Action operads and the free G-monoidal category on n invertible objects

Edward G. Prior

School of Mathematics and Statistics
University of Sheffield

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Abstract

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Chapter 1

Operads and their algebras

1.1 Operads

Definition 1.1. Operads O

Example 1.2 (The symmetric operad).
There is an operad S whose sets of operations $\mathrm{S}(n)$ for each $n \in \mathbb{N}$ are the underlying sets of the symmetric groups S_n. The identity element of this *symmetric operad* is the identity permutation of a single object, $e_1 \in \mathrm{S}_1$, and the operadic multiplication is defined in the following way:

- First, there exist maps $\otimes : \mathrm{S}_m \times \mathrm{S}_n \to \mathrm{S}_{m+n}$ called the *direct sum* or *block sum* of permutations. For any $\sigma \in \mathrm{S}_m$ and $\tau \in \mathrm{S}_n$, these are given by

$$(\sigma \otimes \tau)(i) \quad = \quad \begin{cases} \sigma(i) & 1 \leq i \leq m \\ \tau(i-m)+m & m+1 \leq i \leq m+n \end{cases}$$

  As the name suggests, this direct sum is usually denote by the symbol $\oplus$, but we will stick with $\otimes$ so that our notation here matches all of the other tensor products we will see throughout this paper. Also, notice that the value of these direct sums in general are determined by those specific cases where one of the inputs is an identity permutation:

$$\sigma \otimes \tau \quad = \quad (\sigma \otimes e_n) \cdot (e_m \otimes \tau) \quad = \quad (e_m \otimes \tau) \cdot (\sigma \otimes e_n)$$

- Next, we'll define functions $(\_)_{(k_1,...,k_n)} : \mathrm{S}_n \to \mathrm{S}_{k_1+...+k_n}$ for all $n, k_1, ..., k_n \in \mathbb{N}$. These will act by sending each σ that permutes n individual objects to a corresponding $\sigma_{(k_1,...,k_n)}$ that permutes blocks of objects of size $k_1, ..., k_n$ in the same way. More concretely, if $k_1 + ... + k_{i-1} < j \leq k_1 + ... + k_i$ then

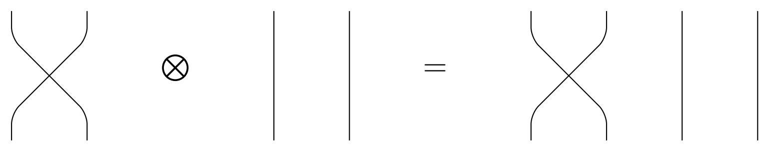
$$\sigma_{(k_1,...,k_n)}(j) \quad = \quad j - k_1 - ... - k_{i-1} + k_{\sigma^{-1}(1)} + ... + k_{\sigma^{-1}(\,\sigma(i)-1\,)}$$

- Finally, the multiplication maps $\mu : \mathrm{S}_n \times \mathrm{S}_{k_1} \times ... \times \mathrm{S}_{k_n} \to \mathrm{S}_{k_1+...+k_n}$ are given by

$$\begin{aligned} \mu(\sigma; \tau_1, ..., \tau_n) \;&:=\; \sigma_{(k_1,...,k_n)} \cdot (\tau_1 \otimes ... \otimes \tau_n) \\ &=\; \big(\tau_{\sigma^{-1}(1)} \otimes ... \otimes \tau_{\sigma^{-1}(n)}\big) \cdot \sigma_{(k_1,...,k_n)} \end{aligned}$$

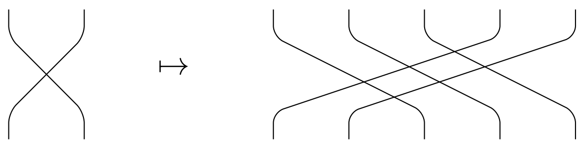
  In other words, the operadic multiplication of permutations comes from both permutating objects within distinct blocks and also permuting the blocks themselves.

If we decide to represent elements of the symmetric operad pictorially — for example as strings which cross over another according to the appropriate permutation — then both $\sigma \otimes \tau$ and $\sigma_{(k_1,...,k_n)}$ have rather nice interpretations. The direct sum of two permutations is just the result of placing two permutations 'next to' each other,



σ τ $\sigma \otimes \tau$

and block permutations are given by expanding string into some number of parallel strings,



σ $\sigma_{(3,2)}$

With a little work, we can actually replace the functions $(\_\!\_)_{(k_1,...,k_n)}$ with an explicit combination of group multiplication and tensor product. This is due to basic fact about the symmetric groups S_n, which is that they possess a presentation in terms of the *elementary transpositions* $(i \;\; i+1)$.

Lemma 1.3. *The group* S_n *is generated by the permutations* $(1\,2), \ldots, (n-1\;\;n)$*, subject to the relations*

$$\begin{array}{rcll} (i\;\;i+1)^2 & = & e & \\ (i-1\;\;i)(i\;\;i+1)(i-1\;\;i) & = & (i\;\;i+1)(i-1\;\;i)(i\;\;i+1) & \\ (i\;\;i+1)(j\;\;j+1) & = & (j\;\;j+1)(i\;\;i+1), & \qquad i+1<j \end{array}$$

Thus if $\sigma \in \mathrm{S}_n$ is a permutation with a decomposition $\sigma = \sigma_m \cdot \ldots \cdot \sigma_1$ in terms of elementary transpositons $\sigma_i \in \mathrm{S}_n$, we can break down the block permutation $\sigma_{(k_1,\ldots,k_n)}$ into the m 'elementary block transpositions' $(\sigma_i)_{(k_1,\ldots,k_n)}$:

$$\begin{array}{rcl} \sigma_{(k_1,\ldots,k_n)}(j) & = & j - k_1 - \ldots - k_{i-1} + k_{\sigma^{-1}(1)} + \ldots + k_{\sigma^{-1}(\sigma(i)-1)} \\ & = & j - k_1 - \ldots - k_{i-1} \\ & & + k_{\sigma_1^{-1}(1)} + \ldots + k_{\sigma_1^{-1}(\sigma_1(i)-1)} \\ & & - k_{\sigma_1^{-1}(1)} - \ldots - k_{\sigma_1^{-1}(\sigma_1(i)-1)} \\ & & + k_{(\sigma_2\sigma_1)^{-1}(1)} + \ldots + k_{(\sigma_2\sigma_1)^{-1}(\sigma_2\sigma_1(i)-1)} \\ & & \vdots \\ & & - k_{(\sigma_{m-1}\ldots\sigma_1^{-1}(1)} - \ldots - k_{(\sigma_{m-1}\ldots\sigma_1)^{-1}(\sigma_{m-1}\ldots\sigma_1(i)-1)} \\ & & + k_{(\sigma_m\ldots\sigma_1)^{-1}(1)} + \ldots + k_{(\sigma_m\ldots\sigma_1)^{-1}(\sigma_m\ldots\sigma_1(i)-1)} \\ & = & \Big((\sigma_m)_{(k_1,\ldots,k_n)} \cdot \ldots \cdot (\sigma_1)_{(k_1,\ldots,k_n)} \Big)(j) \end{array}$$

However, since elementary transpositions only really permute two objects, they can be written as a block sum in the operad S involving the sole transposition of S_2, plus some number of identity permutations.

$$(i\;\;i+1) \quad = \quad e_{i-1} \otimes (1\,2) \otimes e_{n-i-1}$$

This means that the elementary block transpositions are

$$\begin{array}{rcl} (i\;\;i+1)_{(k_1,\ldots,k_n)} & = & (e_{i-1} \otimes (1\,2) \otimes e_{n-i-1})_{(k_1,\ldots,k_n)} \\ & = & e_{k_1+\ldots k_{i-1}} \otimes (1\,2)_{(k_i,k_{i+1})} \otimes e_{k_{i+1}+\ldots k_n} \end{array}$$

So all we need to know to fully understand the functions $(\_)_{(k_1,\ldots,k_n)}$ are the values they take on the transposition $(1\,2)$. These can be defined recursively, via

$$\begin{array}{rclcrcl} (1\,2)_{(0,n)} & = & e_n, & \qquad & (1\,2)_{(m+m',n)} & = & \Big((1\,2)_{(m,n)} \otimes e_{m'} \Big) \cdot \Big(e_m \otimes (1\,2)_{(m',n)} \Big) \\ (1\,2)_{(m,0)} & = & e_m, & & (1\,2)_{(m,n+n')} & = & \Big(e_n \otimes (1\,2)_{(m,n')} \Big) \cdot \Big((1\,2)_{(m,n)} \otimes e_{n'} \Big) \end{array}$$

which all follow from the definition of $(\_)_{(k_1,...,k_n)}$. Therefore all $\sigma_{(k_1,...,k_n)}$ and hence all $\mu(\sigma;\tau_1,...,\tau_n)$ can be expressed in terms of group multiplication $\cdot$ and direct sum $\otimes$, and the elementary permutations which constitute $\sigma,\tau_1,...,\tau_n$.

Example 1.4 (The braid operad).
The *braid groups* B_n are the family of groups that result from taking the symmetric groups and removing the requirement that everything needs to be self-inverse. That is, the group B_n has a presentation on some *elementary braids* $b_1,...,b_{n-1}$, given by the relations

$$b_i b_{i+1} b_i \;=\; b_{i+1} b_i b_{i+1}, \qquad\qquad b_i b_j \;=\; b_j b_i, \quad i+1<j$$

As might be expected, the underlying sets of these groups also form an operad, known as the *braid operad* B, and they do so in a way directly analagous to the operad S. That is, the identity element of B is $e_1 \in \mathrm{B}_1$, and the operadic multiplication is constructed as follows:

- Tensor products $\otimes : B_m \times B_n \to B_{m+n}$ are determined by setting

$$x \otimes y \quad = \quad (x \otimes e_n)\cdot(e_m \otimes y) \quad = \quad (e_m \otimes x)\cdot(y \otimes e_n)$$

  for all $x \in B_m$, $y \in B_n$, and also

$$b_i \quad = \quad e_{i-1} \otimes b \otimes e_{n-i-1}$$

  for any elementary braid $b_i \in B_n$, where b is the only elementary braid in B_2.

- The functions $(\_)_{(k_1,...,k_n)} : B_n \to B_{k_1+...+k_n}$ are first defined recursively on the elementary braid $b \in B_2$ by

$$\begin{array}{rclrcl} b_{(0,n)} & = & e_n, & b_{(m+m',n)} & = & (b_{(m,n)} \otimes e_{m'})\cdot(e_m \otimes b_{(m',n)}) \\ b_{(m,0)} & = & e_m, & b_{(m,n+n')} & = & (e_n \otimes b_{(m,n')})\cdot(b_{(m,n)} \otimes e_{n'}) \end{array}$$

  then on arbitrary elementary braids $b_i \in B_n$ via

$$(b_i)_{(k_1,...,k_n)} \quad = \quad e_{k_1+...k_{i-1}} \otimes b_{(k_i,k_{i+1})} \otimes e_{k_{i+1}+...k_n}$$

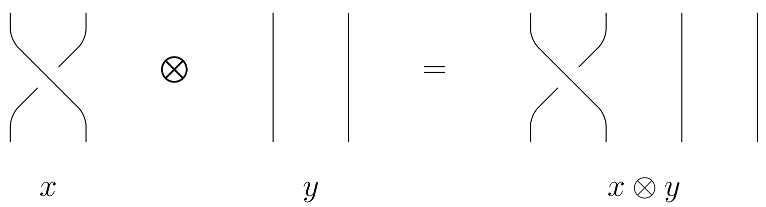
  and finally on all elements of the braid groups by using their presentation in terms of the b_i,

$$\begin{array}{rrcl} & x & = & b_{i_m}\cdot ... \cdot b_{i_1} \\ \Longrightarrow & x_{(k_1,...,k_n)} & = & (b_{i_m})_{(k_1,...,k_n)}\cdot ... \cdot (b_{i_1})_{(k_1,...,k_n)} \end{array}$$

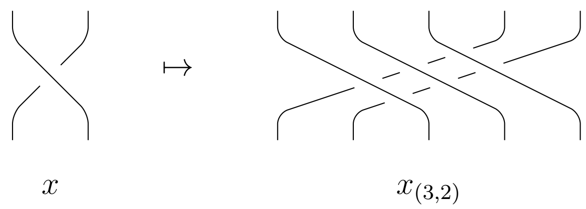
- Then as in the symmetric case, the multiplication maps $\mu : B_n \times B_{k_1} \times ... \times B_{k_n} \to B_{k_1+...+k_n}$ are just

$$\mu(x; y_1, ..., y_n) \quad := \quad x_{(k_1,...,k_n)} \cdot (y_1 \otimes ... \otimes y_n)$$

These operations are exactly what they need to be in order for them to possess the same pictorial representations as the operations in S, but with actual braids replacing simple crossings. That is, the tensor product $x \otimes y$ is the braids x and y laid side-by-side,



and the 'block braids' are multiple strings braided together in parallel,



Definition 1.5. Operads maps

[6] [5]

Definition 1.6. Action operads G, maps, AOp

There are a couple of operads which trivially have the structure of an action operads.

First there is the *terminal operad* T, which has a single operation for each arity, so that $\mathrm{T}(n) = \{e_n\}$. Each of these sets can be seen as the trivial group, and it follows from this that the $\pi^{\mathrm{T}} : \mathrm{T}(n) \to \mathrm{S}_n$ must be the respective zero maps, the terminal homomorphisms in the category of groups. The action operad condition is then

$$\mu(e_n; e_{k_1}, ..., e_{k_n}) \cdot \mu(e_n; e_{k_1}, ..., e_{k_n}) \quad = \quad \mu(e_n; e_{k_1}, ..., e_{k_n})$$

which is really just

$$e_{k_1+...+k_n} \cdot e_{k_1+...+k_n} = e_{k_1+...+k_n}$$

and hence is trivially true. As its name suggests, the terminal operad is the terminal object in the category of operads, but it is also the *initial* object in the category of action operads. This is because for any other G in AOp the zero homomorphisms $\mathrm{T}(n) \to G(n)$ define the unique map of operads $f : \mathrm{T} \to G$.

On the other hand, it is the symmetric operad S itself that functions as the terminal object in AOp. Its action operad structure is just given by the standard group multiplications on the S_n, with the identity maps $\mathrm{id}_{\mathrm{S}_n} : \mathrm{S}_n \to \mathrm{S}_n$ functioning as its π_n. To see terminality, notice that for any other action operad G a valid morphism $f : G \to \mathrm{S}$ in AOp must obey

$$\pi^{\mathrm{S}} \circ f \;=\; \pi^{G} \qquad \Longrightarrow \qquad f \;=\; \pi^{G}$$

Thus there only one map of action operads $G \to \mathrm{S}$, which is the very underlying permutation structure used to define G.

There are more interesting examples of action operads we can look at though. For instance, we know that the braid groups B_n have the same presentation as the symmetric groups, except without the relations $b_i^2 = e$. Thus if we take their quotients by these relations we will obtain a sequence of homomorphisms $B_n \to \mathrm{S}_n$, each sending $b_i \mapsto (i \;\; i+1)$. This provides a natural way to describe the underlying permutation of any braid, and indeed choosing these maps to form π^B gives a valid way of seeing the braid operad as an action operad. Another example can also be built with the so-called ribbon braid groups.

Definition 1.7. For each $n \in \mathbb{N}$, the *ribbon braid group* RB_n is the group whose presentation is the same as that of the braid group B_n, except with the addition of n new generators $t_1, ..., t_n$, known as the *twists*. These twists all commute with one other, and also commute with all braids except in the following cases:

$$b_i \cdot t_i \quad = \quad t_{i+1} \cdot b_i, \qquad\qquad b_i \cdot t_{i+1} \quad = \quad t_i \cdot b_i$$

The *ribbon braid operad* RB is then the operad made up of these groups in a way that extends the definition of the braid operad. In other words, the identity is still $e_1 \in RB_1$, and the operadic multiplication is built up in stages in exactly the same ways as in Example 1.4, but with some additional rules for dealing with twists. For the tensor product, we have that for any twist $t_i \in RB_n$,

$$t_i \quad = \quad e_{i-1} \otimes t \otimes e_{n-i}$$

where t is the sole twist in RB_1, and for the 'block twists' $t_{(m)}$ we again work recursively:

$$t_{(0)} = e_n, \qquad t_{(m+m')} \quad = \quad \left(t_{(m)} \otimes t_{(m')}\right) \cdot b_{(m',m)} \cdot b_{(m,m')}$$

Much as the symmetric groups can be represented by crossings of a collection of strings, and the braid groups by braidings of strings, the ribbon braid groups deal with the ways that one can braid together several flat ribbons, including the ability to twist a ribbon about its own axis by 360 degrees.

b $\qquad$ t

This operad RB is also clearly an action operad, since we can just define $\pi^{RB} : RB_n \to \mathrm{S}_n$ to act like π^B on any braids, at which point the fact that $\pi(t) \in S_1 = \{e_1\}$ will automatically take care of the twists. To learn more about the ribbon braids and their operads, see Natalie Wahl's thesis [13] on the subject, or her subsequent paper with Paolo Salvatore [14].

Lemma 1.8. *For any action operad G, the group $G(0)$ is abelian.*

Definition 1.9. Sub action operads

The most important example of sub action operads are those of the symmetric operad, S. This is because Definition 1.6 itself makes explicit reference to the symmetric groups, and so every action operad will end up related to some sub-operad of S:

Definition 1.10. For an arbitrary action operad G the images of the underlying permutation maps $\pi_n^G : G(n) \to \mathrm{S}_n$ naturally form an action operad $\mathrm{im}(\pi^G)$, where

- the sets of operations are the images of G's sets of operations under the homomorphisms π^G:

$$\mathrm{im}(\pi^G)(n) \quad := \quad \mathrm{im}(\pi_n^G)$$

- the underlying permutation maps are the evident inclusions:

$$\pi_n^{\mathrm{im}(\pi^G)} \; : \; \mathrm{im}(\pi^G)(n) \hookrightarrow \mathrm{S}_n$$

- the operad multiplication is the appropriate restriction of the multiplication of S:

$$\mu^{\mathrm{im}(\pi^G)}(\, g\,;\, h_1, ..., h_n\,) \quad := \quad \mu^{\mathrm{S}}(\, g\,;\, h_1, ..., h_n\,)$$

Clearly this $\mathrm{im}(\pi^G)$ is a sub action operad of the symmetric operad S, and we will call the *underlying permutation operad* of G.

