Action operads and the free G-monoidal category on n invertible objects

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Abstract

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Chapter 1

Operads and their algebras

1.1 Operads

Definition 1.1. Operads O

Example 1.2 (The symmetric operad).
There is an operad S whose sets of operations $\mathrm{S}(n)$ for each $n \in \mathbb{N}$ are the underlying sets of the symmetric groups S_n. The identity element of this *symmetric operad* is the identity permutation of a single object, $e_1 \in \mathrm{S}_1$, and the operadic multiplication is defined in the following way:

- First, there exist maps $\otimes : \mathrm{S}_m \times \mathrm{S}_n \to \mathrm{S}_{m+n}$ called the *direct sum* or *block sum* of permutations. For any $\sigma \in \mathrm{S}_m$ and $\tau \in \mathrm{S}_n$, these are given by

$$(\sigma \otimes \tau)(i) \quad = \quad \begin{cases} \sigma(i) & 1 \leq i \leq m \\ \tau(i-m)+m & m+1 \leq i \leq m+n \end{cases}$$

  As the name suggests, this direct sum is usually denote by the symbol $\oplus$, but we will stick with $\otimes$ so that our notation here matches all of the other tensor products we will see throughout this paper. Also, notice that the value of these direct sums in general are determined by those specific cases where one of the inputs is an identity permutation:

$$\sigma \otimes \tau \quad = \quad (\sigma \otimes e_n) \cdot (e_m \otimes \tau) \quad = \quad (e_m \otimes \tau) \cdot (\sigma \otimes e_n)$$

- Next, we'll define functions $(\_)_{(k_1,\ldots,k_n)} : \mathrm{S}_n \to \mathrm{S}_{k_1+\ldots+k_n}$ for all $n, k_1, \ldots, k_n \in \mathbb{N}$. These will act by sending each σ that permutes n individual objects to a corresponding $\sigma_{(k_1,...,k_n)}$ that permutes blocks of objects of size $k_1, ..., k_n$ in the same way. More concretely, if $k_1 + ... + k_{i-1} < j \leq k_1 + ... + k_i$ then

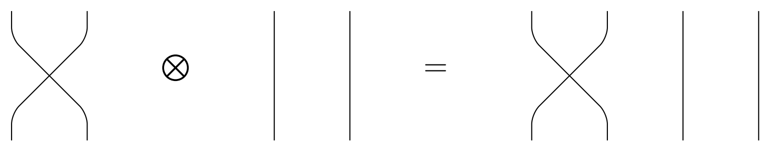
$$\sigma_{(k_1,...,k_n)}(j) \quad = \quad j - k_1 - ... - k_{i-1} + k_{\sigma^{-1}(1)} + ... + k_{\sigma^{-1}(\,\sigma(i)-1\,)}$$

- Finally, the multiplication maps $\mu : \mathrm{S}_n \times \mathrm{S}_{k_1} \times ... \times \mathrm{S}_{k_n} \to \mathrm{S}_{k_1+...+k_n}$ are given by

$$\begin{aligned} \mu(\sigma; \tau_1, ..., \tau_n) \quad &:= \quad \sigma_{(k_1,...,k_n)} \cdot (\tau_1 \otimes ... \otimes \tau_n) \\ &= \quad \left(\tau_{\sigma^{-1}(1)} \otimes ... \otimes \tau_{\sigma^{-1}(n)}\right) \cdot \sigma_{(k_1,...,k_n)} \end{aligned}$$

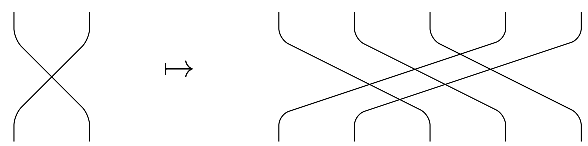
  In other words, the operadic multiplication of permutations comes from both permutating objects within distinct blocks and also permuting the blocks themselves.

If we decide to represent elements of the symmetric operad pictorially — for example as strings which cross over another according to the appropriate permutation — then both $\sigma \otimes \tau$ and $\sigma_{(k_1,...,k_n)}$ have rather nice interpretations. The direct sum of two permutations is just the result of placing two permutations ‘next to’ each other,



σ $\qquad$ τ $\qquad$ $\sigma \otimes \tau$

and block permutations are given by expanding string into some number of parallel strings,



σ $\qquad$ $\sigma_{(3,2)}$

With a little work, we can actually replace the functions $(\_\_)_{(k_1,...,k_n)}$ with an explicit combination of group multiplication and tensor product. This is due to basic fact about the symmetric groups S_n, which is that they possess a presentation in terms of the *elementary transpositions* $(i \;\; i+1)$.

Lemma 1.3. *The group* S_n *is generated by the permutations* $(1\,2), \ldots, (n-1\ \ n)$*, subject to the relations*

$$\begin{array}{rcll} (i\ \ i+1)^2 &=& e & \\ (i-1\ \ i)(i\ \ i+1)(i-1\ \ i) &=& (i\ \ i+1)(i-1\ \ i)(i\ \ i+1) & \\ (i\ \ i+1)(j\ \ j+1) &=& (j\ \ j+1)(i\ \ i+1), & \qquad i+1<j \end{array}$$

Thus if $\sigma \in \mathrm{S}_n$ is a permutation with a decomposition $\sigma = \sigma_m \cdot \ldots \cdot \sigma_1$ in terms of elementary transpositons $\sigma_i \in \mathrm{S}_n$, we can break down the block permutation $\sigma_{(k_1,\ldots,k_n)}$ into the m 'elementary block transpositions' $(\sigma_i)_{(k_1,\ldots,k_n)}$:

$$\begin{array}{rcl} \sigma_{(k_1,\ldots,k_n)}(j) &=& j - k_1 - \ldots - k_{i-1} + k_{\sigma^{-1}(1)} + \ldots + k_{\sigma^{-1}(\,\sigma(i)-1\,)} \\ &=& j - k_1 - \ldots - k_{i-1} \\ && + k_{\sigma_1^{-1}(1)} + \ldots + k_{\sigma_1^{-1}(\,\sigma_1(i)-1\,)} \\ && - k_{\sigma_1^{-1}(1)} - \ldots - k_{\sigma_1^{-1}(\,\sigma_1(i)-1\,)} \\ && + k_{(\sigma_2\sigma_1)^{-1}(1)} + \ldots + k_{(\sigma_2\sigma_1)^{-1}(\,\sigma_2\sigma_1(i)-1\,)} \\ && \vdots \\ && - k_{(\sigma_{m-1}\ldots\sigma_1^{-1}(1)} - \ldots - k_{(\sigma_{m-1}\ldots\sigma_1)^{-1}(\,\sigma_{m-1}\ldots\sigma_1(i)-1\,)} \\ && + k_{(\sigma_m\ldots\sigma_1)^{-1}(1)} + \ldots + k_{(\sigma_m\ldots\sigma_1)^{-1}(\,\sigma_m\ldots\sigma_1(i)-1\,)} \\ &=& \Big((\sigma_m)_{(k_1,\ldots,k_n)} \cdot \ldots \cdot (\sigma_1)_{(k_1,\ldots,k_n)} \Big)(j) \end{array}$$

However, since elementary transpositions only really permute two objects, they can be written as a block sum in the operad S involving the sole transposition of S_2, plus some number of identity permutations.

$$(i\ \ i+1) \quad = \quad e_{i-1} \otimes (1\,2) \otimes e_{n-i-1}$$

This means that the elementary block transpositions are

$$\begin{array}{rcl} (i\ \ i+1)_{(k_1,\ldots,k_n)} &=& (e_{i-1} \otimes (1\,2) \otimes e_{n-i-1})_{(k_1,\ldots,k_n)} \\ &=& e_{k_1+\ldots k_{i-1}} \otimes (1\,2)_{(k_i,k_{i+1})} \otimes e_{k_{i+1}+\ldots k_n} \end{array}$$

So all we need to know to fully understand the functions $(\_\_)_{(k_1,\ldots,k_n)}$ are the values they take on the transposition $(1\,2)$. These can be defined recursively, via

$$\begin{array}{rclrcl} (1\,2)_{(0,n)} &=& e_n, & (1\,2)_{(m+m',n)} &=& \Big((1\,2)_{(m,n)} \otimes e_{m'} \Big) \cdot \Big(e_m \otimes (1\,2)_{(m',n)} \Big) \\ (1\,2)_{(m,0)} &=& e_m, & (1\,2)_{(m,n+n')} &=& \Big(e_n \otimes (1\,2)_{(m,n')} \Big) \cdot \Big((1\,2)_{(m,n)} \otimes e_{n'} \Big) \end{array}$$

which all follow from the definition of $(\_)_{(k_1,...,k_n)}$. Therefore all $\sigma_{(k_1,...,k_n)}$ and hence all $\mu(\sigma;\tau_1,...,\tau_n)$ can be expressed in terms of group multiplication $\cdot$ and direct sum $\otimes$, and the elementary permutations which constitute $\sigma,\tau_1,...,\tau_n$.

Example 1.4 (The braid operad).
The *braid groups* B_n are the family of groups that result from taking the symmetric groups and removing the requirement that everything needs to be self-inverse. That is, the group B_n has a presentation on some *elementary braids* $b_1,...,b_{n-1}$, given by the relations

$$b_ib_{i+1}b_i \;=\; b_{i+1}b_ib_{i+1}, \qquad\qquad b_ib_j \;=\; b_jb_i, \quad i+1<j$$

As might be expected, the underlying sets of these groups also form an operad, known as the *braid operad* B, and they do so in a way directly analagous to the operad S. That is, the identity element of B is $e_1 \in \mathrm{B}_1$, and the operadic multiplication is constructed as follows:

- Tensor products $\otimes : B_m \times B_n \to B_{m+n}$ are determined by setting

  $$x\otimes y \quad = \quad (x\otimes e_n)\cdot(e_m\otimes y) \quad = \quad (e_m\otimes x)\cdot(y\otimes e_n)$$

  for all $x \in B_m$, $y \in B_n$, and also

  $$b_i \quad = \quad e_{i-1}\otimes b\otimes e_{n-i-1}$$

  for any elementary braid $b_i \in B_n$, where b is the only elementary braid in B_2.

- The functions $(\_)_{(k_1,...,k_n)} : B_n \to B_{k_1+...+k_n}$ are first defined recursively on the elementary braid $b \in B_2$ by

  $$\begin{aligned} b_{(0,n)} &= e_n, & b_{(m+m',n)} &= (b_{(m,n)}\otimes e_{m'})\cdot(e_m\otimes b_{(m',n)}) \\ b_{(m,0)} &= e_m, & b_{(m,n+n')} &= (e_n\otimes b_{(m,n')})\cdot(b_{(m,n)}\otimes e_{n'}) \end{aligned}$$

  then on arbitrary elementary braids $b_i \in B_n$ via

  $$(b_i)_{(k_1,...,k_n)} \quad = \quad e_{k_1+...k_{i-1}}\otimes b_{(k_i,k_{i+1})}\otimes e_{k_{i+1}+...k_n}$$

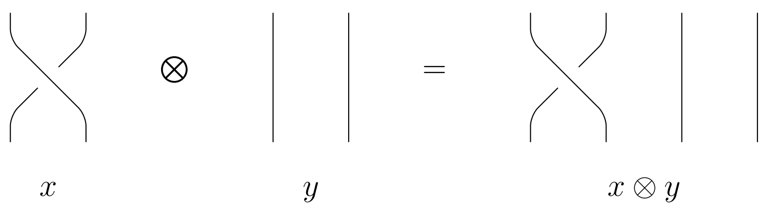
  and finally on all elements of the braid groups by using their presentation in terms of the b_i,

  $$\begin{aligned} x &= b_{i_m}\cdot...\cdot b_{i_1} \\ \Longrightarrow \quad x_{(k_1,...,k_n)} &= (b_{i_m})_{(k_1,...,k_n)}\cdot...\cdot(b_{i_1})_{(k_1,...,k_n)} \end{aligned}$$

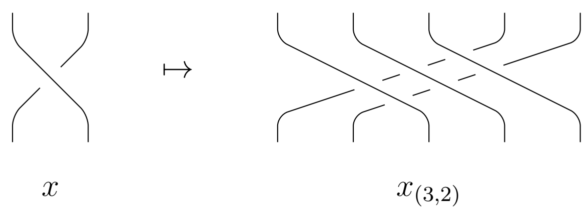
- Then as in the symmetric case, the multiplication maps $\mu : B_n \times B_{k_1} \times ... \times B_{k_n} \to B_{k_1+...+k_n}$ are just

$$\mu(x; y_1, ..., y_n) \quad := \quad x_{(k_1,...,k_n)} \cdot (y_1 \otimes ... \otimes y_n)$$

These operations are exactly what they need to be in order for them to possess the same pictorial representations as the operations in S, but with actual braids replacing simple crossings. That is, the tensor product $x \otimes y$ is the braids x and y laid side-by-side,



and the 'block braids' are multiple strings braided together in parallel,



Definition 1.5. Operads maps

[6] [5]

Definition 1.6. Action operads G, maps, AOp

There are a couple of operads which trivially have the structure of an action operads.

First there is the *terminal operad* T, which has a single operation for each arity, so that $\mathrm{T}(n) = \{e_n\}$. Each of these sets can be seen as the trivial group, and it follows from this that the $\pi^{\mathrm{T}} : \mathrm{T}(n) \to \mathrm{S}_n$ must be the respective zero maps, the terminal homomorphisms in the category of groups. The action operad condition is then

$$\mu(e_n; e_{k_1}, ..., e_{k_n}) \cdot \mu(e_n; e_{k_1}, ..., e_{k_n}) \quad = \quad \mu(e_n; e_{k_1}, ..., e_{k_n})$$

which is really just

$$e_{k_1+...+k_n} \cdot e_{k_1+...+k_n} = e_{k_1+...+k_n}$$

and hence is trivially true. As its name suggests, the terminal operad is the terminal object in the category of operads, but it is also the *initial* object in the category of action operads. This is because for any other G in AOp the zero homomorphisms $\mathrm{T}(n) \to G(n)$ define the unique map of operads $f : \mathrm{T} \to G$.

On the other hand, it is the symmetric operad S itself that functions as the terminal object in AOp. Its action operad structure is just given by the standard group multiplications on the S_n, with the identity maps $\mathrm{id}_{\mathrm{S}_n} : \mathrm{S}_n \to \mathrm{S}_n$ functioning as its π_n. To see terminality, notice that for any other action operad G a valid morphism $f : G \to \mathrm{S}$ in AOp must obey

$$\pi^{\mathrm{S}} \circ f \;=\; \pi^G \qquad \Longrightarrow \qquad f \;=\; \pi^G$$

Thus there only one map of action operads $G \to \mathrm{S}$, which is the very underlying permutation structure used to define G.

There are more interesting examples of action operads we can look at though. For instance, we know that the braid groups B_n have the same presentation as the symmetric groups, except without the relations $b_i^2 = e$. Thus if we take their quotients by these relations we will obtain a sequence of homomorphisms $B_n \to \mathrm{S}_n$, each sending $b_i \mapsto (i \;\; i+1)$. This provides a natural way to describe the underlying permutation of any braid, and indeed choosing these maps to form π^B gives a valid way of seeing the braid operad as an action operad. Another example can also be built with the so-called ribbon braid groups.

Definition 1.7. For each $n \in \mathbb{N}$, the *ribbon braid group* RB_n is the group whose presentation is the same as that of the braid group B_n, except with the addition of n new generators $t_1, ..., t_n$, known as the *twists*. These twists all commute with one other, and also commute with all braids except in the following cases:

$$b_i \cdot t_i \quad = \quad t_{i+1} \cdot b_i, \qquad\qquad b_i \cdot t_{i+1} \quad = \quad t_i \cdot b_i$$

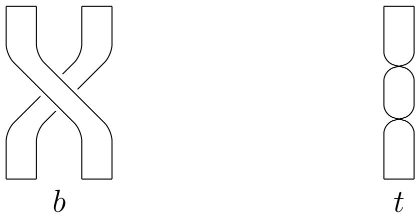
The *ribbon braid operad* RB is then the operad made up of these groups in a way that extends the definition of the braid operad. In other words, the identity is still $e_1 \in RB_1$, and the operadic multiplication is built up in stages in exactly the same ways as in Example 1.4, but with some additional rules for dealing with twists. For the tensor product, we have that for any twist $t_i \in RB_n$,

$$t_i \quad = \quad e_{i-1} \otimes t \otimes e_{n-i}$$

where t is the sole twist in RB_1, and for the 'block twists' $t_{(m)}$ we again work recursively:

$$t_{(0)} = e_n, \qquad t_{(m+m')} \quad = \quad (t_{(m)} \otimes t_{(m')}) \cdot b_{(m',m)} \cdot b_{(m,m')}$$

Much as the symmetric groups can be represented by crossings of a collection of strings, and the braid groups by braidings of strings, the ribbon braid groups deal with the ways that one can braid together several flat ribbons, including the ability to twist a ribbon about its own axis by 360 degrees.



This operad RB is also clearly an action operad, since we can just define $\pi^{RB} : RB_n \to \mathrm{S}_n$ to act like π^B on any braids, at which point the fact that $\pi(t) \in S_1 = \{e_1\}$ will automatically take care of the twists. To learn more about the ribbon braids and their operads, see Natalie Wahl's thesis [13] on the subject, or her subsequent paper with Paolo Salvatore [14].

Lemma 1.8. *For any action operad G, the group $G(0)$ is abelian.*

Proof. □

Definition 1.9. Sub action operads

The most important example of sub action operads are those of the symmetric operad, S. This is because Definition 1.6 itself makes explicit reference to the symmetric groups, and so every action operad will end up related to some sub-operad of S:

Definition 1.10. For an arbitrary action operad G the images of the underlying permutation maps $\pi^G_n : G(n) \to \mathrm{S}_n$ naturally form an action operad $\mathrm{im}(\pi^G)$, where

- the sets of operations are the images of G's sets of operations under the homomorphisms π^G:
$$\mathrm{im}(\pi^G)(n) \quad := \quad \mathrm{im}(\pi^G_n)$$

- the underlying permutation maps are the evident inclusions:

$$\pi_n^{\mathrm{im}(\pi^G)} : \mathrm{im}(\pi^G)(n) \hookrightarrow \mathrm{S}_n$$

- the operad multiplication is the appropriate restriction of the multiplication of S:

$$\mu^{\mathrm{im}(\pi^G)}(\,g\,;\,h_1,...,h_n\,) \quad := \quad \mu^{\mathrm{S}}(\,g\,;\,h_1,...,h_n\,)$$

Clearly this $\mathrm{im}(\pi^G)$ is a sub action operad of the symmetric operad S, and we will call the *underlying permutation operad* of G.

For example, consider the action operad B we just saw in Example 1.4. For a given n, the braid group B_n is generated by $n-1$ elementary braids. But the underlying permutations of these braids are just the $n-1$ adjacent transpositions which generate the symmetric group S_n, and so the underlying permutation maps $\pi^{\mathrm{B}}_n : \mathrm{B}_n \to \mathrm{S}_n$ are all surjective. Thus the underlying permutation operad of B is just the whole symmetric action operad S.

It is even easier to see that S itself will have underlying permutations S, as the maps $\pi^{\mathrm{S}}_n = \mathrm{id} : \mathrm{S}_n \to \mathrm{S}_n$ are obviously surjective. Similarly, the trivial operad T is also its own underlying permutation action operad, as the image of the homomorphims $\pi^{\mathrm{T}}_n : \{e\} \to \mathrm{S}_n$ are trivial. Faced with rather dull examples like these, it might be tempting to try and construct some action operads with more exotic underlying permutations, like maybe the alternating groups $\mathrm{A}_n \subset \mathrm{S}_n$. But it turns out that this is not possible; when it come to their underlying permutation operad, action operads come in exactly two flavours.

Definition 1.11. Let G be an action operad where $\mathrm{im}(\pi)(n)$ is the trivial group for each $n \in \mathbb{N}$. Then we say that G is *non-crossed*, since its operad multiplication will be a true group homomorphism:

$$\begin{aligned}\mu(\,gg'\,;\,h_1h'_1,...,h_nh'_n\,) &= \mu(\,g\,;\,h_{\pi(g')^{-1}(1)},...,h_{\pi(g')^{-1}(n)}\,)\mu(\,g'\,;\,h'_1,...,h'_n\,)\\ &= \mu(\,g\,;\,h_1,...,h_n\,)\mu(\,g'\,;\,h'_1,...,h'_n\,)\end{aligned}$$

Likewise, a *crossed* action operad will refer to any that has a non-trivial underlying permutation operad.

Lemma 1.12. *An action operad G is crossed if and only if it has surjective underlying permutation maps $\pi_n : G(n) \to \mathrm{S}_n$. In other words, the underlying permutations operad of G must be either the trivial operad* T *or the symmetric operad* S.

