract the information we need about the morphisms of $L\mathbb{G}_n$, and so we must try a less straightforward approach.

## 3. Free invertible algebras as coequalizers

In the previous chapter, we made progress towards understanding the strucutre of $L\mathbb{G}_n$ by showing that the algebra was an initial object in a certain comma category. Specifically, we saw that the map $\eta : \mathbb{G}_n \to L\mathbb{G}_n$ is initial among all E$G$-algebra maps $\mathbb{G}_n \to X_{\text{inv}}$. This fact is the rigourous way of expressing a fairly obvious intuition about $L\mathbb{G}_n$ — that we should expect the free algebra on $n$ invertible objects to be like the free algebra on $n$ objects, except that its objects are invertible.

However, this not the only way of thinking about $L\mathbb{G}_n$. Consider for a moment the free E$G$-algebra on $2n$ objects, $\mathbb{G}_{2n}$. Intuitively, if we were to take this algebra and then enforce upon it the extra relations $z_{n+1} = z_1^*, ..., z_{2n} = z_n^*$, then we would be changing it from a structure with $2n$ independent generators into one with $n$ indepedent generators and their inverses. That is, there seems to be a natural way to think about $L\mathbb{G}_n$ as a quotient of the larger algebra $\mathbb{G}_{2n}$. In this chapter we will work towards making this idea precise, and then examine some of its consequences. Together with the information we have already gleaned from the initial object persepctive, this will then provide us with a complete description of the algebra $L\mathbb{G}_n$.

### 3.1. $L\mathbb{G}_n$ as a coequalizer.

We'll begin with some definitions.

**Definition 3.1.** Let $\delta$ be the map of E$G$-algebras defined on generators by

$$\begin{array}{ccrcl} \delta & : & \mathbb{G}_{4n} & \to & \mathbb{G}_{2n} \\ & : & z_i & \mapsto & z_i \\ & : & z_{n+i} & \mapsto & z_{n+i} \\ & : & z_{2n+i} & \mapsto & z_i \otimes z_{n+i} \\ & : & z_{3n+i} & \mapsto & z_{n+i} \otimes z_i \end{array}$$

for $1 \leq i \leq n$. Similarly, let $\zeta$ be the E$G$-algebra map defined by

$$\begin{array}{ccrcl} \zeta & : & \mathbb{G}_{4n} & \to & \mathbb{G}_{2n} \\ & : & z_i & \mapsto & z_i \\ & : & z_{n+i} & \mapsto & z_{n+i} \\ & : & z_{2n+i} & \mapsto & I \\ & : & z_{3n+i} & \mapsto & I \end{array}$$

We will denote by $q : \mathbb{G}_{2n} \to Q$ the coequalizer these two maps.

Note that the above definitions do actually make sense. The given descriptions of $\delta$ and $\zeta$ are enough to specify those maps uniquely because $\mathbb{G}_k$ is the free E$G$-algebra on $k$ objects, and hence algebra maps $\mathbb{G}_k \to \mathbb{G}_{k'}$ are canonically isomorphic to functors $\{z_1, ..., z_k\} \to \mathbb{G}_{k'}$. Also we can be sure that the map $q$ exists, because E$G$Alg$_S$ is a locally finitely presentable category and thus has all finite colimits.

The goal of this approach will be show that $Q$ is in fact that same algebra as $L\mathbb{G}_n$. In order to do this, it would help if we could easily compare $q : \mathbb{G}_{2n} \to Q$ to our initial object $\eta : \mathbb{G}_{2n} \to L\mathbb{G}_n$. In other words, we really want to show that $q$ is an object of $(\mathbb{G}_n \downarrow \text{inv})$ — that $Q$ has only invertible objects. This can be done using the adjunction we found in Proposition 2.10.

**Lemma 3.2.** *The object monoid of $Q$ is $\mathbb{Z}^{*n}$, and the restriction of $q$ to objects $\mathrm{Ob}(q) : \mathrm{Ob}(\mathbb{G}_{2n}) \to \mathrm{Ob}(Q)$ is the monoid homomorphism defined on generators as*

$$\begin{array}{rcrcl} \mathrm{Ob}(q) & : & \mathbb{N}^{*2n} & \to & \mathbb{Z}^{*n} \\ & : & z_i & \mapsto & z_i \\ & : & z_{n+i} & \mapsto & z_i^* \end{array}$$

*Proof.* Consider $\mathrm{Ob}(\delta)$ and $\mathrm{Ob}(\zeta)$, the restrictions on objects of the algebra maps $\delta, \zeta : \mathbb{G}_{4n} \to \mathbb{G}_{2n}$. By Lemma 1.18, these are monoid homomorphisms $\mathbb{N}^{*4n} \to \mathbb{N}^{*2n}$, and since Mon is cocomplete we can take their coequalizer in that category. This will give us a new homomorphism, whose source is $\mathbb{N}^{*2n}$ and whose target is the quotient of $\mathbb{N}^{*2n}$ by the relations $\mathrm{Ob}(\delta)(x) = \mathrm{Ob}(\zeta)(x)$. Remembering Definition 3.1, and that $\mathbb{N}^{*2n}$ is the free monoid on $2n$ generators, this quotient monoid will have the following presentation:

$$\begin{array}{ll} \text{Generators:} & z_1, \dots, z_{2n} \\ \text{Relations:} & z_i \otimes z_{n+i} = I, \\ & z_{n+i} \otimes z_i = I \end{array}$$

This is just the same as

$$\begin{array}{ll} \text{Generators:} & z_1, \dots, z_{2n} \\ \text{Relations:} & z_{n+i} = z_i^*, \end{array}$$

which is the presentation of $\mathbb{Z}^{*n}$.

But by Proposition 2.10, Ob is a left adjoint and hence preserves all colimits. Thus the coequalizer of $\mathrm{Ob}(\delta)$ and $\mathrm{Ob}(\zeta)$ is just the underlying homomorphism $\mathrm{Ob}(q)$ of the coequalizer $q$ of $\delta, \zeta$. Therefore $\mathrm{Ob}(Q) = \mathbb{Z}^{*n}$, and $\mathrm{Ob}(q)$ is the quotient map $\mathbb{N}^{*2n} \to \mathbb{Z}^{*n}$ sending $z_i \mapsto z_i$ and $z_{n+i} \mapsto z_i^*$ for $1 \le i \le n$. □

An immediate corollary of Lemma 3.2 is that every object of the coequalizer algebra $Q$ is invertible. Thus $q : \mathbb{G}_{2n} \to Q$ is an object of the category $(\mathbb{G}_n \downarrow \mathrm{inv})$, and hence we can use the initiality of $\eta$ to determine the following result:

**Proposition 3.3.** *Let $i : \mathbb{G}_n \to \mathbb{G}_{2n}$ be the inclusion of* E$G$*-algebras defined on generators by*

$$i(z_i) = z_i$$

*Then $i \circ q$ is an initial object of $(\mathbb{G}_n \downarrow \mathrm{inv})$. In particular, this means that*

$$Q \cong L\mathbb{G}_n$$

*Proof.* Let $\psi : \mathbb{G}_n \to X$ be an arbitrary object of $(\mathbb{G}_n \downarrow \mathrm{inv})$. The map $\psi^* : \mathbb{G}_n \to X$ which takes values

$$\psi^*(z_i) = \psi(z_i)^*$$

is also an object of $(\mathbb{G}_n \downarrow \mathrm{inv})$, and using these two we can define a new map, $\psi + \psi^*$, using the universal property of the colimit:

$$\begin{tikzcd} & \mathbb{G}_n + \mathbb{G}_n \arrow[dd, dashed, "\psi+\psi^*"] & \\ \mathbb{G}_n \arrow[ur, hook, "i"] \arrow[dr, "\psi"'] & & \mathbb{G}_n \arrow[ul, hook', "i'"'] \arrow[dl, "\psi^*"] \\ & X & \end{tikzcd}$$

But $\mathbb{G}_n$ is the free E$G$-algebra on $n$ objects, and the free functor $F : \mathrm{Cat} \to \mathrm{E}G\mathrm{Alg}_S$ preserves colimits because it is a left adjoint, so clearly

$$\begin{aligned}
\mathbb{G}_n + \mathbb{G}_n &= F(\{z_1, ..., z_n\}) + F(\{z'_1, ..., z'_n\}) \\
&= F(\,\{z_1, ..., z_n\} + \{z'_1, ..., z'_n\}\,) \\
&= F(\{z_1, ..., z_{2n}\}) \\
&= \mathbb{G}_{2n}
\end{aligned}$$

This means that we can compose $\psi + \psi^* : \mathbb{G}_{2n} \to X$ with the maps $\delta, \zeta : \mathbb{G}_{4n} \to \mathbb{G}_{2n}$, though we need to be careful to specify which inclusions we really used in the definition of $\psi + \psi^*$. Suppose that the lefthand inclusion is $i$, the one given in the statement of the proposition, and the other is defined by the assignment $z_i \mapsto z_{i+n}$. Then for $1 \leq i \leq n$,

$$\begin{aligned}
(\psi + \psi^*)\delta(z_i) &= (\psi + \psi^*)(z_i) \\
&= (\psi + \psi^*)\zeta(z_i) \\[1em]
(\psi + \psi^*)\delta(z_{n+i}) &= (\psi + \psi^*)(z_{n+i}) \\
&= (\psi + \psi^*)\zeta(z_i) \\[1em]
(\psi + \psi^*)\delta(z_{2n+i}) &= (\psi + \psi^*)(z_i \otimes z_{n+i}) \\
&= \psi(z_i) \otimes \psi(z_i)^* \\
&= I \\
&= (\psi + \psi^*)(I) \\
&= (\psi + \psi^*)\zeta(z_{2n+i}) \\[1em]
(\psi + \psi^*)\delta(z_{3n+i}) &= (\psi + \psi^*)(z_{n+i} \otimes z_i) \\
&= \psi(z_i)^* \otimes \psi(z_i) \\
&= I \\
&= (\psi + \psi^*)(I) \\
&= (\psi + \psi^*)\zeta(z_{3n+i})
\end{aligned}$$

That is, $(\psi + \psi^*) \circ \delta = (\psi + \psi^*) \circ \zeta$. However, we've already defined $q : \mathbb{G}_{2n} \to Q$ to be the coequalizer of $\delta$ and $\zeta$, the universal map such that its composites with them are equal. Therefore, there must exist a unique E$G$-algebra map $u : Q \to X$ making the righthand triangle below diagram commute:

$$\begin{tikzcd}
\mathbb{G}_n \arrow[r, "i"] \arrow[dr, "\psi"'] & \mathbb{G}_{2n} \arrow[r, "q"] \arrow[d, "\psi+\psi^*"] & Q \arrow[dl, "u"] \\
& X &
\end{tikzcd}$$

The other triangle commutes by the definition of $\psi + \psi^*$, and so together the diagram tells us that for any object $\psi$ of $(\mathbb{G}_n \downarrow \mathrm{inv})$, there exists at least one morphism $u$ in $(\mathbb{G}_n \downarrow \mathrm{inv})$ going from $q \circ i$ to $\psi$.

Next, let $v : Q \to X$ be an arbitrary morphism $q \circ i \to \psi$ in $(\mathbb{G}_n \downarrow \mathrm{inv})$. By definition, this means that

$$\begin{aligned}
& \psi &= \;& vqi \\
\Longrightarrow \quad & \psi + \psi^* &= \;& vqi + (vqi)^*
\end{aligned}$$

Also, for $1 \leq i \leq n$ we have

$$\begin{array}{rcccccccc} q(z_i) \otimes q(z_{n+i}) & = & q(z_{i-n} \otimes z_i) & = & q\delta(z_{2n+i}) & = & q\zeta(z_{2n+i}) & = & I \\ q(z_{n+i}) \otimes q(z_i) & = & q(z_i \otimes z_{n+i}) & = & q\delta(z_{3n+i}) & = & q\zeta(z_{3n+i}) & = & I \\ & \Longrightarrow & q(z_{n+i}) & = & q(z_i)^* & & & & \end{array}$$

Therefore,

$$\begin{aligned} (\psi + \psi^*)(z_i) &= \big(vqi + (vqi)^*\big)(z_i) \\ &= vqi(z_i) \\ &= vq(z_i) \end{aligned}$$

$$\begin{aligned} (\psi + \psi^*)(z_{n+i}) &= \big(vqi + (vqi)^*\big)(z_{n+i}) \\ &= vqi(z_i)^* \\ &= v\big(q(z_i)^*\big) \\ &= vq(z_{n+i}) \end{aligned}$$

or in other words $\psi + \psi^* = v \circ q$ for any morphism $v : q \circ i \to \psi$ in $(\mathbb{G}_n \downarrow \mathrm{inv})$. But this is the property that the map $u$ was supposed to satisfy uniquely, and thus it must be the only morphism $q \circ i \to \psi$ in $(\mathbb{G}_n \downarrow \mathrm{inv})$. Therefore $q \circ i$ is an initial object, and hence it is isomorphic in $(\mathbb{G}_n \downarrow \mathrm{inv})$ to any other initial object, such as $\eta$. It follows that the targets of these two maps, $Q$ and $L\mathbb{G}_n$ respectively, are isomorphic as E$G$-algebras. □

It's worth noting that we have not given a method for actually taking coequalizers in $\mathrm{E}G\mathrm{Alg}_S$, and so Proposition 3.3 doesn't immediately provide an explicit description of $L\mathbb{G}_n$. Nevertheless, we will be able to use this new perspective to fix some of our outstanding problems.