For example, consider the action operad B we just saw in Example 1.4. For a given n, the braid group B_n is generated by $n-1$ elementary braids. But the underlying permutations of these braids are just the $n-1$ adjacent transpositions which generate the symmetric group S_n, and so the underlying permutation maps $\pi_n^{\mathrm{B}} : \mathrm{B}_n \to \mathrm{S}_n$ are all surjective. Thus the underlying permutation operad of B is just the whole symmetric action operad S.

It is even easier to see that S itself will have underlying permutations S, as the maps $\pi_n^{\mathrm{S}} = \mathrm{id} : \mathrm{S}_n \to \mathrm{S}_n$ are obviously surjective. Similarly, the trivial operad T is also its own underlying permutation action operad, as the image of the homomorphims $\pi_n^{\mathrm{T}} : \{e\} \to \mathrm{S}_n$ are trivial. Faced with rather dull examples like these, it might be tempting to try and construct some action operads with more exotic underlying permutations, like maybe the alternating groups $\mathrm{A}_n \subset \mathrm{S}_n$. But it turns out that this is not possible; when it come to their underlying permutation operad, action operads come in exactly two flavours.

Definition 1.11. Let G be an action operad where $\mathrm{im}(\pi)(n)$ is the trivial group for each $n \in \mathbb{N}$. Then we say that G is *non-crossed*, since its operad multiplication will be a true group homomorphism:

$$\begin{aligned} \mu(\, gg' \,;\, h_1h'_1, ..., h_nh'_n \,) &= \mu(\, g \,;\, h_{\pi(g')^{-1}(1)}, ..., h_{\pi(g')^{-1}(n)} \,)\mu(\, g' \,;\, h'_1, ..., h'_n \,) \\ &= \mu(\, g \,;\, h_1, ..., h_n \,)\mu(\, g' \,;\, h'_1, ..., h'_n \,) \end{aligned}$$

Likewise, a *crossed* action operad will refer to any that has a non-trivial underlying permutation operad.

Lemma 1.12. *An action operad G is crossed if and only if it has surjective underlying permutation maps $\pi_n : G(n) \to \mathrm{S}_n$. In other words, the underlying permutations operad of G must be either the trivial operad* T *or the symmetric operad* S*.*

Definition 1.13. G-operads

1.2 Operad algebras

Definition 1.14. Operad algbras

Definition 1.15. G-operad algebras

1.3 EG-algebras

Definition 1.16. The G-operad EG

Definition 1.17. The monad EG

Definition 1.18. EG-algebras

Proposition 1.19. *G-operad algebras are monoidal categories with permutation-like structure*

Corollary 1.20. *Braided monoidal categories are G-operad algebras*

Definition 1.21. A strict monoidal category X is said to be *spacial* if, for any object $x \in \mathrm{Ob}(X)$ and any endomorphism of the unit object $f : I \to I$,

$$f \otimes \mathrm{id}_x = \mathrm{id}_x \otimes f$$

The motivation for the name 'spacial' comes from the context of string diagrams [4]. In a string diagram, the act of tensoring two strings together is represented by placing those strings side by side. Since the defining feature of the unit object is that tensoring it with other objects should have no effect, the unit object is therefore represented diagrammatically by the absense of a string. An endomorphism of the unit thus appears as an entity with no input or output strings, detached from the rest of the diagram. In a real-world version of these diagrams, made out of physical strings arranged in real space, we could use this detachedness to grab these endomorphisms and slide them over or under any strings we please, without affecting anything else in the diagram. This ability is embodied algebraically by the equation above, and hence categories which obey it are called 'spacial'.

Lemma 1.22. *If G is a crossed action operad, then all* EG*-algebras are spacial.*

1.4 The free EG-algebra on n objects

Our goal for the next few chapters will be to understand the free braided monoidal category on an finite number of invertible objects. Thus, now that we have a firm grasp on action operads and their algebras, we should begin to think about the simpler free constructions they can form. We will use this extensively when calculating the invertible case later on.

In the paper [7], Gurski establishes how to contruct free G-operad algebras through the use of the monad $\mathrm{E}G$. What follows in this section is a quick summary of the results which will be useful for our purposes. For a more detailed treatment please refer to [7].

Proposition 1.23. *There exists a free* $\mathrm{E}G$*-algebra on* n *objects. That is, there is an* $\mathrm{E}G$*-algebra* Y *such that for any other* $\mathrm{E}G$*-algebra* X*, we have an isomorphism of categories*

$$\mathrm{E}G\mathrm{Alg}_S(Y, X) \cong X^n$$

Definition 1.24. Let $\{z_1, ..., z_n\}$ be an n-object set, which we will also consider as a discrete category. Then we will denote by $\mathbb{G}_n$ the $\mathrm{E}G$-algebra whose underlying category is $\mathrm{E}G(\{z_1, ..., z_n\})$ and whose action

$$\alpha : \mathrm{E}G\Big(\mathrm{E}G(\{z_1, ..., z_n\})\Big) \to \mathrm{E}G(\{z_1, ..., z_n\})$$

is the appropriate component of the multiplication natural transformation $\mu : \mathrm{E}G \circ \mathrm{E}G \to \mathrm{E}G$ of the 2-monad $\mathrm{E}G$.

Theorem 1.25. $\mathbb{G}_n$ *is the free* $\mathrm{E}G$*-algebra on* n *objects. That is,*

$$F(\{z_1, ..., z_n\}) = \mathbb{G}_n$$

Definition 1.24 is a fairly opaque definition, so we'll spend a little time upacking it. Recall from Definition 1.17 that $\mathrm{E}G(\{z_1, ..., z_n\})$ is the coequalizer of the maps

$$\textstyle\coprod_{m\geq 0} \mathrm{E}G(m) \times G(m) \times \{z_1, ..., z_n\}^m \rightrightarrows \coprod_{m\geq 0} \mathrm{E}G(m) \times \{z_1, ..., z_n\}^m$$

that comes from the action of $G(m)$ on $\mathrm{E}G(m)$ by multiplication on the right,

$$\begin{array}{rcl} \mathrm{E}G(m) \times G(m) & \to & \mathrm{E}G(m) \\ (g, h) & \mapsto & gh \\ (\,! : g \to g', \mathrm{id}_h\,) & \mapsto & ! : gh \to g'h \end{array}$$

and the action of $G(m)$ on $\{z_1, ..., z_n\}^m$ by permutation,

$$\begin{array}{rcl} G(m) \times \{z_1, ..., z_n\}^m & \to & \{z_1, ..., z_n\}^m \\ (\,h\,;\, x_1, ..., x_m\,) & \mapsto & (x_{\pi(h^{-1})(1)}, ..., x_{\pi(h^{-1})(m)}) \\ (\mathrm{id}_h\,;\, \mathrm{id}_{(x_1,...,x_m)}\,) & \mapsto & \mathrm{id}_{(x_{\pi(h^{-1})(1)},...,x_{\pi(h^{-1})(m)})} \end{array}$$

First, objects in this algebra are equivalence classes of tuples $(g; x_1, ..., x_m)$, for $g \in G(m)$ and $x_i \in \{z_1, ..., z_n\}$, under the relation

$$(\, gh\,;\, x_1,\, ...,\, x_m\,) \sim (\, g\,;\, x_{\pi(h)^{-1}(1)},\, ...,\, x_{\pi(h)^{-1}(m)}\,)$$

Notice that using this relation we can rewrite any object uniquely in the form $[e; x_1, ..., x_m]$ for some $m \in \mathbb{N}$ and $x_i \in \{z_1, ..., z_n\}$. This means that each equivalence class is just the tensor product $x_1 \otimes ... \otimes x_m$ in the underlying monoidal category of $\mathbb{G}_n$, for some unique sequence of generators. That is, we can view the objects of $\mathbb{G}_n$ as elements of the monoid freely generated by each of the z_i, or in other words:

Lemma 1.26. $\mathrm{Ob}(\mathbb{G}_n)$ *is the free monoid on n generators,* $\mathbb{N}^{*n}$*, the free product of n copies of* $\mathbb{N}$*.*

Similarly, the morphisms of $\mathbb{G}_n$ are the maps

$$(!; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) : (g; x_1, ..., x_m) \to (g'; x_1, ..., x_m)$$

with $g, g' \in G(m)$ and $x_i \in \{z_1, ..., z_n\}$. Using the relation $\sim$ on objects we can rewrite each of these morphisms in the form

$$[h; \mathrm{id}_{y_1}, ..., \mathrm{id}_{y_m}] \,:\, y_1 \otimes ... \otimes y_m \,\to\, y_{\pi(h^{-1})(1)} \otimes ... \otimes y_{\pi(h^{-1})(m)}$$

where

$$h = g'g^{-1}, \qquad y_i = x_{\pi(g^{-1})(i)}$$

The EG-action of $\mathbb{G}_n$ is permutation and tensor product, and the action on morphisms is given by

$$\alpha(\, g\,;\, [h_1; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_{m_1}}],\, ...,\, [h_k; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_{m_k}}]\,) = [\, \mu(g; h_1, .., h_k)\,;\, \mathrm{id}_{x_1},\, ...,\, \mathrm{id}_{x_{m_k}}\,]$$

Notice that using tensor product notation the object $[e; x]$ is simply x, and so $[e; \mathrm{id}_x] = \mathrm{id}_{[e;x]}$ should be written as id_x. Hence by the above $[g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}]$ is really just $\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$, and so we have the following:

Lemma 1.27. *Every morphism of $\mathbb{G}_n$ can be expressed uniquely as an action morphism*

$$\alpha(\, g\,;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,) \,:\, x_1 \otimes ... \otimes x_m \,\to\, x_{\pi(g)^{-1}(1)} \otimes ... \otimes x_{\pi(g)^{-1}(m)}$$

for some $g, g' \in G(m)$ and $x_i \in \{z_1, ..., z_n\}$.

As an immediate consequence of this, the source and target of a given morphism in $\mathbb{G}_n$ must be related to one another by some permutation of the form $\pi(g)$. In toher words, the connected components of $\mathbb{G}_n$ will depend upon the underlying permutation operad of G, in the following way:

Proposition 1.28. *Considered as a monoid under tensor product,*

$$\pi_0(\mathbb{G}_n) \quad = \quad \begin{cases} \mathbb{N}^n & \textit{if } G \textit{ is crossed} \\ \mathbb{N}^{*n} & \textit{otherwise} \end{cases}$$

Also, the canonical homomorphism sending objects in $\mathbb{G}_n$ to their connected component,

$$[\_] \,:\, \mathrm{Ob}(\mathbb{G}_n) \to \pi_0(\mathbb{G}_n)$$

is the quotient map of abelianisation

$$\mathrm{ab} \,:\, \mathbb{N}^{*n} \to (\mathbb{N}^{*n})^{\mathrm{ab}} \,=\, \mathbb{N}^n$$

when G is crossed, and the identity map $\mathrm{id}_{\mathbb{N}^{*n}}$ otherwise.

Finally, Lemma 1.27 also gives us a complete description of how the morphisms of $\mathbb{G}_n$ interact under tensor product, though we need a little new terminology in order to express it properly.

Definition 1.29. Let G be an action operad. Then we will also the notation G to denote the *underlying monoid* of this action operad. This is the natural way to consider G as a monoid, with its element set being all of its elements together, $\bigsqcup_m G(m)$, and with tensor product as its binary operation, $g \otimes h = \mu(e_2; g, h)$.

Also, note that this monoid comes equipped with a homomorphism $|\_| : G \to \mathbb{N}$, sending each $g \in G$ to the natural number m if and only if g is an element of the group $G(m)$. We'll call this number $|g|$ the *length* of g.

Definition 1.30. Let S be a set and $F(S)$ the free monoid on S, the monoid whose elements are strings of elements of S and whose binary operation is concatenation. Then we will denote by

$$|\_| : F(S) \to \mathbb{N}$$

the monoid homomorphism defined by sending each element of $S \subseteq F(S)$ to 1, and therefore also each concatenation of n elements of S to the natural number n. Again, we will call $|x|$ the *length* of $x \in F(S)$.

Lemma 1.31. *The monoid of morphisms of the algebra $\mathbb{G}_n$ is*

$$\mathrm{Mor}(\mathbb{G}_n) \;\cong\; G \times_{\mathbb{N}} \mathbb{N}^{*n}$$

where this pullback is taken over the respective length homomorphisms,

$$\begin{tikzcd}
G \times_{\mathbb{N}} \mathbb{N}^{*n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{N}^{*n} \arrow[d, "|_|"] \\
G \arrow[r, "|_|"] & \mathbb{N}
\end{tikzcd}$$

using the fact that $\mathbb{N}^{*n}$ is the free monoid $F\Big(\{z_1, ..., z_n\}\Big)$.

Chapter 2

Free invertible algebras as initial objects

In this chapter we will start to consider how to construct free EG-algebras on some number of invertible objects. Specifically, we will begin by showing that such algebras are the initial objects of a particular comma category, in accordance with some well known properties of adjunctions and their units. Using this initial object prespective will allow us to recover all of the data associated with the objects of a given free invertible algebra — what those objects are, how they act under tensor product, and which pairs of objects form the source and target of at least one morphism. Unfortunately, a concrete description of the morphisms themselves will ultimately remain elusive. We can get tantalisingly closer though, and an examiniation of the exact way that this method fails will provide the neccessary insight to motivate a more successful approach in **??**.

2.1 The free algebra on n invertible objects

We saw in Proposition 1.23 that the existence of a free EG-algebra on n objects can be proven by taking the left adjoint of a 2-functor which forgets about the algebra structure. Now we want to extend this idea into the realm of algebras on invertible objects. For the analogous approach, we will need to find a new 2-functor that lets us forget about non-invertible objects, and then hopefully we can find its left adjoint too, and use it to freely add inverses to $\mathbb{G}_n$. First though, we need to make this concept of 'forgetting non-invertible objects' a little more precise.

Definition 2.1. Given an EG-algebra X, we denote by X_{inv} the sub-EG-algebra containing all invertible objects in X and the isomorphisms between them.

Note that this is indeed a well-defined EG-algebra. If $x_1, ..., x_m$ are invertible objects with inverses $x_1^*, ..., x_m^*$, then $\alpha(g; x_1, ..., x_m)$ is an invertible object with inverse $\alpha(g; x_m^*, ..., x_1^*)$, since

$$\begin{aligned} & \alpha(g; x_1, ..., x_m) \otimes \alpha(g; x_m^*, ..., x_1^*) \\ = & \left(x_{\pi(g)^{-1}(1)} \otimes ... \otimes x_{\pi(g)^{-1}(m)} \right) \otimes \left(x^*_{\pi(g)^{-1}(m)} \otimes ... \otimes x^*_{\pi(g)^{-1}(1)} \right) \\ = & \; I \end{aligned}$$

$$\begin{aligned} & \alpha(g; x_m^*, ..., x_1^*) \otimes \alpha(g; x_1, ..., x_m) \\ = & \left(x^*_{\pi(g)^{-1}(m)} \otimes ... \otimes x^*_{\pi(g)^{-1}(1)} \right) \otimes \left(x_{\pi(g)^{-1}(1)} \otimes ... \otimes x_{\pi(g)^{-1}(m)} \right) \\ = & \; I \end{aligned}$$

Likewise, if $f_1, ..., f_m$ are isomorphisms from invertible objects $x_1, ..., x_m$ to invertible objects $y_1, ..., y_m$, then $\alpha(g; f_1, ..., f_m)$ is a map from the invertible object $\alpha(g; x_1, ..., x_m)$ to the invertible object $\alpha(g; y_1, ..., y_m)$, and it has an inverse $\alpha(g^{-1}; f^{-1}_{\pi(g)(1)}, ..., f^{-1}_{\pi(g)(m)})$, since

$$\begin{aligned} & \alpha\left(g^{-1} \, ; \, f^{-1}_{\pi(g)(1)}, \, ..., \, f^{-1}_{\pi(g)(m)} \right) \circ \alpha(\, g \, ; \, f_1, ..., f_m \,) \\ = & \; \alpha\left(g^{-1} g \, ; \, f_1^{-1} f_1, \, ..., \, f_m^{-1} f_m \right) \\ = & \; \mathrm{id}_{x_1 \otimes ... \otimes x_m} \end{aligned}$$

$$\begin{aligned} & \alpha(\, g \, ; \, f_1, ..., f_m \,) \circ \alpha\left(g^{-1} \, ; \, f^{-1}_{\pi(g)(1)}, \, ..., \, f^{-1}_{\pi(g)(m)} \right) \\ = & \; \alpha\left(g g^{-1} \, ; \, f_{\pi(g)(1)} f^{-1}_{\pi(g)(1)}, \, ..., \, f_{\pi(g)(m)} f^{-1}_{\pi(g)(m)} \right) \\ = & \; \mathrm{id}_{y_{\pi(g)(1)} \otimes ... \otimes y_{\pi(g)(m)}} \end{aligned}$$

Clearly then, X_{inv} is the correct algebra for our new forgetful 2-functor to send X to. Knowing this, we can contruct the rest of the functor fairly easily.

Proposition 2.2. *The assignment* $X \mapsto X_{\text{inv}}$ *can be extended to a 2-functor* $(\_)_{\text{inv}} : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$.

Proposition 2.3. *The 2-functor* $(\_)_{\text{inv}} : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$ *has a left adjoint,* $L : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$.

With this new 2-functor $L : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$, we now have the ability to ‘freely add inverses to objects’ in any EG-algebra we want. The algebra $L\mathbb{G}_n$ is then a clear candidate for our free algebra on n invertible objects, and indeed the proof of this is very simple.