Proof. Let $\mathrm{im}(\pi)$ be the underlying permutation operad of G, and let us assume that G is crossed, so that $\mathrm{im}(\pi)$ is not the trivial operad. This means that for some natural number n, the n-ary operations of $\mathrm{im}(\pi)$ include at least one permutation σ which is not the identity element of the relevant symmetric group S_n. Put another way, there must be some σ and some $1 \leq i \leq n$ for which $\sigma(i) \neq i$. But now consider evaluating the expression

$$\mu^{\mathrm{im}(\pi)}(\,\sigma\,;\,e_0,...,e_0,e_1,e_0,....,e_0,e_1,e_0,...,e_0\,)$$

where the e_1's above are appearing in the ith and $\sigma(i)$th coordinates, which we know are distinct. From the definitions of $\mathrm{im}(\pi)(n)$ and of operad multiplication in S, this permutation is really just

$$\mu^{\mathrm{S}}(\,\sigma\,;\,e_0,...,e_0,e_1,e_0,....,e_0,e_1,e_0,...,e_0\,) \quad = \quad (1\,2)$$

the only non-identity element of S_2. This proves that the map $\pi_2 : G(2) \to \mathrm{S}_2$ is indeed surjective, but more than that it shows that $\mathrm{im}(\pi)$ must contain every possible adjacent transposition, since for any $m \in \mathbb{N}$ we have

$$\begin{aligned} & \mu^{\mathrm{im}(\pi)}(\,e_n\,;\,e_1,...,e_1,(1\,2),e_1,....,e_1\,) \\ = \; & \mu^{\mathrm{S}}(\,e_n\,;\,e_1,...,e_1,(1\,2),e_1,....,e_1\,) \\ = \; & (m\;m+1) \in \mathrm{S}_n \end{aligned}$$

Then because adjacent transpositions generate the symmetric groups S_n, it follows that every permutation is actually an operation in $\mathrm{im}(\pi)$, so that it is really just the full symmetric operad S. Thus by only assuming that our action operad G was crossed, we have shown that all of the maps π_n must be surjective. □

Definition 1.13. G-operads

1.2 Operad algebras

Definition 1.14. Operad algbras

Definition 1.15. G-operad algebras

1.3 EG-algebras

Definition 1.16. The G-operad EG

Definition 1.17. The monad EG

Definition 1.18. EG-algebras

Proposition 1.19. *G-operad algebras are monoidal categories with permutation-like structure*

Corollary 1.20. *Braided monoidal categories are G-operad algebras*

Definition 1.21. A strict monoidal category X is said to be *spacial* if, for any object $x \in \mathrm{Ob}(X)$ and any endomorphism of the unit object $f : I \to I$,

$$f \otimes \mathrm{id}_x = \mathrm{id}_x \otimes f$$

The motivation for the name 'spacial' comes from the context of string diagrams [4]. In a string diagram, the act of tensoring two strings together is represented by placing those strings side by side. Since the defining feature of the unit object is that tensoring it with other objects should have no effect, the unit object is therefore represented diagrammatically by the absense of a string. An endomorphism of the unit thus appears as an entity with no input or output strings, detached from the rest of the diagram. In a real-world version of these diagrams, made out of physical strings arranged in real space, we could use this detachedness to grab these endomorphisms and slide them over or under any strings we please, without affecting anything else in the diagram. This ability is embodied algebraically by the equation above, and hence categories which obey it are called 'spacial'.

Lemma 1.22. *If G is a crossed action operad, then all* E*G-algebras are spacial.*

Proof. Let G be a crossed action operad, let X be a EG-algebra, and fix $x \in \mathrm{Ob}(X)$ and $f : I \to I$. From Lemma 1.12 we know that $\pi : G(2) \to S_2$ is surjective, so that the set $\pi^{-1}(\,(1\,2)\,)$ is non-empty, and from the rules for composition of action morphisms we see that for any such $g \in \pi^{-1}(\,(1\,2)\,)$,

$$\begin{aligned} \alpha(\,g\,;\,\mathrm{id}_x,\,\mathrm{id}_I\,) \circ \alpha(\,e_2\,;\,\mathrm{id}_x,\,f\,) &= \alpha(\,g\,;\,\mathrm{id}_x,\,f\,) \\ &= \alpha(\,e_2\,;\,f,\,\mathrm{id}_x\,) \circ \alpha(\,g\,;\,\mathrm{id}_x,\,\mathrm{id}_I\,) \end{aligned}$$

Thus in order to obtain the result we're after, it will suffice to find a particular $g \in \pi^{-1}(\,(1\,2)\,)$ for which

$$\alpha(\,g\,;\,\mathrm{id}_x,\,\mathrm{id}_I\,) = \mathrm{id}_x$$

However, since

$$\begin{aligned} \alpha(\,g\,;\,\mathrm{id}_x,\,\mathrm{id}_I\,) &= \alpha(\,g\,;\,\mathrm{id}_x,\,\alpha(e_0;-)\,) \\ &= \alpha(\,\mu(g;e_1,e_0)\,;\,\mathrm{id}_x\,) \end{aligned}$$

all we really need is to find a $g \in \pi^{-1}(\,(1\,2)\,)$ for which

$$\mu(g; e_1, e_0) = e_1$$

To this end, choose an arbitrary element $h \in \pi^{-1}(\,(1\,2)\,)$. This h probably won't obey the above equation, but we can use it to construct a new element g which does. Specifically, define

$$k \;:=\; \mu(\,h\;;\,e_1,\,e_0\,)$$

and then consider

$$g \;:=\; h \cdot \mu(e_2; k^{-1}, e_1)$$

To see that this is the correct choice of g, first note that we must have $\pi(k) = e_1$, since this is the only element of S_1. Following from that, we have

$$\begin{aligned} \pi\Big(\,\mu(e_2; k^{-1}, e_1)\Big) &= \mu\Big(\,\pi(e_2)\;;\,\pi(k^{-1}),\,\pi(e_1)\,\Big) \\ &= \mu\Big(\,e_2\;;\,e_1,\,e_1\,\Big) \\ &= e_2 \end{aligned}$$

and hence

$$\begin{aligned} \pi(g) &= \pi\Big(h \cdot \mu(e_2; k^{-1}, e_1)\Big) \\ &= \pi(h) \cdot \pi\Big(\mu(e_2; k^{-1}, e_1)\Big) \\ &= (1\,2) \cdot e_2 \\ &= (1\,2). \end{aligned}$$

So g is indeed in $\pi^{-1}(\,(1\,2)\,)$, and furthermore

$$\begin{aligned} \mu(g; e_1, e_0) &= \mu\Big(\,h \cdot \mu(e_2; k^{-1}, e_1)\;;\,e_1,\,e_0\,\Big) \\ &= \mu(\,h\;;\,e_1,\,e_0\,) \cdot \mu\Big(\,\mu(e_2; k^{-1}, e_1)\;;\,e_1,\,e_0\,\Big) \\ &= \mu(\,h\;;\,e_1,\,e_0\,) \cdot \mu\Big(\,e_2\;;\,\mu(k^{-1}; e_1),\,\mu(e_1; e_0)\,\Big) \\ &= \mu(\,h\;;\,e_1,\,e_0\,) \cdot \mu(\,e_2\;;\,k^{-1}, e_0\,) \\ &= k \cdot k^{-1} \\ &= e_1 \end{aligned}$$

Therefore, $h \cdot \mu(e_2; k^{-1}, e_1)$ is exactly the g we were looking for, and so working backwards through the proof we obtain the required result:

$$\begin{array}{rrcl} & \mu(g; e_1, e_0) & = & e_1 \\ \Longrightarrow & \alpha(\, g\, ;\, \mathrm{id}_x,\, \mathrm{id}_I\,) & = & \mathrm{id}_x \end{array}$$

$$\begin{array}{rrcl} \alpha(\, g\, ;\, \mathrm{id}_x,\, \mathrm{id}_I\,) \circ \alpha(\, e_2\, ;\, \mathrm{id}_x,\, f\,) & = & \alpha(\, e_2\, ;\, f,\, \mathrm{id}_x\,) \circ \alpha(\, g\, ;\, \mathrm{id}_x,\, \mathrm{id}_I\,) \\ \Longrightarrow \quad \alpha(\, e_2\, ;\, \mathrm{id}_x,\, f\,) & = & \alpha(\, e_2\, ;\, f,\, \mathrm{id}_I\,) \end{array}$$

□

1.4 The free EG-algebra on n objects

Our goal for the next few chapters will be to understand the free braided monoidal category on an finite number of invertible objects. Thus, now that we have a firm grasp on action operads and their algebras, we should begin to think about the simpler free constructions they can form. We will use this extensively when calculating the invertible case later on.

In the paper [7], Gurski establishes how to contruct free G-operad algebras through the use of the monad EG. What follows in this section is a quick summary of the results which will be useful for our purposes. For a more detailed treatment please refer to [7].

Proposition 1.23. *There exists a free* EG*-algebra on n objects. That is, there is an* EG*-algebra Y such that for any other* EG*-algebra X, we have an isomorphism of categories*

$$\mathrm{E}G\mathrm{Alg}_S(Y, X) \cong X^n$$

Proof. There is an obvious forgetful 2-functor $U : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{Cat}$ sending EG-algebras to their underlying categories. U has a left adjoint, which we call the free 2-functor $F : \mathrm{Cat} \to \mathrm{E}G\mathrm{Alg}_S$ adjoint to it. It follows immediately that

$$\begin{array}{rcl} U(X)^n & = & \mathrm{Cat}(\{z_1, ..., z_n\}, U(X)) \\ & \cong & \mathrm{E}G\mathrm{Alg}_S(F(\{z_1, ..., z_n\}), X) \end{array}$$

where $\{z_1, ..., z_n\}$ is any set with n distinct elements. Since X and $U(X)$ are obviously isomorphic as categories, this shows that $F(\{z_1, ..., z_n\})$ is the free algebra on n objects as required. □

Definition 1.24. Let $\{z_1, ..., z_n\}$ be an n-object set, which we will also consider as a discrete category. Then we will denote by $\mathbb{G}_n$ the EG-algebra whose underlying category is $\mathrm{E}G(\{z_1, ..., z_n\})$ and whose action

$$\alpha : \mathrm{E}G\Big(\mathrm{E}G(\{z_1, ..., z_n\})\Big) \to \mathrm{E}G(\{z_1, ..., z_n\})$$

is the appropriate component of the multiplication natural transformation $\mu : \mathrm{E}G \circ \mathrm{E}G \to \mathrm{E}G$ of the 2-monad EG.

Theorem 1.25. *$\mathbb{G}_n$ is the free* EG*-algebra on n objects. That is,*

$$F(\{z_1, ..., z_n\}) = \mathbb{G}_n$$

Proof. □

Definition 1.24 is a fairly opaque definition, so we'll spend a little time upacking it. Recall from Definition 1.17 that $\mathrm{E}G(\{z_1, ..., z_n\})$ is the coequalizer of the maps

$$\textstyle\coprod_{m\geq 0} \mathrm{E}G(m) \times G(m) \times \{z_1, ..., z_n\}^m \rightrightarrows \coprod_{m\geq 0} \mathrm{E}G(m) \times \{z_1, ..., z_n\}^m$$

that comes from the action of $G(m)$ on $\mathrm{E}G(m)$ by multiplication on the right,

$$\begin{array}{rcl} \mathrm{E}G(m) \times G(m) & \to & \mathrm{E}G(m) \\ (g, h) & \mapsto & gh \\ (\,! : g \to g', \mathrm{id}_h\,) & \mapsto & ! : gh \to g'h \end{array}$$

and the action of $G(m)$ on $\{z_1, ..., z_n\}^m$ by permutation,

$$\begin{array}{rcl} G(m) \times \{z_1, ..., z_n\}^m & \to & \{z_1, ..., z_n\}^m \\ (\,h\,;\, x_1, ..., x_m\,) & \mapsto & \big(x_{\pi(h^{-1})(1)}, ..., x_{\pi(h^{-1})(m)}\big) \\ (\mathrm{id}_h\,;\, \mathrm{id}_{(x_1,...,x_m)}\,) & \mapsto & \mathrm{id}_{(x_{\pi(h^{-1})(1)},...,x_{\pi(h^{-1})(m)})} \end{array}$$

First, objects in this algebra are equivalence classes of tuples $(g; x_1, ..., x_m)$, for $g \in G(m)$ and $x_i \in \{z_1, ..., z_n\}$, under the relation

$$(\,gh\,;\, x_1,\, ...,\, x_m\,) \sim (\,g\,;\, x_{\pi(h)^{-1}(1)},\, ...,\, x_{\pi(h)^{-1}(m)}\,)$$

Notice that using this relation we can rewrite any object uniquely in the form $[e; x_1, ..., x_m]$ for some $m \in \mathbb{N}$ and $x_i \in \{z_1, ..., z_n\}$. This means that each equivalence class is just the tensor product $x_1 \otimes ... \otimes x_m$ in the underlying monoidal category of $\mathbb{G}_n$, for some unique sequence of generators. That is, we can view the objects of $\mathbb{G}_n$ as elements of the monoid freely generated by each of the z_i, or in other words:

Lemma 1.26. $\mathrm{Ob}(\mathbb{G}_n)$ *is the free monoid on n generators, $\mathbb{N}^{*n}$, the free product of n copies of $\mathbb{N}$.*

Similarly, the morphisms of $\mathbb{G}_n$ are the maps

$$(!; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) : (g; x_1, ..., x_m) \to (g'; x_1, ..., x_m)$$

with $g, g' \in G(m)$ and $x_i \in \{z_1, ..., z_n\}$. Using the relation $\sim$ on objects we can rewrite each of these morphisms in the form

$$[h; \mathrm{id}_{y_1}, ..., \mathrm{id}_{y_m}] \; : \; y_1 \otimes ... \otimes y_m \; \to \; y_{\pi(h^{-1})(1)} \otimes ... \otimes y_{\pi(h^{-1})(m)}$$

where

$$h = g'g^{-1}, \qquad y_i = x_{\pi(g^{-1})(i)}$$

The EG-action of $\mathbb{G}_n$ is permutation and tensor product, and the action on morphisms is given by

$$\alpha(\, g\, ;\, [h_1; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_{m_1}}],\, ...,\, [h_k; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_{m_k}}]\,) = [\, \mu(g; h_1, .., h_k)\, ;\, \mathrm{id}_{x_1},\, ...,\, \mathrm{id}_{x_{m_k}}\,]$$

Notice that using tensor product notation the object $[e; x]$ is simply x, and so $[e; \mathrm{id}_x] = \mathrm{id}_{[e;x]}$ should be written as id_x. Hence by the above $[g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}]$ is really just $\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$, and so we have the following:

Lemma 1.27. *Every morphism of $\mathbb{G}_n$ can be expressed uniquely as an action morphism*

$$\alpha(\, g\, ;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,) \; : \; x_1 \otimes ... \otimes x_m \; \to \; x_{\pi(g)^{-1}(1)} \otimes ... \otimes x_{\pi(g)^{-1}(m)}$$

for some $g, g' \in G(m)$ and $x_i \in \{z_1, ..., z_n\}$.

As an immediate consequence of this, the source and target of a given morphism in $\mathbb{G}_n$ must be related to one another by some permutation of the form $\pi(g)$. In toher words, the connected components of $\mathbb{G}_n$ will depend upon the underlying permutation operad of G, in the following way:

Proposition 1.28. *Considered as a monoid under tensor product,*

$$\pi_0(\mathbb{G}_n) \quad = \quad \begin{cases} \mathbb{N}^n & \text{if } G \text{ is crossed} \\ \mathbb{N}^{*n} & \text{otherwise} \end{cases}$$

Also, the canonical homomorphism sending objects in $\mathbb{G}_n$ to their connected component,

$$[\_\,] \,:\, \mathrm{Ob}(\mathbb{G}_n) \to \pi_0(\mathbb{G}_n)$$

is the quotient map of abelianisation

$$\mathrm{ab} \,:\, \mathbb{N}^{*n} \to (\mathbb{N}^{*n})^{\mathrm{ab}} \,=\, \mathbb{N}^n$$

when G is crossed, and the identity map $\mathrm{id}_{\mathbb{N}^{*n}}$ otherwise.

Proof. By Lemma 1.27, all morphisms in $\mathbb{G}_n$ can be written uniquely as $\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$, for some $g \in G(m)$ and $x_i \in \{z_1, ..., z_n\}$, the set of generators of $\mathbb{N}^{*n}$. Since maps of this form have source $x_1 \otimes ... \otimes x_m$ and target $x_{\pi(g^{-1})(1)} \otimes ... \otimes x_{\pi(g^{-1})(m)}$, we see that the only pairs of object which might have a morphism between them are those that can be expanded as tensor products that differ by some permutation.

If our action operad G is crossed, then for any two objects like this — say source $x_1 \otimes ... \otimes x_m$ and target $x_{\sigma^{-1}(1)} \otimes ... \otimes x_{\sigma^{-1}(m)}$ for an arbitrary $\sigma \in \mathrm{S}_m$ — we can always find a map $\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$ between them, because by Lemma 1.12 the underlying permutations maps $\pi_m : G(m) \to S_m$ are all surjective and so there must exist at least one g with $\pi(g) = \sigma$. In particular, for any two generating objects z_i and $z_j$of $\mathbb{G}_n$ there must exist at least morphism between $z_i \otimes z_j$ and $z_j \otimes z_i$, and therefore

$$[z_i] \otimes [z_j] \quad = \quad [z_i \otimes z_j] \quad = \quad [z_j \otimes z_i] \quad = \quad [z_j] \otimes [z_i]$$

Thus the canonical map $[\_\,] : \mathrm{Ob}(\mathbb{G}_n) \to \pi_0(\mathbb{G}_n)$ is the one that makes the free product of $\mathbb{N}^{*n}$ commutative, that is, the quotient map for the abelianisation $\mathrm{ab} : \mathbb{N}^{*n} \to (\mathbb{N}^{*n})^{\mathrm{ab}}$, and so $\pi_0(\mathbb{G}_n) = \mathbb{N}^n$.

Conversely, if G is non-crossed then its underlying permutation operad $\mathrm{im}(\pi)$ is trivial, and so the only morphisms we have in $\mathbb{G}_n$ will be those of the form

$$\alpha(\, e_m\,;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,) \quad = \quad \mathrm{id}_{x_1} \otimes ... \otimes \mathrm{id}_{x_m} \quad = \quad \mathrm{id}_{x_1 \otimes ... \otimes x_m}$$

Therefore the map $[\_]$ just sends each object to its identity morphism, and since that function is one-to-one and onto it follows that

$$\pi_0(\mathbb{G}_n) \quad = \quad \mathrm{Ob}(\mathbb{G}_n) \quad = \quad \mathbb{N}^{*n}, \qquad\qquad [\_] \quad = \quad \mathrm{id}_{\mathbb{N}^{*n}}$$

by Lemma 1.26. □

Finally, Lemma 1.27 also gives us a complete description of how the morphisms of $\mathbb{G}_n$ interact under tensor product, though we need a little new terminology in order to express it properly.

Definition 1.29. Let G be an action operad. Then we will also the notation G to denote the *underlying monoid* of this action operad. This is the natural way to consider G as a monoid, with its element set being all of its elements together, $\bigsqcup_m G(m)$, and with tensor product as its binary operation, $g \otimes h = \mu(e_2; g, h)$.

Also, note that this monoid comes equipped with a homomorphism $|\_| : G \to \mathbb{N}$, sending each $g \in G$ to the natural number m if and only if g is an element of the group $G(m)$. We'll call this number $|g|$ the *length* of g.

Definition 1.30. Let S be a set and $F(S)$ the free monoid on S, the monoid whose elements are strings of elements of S and whose binary operation is concatenation. Then we will denote by

$$|\_| : F(S) \to \mathbb{N}$$

the monoid homomorphism defined by sending each element of $S \subseteq F(S)$ to 1, and therefore also each concatenation of n elements of S to the natural number n. Again, we will call $|x|$ the *length* of $x \in F(S)$.

Lemma 1.31. *The monoid of morphisms of the algebra $\mathbb{G}_n$ is*

$$\mathrm{Mor}(\mathbb{G}_n) \cong G \times_{\mathbb{N}} \mathbb{N}^{*n}$$

where this pullback is taken over the respective length homomorphisms,

$$\begin{tikzcd}
G \times_{\mathbb{N}} \mathbb{N}^{*n} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbb{N}^{*n} \arrow[d, "|_|"] \\
G \arrow[r, "|_|"] & \mathbb{N}
\end{tikzcd}$$

using the fact that $\mathbb{N}^{*n}$ is the free monoid $F\Big(\{z_1, ..., z_n\}\Big)$.

Proof. An element of $G \times_{\mathbb{N}} F\Big(\{z_1, ..., z_n\}\Big)$ is just an element $g \in G(m)$ for some m, together with an m-tuple of objects $(x_1, ..., x_m)$ from the set of generators $\{z_1, ..., z_n\}$. Thus the action on $\mathbb{G}_n$ defines an obvious function

$$\begin{array}{rcrcl} \alpha & : & G \times_{\mathbb{N}} F\Big(\{z_1, ..., z_n\}\Big) & \to & \mathrm{Mor}(\mathbb{G}_n) \\ & : & (g; x_1, ..., x_m) & \mapsto & \alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) \end{array}$$

But by Lemma 1.27, each element of $\mathrm{Mor}(\mathbb{G}_n)$ can be expressed in the form $\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$ for a unique collection $(g; x_1, ..., x_m)$, and so this function α is actually a bijection of sets. Furthermore, this function preserves tensor product, since

$$\begin{array}{rcl} \alpha\Big((g; f_1, ..., f_m) \otimes (g'; f_1', ..., f_m')\Big) & = & \alpha(\, g \otimes g' \,;\, f_1, ..., f_m, f_1', ..., f_m' \,) \\ & = & \alpha(\, g \,;\, f_1, ..., f_m \,) \otimes \alpha(\, g' \,;\, f_1', ..., f_m' \,) \end{array}$$

and hence it is a monoid isomorphism, as required. $\square$

Chapter 2

Free invertible algebras as initial objects

In this chapter we will start to consider how to construct free EG-algebras on some number of invertible objects. Specifically, we will begin by showing that such algebras are the initial objects of a particular comma category, in accordance with some well known properties of adjunctions and their units. Using this initial object prespective will allow us to recover all of the data associated with the objects of a given free invertible algebra — what those objects are, how they act under tensor product, and which pairs of objects form the source and target of at least one morphism. Unfortunately, a concrete description of the morphisms themselves will ultimately remain elusive. We can get tantalisingly closer though, and an examiniation of the exact way that this method fails will provide the neccessary insight to motivate a more successful approach in **??**.

2.1 The free algebra on n invertible objects

We saw in Proposition 1.23 that the existence of a free EG-algebra on n objects can be proven by taking the left adjoint of a 2-functor which forgets about the algebra structure. Now we want to extend this idea into the realm of algebras on invertible objects. For the analogous approach, we will need to find a new 2-functor that lets us forget about non-invertible objects, and then hopefully we can find its left adjoint too, and use it to freely add inverses to $\mathbb{G}_n$. First though, we need to make this concept of 'forgetting non-invertible objects' a little more precise.

Definition 2.1. Given an EG-algebra X, we denote by X_{inv} the sub-EG-algebra containing all invertible objects in X and the isomorphisms between them.