3.2. **Sources and targets in $L\mathbb{G}_n$.** Back in **??**, we saw that the functor $\mathrm{Mor}(\_)^{\mathrm{gp,ab}}$ formed one half of an adjunction between the categories MonCat and Ab. The goal of this section will be to show that we can reconstruct the all of morphisms of $L\mathbb{G}_n$ from just the abelian group $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$, and therefore that this adjunction can actually be used to help find a description of $L\mathbb{G}_n$.

The way we will do this is by splitting $\mathrm{Mor}(L\mathbb{G}_n)$ up as the product of two smaller pieces. The first of these will encode all of the possible combinations of source and target data for morphisms in $L\mathbb{G}_n$, while the second will just be the endomorphisms of the unit object, $L\mathbb{G}_n(I, I)$. In other words, we will see that the monoid $\mathrm{Mor}(L\mathbb{G}_n)$ can be broken down into a context where source and target are the only thing that matters, and another where they are irrelevant. Once we have done this, we can then use the fact that $L\mathbb{G}_n(I, I)$ is always an abelian group to rewrite $\mathrm{Mor}(L\mathbb{G}_n)$ in terms of $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$.

So to get things started, lets consider the source and target information of morphisms in $L\mathbb{G}_n$.

**Definition 3.4.** For any E$G$-algebra $X$, denote by $s : \mathrm{Mor}(X) \to \mathrm{Ob}(X)$ and $t : \mathrm{Mor}(X) \to \mathrm{Ob}X)$ the monoid homomorphisms which send each morphism of $X$ to its source and target, respectively. That is,

$$s(\, f : x \to y) = x, \qquad t(\, f : x \to y) = y$$

If we use the universal property of products, we can combine these two homomorphisms into a single map, $s \times t : \mathrm{Mor}(X) \to \mathrm{Ob}(X) \times \mathrm{Ob}(X)$. The monoid we are interested in finding is the image $L\mathbb{G}_n$ under its instance of this map. In particular, we want to show that $(s \times t)(L\mathbb{G}_n)$ a submonoid of $\mathrm{Mor}(L\mathbb{G}_n)$. This is a little tricky though, since we don't currently know what the morphisms of $L\mathbb{G}_n$ even are. We will be able to sidestep this problem by first proving the analogous statement for all $\mathbb{G}_n$, and then using the coequaliser map $q$ to recover the $L\mathbb{G}_n$ version.

To that end, we need to find a description of the monoids $(s \times t)(\mathbb{G}_n)$ and $\mathrm{Mor}(\mathbb{G}_n)$. The first of these can be done quite generally.

**Lemma 3.5.** *Let $X$ be an $\mathrm{E}G$-algebra, and $s \times t : \mathrm{Mor}(X) \to \mathrm{Ob}(X)^2$ the map built from $s$ and $t$ using the universal property of products. Then the image of this map is*

$$(s \times t)(X) \;=\; \mathrm{Ob}(X) \times_{\pi_0(X)} \mathrm{Ob}(X)$$

*where this pullback is taken over the canonical maps sending objects of $X$ to their connected components:*

$$\begin{tikzcd}
\mathrm{Ob}(X) \times_{\pi_0(X)} \mathrm{Ob}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ob}(X) \arrow[d, "{[_]}"] \\
\mathrm{Ob}(X) \arrow[r, "{[_]}"] & \pi_0(X)
\end{tikzcd}$$

*Proof.* By definition, there exists a morphism $f : x \to y$ between objects $x, y$ of $X$ if and only if they are in the same connected component, $[x] = [y]$. Thus

$$\begin{aligned}
(x, y) \in (s \times t)(X) \quad &\Longleftrightarrow \quad \exists\, f : \quad s(f) = x, \quad t(f) = y \\
&\Longleftrightarrow \quad [x] = [y] \\
&\Longleftrightarrow \quad (x, y) \in \mathrm{Ob}(X) \times_{\pi_0(X)} \mathrm{Ob}(X)
\end{aligned}$$

as required. □

Recalling Lemma 1.18 and Propositions 1.20, 2.12 and 2.16, we can immediately conclude the following:

**Corollary 3.6.**

$$\begin{aligned}
(s \times t)(\mathbb{G}_n) &= \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \\
(s \times t)(L\mathbb{G}_n) &= \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}
\end{aligned}$$

*where these pullbacks are taken over the quotients of abelianisation for $(\mathbb{N}^{*n})^{\mathrm{ab}} = \mathbb{N}^n$ and $(\mathbb{Z}^{*n})^{\mathrm{ab}} = \mathbb{Z}^n$ respectively.*

Next, for the monoid $\mathrm{Mor}(\mathbb{G}_n)$ Definition 1.16 already tells us everything we need to know, but it will be helpful for us to give a nice compact description.

**Definition 3.7.** Let $S$ be a set and $F(S)$ the free monoid on $S$, the monoid whose elements are strings of elements of $S$ and whose binary operation is concatenation. Then we will denote by

$$|\_| : F(S) \to \mathbb{N}$$

the monoid homomorphism defined by sending each element of $S \subseteq F(S)$ to 1, and therefore also each concatenation of $n$ elements of $S$ to the natural number $n$. We will call $|x|$ the *length* of $x \in F(S)$.

**Definition 3.8.** Let $G$ be an action operad. Then we will also the notation $G$ to denote the *underlying monoid* of this action operad. This is the natural way to consider $G$ as a monoid, with its element set being all of its elements together, $\coprod_m G(m)$, and with tensor product as its binary operation, $g \otimes h = \mu(e_2; g, h)$.

Also, note that this monoid comes equipped with a homomorphism $|\_| : G \to \mathbb{N}$, sending each $g \in G$ to the natural number $m$ if and only if $g$ is an element of the group $G(m)$. Again, we'll call this number $|g|$ the *length* of $g$.

**Lemma 3.9.** *The monoid of morphisms of the algebra $\mathbb{G}_n$ is*

$$\mathrm{Mor}(\mathbb{G}_n) \cong G \times_{\mathbb{N}} \mathbb{N}^{*n}$$

*where this pullback is taken over the respective length homomorphisms,*

$$\begin{tikzcd} G \times_{\mathbb{N}} \mathbb{N}^{*n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{N}^{*n} \arrow[d, "|\_|"] \\ G \arrow[r, "|\_|"] & \mathbb{N} \end{tikzcd}$$

*using the fact that $\mathbb{N}^{*n}$ is the free monoid $F\big(\{z_1, ..., z_n\}\big)$.*

*Proof.* An element of $G \times_{\mathbb{N}} F\big(\{z_1, ..., z_n\}\big)$ is just an element $g \in G(m)$ for some $m$, together with an $m$-tuple of objects $(x_1, ..., x_m)$ from the set of generators $\{z_1, ..., z_n\}$. Thus the action on $\mathbb{G}_n$ defines an obvious function

$$\begin{array}{rcrcl} \alpha & : & G \times_{\mathbb{N}} F\big(\{z_1, ..., z_n\}\big) & \to & \mathrm{Mor}(\mathbb{G}_n) \\ & : & (g; x_1, ..., x_m) & \mapsto & \alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) \end{array}$$

But by Lemma 1.19, each element of $\mathrm{Mor}(\mathbb{G}_n)$ can be expressed in the form $\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$ for a unique collection $(g; x_1, ..., x_m)$, and so this function $\alpha$ is actually a bijection of sets. Furthermore, this function preserves tensor product, since

$$\begin{array}{rcl} \alpha\big((g; f_1, ..., f_m) \otimes (g'; f'_1, ..., f'_m)\big) & = & \alpha(\,g \otimes g'\,;\, f_1, ..., f_m, f'_1, ..., f'_m\,) \\ & = & \alpha(\,g\,;\, f_1, ..., f_m\,) \otimes \alpha(\,g'\,;\, f'_1, ..., f'_m\,) \end{array}$$

and hence it is a monoid isomorphism, as required. □

We want to know if $(s \times t)(\mathbb{G}_n)$ can be seen as a submonoid of $\mathrm{Mor}(\mathbb{G}_n)$, which we now see is the same as asking if we can find an injective homomorphism $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \to G \times_{\mathbb{N}} \mathbb{N}^{*n}$. Creating a *function* like this would not be especially hard. For any pair $(w, w') \in \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$, the image of $w$ and $w'$ in the abelian group $\mathbb{N}^n$ is the same, which is to say that the words $w, w' \in \mathbb{N}^{*n}$ are permuations of each other. Since the underlying permutation maps $\pi : G(m) \to \mathrm{S}_m$ in the definition of the action operad $G$ are all surjective, we can therefore always find an element of $g \in G(|w|)$ for which $\pi(g)(w) = w'$. Thus in order to make an injective function $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \to G \times_{\mathbb{N}} \mathbb{N}^{*n}$, all we need to do is make a choice $g_{(w,w')}$ like this for each $(w, w')$, and then set

$$\begin{array}{rcl} \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} & \to & G \times_{\mathbb{N}} \mathbb{N}^{*n} \\ (w, w') & \mapsto & (\,g_{(w,w')}, w\,) \end{array}$$

Injectivity follows from

$$(\,g_{(w,w')}, w\,) \;=\; (\,g_{(v,v')}, v\,) \quad\Longrightarrow\quad \begin{array}{rcl} g_{(w,w')} & = & g_{(v,v')} \\ w & = & v \\ w' & = & \pi(g_{(w,w')})(w) \\ & = & \pi(g_{(v,v')})(v) \\ & = & v' \end{array}$$

So how do we know if we can choose these $g_{(w,w')}$ in such a way that the resulting function is also a monoid homomorphism? If we could find a presentation of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ in terms of generators and relations then this would help a little, since we would only need to pick a $g_{(z,z')}$ for each generator $(z,z')$, and then define all other $g$ by way of products.

$$g_{(vw,v'w')} \;=\; g_{(v,v')}g_{(w,w')}$$

But we would still need to know if our choice of $g_{(z,z')}$ obeyed the necessary relations on the generators of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$. Luckily for us though, this turns out to be no problem at all.

**Proposition 3.10.** *$\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ is a free monoid.*

*Proof.* Given an element $(w,w')$ of the monoid $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$, let $d(w,w')$ be the following set:

$$d(w,w') \;=\; \left\{ \quad (u,u'),(v,v') \in \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \quad : \quad \begin{array}{rcl} (w,w') & = & (u,u') \otimes (v,v'), \\ (u,u') & \neq & (I,I), \\ (v,v') & \neq & (I,I) \end{array} \quad \right\}$$

We can use these sets to recursively define a decomposition of any element $(w,w')$ as a product of other elements of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$. Specifically, if $d(w,w')$ is empty then we say that the decomposition of $(w,w')$ is just $(w,w')$ itself, and otherwise we choose any $\big((u,u'),(v,v')\big) \in d(w,w')$ and say that the decomposition of $(w,w')$ is the concatenation of the decomposition of $(u,u')$ with the decomposition of $(v,v')$. Note that this process definitely terminates, since $|u|$ and $|v|$ are always strictly smaller that $|w|$, and any strictly decreasing sequence of natural numbers is finite.

Of course, we need to check that this decomposition of $(w,w')$ is well-defined, which amounts to checking that the choice of $(u,u'),(v,v')$ we make at each stage won't change the eventual output. To that end, suppose for the sake of contradiction that $(u_1,u'_1),...,(u_m,u'_m)$ and $(v_1,v'_1),...,(v'_m,v'_{m'})$ are distinct decompositions of $(w,w')$ we could arrive at using the above process. Notice that we can assume without loss of generality that $|u_1| < |v_1|$. If instead $|u_1| > |w_1|$, we can just swap the labels of the sequences, and if $|u_1| = |v_1|$ then we can just discard those elements and instead consider the decompositions $(u_2,u'_2),...,(u_m,u'_m)$ and $(v_2,v'_2),...,(v'_m,v'_{m'})$ of $(u_1,u'_1)\otimes ... \otimes (u_m,u'_m) = (v_1,v'_1)\otimes ... \otimes (v'_m,v'_{m'})$. Since $(u_1,u'_1),...,(u_m,u'_m)$ and $(v_1,v'_1),...,(v'_m,v'_{m'})$ were distinct decompositions of $(w,w')$, in this way we will eventually reach some subsequences whose first elements are different; once we have, we can relabel them so that $|u_1| < |v_1|$.