Theorem 2.4. *There exists a free* EG*-algebra on n invertible objects. Specifically, the algebra $L\mathbb{G}_n$ is such that for any other* EG*-algebra X, we have an isomorphism of categories*

$$\mathrm{E}G\mathrm{Alg}_S(L\mathbb{G}_n, X) \quad \cong \quad (X_{\mathrm{inv}})^n$$

2.2 $L\mathbb{G}_n$ as an initial algebra

We have now proven that a free EG-algebra on n invertible objects indeed exists. But this fact on its own is not very helpful. To be able to actually use the free algebra $L\mathbb{G}_n$, we need to know how to contruct it explicitly, in terms of its objects and morphisms. We could do this by finding a detailed characterisation of the 2-functor L, and then applying this to our explicit description of $\mathbb{G}_n$ from Definition 1.24. However, this would probably take far more effort than is required, since it would involve determining the behaviour of L in many situtations that we aren't interested in. Also, we wouldn't be leveraging $\mathbb{G}_n$'s status as a free algebra to make the calculations any easier. We will try a different strategy instead, one that begins by noticing a special property of the functor L.

Proposition 2.5. *For any* EG*-algebra X, we have $L(X)_{\mathrm{inv}} = L(X)$.*

This result is not especially surprising. Intuitively, it just says that when you freely add inverses to an algebra, every object ends up with an inverse. But the upshot of this is that we now have another way of thinking about $L(X)$: as the target object of the unit of our adjunction, $\eta_X : X \to L(X)_{\mathrm{inv}}$. This means that we don't really need to know the entirety of L in order to determine the free algebra $L\mathbb{G}_n$, just its unit. To find this unit directly, we can turn to the following fact about adjunctions, for which a proof can be found in Lemma 2.3.5 of Leinster's *Basic Category Theory* [8].

Proposition 2.6. *Let $F \dashv G : A \to B$ be an adjunction with unit η. For any object a in A, let $(a \downarrow G)$ denote the comma category whose objects are pairs (b, f) consisting of an object* B *from* B *and a morphism $f : a \to G(b)$ from A, and whose morphisms $h : (b, f) \to (b', f')$ are morphisms $f : b \to b'$ from* B *such that $G(f) \circ f = f'$. Then the pair $\big(F(a), \eta_a : a \to GF(a)\big)$ is an initial object of $(a \downarrow G)$.*

Corollary 2.7. *$\eta_{\mathbb{G}_n} : \mathbb{G}_n \to (L\mathbb{G}_n)_{\mathrm{inv}} = L\mathbb{G}_n$ is an initial object of $(\mathbb{G}_n \downarrow \mathrm{inv})$.*

Being able to view $L\mathbb{G}_n$ as the initial object in the comma category $(\mathbb{G}_n \downarrow \mathrm{inv})$ will prove immensely useful in the coming sections. This is because it lets us think about the properties of $L\mathbb{G}_n$ in terms of maps $\psi : \mathbb{G}_n \to X_{\mathrm{inv}}$, and this is exactly the context where we can exploit $\mathbb{G}_n$'s status as a free algebra. As a result, its worth taking some time to think about what exactly this map $\eta_{\mathbb{G}_n}$ is.

Lemma 2.8. *The initial object $\eta_{\mathbb{G}_n} : \mathbb{G}_n \to L\mathbb{G}_n$ is the obvious map from the free* EG*-algebra on n objects into the free* EG*-algebra on n* invertible *objects. That is, $\eta_{\mathbb{G}_n}$ is the algebra map defined by*

$$\begin{array}{rrcl} \eta_{\mathbb{G}_n} & : \quad \mathbb{G}_n & \to & L\mathbb{G}_n \\ & : \quad F(\{z_1, ..., z_n\}) & \to & LF(\{z_1, ..., z_n\}) \\ & : \quad z_i & \mapsto & z_i \end{array}$$

This increadibly simple description makes the map η very easy to work with. For example, we immediately obtain the following property, one which we will use frequently throughout the rest of the paper:

Corollary 2.9. *η is an epimorphism in* EGAlg$_S$*.*

Before moving on, we'll make a small change in notation. From now on, rather than writing objects in $(\mathbb{G}_n \downarrow \mathrm{inv})$ as maps $\psi : \mathbb{G}_n \to Y_{\mathrm{inv}}$, we will instead just let $X = Y_{\mathrm{inv}}$ and speak of maps $\psi : \mathbb{G}_n \to X$. This is purely to prevent the notation from becoming cluttered, and shouldn't be a problem so long as we always remember that the targets of these maps only ever contain invertible objects and morphisms. We'll also drop the subscript from $\eta_{\mathbb{G}_n}$, since it is the only component of the unit we'll ever use.

2.3 The objects of $L\mathbb{G}_n$

So now we know that $L\mathbb{G}_n$ is an initial object in the category $(\mathbb{G}_n \downarrow \mathrm{inv})$. But what does this actually tell us? After all, we do not currently have a method for finding initial objects in an arbitrary collection of EG-algebra maps. Because of this, we'll have to approach the problem step-by-step, using the initiality of η to extract different pieces of information about the algebra $L\mathbb{G}_n$ as we go. We'll begin by tring to find its objects.

Definition 2.10. Denote by Ob : EGAlg$_S$ $\to$ Mon be the functor that sends EG-algebras X to their monoid of objects Ob(X), and algebra maps $F : X \to Y$ to their underlying monoid homomorphism Ob(F) : Ob$(X) \to$ Ob(Y).

In order to find Ob$(L\mathbb{G}_n)$, we'll need to make use of an important result about the nature of Ob.

Definition 2.11. Recall that given a monoid M, the monoidal category $\mathrm{E}M$ is the one whose monoid of objects is M and which has a unique isomorphism between any two objects. We can view $\mathrm{E}M$ as not just a category but an $\mathrm{E}G$-algebra, by letting the action on morphisms take the only possible values it can, given the required source and target. Similarly, for any monoid homomorphisms $h : M \to M'$ we can define a map of $\mathrm{E}G$-algebras

$$\begin{array}{rcrcl} \mathrm{E}h & : & \mathrm{E}M & \to & \mathrm{E}M' \\ & : & m & \mapsto & h(m) \\ & : & m \to m' & \mapsto & h(m) \to h(m') \end{array}$$

This definition of $\mathrm{E}h$ respects composition and identities, and so together with $\mathrm{E}M$ it describes a functor $\mathrm{E} : \mathrm{Mon} \to \mathrm{E}G\mathrm{Alg}_S$.

Proposition 2.12. E *is a right adjoint to the functor* Ob.

What Proposition 2.12 is essentially saying is that the functor Ob provides a way for us to move back and forth between the categories $\mathrm{E}G\mathrm{Alg}_S$ and Mon. By applying this reasoning to the universal property of the initial object η, we can then determine the value of $\mathrm{Ob}(L\mathbb{G}_n)$ in terms of a new universal property of $\mathrm{Ob}(\eta)$ in the category Mon. In particular, the algebras in $(\mathbb{G}_n \downarrow \mathrm{inv})$ are those whose objects are all invertible, and so the induced property of $\mathrm{Ob}(\eta)$ will end up saying something about the relationship between $\mathrm{Ob}(\mathbb{G}_n)$ and groups — those monoids whose elements are all invertible.

Definition 2.13. Let M be a monoid, M^{gp} a group, and $i : M \to M^{\mathrm{gp}}$ a monoid homomorphism between them. Then we say that M^{gp} is the *group completion* of M if for any other group H and homomorphism $h : M \to H$, there exists a unique homomorphism $u : M^{\mathrm{gp}} \to H$ such that $u \circ i = h$.

There are several different ways to actually calculate the group completion of a monoid. One is to use that fact that M^{gp} is the group whose group presentation is the same as the monoid presentation of M. That is, if M is the quotient of the free monoid on generators $\mathcal{G}$ by the relations $\mathcal{R}$, then M^{gp} is the quotient of the free *group* on generators $\mathcal{G}$ by relations $\mathcal{R}$. This makes finding the completion of free monoids particularly simple.

Proposition 2.14. *The object monoid of* $L\mathbb{G}_n$ *is* $\mathbb{Z}^{*n}$*, the group completion of the object monoid of* $\mathbb{G}_n$*. The restriction of* η *on objects,* $\mathrm{Ob}(\eta)$*, is then the obvious inclusion* $\mathbb{N}^{*n} \hookrightarrow \mathbb{Z}^{*n}$.

2.4 The connected components of $L\mathbb{G}_n$

The core result of Proposition 2.14 — that $\mathrm{Ob}(L\mathbb{G}_n)$ is the group completion of $\mathrm{Ob}(\mathbb{G}_n)$ — makes concrete the sense in which the functor L represents 'freely adding inverses' to objects. Extending this same logic to connected components as well, it would seem reasonable to expect that $\pi_0(L\mathbb{G}_n)$ is the group completion of $\pi_0(\mathbb{G}_n)$ as well. This is indeed the case, and the proof proceeds in a way completely analagous to Proposition 2.14.

First, we want to show that the process of taking connected components forms part of an adjunction. To do this we are going to need a category from which we can draw the kind of structures that can act as the components of an EG-algebra. Exactly which category this should be will depend on our choice of action operad G, or more precisely its underlying permutations.

Definition 2.15. For a given action operad G, denote by $\mathrm{im}(\pi)-\mathrm{Mon}$ the full subcategory of Mon on those monoids whose multiplication is invariant under the permutations in $\mathrm{im}(\pi)$. That is, a monoid M is in $\mathrm{im}(\pi)-\mathrm{Mon}$ if and only if

$$m_1, ..., m_n \in M,\ g \in G(n) \quad \Longrightarrow \quad m_1...m_n = m_{\pi(g)^{-1}(1)}...m_{\pi(g)^{-1}(n)}$$

Of course, by Lemma 1.12 there are really only two examples of such a $\mathrm{im}(\pi)-\mathrm{Mon}$. If the underlying permutations of G are trivial, then $\mathrm{im}(\pi)-\mathrm{Mon}$ is just the whole of the category Mon; if instead G is crossed then we are asking for monoids whose multiplication is invariant under arbitrary permutations from S, and so $\mathrm{im}(\pi)-\mathrm{Mon}$ is just the category of *commutative* monoids, CMon. Regardless, when we are working with an arbitrary action operad G, the category $\mathrm{im}(\pi)-\mathrm{Mon}$ is exactly the collection of possible connected components that we were looking for.

Lemma 2.16. *Let G be an action operad and $\mathrm{im}(\pi)$ its underlying permutation action operad. Then there is a functor*

$$\pi_0 : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{im}(\pi)-\mathrm{Mon}$$

which sends each algebra X to its monoid of connected components $\pi_0(X)$, and sends each map of algebras $F : X \to Y$ to its restriction to connected components $\pi_0(F) : \pi_0(X) \to \pi_0(Y)$.

Now that we have a functor which represents the act of finding the connected component monoid of an algebra, we need another functor heading in the opposite direction, so that we can construct an adjunction between them.

Definition 2.17. There exists an inclusion of 2-categories $\mathrm{D} : \mathrm{Set} \hookrightarrow \mathrm{Cat}$ which allows us to view any set S as a *discrete category*, one whose objects are just the elements of S and whose morphisms are all identities. If the given set also happens to be a monoid M, then there is an obvious way to see the discete category $\mathrm{D}M$ as a monoidal category, and so we have a similar inclusion $\mathrm{D} : \mathrm{Mon} \hookrightarrow \mathrm{MonCat}$. Finally, for any action operad G and object M of the category $\mathrm{im}(\pi)-\mathrm{Mon}$, there is a unique way to assign an $\mathrm{E}G$-action to the discete category $\mathrm{D}M$. This works because for any elements $m_1, ..., m_n \in M$ and $g \in G(n)$, the morphism $\alpha(g; \mathrm{id}_{m_1}, ..., \mathrm{id}_{m_n})$ must have source and target

$$m_1 \otimes ... \otimes m_n \quad = \quad m_{\pi(g^{-1})(1)} \otimes ... \otimes m_{\pi(g^{-1})(m)}$$

and therefore it can only be the morphism $\mathrm{id}_{m_1 \otimes ... \otimes m_n}$. This choice of action yields one last inclusion $\mathrm{CMon} \hookrightarrow \mathrm{E}G\mathrm{Alg}_S$, which we shall also call D.

Proposition 2.18. D *is a right adjoint to the functor* π_0.

Now we can utilise Proposition 2.18 to draw out a universal property of $\pi_0(L\mathbb{G}_n)$, just as we did with $\mathrm{Ob}(L\mathbb{G}_n)$ in Proposition 2.12.

Proposition 2.19. *The connected components of* $L\mathbb{G}_n$ *are the group completion of the connected components of* $\mathbb{G}_n$. *Also, the restriction of* η *onto connected components,* $\pi_0(\eta)$, *is the canonical map* $\pi_0(\mathbb{G}_n) \to \pi_0(\mathbb{G}_n)^{\mathrm{gp}}$ *associated with that group completion.*

As we've said before, this result is a reflection of the fact that the functor L is trying to add inverses the objects of $\mathbb{G}_n$ freely, that is, with as little effect on the rest of the algebra as possible. Indeed, if we happen to know whether or not our action operad G is crossed then we can now calculate exactly what the effect on the components will be.

Corollary 2.20. *If* G *is a crossed action algebra then*

- *the connected components of* $L\mathbb{G}_n$ *are the monoid* $\mathbb{Z}^n$
- *the restriction of* η *to components is the obvious inclusion* $\mathbb{N}^n \hookrightarrow \mathbb{Z}^n$
- *the assignment of objects to their component is given by the quotient map of abelianisation* $\mathrm{ab} : \mathbb{Z}^{*n} \to \mathbb{Z}^n$

If instead G *is non-crossed, then*

- *the connected components of* $L\mathbb{G}_n$ *are the monoid* $\mathbb{Z}^{*n}$
- *the restriction of* η *to components is the obvious inclusion* $\mathbb{N}^{*n} \hookrightarrow \mathbb{Z}^{*n}$
- *the assignment of objects to their component is* $\mathrm{id}_{\mathbb{Z}^{*n}}$

2.5 The collapsed morphisms of $L\mathbb{G}_n$

Now that we understand the objects and connected components of the algebra $L\mathbb{G}_n$, the next most obvious thing to look for are its morphisms, $\mathrm{Mor}(L\mathbb{G}_n)$. It would be nice to construct this collection in the same way we constructed $\mathrm{Ob}(L\mathbb{G}_n)$ and $\pi_0(L\mathbb{G}_n)$, by applying the left adjoint of some adjunction to the initial map η. Before we can do this however, we need to ask ourselves a question. What sort of mathematical object is $\mathrm{Mor}(L\mathbb{G}_n)$, exactly?

Given a pair of morphisms $f : x \to y, f' : y' \to z$ in an EG-algebra X, there are two basic binary operations we can perform. First, we can take their tensor product $f \otimes f'$, and this together with the unit map id_I imbues $\mathrm{Mor}(X)$ with the structure of a monoid. Second, if we have $y = y'$ then we can form the composite morphism $f' \circ f$. However, these two operations are not as different as they first appear.

Lemma 2.21. *Let $f : x \to y$ and $f' : y \to z$ be morphisms in some monoidal category, and y is an invertible object of that category. Then*

$$f' \circ f \quad = \quad f' \otimes \mathrm{id}_{y*} \otimes f$$

In other words, composition along invertible objects in X can always be restated in terms of the tensor product. Thus in cases where every object of X is invertible, the monoidal structure together with knowledge of each morphisms source and target will be enough to determine X uniquely. Since all objects in $L\mathbb{G}_n$ are invertible, this means that we could choose to ignore composition of elements of $\mathrm{Mor}(L\mathbb{G}_n)$ for the time being, and focus on its status as a monoid under tensor product.

However, we are trying to extract information about the morphisms of $L\mathbb{G}_n$ by building some sort of left adjoint functor. Presumably we will also be able to apply it to other EG-algebras, some of which won't have all of their objects invertible, and so we can't just use $\mathrm{Mor}(-) : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{Mon}$. What we need is a way to modify the morphism monoid of a category so that both composition and tensor product are recoverable from a single operation. Of course, there is one very easy method for achieving this — simply force $\otimes$ and $\circ$ to be equal.

Definition 2.22. Let $\mathrm{M} : \mathrm{MonCat} \to \mathrm{Mon}$ be the functor which sends monoidal categories X to the quotient of their monoid of morphisms by the relation that sets $\otimes = \circ$.

$$\mathrm{M}X \quad = \quad {\mathrm{Mor}(X)}\big/{f' \circ f \sim f' \otimes f}$$

Each monoidal functors $F : X \to Y$ is then sent to the monoid homomorphism

$$\begin{array}{rcrcl} \mathrm{M}(F) & : & \mathrm{M}X & \to & \mathrm{M}Y \\ & : & \mathrm{M}(f) & \mapsto & \mathrm{M}\Big(F(f) \Big) \end{array}$$

where $\mathrm{M}(f)$ refers to the equivalence class of the map f under the quotient $\mathrm{Mor}(X) \to \mathrm{M}(X)$. This homomorphism is well-defined, since it respects the relation $\otimes = \circ$:

$$\begin{aligned} \mathrm{M}(F)(\, f' \circ f\,) &= \mathrm{M}\Big(F(f' \circ f) \Big) \\ &= \mathrm{M}\Big(F(f') \circ F(f) \Big) \\ &= \mathrm{M}\Big(F(f') \Big) \circ \mathrm{M}\Big(F(f) \Big) \\ &= \mathrm{M}\Big(F(f') \Big) \otimes \mathrm{M}\Big(F(f) \Big) \\ &= \mathrm{M}\Big(F(f') \otimes F(f) \Big) \\ &= \mathrm{M}\Big(F(f' \otimes f) \Big) \\ &= \mathrm{M}(F)(\, f' \otimes f\,) \end{aligned}$$

We will call $\mathrm{M}X$ the *collapsed* morphisms of the X.

From now on we will generally refer to the single operation in $\mathrm{M}X$ as $\otimes$ rather than $\circ$, unless we are focusing on some aspect best understood using compositon. This convention makes it easier to remember that because the tensor product is defined between all pairs of morphisms in X, the equivalence class $\mathrm{M}(f') \otimes \mathrm{M}(f)$ will always contain the morphism $f' \otimes f$, but not neccessarily $f' \circ f$, as it might fail to exist.

Now we need a candidate for the right adjoint to the functor M.