Note that this is indeed a well-defined EG-algebra. If $x_1, ..., x_m$ are invertible objects with inverses $x_1^*, ..., x_m^*$, then $\alpha(g; x_1, ..., x_m)$ is an invertible object with inverse $\alpha(g; x_m^*, ..., x_1^*)$, since

$$\begin{aligned} & \alpha(g; x_1, ..., x_m) \otimes \alpha(g; x_m^*, ..., x_1^*) \\ = & \left(x_{\pi(g)^{-1}(1)} \otimes ... \otimes x_{\pi(g)^{-1}(m)} \right) \otimes \left(x^*_{\pi(g)^{-1}(m)} \otimes ... \otimes x^*_{\pi(g)^{-1}(1)} \right) \\ = & I \end{aligned}$$

$$\begin{aligned} & \alpha(g; x_m^*, ..., x_1^*) \otimes \alpha(g; x_1, ..., x_m) \\ = & \left(x^*_{\pi(g)^{-1}(m)} \otimes ... \otimes x^*_{\pi(g)^{-1}(1)} \right) \otimes \left(x_{\pi(g)^{-1}(1)} \otimes ... \otimes x_{\pi(g)^{-1}(m)} \right) \\ = & I \end{aligned}$$

Likewise, if $f_1, ..., f_m$ are isomorphisms from invertible objects $x_1, ..., x_m$ to invertible objects $y_1, ..., y_m$, then $\alpha(g; f_1, ..., f_m)$ is a map from the invertible object $\alpha(g; x_1, ..., x_m)$ to the invertible object $\alpha(g; y_1, ..., y_m)$, and it has an inverse $\alpha(g^{-1}; f^{-1}_{\pi(g)(1)}, ..., f^{-1}_{\pi(g)(m)})$, since

$$\begin{aligned} & \alpha\Big(g^{-1} \, ; \, f^{-1}_{\pi(g)(1)}, \, ..., \, f^{-1}_{\pi(g)(m)} \Big) \circ \alpha(\, g \, ; \, f_1, ..., f_m \,) \\ = & \alpha\Big(g^{-1} g \, ; \, f_1^{-1} f_1, \, ..., \, f_m^{-1} f_m \Big) \\ = & \mathrm{id}_{x_1 \otimes ... \otimes x_m} \end{aligned}$$

$$\begin{aligned} & \alpha(\, g \, ; \, f_1, ..., f_m \,) \circ \alpha\Big(g^{-1} \, ; \, f^{-1}_{\pi(g)(1)}, \, ..., \, f^{-1}_{\pi(g)(m)} \Big) \\ = & \alpha\Big(g g^{-1} \, ; \, f_{\pi(g)(1)} f^{-1}_{\pi(g)(1)}, \, ..., \, f_{\pi(g)(m)} f^{-1}_{\pi(g)(m)} \Big) \\ = & \mathrm{id}_{y_{\pi(g)(1)} \otimes ... \otimes y_{\pi(g)(m)}} \end{aligned}$$

Clearly then, X_{inv} is the correct algebra for our new forgetful 2-functor to send X to. Knowing this, we can contruct the rest of the functor fairly easily.

Proposition 2.2. *The assignment $X \mapsto X_{\text{inv}}$ can be extended to a 2-functor $(\_)_{\text{inv}}$: $\mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$.*

Proof. Let $F : X \to Y$ be a (strict) map of EG-algebras. If x is an invertible object in X with inverse x^*, then $F(x)$ is an invertible object in Y with inverse $F(x^*)$, by

$$F(x) \otimes F(x^*) = F(x \otimes x^*) = F(I) = I$$

$$F(x^*) \otimes F(x) = F(x^* \otimes x) = F(I) = I$$

Since F sends invertible objects to invertible objects, it will also send isomorphisms of invertible objects to isomorphisms of invertible objects. In other words, the map $F : X \to Y$ can be restricted to a map $F_{\text{inv}} : X_{\text{inv}} \to Y_{\text{inv}}$. Moreover, we have that

$$(F \circ G)_{\text{inv}}(x) = F \circ G(x) = F_{\text{inv}} \circ G_{\text{inv}}(x)$$

$$(F \circ G)_{\text{inv}}(f) = F \circ G(f) = F_{\text{inv}} \circ G_{\text{inv}}(f)$$

and so the assignment $F \mapsto F_{\text{inv}}$ is clearly functorial. Next, let $\theta : F \Rightarrow G$ be an EG-monoidal natural transformation. Choose an invertible object x from X, and consider the component map of its inverse, $\theta_{x^*} : F(x^*) \to G(x^*)$. Since θ is monoidal, we have $\theta_{x^*} \otimes \theta_x = \theta_I = I$ and $\theta_x \otimes \theta_{x^*} = I$, or in other words that θ_{x^*} is the monoidal inverse of θ_x. We can use this fact to construct a compositional inverse as well, namely $\mathrm{id}_{F(x)} \otimes \theta_{x^*} \otimes \mathrm{id}_{G(x)}$, which can be seen as follows:

$$\begin{aligned}\left(\mathrm{id}_{F(x)} \otimes \theta_{x^*} \otimes \mathrm{id}_{G(x)}\right) \circ \theta_x &= \theta_x \otimes \theta_{x^*} \otimes \mathrm{id}_{G(x)} \\ &= \mathrm{id}_{G(x)}\end{aligned}$$

$$\begin{aligned}\theta_x \circ \left(\mathrm{id}_{F(x)} \otimes \theta_{x^*} \otimes \mathrm{id}_{G(x)}\right) &= \mathrm{id}_{F(x)} \otimes \theta_{x^*} \otimes \theta_x \\ &= \mathrm{id}_{F(x)}\end{aligned}$$

Therefore, we see that all the components of our transformation on invertible objects are isomorphisms, and hence we can define a new transformation $\theta_{\text{inv}} : F_{\text{inv}} \Rightarrow G_{\text{inv}}$ whose components are just $(\theta_{\text{inv}})_x = \theta_x$. The assignment $\theta \mapsto \theta_{\text{inv}}$ is also clearly functorial, and thus we have a complete 2-functor $(\_)_{\text{inv}} : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$. □

Proposition 2.3. *The 2-functor* $(\_)_{\text{inv}} : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$ *has a left adjoint,* $L : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$.

Proof. To begin, consider the 2-monad E$G(_)$. This is a finitary monad, that is it preserves all filtered colimits, and it is a 2-monad over Cat, which is locally finitely presentable. It follows from this that $\mathrm{E}G\mathrm{Alg}_S$ is itself locally finitely presentable. Thus if we want to prove $(\_)_{\text{inv}}$ has a left adjoint, we can use the Adjoint Functor Theorem for locally finitely presentable categories, which amounts to showing that $(\_)_{\text{inv}}$ preserves both limits and filtered colimits.

- Given an indexed collection of EG-algebras X_i, the EG-action of their product $\prod X_i$ is defined componentwise. In particular, this means that the tensor product of two objects in $\prod X_i$ is just the collection of the tensor products of their components in each of the X_i. An invertible object in $\prod X_i$ is thus simply a family of invertible objects from the X_i — in other words, $(\prod X_i)_{\mathrm{inv}} = \prod (X_i)_{\mathrm{inv}}$.

- Given maps of EG-algebras $F : X \to Z$, $G : Y \to Z$, the EG-action of their pullback $X \times_Z Y$ is also defined componentwise. It follows that an invertible object in $X \times_Z Y$ is just a pair of invertible objects (x, y) from X and Y, such that $F(x) = G(y)$. But this is the same as asking for a pair of objects (x, y) from X_{inv} and Y_{inv} such that $F_{\mathrm{inv}}(x) = G_{\mathrm{inv}}(y)$, and hence $(X \times_Z Y)_{\mathrm{inv}} = X_{\mathrm{inv}} \times_{Z_{\mathrm{inv}}} Y_{\mathrm{inv}}$.
- Given a filtered diagram D of EG-algebras, the EG-action of their colimit $\mathrm{colim}(D_n)$ is defined in the following way: use filteredness to find an algebra which contains (representatives of the classes of) all the things you want to act on, then apply the action of that algebra. In the case of tensor products this means that $[x] \otimes [y] = [x \otimes y]$, and thus an invertible object in $\mathrm{colim}(D_n)$ is just (the class of) an invertible object in one of the algebras of D. In other words, $\mathrm{colim}(D_n)_{\mathrm{inv}} = \mathrm{colim}(D_{\mathrm{inv}})$.

Preservation of products and pullbacks gives preservation of limits, and preservation of limits and filtered colimits gives the result. □

With this new 2-functor $L : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$, we now have the ability to 'freely add inverses to objects' in any EG-algebra we want. The algebra $L\mathbb{G}_n$ is then a clear candidate for our free algebra on n invertible objects, and indeed the proof of this is very simple.

Theorem 2.4. *There exists a free* EG*-algebra on n invertible objects. Specifically, the algebra $L\mathbb{G}_n$ is such that for any other* EG*-algebra X, we have an isomorphism of categories*

$$\mathrm{E}G\mathrm{Alg}_S(L\mathbb{G}_n, X) \quad \cong \quad (X_{\mathrm{inv}})^n$$

Proof. Using the adjunction from Proposition 2.3 along with the one from Proposition 1.23, we see that

$$\begin{aligned} U(X_{\mathrm{inv}})^n &= \mathrm{Cat}(\{z_1, ..., z_n\}, U(X_{\mathrm{inv}})) \\ &\cong \mathrm{E}G\mathrm{Alg}_S(F(\{z_1, ..., z_n\}), X_{\mathrm{inv}}) \\ &\cong \mathrm{E}G\mathrm{Alg}_S(LF(\{z_1, ..., z_n\}), X) \end{aligned}$$

As before, X_{inv} and $U(X_{\mathrm{inv}})$ are obviously isomorphic as categories, and so $LF(\{z_1, ..., z_n\}) = L\mathbb{G}_n$ satisfies the requirements for the free algebra on n invertible objects. □

2.2 $L\mathbb{G}_n$ as an initial algebra

We have now proven that a free EG-algebra on n invertible objects indeed exists. But this fact on its own is not very helpful. To be able to actually use the free algebra $L\mathbb{G}_n$, we need to know how to contruct it explicitly, in terms of its objects and morphisms. We could do this by finding a detailed characterisation of the 2-functor L, and then applying this to our explicit description of $\mathbb{G}_n$ from Definition 1.24. However, this would probably take far more effort than is required, since it would involve determining the behaviour of L in many situtations that we aren't interested in. Also, we wouldn't be leveraging $\mathbb{G}_n$'s status as a free algebra to make the calculations any easier. We will try a different strategy instead, one that begins by noticing a special property of the functor L.

Proposition 2.5. *For any* EG*-algebra X, we have $L(X)_{\mathrm{inv}} = L(X)$.*

Proof. From the definition of adjunctions, the isomorphisms

$$\mathrm{E}G\mathrm{Alg}_S(LX, Y) \quad \cong \quad \mathrm{E}G\mathrm{Alg}_S(X, Y_{\mathrm{inv}})$$

are subject to certain naturality conditions. Specifically, given $F : X' \to X$ and $G : Y \to Y'$ we get a commutative diagram

$$\begin{tikzcd}
\mathrm{E}G\mathrm{Alg}_S(LX, Y) \arrow[r, "\sim"] \arrow[d, "G\circ_\circ LF"'] & \mathrm{E}G\mathrm{Alg}_S(X, Y_{\mathrm{inv}}) \arrow[d, "G_{\mathrm{inv}}\circ_\circ F"] \\
\mathrm{E}G\mathrm{Alg}_S(LX', Y') \arrow[r, "\sim"] & \mathrm{E}G\mathrm{Alg}_S(X', Y'_{\mathrm{inv}})
\end{tikzcd}$$

Consider the case where F is the identity map $\mathrm{id}_X : X \to X$ and G is the inclusion $j : L(X)_{\mathrm{inv}} \to L(X)$. Note that because j is an inclusion, the restriction $j_{\mathrm{inv}} : (L(X)_{\mathrm{inv}})_{\mathrm{inv}} \to L(X)_{\mathrm{inv}}$ is also an inclusion, but since $((\_)_{\mathrm{inv}})_{\mathrm{inv}} = (\_)_{\mathrm{inv}}$, we have that $j_{\mathrm{inv}} = \mathrm{id}$. It follows that

$$\begin{tikzcd}
\mathrm{E}G\mathrm{Alg}_S(LX, LX_{\mathrm{inv}}) \arrow[r, "\sim"] \arrow[d, "j\circ_"'] & \mathrm{E}G\mathrm{Alg}_S(X, LX_{\mathrm{inv}}) \arrow[d, equal] \\
\mathrm{E}G\mathrm{Alg}_S(LX, LX) \arrow[r, "\sim"] & \mathrm{E}G\mathrm{Alg}_S(X, LX_{\mathrm{inv}})
\end{tikzcd}$$

Therefore, for any map $f : LX \to LX$ there exists a unique $g : LX \to LX_{\text{inv}}$ such that $j \circ g = f$. But this means that for any such f, we must have $\text{im}(f) \subseteq L(X)_{\text{inv}}$, and so in particular $L(X) = \text{im}(\text{id}_{LX}) \subseteq L(X)_{\text{inv}}$. Since $L(X)_{\text{inv}} \subseteq L(X)$ by definition, we obtain the result. □

This result is not especially surprising. Intuitively, it just says that when you freely add inverses to an algebra, every object ends up with an inverse. But the upshot of this is that we now have another way of thinking about $L(X)$: as the target object of the unit of our adjunction, $\eta_X : X \to L(X)_{\text{inv}}$. This means that we don't really need to know the entirety of L in order to determine the free algebra $L\mathbb{G}_n$, just its unit. To find this unit directly, we can turn to the following fact about adjunctions, for which a proof can be found in Lemma 2.3.5 of Leinster's *Basic Category Theory* [8].

Proposition 2.6. *Let $F \dashv G : A \to B$ be an adjunction with unit η. For any object a in A, let $(a \downarrow G)$ denote the comma category whose objects are pairs (b, f) consisting of an object* B *from* B *and a morphism $f : a \to G(b)$ from A, and whose morphisms $h : (b, f) \to (b', f')$ are morphisms $f : b \to b'$ from* B *such that $G(f) \circ f = f'$. Then the pair $\Big(F(a), \eta_a : a \to GF(a)\Big)$ is an initial object of $(a \downarrow G)$.*

Corollary 2.7. *$\eta_{\mathbb{G}_n} : \mathbb{G}_n \to (L\mathbb{G}_n)_{\text{inv}} = L\mathbb{G}_n$ is an initial object of $(\mathbb{G}_n \downarrow \text{inv})$.*

Being able to view $L\mathbb{G}_n$ as the initial object in the comma category $(\mathbb{G}_n \downarrow \text{inv})$ will prove immensely useful in the coming sections. This is because it lets us think about the properties of $L\mathbb{G}_n$ in terms of maps $\psi : \mathbb{G}_n \to X_{\text{inv}}$, and this is exactly the context where we can exploit $\mathbb{G}_n$'s status as a free algebra. As a result, its worth taking some time to think about what exactly this map $\eta_{\mathbb{G}_n}$ is.

Lemma 2.8. *The initial object $\eta_{\mathbb{G}_n} : \mathbb{G}_n \to L\mathbb{G}_n$ is the obvious map from the free* EG*-algebra on n objects into the free* EG*-algebra on n* invertible *objects. That is, $\eta_{\mathbb{G}_n}$ is the algebra map defined by*

$$\begin{array}{rrcl} \eta_{\mathbb{G}_n} & : \quad \mathbb{G}_n & \to & L\mathbb{G}_n \\ & : \quad F(\{z_1, ..., z_n\}) & \to & LF(\{z_1, ..., z_n\}) \\ & : \quad z_i & \mapsto & z_i \end{array}$$

Proof. Consider the n-tuple $(z_1, ..., z_n)$ in $(\mathbb{G}_n)^n$. Clearly the image of $(z_1, ..., z_n)$ under the functor L is just the object $(z_1, ..., z_n)$ in the algebra

$$L\Big((\mathbb{G}_n)^n\Big) \quad = \quad (L\mathbb{G}_n)^n \quad = \quad LF(\{z_1, ..., z_n\})^n$$

But the image of $(z_1, ..., z_n) \in (\mathbb{G}_n)^n$ under the isomorphism

$$\mathrm{E}G\mathrm{Alg}_S(\,\mathbb{G}_n, \mathbb{G}_n\,) \quad \cong \quad (\mathbb{G}_n)^n$$

is just the identity map $\mathrm{id}_{\mathbb{G}_n}$. Thus by functoriality of L, the map $L(\mathrm{id}_{\mathbb{G}_n}) = \mathrm{id}_{L\mathbb{G}_n}$ must be the one which corresponds to the n-tuple $(z_1, ..., z_n) \in (L\mathbb{G}_n)^n$ image via the isomorphism

$$\mathrm{E}G\mathrm{Alg}_S(\,L\mathbb{G}_n, L\mathbb{G}_n\,) \quad \cong \quad (L\mathbb{G}_n)^n$$

Furthermore, the $\mathbb{G}_n$ component of the unit η is by definition the image of the identity map $\mathrm{id}_{L\mathbb{G}_n}$ under the isomorphism

$$\mathrm{E}G\mathrm{Alg}_S(\,L\mathbb{G}_n, L\mathbb{G}_n\,) \quad \cong \quad \mathrm{E}G\mathrm{Alg}_S(\,\mathbb{G}_n, L\mathbb{G}_n\,)$$

Hence it follows that $\eta_{\mathbb{G}_n}$ is the map that corresponds to $(z_1, ..., z_n)$ via

$$\mathrm{E}G\mathrm{Alg}_S(\,\mathbb{G}_n, L\mathbb{G}_n\,) \quad \cong \quad (L\mathbb{G}_n)^n$$

which is exactly the definition given in the statement of the lemma. □

This increadibly simple description makes the map η very easy to work with. For example, we immediately obtain the following property, one which we will use frequently throughout the rest of the paper:

Corollary 2.9. *η is an epimorphism in* $\mathrm{E}G\mathrm{Alg}_S$*.*

Proof. Let $\phi, \psi : L\mathbb{G}_n \to X$ be a pair of algebra maps for which $\phi \circ \eta = \psi \circ \eta$. Then on the generators of $L\mathbb{G}_n$ we have

$$\begin{array}{ccccccc} \phi(z_i) & = & \phi\eta(z_i) & = & \psi\eta(z_i) & = & \psi(z_i) \\ \Longrightarrow & & \phi_{\mathrm{inv}}(z_i) & = & \psi_{\mathrm{inv}}(z_i) & & \end{array}$$

But $L\mathbb{G}_n$ is the free EG-algebra on n invertible objects, so maps $L\mathbb{G}_n \to X_{\mathrm{inv}}$ are determined uniquely by where they those generating objects. It follows that $\phi_{\mathrm{inv}} = \psi_{\mathrm{inv}}$, and if $i : X_{\mathrm{inv}} \to X$ is the obvious inclusion,

$$\phi \quad = \quad i\phi_{\mathrm{inv}} \quad = \quad i\psi_{\mathrm{inv}} \quad = \quad \psi$$

□

Before moving on, we'll make a small change in notation. From now on, rather than writing objects in $(\mathbb{G}_n \downarrow \mathrm{inv})$ as maps $\psi : \mathbb{G}_n \to Y_{\mathrm{inv}}$, we will instead just let $X = Y_{\mathrm{inv}}$

and speak of maps $\psi : \mathbb{G}_n \to X$. This is purely to prevent the notation from becoming cluttered, and shouldn't be a problem so long as we always remember that the targets of these maps only ever contain invertible objects and morphisms. We'll also drop the subscript from $\eta_{\mathbb{G}_n}$, since it is the only component of the unit we'll ever use.

2.3 The objects of $L\mathbb{G}_n$

So now we know that $L\mathbb{G}_n$ is an initial object in the category $(\mathbb{G}_n \downarrow \mathrm{inv})$. But what does this actually tell us? After all, we do not currently have a method for finding initial objects in an arbitrary collection of EG-algebra maps. Because of this, we'll have to approach the problem step-by-step, using the initiality of η to extract different pieces of information about the algebra $L\mathbb{G}_n$ as we go. We'll begin by tring to find its objects.

Definition 2.10. Denote by $\mathrm{Ob} : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{Mon}$ be the functor that sends EG-algebras X to their monoid of objects $\mathrm{Ob}(X)$, and algebra maps $F : X \to Y$ to their underlying monoid homomorphism $\mathrm{Ob}(F) : \mathrm{Ob}(X) \to \mathrm{Ob}(Y)$.

In order to find $\mathrm{Ob}(L\mathbb{G}_n)$, we'll need to make use of an important result about the nature of Ob.

Definition 2.11. Recall that given a monoid M, the monoidal category EM is the one whose monoid of objects is M and which has a unique isomorphism between any two objects. We can view EM as not just a category but an EG-algebra, by letting the action on morphisms take the only possible values it can, given the required source and target. Similarly, for any monoid homomorphisms $h : M \to M'$ we can define a map of EG-algebras

$$\begin{array}{rcrcl} \mathrm{E}h & : & \mathrm{E}M & \to & \mathrm{E}M' \\ & : & m & \mapsto & h(m) \\ & : & m \to m' & \mapsto & h(m) \to h(m') \end{array}$$

This definition of Eh respects composition and identities, and so together with EM it describes a functor $\mathrm{E} : \mathrm{Mon} \to \mathrm{E}G\mathrm{Alg}_S$.