Then by definition,

$$u_1 \otimes \Big(\bigotimes_{i=2}^{m} u_i\Big) \;=\; w \;=\; v_1 \otimes \Big(\bigotimes_{i=2}^{m'} v_i\Big)$$

But $w, u_1, v_1, \bigotimes_{i=2}^{m} u_i, \bigotimes_{i=2}^{m'} v_i$ are all elements of $\mathbb{N}^{*n}$, which is a free monoid, and so they each have a unique decomposition as products of the generators $\{z_1,...,z_n\}$, and these all respect tensor products. Therefore, since $|u_1| < |v_1|$, there must exist some element $a$ of $\mathbb{N}^{*n}$ such that

$$w \;=\; u_1 \otimes a \otimes \Big(\bigotimes_{i=2}^{m'} v_i\Big) \quad \Longrightarrow \quad v_1 \;=\; u_1 \otimes a$$

Since
$$|u_1'| = |u_1| < |v_1| = |v_1'|$$
we can also use exactly the same reasoning to find an $a'$ in $\mathbb{N}^{*n}$ with $v_1' = u_1' \otimes a'$, and hence $(v_1, v_1') = (u_1, u_1') \otimes (a, a')$. Moreover, this $(a, a')$ is an element of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$, because

$$\begin{array}{rrcccl} & v_1 & = & u_1 \otimes a & & \\ \Longrightarrow & [v_1] & = & [u_1 \otimes a] & = & [u_1] + [a] \\ & & & & & \\ & v_1' & = & u_1' \otimes a' & & \\ \Longrightarrow & [v_1'] & = & [u_1' \otimes a'] & = & [u_1'] + [a'] \\ & & & & & \\ \Longrightarrow & [a] & = & [v_1] - [u_1] & & \\ & & & [v_1'] - [u_1'] & = & [a'] \end{array}$$

In other words, we have shown that the pair $\big((u_1, u_1')(a, a')\big)$ is an element of $d(v_1, v_1')$. But by assumption $(v_1, v_1'), ..., (v_m', v_{m'}')$ was a decomposition of $(w, w')$, and hence the $d(v_i, v_i')$ were supposed to be empty for each $i$, since that is when the decomposition finding process terminates. This is a contradiction, and hence our assumption that $(u_1, u_1'), ..., (u_m, u_m')$ and $(v_1, v_1'), ..., (v_m', v_{m'}')$ were distinct decompositions of $(w, w')$ is false. Therefore, each $(w, w')$ in $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ has a unique decomposition in terms of elements $(v_i, v_i')$ for which $d(v_i, v_i')$ is empty, and so $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ is the free monoid whose generators are all such elements. □

It follows immediately from this that our contruction of an injective function $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \to G \times_{\mathbb{N}} \mathbb{N}^{*n}$ can be extended to be an inclusion of monoids.

**Proposition 3.11.** *$(s \times t)(\mathbb{G}_n)$ is (isomorphic to) a submonoid of* $\mathrm{Mor}(\mathbb{G}_n)$

*Proof.* For each generator $(z, z')$ of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$, choose an element of $g_{(z,z')} \in G(|z|)$ with the property that $\pi(g_{(z,z')})(z) = z'$. This is always possible, since $(z, z') \in \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ implies that the words $z, z' \in \mathbb{N}^{*n}$ are permuations of each other, and the maps $\pi : G(m) \to \mathrm{S}_m$ are always surjective. Then we can define the homomorphism $i$ to be

$$\begin{array}{rcccl} i & : & \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} & \to & G \times_{\mathbb{N}} \mathbb{N}^{*n} \\ & : & (z, x') & \mapsto & (\, g_{(z,z')}, z \,) \end{array}$$

This is well-defined monoid homomorphism, because by Proposition 3.10 $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ is free. Moreover, for any two generators $(z_1, z_1'), (z_2, z_2')$, we have

$$(\, g_{(z_1,z_1')}, z_1 \,) \;=\; (\, g_{(z_2,z_2')}, z_2 \,) \;\Longrightarrow\; \begin{array}{rcl} g_{(z_1,z_1')} & = & g_{(z_2,z_2')} \\ z_1 & = & z_2 \\ z_1' & = & \pi(g_{(z_1,z_1')})(z_1) \\ & = & \pi(g_{(z_2,z_2')})(z_2) \\ & = & z_2' \end{array}$$

and thus $i$ is injective. Therefore the image of $i$ is a submonoid of $G \times_{\mathbb{N}} \mathbb{N}^{*n}$ which is isomorphic to $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$, which we can see as a submonoid of $\mathrm{Mor}(\mathbb{G}_n)$ isomorphic to $(s \times t)(\mathbb{G}_n)$, as required. □

So now we know that $(s \times t)(\mathbb{G}_{2n}) \subseteq \mathrm{Mor}(\mathbb{G}_{2n})$ and $(s \times t)(\mathbb{G}_{4n}) \subseteq \mathrm{Mor}(\mathbb{G}_{4n})$. If we want to use the map $q$ to turn these into the statement we actually care about, $(s \times t)(L\mathbb{G}_n) \subseteq \mathrm{Mor}(L\mathbb{G}_n)$, then we need to show that the coequalizer description of $L\mathbb{G}_n$ interacts nicely with the map $s \times t$.

**Proposition 3.12.** *The homomorphism* $(s\times t)(q) : (s\times t)(\mathbb{G}_{2n}) \to (s\times t)(L\mathbb{G}_n)$ *is the coequalizer of the maps* $(s\times t)(\delta) : (s\times t)(\mathbb{G}_{4n}) \to (s\times t)(\mathbb{G}_{2n})$ *and* $(s\times t)(\zeta) : (s\times t)(\mathbb{G}_{4n}) \to (s\times t)(\mathbb{G}_{2n})$ *in the category of monoids.*

*Proof.* The maps $(s\times t)(\delta)$ and $(s\times t)(\zeta)$ are given by

First, consider the monoid homomorphisms $\mathrm{Ob}(\delta), \mathrm{Ob}(\zeta) : \mathrm{Ob}(\mathbb{G}_{4n}) \to \mathrm{Ob}(\mathbb{G}_{4n})$. We know from Proposition 2.10 that Ob is a left adjoint functor, and hence that it must preserves colimits. Therefore

$$\mathrm{coeq}\big(\,\mathrm{Ob}(\delta), \mathrm{Ob}(\zeta)\,\big) \;=\; \mathrm{Ob}\big(\,\mathrm{coeq}(\delta,\zeta)\,\big) \;=\; \mathrm{Ob}(q)$$

Similarly, for $\pi_0(\delta), \pi_0(\zeta) : \pi_0(\mathbb{G}_{4n}) \to \pi_0(\mathbb{G}_{4n})$ it follows from Proposition 2.15 that

$$\mathrm{coeq}\big(\,\pi_0(\delta), \pi_0(\zeta)\,\big) \;=\; \pi_0\big(\,\mathrm{coeq}(\delta,\zeta)\,\big) \;=\; \pi_0(q)$$

From this, along with functoriality of $\delta$, $\zeta$ and $q$, we can conclude that every square in the diagram below commutes:

$$\begin{tikzcd}
\mathrm{Ob}(\mathbb{G}_{4n}) \arrow[dd, shift right, "\mathrm{Ob}(\delta)"'] \arrow[dd, shift left, "\mathrm{Ob}(\zeta)"] \arrow[dr, "{[\,\text{-}\,]}"] & & \mathrm{Ob}(\mathbb{G}_{4n}) \arrow[dd, shift right, "\mathrm{Ob}(\delta)"'] \arrow[dd, shift left, "\mathrm{Ob}(\zeta)"] \arrow[dl, "{[\,\text{-}\,]}"'] \\
& \pi_0(\mathbb{G}_{4n}) \arrow[dd, shift right, "\pi_0(\delta)"'] \arrow[dd, shift left, "\pi_0(\zeta)"] & \\
\mathrm{Ob}(\mathbb{G}_{2n}) \arrow[dd, "\mathrm{Ob}(q)"'] \arrow[dr, "{[\,\text{-}\,]}"'] & & \mathrm{Ob}(\mathbb{G}_{2n}) \arrow[dd, "\mathrm{Ob}(q)"] \arrow[dl, "{[\,\text{-}\,]}"] \\
& \pi_0(\mathbb{G}_{2n}) \arrow[dd, "\pi_0(q)"] & \\
\mathrm{Ob}(L\mathbb{G}_n) \arrow[dr, "{[\,\text{-}\,]}"'] & & \mathrm{Ob}(L\mathbb{G}_n) \arrow[dl, "{[\,\text{-}\,]}"] \\
& \pi_0(L\mathbb{G}_n) &
\end{tikzcd}$$

□

**Proposition 3.13.** $(s\times t)(L\mathbb{G}_n)$ *is a submonoid of* $\mathrm{Mor}(L\mathbb{G}_n)$

*Proof.* □

**Proposition 3.14.**

$$\mathrm{Mor}(L\mathbb{G}_n) \quad = \quad \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \times \left. \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \middle/ \mathbb{Z}^n \right.$$

*Proof.*

$$\mathrm{Mor}(L\mathbb{G}_n) \;=\; L\mathbb{G}_n(I,I) \times (s\times t)(L\mathbb{G}_n)$$

□

Comment about $(s\times t)(L\mathbb{G}_n)^{\mathrm{ab}} = \mathrm{Ob}(L\mathbb{G}_n)^{\mathrm{ab}}$, or more explicitly $\langle \mathrm{id} \rangle$.

3.3. **$L\mathbb{G}_n$ as a reflexive coequalizer.** Since left adjoint functor preserve colomits, Propositions 2.10 and 2.15 both imply results about the partial surjectivity of this new map $q$. The former says that since $\mathrm{Ob}(q)$ is a coequalizer map of monoids, and hence that every object of $L\mathbb{G}_n$ is the image under $q$ of some object of $\mathbb{G}_{2n}$; the latter says a similar thing for connected components. From this one might guess that $q$ is will just turn out to be a surjective map of E$G$-algebras, and indeed this is the case. Moreover, much as Propositions 2.12 and 2.16 are analogues of Lemma 1.18 and Proposition 1.20 respectively, the fact that $q$ is surjective on morphisms means that there is a result analagous to Lemma 1.19 as well. That is, since every morphism of $\mathbb{G}_{2n}$ is an action morphism, and since E$G$-algebra maps always send action morphisms to action morphisms, if $q$ is surjective then every morphism of $L\mathbb{G}_n$ is also an action morphism.

However, the proof of this is not so simple. Unfortunately, we can not go about proving that $q$ is surjective on morphisms by using the adjunction from Proposition 2.19, since this will only tell us about the map $\mathrm{Mor}(q)^{gp,ab}$. Without this, it is not obvious how the morphisms of some algebras should be related to the morphisms of their coequalizer in this way. In particular, for it to make sense that $q$ is surjective, we would need the image of $q$ to be closed under composition, since it is supposed to be the whole algebra $L\mathbb{G}_n$. Again by Lemma 1.19, this is equivalent to saying that the set of all action morphisms of $L\mathbb{G}_n$ would have to be closed under composition, a statement which is not at all obvious. In the case of $\mathbb{G}_{2n}$, we know that the action morphism are closed because maps $\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$, $\alpha(g'; \mathrm{id}_{x'_1}, ..., \mathrm{id}_{x'_m})$, $x_i, x'_i \in \{z_1, ..., z_{2n}\}$ are composable only if the target of the first is equal to the source of the second, and hence

$$\begin{array}{rrcll} & x_{\pi(g^{-1})(1)} \otimes ... \otimes x_{\pi(g^{-1})(m)} & = & x'_1 \otimes ... \otimes x'_m, & x_i, x'_i \in \{z_1, ..., z_{2n}\} \\ \Longrightarrow & x_{\pi(g^{-1})(i)} & = & x'_i, & 1 \leq i \leq m \\ \Longrightarrow & \alpha(gg'; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) & = & \alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) \circ \alpha(g'; \mathrm{id}_{x'_1}, ..., \mathrm{id}_{x'_m}) & \end{array}$$

However, in $L\mathbb{G}_n$ this line of reasoning does not work, because for example

$$z_1 \otimes z_1^* \quad = \quad I \quad = \quad z_2 \otimes z_2^*$$

but this does not imply that $z_1 = z_2$.

The key to soving this problem is noticing that the maps $\delta$ and $\zeta$ form a reflexive pair — parallel maps which share a right-inverse.

**Lemma 3.15.** *Let $\iota : \mathbb{G}_{2n} \to \mathbb{G}_{4n}$ be the inclusion defined on generators by $z_i \mapsto z_i$. Then $\iota$ is a right-inverse of both $\delta$ and $\zeta$.*

*Proof.* For $1 \leq i \leq 2n$,

$$\begin{array}{rcccl} \delta\iota(z_i) & = & \delta(z_i) & = & z_i \\ \zeta\iota(z_i) & = & \zeta(z_i) & = & z_i \end{array}$$

$$\Longrightarrow \quad \delta \circ \iota \quad = \quad \mathrm{id}_{\mathbb{G}_{2n}} \quad = \quad \zeta \circ \iota$$

□

In other words, $q$ is a reflexive coequalizer. Using this property of $q$, we will eventually be able to prove that the image of $q$ is closed under composition. First though, we need some intermediate results.