Definition 2.23. For a given monoid M, let $\mathrm{B}M$ represent the one-object category whose morphisms are the elements of M, with monoid multiplication as composition. Likewise, for any monoid homomorphism $h : M \to M'$ between abelian groups, denote by $\mathrm{B}h : \mathrm{B}M \to \mathrm{B}M'$ the obvious monoidal functor which acts like h on morphisms. This defines a functor $\mathrm{B} : \mathrm{Mon} \to \mathrm{Cat}$ from the category of monoids onto the category of small categories.

Moreover, let C be a commutative monoid. Then we can view $\mathrm{B}C$ as a monoidal category, with the tensor product also given by the multiplication in C, and the sole object as the unit I. Clearly for any homomorphism between commutative monoids $h : C \to C'$ the corresponding functor $\mathrm{B}h : \mathrm{B}C \to \mathrm{B}C'$ will preserve this monoidal structure, as it is already preserving it as compositon. Thus the restriction of B to commutative monoids also gives a functor $\mathrm{CMon} \to \mathrm{MonCat}$, which we will still call B.

Commutativity is required in order for BC to be a well-defined monoidal category because we need its operations $\circ$ and $\otimes$ to obey the interchange law for monoidal categories:

$$\begin{array}{rrcl} & (\mathrm{id}_I \circ f) \otimes (f' \otimes \mathrm{id}_I) & = & (\mathrm{id}_I \otimes f') \circ (f \otimes \mathrm{id}_I) \\ \implies & \mathrm{id}_I \cdot f \cdot f' \cdot \mathrm{id}_I & = & \mathrm{id}_I \cdot f' \cdot f \cdot \mathrm{id}_I \\ \implies & f \cdot f' & = & f' \cdot f \end{array}$$

Proposition 2.24. B *is a right adjoint to the functor* $\mathrm{M}(\_)^{\mathrm{ab}} : \mathrm{MonCat} \to \mathrm{CMon}$.

Proposition 2.24 seems at first glance very similar to Propositions 2.12 and 2.18. However, our goal was to discover the relationship between the morphisms of $\mathbb{G}_n$ and $L\mathbb{G}_n$, paralleling what we did in Propositions 2.14 and 2.19, and in that regard M falls short in two very important ways.

1. What we really wanted to have was an adjunction involving $\mathrm{E}G\mathrm{Alg}_S$, not MonCat. This is because our previous methodology involved applying our left adjoint functors to η and then using its initial property to factor various maps through $L\mathbb{G}_n$. But η is an initial object in $(\mathbb{G}_n \downarrow \mathrm{inv})$, and so we only know how to use it to factor *algebra* maps $\mathbb{G}_n \to X_{\mathrm{inv}}$, and not general monoidal functors.

2. Even if we do find a way to use this adjunction to extract information about $L\mathbb{G}_n$, it will not be the monoid $\mathrm{Mor}(L\mathbb{G}_n)$ we were originally after, only a strange abelianised version where tensor product and composition coincide.

Unfortunately, this adjunction seems to be the best we can do. The only general method for assigning an EG-action to the monoidal category BC for all C is to set all of its action morphisms $\alpha(g; \mathrm{id}_I, ..., \mathrm{id}_I)$ to be id_I. This would then cause the homomorphism M$X \to C$ corresponding to any algebra map $X \to$ BC to be the zero map if X has only action morphisms. Given Lemma 1.27, this is clearly no use. However, it turns out that this approach is fixable. To that end, we will spend the bulk of the next two chapters directly addressing problems 1 and 2.

For now though, we will make one last small alteration to our plan going forward. Instead of working directly with the functor $\mathrm{M}(\_)^{\mathrm{ab}} : \mathrm{MonCat} \to \mathrm{CMon}$, we will instead focus on its composite with the group completion functor, $(\_)^{\mathrm{gp}} : \mathrm{CMon} \to \mathrm{Ab}$. It may not be clear yet why we would choose to do this, but over the next couple of chapters we will frequently find ourselves having to forming quotients of certain algebraic objects. If we were to stick with the functor M these would all be commutative monoid quotients, whereas by making the switch to $\mathrm{M}(\_)^{\mathrm{gp,ab}}$ they will be abelian groups instead, which are far easier to work with. Also, notice that since the process of group completion is left adjoint to the forgetful functor $\mathrm{Ab} \to \mathrm{CMon}$, its composite with the left adjoint $\mathrm{M}(\_)^{\mathrm{ab}}$ will be a left adjoint functor too. Thus with this new functor we will be able use all of the same important properties that we would have done with $\mathrm{M}(\_)^{\mathrm{ab}}$, such as the preservation of colimits. Moreover, while we won't prove this for some time, it turns out that the morphisms of $L\mathbb{G}_n$ actually form a group under tensor product. This means that whatever method we would have used to recover $\mathrm{Mor}(L\mathbb{G}_n)$ from $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{ab}}$ should still let us recover $\mathrm{Mor}(L\mathbb{G}_n) = \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp}}$ from $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$.

Before we move on, we should spend a little time thinking about this new functor $\mathrm{M}(\_)^{\mathrm{gp,ab}}$. Specifically, we might ask in what order we have to carry out its constituent parts: the collapsing of $\circ$ and $\otimes$ into a single operation, group completion, and abelianisation. It is a well known fact that group completion and abelianisation commute:

$$\begin{tikzcd}
\mathrm{Mon} \arrow[r, "(_)^{\mathrm{gp}}"] \arrow[d, "(_)^{\mathrm{ab}}"'] & \mathrm{Grp} \arrow[d, "(_)^{\mathrm{ab}}"] \\
\mathrm{CMon} \arrow[r, "(_)^{\mathrm{gp}}"] & \mathrm{Ab}
\end{tikzcd}$$

Indeed, we already assume this when talking of 'the' canonical map $\mathrm{M}(X)^{\mathrm{gp,ab}}$. But a more interesting question is whether it matters if we choose to group complete or abelianise the tensor product of a monoidal category before or after we collapse its morphisms.

Lemma 2.25. *For any monoidal category X, define*

$$\mathrm{M}_{\mathrm{gp}}(X) \cong {\mathrm{Mor}(X)^{\mathrm{gp}}}\Big/{\mathrm{gp}(f' \circ f) \sim \mathrm{gp}(f' \otimes f)}$$

$$\mathrm{M}_{\mathrm{ab}}(X) \cong {\mathrm{Mor}(X)^{\mathrm{ab}}}\Big/{\mathrm{ab}(f' \circ f) \sim \mathrm{ab}(f' \otimes f)}$$

Then

$$\mathrm{M}_{\mathrm{gp}}(X) = \mathrm{M}(X)^{\mathrm{gp}}, \qquad \mathrm{M}_{\mathrm{ab}}(X) = \mathrm{M}(X)^{\mathrm{ab}}$$

In other words, we do not need to worry about order of operations when using the left adjoint functor $\mathrm{M}(\_)^{\mathrm{gp,ab}}$. This is very convenient, and later on when we actually need to evalute particular $\mathrm{M}(X)^{\mathrm{gp,ab}}$, we will use this fact to carry out the calculation in whichever order proves easiest.

Chapter 3

Free invertible algebras as colimits

In the previous chapter, we made progress towards understanding the structure of $L\mathbb{G}_n$ by showing that the algebra was an initial object in a certain comma category. Specifically, we saw that the map $\eta : \mathbb{G}_n \to L\mathbb{G}_n$ is initial among all EG-algebra maps $\mathbb{G}_n \to X_{\text{inv}}$. This fact is the rigourous way of expressing a fairly obvious intuition about $L\mathbb{G}_n$ — that we should expect the free algebra on n invertible objects to be like the free algebra on n objects, except that its objects are invertible.

However, this not the only way of thinking about $L\mathbb{G}_n$. Consider for a moment the free EG-algebra on $2n$ objects, $\mathbb{G}_{2n}$. Intuitively, if we were to take this algebra and then enforce upon it the extra relations $z_{n+1} = z_1^*, ..., z_{2n} = z_n^*$, then we would be changing it from a structure with $2n$ independent generators into one with n indepedent generators and their inverses. That is, there seems to be a natural way to think about $L\mathbb{G}_n$ as a quotient of the larger algebra $\mathbb{G}_{2n}$. In this chapter we will work towards making this idea precise, and then examine some of its consequences, the most important of which will be allowing us to describe the group $\mathrm{M}(L\mathbb{G}_n)^{\text{gp,ab}}$.

3.1 $L\mathbb{G}_n$ as a cokernel in $\mathrm{E}G\mathrm{Alg}_S$

We'll begin with some definitions.

Definition 3.1. Let δ be the map of EG-algebras defined on generators by

$$\begin{array}{rcrcl} \delta & : & \mathbb{G}_{2n} & \to & \mathbb{G}_{2n} \\ & : & z_i & \mapsto & z_i \otimes z_{n+i} \\ & : & z_{n+i} & \mapsto & z_{n+i} \otimes z_i \end{array}$$

for $1 \leq i \leq n$. We will also denote by $q : \mathbb{G}_{2n} \to Q$ the cokernel this map.

Note that the above definition does actually make sense. The given descriptions of δ is enough to specify it uniquely because $\mathbb{G}_{2n}$ is the free EG-algebra on $2n$ objects, and hence algebra maps $\mathbb{G}_{2n} \to \mathbb{G}_{2n}$ are canonically isomorphic to functions $\{z_1, ..., z_{2n}\} \to \mathrm{ob}(\mathbb{G}_{2n})$. Also we can be sure that the map q exists, because E$G\mathrm{Alg}_S$ is a locally finitely presentable category and thus has all finite colimits.

The goal of this approach will be show that Q is in fact that same algebra as $L\mathbb{G}_n$. In order to do this, it would help if we could easily compare $q : \mathbb{G}_{2n} \to Q$ to our initial object $\eta : \mathbb{G}_{2n} \to L\mathbb{G}_n$. In other words, we really want to show that q is an object of $(\mathbb{G}_n \downarrow \mathrm{inv})$ — that Q has only invertible objects. This can be done using the adjunction we found in Proposition 2.12.

Proposition 3.2. *The object monoid of Q is $\mathbb{Z}^{*n}$, and the restriction of q to objects $\mathrm{Ob}(q) : \mathrm{Ob}(\mathbb{G}_{2n}) \to \mathrm{Ob}(Q)$ is the monoid homomorphism defined on generators as*

$$\begin{array}{rcrcl} \mathrm{Ob}(q) & : & \mathbb{N}^{*2n} & \to & \mathbb{Z}^{*n} \\ & : & z_i & \mapsto & z_i \\ & : & z_{n+i} & \mapsto & z_i^* \end{array}$$

An immediate corollary of Proposition 3.2 is that every object of the cokernel algebra Q is invertible. Thus $q : \mathbb{G}_{2n} \to Q$ is an object of the category $(\mathbb{G}_n \downarrow \mathrm{inv})$, and hence we can use the initiality of η to determine the following result:

Proposition 3.3. *Let $i : \mathbb{G}_n \to \mathbb{G}_{2n}$ be the inclusion of* EG*-algebras defined on generators by $i(z_i) = z_i$. Then $i \circ q$ is an initial object of $(\mathbb{G}_n \downarrow \mathrm{inv})$. In particular, this means that*

$$Q \quad \cong \quad L\mathbb{G}_n$$

It's worth noting that we have not given a method for actually taking cokernels in E$G\mathrm{Alg}_S$, and so Proposition 3.3 doesn't immediately provide an explicit description for the whole of $L\mathbb{G}_n$. However, it does offer us another way to extract partial information, like what we were doing in Chapter 2. Consider Proposition 3.2; now that we know that Q is actually $L\mathbb{G}_n$, the statement of this proposition is just the same as that of Proposition 2.14. But the proof of the former uses the ability of cokernels to preserve left adjoint functors, rather than any of the initial algebra and group completion properties that appear in the latter.

Of course, by Proposition 3.3 the fact that q is a cokernel is equivalent to it being initial, and so while they may not look it at first glance, these two approaches are secretly the same. Thus from now on whenever we are trying to determine some aspect of $L\mathbb{G}_n$, we will make sure to take a look at both methods, just in case there are some properties of our free algebra which are more readily apparent from one description than another.

3.2 $L\mathbb{G}_n$ as a surjective coequaliser

An immediate consequence our new cokernel perspective of $L\mathbb{G}_n$ is that, since left adjoint functor all preserve colimits, Propositions 2.12 and 2.18 now both imply results about the partial surjectivity of this new map q. The former says that since $\mathrm{Ob}(q)$ is a cokernel map of monoids, and hence that every object of $L\mathbb{G}_n$ is the image under q of some object of $\mathbb{G}_{2n}$; the latter says a similar thing for connected components. From this one might guess that q is will just turn out to be a surjective map of EG-algebras, and indeed this is the case.

Unfortunately, we can not go about proving that q is surjective on morphisms by a similar adjunction technique, since this best we have is the one from Proposition 2.24 and it will only tell us about the map $\mathrm{M}(q)^{\mathrm{gp,ab}}$. However, there is a general result about the coequalisers of EG-algebras that we can prove to get us around this.

Proposition 3.4. *Let $\phi, \phi' : X \to Y$ be a pair of parallel* EG*-algebra maps, and $k : Y \to Z$ their coequalizer in* $\mathrm{E}G\mathrm{Alg}_S$*. If the monoid* $\mathrm{Ob}(Z)$ *is also a group, then the functor k is surjective.*

Because the cokernel of a morphism is just its coequaliser with the zero map, and since we know that the objects of $L\mathbb{G}_n$ form a group, we can immediately apply this result to the functor q.

Corollary 3.5. *The cokernel map $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$ is surjective.*

This is probably the single most important step in our effort to determine the morphisms of $L\mathbb{G}_n$, in the sense of how many of the results hereafter rely on this relatively simple property. Indeed this result is so strong that after a cursory glance, one might be forgiven for thinking that it will immediately provide for us the main thing we have been working towards this chapter — the value of $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$.

After all, every surjective functor is an epimorphism in the category MonCat. We know that left adjoint functors preserve epimorphisms, and that $\mathrm{M}(\_)^{\mathrm{gp,ab}}$ is a left adjoint, so from Corollary 3.5 we can surmise that $\mathrm{M}(q)^{\mathrm{gp,ab}}$ is also an epimorphism, this time in Ab. But an epimorphic map of abelian groups is nothing other than a surjective homomorphism, and thus we may apply the First Isomorphism Theorem of groups to get the following:

$$\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \quad = \quad {\mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}}} \Big/ {\ker\Big(\,\mathrm{M}(q)^{\mathrm{gp,ab}}\,\Big)}$$

So if we knew what the kernel of $\mathrm{M}(q)^{\mathrm{gp,ab}}$ was, we would be done. And it seems like we *should* know this; q was defined to be the cokernel of δ, and by preservation of this colimits means that $\mathrm{M}(q)^{\mathrm{gp,ab}}$ is the cokernel of $\mathrm{M}(\delta)^{\mathrm{gp,ab}}$. Then since we are working with abelian groups, kernels and cokernels interact in a nice way:

$$\ker\operatorname{coker}\Big(\,\mathrm{M}(\delta)^{\mathrm{gp,ab}}\,\Big) \quad = \quad \mathrm{im}\Big(\,\mathrm{M}(\delta)^{\mathrm{gp,ab}}\,\Big)$$

However, this last step doesn't actually work — q was defined to be $\mathrm{coker}(\delta)$, but only in the category of $\mathrm{E}G$-algebras. In general this will *not* be the same thing as the cokernel of δ in MonCat, which is what we would really need in order for $\mathrm{M}(\_)^{\mathrm{gp,ab}}$ to preserve it.

Still, this is a pretty reasonable guess for what $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$ is, and provides an indication of how we should proceed in order to find its true value. We will pick up on this idea again in Section 3.4.

3.3 Action morphisms of $L\mathbb{G}_n$

One important consequence of the surjectivity of q is that it will allow us to import certain results about the free algebra $\mathbb{G}_{2n}$ into the free invertible algebra $L\mathbb{G}_n$. In fact, we have done this once already; looking back at Proposition 3.2 with our current knowledge that $Q = L\mathbb{G}_n$, we can see that it is a direct analogue of Lemma 1.26, using the fact that q is surjective on objects.

In that same vein, one might ask if we can take Lemma 1.27, a statement about the morphisms $\mathbb{G}_{2n}$, and extend it to an analagous result on $L\mathbb{G}_n$, using surjectivity of q on morphisms instead. That is, since every morphism of $\mathbb{G}_{2n}$ is an action morphism, and since $\mathrm{E}G$-algebra maps always send action morphisms to action morphisms, we should be able to use q to identify every morphism of $L\mathbb{G}_n$ as an action morphism. This is indeed pretty simple to show.

Lemma 3.6. *Every morphism in $L\mathbb{G}_n$ can be expressed as $\alpha_{L\mathbb{G}_n}(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$, for some $g \in G(m)$ and $x_i \in \{z_1, ..., z_n, z_1^*, ..., z_n^*\}$.*

Lemma 3.6 formalises a certain intuition about how the functor L should act on algebras, the idea that a 'free' structure really shouldn't have any 'superfluous' components, only whatever data is absolutely required for it to be well-defined. In the case of $L\mathbb{G}_n$, we have proven that the only morphisms contained in the free EG-algebra on invertible objects are EG-action morphisms. However, while this is very similar to what we have in the non-invertible case it should be stressed that Lemma 3.6 does *not* prove that the morphisms of $L\mathbb{G}_n$ have *unique* representations $\alpha(g; \mathrm{id}_{w_1}, ..., \mathrm{id}_{w_m})$, as morphisms of $\mathbb{G}_n$ do.

Also, notice that when we eventually find a complete description of $L\mathbb{G}_n$ as a monoidal category, we will be able to use the surjective algebra map q to determine it's EG-action as well. This follows from the same reasoning we used to prove Lemma 3.6, but in reverse:

$$\begin{aligned} \alpha_{L\mathbb{G}_n}(\, g\, ;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,) &= \alpha_{L\mathbb{G}_n}(\, g\, ;\, \mathrm{id}_{q(x'_1)}, ..., \mathrm{id}_{q(x'_m)}\,) \\ &= q\Big(\, \alpha_{\mathbb{G}_{2n}}(\, g\, ;\, \mathrm{id}_{x'_1}, ..., \mathrm{id}_{x'_m}\,)\Big) \end{aligned}$$

In fact, since we do know that q is a cokernel of the map δ, we can even extract some information about this action right away, before we have built an understanding of the morphisms of $L\mathbb{G}_n$.