Proposition 2.12. E *is a right adjoint to the functor* Ob*.*

Proof. For any EG-algebra X, a map $F : X \to \mathrm{E}M$ is determined entirely by its restriction to objects, the monoid homomorphism $\mathrm{Ob}(F) : \mathrm{Ob}(X) \to M$. This is because functorality of F ensures that any map $x \to x'$ in X must be sent to a map $F(x) \to F(x')$ in $\mathrm{E}M$, and by the definition of E there is always exactly one of these to choose from. In other words, we have an isomorphism between the homsets

$$\mathrm{E}G\mathrm{Alg}_S(\, X,\, \mathrm{E}M\,) \quad \cong \quad \mathrm{Mon}(\,\mathrm{Ob}(X),\, M\,)$$

Additionally, this isomorphism is natural in both coordinates. That is, for any $G : X \to X'$ in $\mathrm{E}G\mathrm{Alg}_S$ and $h : M \to M'$ in Mon, the diagram

$$\begin{tikzcd}
\mathrm{E}G\mathrm{Alg}_S(X, \mathrm{E}M) \arrow[r, "\sim"] \arrow[d, "\mathrm{E}h\circ_\circ G"'] & \mathrm{Mon}(\mathrm{Ob}(X), M) \arrow[d, "h\circ_\circ\mathrm{Ob}(G)"] \\
\mathrm{E}G\mathrm{Alg}_S(X', \mathrm{E}M') \arrow[r, "\sim"] & \mathrm{Mon}(\mathrm{Ob}(X'), M')
\end{tikzcd}$$

commutes, because

$$\mathrm{Ob}(\,\mathrm{E}h \circ F \circ G\,) \quad = \quad \mathrm{Ob}(Eh) \circ \mathrm{Ob}(F) \circ \mathrm{Ob}(G) \quad = \quad h \circ \mathrm{Ob}(F) \circ \mathrm{Ob}(G)$$

Therefore, $\mathrm{Ob} \dashv \mathrm{E}$. □

What Proposition 2.12 is essentially saying is that the functor Ob provides a way for us to move back and forth between the categories $\mathrm{E}G\mathrm{Alg}_S$ and Mon. By applying this reasoning to the universal property of the initial object η, we can then determine the value of $\mathrm{Ob}(L\mathbb{G}_n)$ in terms of a new universal property of $\mathrm{Ob}(\eta)$ in the category Mon. In particular, the algebras in $(\mathbb{G}_n \downarrow \mathrm{inv})$ are those whose objects are all invertible, and so the induced property of $\mathrm{Ob}(\eta)$ will end up saying something about the relationship between $\mathrm{Ob}(\mathbb{G}_n)$ and groups — those monoids whose elements are all invertible.

Definition 2.13. Let M be a monoid, M^{gp} a group, and $i : M \to M^{\mathrm{gp}}$ a monoid homomorphism between them. Then we say that M^{gp} is the *group completion* of M if for any other group H and homomorphism $h : M \to H$, there exists a unique homomorphism $u : M^{\mathrm{gp}} \to H$ such that $u \circ i = h$.

There are several different ways to actually calculate the group completion of a monoid. One is to use that fact that M^{gp} is the group whose group presentation is the same as the monoid presentation of M. That is, if M is the quotient of the free monoid on generators $\mathcal{G}$ by the relations $\mathcal{R}$, then M^{gp} is the quotient of the free *group* on generators $\mathcal{G}$ by relations $\mathcal{R}$. This makes finding the completion of free monoids particularly simple.

Proposition 2.14. *The object monoid of $L\mathbb{G}_n$ is $\mathbb{Z}^{*n}$, the group completion of the object monoid of $\mathbb{G}_n$. The restriction of η on objects,* $\mathrm{Ob}(\eta)$*, is then the obvious inclusion* $\mathbb{N}^{*n} \hookrightarrow \mathbb{Z}^{*n}$.

Proof. Let H be a group, and $h : \mathrm{Ob}(\mathbb{G}_n) \to H$ a monoid homomorphism. By Proposition 2.12 we have an isomorphism of homsets

$$\mathrm{E}G\mathrm{Alg}_S(\, \mathbb{G}_n, \, \mathrm{E}H\,) \quad \cong \quad \mathrm{Mon}(\, \mathrm{Ob}(\mathbb{G}_n), \, H\,)$$

Denote by $h' : \mathbb{G}_n \to \mathrm{E}H$ the map of $\mathrm{E}G$-algebras corresponding to h under this isomorphism. Since H is a group, every object in $\mathrm{E}H$ is invertible, and so h' is an object of $(\mathbb{G}_n \downarrow \mathrm{inv})$. Thus, by initiality of η, there must exist a unique map $u : L\mathbb{G}_n \to \mathrm{E}G$ making the lefthand traingle below commute:

$$\begin{tikzcd} \mathbb{G}_n \arrow[d, "\eta"'] \arrow[dr, "h'"] & \\ L\mathbb{G}_n \arrow[r, "u"'] & \mathrm{E}H \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Ob}(\mathbb{G}_n) \arrow[d, "\mathrm{Ob}(\eta)"'] \arrow[dr, "h"] & \\ \mathrm{Ob}(L\mathbb{G}_n) \arrow[r, "\mathrm{Ob}(u)"'] & H \end{tikzcd}$$

It follows that the righthand triangle — which is the image of the first under Ob — also commutes. Hence for any group H and homomorphism $h : \mathrm{Ob}(\mathbb{G}_n) \to H$, there is at least one map which factors h through $\mathrm{Ob}(\eta)$.

But now recall from Corollary 2.9 that η is an epimorphism. Left adjoint functors preseve epimorphisms, which means that $\mathrm{Ob}(\eta)$ is one too, and so for any $v : \mathrm{Ob}(L\mathbb{G}_n) \to H$,

$$\begin{aligned} v \circ \mathrm{Ob}(\eta) \;=\; h \quad &\Longrightarrow \quad v \circ \mathrm{Ob}(\eta) \;=\; \mathrm{Ob}(u) \circ \mathrm{Ob}(\eta) \\ &\Longrightarrow \qquad\qquad v \;=\; \mathrm{Ob}(u) \end{aligned}$$

Thus there is actually only one possible map which factors h through $\mathrm{Ob}(\eta)$, and therefore every homomorphism from $\mathrm{Ob}(\mathbb{G}_n)$ onto a group factors uniquely through the group $\mathrm{Ob}(L\mathbb{G}_n)$. In other words, $\mathrm{Ob}(L\mathbb{G}_n)$ is the group completion $\mathrm{Ob}(\mathbb{G}_n)^{\mathrm{gp}}$. Since by Lemma 1.26 the object monoid of $\mathbb{G}_n$ is $\mathbb{N}^{*n}$, the free monoid on n generators, we can conclude that

$$\mathrm{Ob}(L\mathbb{G}_n) \quad = \quad \mathrm{Ob}(\mathbb{G}_n)^{\mathrm{gp}} \quad = \quad (\mathbb{N}^{*n})^{\mathrm{gp}} \quad = \quad \mathbb{Z}^{*n}$$

the free group on n generators. Moreover, the map $\mathrm{Ob}(\eta)$ is then the inclusion of $\mathrm{Ob}(\mathbb{G}_n)$ into its completion, which is just $\mathbb{N}^{*n} \hookrightarrow \mathbb{Z}^{*n}$. □

2.4 The connected components of $L\mathbb{G}_n$

The core result of Proposition 2.14 — that $\mathrm{Ob}(L\mathbb{G}_n)$ is the group completion of $\mathrm{Ob}(\mathbb{G}_n)$ — makes concrete the sense in which the functor L represents 'freely adding inverses' to objects. Extending this same logic to connected components as well, it would seem reasonable to expect that $\pi_0(L\mathbb{G}_n)$ is the group completion of $\pi_0(\mathbb{G}_n)$ as well. This is indeed the case, and the proof proceeds in a way completely analagous to Proposition 2.14.

First, we want to show that the process of taking connected components forms part of an adjunction. To do this we are going to need a category from which we can draw the kind of structures that can act as the components of an EG-algebra. Exactly which category this should be will depend on our choice of action operad G, or more precisely its underlying permutations.

Definition 2.15. For a given action operad G, denote by $\mathrm{im}(\pi)-\mathrm{Mon}$ the full subcategory of Mon on those monoids whose multiplication is invariant under the permutations in $\mathrm{im}(\pi)$. That is, a monoid M is in $\mathrm{im}(\pi)-\mathrm{Mon}$ if and only if

$$m_1, ..., m_n \in M,\ g \in G(n) \quad \Longrightarrow \quad m_1...m_n \ = \ m_{\pi(g)^{-1}(1)}...m_{\pi(g)^{-1}(n)}$$

Of course, by Lemma 1.12 there are really only two examples of such a $\mathrm{im}(\pi)-\mathrm{Mon}$. If the underlying permutations of G are trivial, then $\mathrm{im}(\pi)-\mathrm{Mon}$ is just the whole of the category Mon; if instead G is crossed then we are asking for monoids whose multiplication is invariant under arbitrary permutations from S, and so $\mathrm{im}(\pi)-\mathrm{Mon}$ is just the category of *commutative* monoids, CMon. Regardless, when we are working with an arbitrary action operad G, the category $\mathrm{im}(\pi)-\mathrm{Mon}$ is exactly the collection of possible connected components that we were looking for.

Lemma 2.16. *Let G be an action operad and* $\mathrm{im}(\pi)$ *its underlying permutation action operad. Then there is a functor*

$$\pi_0 : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{im}(\pi)-\mathrm{Mon}$$

which sends each algebra X to its monoid of connected components $\pi_0(X)$, and sends each map of algebras $F : X \to Y$ to its restriction to connected components $\pi_0(F) : \pi_0(X) \to \pi_0(Y)$.

Proof. Let $x_1, ..., x_n$ be an arbitrary collection of objects from the algebra X, and g an element of the group $G(n)$. Then the action of G guarantees the existence of a morphism

$$\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_n}) \, : \, x_1 \otimes ... \otimes x_n \to x_{\pi(g^{-1})(1)} \otimes ... \otimes x_{\pi(g^{-1})(n)}$$

By definition the source and target of this morphism belong to the same connected component, and hence

$$\begin{array}{rrcl} & [\, x_1 \otimes ... \otimes x_n \,] & = & [\, x_{\pi(g^{-1})(1)} \otimes ... \otimes x_{\pi(g^{-1})(n)} \,] \\ \Longrightarrow & [x_1] \otimes ... \otimes [x_n] & = & [x_{\pi(g^{-1})(1)}] \otimes ... \otimes [x_{\pi(g^{-1})(n)}] \end{array}$$

But since the x_i are just arbitrary objects of X, the components $[x_i]$ are an arbitrary collection of elements from $\pi_0(X)$, and likewise for the group element g and the permutation $\pi(g)$. Therefore multiplication in the monoid $\pi_0(X)$ is invariant under all permutations in the images of the homomorphisms $\pi_n : G(n) \to S_n$, and thus $\pi_0(X)$ is an object of $\mathrm{im}(\pi)-\mathrm{Mon}$, as required. Well-definedness of the functor π_0 on morphisms then follows immediately from the fullness of $\mathrm{im}(\pi)-\mathrm{Mon}$. □

Now that we have a functor which represents the act of finding the connected component monoid of an algebra, we need another functor heading in the opposite direction, so that we can construct an adjunction between them.

Definition 2.17. There exists an inclusion of 2-categories $\mathrm{D} : \mathrm{Set} \hookrightarrow \mathrm{Cat}$ which allows us to view any set S as a *discrete category*, one whose objects are just the elements of S and whose morphisms are all identities. If the given set also happens to be a monoid M, then there is an obvious way to see the discete category $\mathrm{D}M$ as a monoidal category, and so we have a similar inclusion $\mathrm{D} : \mathrm{Mon} \hookrightarrow \mathrm{MonCat}$. Finally, for any action operad G and object M of the category $\mathrm{im}(\pi)-\mathrm{Mon}$, there is a unique way to assign an $\mathrm{E}G$-action to the discete category $\mathrm{D}M$. This works because for any elements $m_1, ..., m_n \in M$ and $g \in G(n)$, the morphism $\alpha(g; \mathrm{id}_{m_1}, ..., \mathrm{id}_{m_n})$ must have source and target

$$m_1 \otimes ... \otimes m_n \quad = \quad m_{\pi(g^{-1})(1)} \otimes ... \otimes m_{\pi(g^{-1})(m)}$$

and therefore it can only be the morphism $\mathrm{id}_{m_1 \otimes ... \otimes m_n}$. This choice of action yields one last inclusion $\mathrm{CMon} \hookrightarrow \mathrm{E}G\mathrm{Alg}_S$, which we shall also call D.

Proposition 2.18. D *is a right adjoint to the functor* π_0.

Proof. Consider a map of $F : X \to \mathrm{D}C$ from some EG-algebra X onto the discrete EG-algebra for a monoid M in im(π)−Mon. For any $f : x \to x'$ in X, the morphism $F(f)$ must be an identity map in DM, since these are the only morphisms that DM has. It follows that x and x' being in the same connected component will imply $F(x) = F(x')$, and so F is determined entirely by its restriction to connected components, the monoid homomorphism $\pi_0(F) : \pi_0(X) \to M$. In other words, we have an isomorphism between the homsets

$$\mathrm{E}G\mathrm{Alg}_S(\, X, \mathrm{D}M\,) \quad \cong \quad \mathrm{im}(\pi)\text{−}\mathrm{Mon}(\, \pi_0(X), M\,)$$

This isomorphism is natural in both coordinates, since for any $G : X \to X'$ in EGAlg$_S$ and $h : M \to M'$ in im(π)−Mon,

$$\pi_0(\, \mathrm{D}h \circ F \circ G\,) \quad = \quad \pi_0(\mathrm{D}h) \circ \pi_0(F) \circ \pi_0(G) \quad = \quad h \circ \pi_0(F) \circ \pi_0(G)$$

and so the diagram

$$\begin{tikzcd}
\mathrm{E}G\mathrm{Alg}_S(X, \mathrm{D}M) \arrow[r, "\sim"] \arrow[d, "\mathrm{D}h\circ_\circ G"'] & \mathrm{im}(\pi)\text{−}\mathrm{Mon}\Big(\pi_0(X), M\Big) \arrow[d, "h\circ_\circ\pi_0(G)"] \\
\mathrm{E}G\mathrm{Alg}_S(X', \mathrm{D}M') \arrow[r, "\sim"] & \mathrm{im}(\pi)\text{−}\mathrm{Mon}\Big(\pi_0(X'), M'\Big)
\end{tikzcd}$$

commutes. Therefore, $\pi_0 \dashv \mathrm{D}$. ☐

Now we can utilise Proposition 2.18 to draw out a universal property of $\pi_0(L\mathbb{G}_n)$, just as we did with Ob$(L\mathbb{G}_n)$ in Proposition 2.12.

Proposition 2.19. *The connected components of* $L\mathbb{G}_n$ *are the group completion of the connected components of* $\mathbb{G}_n$. *Also, the restriction of* η *onto connected components,* $\pi_0(\eta)$, *is the canonical map* $\pi_0(\mathbb{G}_n) \to \pi_0(\mathbb{G}_n)^{\mathrm{gp}}$ *associated with that group completion.*

Proof. Let H be a group which is also an object of im(π)−Mon, and let $h : \pi_0(\mathbb{G}_n) \to H$ be a monoid homomorphism. By Proposition 2.18 there is a homset isomorphism

$$\mathrm{E}G\mathrm{Alg}_S(\, \mathbb{G}_n,\, \mathrm{D}H\,) \quad \cong \quad \mathrm{im}(\pi)\text{−}\mathrm{Mon}(\, \pi_0(\mathbb{G}_n),\, H\,)$$

and thus some EG-algebra map $h' : \mathbb{G}_n \to \mathrm{D}H$ corresponding to h. As H is a group, every object of DH is invertible, and so h' is an object of $(\mathbb{G}_n \downarrow \mathrm{inv})$. It follows that there exists a unique map $u : L\mathbb{G}_n \to \mathrm{D}M$ which factors h' through the initial object η:

$$\begin{tikzcd} \mathbb{G}_n \arrow[d, "\eta"'] \arrow[dr, "h'"] & \\ L\mathbb{G}_n \arrow[r, "u"'] & \mathrm{D}H \end{tikzcd} \qquad \begin{tikzcd} \pi_0(\mathbb{G}_n) \arrow[d, "\pi_0(\eta)"'] \arrow[dr, "h"] & \\ \pi_0(L\mathbb{G}_n) \arrow[r, "\pi_0(u)"'] & H \end{tikzcd}$$

Applying the functor π_0 everywhere, we see that $\pi_0(u)$ must also factor h through the homomorphism $\pi_0(\eta)$. Moreover, since η is an epimorphism and π_0 a left adjoint functor, $\pi_0(\eta)$ is an epimorphism too, and so $\pi_0(u)$ is the only map with this property. Therefore, any monoid homomorphism $\pi_0(\mathbb{G}_n) \to H$ will factor uniquely through $\pi_0(L\mathbb{G}_n)$, so long as H is in $\mathrm{im}(\pi)-\mathrm{Mon}$.

Now consider another monoid homomorphism $k : \pi_0(\mathbb{G}_n) \to K$, where this time K is still a group but not neccessarily in $\mathrm{im}(\pi)-\mathrm{Mon}$. From Lemma 2.16, we know that $\pi_0(\mathbb{G}_n)$ is still an object of $\mathrm{im}(\pi)-\mathrm{Mon}$, and from this we can conclude that the image $\mathrm{im}(k)$ will be too:

$$\begin{array}{rrcl} x_1, ..., x_m \in \pi_0(\mathbb{G}_n),\ g \in G(n) & \Longrightarrow \quad x_1 \otimes ... \otimes x_m & = & x_{\pi(g)(1)} \otimes ... \otimes x_{\pi(g)(m)} \\ & \Longrightarrow \quad k(\, x_1 \otimes ... \otimes x_m \,) & = & k(\, x_{\pi(g)(1)} \otimes ... \otimes x_{\pi(g)(m)} \,) \\ & \Longrightarrow \quad k(x_1) \otimes ... \otimes k(x_m) & = & k(x_{\pi(g)(1)}) \otimes ... \otimes k(x_{\pi(g)(m)}) \end{array}$$

Also, since $\mathrm{im}(k)$ is a submonoid of the group K, it is a group as well. Thus if we denote by $k_{\mathrm{im}} : \mathrm{Ob}(\mathbb{G}_n) \to \mathrm{im}(k)$ the restriction of k to it image, then k_{im} is a map in $\mathrm{im}(\pi)-\mathrm{Mon}$ out of $\mathrm{Ob}(\mathbb{G}_n)$ and onto a group, and therefore by what we showed earlier there exists a unique homomorphism $v : \mathrm{Ob}(L\mathbb{G}_n) \to \mathrm{im}(k)$ with the property $v \circ \pi_0(\eta) = k_{\mathrm{im}}$. Composing this v with the inclusion $i : \mathrm{im}(k) \hookrightarrow K$, we see that

$$i \circ v \circ \pi_0(\eta) \ = \ i \circ k_{\mathrm{im}} \ = \ k$$

and $i \circ v$ must be the only map for which this is true, for restricting this equation back on $\mathrm{im}(k)$ yields the unique property of v again. Thus $\pi_0(\eta)$ will actually take any homomorphism from $\mathrm{Ob}(\mathbb{G}_n)$ onto a group and factor it through $\pi_0(L\mathbb{G}_n)$ in a unique way, not just those homomorphisms in $\mathrm{im}(\pi)-\mathrm{Mon}$. In other words,

$$\pi_0(L\mathbb{G}_n) \quad = \quad \pi_0(\mathbb{G}_n)^{\mathrm{gp}}$$

and $\pi_0(\eta)$ is the canonical map of this group completion. $\square$

As we've said before, this result is a reflection of the fact that the functor L is trying to add inverses the objects of $\mathbb{G}_n$ freely, that is, with as little effect on the rest of the algebra as possible. Indeed, if we happen to know whether or not our action operad G is crossed then we can now calculate exactly what the effect on the components will be.

Corollary 2.20. *If G is a crossed action algebra then*

- *the connected components of $L\mathbb{G}_n$ are the monoid $\mathbb{Z}^n$*
- *the restriction of η to components is the obvious inclusion $\mathbb{N}^n \hookrightarrow \mathbb{Z}^n$*
- *the assignment of objects to their component is given by the quotient map of abelianisation* $\mathrm{ab} : \mathbb{Z}^{*n} \to \mathbb{Z}^n$

If instead G is non-crossed, then

- *the connected components of $L\mathbb{G}_n$ are the monoid $\mathbb{Z}^{*n}$*
- *the restriction of η to components is the obvious inclusion $\mathbb{N}^{*n} \hookrightarrow \mathbb{Z}^{*n}$*
- *the assignment of objects to their component is* $\mathrm{id}_{\mathbb{Z}^{*n}}$

Proof. Combining Propositions 1.28 and 2.19, we see that

$$\pi_0(L\mathbb{G}_n) \quad = \quad \pi_0(\mathbb{G}_n)^{\mathrm{gp}} \quad = \quad \begin{cases} (\mathbb{N}^n)^{\mathrm{gp}} \quad = \quad \mathbb{Z}^n & \text{if } G \text{ is crossed} \\ (\mathbb{N}^{*n})^{\mathrm{gp}} \quad = \quad \mathbb{Z}^{*n} & \text{otherwise} \end{cases}$$

Moreover, Proposition 2.19 says that restriction of η to connected components, $\pi_0(\eta)$, will be the homomorphism associated with these group completion, which means the inclusion $\mathbb{N}^n \hookrightarrow \mathbb{Z}^n$ when G is crossed and $\mathbb{N}^{*n} \hookrightarrow \mathbb{Z}^{*n}$ when it is not.