**Definition 3.16.** Let $x$ be an object of $\mathbb{G}_m$ for some $m \in \mathbb{N}$, and hence an element of $\mathbb{N}^{*m}$, the free monoid on $m$ generators $\{z_1, ..., z_m\}$. Then by the definition of the free product of monoids, $x$ will have a unique decomposition of the form

$$x = \bigotimes_{i=1}^{|x|} g(x,i)$$

where the upper bound $|x|$ is the length of the element $x$ as in Definition 3.7, and each $g(x,i)$ is a generator of one of the $m$ copies of $\mathbb{N}$, so $g(x,i) \in \{z_1, ..., z_m\}$.

Note that this decomposition shows that the length of $x$ is just the sum of all of the coordinates of its connected component, $[x] \in \mathbb{N}^m$:

**Lemma 3.17.**

$$|x| = \sum_{i=1}^{m} [x]_i$$

*Proof.* For any generator $z_j$ its connected component has coordinates

$$[z_j]_i = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{otherwise} \end{cases}$$

and hence

$$\begin{aligned} \textstyle\sum_{i=1}^{m} [x]_i &= \textstyle\sum_{i=1}^{m} \left[ \bigotimes_{j=1}^{|x|} g(x,j) \right]_i \\ &= \textstyle\sum_{i=1}^{m} \sum_{j=1}^{|x|} [\, g(x,j)\, ]_i \\ &= \textstyle\sum_{j=1}^{|x|} \sum_{i=1}^{m} [\, g(x,j)\, ]_i \\ &= \textstyle\sum_{j=1}^{|x|} 1 \\ &= |x| \end{aligned}$$

□

**Proposition 3.18.** *Let $w$ be an object of $\mathbb{G}_{2n}$. Then there exist objects $w_1, ..., w_k$ in $\mathbb{G}_{2n}$ and $u_1, ..., u_k$ in $\mathbb{G}_{4n}$, for some value of $k \in \mathbb{N}$, such that*

$$w_1 = w, \qquad \zeta(u_{i-1}) = w_i = \delta(u_i), \qquad u_k = \iota(w_k)$$

*for $1 \le i \le k$, and for any object $u$ of $\mathbb{G}_{4n}$,*

$$\delta(u) = w_k \quad \Longrightarrow \quad u = u_k$$

*Proof.* From Definitions 3.1 and 3.7, we know that for any generator $z_i$ of $\mathbb{G}_{4n}$,

$$|\delta(z_i)| = \left\{ \begin{matrix} 1 & \text{if} & 1 \le i \le 2n \\ 2 & \text{if} & 2n+1 \le i \le 4n \end{matrix} \right\} \ge 1$$

$$|\zeta(z_i)| = \left\{ \begin{matrix} 1 & \text{if} & 1 \le i \le 2n \\ 0 & \text{if} & 2n+1 \le i \le 4n \end{matrix} \right\} \le 1$$

Also these lengths are additive across tensor products, since $|\_|$ is a monoid homomorphism $\mathbb{G}_{2n} \to \mathbb{N}$. Thus for any object $u$ in $\mathbb{G}_{4n}$, we can conclude that

$$\begin{aligned} |\delta(u)| &= |\,\delta\big(\bigotimes_{i=1}^{|u|} g(u,i)\big)\,| &= |\bigotimes_{i=1}^{|u|} \delta\big(g(u,i)\big)\,| &= \sum_{i=1}^{|u|} |\,\delta\big(g(u,i)\big)\,| &\ge |u| \\ |\zeta(u)| &= |\,\zeta\big(\bigotimes_{i=1}^{|u|} g(u,i)\big)\,| &= |\bigotimes_{i=1}^{|u|} \zeta\big(g(u,i)\big)\,| &= \sum_{i=1}^{|u|} |\,\zeta\big(g(u,i)\big)\,| &\le |u| \end{aligned}$$

Also, since the only generators that have $|\delta(z_i)| = |\zeta(z_i)| = 1$ are those from the $\mathbb{G}_{2n}$ subalgebra associated with $\iota$, the inequality above becomes an equality if and only if $u$ is in the image of $\iota$. That is,

$$|\zeta(u)| = |u| = |\delta(u)| \quad \Longleftrightarrow \quad \exists\, v \in \mathbb{N}^{*2n} \,:\, u = \iota(v)$$

Next, consider the set

$$\delta^{-1}(w) = \{\, u \in \mathrm{N}^{*4n} : \delta(u) = w \,\}$$

of all objects in $\mathbb{G}_{4n}$ which $\delta$ sends to $w$. This set is always nonempty, since by Lemma 3.15 $\iota$ is a right-inverse of $\delta$:

$$\delta\iota(w) = w \quad \Longrightarrow \quad \iota(w) \in \delta^{-1}(w)$$

Moreover, $\iota(w)$ is the only element of $\delta^{-1}(w)$ which can be expessed as $\iota(v)$ for some object $v$ in $\mathbb{G}_{2n}$, because

$$\delta\big(\,\iota(v)\,\big) = w \quad \Longrightarrow v = w$$

With all of this now in place, we can begin constructing the sequences $w_1, ..., w_k$ and $u_1, ..., u_k$. Start by setting $w_1 = w$ and $i = 1$, then apply the following algorithm:

(1) If $\delta^{-1}(w_i)$ is just the set $\{\iota(w_i)\}$, choose $u_i = \iota(w_i)$, set $k$ to be the current value of $i$, and terminate.
(2) Otherwise, choose $u_i$ to be any element of $\delta^{-1}(w_i)$ other than $\iota(w_i)$.
(3) Set $w_{i+1} = \zeta(u_i)$.
(4) Increase the value of $i$ by 1, then return to step 1.

By design, none of the $u_i$ produced by this process can be expressed as $u_i = \iota(v)$ for some $v$ in $\mathbb{G}_{2n}$, with the possible exception of $u_k$ if the algorithm terminates. This is because $\iota(w_i)$ is the only element of $\delta^{-1}(w_i)$ that can be expressed that way, and the above process will terminate the first time it has to pick $u_i = \iota(w_i)$, at which point $i$ is set equal to $k$. Therefore, for any $i \neq k$, we will have the following strict inequalities:

$$|w_{i+1}| = |\zeta(u_i)| < |u_i| < |\delta(u_i)| = |w_i|$$

That is, the $w_i$ generated by this algorithm form a sequence with strictly decreasing length. However, it is impossible to have a infinite sequence of strictly decreasing natural numbers, and hence we can be sure that this process will terminate at some finite $k$.

But in order for the algorithm to have terminate, it must be the case that

$$\delta^{-1}(w_k) = \{\iota(w_k)\}$$

and hence

$$\delta(u) = w_k \quad \Longrightarrow \quad u = \iota(w_k) = u_k$$

Thus the sequences $w_1, ..., w_k$ and $u_1, ..., u_k$ satisfy all of the conditions in the statement of the lemma. □

The intuition behind Proposition 3.18 is that we are successively removing parts of the object $w$, without changing its image under $q$. The map $\delta$ sends $z_{2n+i} \mapsto z_i \otimes z_{n+i}$ and $z_{3n+i} \mapsto z_{n+1} \otimes z_i$ while $\zeta$ sends these all to $I$, and so for any $u$ in $\mathbb{G}_{4n}$ the object $\zeta(u)$ will look like $\delta(u)$ except missing some number of $z_i \otimes z_{n+i}$ or $z_{n+1} \otimes z_i$ substrings. But since $q$ sends $z_{n+i} \mapsto z_i^*$, these are exactly the sort of omissions which the coequaliser doesn't care about. If we repeat this process then it will eventually terminate at $u_k = \iota(w_k)$, so we really have a method for removing *all* of the relevent substrings from objects of $\mathbb{G}_{2n}$. In other words, $w_k$ has the smallest possible length while still having $q(w_k) = q(w)$. In fact, we will show that it is the unique shortest object of $\mathbb{G}_{2n}$ with this property.

**Definition 3.19.** Let $x$ be an object of $L\mathbb{G}_m$ for some $m \in \mathbb{N}$, and hence an element of $\mathbb{Z}^{*m}$, the free monoid on $m$ invertible generators $\{z_1, ..., z_m\}$. Then as in Definition 3.16 we can use the definition of the free product of monoids to write $x$ uniquely in the form

$$x = \bigotimes_{i=1}^{|x|} g(x,i)$$

This time each $g(x,i)$ is either a generator of one of the copies of $\mathbb{N}$ or the inverse of a generator, $g(x,i) \in \{z_1, ..., z_m, z_1^*, ..., z_m^*\}$, and for any $1 \le i < |x|$ we will always have $g(x, i+1) \ne g(x,i)^*$. As before, we will call the upper bound $|x|$ of this tensor product the *length* of the element $x$.

**Proposition 3.20.** *Let $w$, $w'$ be objects of $\mathbb{G}_{2n}$ such that $q(w) = q(w')$. If $w_1, ..., w_k$ and $u_1, ..., u_k$ are sequences generated from $w$ via Proposition 3.18, and likewise $w'_1, ..., w'_{k'}$ and $u'_1, ..., u'_{k'}$ from $w'$, then $w_k = w'_{k'}$ and $u_k = u'_{k'}$.*

*Proof.* Consider the decomposition of the object $w_k \in \mathbb{N}^{*2n}$ as in Definition 3.16. Assume, for the sake of contradiction, that there exist $1 \le j < |w_k|$ and $1 \le m \le n$ such that

$$g(w_k, j) = z_m, \qquad g(w_k, j+1) = z_{n+m}$$

Then we can use $j, m$ to contruct an element $u \in \mathbb{N}^{*4n}$, defined by

$$|u| = |w| - 1, \qquad g(u,i) = \begin{cases} \iota\big(\,g(w_k,i)\,\big) & \text{if} \quad 1 \le i < j \\ z_{2n+m} & \text{if} \quad i = j \\ \iota\big(\,g(w_k, i+1)\,\big) & \text{if} \quad j < i \le |u| \end{cases}$$

This $u$ will then have the property that

$$\begin{aligned}
\delta(u) &= \delta\big(\bigotimes_{i=1}^{|u|} g(u,i)\,\big) \\
&= \bigotimes_{i=1}^{|u|} \delta\big(\,g(u,i)\,\big) \\
&= \bigotimes_{i=1}^{j-1} \delta\iota\big(\,g(w_k,i)\,\big) \otimes \delta(z_{2n+m}) \otimes \bigotimes_{i=j+1}^{|u|} \delta\iota\big(\,g(w_k,i+1)\,\big) \\
&= \bigotimes_{i=1}^{j-1} g(w_k,i) \otimes g(w_k,j) \otimes g(w_k,j+1) \otimes \bigotimes_{i=j+2}^{|u|+1} g(w_k,i) \\
&= w_k
\end{aligned}$$

But this is impossible, since by Proposition 3.18 $u_k$ is the only object of $\mathbb{G}_{4n}$ whose image under $\delta$ is $w_k$, and this $u$ we have constructed is manifestly not $w_k$. Thus we can conclude that there are no values of $j$ and $m$ for which

$$g(w_k, j) = z_m, \qquad g(w_k, j+1) = z_{n+m}$$

An analogous line of reasoning — using $z_{3n+m}$ rather than $z_{2n+m}$ in the definition of $u$ — demonstrates that there are also no $j, m$ with

$$g(w_k, j) \;=\; z_{n+m}, \qquad g(w_k, j+1) \;=\; z_m$$

As a result, for all $1 \leq i < |w_k|$

$$q\big(\, g(w_k, i+1)\,\big) \;\neq\; q\big(\, g(w_k, i)\,\big)^*$$

and this combined with the fact that

$$\bigotimes_{i=1}^{|w_k|} q\big(\, g(w_k, i)\,\big) \;=\; q\big(\bigotimes_{i=1}^{|w_k|} g(w_k, i)\,\big) \;=\; q(w_k)$$

shows that the unique decomposition of $q(w_k)$ as in Definition 3.19 is given by

$$|q(w_k)| \;=\; |w_k|, \qquad g\big(\, q(w_k), i\,\big) \;=\; q\big(\, g(w_k, i)\,\big)$$

Next, let $s$ be a function defined by

$$\begin{array}{cclcl} s & : & \mathbb{Z}^{*n} & \to & \mathbb{N}^{*2n} \\ & : & z_i & \mapsto & z_i \\ & : & z_i^* & \mapsto & z_{n+i} \\ & : & x & \mapsto & \bigotimes_{i=1}^{|x|} s\big(\, g(x, i)\,\big) \end{array}$$

Then for $1 \leq i \leq n$,

$$sq(z_i) \;=\; s(z_i) \;=\; z_i, \qquad sq(z_{n+i}) \;=\; s(z_i^*) \;=\; z_{n+i}$$

and so it follows that

$$\begin{aligned} sq(w_k) \;&=\; \bigotimes_{i=1}^{|w_k|} s\Big(\, g\big(\, q(w_k), i\,\big)\Big) \\ &=\; \bigotimes_{i=1}^{|w_k|} sq\big(\, g(w_k, i)\,\big) \\ &=\; \bigotimes_{i=1}^{|w_k|} g(w_k, i) \\ &=\; w_k \end{aligned}$$