Lemma 3.7. *For any element $g \in G(m), m \in \mathbb{N}$ of an action operad G,*

$$\alpha_{L\mathbb{G}_n}(\, g\, ;\, \mathrm{id}_I, ..., \mathrm{id}_I\,) \quad = \quad \mathrm{id}_I$$

Equivalently, for any element $h \in G(0)$,

$$\alpha_{L\mathbb{G}_n}(\, h\, ;\, -\,) \quad = \quad \mathrm{id}_I$$

This is a pretty interesting result. By Lemma 1.27, morphisms of the form $\alpha_{\mathbb{G}_n}(g; \mathrm{id}_I, ..., \mathrm{id}_I)$ make up the entirety of the homset $\mathbb{G}_n(I, I)$. Now we see that their image under the algebra map $\eta : \mathbb{G}_n \to L\mathbb{G}_n$ is always id_I, and so it follows that the unit endomorphisms of free algebras are wholly unrelated to the unit endomorphisms of the corresponding free *invertible* algebras. That is, when constructing $L\mathbb{G}_n$ it seems like it should not matter whether our chosen action operad G has nontrivial $G(0)$, since all morphisms $\alpha_{L\mathbb{G}_n}(g; -)$ for $g \in G(0)$ are going to end up as the identity regardless. In order to state this idea more concretely though, we need some way of 'removing' the group $G(0)$ from G.

Proposition 3.8. *Let G be a crossed action operad. Then there exists another crossed action operad G' which has $G'(m) = G(m)/G(0)$ for all $m \in$* N.

For crossed G, this notion of quotient by $G(0)$ does exactly what we want it to do — remove certain information which is unnecessary for forming the algebra $L\mathbb{G}_n$.

Proposition 3.9. *Let G be a crossed action operad, and let G' be the action operad with $G'(m) = G(m)/G(0)$ for all $m \in \mathrm{N}$. Then for any $n \in \mathrm{N}$,*

$$L\mathbb{G}'_n \quad \cong \quad L\mathbb{G}_n$$

both as EG*-algebras and as* EG'*-algebras. That is, every free invertible algebra over a crossed action operad is the same as one over an action operad with trivial $G(0)$.*

For noncrossed G we cannot so easily remove the group $G(0)$ like this, as without being spacial we have no way to draw its elements out from inbetween elements of the higher $G(m)$. Still, there is one more thing about the morphisms of $L\mathbb{G}_n$ that we can deduce from Lemma 3.7.

Definition 3.10. Let G be a noncrossed action operad in which every element of each $G(m)$ can be written as $\mu(g; e_m)$ for some $G \in G(1)$. Then we say that G is a $G(1)$-*generated* action operad.

Lemma 3.11. *If G is a $G(1)$-generated action operad, then $L\mathbb{G}_n(I, I)$ is the trivial group.*

Ultimately, we will see that there is very little we can say for sure about the unit endomorphisms of $L\mathbb{G}_n$ when G is not crossed, other than Lemma 3.11. For this reason, the main theorems of this paper will end up describing only those invertible EG-algebras whose actionoperads are either crossed or $G(1)$-generated.

3.4 $L\mathbb{G}_n$ as a coequaliser in MonCat

Looking back at the proof of Proposition 3.4, notice that we never needed to use the fact that ϕ, ϕ' and k were maps of EG-algebras, only that they were monoidal functors. Because we had assumed from the beginning that we were working in EGAlg$_S$, we did at one point have to show that the category $k(Y)$ was an algebra, so that we could then use the universal property of k in EGAlg$_S$, but if k had just been a coequaliser in MonCat from the start then this part would not have been necessary. We also had to invoke Proposition 2.12 — which says that Ob : EGAlg$_S$ $\to$ Mon is a left adjoint — so that we could exploit preservation of colimits. But since Ob clearly doesn't care about the morphisms of an algebra, it doesn't really matter whether we are applying it to an algebra in the first place. The actions of X, Y and Z just never came into play.

With that in mind, we can co-opt all of these previous proofs about EG-algebra maps to prove the analagous statements about monoidal functors.

Proposition 3.12. *Let the functors*

$$\mathrm{Ob} : \mathrm{MonCat} \to \mathrm{Mon}, \qquad \mathrm{E} : \mathrm{Mon} \to \mathrm{MonCat}$$

be defined exactly as those from Definitions 2.10 and 2.11, except without the requirement that the monoidal categories be EG*-algebras. Then* E *is a right adjoint to the functor* Ob*.*

Proposition 3.13. *Let $\phi, \phi' : X \to Y$ be a pair of parallel monoidal functors, and $k : Y \to Z$ their coequalizer in* MonCat*. If the monoid* $\mathrm{Ob}(Z)$ *is also a group, then the functor k is surjective.*

Further, these new propositions prove a surjectivity statement just like Corollary 3.5.

Definition 3.14. Let the monoidal functor $c : \mathbb{G}_{2n} \to C$ onto some monoidal category C be the cokernel of the underlying monoidal functor of δ in MonCat. This map definitely exists because MonCat is cocomplete, and like with q we can show that its target has a group of objects.

Proposition 3.15. *The object monoid of C is $\mathbb{Z}^{*n}$, and the restriction of c to objects* $\mathrm{Ob}(c) : \mathrm{Ob}(\mathbb{G}_{2n}) \to \mathrm{Ob}(C)$ *is the monoid homomorphism defined on generators as*

$$\begin{array}{rcrcl} \mathrm{Ob}(c) & : & \mathbb{N}^{*2n} & \to & \mathbb{Z}^{*n} \\ & : & z_i & \mapsto & z_i \\ & : & z_{n+i} & \mapsto & z_i^* \end{array}$$

Propositions 3.13 and 3.15 then immediately combine to give:

Corollary 3.16. *The cokernel map $c : \mathbb{G}_{2n} \to C$ is surjective.*

This statement is actually pretty unusual. In Corollary 3.5 it made sense that q would be surjective, but that was because its source and target were special. $\mathbb{G}_{2n}$ is the free EG-algebra on $2n$ objects, and $L\mathbb{G}_n$ is the free EG-algebra on n objects and their n inverses, and so intuitively the map identifying those sets generators would tell us everything we need to know about the algebra structure of $L\mathbb{G}_n$. And since by freeness we expect algebra maps to be all there really is to $L\mathbb{G}_n$, it was a safe bet that q was going to be surjective.

But none of that is true for c. The underlying monoidal category of $\mathbb{G}_{2n}$ is not anything special in MonCat, and neither is C. So what is going on here? The answer is that category C is *almost* the algebra $L\mathbb{G}_n$, and likewise the functor c is *almost* the map q. To see this, consider the following niave method for assigning an EG-action α_C to C:

$$\alpha_C(\,g\,;\,c(f_1),...,c(f_m)\,) \quad := \quad c\Big(\,\alpha_{\mathbb{G}_{2n}}(\,g\,;\,f_1,...,f_m\,)\Big)$$

Any action on C that made c into a map of EG-algebras would have to satify this condition, of course. But because c is surjective, every collection of morphisms in C can be written as $c(f_1),...,c(f_m)$, and this forces α_C to take a unique value everywhere, assuming it is well-defined. Then, since the the cokernel of δ in MonCat would be an EG-algebra map, we could conclude that it was also the cokernel of δ in EGAlg$_S$ too. However, 'assuming it is well-defined' is where the problems lie. In particular, since c is not injective on objects we can find $w_1,...,w_m$ and $w'_1,...,w'_m$ in $\mathbb{G}_{2n}$ for which $c(w_i) = c(w'_i)$, and so α^C would only be well-defined if

$$c\Big(\,\alpha_{\mathbb{G}_{2n}}(\,g\,;\,\mathrm{id}_{w_1},...,\mathrm{id}_{w_m}\,)\Big) \quad = \quad c\Big(\,\alpha_{\mathbb{G}_{2n}}(\,g\,;\,\mathrm{id}_{w'_1},...,\mathrm{id}_{w'_m}\,)\Big)$$

which we have no reason to believe is true.

To fix this issue, what we need is a way of describing the map q as a colimit of a slightly different diagram in EG-algebras, one whose colimit in MonCat will have all of the same properties that c does but will also satisfy the condition above. To that end, consider the following EG-algebra maps:

Definition 3.17. Let $\tilde{\delta} := \mathrm{id}_{\mathbb{G}_{2n}} + \delta$ be the map defined from δ and the identity by using the universal property of the coproduct $\mathbb{G}_{4n} = \mathbb{G}_{2n} + \mathbb{G}_{2n}$ in EGAlg$_S$. That is, $\mathrm{id}_{\mathbb{G}_{2n}} + \delta$ is the map of EG-algebras which acts on generators by

$$\begin{array}{cccccc}
\tilde{\delta} & : & \mathbb{G}_{4n} & \to & \mathbb{G}_{2n} \\
 & : & z_i & \mapsto & z_i \\
 & : & z_{n+i} & \mapsto & z_{n+i} \\
 & : & z_{2n+i} & \mapsto & z_i \otimes z_{n+i} \\
 & : & z_{3n+i} & \mapsto & z_{n+i} \otimes z_i
\end{array}$$

for $1 \leq i \leq n$. Similarly, let $\tilde{I} := \mathrm{id}_{\mathbb{G}_{2n}} + I$ be the EG-algebra map defined in the same way but from the constant map on the unit I instead of δ:

$$\begin{array}{rcrcl} \tilde{I} & : & \mathbb{G}_{4n} & \to & \mathbb{G}_{2n} \\ & : & z_i & \mapsto & z_i \\ & : & z_{n+i} & \mapsto & z_{n+i} \\ & : & z_{2n+i} & \mapsto & I \\ & : & z_{3n+i} & \mapsto & I \end{array}$$

Lemma 3.18. *q is the coequaliser of $\tilde{\delta}$ and $\tilde{I}$ in* EGAlg$_S$.

While this proof may seem rather trivial, notice that it does rely on the fact that the $+$ here represents the coproduct in the category of EG-algebras. There is no reason to expect that the coequaliser of the underlying monoidal functors of these maps would also be equal the cokernel of the underlying monoidal functor of δ. Thus these new maps will give rise to a new map which is distinct from the cokernel functor c, yet possesses many of the same properties.

Definition 3.19. Denote by $\tilde{c} : \mathbb{G}_{2n} \to \tilde{C}$ the coequaliser of $\tilde{\delta}$ and $\tilde{I}$ in the category MonCat.

Lemma 3.20. *The object monoid of $\tilde{C}$ is*

$$\mathrm{Ob}(\tilde{C}) \quad = \quad \mathrm{Ob}(C) \quad = \quad \mathbb{Z}^{*n}$$

and the restriction of $\tilde{c}$ to objects $\mathrm{Ob}(\tilde{c}) : \mathrm{Ob}(\mathbb{G}_{2n}) \to \mathrm{Ob}(\tilde{C})$ is just the monoid homomorphism $\mathrm{Ob}(c) : \mathbb{N}^{*2n} \to \mathbb{Z}^{*n}$ from Proposition 3.15.

Corollary 3.21. *The coequaliser map $\tilde{c} : \mathbb{G}_{2n} \to \tilde{C}$ is surjective.*

So why bother with any of this? What features do $\tilde{\delta}$ and $\tilde{I}$ have that will make an action possible on $\tilde{C}$ when it wasn't on C? The answer is that unlike δ and I, these new maps form a *reflexive pair* — a parallel pair of functors which share a right-inverse.

Lemma 3.22. *Let $\iota : \mathbb{G}_{2n} \to \mathbb{G}_{4n}$ be the inclusion of algebras defined on generators by $z_i \mapsto z_i$. Then ι is a right-inverse of both $\tilde{\delta}$ and $\tilde{I}$.*

In other words, $\tilde{c}$ is a *reflexive coequalizer* in the category MonCat. This is the key difference which will eventually let us prove that $\tilde{c}$ respects action morphisms in the way that we need it to. First though, we will need a few intermediate results.

Definition 3.23. If w is an element of $\mathbb{N}^{*m}$, then we can use the definition of the free product of groups to decompose it uniquely as a tensor product of the m generators $z_1, ..., z_m$. We'll denote this by

$$w \quad =: \quad \bigotimes_{i=1}^{|w|} d(w,i), \qquad\qquad d(w,i) \in \{z_1, ..., z_m\}$$

If instead w is an element of $\mathbb{Z}^{*m}$, then we can use the definition of the free product of groups to decompose x uniquely as a tensor product, but this time one made up of the m generators $z_1, ..., z_m$ and their inverses $z_1^*, ..., z_m^*$. As before we'll denote this by

$$w \quad = \quad \bigotimes_{i=1}^{|w|} d(w,i)$$

where $d(w,i) \in \{z_1, ..., z_m, z_1^*, ..., z_m^*\}$, and also for any $1 \leq i < |w|$ we will always have $d(w,i+1) \neq d(w,i)^*$. By analogy with Definition 1.30, we will call the upper bound of this tensor product the *length* of the element w, and denote it by $|w|$, but be aware that this number is the one that comes from the *monoid* homomorphism

$$F\Big(\{z_1, ..., z_m, z_1^*, ..., z_m^*\}\Big) \to \mathbb{N}$$

that sends each generator to 1, and not the perhaps more obvious *group* homomorphism

$$F\Big(\{z_1, ..., z_m\}\Big) \to \mathbb{Z}$$

Proposition 3.24. *Let w be an object of $\mathbb{G}_{2n}$. Then there exist objects $w^{(1)}, ..., w^{(k)}$ in $\mathbb{G}_{2n}$ and $u^{(1)}, ..., u^{(k)}$ in $\mathbb{G}_{4n}$, for some value of $k \in \mathbb{N}$, such that*

$$w^{(1)} \;=\; w, \qquad u^{(k)} \;=\; \iota(w^{(k)}), \qquad\quad \tilde{I}(u^{(i-1)}) \quad = \quad w^{(i)} \quad = \quad \tilde{\delta}(u^{(i)})$$

for $1 \leq i \leq k$, and for any object u of $\mathbb{G}_{4n}$,

$$\tilde{\delta}(u) \;=\; w^{(k)} \qquad \Longleftrightarrow \qquad u \;=\; u^{(k)}$$

The intuition behind Proposition 3.24 is that we are successively removing parts of the object w, without changing its image under $\tilde{c}$. The map $\tilde{\delta}$ sends $z_{2n+i} \mapsto z_i \otimes z_{n+i}$ and $z_{3n+i} \mapsto z_{n+1} \otimes z_i$ while $\tilde{I}$ sends these all to I, and so for any u in $\mathbb{G}_{4n}$ the object $\tilde{I}(u)$ will look like $\tilde{\delta}(u)$ except missing some number of $z_i \otimes z_{n+i}$ or $z_{n+1} \otimes z_i$ substrings. But since $\tilde{c}$ sends $z_{n+i} \mapsto z_i^*$, these are exactly the sort of omissions which the coequaliser doesn't care about. If we repeat this process then it will eventually terminate at $u^{(k)} = \iota(w^{(k)})$, so we really have a method for removing *all* of the relevent substrings from objects of $\mathbb{G}_{2n}$. In other words, $w^{(k)}$ has the smallest possible length while still having $\tilde{c}(w^{(k)}) = \tilde{c}(w)$. In fact, we will show that it is the unique shortest object of $\mathbb{G}_{2n}$ with this property.

Proposition 3.25. *Let w, w' be objects of $\mathbb{G}_{2n}$ such that $\tilde{c}(w) = \tilde{c}(w')$. If $w^{(1)}, ..., w^{(k)}$ and $u^{(1)}, ..., u^{(k)}$ are the sequences generated from w via Proposition 3.24, and likewise ${w'}^{(1)}, ..., {w'}^{(k')}$ and ${u'}^{(1)}, ..., {u'}^{(k')}$ from w', then $w^{(k)} = {w'}^{(k')}$ and $u^{(k)} = {u'}^{(k')}$.*

It is this special property — shared by all w, w' for which $\tilde{c}(w) = \tilde{c}(w')$ — that will now let us prove that the coequaliser $\tilde{c}$ satisfies the condition which we couldn't prove about the cokernel c. In other words, with Propositions 3.24 and 3.25 we can now contruct a valid $\mathrm{E}G$-action on the monoidal category $\tilde{C}$.

Proposition 3.26. *There is a unique action $\alpha_{\tilde{C}}$ making the category $\tilde{C}$ into $\mathrm{E}G$-algebra and the functor $\tilde{c} : \mathbb{G}_{2n} \to \tilde{C}$ into a map of $\mathrm{E}G$-algebras.*

3.5 Extracting $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$ from $\mathbb{G}_{2n}$

We are now finally ready to address problem 1 from the end of the previous chapter: how can we deal with the fact that our adjunction $\mathrm{M}(\_)^{\mathrm{gp,ab}} \dashv C$ involves monoidal categories rather than full $\mathrm{E}G$-algebras? It turns out that this is all we really needed, as despite us originally conceiving of $L\mathbb{G}_n$ as a colimit in $\mathrm{E}G\mathrm{Alg}_S$ it can equally be viewed as a slightly more complicated colimit in MonCat.

Proposition 3.27. *The coequaliser functor $\tilde{c} : \mathbb{G}_{2n} \to \tilde{C}$ defined in Definition 3.19 is isomorphic as a map of $\mathrm{E}G$-algebras to $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$, the cokernel of δ in $\mathrm{E}G\mathrm{Alg}_S$.*

With our newfound ability to express the map $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$ as a colimit of monoidal categories, we can now set about using the adjunction from Proposition 2.24 to calculate $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$. The most obvious way to do this is to mimic what we did in Proposition 3.2 — apply the left adjoint functor to q and then commute it with the colimit to get a formula in terms of the known monoid $\mathrm{Mor}(\mathbb{G}_{2n})$.