Next, by Proposition 1.28 we know that the map $[\_] : \mathrm{Ob}(\mathbb{G}_n) \to \pi_0(\mathbb{G}_n)$ sending objects of $\mathbb{G}_n$ to their connected component is either the quotient map of abelianisation $\mathbb{N}^{*n} \to \mathbb{N}^n$ or the identity on $\mathbb{N}^{*n}$, depending on whether or not it is crossed. If we also use $[\_]$ to denote the map sending objects of $L\mathbb{G}_n$ to their components, it then follows from functoriality of η that the corresponding choice of the followings two diagrams will commute:

$$\begin{tikzcd}
\mathbb{N}^{*n} \arrow[r, hook, "\mathrm{Ob}(\eta)"] \arrow[d, "{[_]}"'] & \mathbb{Z}^{*n} \arrow[d, "{[_]}"] \\
\mathbb{N}^{n} \arrow[r, hook, "\pi_0(\eta)"'] & \mathbb{Z}^{n}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{N}^{*n} \arrow[r, hook, "\mathrm{Ob}(\eta)"] \arrow[d, equal, "{[_]}"'] & \mathbb{Z}^{*n} \arrow[d, "{[_]}"] \\
\mathbb{N}^{*n} \arrow[r, hook, "\pi_0(\eta)"'] & \mathbb{Z}^{*n}
\end{tikzcd}$$

Using the values of [_] from Proposition 1.28, $\mathrm{Ob}(\eta)$ from Proposition 2.14, and $\pi_0(\eta)$ from earlier in this proof, it follows that for any generator z_i of $\mathbb{Z}^{*n}$,

$$[z_i] \quad = \quad [\mathrm{Ob}(\eta)(z_i)] \quad = \quad \pi_0(\eta)([z_i]) \quad = \quad \pi_0(\eta)(z_i) \quad = \quad z_i$$

But this description of $[\_] : \mathrm{Ob}(L\mathbb{G}_n) \to \pi_0(L\mathbb{G}_n)$ on generators is either the definition of the quotient map $\mathrm{ab} : \mathbb{Z}^{*n} \to (\mathbb{Z}^{*n})^{\mathrm{ab}}$ or the identity $\mathrm{id} : \mathbb{Z}^{*n} \to \mathbb{Z}^{*n}$, depending on the value of target monoid, as required. □

2.5 The collapsed morphisms of $L\mathbb{G}_n$

Now that we understand the objects and connected components of the algebra $L\mathbb{G}_n$, the next most obvious thing to look for are its morphisms, $\mathrm{Mor}(L\mathbb{G}_n)$. It would be nice to construct this collection in the same way we constructed $\mathrm{Ob}(L\mathbb{G}_n)$ and $\pi_0(L\mathbb{G}_n)$, by applying the left adjoint of some adjunction to the initial map η. Before we can do this however, we need to ask ourselves a question. What sort of mathematical object is $\mathrm{Mor}(L\mathbb{G}_n)$, exactly?

Given a pair of morphisms $f : x \to y, f' : y' \to z$ in an EG-algebra X, there are two basic binary operations we can perform. First, we can take their tensor product $f \otimes f'$, and this together with the unit map id_I imbues $\mathrm{Mor}(X)$ with the structure of a monoid. Second, if we have $y = y'$ then we can form the composite morphism $f' \circ f$. However, these two operations are not as different as they first appear.

Lemma 2.21. *Let $f : x \to y$ and $f' : y \to z$ be morphisms in some monoidal category, and y is an invertible object of that category. Then*

$$f' \circ f \quad = \quad f' \otimes \mathrm{id}_{y*} \otimes f$$

Proof. By the interchange law for monoidal categories,

$$\begin{aligned} f' \circ f &= (f' \otimes \mathrm{id}_I) \circ (\mathrm{id}_I \otimes f) \\ &= (f' \otimes \mathrm{id}_{y*} \otimes \mathrm{id}_y) \circ (\mathrm{id}_y \otimes \mathrm{id}_{y*} \otimes f) \\ &= (f' \circ \mathrm{id}_y) \otimes (\mathrm{id}_{y*} \circ \mathrm{id}_{y*}) \otimes (\mathrm{id}_y \circ f) \\ &= f' \otimes \mathrm{id}_{y*} \otimes f \end{aligned}$$

□

In other words, composition along invertible objects in X can always be restated in terms of the tensor product. Thus in cases where every object of X is invertible, the monoidal structure together with knowledge of each morphisms source and target will be enough to determine X uniquely. Since all objects in $L\mathbb{G}_n$ are invertible, this means that we could choose to ignore composition of elements of $\mathrm{Mor}(L\mathbb{G}_n)$ for the time being, and focus on its status as a monoid under tensor product.

However, we are trying to extract information about the morphisms of $L\mathbb{G}_n$ by building some sort of left adjoint functor. Presumably we will also be able to apply it to other EG-algebras, some of which won't have all of their objects invertible, and so we can't just use $\mathrm{Mor}(-) : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{Mon}$. What we need is a way to modify the morphism monoid of a category so that both composition and tensor product are recoverable from a single operation. Of course, there is one very easy method for achieving this — simply force $\otimes$ and $\circ$ to be equal.

Definition 2.22. Let $\mathrm{M} : \mathrm{MonCat} \to \mathrm{Mon}$ be the functor which sends monoidal categories X to the quotient of their monoid of morphisms by the relation that sets $\otimes = \circ$.

$$\mathrm{M}X \quad = \quad \mathrm{Mor}(X) \big/ {f' \circ f \sim f' \otimes f}$$

Each monoidal functors $F : X \to Y$ is then sent to the monoid homomorphism

$$\begin{array}{rcrcl} \mathrm{M}(F) & : & \mathrm{M}X & \to & \mathrm{M}Y \\ & : & \mathrm{M}(f) & \mapsto & \mathrm{M}\Big(F(f) \Big) \end{array}$$

where $\mathrm{M}(f)$ refers to the equivalence class of the map f under the quotient $\mathrm{Mor}(X) \to \mathrm{M}(X)$. This homomorphism is well-defined, since it respects the relation $\otimes = \circ$:

$$\begin{aligned} \mathrm{M}(F)(\, f' \circ f\,) &= \mathrm{M}\Big(F(f' \circ f) \Big) \\ &= \mathrm{M}\Big(F(f') \circ F(f) \Big) \\ &= \mathrm{M}\Big(F(f') \Big) \circ \mathrm{M}\Big(F(f) \Big) \\ &= \mathrm{M}\Big(F(f') \Big) \otimes \mathrm{M}\Big(F(f) \Big) \\ &= \mathrm{M}\Big(F(f') \otimes F(f) \Big) \\ &= \mathrm{M}\Big(F(f' \otimes f) \Big) \\ &= \mathrm{M}(F)(\, f' \otimes f\,) \end{aligned}$$

We will call $\mathrm{M}X$ the *collapsed* morphisms of the X.

From now on we will generally refer to the single operation in $\mathrm{M}X$ as $\otimes$ rather than $\circ$, unless we are focusing on some aspect best understood using compositon. This convention makes it easier to remember that because the tensor product is defined between all pairs of morphisms in X, the equivalence class $\mathrm{M}(f') \otimes \mathrm{M}(f)$ will always contain the morphism $f' \otimes f$, but not neccessarily $f' \circ f$, as it might fail to exist.

Now we need a candidate for the right adjoint to the functor M.

Definition 2.23. For a given monoid M, let $\mathrm{B}M$ represent the one-object category whose morphisms are the elements of M, with monoid multiplication as composition. Likewise, for any monoid homomorphism $h : M \to M'$ between abelian groups, denote by $\mathrm{B}h : \mathrm{B}M \to \mathrm{B}M'$ the obvious monoidal functor which acts like h on morphisms. This defines a functor $\mathrm{B} : \mathrm{Mon} \to \mathrm{Cat}$ from the category of monoids onto the category of small categories.

Moreover, let C be a commutative monoid. Then we can view $\mathrm{B}C$ as a monoidal category, with the tensor product also given by the multiplication in C, and the sole object as the unit I. Clearly for any homomorphism between commutative monoids $h : C \to C'$ the corresponding functor $\mathrm{B}h : \mathrm{B}C \to \mathrm{B}C'$ will preserve this monoidal structure, as it is already preserving it as compositon. Thus the restriction of B to commutative monoids also gives a functor $\mathrm{CMon} \to \mathrm{MonCat}$, which we will still call B.

Commutativity is required in order for $\mathrm{B}C$ to be a well-defined monoidal category because we need its operations $\circ$ and $\otimes$ to obey the interchange law for monoidal categories:

$$\begin{array}{rrcl} & (\mathrm{id}_I \circ f) \otimes (f' \otimes \mathrm{id}_I) & = & (\mathrm{id}_I \otimes f') \circ (f \otimes \mathrm{id}_I) \\ \Longrightarrow & \mathrm{id}_I \cdot f \cdot f' \cdot \mathrm{id}_I & = & \mathrm{id}_I \cdot f' \cdot f \cdot \mathrm{id}_I \\ \Longrightarrow & f \cdot f' & = & f' \cdot f \end{array}$$

Proposition 2.24. B *is a right adjoint to the functor* $\mathrm{M}(\_)^{\mathrm{ab}} : \mathrm{MonCat} \to \mathrm{CMon}$.

Proof. Let X be a monoidal category, C a commutative monoid, and $F : X \to \mathrm{B}C$ a monoidal functor. For any $f : x \to x'$ in X, the morphism $F(f)$ is just an element of the monoid C, and so F can be used to define a function

$$\begin{array}{rcrcl} F' & : & \mathrm{M}(X)^{\mathrm{ab}} & \to & C \\ & : & \mathrm{ab} \circ \mathrm{M}(f) & \mapsto & F(f) \end{array}$$

where ab is the quotient map of abelianisation $\mathrm{M}(X) \to \mathrm{M}(X)^{\mathrm{ab}}$. This F' is a well-defined monoid homomorphism; it preserves multiplication and respects the relation $\otimes = \circ$ because the monoid multiplication of C is acts as both tensor product and composition in BC.

$$\begin{array}{rcl} F'\Big(\mathrm{abM}(f' \circ f)\Big) & = & F(f' \circ f) \\ & = & F(f') \circ F(f) \\ & = & F(f') \cdot F(f) \\ & = & F(f') \otimes F(f) \\ & = & F(f' \otimes f) \\ & = & F'\Big(\mathrm{abM}(f' \otimes f)\Big) \end{array}$$

Conversely, if $h : \mathrm{M}(X)^{\mathrm{ab}} \to C$ is a monoid homomorphism, we can define from it a monoidal functor

$$\begin{array}{rrcl} h' & : & X & \mapsto & \mathrm{B}C \\ & : & x & \mapsto & I \\ & : & f : x \to y & \mapsto & h\Big(\mathrm{abM}(f)\Big) : I \to I \end{array}$$

Yet again, the monoidal functor h' is well-defined because the fact that $\otimes = \circ$ in BC forces h' to respect that relation.

$$\begin{array}{rcl} h'(f' \circ f) & = & h\Big(\mathrm{abM}(f' \circ f)\Big) \\ & = & h\Big(\mathrm{abM}(f') \circ \mathrm{M}(f')\Big) \\ & = & h\Big(\mathrm{abM}(f')\Big) \circ h\Big(\mathrm{abM}(f')\Big) \\ & = & h\Big(\mathrm{abM}(f')\Big) \cdot h\Big(\mathrm{abM}(f')\Big) \\ & = & h\Big(\mathrm{abM}(f')\Big) \otimes h\Big(\mathrm{abM}(f')\Big) \\ & = & h\Big(\mathrm{abM}(f') \otimes \mathrm{abM}(f')\Big) \\ & = & h\Big(\mathrm{abM}(f' \otimes f')\Big) \\ & = & h'(f' \otimes f) \end{array}$$

But these assignments $F \mapsto F'$ and $h \mapsto h'$ are clearly inverse to one another. For any $F : X \to \mathrm{B}C$ applying them twice gives

$$\begin{array}{rrcll} F'' & : & X & \to & \mathrm{B}C \\ & : & x & \mapsto & I \\ & : & f : x \to y & \mapsto & F'\Big(\mathrm{abM}(f)\Big) : I \to I \quad = \quad F(f) \end{array}$$

and similarly for $h : \mathrm{M}X \to C$ we get

$$\begin{array}{rcccccl} h'' & : & \mathrm{M}(X)^{\mathrm{ab}} & \to & C & & \\ & : & \mathrm{abM}(f) & \mapsto & h'(f) & = & h\Big(\mathrm{abM}(f)\Big) \end{array}$$

In other words, we have an isomorphism between the homsets

$$\mathrm{MonCat}(\, X, \mathrm{B}C\,) \quad \cong \quad \mathrm{CMon}(\, \mathrm{M}(X)^{\mathrm{ab}}, C\,)$$

This isomorphism is natural in both coordinates, as for any monoidal functor $G : X \to X'$ and homomorphism $h : C \to C'$ between commutative monoids,

$$\mathrm{abM}(\, \mathrm{B}h \circ F \circ G\,) \quad = \quad \mathrm{abM}(\mathrm{B}h) \circ \mathrm{abM}(F) \circ \mathrm{abM}(G) \quad = \quad h \circ \mathrm{abM}(F) \circ \mathrm{abM}(G)$$

and so the diagram

$$\begin{tikzcd} \mathrm{MonCat}(X, \mathrm{B}C) \arrow[r, "\sim"] \arrow[d, "\mathrm{B}h \circ \_ \circ G"'] & \mathrm{CMon}\Big(\, \mathrm{M}(X)^{\mathrm{ab}}, C\,\Big) \arrow[d, "h \circ \_ \circ \mathrm{abM}G"] \\ \mathrm{MonCat}(X', \mathrm{B}C') \arrow[r, "\sim"] & \mathrm{CMon}\Big(\, \mathrm{M}(X')^{\mathrm{ab}}, M'\,\Big) \end{tikzcd}$$

commutes. Therefore, $\mathrm{M}(\,\_\,)^{\mathrm{ab}} \dashv \mathrm{B}$. $\square$

Proposition 2.24 seems at first glance very similar to Propositions 2.12 and 2.18. However, our goal was to discover the relationship between the morphisms of $\mathbb{G}_n$ and $L\mathbb{G}_n$, paralleling what we did in Propositions 2.14 and 2.19, and in that regard M falls short in two very important ways.

1. What we really wanted to have was an adjunction involving $\mathrm{E}G\mathrm{Alg}_S$, not MonCat. This is because our previous methodology involved applying our left adjoint functors to η and then using its initial property to factor various maps through $L\mathbb{G}_n$. But η is an initial object in $(\mathbb{G}_n \downarrow \mathrm{inv})$, and so we only know how to use it to factor *algebra* maps $\mathbb{G}_n \to X_{\mathrm{inv}}$, and not general monoidal functors.

2. Even if we do find a way to use this adjunction to extract information about $L\mathbb{G}_n$, it will not be the monoid $\mathrm{Mor}(L\mathbb{G}_n)$ we were originally after, only a strange abelianised version where tensor product and composition coincide.

Unfortunately, this adjunction seems to be the best we can do. The only general method for assigning an $\mathrm{E}G$-action to the monoidal category $\mathrm{B}C$ for all C is to set all of its action morphisms $\alpha(g; \mathrm{id}_I, ..., \mathrm{id}_I)$ to be id_I. This would then cause the homomorphism $\mathrm{M}X \to C$ corresponding to any algebra map $X \to \mathrm{B}C$ to be the zero map if X has only action morphisms. Given Lemma 1.27, this is clearly no use. However, it turns out that this approach is fixable. To that end, we will spend the bulk of the next two chapters directly addressing problems 1 and 2.

For now though, we will make one last small alteration to our plan going forward. Instead of working directly with the functor $\mathrm{M}(\_)^{\mathrm{ab}} : \mathrm{MonCat} \to \mathrm{CMon}$, we will instead focus on its composite with the group completion functor, $(\_)^{\mathrm{gp}} : \mathrm{CMon} \to \mathrm{Ab}$. It may not be clear yet why we would choose to do this, but over the next couple of chapters we will frequently find ourselves having to forming quotients of certain algebraic objects. If we were to stick with the functor M these would all be commutative monoid quotients, whereas by making the switch to $\mathrm{M}(\_)^{\mathrm{gp,ab}}$ they will be abelian groups instead, which are far easier to work with. Also, notice that since the process of group completion is left adjoint to the forgetful functor $\mathrm{Ab} \to \mathrm{CMon}$, its composite with the left adjoint $\mathrm{M}(\_)^{\mathrm{ab}}$ will be a left adjoint functor too. Thus with this new functor we will be able use all of the same important properties that we would have done with $\mathrm{M}(\_)^{\mathrm{ab}}$, such as the preservation of colimits. Moreover, while we won't prove this for some time, it turns out that the morphisms of $L\mathbb{G}_n$ actually form a group under tensor product. This means that whatever method we would have used to recover $\mathrm{Mor}(L\mathbb{G}_n)$ from $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{ab}}$ should still let us recover $\mathrm{Mor}(L\mathbb{G}_n) = \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp}}$ from $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$.

Before we move on, we should spend a little time thinking about this new functor $\mathrm{M}(\_)^{\mathrm{gp,ab}}$. Specifically, we might ask in what order we have to carry out its constituent parts: the collapsing of $\circ$ and $\otimes$ into a single operation, group completion, and abelianisation. It is a well known fact that group completion and abelianisation commute:

$$\begin{tikzcd} \mathrm{Mon} \arrow[r, "(\_)^{\mathrm{gp}}"] \arrow[d, "(\_)^{\mathrm{ab}}"'] & \mathrm{Grp} \arrow[d, "(\_)^{\mathrm{ab}}"] \\ \mathrm{CMon} \arrow[r, "(\_)^{\mathrm{gp}}"] & \mathrm{Ab} \end{tikzcd}$$

Indeed, we already assume this when talking of 'the' canonical map $\mathrm{M}(X)^{\mathrm{gp,ab}}$. But a more interesting question is whether it matters if we choose to group complete or abelianise the tensor product of a monoidal category before or after we collapse its morphisms.

Lemma 2.25. *For any monoidal category X, define*

$$\mathrm{M}_{\mathrm{gp}}(X) \cong \mathrm{Mor}(X)^{\mathrm{gp}} \big/ \mathrm{gp}(f' \circ f) \sim \mathrm{gp}(f' \otimes f)$$
$$\mathrm{M}_{\mathrm{ab}}(X) \cong \mathrm{Mor}(X)^{\mathrm{ab}} \big/ \mathrm{ab}(f' \circ f) \sim \mathrm{ab}(f' \otimes f)$$

Then

$$\mathrm{M}_{\mathrm{gp}}(X) = \mathrm{M}(X)^{\mathrm{gp}}, \qquad \mathrm{M}_{\mathrm{ab}}(X) = \mathrm{M}(X)^{\mathrm{ab}}$$

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
 & \mathrm{M}(X) \arrow[r, "\mathrm{gp}"] \arrow[ddr, dashed, "v"] & \mathrm{M}(X)^{\mathrm{gp}} \arrow[dd, dashed, shift left, "u'"] \\
\mathrm{Mor}(X) \arrow[ur, "\mathrm{M}"] \arrow[dr, "\mathrm{gp}"'] & & \\
 & \mathrm{Mor}(X)^{\mathrm{gp}} \arrow[r, "\mathrm{M}"'] \arrow[uur, dashed, "u"'] & \mathrm{M}_{\mathrm{gp}}(X) \arrow[uu, dashed, shift left, "v'"]
\end{tikzcd}$$

Here all of the solid arrows are the respective canonical homomorphisms.

Starting from the left, the top edge of the diagram is a map coming out of $\mathrm{Mor}(X)$ and going into a group, and so by the universal property of the group completion there is a unique homomorphism u factoring it through $\mathrm{Mor}(X)^{\mathrm{gp}}$. But now this u is a map out of $\mathrm{Mor}(X)^{\mathrm{gp}}$ and into group where tensor product and compositon are equal, and so by the universal property of the quotient this factors once more through the map u'. On the other hand, the bottom edge of the diagram will factor through the map v because of the collapsed morphisms property, and then through the map v' due to the group completion property. Then this diagram says that

$$\begin{aligned}
v' \circ u' \circ \mathrm{gp} \circ \mathrm{M} &= v' \circ u' \circ u \circ \mathrm{gp} \\
&= v' \circ \mathrm{M} \circ \mathrm{gp} \\
&= u \circ \mathrm{gp} \\
&= \mathrm{gp} \circ \mathrm{M}
\end{aligned}$$

But $\mathrm{M} : \mathrm{Mor}(X) \to \mathrm{M}(X)$ is the map associated with a quotient, and so it is an epimorphism. Thus we can cancel it out on the right, leaving just

$$v' \circ u' \circ \mathrm{gp} = \mathrm{gp}$$

Then from this we can conclude that for any $\mathrm{M}(f) \in \mathrm{M}(X)$,

$$\begin{aligned} v'u'\Big(\,\mathrm{gpM}(f)\,\Big) &= \quad \mathrm{gpM}(f) \\ v'u'\Big(\,\mathrm{gpM}(f)^*\,\Big) &= v'u'\Big(\,\mathrm{gpM}(f)\,\Big)^* = \mathrm{gpM}(f)^* \end{aligned}$$

All elements of $\mathrm{M}(X)^{\mathrm{gp}}$ can be written as $\mathrm{gpM}(f)$ or $\mathrm{gpM}(f)^*$ for at least one f, so this really says that $v' \circ u'$ is the identity homomorphisms on $\mathrm{M}(X)^{\mathrm{gp}}$.

A completely analagous argument can also be by made starting from the bottom edge of the diagram instead, and then concluding that $u' \circ v' = \mathrm{id}_{\mathrm{M}_{\mathrm{gp}}(X)}$. Furthermore, we can construct another diagram using the universal property of the abelianisation,

$$\begin{tikzcd} & \mathrm{M}(X) \arrow[r, "\mathrm{ab}"] \arrow[dr, dashed, "v''"] & \mathrm{M}(X)^{\mathrm{ab}} \arrow[d, dashed, shift left, "u'''"] \\ \mathrm{Mor}(X) \arrow[ur, "\mathrm{M}"] \arrow[dr, "\mathrm{ab}"'] & & \\ & \mathrm{Mor}(X)^{\mathrm{ab}} \arrow[r, "\mathrm{M}"'] \arrow[uur, dashed, "u''"'] & \mathrm{M}_{\mathrm{ab}}(X) \arrow[u, dashed, shift left, "v'''"] \end{tikzcd}$$

and then through a series of analagous arguments conclude that $v''' \circ u''' = \mathrm{id}_{\mathrm{M}(X)^{\mathrm{ab}}}$ and $u''' \circ v''' = \mathrm{id}_{\mathrm{M}_{\mathrm{ab}}(X)}$. All together, these yield the two isomorphisms given in the statement of the proposition. $\square$

In other words, we do not need to worry about order of operations when using the left adjoint functor $\mathrm{M}(\_\_)^{\mathrm{gp,ab}}$. This is very convenient, and later on when we actually need to evalute particular $\mathrm{M}(X)^{\mathrm{gp,ab}}$, we will use this fact to carry out the calculation in whichever order proves easiest.