Finally, notice that the exact same logic as we've used above will also work for $w'_{k'}$, so that $sq(w'_{k'}) = w'_{k'}$. Therefore,

$$\begin{aligned}
w_k &= sq(w_k)\\
&= sq\zeta(u_{k-1})\\
&= sq\delta(u_{k-1})\\
&= sq(w_{k-1})\\
&\vdots\\
&= sq(w)\\
&= sq(w')\\
&= sq\delta(u'_1)\\
&= sq\zeta(u'_1)\\
&= sq(w'_2)\\
&\vdots\\
&= sq(w'_{k'})\\
&= w'_{k'}
\end{aligned}$$

as required. □

Next, we need to take things one step further. We've already shown that we can find an object $w_k$ which is like $w$ but with all $z_i \otimes z_{n+i}$ and $z_{n+i} \otimes z_i$ substrings removed. In order to prove the closure of $\mathrm{im}(q)$, we will need to construct a morphism of $\mathbb{G}_{2n}$ which witnesses this removal — that is, a map $h : w \to v \otimes w_k$ where $v$ is made up of all of the $z_i \otimes z_{n+i}$ that need to be taken out of $w$ to make $w_k$. Likewise, the fact that $q(w_k) = q(w)$ will be embodied by the fact that this $h$ becomes an identity morphism under $q$. This map $h$ will need to be built out of smaller pieces, using the sequence $u_1, ..., u_k$ and the following proposition:

**Proposition 3.21.** *Let $u$ be an object of $\mathbb{G}_{4n}$. Then there exists another object $x$ in $\mathbb{G}_{4n}$ and a morphism $h : \delta(u) \to \delta(x) \otimes \zeta(u)$ in $\mathbb{G}_{2n}$ such that $q(h) = \mathrm{id}_{q\delta(u)}$.*

*Proof.* First, consider the object $\iota\zeta(u) \in \mathbb{N}^{*4n}$. Because $\zeta$ acts by sending the generator $z_{2n+i}$ to $I$ for each $1 \le i \le n$, and since $\iota$ is an inclusion, the connected component $[\iota\zeta(u)] \in \mathbb{N}^{4n}$ will have $i$th coordinate

$$[\,\iota\zeta(u)\,]_i \;=\; [\,\zeta(u)\,]_i \;=\; \begin{cases} [u]_i & \text{if} \quad 1 \le i \le 2n \\ 0 & \text{if} \quad 2n+1 \le i \le 4n \end{cases}$$

Intuitively, $\iota\zeta(u)$ is what would be left if we were to remove all of the $z_{2n+i}$ generators from $u$. Conversely, the object $x \in \mathbb{G}_{4n}$ defined as

$$x \;=\; {z_{2n+1}}^{\otimes [u]_{2n+1}} \otimes \ldots \otimes {z_{4n}}^{\otimes [u]_{4n}}$$

is like a reordered list of all of the $z_{2n+i}$ that we would have to remove from $u$ in order to make $\iota\zeta(u)$, and will have the property that

$$[x]_i \;=\; \begin{cases} 0 & \text{if} \quad 1 \le i \le 2n \\ [u]_i & \text{if} \quad 2n+1 \le i \le 4n \end{cases}$$

Thus the object $x \otimes \iota\zeta(u)$ is simply a reordered version of $u$, or more concretely

$$[\,x \otimes \iota\zeta(u)\,]_i \;=\; [x]_i + [\,\iota\zeta(u)\,]_i \;=\; \left.\begin{cases} 0 + [u]_i & \text{if} \quad 1 \le i \le 2n \\ [u]_i + 0 & \text{if} \quad 2n+1 \le i \le 4n \end{cases}\right\} \;=\; [u]_i$$

for all $i$, and hence $u$ and $x \otimes \iota\zeta(u)$ are part of the same connected component.

Knowing this, we can now choose an arbitrary morphism $f : u \to x \otimes \iota\zeta(u)$ from $\mathbb{G}_{4n}$. The map $\zeta(f)$ will then have source $\zeta(u)$ and target

$$\zeta\big(\, x \otimes \iota\zeta(u) \,\big) \;=\; \zeta(x) \otimes \zeta\iota\zeta(u) \;=\; \zeta(u)$$

because $\iota$ is a right inverse of $\zeta$, and $x$ was defined in such a way that $\zeta(x) = I$. It follows that $\iota\zeta(f)$ is a map $\iota\zeta(u) \to \iota\zeta(u)$, and therefore it is possible to form the composite

$$u \xrightarrow{\;f\;} x \otimes \iota\zeta(u) \xrightarrow{\;\mathrm{id}_x \otimes \iota\zeta(f)^{-1}\;} x \otimes \iota\zeta(u)$$

which we'll call $g$. Effectively, what $g$ does is to first apply the map $f$, and then 'undo' its effect on the generators $z_1, ..., z_{2n}$, while leaving the $z_{2n+1}, ..., z_{3n}$ in $x$ untouched. It should not be surprising then that if we take the image of $g$ under $\zeta$, we get

$$\begin{array}{rcl} \zeta(g) & = & \zeta\Big(\, \big(\, \mathrm{id}_x \otimes \iota\zeta(f)^{-1} \,\big) \circ f \,\Big) \\ & = & \big(\, \zeta(\mathrm{id}_x) \otimes \zeta\iota\zeta(f)^{-1} \,\big) \circ \zeta(f) \\ & = & \zeta(f)^{-1} \circ \zeta(f) \\ & = & \mathrm{id}_{\zeta(u)} \end{array}$$

Finally, consider the morphism $\delta(g)$. This has source $\delta(u)$ and target

$$\delta\big(\, x \otimes \iota\zeta(u) \,\big) \;=\; \delta(x) \otimes \delta\iota\zeta(u) \;=\; \zeta(u)$$

because $\iota$ is also the right inverse of $\delta$, and so we can choose $\delta(g)$ to be the map $h$ that we are looking for. If we do, then since $q$ is the coequalizer of $\delta$ and $\zeta$ we'll have

$$q(h) \;=\; q\delta(g) \;=\; q\zeta(g) \;=\; q(\mathrm{id}_{\zeta(u)}) \;=\; \mathrm{id}_{q\zeta(u)} \;=\; \mathrm{id}_{q\delta(u)}$$

as required. $\square$

Hopefully it is clear that nothing in the proof of Proposition 3.20 really depended on the order in which we tensored together $\delta(x) \otimes \zeta(u)$. Consequently, the same basic proof will also work when this order is reversed:

**Corollary 3.22.** *Let $u$ be an object of $\mathbb{G}_{4n}$. Then there exists another object $x$ in $\mathbb{G}_{4n}$ and a morphism $h : \delta(u) \to \zeta(u) \otimes \delta(x)$ in $\mathbb{G}_{2n}$ such that $q(h) = \mathrm{id}_{q\delta(u)}$.*

Now we finally have enough results in place to show that the image of $q$ is closed under composition.

**Proposition 3.23.** *Let $f : v \to w$ and $f' : w' \to v'$ be two morphisms of $\mathbb{G}_{2n}$ such that $q(f)$ and $q(f')$ are composable. Then there exist objects $y, y'$ and a morphism $h : y \to y'$ in $\mathbb{G}_{2n}$ such that $q(h) = q(f') \circ q(f)$.*

*Proof.* To begin, we'll use Proposition 3.18 to contruct from $w$ the sequences of objects $w_1, ..., w_k$ in $\mathbb{G}_{2n}$ and $u_1, ..., u_k$ in $\mathbb{G}_{4n}$, for some $k \in \mathbb{N}$. These have

$$w_1 \;=\; w, \qquad \zeta(u_{i-1}) \;=\; w_i \;=\; \delta(u_i), \qquad \iota(w_k) \;=\; u_k$$

Then we can apply Proposition 3.21 to each $u_i$ in turn, producing a new sequence of object $x_1, ..., x_{k-1}$ in $\mathbb{G}_{4n}$ and morphisms

$$\begin{array}{rcrcl} h_i & : & \delta(u_i) & \to & \delta(x_i) \otimes \zeta(u_i) \\ & : & w_i & \to & \delta(x_i) \otimes w_{i+1} \end{array}$$

$$\begin{tikzcd}
v \otimes \delta(x'_{k'-1}) \otimes \ldots \otimes \delta(x'_1) \arrow[d, "f \otimes \mathrm{id}"'] \\
w \otimes \delta(x'_{k'-1}) \otimes \ldots \otimes \delta(x'_1) \arrow[d, "h_1 \otimes \mathrm{id}"'] \\
\delta(x_1) \otimes w_2 \otimes \delta(x'_{k'-1}) \otimes \ldots \otimes \delta(x'_1) \arrow[d, "\mathrm{id} \otimes h_2 \otimes \mathrm{id}"'] \\
\delta(x_1) \otimes \delta(x_2) \otimes w_3 \otimes \delta(x'_{k'-1}) \otimes \ldots \otimes \delta(x'_1) \arrow[d, "\mathrm{id} \otimes h_3 \otimes \mathrm{id}"'] \\
\vdots \arrow[d, "\mathrm{id} \otimes h_{k-1} \otimes \mathrm{id}"'] \\
\delta(x_1) \otimes \ldots \otimes \delta(x_{k-1}) \otimes w_k \otimes \delta(x'_{k'-1}) \otimes \ldots \otimes \delta(x'_1) \arrow[d, equal] \\
\delta(x_1) \otimes \ldots \otimes \delta(x_{k-1}) \otimes w'_{k'} \otimes \delta(x'_{k'-1}) \otimes \ldots \otimes \delta(x'_1) \arrow[d, "\mathrm{id} \otimes h'_{k'-1} \otimes \mathrm{id}"] \\
\vdots \arrow[d, "\mathrm{id} \otimes h'_1"] \\
\delta(x_1) \otimes \ldots \otimes \delta(x_{k-1}) \otimes w' \arrow[d, "\mathrm{id} \otimes f'"] \\
\delta(x_1) \otimes \ldots \otimes \delta(x_{k-1}) \otimes v'
\end{tikzcd}$$

FIGURE 1. The composite $h$ from Proposition 3.23

for each $1 \leq i < k$, with the properties

$$\zeta(x_i) = I, \qquad q(h_i) = \mathrm{id}_{q(w_i)}$$

We can do a similar thing with $w'$, first using Proposition 3.18 to get sequences $w'_1, ..., w'_{k'}$ and $u'_1, ..., u'_{k'}$ with the appropriate properties, but this time applying Corollary 3.22 to find objects $x'_1, ..., x'_{k'-1}$ and morphisms $h'_i : w'_i \to w_{i+1} \otimes \delta(x_i)$. Moreover, since the morphisms $q(f) : q(v) \to q(w)$ and $q(f') : q(w') \to q(v')$ are-composable, it must be the case that $q(w)$ and $q(w')$ are equal. By Proposition 3.20, this means that $w_k = w'_{k'}$

Putting all of this together, we can form the composite morphism shown in Figure 1. Here the subscripts of the identity maps have been suppressed for clarity. This morphism will be our choice of $h : y \to y'$. To complete the proof, notice that

$$\begin{array}{rccccccccc} & q\delta(x_i) & = & q\zeta(x_i) & = & I & = & q\zeta(x'_i) & = & q\delta(x'_i) \\ \Longrightarrow & q(\mathrm{id}_{\delta(x_i)}) & = & \mathrm{id}_{q\delta(x_i)} & = & \mathrm{id}_I & = & \mathrm{id}_{q\delta(x'_i)} & = & q(\mathrm{id}_{\delta(x'_i)}) \end{array}$$

for each value of $i$, and so when we take the image of the composite $h$ under $q$ all of the $\mathrm{id}_{\delta(x_i)}$ and $\mathrm{id}_{\delta(x'_i)}$ terms will cancel out, leaving just

$$\begin{array}{rcl} q(h) & = & q(f') \circ q(h'_1) \circ ... \circ q(h'_{k'-1}) \circ q(h_{k-1}) \circ ... \circ q(h_1) \circ q(f) \\ & = & q(f') \circ \mathrm{id}_{q(w)} \circ ... \circ \mathrm{id}_{q(w)} \circ \mathrm{id}_{q(w)} \circ ... \circ \mathrm{id}_{q(w)} \circ q(f) \\ & = & q(f') \circ q(f) \end{array}$$

as required. □

Finally, we can return to how we began this section. We wished to show that the map $q$ is surjective, and from this conclude that all morphisms of $L\mathbb{G}_n$ are action morphisms. However, it was not obvious that such a statement would even make sense, since the image of $q$ would consist entirely of morphisms of the form $\alpha_{L\mathbb{G}_n}(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$, and these are not a priori closed under composition. But with Proposition 3.23 we now know that the image of $q$ is indeed closed, and we can immediately use this to demonstrate surjectivity.