Proposition 3.28. *Let Δ be the subgroup of $\mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}}$ generated by elements of the form*

$$\mathrm{M}(\tilde{\delta})^{\mathrm{gp,ab}}(f) \otimes \mathrm{M}(\tilde{I})^{\mathrm{gp,ab}}(f)^*, \qquad f \in \mathrm{M}(\mathbb{G}_{4n})^{\mathrm{gp,ab}}$$

Then the abelianisation of the group completion of the collapsed morphisms of $L\mathbb{G}_n$ *is*

$$\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \quad = \quad {\mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}}} \Big/ {\Delta}$$

with $\mathrm{M}(q)^{\mathrm{gp,ab}}$ *acting as the appropriate quotient map.*

Notice that the subgroup Δ contains all of $\mathrm{im}(\mathrm{M}(\delta)^{\mathrm{gp,ab}})$, but in general these two groups are not the same. This means that the effort we put into avoiding the naive mistake we could have made at the end of Section 3.2 was indeed worth it.

Now, at some point later on we will want to actually evaluate the quotient in for particular values of action operad G. This would be fairly tricky without an explicit desciption of the elements of Δ, so we need to take a moment to think about what we really mean by $\mathrm{M}(\tilde{\delta})^{\mathrm{gp,ab}}(f) \otimes \mathrm{M}(\tilde{I})^{\mathrm{gp,ab}}(f)^*$.

Lemma 3.29. Δ *is the subgroup of* $\mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}}$ *whose elements are tensor products of equivalence classes*

$$\begin{gathered} \Big[\, \alpha_{\mathbb{G}_{2n}}\Big(\, \mu(\, g\, ;\, e_{|\tilde{\delta}(x_1)|}, ..., e_{|\tilde{\delta}(x_m)|}\,)\, ;\, \mathrm{id}_{x'_1}, ..., \mathrm{id}_{x'_{m'}}\,\Big)\,\Big] \\ \otimes \\ \Big[\, \alpha_{\mathbb{G}_{2n}}\Big(\, \mu(\, g\, ;\, e_{|\tilde{I}(x_1)|}, ..., e_{|\tilde{I}(x_m)|}\,)\, ;\, \mathrm{id}_{x''_1}, ..., \mathrm{id}_{x''_{m''}}\,\Big)\,\Big]^* \end{gathered}$$

where $g \in G(m)$*, the* x_i *are generators of* $\mathbb{N}^{*4n}$*, the* x'_i, x''_i *are generators of* $\mathbb{N}^{*2n}$*, and*

$$\begin{aligned} \tilde{\delta}(x_1 \otimes ... \otimes x_m) &= x'_1 \otimes ... \otimes x'_{m'} \\ \tilde{I}(x_1 \otimes ... \otimes x_m) &= x''_1 \otimes ... \otimes x''_{m''} \end{aligned}$$

Chapter 4

Morphisms of free invertible algebras

The goal of this chapter will be to show that we can reconstruct all of the morphisms of $L\mathbb{G}_n$ from the abelian group $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$, and therefore that we can actually use the adjunction from Proposition 2.24 to help find a description of the free $\mathrm{E}G$-algebra on n invertible objects.

The first step towards this goal will involve splitting $\mathrm{Mor}(L\mathbb{G}_n)$ up as the product of two other monoids. The first of these will encode all of the possible combinations of source and target data for morphisms in $L\mathbb{G}_n$, while the second will just be the endomorphisms of the unit object, $L\mathbb{G}_n(I, I)$. In other words, we will see that the monoid $\mathrm{Mor}(L\mathbb{G}_n)$ can be broken down into a context where source and target are the only thing that matters, and another where they are irrelevant.

Once we have done this, we can then use the fact that $L\mathbb{G}_n(I, I)$ is always an abelian group to rewrite $\mathrm{Mor}(L\mathbb{G}_n)$ in terms of its abelian group completion, $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$. This is not quite the same thing as $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$, but they are close enough that we can find a simple equation linking the two, which will in turn allow us to frame the former in terms of the quotient of $\mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}}$ we described last chapter. All together, this will constitute an expression for $\mathrm{Mor}(L\mathbb{G}_n)$ that is built up of pieces which we know how to calculate.

4.1 Sources and targets in $L\mathbb{G}_n$

To get things started, we will spend this section considering the source and target information of morphisms in $L\mathbb{G}_n$.

Definition 4.1. For any EG-algebra X, denote by $s : \mathrm{Mor}(X) \to \mathrm{Ob}(X)$ and $t : \mathrm{Mor}(X) \to \mathrm{Ob}X)$ the monoid homomorphisms which send each morphism of X to its source and target, respectively. That is,

$$s(f : x \to y) = x, \qquad t(f : x \to y) = y$$

If we use the universal property of products, we can combine these source and target homomorphisms into a single map, $s \times t : \mathrm{Mor}(X) \to \mathrm{Ob}(X) \times \mathrm{Ob}(X)$. The monoid we are interested in finding is the image $L\mathbb{G}_n$ under its instance of this map.

Lemma 4.2. *Let X be an* EG*-algebra, and $s \times t : \mathrm{Mor}(X) \to \mathrm{Ob}(X)^2$ the map built from s and t using the universal property of products. Then the image of this map is*

$$(s \times t)(X) = \mathrm{Ob}(X) \times_{\pi_0(X)} \mathrm{Ob}(X)$$

where this pullback is taken over the canonical maps sending objects of X to their connected components:

$$\begin{tikzcd}
\mathrm{Ob}(X) \times_{\pi_0(X)} \mathrm{Ob}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ob}(X) \arrow[d, "{[_]}"] \\
\mathrm{Ob}(X) \arrow[r, "{[_]}"] & \pi_0(X)
\end{tikzcd}$$

Recalling Lemma 1.26, Propositions 1.28 and 2.14, and Corollary 2.20, we can immediately conclude the following:

Corollary 4.3.

$$(s \times t)(\mathbb{G}_n) = \begin{cases} \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} & \textit{if } G \textit{ is crossed} \\ \mathbb{N}^{*n} & \textit{otherwise} \end{cases}$$

$$(s \times t)(L\mathbb{G}_n) = \begin{cases} \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} & \textit{if } G \textit{ is crossed} \\ \mathbb{Z}^{*n} & \textit{otherwise} \end{cases}$$

where the pullbacks are taken over the quotients of abelianisation for $(\mathbb{N}^{*n})^{\mathrm{ab}} = \mathbb{N}^n$ and $(\mathbb{Z}^{*n})^{\mathrm{ab}} = \mathbb{Z}^n$ respectively.

Next, we want to show that this $(s \times t)(L\mathbb{G}_n)$ we have described is in fact a submonoid of $\mathrm{Mor}(L\mathbb{G}_n)$. This is a little tricky though, since we don't currently know what the morphisms of $L\mathbb{G}_n$ even are. We will sidestep this problem by first proving the analogous statement for all $\mathbb{G}_n$, and then recovering the $L\mathbb{G}_n$ version from it later.

Now, by Lemma 1.31 we know that wanting $(s \times t)(\mathbb{G}_n)$ to be a submonoid of $\mathrm{Mor}(\mathbb{G}_n)$ is the same as asking if we can find an injective homomorphism $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \to G \times_{\mathbb{N}} \mathbb{N}^{*n}$, assuming G is crossed, or $\mathbb{N}^{*n} \to G \times_{\mathbb{N}} \mathbb{N}^{*n}$ if it is not. The latter case is pretty obvious, so we'll focus on crossed G for the moment. Creating a injective *function* $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \to G \times_{\mathbb{N}} \mathbb{N}^{*n}$ is not especially hard. For any pair $(w, w') \in \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$, the image of w and w' in the abelian group $\mathbb{N}^n$ is the same, which is to say that if $x_1, ..., x_m \in \{z_1, ..., z_n\}$ are the collection of generators for which

$$w \quad = \quad x_1 \otimes ... \otimes x_m$$

and there exists at least one permutation $\sigma \in S_m$ such that

$$w' \quad = \quad x_{\sigma(1)} \otimes ... \otimes x_{\sigma(m)}$$

Then since the underlying permutation maps $\pi : G(m) \to \mathrm{S}_m$ of a crossed action operad G are all surjective, we can always find an element of $g \in G(m)$ for which $\pi(g) = \sigma$. Thus in order to make our injective function all we need to do is make a choice $g =: \rho(w, w')$ like this to represent each (w, w'), and then set

$$\begin{array}{rcl} \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} & \to & G \times_{\mathbb{N}} \mathbb{N}^{*n} \\ (w, w') & \mapsto & (\,\rho(w, w'), w\,) \end{array}$$

Injectivity follows because given a specific (g, w), the only element that could map onto it is $(w, \pi(g)(w))$.

So how do we know if we can choose these representatives $\rho(w, w')$ in such a way that the resulting function i is also a monoid homomorphism? If we could find a presentation of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ in terms of generators and relations then this would help a little, since we would only need to pick a $\rho(z, z')$ for each generator (z, z'), and then define all other $\rho(w, w')$ by way of tensor products:

$$\rho(v \otimes w, v' \otimes w') \quad = \quad \rho(v, v') \otimes \rho(w, w')$$

But then we would still need make sure that our choice of $\rho(z, z')$ obeyed the necessary relations on the generators of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$. Luckily for us though, this turns out to be no problem at all.

Proposition 4.4. $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ *is a free monoid.*

It follows immediately from this that our earlier contruction of an injective function $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \to G \times_{\mathbb{N}} \mathbb{N}^{*n}$ can always be extended to be an inclusion of monoids.

Proposition 4.5. *$(s \times t)(\mathbb{G}_n)$ is (isomorphic to) a submonoid of* $\mathrm{Mor}(\mathbb{G}_n)$

In other words, this result says that the source and target data of $\mathbb{G}_n$ is isomorphic to the monoid made up of action morphisms

$$\alpha\Big(\rho(x_1 \otimes ... \otimes x_m, x_{\sigma(1)} \otimes ... \otimes x_{\sigma(1)})\,;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\Big)$$

when G is crossed, and

$$\alpha(e_m; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) \quad = \quad \mathrm{id}_{x_1 \otimes ... \otimes x_m}$$

otherwise, for $\sigma \in S_m$, $x_1, ..., x_m \in \{z_1, ..., z_n\}$. Now, in theory the map $\rho : \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \longrightarrow G$ that we use to choose representatives can be any valid homomorphism between those monoids for which

$$\pi(\rho(w, w'))(w) \quad = \quad w'$$

but later on we will be able to make things easier on ourselves by making a more systematic choice.

So now we have shown that $(s \times t)(\mathbb{G}_n)$ is a submonoid of $\mathrm{Mor}(\mathbb{G}_n)$, but what we were really interested in is whether or not $(s \times t)(\mathbb{G}_n)$ is a submonoid of $\mathrm{Mor}(\mathbb{G}_n)$. To recover the latter result from the former, we will use our cokernel map $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$. In particular, the surjectivity of q combined with the case $(s \times t)(\mathbb{G}_{2n}) \subseteq \mathrm{Mor}(\mathbb{G}_{2n})$ from Proposition 4.5, immediately gives us what we need.

Corollary 4.6. *$(s \times t)(L\mathbb{G}_n)$ is (isomorphic to) a submonoid of* $\mathrm{Mor}(L\mathbb{G}_n)$

4.2 Unit endomorphisms of $L\mathbb{G}_n$

To help us understand $\mathrm{Mor}(L\mathbb{G}_n)$, we decided to break it down into two smaller pieces. The first of these was the source/target data $(s \times t)(L\mathbb{G}_n)$, which we explored in the previous section. The other piece that we now have to consider is the monoid of unit endomorphisms, $L\mathbb{G}_n(I, I)$.

This is a particularly important submonoid of the morphisms $\mathrm{Mor}(L\mathbb{G}_n)$, since it is the only submonoid which is also a homset of the category $L\mathbb{G}_n$. Moreover, because the maps in $L\mathbb{G}_n(I, I)$ all share the same source and target, what we have is not just a monoid under tensor product but also under composition as well. This fact leads to a series of special properties for $L\mathbb{G}_n(I, I)$, the first of which is just another instance of the classic Eckmann-Hilton argument.

Lemma 4.7. *$L\mathbb{G}_n(I, I)$ is a commutative monoid under both tensor product and composition, with $f \otimes f' = f \circ f'$.*

In fact, since we already proved that the morphisms of $L\mathbb{G}_n$ are all actions morphisms, we can take this one step further.

Proposition 4.8. *$L\mathbb{G}_n(I, I)$ is an abelian group.*

Indeed, by using a slightly broader argument we can extend this result to every morphism of $L\mathbb{G}_n$.

Proposition 4.9. *Every morphism $f : w \to v$ in $L\mathbb{G}_n$ has an inverse under tensor product, $f^* : w^* \to v^*$. That is, the monoid $\mathrm{Mor}(L\mathbb{G}_n)$ is actually a group.*

So $\mathrm{Mor}(L\mathbb{G}_n)$ and $L\mathbb{G}_n(I, I)$ both turn out to be groups under tensor product. Obviously it follows from this that $L\mathbb{G}_n(I, I)$ is a not just a submonoid of $\mathrm{Mor}(L\mathbb{G}_n)$ but a subgroup — in particular an abelian subgroup, going by Proposition 4.8. But $L\mathbb{G}_n(I, I)$ is actually an even more special subgroup than this.

Proposition 4.10. *$L\mathbb{G}_n(I, I)$ is a normal subgroup of $\mathrm{Mor}(L\mathbb{G}_n)$. Moreover, if G is a crossed action operad, then $L\mathbb{G}_n(I, I)$ is a subgroup of the centre of $\mathrm{Mor}(L\mathbb{G}_n)$.*

4.3 The morphisms of $L\mathbb{G}_n$

We have finally described all of the important properties of $(s \times t)(L\mathbb{G}_n)$ and $L\mathbb{G}_n(I, I)$ that we will need going forward. Putting all of these results together will allow us to characterize the morphisms of $L\mathbb{G}_n$ as a product of groups, as was promised at the beginning of this chapter. Before we do so though, it will be worth going over a few well-known pieces of group theory that we will be using in the proof of Proposition 4.14.

Definition 4.11. Let H, K and N be groups. Then we say that H is a *group extension* of K by N if there exists a short exact sequence

$$0 \longrightarrow N \overset{i}{\hookrightarrow} H \xrightarrow{p} K \longrightarrow 0$$

In other words, H is an extension of K by N whenever we have $K = H/N$. Moreover, if N is a subgroup of the centre of H, we say that this is a *central* extension, and if the map p has a right-inverse, $r : K \to H$, $p \circ r = \mathrm{id}_K$, then we say that it is a *split* extension.

Definition 4.12. Let H be a group with subgroup K and normal subgroup N. Then we say that H is a *semidirect product* $K \ltimes N$ if the underlying set of H is the same as underlying set of $K \times N$, but multiplication is given by

$$(k, n) \cdot (k', n') \quad = \quad (\, kk',\, nkn'k^{-1}\,)$$

Lemma 4.13. **??** *If H is a split extension of K by N then $H = K \ltimes N$, with $r : K \to H$ acting as the appropriate subgroup inclusion. Further, if H is a both split and central, then $H \cong K \times N$.*

With that out of the way, we can now produce an expression for the morphisms of $L\mathbb{G}_n$.

Proposition 4.14. *For any action operad G,*

$$\mathrm{Mor}(L\mathbb{G}_n) \quad \cong \quad (s \times t)(L\mathbb{G}_n) \ltimes L\mathbb{G}_n(I, I)$$

Moreover, if G is a crossed action operad, then

$$\mathrm{Mor}(L\mathbb{G}_n) \quad \cong \quad (s \times t)(L\mathbb{G}_n) \times L\mathbb{G}_n(I, I)$$

In certain select cases, Proposition 4.14 will actually be sufficient to fully determine $\mathrm{Mor}(L\mathbb{G}_n)$ — specifically, whenever we know that the unit endomorphisms of $L\mathbb{G}_n$ are trivial. We already know of two examples like this, due to Proposition 3.9 and Lemma 3.11.

Corollary 4.15. *If G is a crossed action operad with $G(m) = G(0)$ for all $m \in \mathbb{N}$, then*

$$\mathrm{Mor}(L\mathbb{G}_n) \quad = \quad (s \times t)(L\mathbb{G}_n) \quad = \quad \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$$

Instead if G is a $G(1)$-generated action operad, then

$$\mathrm{Mor}(L\mathbb{G}_n) \quad = \quad (s \times t)(L\mathbb{G}_n) \quad = \quad \mathrm{Ob}(L\mathbb{G}_n) \quad = \quad \mathbb{Z}^{*n}$$

In the latter case, what this is saying is that every morphism in $L\mathbb{G}_n$ is just the identity element of some object.

But what about for more general $L\mathbb{G}_n$ with nontrivial unit endomorphisms? For crossed G, the key insight is that one half of the product in Proposition 4.14, $L\mathbb{G}_n(I, I)$, is always an abelian group. This means that it will remain untouched if we were to abelianise the entire product, thus providing a link between $\mathrm{Mor}(L\mathbb{G}_n)$ before and after abelianisation.

Proposition 4.16. *Let G be a crossed action operad. Then the endomorphisms of the unit object of $L\mathbb{G}_n$ are*

$$L\mathbb{G}_n(I, I) \quad = \quad {\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}}} \big/ {(s \times t)(L\mathbb{G}_n)^{\mathrm{ab}}}$$

and therefore

$$\mathrm{Mor}(L\mathbb{G}_n) \quad = \quad (s \times t)(L\mathbb{G}_n) \times {\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}}} \big/ {(s \times t)(L\mathbb{G}_n)^{\mathrm{ab}}}$$

Unfortunately, there is no general version of Proposition 4.16 for when G is not crossed. If we try to abelianise the semideirect product from Proposition 4.14, we will arrive at a product of the relevant abelian group, but a new term will also appear indicating the degree to which $L\mathbb{G}_n(I, I)$ and $(s \times t)(L\mathbb{G}_n)$ fail to commute.

Lemma 4.17. *If H is semidirect product $K \ltimes N$, then its abelianisation is*

$$H^{\mathrm{ab}} \quad = \quad K^{\mathrm{ab}} \times {N^{\mathrm{ab}}} \big/ {[N, K]}$$

where $[N, K]$ is the commutator of N with K.