Chapter 3

Free invertible algebras as colimits

In the previous chapter, we made progress towards understanding the structure of $L\mathbb{G}_n$ by showing that the algebra was an initial object in a certain comma category. Specifically, we saw that the map $\eta : \mathbb{G}_n \to L\mathbb{G}_n$ is initial among all EG-algebra maps $\mathbb{G}_n \to X_{\text{inv}}$. This fact is the rigourous way of expressing a fairly obvious intuition about $L\mathbb{G}_n$ — that we should expect the free algebra on n invertible objects to be like the free algebra on n objects, except that its objects are invertible.

However, this not the only way of thinking about $L\mathbb{G}_n$. Consider for a moment the free EG-algebra on $2n$ objects, $\mathbb{G}_{2n}$. Intuitively, if we were to take this algebra and then enforce upon it the extra relations $z_{n+1} = z_1^*, ..., z_{2n} = z_n^*$, then we would be changing it from a structure with $2n$ independent generators into one with n indepedent generators and their inverses. That is, there seems to be a natural way to think about $L\mathbb{G}_n$ as a quotient of the larger algebra $\mathbb{G}_{2n}$. In this chapter we will work towards making this idea precise, and then examine some of its consequences, the most important of which will be allowing us to describe the group $\mathrm{M}(L\mathbb{G}_n)^{\text{gp,ab}}$.

3.1 $L\mathbb{G}_n$ as a cokernel in $\mathrm{E}G\mathrm{Alg}_S$

We'll begin with some definitions.

Definition 3.1. Let δ be the map of EG-algebras defined on generators by

$$\begin{array}{rcrcl} \delta & : & \mathbb{G}_{2n} & \to & \mathbb{G}_{2n} \\ & : & z_i & \mapsto & z_i \otimes z_{n+i} \\ & : & z_{n+i} & \mapsto & z_{n+i} \otimes z_i \end{array}$$

for $1 \leq i \leq n$. We will also denote by $q : \mathbb{G}_{2n} \to Q$ the cokernel this map.

Note that the above definition does actually make sense. The given descriptions of δ is enough to specify it uniquely because $\mathbb{G}_{2n}$ is the free EG-algebra on $2n$ objects, and hence algebra maps $\mathbb{G}_{2n} \to \mathbb{G}_{2n}$ are canonically isomorphic to functions $\{z_1, ..., z_{2n}\} \to \mathrm{ob}(\mathbb{G}_{2n})$. Also we can be sure that the map q exists, because EGAlg$_S$ is a locally finitely presentable category and thus has all finite colimits.

The goal of this approach will be show that Q is in fact that same algebra as $L\mathbb{G}_n$. In order to do this, it would help if we could easily compare $q : \mathbb{G}_{2n} \to Q$ to our initial object $\eta : \mathbb{G}_{2n} \to L\mathbb{G}_n$. In other words, we really want to show that q is an object of $(\mathbb{G}_n \downarrow \mathrm{inv})$ — that Q has only invertible objects. This can be done using the adjunction we found in Proposition 2.12.

Proposition 3.2. *The object monoid of Q is $\mathbb{Z}^{*n}$, and the restriction of q to objects $\mathrm{Ob}(q) : \mathrm{Ob}(\mathbb{G}_{2n}) \to \mathrm{Ob}(Q)$ is the monoid homomorphism defined on generators as*

$$\begin{array}{rcrcl} \mathrm{Ob}(q) & : & \mathbb{N}^{*2n} & \to & \mathbb{Z}^{*n} \\ & : & z_i & \mapsto & z_i \\ & : & z_{n+i} & \mapsto & z_i^* \end{array}$$

Proof. Consider $\mathrm{Ob}(\delta)$, the restrictions on objects of the algebra maps $\delta : \mathbb{G}_{2n} \to \mathbb{G}_{2n}$. By Lemma 1.26, this is a monoid homomorphism $\mathbb{N}^{*2n} \to \mathbb{N}^{*2n}$, and since Mon is cocomplete it too must have a cokernel. This will be a new homomorphism whose source is $\mathbb{N}^{*2n}$ and whose target is the quotient of $\mathbb{N}^{*2n}$ by the relations $\mathrm{Ob}(\delta)(x) = I$. Remembering Definition 3.1, and that $\mathbb{N}^{*2n}$ is the free monoid on $2n$ generators, this quotient monoid will have the following presentation:

$$\begin{array}{ll} \text{Generators:} & z_1, \dots, z_{2n} \\ \text{Relations:} & z_i \otimes z_{n+i} = I, \\ & z_{n+i} \otimes z_i = I \end{array}$$

This is just the same as

$$\begin{array}{ll} \text{Generators:} & z_1, \dots, z_{2n} \\ \text{Relations:} & z_{n+i} = z_i^*, \end{array}$$

which is the presentation of $\mathbb{Z}^{*n}$.

But by Proposition 2.12, Ob is a left adjoint and hence preserves all colimits. Thus the cokernel of $\mathrm{Ob}(\delta)$ is just the underlying homomorphism of the cokernel of δ. Therefore $\mathrm{Ob}(Q) = \mathbb{Z}^{*n}$, and $\mathrm{Ob}(q)$ is the quotient map $\mathbb{N}^{*2n} \to \mathbb{Z}^{*n}$ sending $z_i \mapsto z_i$ and $z_{n+i} \mapsto z_i^*$ for $1 \leq i \leq n$. $\square$

An immediate corollary of Proposition 3.2 is that every object of the cokernel algebra Q is invertible. Thus $q : \mathbb{G}_{2n} \to Q$ is an object of the category $(\mathbb{G}_n \downarrow \mathrm{inv})$, and hence we can use the initiality of η to determine the following result:

Proposition 3.3. *Let $i : \mathbb{G}_n \to \mathbb{G}_{2n}$ be the inclusion of* EG*-algebras defined on generators by $i(z_i) = z_i$. Then $i \circ q$ is an initial object of* $(\mathbb{G}_n \downarrow \mathrm{inv})$*. In particular, this means that*

$$Q \quad \cong \quad L\mathbb{G}_n$$

Proof. Let $\psi : \mathbb{G}_n \to X$ be an arbitrary object of $(\mathbb{G}_n \downarrow \mathrm{inv})$. Since $\mathbb{G}_n$ is the free EG-algebra on n objects, we can use it and ψ to define a new map, $\psi^* : \mathbb{G}_n \to X$, which takes the values

$$\psi^*(z_i) \quad := \quad \psi(z_i)^*$$

on generators. Using these two functors we can define a new map, $\psi + \psi^*$, via the universal property of the coproduct:

$$\begin{tikzcd} & \mathbb{G}_n + \mathbb{G}_n \arrow[dd, dashed, "\psi+\psi^*"] & \\ \mathbb{G}_n \arrow[ur, hook, "i"] \arrow[dr, "\psi"'] & & \mathbb{G}_n \arrow[ul, hook', "i'"'] \arrow[dl, "\psi^*"] \\ & X & \end{tikzcd}$$

But because $\mathbb{G}_n$ is the free algebra on n objects, and the free functor $F : \mathrm{Cat} \to$ EGAlg$_S$ is a left adjoint and thus preserves colimits, we must have

$$\begin{aligned} \mathbb{G}_n + \mathbb{G}_n &= F(\{z_1, ..., z_n\}) + F(\{z_1', ..., z_n'\}) \\ &= F(\,\{z_1, ..., z_n\} + \{z_1', ..., z_n'\}\,) \\ &= F(\{z_1, ..., z_{2n}\}) \\ &= \mathbb{G}_{2n} \end{aligned}$$

This means that we can compose $\psi + \psi^* : \mathbb{G}_{2n} \to X$ with the map $\delta : \mathbb{G}_{2n} \to \mathbb{G}_{2n}$, though we need to be careful to specify exactly which inclusions we used in the definition of $\psi + \psi^*$. Suppose that the lefthand inclusion is i, the one given in the statement of the proposition, and the other is defined by the assignment $z_i \mapsto z_{i+n}$. Then for

$1 \leq i \leq n,$

$$\begin{aligned}
(\psi + \psi^*)\delta(z_i) &= (\psi + \psi^*)(z_i \otimes z_{n+i}) \\
&= \psi(z_i) \otimes \psi(z_i)^* \\
&= I
\end{aligned}$$

$$\begin{aligned}
(\psi + \psi^*)\delta(z_{n+i}) &= (\psi + \psi^*)(z_{n+i} \otimes z_i) \\
&= \psi(z_i)^* \otimes \psi(z_i) \\
&= I
\end{aligned}$$

That is, $(\psi + \psi^*) \circ \delta = I$. But we've already defined $q : \mathbb{G}_{2n} \to Q$ to be the cokernel of δ, the universal map with this property, and so there must exist a unique EG-algebra map $u : Q \to X$ making the righthand triangle below diagram commute:

$$\begin{tikzcd}
\mathbb{G}_n \arrow[r, hook, "i"] \arrow[dr, "\psi"'] & \mathbb{G}_{2n} \arrow[r, "q"] \arrow[d, "\psi+\psi^*"] & Q \arrow[dl, "u"] \\
& X &
\end{tikzcd}$$

The other triangle commutes by the definition of $\psi + \psi^*$, and so together the diagram tells us that for any object ψ of $(\mathbb{G}_n \downarrow \text{inv})$, there exists at least one morphism u in $(\mathbb{G}_n \downarrow \text{inv})$ going from $q \circ i$ to ψ.

Next, let $v : Q \to X$ be an arbitrary morphism $q \circ i \to \psi$ in $(\mathbb{G}_n \downarrow \text{inv})$. By definition, this means that

$$\begin{aligned}
\psi &= vqi \\
\Longrightarrow \quad \psi + \psi^* &= vqi + (vqi)^*
\end{aligned}$$

Also, for $1 \leq i \leq n$ we have

$$\begin{aligned}
q(z_i) \otimes q(z_{n+i}) \quad &= \quad q(z_i \otimes z_{n+i}) \quad = \quad q\delta(z_i) \quad = \quad I \\
q(z_{n+i}) \otimes q(z_i) \quad &= \quad q(z_{n+i} \otimes z_i) \quad = \quad q\delta(z_{n+i}) \quad = \quad I \\
\Longrightarrow \quad & \quad q(z_{n+i}) \quad = \quad q(z_i)^*
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\psi + \psi^*)(z_i) &= \big(vqi + (vqi)^*\big)(z_i) \\
&= vqi(z_i) \\
&= vq(z_i)
\end{aligned}$$

$$\begin{aligned}(\psi+\psi^*)(z_{n+i}) &= \Big(vqi+(vqi)^*\Big)(z_{n+i})\\ &= vqi(z_i)^*\\ &= v\Big(q(z_i)^*\Big)\\ &= vq(z_{n+i})\end{aligned}$$

or in other words $\psi+\psi^* = v\circ q$ for any morphism $v : q\circ i \to \psi$ in $(\mathbb{G}_n \downarrow \mathrm{inv})$. But this is the property that the map u was supposed to satisfy uniquely, and thus it must be the only morphism $q\circ i \to \psi$ in $(\mathbb{G}_n \downarrow \mathrm{inv})$. Therefore $q\circ i$ is an initial object, and hence it is isomorphic in $(\mathbb{G}_n \downarrow \mathrm{inv})$ to any other initial object, such as η. It follows that the targets of these two maps, Q and $L\mathbb{G}_n$ respectively, are isomorphic as EG-algebras. □

It's worth noting that we have not given a method for actually taking cokernels in EGAlg$_S$, and so Proposition 3.3 doesn't immediately provide an explicit description for the whole of $L\mathbb{G}_n$. However, it does offer us another way to extract partial information, like what we were doing in Chapter 2. Consider Proposition 3.2; now that we know that Q is actually $L\mathbb{G}_n$, the statement of this proposition is just the same as that of Proposition 2.14. But the proof of the former uses the ability of cokernels to preserve left adjoint functors, rather than any of the initial algebra and group completion properties that appear in the latter.

Of course, by Proposition 3.3 the fact that q is a cokernel is equivalent to it being initial, and so while they may not look it at first glance, these two approaches are secretly the same. Thus from now on whenever we are trying to determine some aspect of $L\mathbb{G}_n$, we will make sure to take a look at both methods, just in case there are some properties of our free algebra which are more readily apparent from one description than another.

3.2 $L\mathbb{G}_n$ as a surjective coequaliser

An immediate consequence our new cokernel perspective of $L\mathbb{G}_n$ is that, since left adjoint functor all preserve colimits, Propositions 2.12 and 2.18 now both imply results about the partial surjectivity of this new map q. The former says that since $\mathrm{Ob}(q)$ is a cokernel map of monoids, and hence that every object of $L\mathbb{G}_n$ is the image under q of some object of $\mathbb{G}_{2n}$; the latter says a similar thing for connected components. From this one might guess that q is will just turn out to be a surjective map of EG-algebras, and indeed this is the case.

Unfortunately, we can not go about proving that q is surjective on morphisms by a similar adjunction technique, since this best we have is the one from Proposition 2.24 and it will only tell us about the map $\mathrm{M}(q)^{\mathrm{gp,ab}}$. However, there is a general result about the coequalisers of EG-algebras that we can prove to get us around this.

Proposition 3.4. *Let $\phi, \phi' : X \to Y$ be a pair of parallel* EG*-algebra maps, and $k : Y \to Z$ their coequalizer in* $\mathrm{E}G\mathrm{Alg}_S$*. If the monoid* $\mathrm{Ob}(Z)$ *is also a group, then the functor k is surjective.*

Proof. We begin by mirroring the proof of Proposition 3.2. We know that the functor $\mathrm{Ob} : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{Mon}$ is a left adjoint, by Proposition 2.12, and thus preserves all colimits. It follows that the monoid homomorphism $\mathrm{Ob}(k) : \mathrm{Ob}(Y) \to \mathrm{Ob}(Z)$ is the coequaliser of the parallel pair $\mathrm{Ob}(\phi), \mathrm{Ob}(\phi') : \mathrm{Ob}(X) \to \mathrm{Ob}(Y)$ in Mon, or in other words

$$\mathrm{Ob}(Z) \quad = \quad {\mathrm{Ob}(Y)}\big/{\sim}$$

where $\sim$ is the relation defined by

$$\mathrm{Ob}(\phi)(y) \sim \mathrm{Ob}(\phi')(y), \qquad a \sim a', b \sim b' \implies ab \sim a'b'$$

The map $\mathrm{Ob}(k) : \mathrm{Ob}(Y) \to \mathrm{Ob}(Y)/\sim$ is then clearly surjective.

Next, let $f : v \to w$ and $f' : w' \to v'$ be any two morphisms of the algebra Y for which $k(f)$ and $k(f')$ are composable in Z. Since these maps are composable we know that $k(w)$ and $k(w')$ must be the same object of Z, and since Z is a group we know this object has an inverse $k(w)^* = k(w')^*$. So by the surjectivity of k we can find another object y of Y for which $k(y) = k(w)^*$. Using this, define the morphism $h : x \to x'$ to be the tensor product $f' \otimes \mathrm{id}_y \otimes f$. Then

$$\begin{aligned} k(h) &= k(f' \otimes \mathrm{id}_y \otimes f) \\ &= k(f') \otimes \mathrm{id}_{k(y)} \otimes k(f) \\ &= k(f') \otimes \mathrm{id}_{k(w)^*} \otimes k(f) \end{aligned}$$

But by Lemma 2.21, this is really just the composite $k(f') \circ k(f)$. Thus the set of morphisms of Z which are images of morphisms of Y is closed under composition.

So now consider $k(Y)$, the subcategory of Z that contains every object x' for which there exists x in Y with $k(x) = x'$, and every morphism f' for which there exists f in Y with $q(f) = f'$. We know that the morphisms of $k(Y)$ are closed under composition, and so this is indeed a well-defined category. Moreover, for any collection of morphisms

$f'_1, ..., f'_m$ of $k(Y)$ we'll have

$$\begin{aligned}\alpha_Z(g; f'_1, ..., f'_m) &= \alpha_Z\Big(\, g\,;\, k(f_1), ..., k(f_m)\,\Big)\\ &= k\Big(\, \alpha_Y(g; f_1, ..., f_m)\,\Big)\\ &\in k(Y)\end{aligned}$$

for some $f_1, ..., f_m$, since k is a map of EG-algebras. Thus $k(Y)$ is also a well-defined sub-EG-algebra of Z. There is also clearly a canonical map $k' : Y \to k(Y)$, the unique surjective map of EG-algebras with the property that $k'(x) = k(x)$ for any object x and $k'(f) = k(f)$ for any morphism f. If we denote by i the evident inclusion of algebras $i : k(Y) \hookrightarrow Z$, then these maps are related by the fact that $i \circ k' = k$.

$$\begin{tikzcd} & X \arrow[d, bend right, "\phi"'] \arrow[d, bend left, "\phi'"] & \\ & Y \arrow[dl, "k'"'] \arrow[d, "k"] \arrow[dr, "j"] & \\ k(Y) \arrow[r, hook, "i"] & Z \arrow[r, "u"] & U \end{tikzcd}$$

Given all of this, let $j : Y \to U$ be any map of EG-algebras with the property that $j \circ \phi = j \circ \phi'$. Since h is the coequaliser of ϕ and ϕ', it follows that there exists a unique map $u : Y \to U$ such that $j = u \circ k$. This means that $j = u \circ i \circ k'$, and hence there is obviously at least one map, $u \circ i$, which lets us factors j through k'. But for any other map $v : k(Y) \to U$ that factors j like this, we'll have

$$\begin{aligned} v \circ k' &= j \\ &= u \circ i \circ k' \\ \Longrightarrow \qquad v &= u \circ i \end{aligned}$$

because k' is surjective, and thus $u \circ i$ is the unique map with this property. That is, k' is also a coequaliser of ϕ and ϕ'. But colimits are always unique up to a unique isomorphism, and so there should be a unique invertible map $k(Y) \to Z$ factoring k through k'. This is clearly just the inclusion i, and as a result $k(Y) = Z$ and $k' = k$. In other words, the map coequaliser mapm k is surjective. $\square$

Because the cokernel of a morphism is just its coequaliser with the zero map, and since we know that the objects of $L\mathbb{G}_n$ form a group, we can immediately apply this result to the functor q.

Corollary 3.5. *The cokernel map $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$ is surjective.*

This is probably the single most important step in our effort to determine the morphisms of $L\mathbb{G}_n$, in the sense of how many of the results hereafter rely on this relatively simple property. Indeed this result is so strong that after a cursory glance, one might be forgiven for thinking that it will immediately provide for us the main thing we have been working towards this chapter — the value of $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$.

After all, every surjective functor is an epimorphism in the category MonCat. We know that left adjoint functors preserve epimorphisms, and that $\mathrm{M}(\_)^{\mathrm{gp,ab}}$ is a left adjoint, so from Corollary 3.5 we can surmise that $\mathrm{M}(q)^{\mathrm{gp,ab}}$ is also an epimorphism, this time in Ab. But an epimorphic map of abelian groups is nothing other than a surjective homomorphism, and thus we may apply the First Isomorphism Theorem of groups to get the following:

$$\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \quad = \quad \mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \Big/ \ker\Big(\mathrm{M}(q)^{\mathrm{gp,ab}}\Big)$$

So if we knew what the kernel of $\mathrm{M}(q)^{\mathrm{gp,ab}}$ was, we would be done. And it seems like we *should* know this; q was defined to be the cokernel of δ, and by preservation of this colimits means that $\mathrm{M}(q)^{\mathrm{gp,ab}}$ is the cokernel of $\mathrm{M}(\delta)^{\mathrm{gp,ab}}$. Then since we are working with abelian groups, kernels and cokernels interact in a nice way:

$$\ker\operatorname{coker}\Big(\mathrm{M}(\delta)^{\mathrm{gp,ab}}\Big) \quad = \quad \mathrm{im}\Big(\mathrm{M}(\delta)^{\mathrm{gp,ab}}\Big)$$

However, this last step doesn't actually work — q was defined to be $\operatorname{coker}(\delta)$, but only in the category of $\mathrm{E}G$-algebras. In general this will *not* be the same thing as the cokernel of δ in MonCat, which is what we would really need in order for $\mathrm{M}(\_)^{\mathrm{gp,ab}}$ to preserve it.

Still, this is a pretty reasonable guess for what $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$ is, and provides an indication of how we should proceed in order to find its true value. We will pick up on this idea again in Section 3.4.

3.3 Action morphisms of $L\mathbb{G}_n$

One important consequence of the surjectivity of q is that it will allow us to import certain results about the free algebra $\mathbb{G}_{2n}$ into the free invertible algebra $L\mathbb{G}_n$. In fact, we have done this once already; looking back at Proposition 3.2 with our current knowledge that $Q = L\mathbb{G}_n$, we can see that it is a direct analogue of Lemma 1.26, using the fact that q is surjective on objects.

In that same vein, one might ask if we can take Lemma 1.27, a statement about the morphisms $\mathbb{G}_{2n}$, and extend it to an analagous result on $L\mathbb{G}_n$, using surjectivity of q on morphisms instead. That is, since every morphism of $\mathbb{G}_{2n}$ is an action morphism, and since EG-algebra maps always send action morphisms to action morphisms, we should be able to use q to identify every morphism of $L\mathbb{G}_n$ as an action morphism. This is indeed pretty simple to show.