**Proposition 3.24.** *The quotient map $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$ is surjective. Therefore, every morphism in $L\mathbb{G}_n$ can be expressed as $\alpha_{L\mathbb{G}_n}(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$ for some $g \in G(m)$ and $x_i \in \{z_1, ..., z_n, z_1^*, ..., z_n^*\}$.*

*Proof.* Consider $q(\mathbb{G}_{2n})$, the subcategory of $L\mathbb{G}_n$ that contains every object $x'$ for which there exists $x$ in $\mathbb{G}_{2n}$ with $q(x) = x'$, and every morphism $f'$ for which there exists $f$ in $\mathbb{G}_{2n}$ with $q(f) = f'$. By Proposition 3.23 the morphisms of $q(\mathbb{G}_{2n})$ are closed under composition, and so this is a well-defined category. Moreover, since $q$ is a map of E$G$-algebras then for any morphisms $f'_1, ..., f'_m$ of $q(\mathbb{G}_{2n})$ we'll have

$$\begin{array}{rcl} \alpha_{L\mathbb{G}_n}(g; f'_1, ..., f'_m) & = & \alpha_{L\mathbb{G}_n}\big(\, g\,;\, q(f_1), ..., q(f_m)\,\big) \\ & = & q\big(\, \alpha_{\mathbb{G}_{2n}}(g; f_1, ..., f_m)\,\big) \\ & \in & q(\mathbb{G}_{2n}) \end{array}$$

Thus $q(\mathbb{G}_{2n})$ is also a well-defined sub-E$G$-algebra of $L\mathbb{G}_n$.

Next, let $q' : \mathbb{G}_{2n} \to q(\mathbb{G}_{2n})$ be the unique surjective map with the property that $q'(x) = q(x)$ for any object $x$ and $q'(f) = q(f)$ for any morphism $f$. Also denote by

$i$ the evident inclusion of algebras $q(\mathbb{G}_{2n}) \hookrightarrow L\mathbb{G}_n$, so that for instance $i \circ q' = q$.

$$\begin{tikzcd}
& \mathbb{G}_{4n} \arrow[d, bend right, "\delta"'] \arrow[d, bend left, "\zeta"] & \\
& \mathbb{G}_{2n} \arrow[dl, "q'"'] \arrow[d, "q"] \arrow[dr, "p"] & \\
q(\mathbb{G}_{2n}) \arrow[r, hook, "i"] & L\mathbb{G}_n \arrow[r, "u"] & X
\end{tikzcd}$$

Further, let $p : \mathbb{G}_{2n} \to X$ be any map of E$G$-algebras with the property that $p \circ \delta = p \circ \zeta$. Since $q$ is the coequalizer of $\delta$ and $\zeta$, there will then exist a unique map $u : L\mathbb{G}_n \to X$ such that $p = u \circ q$. This means that $p = u \circ i \circ q'$, and hence there is obviously at least one map, $u \circ i$, which lets us factors $p$ through $q'$. But for any other map $v : q(\mathbb{G}_{2n}) \to X$ that factors $p$ like this, we have

$$\begin{array}{rrcl} & v \circ q' & = & p \\ & & = & u \circ i \circ q' \\ \Longrightarrow & v & = & u \circ i \end{array}$$

because $q'$ is surjective, and thus $u \circ i$ is the unique map with this property. That is, $q'$ is also a coequalizer of $\delta$ and $\zeta$. But colimits are always unique up to a unique isomorphism, and so there should be a unique invertible map $j : q(\mathbb{G}_{2n}) \to L\mathbb{G}_n$ such that $q = j \circ q'$. This is clearly just the inclusion $i$, and as a result $q(\mathbb{G}_{2n}) = L\mathbb{G}_n$ and $q' = q$. In other words, $q$ is surjective.

Finally, let $f'$ be an arbitrary morphism in $L\mathbb{G}_n$. By surjectivity there exists at least one morphism $f$ in $\mathbb{G}_{2n}$ such that $q(f) = f'$, and from Lemma 1.19 we know that this $f$ can be expressed uniquely as $\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$ for some $g \in G(m)$ and $x_i \in \{z_1, ..., z_{2n}\}$. Thus, because $q$ is a map of E$G$-algebras, we will have

$$f' = q(f) = q\big(\alpha_{\mathbb{G}_{2n}}(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})\big) = \alpha_{L\mathbb{G}_n}(g; \mathrm{id}_{q(x_1)}, ..., \mathrm{id}_{q(x_m)})$$

But by Lemma 3.2, for each generator $x_i \in \{z_1, ..., z_{2n}\}$ the object $q(x_i)$ is either one of the generators $z_1, ..., z_n$ of $L\mathbb{G}_n$ or one of their inverse $z_1^*, ..., z_n^*$. Therefore, there is at least one collection of $x_i' = q(x_i)$ for which the statement of the proposition holds. □

Proposition 3.24 formalises a certain intuition about how the functor $L$ should act on algebras, the idea that a 'free' structure really shouldn't have any 'superfluous' components, only whatever data is absolutely required for it to be well-defined. In the case of $L\mathbb{G}_n$, we have proven that the only morphisms contained in the free E$G$-algebra on invertible objects are E$G$-action morphisms. However, while this is very similar to what we have in the non-invertible case it should be stressed that Proposition 3.24 does *not* prove that the morphisms of $L\mathbb{G}_n$ have *unique* representations $\alpha(g; \mathrm{id}_{w_1}, ..., \mathrm{id}_{w_m})$, as morphisms of $\mathbb{G}_n$ do.

### 3.4. **The morphisms of $L\mathbb{G}_n$.**

**Proposition 3.25.** *Propositions in previous section do not depend on $q$ having any property except $q \circ \delta = q \circ \zeta$.*

*Proof.* □

**Proposition 3.26.** *The coequalizer of* $\delta$ *and* $\zeta$ *in* MonCat *is just* $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$*, their coequalizer in* $EG\mathrm{Alg}_S$.

*Proof.* Let the monoidal functor $c : \mathbb{G}_{2n} \to C$ onto some monoidal category $C$ be the coequalizer of $\delta : \mathbb{G}_{4n} \to \mathbb{G}_{2n}$ and $\zeta : \mathbb{G}_{4n} \to \mathbb{G}_{2n}$ in MonCat. Then we can consider $c(\mathbb{G}_{2n})$, the subcategory of $C$ containing the objects $c(x)$ and morphisms $c(f)$, for each object $x$ and morphism $f$ in $\mathbb{G}_{2n}$. Note that this is indeed a well-defined category, as by **??** the morphisms of $c(\mathbb{G}_{2n})$ are closed under composition.

We can use this subcategory to prove that the map $c$ is surjective, in the same way we did for $q$ in Proposition 3.24. Let $c' : \mathbb{G}_{2n} \to c(\mathbb{G}_{2n})$ be the obvious surjective functor which acts on objects and morphism exactly as $c$ does, $i : c(\mathbb{G}_{2n}) \hookrightarrow C$ the appropriate inclusion, and $d : \mathbb{G}_{2n} \to X$ any monoidal functor for which $d \circ \delta = d \circ \zeta$. Then since $c$ is the coequalizer of $\delta$ and $\zeta$, we will have a unique monoidal functor $u : C \to X$ with the property that $d = u \circ c$.

$$\begin{tikzcd}
 & \mathbb{G}_{4n} \arrow[d, bend right, "\delta"'] \arrow[d, bend left, "\zeta"] & \\
 & \mathbb{G}_{2n} \arrow[dl, "c'"'] \arrow[d, "c"] \arrow[dr, "d"] & \\
c(\mathbb{G}_{2n}) \arrow[r, hook, "i"] & C \arrow[r, "u"] & X
\end{tikzcd}$$

It follows that $d = u \circ i \circ c'$, and hence that there is at least one map, $u \circ i$, which factors $d$ through $c'$. Moreover, $u \circ i$ is the only functor that does this, since for any other $v : c(\mathbb{G}_{2n}) \to X$,

$$\begin{array}{rclcrcl} v \circ c' & = & d & \implies & v \circ c' & = & u \circ i \circ c' \\ & & & \implies & v & = & u \circ i \end{array}$$

by surjectivity of $c'$. Thus $c'$ is also a coequalizer of $\delta$ and $\zeta$, and since colimits are always unique up to unique isomorphism we can conclude that $c(\mathbb{G}_{2n}) = C$, $c' = c$, and that $c$ is surjective.

By cref???, a surjective monoidal functor out of an $EG$-algebra will always induce an algebra structure upon its target category, such that the original functor becomes a map of algebras between them. Thus it follows immediately that any coequalizer $c : \mathbb{G}_{2n} \to C$ of $\delta$ and $\zeta$ in MonCat is also a map in $EG\mathrm{Alg}_S$. So $c$ is an algebra map with $c \circ \delta = c \circ \zeta$, while $q$ is the universal map of algebras with this property. Likewise, $q$ has an underlying monoidal functor for which $q \circ \delta = q \circ \zeta$, while $c$ is the universal monoidal functor with that property. It follows that there exists a unique pair of maps, $u : L\mathbb{G}_n \to C$ in $EG\mathrm{Alg}_S$ and $u' : C \to L\mathbb{G}_n$ in MonCat, such that $c = u \circ q$ and $q = u' \circ c$. But then, viewing $u$ as its underlying monoidal functor, we have

$$\begin{array}{rclclcrcl} c & = & u \circ q & = & u \circ u' \circ c & \implies & u \circ u' & = & \mathrm{id}_C \\ q & = & u' \circ q & = & u' \circ u \circ q & \implies & u' \circ u & = & \mathrm{id}_{L\mathbb{G}_n} \end{array}$$

in MonCat, by surjectivity of $q$ and $c$. Therefore $L\mathbb{G}_n$ and $C$ are isomorphic as monoidal categories, and since coequalizers are only unique up to isomorphism, $q = u' \circ c$ is also a coequalizer of $\delta$ and $\zeta$ in MonCat. $\square$

**Proposition 3.27.**

$$\mathrm{Mor}(L\mathbb{G}_n) \quad = \quad \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \times \mathrm{Mor}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \Big/ \mathrm{im}\big(\,\mathrm{Mor}(\delta)^{\mathrm{gp,ab}} - \mathrm{Mor}(\zeta)^{\mathrm{gp,ab}}\,\big)$$

*Proof.* From Proposition 2.19, we know that $\mathrm{Mor}(\_)^{\mathrm{gp,ab}} : \mathrm{MonCat} \to \mathrm{Ab}$ is a left adjoint functor. This means that it preserves all colimits in MonCat, which by Proposition 3.26 will include the coequalizer $q$. Thus

$$\mathrm{coeq}\big(\,\mathrm{Mor}(\delta)^{\mathrm{gp,ab}}, \mathrm{Mor}(\zeta)^{\mathrm{gp,ab}}\,\big) \;=\; \mathrm{Mor}\big(\,\mathrm{coeq}(\delta,\zeta)\,\big)^{\mathrm{gp,ab}} \;=\; \mathrm{Mor}(q)^{\mathrm{gp,ab}}$$

In other words, the following is a coequalizer diagram in the category of abelian groups:

$$\begin{tikzcd}
\mathrm{Mor}(\mathbb{G}_{4n})^{\mathrm{gp,ab}} \arrow[d, shift right, "\mathrm{Mor}(\delta)^{\mathrm{gp,ab}}"'] \arrow[d, shift left, "\mathrm{Mor}(\zeta)^{\mathrm{gp,ab}}"] \\
\mathrm{Mor}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \arrow[d, "\mathrm{Mor}(q)^{\mathrm{gp,ab}}"] \\
\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}}
\end{tikzcd}$$

But the coequalizer of two abelian group homomorphisms $h_1, h_2$ is just the quotient of their target group by the image of their difference map $h_1 - h_2$, the homomorphism that sends $x$ to $h_1(x)h_2(x)^*$. Hence in this case we have

$$\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \quad = \quad \mathrm{Mor}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \Big/ \mathrm{im}\big(\,\mathrm{Mor}(\delta)^{\mathrm{gp,ab}} - \mathrm{Mor}(\zeta)^{\mathrm{gp,ab}}\,\big)$$

Next, recall that we saw in Proposition 3.14 how to express the full morphism monoid of $L\mathbb{G}_n$ in terms of $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$. In particular, we had

$$\mathrm{Mor}(L\mathbb{G}_n) \quad = \quad \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \times \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \Big/ \mathbb{Z}^n$$

where the $\mathbb{Z}^n$ subgroup of $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$ was the one generated by the identity maps. However each of these maps is already an element of the group $\mathrm{im}\big(\,\mathrm{Mor}(\delta)^{\mathrm{gp,ab}} - \mathrm{Mor}(\zeta)^{\mathrm{gp,ab}}\,\big)$, and so really this last quotient is unnecessary. Therefore, substituting the value for $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$ into the above equation yields the identity given in the statement of the proposition. $\square$

## 3.5. **The action of $L\mathbb{G}_n$.**

**Proposition 3.28.** *The action of $L\mathbb{G}_n$ is given by the following map:*

*Proof.* $\square$

3.6. **A full description of $L\mathbb{G}_n$.** With this last proposition proven, the results in these previous two chapters now collectively describe how to construct free E$G$-algebras on $n$ invertible objects. However, since this charachterization was discovered by us in such a piecemeal fashion, it would be best to restate the complete conclusion all in one place.