If we stick to working with crossed action operads however, we are now only one step away from our full expression for $\mathrm{Mor}(L\mathbb{G}_n)$. The last term whose value we do not know is $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}} = \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}}$, and as one might expect this is related to the value that the algebra takes under the collapsed morphism left adjoint, $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}} = \mathrm{M}(L\mathbb{G}_n)^{\mathrm{ab}}$

Proposition 4.18. *Let X be any monoidal category whose objects are morphisms are all invertible under tensor product. Then the group completion of the abelianisation of the collapsed morphisms of X are*

$$\mathrm{M}(X)^{\mathrm{ab}} \quad \cong \quad {\mathrm{Mor}(X)^{\mathrm{ab}}} \big/ {\mathrm{Ob}(X)^{\mathrm{ab}}}$$

where we are viewing $\mathrm{Ob}(X)$ *as a subgroup of* $\mathrm{Mor}(X)$ *under tensor product by using the inclusion*

$$\begin{array}{rcl} \mathrm{Ob}(X) & \to & \mathrm{Mor}(X) \\ x & \mapsto & \mathrm{id}_x \end{array}$$

Now it just remains to chain together all of our previous results.

Proposition 4.19. *For crossed action operads G, the morphism monoid of $L\mathbb{G}_n$ is equal to*

$$\mathrm{Mor}(L\mathbb{G}_n) \quad = \quad \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \times \frac{\left(\mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \big/ \Delta\right)}{\left((\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}} \big/ \mathbb{Z}^n\right)}$$

4.4 Abelianising sources and targets

To say that the expression for $\mathrm{Mor}(L\mathbb{G}_n)$ we just found is 'complicated' would probably be an understament. If we are to have any hope of eventually being able to use Proposition 4.19, we need to work out a more explicit presentation for its quotient part. We'll start by trying to find the value of $(s \times t)(L\mathbb{G}_n)^{\mathrm{ab}}$ for crossed G, the abelian group $(\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}}$. This will require some careful consideration, since in general limits such as the pullback do not interact with abelianisation in a simple way. What would help is a suitable presentation of $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$ in terms of generators and relations.

Proposition 4.20. *The group $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$ is generated by the elements*

$$\langle x \rangle \quad := \quad (x, x) \qquad \text{and} \qquad \langle xy \rangle \quad := \quad (xy, yx)$$

where $x, y \in \{z_1, ..., z_n, z_1^*, ..., z_n^*\}$ are generators of the free group $\mathbb{Z}^{*n}$ or their inverses. These are subject to the relations

$$\langle x \rangle^{-1} \quad = \quad \langle x^* \rangle, \qquad \langle xy \rangle^{-1} \quad = \quad \langle y^* x^* \rangle$$

$$\langle xx^* \rangle \quad = \quad e \quad = \quad \langle x^* x \rangle, \qquad \langle xx \rangle \quad = \quad \langle x \rangle^2$$

$$\langle xy \rangle \langle x^* \rangle \langle xy^* \rangle \quad = \quad \langle x \rangle$$

$$\langle xy \rangle \langle x^* \rangle \langle y^* \rangle \langle yx \rangle \quad = \quad \langle x \rangle \langle y \rangle \quad = \quad \langle yx \rangle \langle x^* \rangle \langle y^* \rangle \langle xy \rangle$$

$$\langle xy \rangle \langle x^* \rangle \langle xz \rangle \langle x^* \rangle \langle z^* \rangle \langle y^* \rangle \langle yz \rangle \langle y^* \rangle \langle yx \rangle \langle y \rangle \langle x^* \rangle \langle z^* \rangle^{-1} \langle zx \rangle \langle z^* \rangle \langle zy \rangle \quad = \quad \langle x \rangle \langle y \rangle \langle z \rangle$$

Of course, the collection of relations we just gave in Proposition 4.20 are nowhere near minimal. Many of them clearly interact with each other in ways that would let us simplify or cancel some relations, or even generators. However, we will not expend any effort trying to do this, because we do not need to. With this inefficient presentation of $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$ in hand, we have in a sense already found its abelianisation. After all, for any presentation of some group H, the group H^{ab} possesses a presentation consisting of the exact same generators, subject to the same relations, plus a commutativity condition. This too will not normally be the most efficient description of the new group, but that remains true even if the presentation of H we started with was minimal, and so any time spent finding one will just be wasted. Instead, we'll suppress the urge to simplify Proposition 4.20 and move straight on to tackling $(\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}}$.

Proposition 4.21.

$$(\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}} \quad = \quad \mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}}$$

From this presentation, it should be immediately obvious how to calculate the denominator from Proposition 4.19.

Corollary 4.22.

$$(\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}} \big/ \mathbb{Z}^n \quad = \quad \mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}} \big/ \mathbb{Z}^n \quad = \quad \mathbb{Z}^{\binom{n}{2}}$$

Before moving on, we should be clear about exactly which $\mathbb{Z}^{\binom{n}{2}}$ subgroup of $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{ab}}$ we have just identified — after all, we will eventually need to perform a quotient involving it. In Proposition 4.20 we defined the generators $\langle z_i z_j \rangle$ to be the elements $(z_i \otimes z_j, z_j \otimes z_i)$ of the monoid $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$, which are the source/target combinations of morphisms of $L\mathbb{G}_n$. Using Corollary 4.6 we can identify this with a particular submonoid of the morphisms of $L\mathbb{G}_n$, specifically the image under q of the submonoid $\mathbb{N}^{*2n} \times_{\mathbb{N}^{2n}} \mathbb{N}^{*2n} = (s \times t)(\mathbb{G}_{2n}) \subseteq \mathrm{Mor}(\mathbb{G}_{2n})$ we chose in Proposition 4.5. In particular, since on objects we have $q(z_i) = z_i$ for all $1 \leq i \leq n$, the generators $(z_i \otimes z_j, z_j \otimes z_i)$ of $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$ are clearly the image of the generators $(z_i \otimes z_j, z_j \otimes z_i)$ of $\mathbb{N}^{*2n} \times_{\mathbb{N}^{2n}} \mathbb{N}^{*2n}$.

Thus, consider the following commutative diagram, whose top-left region comes from Corollary 4.6, bottom-left from the naturality of the adjoint functor $\mathrm{M}(\_\_)^{\mathrm{gp,ab}}$, and right-hand square from Proposition 4.18.

$$\begin{tikzcd}
& (s\times t)(\mathbb{G}_{2n}) \arrow[rr, "q"] \arrow[dl, hook] & & (s\times t)(L\mathbb{G}_n) \arrow[dl, hook] \arrow[dr] & \\
\mathrm{Mor}(\mathbb{G}_{2n}) \arrow[rr, "q"] \arrow[dr] & & \mathrm{Mor}(L\mathbb{G}_n) \arrow[dr] & & \dfrac{(s\times t)(L\mathbb{G}_n)^{\mathrm{ab}}}{\mathrm{Ob}(L\mathbb{G}_n)^{\mathrm{ab}}} \arrow[dl, hook] \\
& \mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \arrow[rr, "\mathrm{M}(q)^{\mathrm{gp,ab}}"] & & \mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}} &
\end{tikzcd}$$

What we've just said that if we start with the element $(z_i \otimes z_j, z_j \otimes z_i)$ of $(s \times t)(\mathbb{G}_{2n})$, moving clockwise around the diagram will send it to the generator $\langle z_i z_j \rangle$ in $(s \times t)(L\mathbb{G}_n)^{\mathrm{ab}}/\mathrm{Ob}(L\mathbb{G}_n)^{\mathrm{ab}} = \mathbb{Z}^{\binom{n}{2}}$. If we instead move anticlockwise, then we will first pass to our chosen representative $\alpha_{\mathbb{G}_{2n}}(\rho(z_i \otimes z_j, z_j \otimes z_i); \mathrm{id}_{z_i}, \mathrm{id}_{z_j})$ in $\mathrm{Mor}(\mathbb{G}_{2n})$, then its equivalence class in $\mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}}$, then its equivalence class in $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$, using the fact that $\mathrm{M}(q)^{\mathrm{gp,ab}}$ is the canonical map associated with the quotient

$$\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \quad = \quad \mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \big/ \Delta$$

which we proved back in Section 3.5. Since the bottom-right inclusion completes this circuit, we see that the specific subgroup we are talking about in **??** is

$$\mathbb{Z}^{\binom{n}{2}} \;=\; \Big\{ \Big[\alpha_{\mathbb{G}_{2n}}\Big(\rho(z_i \otimes z_j, z_j \otimes z_i)\,;\, \mathrm{id}_{z_i}, \mathrm{id}_{z_j} \Big) \Big] \;:\; 1 \leq i < j \leq n \Big\} \;\subseteq\; \mathrm{M}(L\mathbb{G}_n)^{\mathrm{ab}}$$

Of course, ρ was an arbitrary permutation-preserving map $\mathbb{N}^{*n} \times_{\mathbb{N}} \mathbb{N}^{*n} \to G$, chosen using the freeness of its source monoid. Thus if we wanted to we could just pick the same element $\rho(2) \in \pi^{-1}((1\,2))$ to act as $\rho(z_i \otimes z_j, z_j \otimes z_i)$ for all i, j, and for simplicity's sake we will indeed assume this from now on.

4.5 Group completion of action operads

The next group we are interested in understanding a little better is $\mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}}$. Per **??**, the operations needed to produce this group out of $\mathrm{Mor}(\mathbb{G}_{2n}) = G \times_{\mathbb{N}} \mathbb{N}^{*2n}$ can be done in any order we choose, and so we will save the identification of $\otimes$ and $\circ$ until last. This will let us keep the tensor product as simple as possible whilst we are in the process of group completing and abelianising it.

So the obvious place to start is to ask how to simplify the expression $(G \times_{\mathbb{N}} \mathbb{N}^{*2n})^{\mathrm{gp}}$. In principle we might not be able to, since for generic G we lack any sort of a presentation by generators and relations. It would help if we at least knew that the group completion map $\mathrm{gp} : G \to G^{\mathrm{gp}}$ was injective — or equivalently, that there exists any group H and injective homomorphism $G \to H$ — but proving this kind of statement is notoriously tricky. In 1935, a paper by Anton Sushkevich 'proved' that a semigroup, and thus a monoid, could be embedded into a group if and only if it was cancellative.

Definition 4.23. We say that a monoid M is *left-cancellative* if for any $a, b, c \in M$, we have

$$ab = ac \quad \Longrightarrow \quad b = c$$

That is, common factors may be cancelled out on the left. Similarly, we call M *right-cancellative* if common factors can be cancelled on the right:

$$ac = bc \quad \Longrightarrow \quad a = b$$

A monoid that is both left- and right-cancellative is simply referred to as *cancellative.*

However, just two years later Anatoly Malcev published a simple counterexample [11] to Sushkevich's proposition. To make matters worse, in 1939 Malcev would go on to show that the actual set of neccessary and sufficient conditions for a semigroup to be embeddible in a group consisted of an infinite collection of independent relations [12]. Thus the requirement that the group completion of monoid be injective is a deceptively complicated one.

Luckily for us though, there does exist a much simpler set of sufficient-but-not-neccessary conditions for embeddibility which all action operads G happen to satisfy. These come from a 1948 paper by Raouf Doss [10], and in addition to cancellativity they depend on the way that a monoid deals with multiples of different elements being equal.

Definition 4.24. An element a of a monoid M is said to be *regular on the left* if it shares a common left-multiple with every other element of M. That is,

$$\forall\, b \in M, \quad \exists\, c, d \in M \quad : \quad ca = db$$

The monoid as a whole is said to be *regular on the left* if all of its elements are, but we can also define a notion of M being *quasi-regular on the left.* This means that any two elements a, b of M will share a common left-multiple if and only if

$$\exists\, c, d \in M \quad : \quad ca \;=\; db, \qquad c \text{ or } d \text{ is regular in } M$$

Again, we can define a similar condition for being quasi-regular on the right, and we say that a monoid is *quasi-regular* when it is both.

Proposition 4.25. *If a monoid M is cancellative and quasi-regular on the left, then it can be embedded into a group.*

For a given action operad, both of these conditions will follow from the way that operadic multiplication interacts with the elements of the abelian group $G(0)$.

Proposition 4.26. *Every action operad G is both cancellative and quasi-regular as a monoid under tensor product.*

Corollary 4.27. *The canonical map* $\mathrm{gp} : G \to G^{\mathrm{gp}}$ *associated with the group completion of G is an inclusion.*

From now on we'll just write g for $\mathrm{gp}(g)$ and g^* for $\mathrm{gp}(g)^*$, in order to save on space.

4.6 Freely generated action operads

Knowing that the monoid $G \times_{\mathbb{N}} \mathbb{N}^{*n}$ always has a particularly well-behaved group completion is a good first step towards finding a description for said completion. However, it is worth noting that **??** is true for all action operads G, which is more than we really need. After all, the only reason we care about $\mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}}$ is that we know from Proposition 4.19 that it is crucial to understanding the morphisms of *crossed* action operads. Thus it would be nice if we could use some consequence of crossed-ness to tell us even more about the inclusion map $\mathrm{gp} : G \times_{\mathbb{N}} \mathbb{N}^{*n} \to (G \times_{\mathbb{N}} \mathbb{N}^{*n})^{\mathrm{gp}}$.

One such consequence was given back in Proposition 3.9. If G is a crossed action operad, then the action operad G' defined by $G'(m) = G(m)/G(0)$ possesses the same free algebra on invertible algebra that G does. In other words, we don't even need to worry about finding $\mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}}$ for all crossed G, merely those which have a trivial $G(0)$. As it turns out, this fact is hugely relevant to our search for group completions, since elements of $G(0)$ are the only ones in G which might already have an inverse under tensor product. This follows from additivity of lengths:

$$\begin{array}{rclcrclcl} g \otimes h & = & e_0 & \implies & |g| + |h| & = & |e_0| & = & 0 \\ & & & \implies & & & |g| & = & -|h|, \qquad |g|, |h| \in \mathbb{N} \\ & & & \implies & |g| & = & |h| & = & 0 \end{array}$$

Cancellativity, quasi-regularity, and lack of invertible objects then combine to give something much stronger than mere injectivity of the group completion map.

Proposition 4.28. *If G is an action operad with trivial $G(0)$, then G is a free monoid under tensor product.*

Whenever we can be sure of that G is a free monoid — whether by using Proposition 4.28 or some other method — this freeness will carry over directly to the algebras $\mathbb{G}_n$, giving us a new way to represent their morphisms.

Proposition 4.29. *Let $\mathcal{G}$ be a set that freely generates the action operad G under tensor product, and for each $m \in \mathbb{N}$ define $\mathcal{G}_m := \mathcal{G} \cap G(m)$, the subset of $\mathcal{G}$ containing all elements of length m. Then the monoid* $\mathrm{Mor}(\mathbb{G}_n)$ *is*

$$G \times_{\mathbb{N}} \mathbb{N}^{*n} \quad = \quad \mathbb{N}^{*(\,|\mathcal{G}_0| + n|\mathcal{G}_1| + n^2|\mathcal{G}_2| + \dots\,)}$$

This obviously makes the group completion and abelianisation which we want to do trivial.

Corollary 4.30. *If $\mathcal{G}$ is a set that freely generates G under tensor product, and $\mathcal{G}_m := \mathcal{G} \cap G(m)$, then the abelian group* $\mathrm{Mor}(\mathbb{G}_n)^{\mathrm{gp,ab}}$ *is*

$$(G \times_{\mathbb{N}} \mathbb{N}^{*n})^{\mathrm{gp,ab}} \quad = \quad \mathbb{Z}^{|\mathcal{G}_0| + n|\mathcal{G}_1| + n^2|\mathcal{G}_2| + \dots}$$

Now all that remains is to account for what happens when we collapse the morphisms of $\mathbb{G}_n$ — that is, evaluate the quotient

$$\mathrm{M}(\mathbb{G}_n)^{\mathrm{gp,ab}} \quad = \quad \mathbb{Z}^{|\mathcal{G}_0| + n|\mathcal{G}_1| + n^2|\mathcal{G}_2| + \dots} \Big/ \otimes \sim \circ$$

Unfortunately, because this will depend on the exact multiplicative structure of the operad groups $G(m)$, there is no way to carry out this computation in general. The best we can say is that as composition in $\mathrm{Mor}(\mathbb{G}_n)$ is partly determined by the group multiplication of the $G(m)$, then in place of $\mathcal{G}$ in the quotient in Corollary 4.30 it would suffice to have some collection of elements which generate G when using multiplication as well as tensor product.

Lemma 4.31. *Let $\mathcal{G}$ be a subset of the action operad G that freely generates it under tensor product, and let $\mathcal{G}'$ be a subset of $\mathcal{G}$ which generates G under a combination of tensor product and group multiplication. Also let $\mathcal{G}_m := \mathcal{G} \cap G(m)$ and $\mathcal{G}'_m := \mathcal{G}' \cap G(m)$. Then*

$$\mathbb{Z}^{|\mathcal{G}_0|+n|\mathcal{G}_1|+n^2|\mathcal{G}_2|+\dots} \Big/ \otimes \sim \circ \qquad = \qquad \mathbb{Z}^{|\mathcal{G}'_0|+n|\mathcal{G}'_1|+n^2|\mathcal{G}'_2|+\dots} \Big/ \otimes \sim \circ$$

Beyond this, the value of this quotient will have to be found seperately for each individual action operad.

Chapter 5

Complete descriptions of free invertible algebras

At last, we finally have an expression for the morphisms of $L\mathbb{G}_n$, one built out of smaller parts which we know how to calculate. This means that it is almost time to draw together everything we have done over the past three chapters into a single, complete description of free invertible EG-algebras — at least, in cases where G is crossed or $G(1)$-generated.

5.1 The action of $L\mathbb{G}_n$

At this stage, there is only one part of this EG-algebra that we have yet to find — its action, $\alpha_{L\mathbb{G}_n}$. When our action operad G is $G(1)$-generated, everything is so simple that there is really only one thing the action could be.

Lemma 5.1. *Let G be a $G(1)$-generated action operad, g an element of $G(m)$ for some $m \in \mathbb{N}$, and $x_1, ..., x_m$ elements of $\mathbb{Z}^{*n}$. Then the action of $L\mathbb{G}_n$ is given by*

$$\alpha_{L\mathbb{G}_n}(\, g\, ;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,) \quad = \quad \mathrm{id}_{x_1 \otimes ... \otimes x_m}$$

For crossed G, things are more complicated. What we need to do is employ the trick that was previously mentioned in Section 3.3, where we exploit the surjectivity of the algebra map $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$. This will allow us to express $\alpha_{L\mathbb{G}_n}$ in terms of the action $\alpha_{\mathbb{G}_{2n}}$.