Lemma 3.6. *Every morphism in $L\mathbb{G}_n$ can be expressed as $\alpha_{L\mathbb{G}_n}(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$, for some $g \in G(m)$ and $x_i \in \{z_1, ..., z_n, z_1^*, ..., z_n^*\}$.*

Proof. Let f be an arbitrary morphism in $L\mathbb{G}_n$. By surjectivity of q, there must exist at least one morphism f' in $\mathbb{G}_{2n}$ such that $q(f') = f$, and from Lemma 1.27 we know that this f' can be expressed uniquely as $\alpha(g; \mathrm{id}_{x'_1}, ..., \mathrm{id}_{x'_m})$ for some $g \in G(m)$ and $x'_i \in \{z_1, ..., z_{2n}\}$. Thus, because q is a map of EG-algebras, we will have

$$\begin{aligned} f &= q(f') \\ &= q\Big(\alpha_{\mathbb{G}_{2n}}(\,g\,;\,\mathrm{id}_{x'_1}, ..., \mathrm{id}_{x'_m}\,)\Big) \\ &= \alpha_{L\mathbb{G}_n}(\,g\,;\,\mathrm{id}_{q(x'_1)}, ..., \mathrm{id}_{q(x'_m)}\,) \end{aligned}$$

Therefore there is at least one collection of $x_i = q(x'_i)$ for which the statement of the proposition holds. □

Lemma 3.6 formalises a certain intuition about how the functor L should act on algebras, the idea that a 'free' structure really shouldn't have any 'superfluous' components, only whatever data is absolutely required for it to be well-defined. In the case of $L\mathbb{G}_n$, we have proven that the only morphisms contained in the free EG-algebra on invertible objects are EG-action morphisms. However, while this is very similar to what we have in the non-invertible case it should be stressed that Lemma 3.6 does *not* prove that the morphisms of $L\mathbb{G}_n$ have *unique* representations $\alpha(g; \mathrm{id}_{w_1}, ..., \mathrm{id}_{w_m})$, as morphisms of $\mathbb{G}_n$ do.

Also, notice that when we eventually find a complete description of $L\mathbb{G}_n$ as a monoidal category, we will be able to use the surjective algebra map q to determine it's

EG-action as well. This follows from the same reasoning we used to prove Lemma 3.6, but in reverse:

$$\begin{aligned}\alpha_{L\mathbb{G}_n}(\,g\,;\,\mathrm{id}_{x_1},...,\mathrm{id}_{x_m}\,) &= \alpha_{L\mathbb{G}_n}(\,g\,;\,\mathrm{id}_{q(x'_1)},...,\mathrm{id}_{q(x'_m)}\,)\\ &= q\Big(\,\alpha_{\mathbb{G}_{2n}}(\,g\,;\,\mathrm{id}_{x'_1},...,\mathrm{id}_{x'_m}\,)\Big)\end{aligned}$$

In fact, since we do know that q is a cokernel of the map δ, we can even extract some information about this action right away, before we have built an understanding of the morphisms of $L\mathbb{G}_n$.

Lemma 3.7. *For any element $g \in G(m), m \in \mathbb{N}$ of an action operad G,*

$$\alpha_{L\mathbb{G}_n}(\,g\,;\,\mathrm{id}_I,...,\mathrm{id}_I\,) \quad = \quad \mathrm{id}_I$$

Equivalently, for any element $h \in G(0)$,

$$\alpha_{L\mathbb{G}_n}(\,h\,;\,-\,) \quad = \quad \mathrm{id}_I$$

Proof. First, let $g \in G(m)$. Then because q is the cokernel of δ in EGAlg$_S$,

$$\begin{aligned}\alpha_{L\mathbb{G}_n}(\,g\,;\,\mathrm{id}_I,...,\mathrm{id}_I\,) &= \alpha_{L\mathbb{G}_n}(\,g\,;\,\mathrm{id}_{q(I)},...,\mathrm{id}_{q(I)}\,)\\ &= q\Big(\,\alpha_{\mathbb{G}_{2n}}(\,g\,;\,\mathrm{id}_I,...,\mathrm{id}_I\,)\Big)\\ &= q\Big(\,\alpha_{\mathbb{G}_{2n}}(\,g\,;\,\mathrm{id}_{\delta(I)},...,\mathrm{id}_{\delta(I)}\,)\Big)\\ &= q\delta\Big(\,\alpha_{\mathbb{G}_{2n}}(\,g\,;\,\mathrm{id}_I,...,\mathrm{id}_I\,)\Big)\\ &= \mathrm{id}_I\end{aligned}$$

Clearly this result implies that

$$\alpha_{L\mathbb{G}_n}(\,h\,;\,-\,) \quad = \quad \mathrm{id}_I$$

for any element $h \in G(0)$, but the implication also goes the other way, since

$$\begin{aligned}\alpha(\,g\,;\,\mathrm{id}_I,...,\mathrm{id}_I\,) &= \alpha\Big(\,g\,;\,\alpha(e_0;-),...,\alpha(e_0;-)\Big)\\ &= \alpha\Big(\,\mu(g;e_0,...,e_0)\,;\,-\,\Big)\end{aligned}$$

and $\mu(g;e_0,...,e_0) \in G(0)$. $\square$

This is a pretty interesting result. By Lemma 1.27, morphisms of the form $\alpha_{\mathbb{G}_n}(g;\mathrm{id}_I,...,\mathrm{id}_I)$ make up the entirety of the homset $\mathbb{G}_n(I,I)$. Now we see that their image under the algebra map $\eta : \mathbb{G}_n \to L\mathbb{G}_n$ is always id_I, and so it follows that the unit endomorphisms of free algebras are wholly unrelated to the unit endomorphisms of the corresponding free *invertible* algebras. That is, when constructing $L\mathbb{G}_n$ it seems like it should not matter whether our chosen action operad G has nontrivial $G(0)$, since all morphisms $\alpha_{L\mathbb{G}_n}(g;-)$ for $g \in G(0)$ are going to end up as the identity regardless. In order to state this idea more concretely though, we need some way of 'removing' the group $G(0)$ from G.

Proposition 3.8. *Let G be a crossed action operad. Then there exists another crossed action operad G' which has $G'(m) = G(m)/G(0)$ for all $m \in \mathbb{N}$.*

Proof. For any elements $g \in G(m)$ and $h \in G(0)$, their tensor product $h \otimes g := \mu(e_2; h, g)$ is also an element of $G(m)$. This defines a map $G(0) \times G(m) \to G(m)$, which is both a group homomorphism and a group action:

$$\begin{aligned}
(hh') \otimes (gg') &= \mu(\, e_2\, ;\, hh', gg'\,) \\
&= \mu(\, e_2\, ;\, h, g\,) \cdot \mu(\, e_2\, ;\, h', g'\,) \\
&= (h \otimes g) \cdot (h' \otimes g')
\end{aligned}$$

$$e_0 \otimes g = g$$

$$\begin{aligned}
h' \otimes (h \otimes g) &= (h' \otimes h) \otimes g \\
&= (h'h) \otimes g
\end{aligned}$$

The last step here uses the fact that tensor product and group multiplication coincide on $G(0)$, by Lemma 1.8. We can thus take the quotient of each $G(m)$ by the action of $G(0)$, which will amount to quotienting out the image in $G(m)$ of the subgroup $G(0) \cong G(0) \times \{e_m\} \subseteq G(0) \times G(m)$.

In order for these new groups $G'(m) = G(m)/G(0)$ to form an action operad, we'll need operadic multiplication maps $\mu^{G'}$ and underlying permuation maps $\pi^{G'}$. These will be defined from μ^G and π^G using the universal property of the quotient. Specifically, let $h, h_1, ..., h_m \in G(0)$ and $k_1, ..., k_m \in \mathbb{N}$. Then we have

$$\begin{aligned}
\mu^G(\, h \otimes e_m\, ;\, h_1 \otimes e_{k_1}, ..., h_m \otimes e_{k_m}\,) &= \mu^G\Big(\, \mu^G(e_2; h, e_m)\, ;\, h_1 \otimes e_{k_1}, ..., h_m \otimes e_{k_m} \Big) \\
&= \mu^G\Big(\, e_2\, ;\, \mu^G(h; -), \mu^G(e_m; h_1 \otimes e_{k_1}, ..., h_m \otimes e_{k_m}) \Big) \\
&= \mu^G(h; -) \otimes \mu^G(e_m; h_1 \otimes e_{k_1}, ..., h_m \otimes e_{k_m}) \\
&= h \otimes h_1 \otimes e_{k_1} \otimes ... \otimes h_m \otimes e_{k_m} \\
&= e_{k_1} \otimes ... \otimes e_{k_m} \otimes h \otimes h_1 ... \otimes h_m \\
&= e_{k_1 + ... + k_m} \otimes h \otimes h_1 ... \otimes h_m
\end{aligned}$$

since by Lemma 1.22 our crossed G is spacial, and so the e_k commute with elements of $G(0)$. In other words, we know that the upper square in the diagram below commutes:

$$\begin{tikzcd}
G(0) \times G(0) \times ... \times ...G(0) \arrow[r, "\otimes"] \arrow[d, hook] & G(0) \arrow[d, hook] \\
G(m) \times G(k_1) \times ... \times G(k_m) \arrow[r, "\mu^G_m"] \arrow[d, "[_]\times...\times[_]"'] & G(k_1 + ... + k_m) \arrow[d, "[_]"] \\
G(m)/G(0) \times G(k_1)/G(0) \times ... \times G(k_m)/G(0) \arrow[r, "\mu^{G'}_m"] & G(k_1 + ... + k_m)/G(0)
\end{tikzcd}$$

Now, the composite on the right-hand side of the diagram is by definition the zero map, and so too is its composite with the $(m+1)$-fold tensor product $G(0)^{m+1} \to G(0)$. Using commutativity of the upper square, it follows that the composite of the inclusion on the left and the upper-right path in the bottom square is also zero, and so this upper-right path will factor uniquely through the quotient of that inclusion. The resulting homomorphism $\mu^{G'}_m$ is then exactly the operadic multiplication map we are looking for; the identity and associativity conditions are immediate consequences of the corresponding conditions for μ^G.

$$\begin{array}{rrcccl}
 & \mu^G(\, g\, ;\, e_1, ..., e_1\,) & = & g & = & \mu^G(\, e_1\, ;\, g\,) \\
\Longrightarrow & \Big[\, \mu^G(\, g\, ;\, e_1, ..., e_1\,)\Big] & = & [g] & = & \Big[\, \mu^G(\, e_1\, ;\, g\,)\Big] \\
\Longrightarrow & \mu^{G'}\Big(\, [g]\, ;\, [e_1], ..., [e_1]\Big) & = & [g] & = & \mu^{G'}\Big(\, [e_1]\, ;\, [g]\Big)
\end{array}$$

$$\begin{aligned}
& \mu^{G'}\Big(\, [g]\, ;\, \mu^{G'}\Big(\, [g_1]\, ;\, [h_{1,1}], ..., [h_{1,k_1}]\Big), ..., \mu^{G'}\Big(\, [g_m]\, ;\, [h_{m,1}], ..., [h_{m,k_m}]\Big)\Big) \\
= \;& \mu^{G'}\Big(\, [g]\, ;\, \Big[\, \mu^G(g_1; h_{1,1}, ..., h_{1,k_1})\Big], ..., \Big[\, \mu^G(g_m; h_{m,1}, ..., h_{m,k_m})\Big]\Big) \\
= \;& \Big[\, \mu^G\Big(\, g\, ;\, \mu^G(g_1; h_{1,1}, ..., h_{1,k_1}), ..., \mu^G(g_m; h_{m,1}, ..., h_{m,k_m})\Big)\Big] \\
= \;& \Big[\, \mu^G\Big(\, \mu^G(g; g_1, ..., g_m)\, ;\, h_{1,1}, ..., h_{1,k_1}, ..., h_{m,1}, ..., h_{m,k_m})\Big)\Big] \\
= \;& \mu^{G'}\Big(\, \Big[\, \mu^G(g; g_1, ..., g_m)\Big]\, ;\, [h_{1,1}], ..., [h_{1,k_1}], ..., [h_{m,1}], ..., [h_{m,k_m}]\Big) \\
= \;& \mu^{G'}\Big(\, \mu^{G'}\Big(\, [g]\, ;\, [g_1], ..., [g_m]\Big)\, ;\, [h_{1,1}], ..., [h_{1,k_1}], ..., [h_{m,1}], ..., [h_{m,k_m}]\Big)
\end{aligned}$$

Similarly, for any $h \in G(0)$ and $m \in \mathbb{N}$ we know that

$$\pi^G(h \otimes e_m) \quad = \quad \pi^G(h) \otimes \pi^G(e_m) \quad = \quad e_0 \otimes e_m \quad = \quad e_m$$

and so the top square in the diagram below will commute:

$$\begin{tikzcd}
G(0) \arrow[r] \arrow[d, hook] & S_0 \arrow[d, hook] \\
G(m) \arrow[r, "\pi^G_m"] \arrow[d, "{[_]}"'] & S_m \arrow[d, equal] \\
G(m)\diagup G(0) \arrow[r, "\pi^{G'}_m"] & S_m
\end{tikzcd}$$

Using the same reasoning as before this will define the homomorphisms $\pi^{G'}_m$ uniquely, and the conditions for them to be underlying permutation maps of an action operad follow from those of π^G.

$$\pi^{G'}\Big([e_1]\Big) \quad = \quad \pi^G(e_1) \quad = \quad e_1$$

$$\begin{aligned}
\pi^{G'}\Bigg(\mu^{G'}\Big([g]\,;\,[h_1],...,[h_m]\Big)\Bigg) &= \pi^{G'}\Bigg(\Big[\mu^G(g;h_1,...,h_m)\Big]\Bigg) \\
&= \pi^G\Big(\mu^G(g;h_1,...,h_m)\Big) \\
&= \mu^S\Big(\pi^G(g)\,;\,\pi^G(h_1),...,\pi^G(h_m)\Big) \\
&= \mu^S\Bigg(\pi^{G'}\Big([g]\Big)\,;\,\pi^{G'}\Big([h_1]\Big),...,\pi^{G'}\Big([h_m]\Big)\Bigg)
\end{aligned}$$

$$\begin{aligned}
& \mu^{G'}\Big([g]\,;[h_1],...,[h_m]\Big)\cdot\mu^{G'}\Big([g']\,;[h'_1],...,[h'_m]\Big) \\
= \;& \Big[\mu^G(g;h_1,...,h_m)\Big]\cdot\Big[\mu^G(g';h'_1,...,h'_m)\Big] \\
= \;& \Big[\mu^G(g;h_1,...,h_m)\cdot\mu^G(g';h'_1,...,h'_m)\Big] \\
= \;& \Bigg[\mu^G\Big(gg'\,;h_{\pi^G(g')(1)}h'_1,...,h_{\pi^G(g')(m)}h'_m\Big)\Bigg] \\
= \;& \mu^{G'}\Big([gg']\,;[h_{\pi^G(g')(1)}h'_1],...,[h_{\pi^G(g')(m)}h'_m]\Big) \\
= \;& \mu^{G'}\Big([g]\cdot[g']\,;[h_{\pi^G(g')(1)}]\cdot[h'_1],...,[h_{\pi^G(g')(m)}]\cdot[h'_m]\Big) \\
= \;& \mu^{G'}\Big([g]\cdot[g']\,;[h_{\pi^{G'}([g'])(1)}]\cdot[h'_1],...,[h_{\pi^G(g')(m)}]\cdot[h'_m]\Big)
\end{aligned}$$

Thus G' really is a well-defined action operad. $\square$

For crossed G, this notion of quotient by $G(0)$ does exactly what we want it to do — remove certain information which is unnecessary for forming the algebra $L\mathbb{G}_n$.

Proposition 3.9. *Let G be a crossed action operad, and let G' be the action operad with $G'(m) = G(m)/G(0)$ for all $m \in \mathrm{N}$. Then for any $n \in \mathrm{N}$,*

$$L\mathbb{G}'_n \quad \cong \quad L\mathbb{G}_n$$

both as $\mathrm{E}G$*-algebras and as* $\mathrm{E}G'$*-algebras. That is, every free invertible algebra over a crossed action operad is the same as one over an action operad with trivial* $G(0)$*.*

Proof. It is fairly easy to see that the maps $[\_] : G(m) \to G(m)/G(0)$ sending elements to their equivalence class under the quotient must be surjective. Because of this, we will be able to use the action $\alpha_{L\mathbb{G}'_n}$ of $L\mathbb{G}'_n$ not just as an $\mathrm{E}G'$-action, but also as an $\mathrm{E}G$-action, which we'll call $\tilde{\alpha}_{L\mathbb{G}'_n}$ for the same of keeping the two concepts distinct. That is,

$$\tilde{\alpha}_{L\mathbb{G}'_n}(\, g\,;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,) \quad := \quad \alpha_{L\mathbb{G}'_n}\Big(\, [g]\,;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,\Big)$$

Likewise, the $\mathrm{E}G$-action of $L\mathbb{G}_n$ is also an $\mathrm{E}G'$-action, via

$$\tilde{\alpha}_{L\mathbb{G}_n}\Big(\, [g]\,;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,\Big) \quad := \quad \alpha_{L\mathbb{G}_n}(\, g\,;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,)$$

Lemma 3.7 ensures that this statement makes sense; whenever we have $[g] = [g']$ it is because there is some $h \in G(0)$ for which $g' = h \otimes g$, and so

$$\begin{aligned}
\alpha_{L\mathbb{G}_n}(\, g'\,;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,) &= \alpha_{L\mathbb{G}_n}(\, h \otimes g\,;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,) \\
&= \alpha_{L\mathbb{G}_n}\Big(\, \mu(e_2; h, g)\,;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,\Big) \\
&= \alpha_{L\mathbb{G}_n}\Big(\, e_2\,;\, \alpha_{L\mathbb{G}_n}(h; -), \alpha_{L\mathbb{G}_n}(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})\,\Big) \\
&= \alpha_{L\mathbb{G}_n}(h; -) \otimes \alpha_{L\mathbb{G}_n}(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) \\
&= \mathrm{id}_I \otimes \alpha_{L\mathbb{G}_n}(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) \\
&= \alpha_{L\mathbb{G}_n}(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})
\end{aligned}$$

By Proposition 2.14 we already know that $L\mathbb{G}_n$ and $L\mathbb{G}'_n$ have isomorphic object sets, and so by using the universal properties of $\mathbb{G}_n$ and $\mathbb{G}'_n$ we can produce maps

$$\mathbb{G}_n \longrightarrow L\mathbb{G}'_n \qquad \text{and} \qquad \mathbb{G}'_n \longrightarrow L\mathbb{G}_n$$

which correspond to the same choices of n invertible objects that the maps η^G and $\eta^{G'}$ do. The universal properties of $L\mathbb{G}_n$ and $L\mathbb{G}'_n$ will then make these new maps factor through the respective η's, and so there must exist an EG-algebra map

$$\begin{array}{rcl} L\mathbb{G}_n & \to & L\mathbb{G}'_n \\ x & \mapsto & x \\ \alpha_{L\mathbb{G}_n}(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) & \mapsto & \tilde{\alpha}_{L\mathbb{G}'_n}(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) \\ & & = \alpha_{L\mathbb{G}'_n}\Big([g] \,;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m} \Big) \end{array}$$

and an EG'-algebra map

$$\begin{array}{rcl} L\mathbb{G}'_n & \to & L\mathbb{G}_n \\ x & \mapsto & x \\ \alpha_{L\mathbb{G}'_n}\Big([g] \,; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m} \Big) & \mapsto & \tilde{\alpha}_{L\mathbb{G}_n}\Big([g] \,;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m} \Big) \\ & & = \alpha_{L\mathbb{G}_n}(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) \end{array}$$

These functors are clearly inverses, and also algebra maps for both G and G'. Therefore

$$L\mathbb{G}'_n \quad \cong \quad L\mathbb{G}_n$$

in both senses, as required. $\square$

For noncrossed G we cannot so easily remove the group $G(0)$ like this, as without being spacial we have no way to draw its elements out from inbetween elements of the higher $G(m)$. Still, there is one more thing about the morphisms of $L\mathbb{G}_n$ that we can deduce from Lemma 3.7.

Definition 3.10. Let G be a noncrossed action operad in which every element of each $G(m)$ can be written as $\mu(g; e_m)$ for some $G \in G(1)$. Then we say that G is a $G(1)$*-generated* action operad.

Lemma 3.11. *If G is a $G(1)$-generated action operad, then $L\mathbb{G}_n(I, I)$ is the trivial group.*

Proof. First we need to check that this claim makes sense, that elements of the required form are indeed closed under operadic multiplication so that they may make up a valid

G. This is the case, as we have

$$\begin{aligned}
\mu\Big(\mu(g;e_m)\,;\,\mu(h_1;e_{k_1}),...,\mu(h_1;e_{k_m})\Big) &= \mu\Big(g\,;\,\mu\Big(e_m\,;\,\mu(h_1;e_{k_1}),...,\mu(h_m;e_{k_m})\Big)\Big) \\
&= \mu\Big(g\,;\,\mu\Big(\mu(e_m;h_1,...,h_m)\,;\,e_{k_1},...,e_{k_m}\Big)\Big) \\
&= \mu\Big(g\,;\,\mu\Big(\mu(h;e_m)\,;\,e_{k_1},...,e_{k_m}\Big)\Big) \\
&= \mu\Big(g\,;\,\mu\Big(h\,;\,\mu(e_m;e_{k_1},...,e_{k_m})\Big)\Big) \\
&= \mu\Big(g\,;\,\mu(\,h\,;\,e_{k_1+...+k_m}\,)\Big) \\
&= \mu\Big(\mu(g;h)\,;\,e_{k_1+...+k_m}\Big)
\end{aligned}$$

where $\mu(h;e_m)$ is any way of writing $\mu(e_m;h_1,...,h_m) = h_1 \otimes ... \otimes h_m$ in the required form.