**Theorem 3.29.** *Let $\mathbb{G}_n$ be the free* E$G$*-algebra on $n$ objects. Then the free* E$G$*-algebra on $n$ invertible objects, $L\mathbb{G}_n$, is the algebra described by*

*Proof.* □

With Theorem 3.29 proven we can now finally achieve the first main goal of this paper — to describe the free braided monoidal category on $n$ invertible objects. In addition, this section will provide a few other simple applications of the theorem, in an effort to build up to the main result more gently. The definition of $L\mathbb{G}_n$ given in 3.29 is after all a little difficult to parse on first reading, because of the fairly abstract way it is presented, and hopefully the following concrete examples should allow the braided case to be properly understood.

**Proposition 3.30.** *One object case*

**Proposition 3.31.** *Symmetric case*

**Proposition 3.32.** *Braided case*

**Proposition 3.33.** *Cactus group case*

. . . . .

3.7. **The free algebra on $n$ weakly invertible objects.** Up until now, we've been working under the convention that by 'invertible' objects we mean stictly invertible — $x \otimes x^* = I$. As an additional exercise, we can ask ourselves how all of this would change if we permitted our objects to be only weakly invertible, that is $x \otimes x^* \cong I$. The situation is actually quite elegant, in that the effect of weakening in our objects can be offset completely by the effect of also weakening our algebra homomorphisms, such that we won't need to calculate any new free algebras other than those given by Theorem 3.29. Before proving this though, we first to need to set out some definitions.

**Definition 3.34.** Given an E$G$-algebra $X$, we denote by $X_{\text{wkinv}}$ the category whose

- objects are tuples $(x, x^*, \eta, \epsilon)$, where $x$ and $x^*$ are objects of $X$ and $\eta : I \to x^* \otimes x$ and $\epsilon : x \otimes x^* \to I$ are morphisms such that the composites

$$x \xrightarrow{\mathrm{id}\otimes\eta} x \otimes x^* \otimes x \xrightarrow{\epsilon\otimes\mathrm{id}} x \qquad\qquad x^* \xrightarrow{\eta\otimes\mathrm{id}} x^* \otimes x \otimes x^* \xrightarrow{\mathrm{id}\otimes\epsilon} x^*$$

  are identity morphisms.
- maps $(f, f^*) : (x, x^*, \eta_x, \epsilon_x) \to (y, y^*, \eta_y, \epsilon_y)$ are pairs $f : x \to y$, $f^* : x^* \to y^*$ of morphisms such that the diagrams

$$\begin{tikzcd} & I \arrow[dl, "\eta_x"'] \arrow[dr, "\eta_y"] & \\ x^* \otimes x \arrow[rr, "f^*\otimes f"] & & y \otimes y^* \end{tikzcd} \qquad \begin{tikzcd} x \otimes x^* \arrow[rr, "f\otimes f^*"] \arrow[dr, "\epsilon_x"'] & & y \otimes y^* \arrow[dl, "\epsilon_y"] \\ & I & \end{tikzcd}$$

  commute.

**Definition 3.35.** Let $(X,\alpha)$ and $(Y,\beta)$ be E$G$-algebras. A *weak* E$G$*-algebra homomorphism* between them is a weak monoidal functor $\psi : X \to Y$ such that all diagrams of the form

$$\begin{tikzcd}
\psi(x_1 \otimes ... \otimes x_m) \arrow[r, "\sim"] \arrow[d, "\psi(\alpha(g;h_1,...,h_m))"'] & \psi(x_1) \otimes ... \otimes \psi(x_m) \arrow[d, "\beta(g;\psi(h_1),...,\psi(h_m))"] \\
\psi(y_{\pi(g)^{-1}(1)} \otimes ... \otimes y_{\pi(g)^{-1}(m)}) \arrow[r, "\sim"] & \psi(y_{\pi(g)^{-1}(1)}) \otimes ... \otimes \psi(y_{\pi(g)^{-1}(m)})
\end{tikzcd}$$

commute.

**Definition 3.36.** We denote by E$G$Alg$_W$ the 2-category of E$G$-algebras, weak E$G$-algebra homomorphisms, and weak monoidal transformations.

Now we can properly express what we mean by the free algebras on weakly invertible objects being the same as those in the strict case.

**Theorem 3.37.** *The algebra $L\mathbb{G}_n$ is also the free* E$G$*-algebra on $n$ weakly invertible objects. Specifically, for any other* E$G$*-algebra $X$ there is an equivalence of categories*

$$\mathrm{E}G\mathrm{Alg}_W(L\mathbb{G}_n, X) \simeq (X_{\mathrm{wkinv}})^n$$

*Proof.* We begin by defining a functor $F : \mathrm{E}G\mathrm{Alg}_W(L\mathbb{G}_n, X) \to (X_{\mathrm{wkinv}})^n$. On weak maps, $F$ acts as

$$F(\,\psi : L\mathbb{G}_n \to X\,) = \left\{\,(\,\psi(z_i),\, \psi(z_i^*),\, I \xrightarrow{\sim} \psi(I) \xrightarrow{\sim} \psi(z_i^*)\psi(z_i),\, \psi(z_i)\psi(z_i^*) \xrightarrow{\sim} \psi(I) \xrightarrow{\sim} I\,)\,\right\}_{i\in\{z_1,...,z_n\}}$$

where the $z_i$ are the generators of $\mathbb{Z}^{*n}$ and the isomorphisms are those given by $\psi$ being a weak moniodal functor. On weak monoidal transformations, $F$ acts as

$$F(\,\theta : \psi \to \chi\,) = \left\{\,(\,\theta_{z_i},\, \theta_{z_i^*}\,)\,\right\}_{i\in\{z_1,...,z_n\}}$$

This choice does satisfy the condition on morphisms of $(X_{\mathrm{wkinv}})^n$, since we can build the required commuting diagrams out of smaller ones given by $\theta$ being a weak monoidal transfomation:

$$\begin{tikzcd}
 & I \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\
\psi(I) \arrow[rr, "\theta_I"] \arrow[d, "\sim"'] & & \chi(I) \arrow[d, "\sim"] \\
\psi(z_i^*) \otimes \psi(z_i) \arrow[rr, "\theta_{z_i^*} \otimes \theta_{z_i}"] & & \chi(z_i^*) \otimes \chi(z_i)
\end{tikzcd}
\qquad
\begin{tikzcd}
\psi(z_i) \otimes \psi(z_i^*) \arrow[rr, "\theta_{z_i} \otimes \theta_{z_i^*}"] \arrow[d, "\sim"'] & & \chi(z_i) \otimes \chi(z_i^*) \arrow[d, "\sim"] \\
\psi(I) \arrow[rr, "\theta_I"] \arrow[dr, "\sim"'] & & \chi(I) \arrow[dl, "\sim"] \\
 & I &
\end{tikzcd}$$

Now we need to check if $F$ is an equivalence of categories. First, let $\left\{(x_i, x_i^*, \eta_i, \epsilon_i)\right\}_{i\in\{z_1,...,z_n\}}$ be an arbitrary object of $(X_{\mathrm{wkinv}})^n$. We can construct a weak algebra map $\psi : L\mathbb{G}_n \to X$ from it as follows. Define

$$\psi(I) = I, \quad \psi(z_i) = x_i, \quad \psi(z_i^*) = x_i^*$$

and choose the isomorphisms

$$\begin{array}{rclcccl}
\psi_I & : & I \to \psi(I) & = & \mathrm{id}_I & : & I \to I \\
\psi_{z_i,z_i^*} & : & \psi(z_i) \otimes \psi(z_i^*) \to \psi(I) & = & \epsilon_i & : & x_i \otimes x_i^* \to I \\
\psi_{z_i^*,z_i} & : & \psi(z_i^*) \otimes \psi(z_i) \to \psi(I) & = & \eta_i^{-1} & : & x_i^* \otimes x_i \to I
\end{array} .$$

Then for any $w, w' \in \mathrm{Ob}(L\mathbb{G}_n)$ such that $d(w \otimes w') = d(w) \otimes d(w')$, where $d(-)$ is the minimal generator decomposition from **??**, set

$$\psi(w \otimes w') = \psi(w) \otimes \psi(w'), \qquad \psi_{w,w'} = \mathrm{id}_{\psi(w) \otimes \psi(w')}$$

This is enough to determine the value of $\psi$ on all of the remaining objects, via successive decompositions. For the isomorphisms, first note that the ones we have already defined satisfy the associativity and unitality required of weak monoidal functors. Now consider some $w, w'$ with $d(w \otimes w') \neq d(w) \otimes d(w')$. The fact that they differ implies that tensoring $w$ with $w'$ causes some cancellation of inverses to occur where the end of one sequence meets the beginning of another. In particular, if we let $b$ be the last term in the minimal generator decomposition of $w$, and let $c = w'$, then we conclude that the length $d(b \otimes c)$ is smaller than the length of $d(c)$. Let $a$ be the product of the rest of $d(w)$, so that $a \otimes b = w$. Then we can use requirement for associativity,

$$\begin{tikzcd}
\psi(a) \otimes \psi(b) \otimes \psi(c) \arrow[r, "\mathrm{id} \otimes \psi_{b,c}"] \arrow[d, "\psi_{a,b} \otimes \mathrm{id}"'] & \psi(a) \otimes \psi(b \otimes c) \arrow[d, "\psi_{a, b \otimes c}"] \\
\psi(a \otimes b) \otimes \psi(c) \arrow[r, "\psi_{a \otimes b, c}"] & \psi(a \otimes b \otimes c)
\end{tikzcd}$$

to define $\psi_{w,w'} = \psi a \otimes b, c$ in terms of three other isomorphisms that each have strictly smaller decompositions. Repeating this process will therefore eventually yield a definition in terms of our previous isomorphisms.

By Proposition 3.24, every morphism in $L\mathbb{G}_n$ can be written as $\alpha(g; \mathrm{id}_{w_1}, ..., \mathrm{id}_{w_m})$ for some $g \in G(m)$, $w_i \in \mathbb{Z}^{*n}$. The action of $\psi$ on morphisms is thus determined by the diagram in Definition 3.35, that is

$$\psi(\alpha(g; w_1, ...w_m)) \; = \; \psi_{\mathbf{w}_{\pi(g)^{-1}}} \circ \beta(\, g \,;\, \mathrm{id}_{\psi(w_1)},\, ...,\, \mathrm{id}_{\psi(w_m)} \,) \circ \psi_{\mathbf{w}}^{-1}$$

However, morphisms do not have a unique representation of this form, so we must check that whenever we have different representations of the same morphism

$$\alpha(g; \mathrm{id}_{w_1}, ..., \mathrm{id}_{w_m}) = \alpha(g'; \mathrm{id}_{w'_1}, ..., \mathrm{id}_{w'_{m'}})$$

their diagrams give the same image under $\psi$. There are two cases to consider here;

$$\alpha(g; \mathrm{id}_{w_1}, ..., \mathrm{id}_{w_m}) = \alpha(\, g \otimes e_k \,;\, \mathrm{id}_{w_1},\, ...,\, \mathrm{id}_{w_m},\, \mathrm{id}_{v_1},\, ...,\, \mathrm{id}_{v_k} \,)$$

when $v_1 \otimes ... \otimes v_k = 0$, which comes from the edges of the colimit diagram $D_n$ in **??**; and

$$\begin{aligned}
\alpha(g; \mathrm{id}_{w_1}, ..., \mathrm{id}_{w_m}) \quad &= \quad \alpha(\, h \,;\, \mathrm{id}_{w'_1},\, ...,\, \mathrm{id}_{w_{m'}} \,) \\
&\quad\; \circ\, \alpha(\, j \,;\, \mathrm{id}_{w''_1},\, ...,\, \mathrm{id}_{w''_{m''}} \,) \\
&\quad\; \circ\, \alpha(\, h^{-1} \,;\, \mathrm{id}_{w'_1},\, ...,\, \mathrm{id}_{w'_{m'}} \,) \\
&\quad\; \circ\, \alpha(\, j^{-1} \,;\, \mathrm{id}_{w''_1},\, ...,\, \mathrm{id}_{w''_{m''}} \,) \\
&= \quad \mathrm{id}_{w_1 \otimes ... \otimes w_m}
\end{aligned}$$

for $\alpha(\, h \,;\, \mathrm{id}_{w'_1},\, ...,\, \mathrm{id}_{w_{m'}} \,), \alpha(\, j \,;\, \mathrm{id}_{w''_1},\, ...,\, \mathrm{id}_{w''_{m''}} \,) \in \mathbb{G}_n(w_1 \otimes ... \otimes w_m, w_1 \otimes ... \otimes w_m)$, which comes from the abelianisation of the vertices of $D_n$. All other ways for a morphism to have different representations must be generated by successive examples of these cases, since otherwise they wouldn't be coequalised by the colimit in **??**. In the first case we just have