Proposition 5.2. *Let G be a crossed action operad, and for some $m \in \mathbb{N}$ choose an element $g \in G(m)$ and morphisms $(x_1, y_1, h_1), ..., (x_m, y_m, h_m)$ in $L\mathbb{G}_n$. That is, the*

(x_i, y_i) are pairs of objects from $(s \times t)(L\mathbb{G}_n)$, and the h_i are morphisms in $L\mathbb{G}_n(I, I)$. Then the action of $L\mathbb{G}_n$ is given by

$$\alpha_{L\mathbb{G}_n}\Big(g\,;\,(x_1, y_1, h_1), ..., (x_m, y_m, h_m) \Big)$$
$$=$$
$$\Big(\bigotimes_i x_i, \quad \bigotimes_i y_{\pi(g^{-1})(i)}, \quad \Psi\alpha_{\mathbb{G}_{2n}}(\, g\,;\,\mathrm{id}_{q^{-1}(y_1)}, ..., \mathrm{id}_{q^{-1}(y_m)}\,) \otimes (\bigotimes_i h_i) \Big)$$

Here $q^{-1} : \mathrm{Ob}(L\mathbb{G}_n) \to \mathrm{Ob}(\mathbb{G}_{2n})$ is the function

$$\begin{array}{rcrcl} q^{-1} & : & \mathbb{Z}^{*n} & \to & \mathbb{N}^{*2n} \\ & : & z_i & \mapsto & z_i \\ & : & z_i^* & \mapsto & z_{n+1} \\ & : & w & \mapsto & \bigotimes_{i=1}^{|w|} q^{-1}\Big(d(w,i) \Big) \end{array}$$

with $\bigotimes_{i=1}^{|w|} d(w,i)$ the decomposition of w given in Definition 3.23, and $\Psi : \mathrm{Mor}(\mathbb{G}_{2n}) \to L\mathbb{G}_n(I, I)$ is the canonical map associated with the repeated quotient

$$\begin{array}{ccccc} \mathrm{Mor}(\mathbb{G}_{2n}) & \longrightarrow & \mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} & \longrightarrow & \mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \big/ \Delta & & \\ & & & & \| & & \\ & & & & \mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}} & \longrightarrow & \mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \big/ \mathbb{Z}^{\binom{n}{2}} \\ & & & & & & \| \\ & & & & & & L\mathbb{G}_n(I,I) \end{array}$$

5.2 A full description of $L\mathbb{G}_n$

With this last proposition proven, the results in this paper now collectively describe how to construct the free EG-algebras on n invertible objects for most values of G. However, since this characterization was discovered by us in such a piecemeal fashion, we will now restate everything in one place, for ease of reading. We'll begin with the uncrossed case, or as much of it as we were able to draw a complete conclusion about.

Theorem 5.3. *Let G be a $G(1)$-generated action operad. Then the free* EG*-algebra on n invertible objects is just the discrete category*

$$L\mathbb{G}_n \quad = \quad \mathbb{Z}^{*n}$$

equipped with a tensor product which is the usual monoid multiplication, and an EG*-action given by*

$$\alpha_{L\mathbb{G}_n}(\, g\,;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,) \quad = \quad \mathrm{id}_{x_1 \otimes ... \otimes x_m}$$

It is a shame that we were not able to find a formulation for uncrossed $L\mathbb{G}_n$ in full generality; this will have to be the subject of future research. For crossed action operads however, we were able to achieve this.

Theorem 5.4. *Let G be a crossed action algebra, and let G' be the action operad defined by $G'(m) := G(m)/G(0)$. Choose a subset $\mathcal{G}$ that generates G' under a combination of tensor product and group multiplication, which itself has subsets $\mathcal{G}_m := \mathcal{G} \cap G'(m)$. Then denote by A the abelian group obtained from the free abelian group*

$$\mathrm{F}(\mathcal{G} \times_{\mathbb{N}} \mathbb{N}^{*2n}) \quad = \quad \mathbb{Z}^{2n|\mathcal{G}_1| + (2n)^2 |\mathcal{G}_2| + ...}$$

via the following steps:

1. *For all $g, g' \in G(m)$ and $w \in \mathbb{N}^{*2n}$ with $|w| = m$, quotient out by the relation*

$$(g, w) \otimes \Big(g', \pi(g^{-1})(w)\Big) \quad \sim \quad (g \cdot g', w)$$

2. *Quotient out by the subgroup Δ, which is generated by the equivalence classes of elements of the form*

$$\begin{gathered}\Big(\mu(\, g\,;\, e_{|\tilde{\delta}(x_1)|}, ..., e_{|\tilde{\delta}(x_m)|}\,),\, \tilde{\delta}(\, x_1 \otimes ... \otimes x_m\,)\Big) \\ \otimes \\ \Big(\mu(\, g\,;\, e_{|\tilde{I}(x_1)|}, ..., e_{|\tilde{I}(x_m)|}\,),\, \tilde{I}(\, x_1 \otimes ... \otimes x_m\,)\Big)^*\end{gathered}$$

where $g \in G(m)$, the x_i are generators of $\mathbb{N}^{*4n}$, and for all $1 \leq i \leq n$,

$$\begin{aligned} \tilde{\delta}(z_i) &= z_i, & \tilde{I}(z_i) &= z_i, \\ \tilde{\delta}(z_{n+i}) &= z_{n+i}, & \tilde{I}(z_{n+i}) &= z_{n+i}, \\ \tilde{\delta}(z_{2n+i}) &= z_i \otimes z_{n+i}, & \tilde{I}(z_{2n+i}) &= I, \\ \tilde{\delta}(z_{3n+i}) &= z_{n+i} \otimes z_i & \tilde{I}(z_{3n+i}) &= I \end{aligned}$$

3. Choose any $\rho(2) \in \pi^{-1}((1\,2))$, and then quotient out by the $\mathbb{Z}^{\binom{n}{2}}$ subgroup generated by the equivalence classes of the elements

$$\Big(\rho(2)\,;\,z_i, z_j\Big), \qquad 1 \leq i < j \leq n$$

Also, denote by $\Psi : G \times_{\mathbb{N}} \mathbb{N}^{*2n} \to A$ the corresponding quotient map. Then the free EG*-algebra on n invertible objects is the category*

$$L\mathbb{G}_n \quad = \quad \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \;\times\; \mathrm{B}A$$

equipped with a tensor product given by componentwise monoid multiplication,

$$(x', y', h') \otimes (x, y, h) \;=\; (\,x' \otimes x,\; y' \otimes y,\; h'h\,)$$

and an EG*-action given by*

$$\alpha_{L\mathbb{G}_n}\Big(\,g\,;\,(x_1, y_1, h_1), ..., (x_m, y_m, h_m)\Big)$$
$$=$$
$$\Big(\;\textstyle\bigotimes_i x_i, \quad \bigotimes_i y_{\pi(g^{-1})(i)}, \quad \Psi\alpha_{\mathbb{G}_{2n}}(\,g\,;\,\mathrm{id}_{q^{-1}(y_1)}, ..., \mathrm{id}_{q^{-1}(y_m)}\;) \otimes (\bigotimes_i h_i)\Big)$$

where q^{-1} is the function

$$\begin{array}{rrcl} q^{-1} & : \;\; \mathbb{Z}^{*n} & \to & \mathbb{N}^{*2n} \\ & : \;\; z_i & \mapsto & z_i \\ & : \;\; z_i^* & \mapsto & z_{n+1} \\ & : \;\; w & \mapsto & \bigotimes_{i=1}^{|w|} q^{-1}\Big(d(w,i)\Big) \end{array}$$

5.3 Free symmetric monoidal categories on invertible objects

Even collected all together, Theorem 5.4 is still a fairly opaque result. In the next couple of sections we will work through some specific applications of the theorem, which will hopefully prove enlightening in this regard. A good place to start will be with the simplest of all the crossed action operads, the symmetric operad S. As one might expect, the free invertible algebras $L\mathbb{S}_n$ have a particularly straightforward form when viewed as monoidal categories.

Proposition 5.5. *The underlying monoidal category of the free* ES*-algebra on n invertible objects is*

$$L\mathbb{S}_n \quad = \quad \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \times \mathrm{B}\mathbb{Z}_2^n$$

with componentwise tensor product.

If we are to understand $L\mathbb{S}_n$'s role as a *symmetric* monoidal category, we now just need to use the rest of Theorem 5.4 to find its ES-action. This operation too is incredibly simple.

Proposition 5.6. *The action of $L\mathbb{S}_n$ is fully determined by the values*

$$\alpha\Big((1\,2)\,;\, \mathrm{id}_{z_i}, \mathrm{id}_{z_j} \Big) \quad = \begin{cases} \Big(z_i \otimes z_j,\; z_j \otimes z_i,\; (0,...,0) \Big) & \text{if} \quad i \neq j \\ \Big(z_i \otimes z_i,\; z_i \otimes z_i,\; (0,...,0,1,0,...,0) \Big) & \text{if} \quad i = j \end{cases}$$

where the 1 *appears in the ith coordinate of $\mathbb{Z}_2^n$, along with the identities*

$$\begin{aligned} \alpha\Big((1\,2)\,;\, \mathrm{id}_{z_i}, \mathrm{id}_{z_j} \Big) \quad &= \quad \alpha\Big((1\,2)\,;\, \mathrm{id}_{z_i^*}, \mathrm{id}_{z_j} \Big) \\ &= \quad \alpha\Big((1\,2)\,;\, \mathrm{id}_{z_i}, \mathrm{id}_{z_j^*} \Big) \\ &= \quad \alpha\Big((1\,2)\,;\, \mathrm{id}_{z_i^*}, \mathrm{id}_{z_j^*} \Big) \end{aligned}$$

Thus we see that in the free symmetric monoidal category on n invertible objects, every morphism can be expressed as a composite of tensor products of identities and symmetries maps

$$\beta_{z_i,z_j} \quad = \quad \alpha\Big((1\,2)\,;\, \mathrm{id}_{z_i}, \mathrm{id}_{z_j} \Big)$$

Moreover, two parallel morphisms in $L\mathbb{S}_n$ are equal if and only if the number of symmetries from

$$\Big\{ \beta_{z_i,z_i}, \beta_{z_i^*,z_i}, \beta_{z_i,z_i^*}, \beta_{z_i^*,z_i^*} \Big\}$$

appearing in these two expressions has the same parity, for each $1 \leq i \leq n$.

5.4 Free braided monoidal categories on invertible objects

Having successfully understood the symmetric monoidal case, we should now be ready to tackle the very similar world of braided monoidal categories. Indeed, since the only difference between the braid groups B_n and the symmetry groups S_n is the presense or absense of a self-invertibility condition, the abelian group $L\mathbb{B}_n(I, I)$ is simply the value we would gotten for $L\mathbb{S}_n(I, I)$ if we had never set $((1\,2); z_i, z_j) \otimes ((1\,2); z_i, z_j) \sim I$.

Proposition 5.7. *The underlying monoidal category of the free* EB*-algebra on n invertible objects is*

$$L\mathbb{B}_n \quad = \quad \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \, \times \, \mathrm{B}(\mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}})$$

with componentwise tensor product.

Just to be clear, the first n generators of this group $\mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}}$ are the images under $q : \mathbb{B}_{2n} \to L\mathbb{B}_n$ of the action morphisms $\alpha_{\mathbb{B}_{2n}}(b; \mathrm{id}_{z_i}, \mathrm{id}_{z_i})$, and the other $\binom{n}{2}$ come from the $\alpha_{\mathbb{B}_{2n}}(b; \mathrm{id}_{z_i}, \mathrm{id}_{z_j})$ for $i > j$. This seems a little strange at first — why would $L\mathbb{B}_n$ have this kind of directionality to it, where the $i < j$ generators have been cancelled out but the $i > j$ remain? The important thing to realise is this group is representing the unit endomorphisms $L\mathbb{B}_n(I, I)$, which have the same source and target. By contrast, if $i \neq j$ then $\alpha_{\mathbb{B}_{2n}}(b; \mathrm{id}_{z_i}, \mathrm{id}_{z_j})$ will have distinct source and target $z_i \otimes z_j \neq z_j \otimes z_i$, and thus the only way we can add it onto a composite without changing the source and target is to also add in the corresponding $\alpha_{\mathbb{B}_{2n}}(b; \mathrm{id}_{z_j}, \mathrm{id}_{z_i})$ somewhere. Therefore we really only need to keep track of one of these two kinds of morphisms, such as all of the ones where $i > j$. This is also reflected in the action of this algebra.

Proposition 5.8. *The action of $L\mathbb{B}_n$ is fully determined by the values*

$$\alpha(\, b \,;\, \mathrm{id}_{z_i}, \mathrm{id}_{z_j} \,) \quad = \begin{cases} \Big(z_i \otimes z_j,\; z_j \otimes z_i,\; (0, ..., 0) \Big) & \textit{if} \quad i < j \\ \Big(z_i \otimes z_j,\; z_j \otimes z_i,\; (0, ..., 0, 1, 0, ..., 0) \Big) & \textit{if} \quad i \geq j \end{cases}$$

where the 1 *appears in the ith coordinate of $\mathbb{Z}^n$ when $i = j$, and the (i, j)th coordinate of $\mathbb{Z}^{\binom{n}{2}}$ when $i > j$, and also*

$$\begin{aligned} \alpha(\, b \,;\, \mathrm{id}_{z_i}, \mathrm{id}_{z_j} \,) \quad &= \quad \alpha(\, b \,;\, \mathrm{id}_{z_i^*}, \mathrm{id}_{z_j} \,)^* \\ &= \quad \alpha(\, b \,;\, \mathrm{id}_{z_i}, \mathrm{id}_{z_j^*} \,)^* \\ &= \quad \alpha(\, b \,;\, \mathrm{id}_{z_i^*}, \mathrm{id}_{z_j^*} \,) \end{aligned}$$

To put this in a more categorical perspective, suppose that we decide to call the following kinds of braiding isomorphisms ‘positive’,

$$\begin{aligned} \beta_{z_i, z_j} \quad &= \quad \alpha(\, b \,;\, \mathrm{id}_{z_i}, \mathrm{id}_{z_j} \,), & \beta^{-1}_{z_i^*, z_j} \quad &= \quad \alpha(\, b \,;\, \mathrm{id}_{z_i^*}, \mathrm{id}_{z_j} \,)^{-1}, \\ \beta^{-1}_{z_i, z_j^*} \quad &= \quad \alpha(\, b \,;\, \mathrm{id}_{z_i}, \mathrm{id}_{z_j^*} \,)^{-1}, & \beta_{z_i^*, z_j^*} \quad &= \quad \alpha(\, b \,;\, \mathrm{id}_{z_i^*}, \mathrm{id}_{z_j^*} \,) \end{aligned}$$

and likewise call their inverses 'negative'. Then what Proposition 5.8 is saying is that in the free braided monoidal category on n invertible objects, parallel morphisms coincide only when the number of positive braidings minus the number of negative braidings they contain is the same.

Something else to notice about $L\mathbb{B}_n$ is that we've actually seen its unit endomorphism group before. Back in Proposition 4.21 we proved that for any crossed action operad G,

$$(s \times t)(L\mathbb{G}_n)^{\mathrm{ab}} \quad = \quad (\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}} \quad = \quad \mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}}$$

This means that in the case of the braid operad, we have the unusual identity

$$(s \times t)(L\mathbb{B}_n)^{\mathrm{ab}} \quad \cong \quad L\mathbb{B}_n(I, I)$$

What is the significance of this fact? It is not entirely clear, though certainly the isomorphism involved is highly nontrivial. For example the $\mathbb{Z}^n$ subgroup of $(s\times t)(L\mathbb{B}_n)^{\mathrm{ab}}$ has generators representing maps with source and target $z_i \to z_i$, $1 \leq i \leq n$, while the same generators of $\mathbb{Z}^n \subseteq L\mathbb{B}_n(I, I)$ represent the braidings $\beta_{z_i,z_i} = \alpha(b; \mathrm{id}_{z_i}, \mathrm{id}_{z_i})$. Of course, it is possible that this connection between the groups that make up $\mathrm{Mor}(L\mathbb{B}_n)$ could simply be a conincidence. It would help if we could compare B to another action operad which shares this property — either another crossed G whose algebra has the same underlying category as the $L\mathbb{B}_n$, or an uncrossed G whose algebra has $L\mathbb{G}_n(I, I) = (\mathbb{Z}^{*n})^{\mathrm{ab}} = \mathbb{Z}^n$ — but none of these are currently known to the author.

5.5 Free ribbon braided monoidal categories on invertible objects

The last action operad whose invertible algebras we will calculate explicitly is the ribbon braid operad, RB. The details will prove largely similar to those we saw for the braided case in Proposition 5.7, much as the braided case itself was built upon the symmatric case with a few small changes.

Proposition 5.9. *The underlying monoidal category of the free* ERB*-algebra on n invertible objects is*

$$L\mathbb{RB}_n \quad = \quad \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \; \times \; \mathrm{B}(\mathbb{Z}^n \times \mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}})$$

with componentwise tensor product. Moreover, the action of $L\mathbb{RB}_n$ is determined by its restriction to the subcategory $L\mathbb{B}_n \subseteq L\mathbb{RB}_n$, plus the values

$$\alpha(\,t\,;\,\mathrm{id}_{z_i}\,) \quad = \quad \Big(\,z_i,\; z_i,\; (0,...,0,1,0,...,0)\,\Big)$$

where the 1 *appears in the* i*th coordinate of the copy of* $\mathbb{Z}^n$ *which is not shared with* $L\mathbb{B}_n$*, and*

$$\alpha(\,t\,;\,\mathrm{id}_{z_i^*}\,) \quad = \quad \alpha(\,t\,;\,\mathrm{id}_{z_i}\,)^* \otimes \alpha(\,b\,;\,\mathrm{id}_{z_i},\mathrm{id}_{z_i}\,)^{\otimes 2}$$

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