$$\begin{array}{rl} & \psi(\,\alpha(\,g\otimes e_k\,;\,\mathrm{id}_{w_1},\,...,\,\mathrm{id}_{w_m},\,\mathrm{id}_{v_1},\,...,\,\mathrm{id}_{v_k}\,)\,) \\ = & \psi_{\mathbf{w}_{\pi(g)^{-1}},\mathbf{v}}\circ\beta(\,g\otimes e_k\,;\,\mathrm{id}_{\psi(w_1)},\,...,\,\mathrm{id}_{\psi(w_m)},\,\mathrm{id}_{\psi(v_1)},\,...,\,\mathrm{id}_{\psi(v_k)}\,)\circ\psi^{-1}_{\mathbf{w},\mathbf{v}} \\ = & \big(\psi_{\mathbf{w}_{\pi(g)^{-1}}}\otimes\psi_{\mathbf{v}}\big)\circ\big(\beta(g;\mathrm{id}_{\psi(w_1)},...,\mathrm{id}_{\psi(w_m)})\otimes\mathrm{id}_{\psi(\mathbf{v})}\big)\circ\big(\psi^{-1}_{\mathbf{w}}\otimes\psi^{-1}_{\mathbf{v}}\big) \\ = & \big(\psi_{\mathbf{w}_{\pi(g)^{-1}}}\circ\beta(g;\mathrm{id}_{\psi(w_1)},...,\mathrm{id}_{\psi(w_m)})\circ\psi^{-1}_{\mathbf{w}}\big)\otimes\big(\psi_{\mathbf{v}}\circ\mathrm{id}_{\psi(\mathbf{v})}\circ\psi^{-1}_{\mathbf{v}}\big) \\ = & \psi_{\mathbf{w}_{\pi(g)^{-1}}}\circ\beta(g;\mathrm{id}_{\psi(w_1)},...,\mathrm{id}_{\psi(w_m)})\circ\psi^{-1}_{\mathbf{w}} \\ = & \psi(\,\alpha(g;\mathrm{id}_{w_1},...,\mathrm{id}_{w_m})\,) \end{array}$$

as required. The second case is more subtle. We begin by expanding

$$\begin{array}{rl} & \psi(\,\alpha(\,g\,;\,\mathrm{id}_{w_1},\,...,\,\mathrm{id}_{w_m}\,) \\ = & \psi(\,\alpha(\,h\,;\,\mathrm{id}_{w'_1},\,...,\,\mathrm{id}_{w_{m'}}\,)\,) \\ & \circ\,\psi(\,\alpha(\,j\,;\,\mathrm{id}_{w''_1},\,...,\,\mathrm{id}_{w''_{m''}}\,)\,) \\ & \circ\,\psi(\,\alpha(\,h^{-1}\,;\,\mathrm{id}_{w'_1},\,...,\,\mathrm{id}_{w'_{m'}}\,)\,) \\ & \circ\,\psi(\,\alpha(\,j^{-1}\,;\,\mathrm{id}_{w''_1},\,...,\,\mathrm{id}_{w''_{m''}}\,)\,) \\ = & \psi_{\mathbf{w}'}\circ\beta(\,h\,;\,\mathrm{id}_{\psi(w'_1)},\,...,\,\mathrm{id}_{\psi(w_{m'})}\,)\circ\psi^{-1}_{\mathbf{w}'} \\ & \circ\,\psi_{\mathbf{w}''}\circ\beta(\,j\,;\,\mathrm{id}_{\psi(w''_1)},\,...,\,\mathrm{id}_{\psi(w''_{m''})}\,)\circ\psi^{-1}_{\mathbf{w}''} \\ & \circ\,\psi_{\mathbf{w}'}\circ\beta(\,h^{-1}\,;\,\mathrm{id}_{\psi(w'_1)},\,...,\,\mathrm{id}_{\psi(w'_{m'})}\,)\circ\psi^{-1}_{\mathbf{w}'} \\ & \circ\,\psi_{\mathbf{w}''}\circ\beta(\,j^{-1}\,;\,\mathrm{id}_{\psi(w''_1)},\,...,\,\mathrm{id}_{\psi(w''_{m''})}\,)\circ\psi^{-1}_{\mathbf{w}''} \end{array}$$

Here the objects $w_i, w'_i, w''_i$ are all in $\mathbb{G}_n \subseteq L\mathbb{G}_n$, and so we know their minimal generator decompositions are also in $\mathbb{G}_n$. It follows that $d(w_i\otimes w_j) = d(w_i)\otimes d(w_j)$ for all $i, j$, and hence by our definition of $\psi$ we have $\psi(w_i \otimes w_j) = \psi(w_i)\otimes\psi(w_j)$ and also $\psi_{\mathbf{w}_\sigma} = id$ for any permuation $\sigma$ — and the same for $\mathbf{w}'$ and $\mathbf{w}''$. Also, note that since we are working in $\mathbb{G}_n(w_1 \otimes ... \otimes w_m, w_1 \otimes ... \otimes w_m)$, all of the action morphisms in the above composite have the same source and target, $\psi(w_1 \otimes ... \otimes w_m)$. This object is weakly invertible, because each of the $w_i$ are invertible. However, the automorphisms of any weakly invertible object are isomorphic to the automorphisms of the unit object, as in the proof of **??**, and hence form an abelian group, by an Eckmann-Hilton argument like in the proof of **??**. Therefore we may permute these action morphisms freely, and so

$$\begin{array}{rl} & \psi(\,\alpha(\,g\,;\,\mathrm{id}_{w_1},\,...,\,\mathrm{id}_{w_m}\,) \\ = & \beta(\,h\,;\,\mathrm{id}_{\psi(w'_1)},\,...,\,\mathrm{id}_{\psi(w_{m'})}\,) \\ & \circ\,\beta(\,h^{-1}\,;\,\mathrm{id}_{\psi(w'_1)},\,...,\,\mathrm{id}_{\psi(w'_{m'})}\,) \\ & \circ\,\beta(\,j\,;\,\mathrm{id}_{\psi(w''_1)},\,...,\,\mathrm{id}_{\psi(w''_{m''})}\,) \\ & \circ\,\beta(\,j^{-1}\,;\,\mathrm{id}_{\psi(w''_1)},\,...,\,\mathrm{id}_{\psi(w''_{m''})}\,) \\ = & \mathrm{id}_{\psi(w_1)\otimes...\otimes\psi(w_m)} \\ = & \psi_{\mathbf{w}}\circ\beta(\,e_m\,;\,\mathrm{id}_{\psi(w_1)},\,...,\,\mathrm{id}_{\psi(w_m)}\,)\circ\psi^{-1}_{\mathbf{w}} \end{array}$$

as required.

With $\psi$ now fully defined, notice that

$$\begin{array}{rcl} F(\psi) & = & \big\{\,(\,\psi(z_i),\,\psi(z_i^*),\,I\xrightarrow{\sim}\psi(I)\xrightarrow{\sim}\psi(z_i^*)\psi(z_i),\,\psi(z_i)\psi(z_i^*)\xrightarrow{\sim}\psi(I)\xrightarrow{\sim}I\,)\,\big\}_{i\in\{z_1,\ldots,z_n\}} \\ & = & \big\{\,(\,x_i,\,x_i^*,\,\eta_i,\,\epsilon_i\,)\,\big\}_{i\in\{z_1,\ldots,z_n\}} \end{array}$$

which was our arbitrary object in $(X_{\mathrm{wkinv}})^n$. Therefore, $F$ is surjective on objects.

Next, choose an arbitrary monoidal transformation $\theta : \psi \to \chi$ from $\mathrm{E}G\mathrm{Alg}_W(L\mathbb{G}_n, X)$. By naturality, for any $w, w' \in \mathrm{Ob}(L\mathbb{G}_n)$ we have that

$$\begin{tikzcd} \psi(w)\otimes\psi(w') \arrow[r, "\sim"] \arrow[d, "\theta_w\otimes\theta_{w'}"'] & \psi(w\otimes w') \arrow[d, "\theta_{w\otimes w'}"] \\ \chi(w)\otimes\chi(w') \arrow[r, "\sim"] & \chi(w\otimes w') \end{tikzcd}$$

or equivalently, $\theta_{w\otimes w'} = \chi_{w,w'} \circ (\theta_w \otimes \theta_{w'}) \circ \psi^{-1}_{w,w'}$. It follows from this that the components of $\theta$ are generated by the components on the generators of $\mathrm{Ob}(L\mathbb{G}_n)$, namely $\{\,(\theta_{z_i}, \theta_{z_i^*})\,\}_{i\in\{z_1,\dots,z_n\}}$. But this is just $F(\theta)$, and thus any monoidal transformation $\theta$ is determined uniquely by its image under $F$, or in other words $F$ is faithful.

Finally, let $\psi, \chi$ be objects of $\mathrm{E}G\mathrm{Alg}_W(L\mathbb{G}_n, X)$, and choose an arbitrary map $\{\,(f_i, f_i^*)\,\}_{i\in\{z_1,\dots,z_n\}} : F(\psi) \to F(\chi)$ from $(X_{\mathrm{wkinv}})^n$. We can use this to construct a monoidal transformation $\theta : \psi \to \chi$ via the reverse of process we just used. Specifically, if we define

$$\theta_I = \chi_I \circ \psi_I^{-1}, \qquad \theta_{z_i} = f_i, \qquad \theta_{z_i^*} = f_i^*$$

then these will automatically form the naturality squares

$$\begin{tikzcd} \psi(z_i)\otimes\psi(z_i^*) \arrow[r, "\psi_{z_i,z_i^*}"] \arrow[dd, "f_i\otimes f_i^*"'] & \psi(I) \arrow[d, "\psi_I^{-1}"] \\ & I \arrow[d, "\chi_I"] \\ \chi(z_i)\otimes\chi(z_i^*) \arrow[r, "\chi_{z_i,z_i^*}"] & \chi(I) \end{tikzcd} \qquad \begin{tikzcd} \psi(z_i^*)\otimes\psi(z_i) \arrow[r, "\psi_{z_i^*,z_i}"] \arrow[dd, "f_i^*\otimes f_i"'] & \psi(I) \arrow[d, "\psi_I^{-1}"] \\ & I \arrow[d, "\chi_I"] \\ \chi(z_i^*)\otimes\chi(z_i) \arrow[r, "\chi_{z_i^*,z_i}"] & \chi(I) \end{tikzcd}$$

since these are just the conditions for $\{\,(f_i, f_i^*)\,\}_{i\in\{z_1,\dots,z_n\}}$ to be a map $F(\psi) \to F(\chi)$ in $(X_{\mathrm{wkinv}})^n$. Repeatedly applying the naturality condition $\theta_{w\otimes w'} = \chi_{w,w'} \circ (\theta_w \otimes \theta_{w'}) \circ \psi^{-1}_{w,w'}$ will then generate all of the other components of $\theta$, in a way that clearly satisfies naturality. Thus we have a well-defined monoidal transformation $\theta : \psi \to \chi$, and applying $F$ to it gives

$$\begin{aligned} F(\theta) &= \big\{\,(\theta_{z_i}, \theta_{z_i^*})\,\big\}_{i\in\{z_1,\dots,z_n\}} \\ &= \big\{\,(f_i, f_i^*)\,\big\}_{i\in\{z_1,\dots,z_n\}}, \end{aligned}$$

our arbitrary map. Therefore $F$ is full and, putting this together with the previous results, is an equivalence of categories. □

## 4. The classification of 2-groups

See [1] for more detail.

### 4.1. $n$-groups.

**Definition 4.1.** Weak $n$-categories
**Definition 4.2.** $n$-groupoids
**Definition 4.3.** $n$-groups
**Example 4.4.** Examples

### 4.2. Classifying 2-groups.

**Definition 4.5.** 2-groups
**Definition 4.6.** Coherent 2-groups

**Theorem 4.7.** *The classification of 2-groups*

### 4.3. Group cohomology.

**Definition 4.8.** Cochain complex of groups
**Definition 4.9.** Cocycles, coboundaries, and cohomology classes
**Definition 4.10.** Group cohomology

### 4.4. Proof of the classification theorem.

*Proof of Theorem 4.7.* □

### 4.5. Generalizing to 3-groups.

## 5. 2-group cohomology

See [2] and [3] for more details.

### 5.1. Cohomology of categories.

**Definition 5.1.** Cochain complex of categories
**Definition 5.2.** Cocycles, coboundaries, and cohomology classes

### 5.2. Symmetric 2-group cohomology.

**Definition 5.3.** Symmetric 2-group cohomology

**Proposition 5.4.** *Coherence of symmetric 2-group cohomology*

### 5.3. Braided 2-group cohomology.

**Definition 5.5.** Braided 2-group cohomology

**Proposition 5.6.** *Coherence of braided 2-group cohomology*

## 6. Classifying 3-groups

**Proposition 6.1.** *3-group coherence data as elements of cohomology classes*

**Theorem 6.2.** *The classification of 3-groups*

**Example 6.3.** Examples

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