Now let f be an arbitrary element of $L\mathbb{G}_n(I,I)$. By Lemma 3.6 there must be some objects $x_1,...,x_m$ such that

$$f \quad = \quad \alpha(g;\mathrm{id}_{x_1},...,\mathrm{id}_{x_m})$$

Then by assumption there must also exist some $h \in G(1)$ for which $g = \mu(h;e_m)$. With this in mind, we see that

$$\begin{aligned}
\alpha(g;\mathrm{id}_{x_1},...,\mathrm{id}_{x_m}) &= \alpha\Big(\mu(h;e_m)\,;\,\mathrm{id}_{x_1},...,\mathrm{id}_{x_m}\Big) \\
&= \alpha\Big(h\,;\,\mu(e_m;\mathrm{id}_{x_1},...,\mathrm{id}_{x_m})\Big) \\
&= \alpha(\,h\,;\,\mathrm{id}_{x_1\otimes...\otimes x_m})
\end{aligned}$$

But this is supposed to be a morphism $f : I \to I$, so we know that $x_1 \otimes ... \otimes x_m = I$, and therefore by Lemma 3.7

$$f \quad = \quad \alpha(\,h\,;\,\mathrm{id}_I) \quad = \quad \mathrm{id}_I$$

As f was chosen arbitrarily, it follows that $L\mathbb{G}_n(I,I) = \{\mathrm{id}_I\}$. □

Ultimately, we will see that there is very little we can say for sure about the unit endomorphisms of $L\mathbb{G}_n$ when G is not crossed, other than Lemma 3.11. For this reason, the main theorems of this paper will end up describing only those invertible EG-algebras whose actionoperads are either crossed or $G(1)$-generated.

3.4 $L\mathbb{G}_n$ as a coequaliser in MonCat

Looking back at the proof of Proposition 3.4, notice that we never needed to use the fact that ϕ, ϕ' and k were maps of EG-algebras, only that they were monoidal functors. Because we had assumed from the beginning that we were working in $\mathrm{E}G\mathrm{Alg}_S$, we did at one point have to show that the category $k(Y)$ was an algebra, so that we could then use the universal property of k in $\mathrm{E}G\mathrm{Alg}_S$, but if k had just been a coequaliser in MonCat from the start then this part would not have been necessary. We also had to invoke Proposition 2.12 — which says that $\mathrm{Ob} : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{Mon}$ is a left adjoint — so that we could exploit preservation of colimits. But since Ob clearly doesn't care about the morphisms of an algebra, it doesn't really matter whether we are applying it to an algebra in the first place. The actions of X, Y and Z just never came into play.

With that in mind, we can co-opt all of these previous proofs about EG-algebra maps to prove the analagous statements about monoidal functors.

Proposition 3.12. *Let the functors*

$$\mathrm{Ob} : \mathrm{MonCat} \to \mathrm{Mon}, \qquad \mathrm{E} : \mathrm{Mon} \to \mathrm{MonCat}$$

be defined exactly as those from Definitions 2.10 and 2.11, except without the requirement that the monoidal categories be EG*-algebras. Then* E *is a right adjoint to the functor* Ob.

Proof. The same as the proof of Proposition 2.12. □

Proposition 3.13. *Let $\phi, \phi' : X \to Y$ be a pair of parallel monoidal functors, and $k : Y \to Z$ their coequalizer in* MonCat. *If the monoid* $\mathrm{Ob}(Z)$ *is also a group, then the functor k is surjective.*

Proof. The same as the proof of Proposition 3.4, but with Proposition 3.12 in place of Proposition 2.12, and no reference to $k(Y)$ being a sub-EG-algebra. □

Further, these new propositions prove a surjectivity statement just like Corollary 3.5.

Definition 3.14. Let the monoidal functor $c : \mathbb{G}_{2n} \to C$ onto some monoidal category C be the cokernel of the underlying monoidal functor of δ in MonCat. This map definitely exists because MonCat is cocomplete, and like with q we can show that its target has a group of objects.

Proposition 3.15. *The object monoid of C is $\mathbb{Z}^{*n}$, and the restriction of c to objects* $\mathrm{Ob}(c) : \mathrm{Ob}(\mathbb{G}_{2n}) \to \mathrm{Ob}(C)$ *is the monoid homomorphism defined on generators as*

$$\begin{array}{rcrcl} \mathrm{Ob}(c) & : & \mathbb{N}^{*2n} & \to & \mathbb{Z}^{*n} \\ & : & z_i & \mapsto & z_i \\ & : & z_{n+i} & \mapsto & z_i^* \end{array}$$

Proof. The same as the proof of Proposition 3.2, but with $c : \mathbb{G}_{2n} \to C$ in place of $q : \mathbb{G}_{2n} \to Q$ and Proposition 3.12 in place of Proposition 2.12. □

Propositions 3.13 and 3.15 then immediately combine to give:

Corollary 3.16. *The cokernel map $c : \mathbb{G}_{2n} \to C$ is surjective.*

This statement is actually pretty unusual. In Corollary 3.5 it made sense that q would be surjective, but that was because its source and target were special. $\mathbb{G}_{2n}$ is the free EG-algebra on $2n$ objects, and $L\mathbb{G}_n$ is the free EG-algebra on n objects and their n inverses, and so intuitively the map identifying those sets generators would tell us everything we need to know about the algebra structure of $L\mathbb{G}_n$. And since by freeness we expect algebra maps to be all there really is to $L\mathbb{G}_n$, it was a safe bet that q was going to be surjective.

But none of that is true for c. The underlying monoidal category of $\mathbb{G}_{2n}$ is not anything special in MonCat, and neither is C. So what is going on here? The answer is that category C is *almost* the algebra $L\mathbb{G}_n$, and likewise the functor c is *almost* the map q. To see this, consider the following niave method for assigning an EG-action α_C to C:

$$\alpha_C(\, g\, ;\, c(f_1), ..., c(f_m)\,) \quad := \quad c\Big(\, \alpha_{\mathbb{G}_{2n}}(\, g\, ;\, f_1, ..., f_m\,)\Big)$$

Any action on C that made c into a map of EG-algebras would have to satify this condition, of course. But because c is surjective, every collection of morphisms in C can be written as $c(f_1), ..., c(f_m)$, and this forces α_C to take a unique value everywhere, assuming it is well-defined. Then, since the the cokernel of δ in MonCat would be an EG-algebra map, we could conclude that it was also the cokernel of δ in E$G\mathrm{Alg}_S$ too. However, 'assuming it is well-defined' is where the problems lie. In particular, since c is not injective on objects we can find $w_1, ..., w_m$ and $w'_1, ..., w'_m$ in $\mathbb{G}_{2n}$ for which $c(w_i) = c(w'_i)$, and so α^C would only be well-defined if

$$c\Big(\, \alpha_{\mathbb{G}_{2n}}(\, g\, ;\, \mathrm{id}_{w_1}, ..., \mathrm{id}_{w_m}\,)\Big) \quad = \quad c\Big(\, \alpha_{\mathbb{G}_{2n}}(\, g\, ;\, \mathrm{id}_{w'_1}, ..., \mathrm{id}_{w'_m}\,)\Big)$$

which we have no reason to believe is true.

To fix this issue, what we need is a way of describing the map q as a colimit of a slightly different diagram in EG-algebras, one whose colimit in MonCat will have all of the same properties that c does but will also satisfy the condition above. To that end, consider the following EG-algebra maps:

Definition 3.17. Let $\tilde{\delta} := \mathrm{id}_{\mathbb{G}_{2n}} + \delta$ be the map defined from δ and the identity by using the universal property of the coproduct $\mathbb{G}_{4n} = \mathbb{G}_{2n} + \mathbb{G}_{2n}$ in EGAlg$_S$. That is, $\mathrm{id}_{\mathbb{G}_{2n}} + \delta$ is the map of EG-algebras which acts on generators by

$$\begin{array}{rcrcl} \tilde{\delta} & : & \mathbb{G}_{4n} & \to & \mathbb{G}_{2n} \\ & : & z_i & \mapsto & z_i \\ & : & z_{n+i} & \mapsto & z_{n+i} \\ & : & z_{2n+i} & \mapsto & z_i \otimes z_{n+i} \\ & : & z_{3n+i} & \mapsto & z_{n+i} \otimes z_i \end{array}$$

for $1 \leq i \leq n$. Similarly, let $\tilde{I} := \mathrm{id}_{\mathbb{G}_{2n}} + I$ be the EG-algebra map defined in the same way but from the constant map on the unit I instead of δ:

$$\begin{array}{rcrcl} \tilde{I} & : & \mathbb{G}_{4n} & \to & \mathbb{G}_{2n} \\ & : & z_i & \mapsto & z_i \\ & : & z_{n+i} & \mapsto & z_{n+i} \\ & : & z_{2n+i} & \mapsto & I \\ & : & z_{3n+i} & \mapsto & I \end{array}$$

Lemma 3.18. *q is the coequaliser of $\tilde{\delta}$ and $\tilde{I}$ in* EGAlg$_S$.

Proof. Let $\psi : \mathbb{G}_{2n} \to X$ be an map of EG-algebras. Then

$$\psi \circ (\mathrm{id}_{\mathbb{G}_{2n}} + \delta)(z_i) \quad = \quad \psi \circ (\mathrm{id}_{\mathbb{G}_{2n}} + I)$$
$$\Longleftrightarrow$$
$$\psi \circ \mathrm{id}_{\mathbb{G}_{2n}} \quad = \quad \psi \circ \mathrm{id}_{\mathbb{G}_{2n}}, \qquad \psi \circ \delta \quad = \quad \psi \circ I$$

and hence

$$\mathrm{coeq}(\,\mathrm{id}_{\mathbb{G}_{2n}} + \delta, \mathrm{id}_{\mathbb{G}_{2n}} + I\,) \quad = \quad \mathrm{coeq}(\delta, I) \quad = \quad \mathrm{coker}(\delta) \quad = \quad q$$

□

While this proof may seem rather trivial, notice that it does rely on the fact that the + here represents the coproduct in the category of EG-algebras. There is no reason to expect that the coequaliser of the underlying monoidal functors of these maps would also be equal the cokernel of the underlying monoidal functor of δ. Thus these new maps will give rise to a new map which is distinct from the cokernel functor c, yet possesses many of the same properties.

Definition 3.19. Denote by $\tilde{c} : \mathbb{G}_{2n} \to \tilde{C}$ the coequaliser of $\tilde{\delta}$ and $\tilde{I}$ in the category MonCat.

Lemma 3.20. *The object monoid of $\tilde{C}$ is*

$$\mathrm{Ob}(\tilde{C}) \quad = \quad \mathrm{Ob}(C) \quad = \quad \mathbb{Z}^{*n}$$

and the restriction of $\tilde{c}$ to objects $\mathrm{Ob}(\tilde{c}) : \mathrm{Ob}(\mathbb{G}_{2n}) \to \mathrm{Ob}(\tilde{C})$ is just the monoid homomorphism $\mathrm{Ob}(c) : \mathbb{N}^{*2n} \to \mathbb{Z}^{*n}$ from Proposition 3.15.

Proof. Consider the monoid homomorphisms $\mathrm{Ob}(\tilde{\delta}) : \mathbb{N}^{*4n} \to \mathbb{N}^{*2n}$ and $\mathrm{Ob}(\tilde{I}) : \mathbb{N}^{*4n} \to \mathbb{N}^{*2n}$. These are fully determined by the descriptions of the corresponding algebra maps in **??**, and as such they are obviously just

$$\mathrm{Ob}(\mathrm{id}_{\mathbb{G}_{2n}} + \delta) \quad = \quad \mathrm{id}_{\mathbb{N}^{*2n}} + \mathrm{Ob}(\delta) \qquad\qquad \begin{aligned} \mathrm{Ob}(\mathrm{id}_{\mathbb{G}_{2n}} + I) \quad &= \quad \mathrm{id}_{\mathbb{N}^{*2n}} + \mathrm{Ob}(I) \\ &= \quad \mathrm{id}_{\mathbb{N}^{*2n}} + I \end{aligned}$$

where the $+$ on the righthand side of the equations means the coproduct in the category of monoids. Therefore

$$\mathrm{coeq}\Big(\mathrm{Ob}(\mathrm{id}_{\mathbb{G}_{2n}} + \delta), \mathrm{Ob}(\mathrm{id}_{\mathbb{G}_{2n}} + I)\,\Big) \quad = \quad \mathrm{coeq}\Big(\mathrm{Ob}(\delta), I\Big) \quad = \quad \mathrm{Ob}(c)$$

and thus $\mathrm{Ob}(\tilde{C}) = \mathrm{Ob}(C)$. □

Corollary 3.21. *The coequaliser map $\tilde{c} : \mathbb{G}_{2n} \to \tilde{C}$ is surjective.*

Proof. Lemma 3.20 says that the monoid $\tilde{C}$ is a group, so we may apply Proposition 3.13. □

So why bother with any of this? What features do $\tilde{\delta}$ and $\tilde{I}$ have that will make an action possible on $\tilde{C}$ when it wasn't on C? The answer is that unlike δ and I, these new maps form a *reflexive pair* — a parallel pair of functors which share a right-inverse.

Lemma 3.22. *Let $\iota : \mathbb{G}_{2n} \to \mathbb{G}_{4n}$ be the inclusion of algebras defined on generators by $z_i \mapsto z_i$. Then ι is a right-inverse of both $\tilde{\delta}$ and $\tilde{I}$.*

Proof. For $1 \leq i \leq 2n$,

$$\begin{array}{ccccccccc} \tilde{\delta}\iota(z_i) & = & \tilde{\delta}(z_i) & = & z_i & = & \tilde{I}(z_i) & = & \tilde{I}\iota(z_i) \\ \implies & & \tilde{\delta} \circ \iota & = & \mathrm{id}_{\mathbb{G}_{2n}} & = & \tilde{I} \circ \iota & & \end{array}$$

□

In other words, $\tilde{c}$ is a *reflexive coequalizer* in the category MonCat. This is the key difference which will eventually let us prove that $\tilde{c}$ respects action morphisms in the way that we need it to. First though, we will need a few intermediate results.

Definition 3.23. If w is an element of $\mathbb{N}^{*m}$, then we can use the definition of the free product of groups to decompose it uniquely as a tensor product of the m generators $z_1, ..., z_m$. We'll denote this by

$$w \quad =: \quad \bigotimes_{i=1}^{|w|} d(w,i), \qquad\qquad d(w,i) \in \{z_1, ..., z_m\}$$

If instead w is an element of $\mathbb{Z}^{*m}$, then we can use the definition of the free product of groups to decompose x uniquely as a tensor product, but this time one made up of the m generators $z_1, ..., z_m$ and their inverses $z_1^*, ..., z_m^*$. As before we'll denote this by

$$w \quad = \quad \bigotimes_{i=1}^{|w|} d(w,i)$$

where $d(w,i) \in \{z_1, ..., z_m, z_1^*, ..., z_m^*\}$, and also for any $1 \leq i < |w|$ we will always have $d(w, i+1) \neq d(w,i)^*$. By analogy with Definition 1.30, we will call the upper bound of this tensor product the *length* of the element w, and denote it by $|w|$, but be aware that this number is the one that comes from the *monoid* homomorphism

$$F\Big(\{z_1, ..., z_m, z_1^*, ..., z_m^*\} \Big) \to \mathbb{N}$$

that sends each generator to 1, and not the perhaps more obvious *group* homomorphism

$$F\Big(\{z_1, ..., z_m\} \Big) \to \mathbb{Z}$$

Proposition 3.24. *Let w be an object of $\mathbb{G}_{2n}$. Then there exist objects $w^{(1)}, ..., w^{(k)}$ in $\mathbb{G}_{2n}$ and $u^{(1)}, ..., u^{(k)}$ in $\mathbb{G}_{4n}$, for some value of $k \in \mathbb{N}$, such that*

$$w^{(1)} = w, \qquad u^{(k)} = \iota(w^{(k)}), \qquad \tilde{I}(u^{(i-1)}) \;=\; w^{(i)} \;=\; \tilde{\delta}(u^{(i)})$$

for $1 \leq i \leq k$, and for any object u of $\mathbb{G}_{4n}$,

$$\tilde{\delta}(u) = w^{(k)} \quad \Longleftrightarrow \quad u = u^{(k)}$$

Proof. From Definitions 1.30 and 3.17, we know that for any generator z_i of $\mathbb{G}_{4n}$,

$$|\tilde{\delta}(z_i)| \;=\; \left\{ \begin{array}{lll} 1 & \text{if} & 1 \leq i \leq 2n \\ 2 & \text{if} & 2n+1 \leq i \leq 4n \end{array} \right\} \geq 1$$

$$|\tilde{I}(z_i)| \;=\; \left\{ \begin{array}{lll} 1 & \text{if} & 1 \leq i \leq 2n \\ 0 & \text{if} & 2n+1 \leq i \leq 4n \end{array} \right\} \leq 1$$

Also these lengths are additive across tensor products, since $|\_|$ is a monoid homomorphism $\mathbb{G}_{2n} \to \mathbb{N}$. Thus for any object u in $\mathbb{G}_{4n}$, we can conclude that

$$\begin{array}{lllllllll} |\tilde{\delta}(u)| & = & |\,\tilde{\delta}\Big(\bigotimes_{i=1}^{|u|} d(u,i)\Big)\,| & = & \sum_{i=1}^{|u|} |\,\tilde{\delta}\Big(\,d(u,i)\,\Big)\,| & \geq & \sum_{i=1}^{|u|} 1 & = & |u| \\ |\tilde{I}(u)| & = & |\,\tilde{I}\Big(\bigotimes_{i=1}^{|u|} d(u,i)\Big)\,| & = & \sum_{i=1}^{|u|} |\,\tilde{I}\Big(\,d(u,i)\,\Big)\,| & \leq & \sum_{i=1}^{|u|} 1 & = & |u| \end{array}$$

Also, since the only generators that have $|\tilde{\delta}(z_i)| = |\tilde{I}(z_i)| = 1$ are those from the $\mathbb{G}_{2n}$ subalgebra associated with ι, the inequalities above becomes equalities if and only if u is in the image of ι. That is,

$$|\tilde{I}(u)| \;=\; |u| \;=\; |\tilde{\delta}(u)| \quad \Longleftrightarrow \quad \exists\, v \in \mathbb{N}^{*2n} \;:\; u \;=\; \iota(v)$$

Next, consider the set

$$\tilde{\delta}^{-1}(w) \quad := \quad \{\, u \in \mathbb{N}^{*4n} : \tilde{\delta}(u) = w \,\}$$

of all objects in $\mathbb{G}_{4n}$ which $\tilde{\delta}$ sends to w. This set is always nonempty, since by Lemma 3.22 ι is a right-inverse of δ:

$$\tilde{\delta}\iota(w) = w \quad \Longrightarrow \quad \iota(w) \in \tilde{\delta}^{-1}(w)$$

Moreover, $\iota(w)$ is the only element of $\tilde{\delta}^{-1}(w)$ which can be expessed as $\iota(v)$ for some object v in $\mathbb{G}_{2n}$, because

$$\tilde{\delta}\Big(\iota(v)\Big) = w \quad \Longrightarrow \quad v = w$$

With all of this now in place, we can begin constructing the sequences $w^{(1)}, ..., w^{(k)}$ and $u^{(1)}, ..., u^{(k)}$. Start by setting $w^{(1)} = w$ and $i = 1$, then apply the following algorithm:

1. If $\tilde{\delta}^{-1}(w^{(i)})$ is just the set $\{\iota(w^{(i)})\}$, choose $u^{(i)} = \iota(w^{(i)})$, set k to be the current value of i, and terminate.
2. Otherwise, choose $u^{(i)}$ to be any element of $\tilde{\delta}^{-1}(w^{(i)})$ other than $\iota(w^{(i)})$.
3. Set $w^{(i+1)} = \tilde{I}(u^{(i)})$.
4. Increase the value of i by 1, then return to step 1.

By design, none of the $u^{(i)}$ produced by this process can be expressed as $u^{(i)} = \iota(v)$ for some v in $\mathbb{G}_{2n}$, with the possible exception of u_k if the algorithm terminates. This is because $\iota(w^{(i)})$ is the only element of $\delta^{-1}(w^{(i)})$ that can be expressed that way, and the above process will terminate the first time it has to pick $u^{(i)} = \iota(w^{(i)})$, at which point i is set equal to k. Thus given what we found earlier in the proof, for any $i \neq k$ we must have the following *strict* inequalities:

$$|w^{(i+1)}| \quad = \quad |\tilde{I}(u^{(i)})| \quad < \quad |u^{(i)}| \quad < \quad |\tilde{\delta}(u^{(i)})| \quad = \quad |w^{(i)}|$$

That is, the $w^{(i)}$ produced by this algorithm form a sequence with strictly decreasing length. However, it is impossible to have a infinite sequence of strictly decreasing natural numbers, and hence we can be sure that this process will terminate at some finite k.

But in order for the algorithm to terminate, it must be the case that

$$\tilde{\delta}^{-1}(w^{(k)}) \quad = \quad \{\iota(w^{(k)})\}$$

and hence

$$\tilde{\delta}(u) = w^{(k)} \quad \Longleftrightarrow \quad u = \iota(w^{(k)}) = u^{(k)}$$

Thus the sequences $w^{(1)}, ..., w^{(k)}$ and $u^{(1)}, ..., u^{(k)}$ satisfy all of the conditions in the statement of the lemma. □

The intuition behind Proposition 3.24 is that we are successively removing parts of the object w, without changing its image under $\tilde{c}$. The map $\tilde{\delta}$ sends $z_{2n+i} \mapsto z_i \otimes z_{n+i}$ and $z_{3n+i} \mapsto z_{n+1} \otimes z_i$ while $\tilde{I}$ sends these all to I, and so for any u in $\mathbb{G}_{4n}$ the object $\tilde{I}(u)$ will look like $\tilde{\delta}(u)$ except missing some number of $z_i \otimes z_{n+i}$ or $z_{n+1} \otimes z_i$ substrings. But since $\tilde{c}$ sends $z_{n+i} \mapsto z_i^*$, these are exactly the sort of omissions which the coequaliser doesn't care about. If we repeat this process then it will eventually terminate at $u^{(k)} = \iota(w^{(k)})$, so we really have a method for removing *all* of the relevent substrings from objects of $\mathbb{G}_{2n}$. In other words, $w^{(k)}$ has the smallest possible length while still having $\tilde{c}(w^{(k)}) = \tilde{c}(w)$. In fact, we will show that it is the unique shortest object of $\mathbb{G}_{2n}$ with this property.

Proposition 3.25. *Let w, w' be objects of $\mathbb{G}_{2n}$ such that $\tilde{c}(w) = \tilde{c}(w')$. If $w^{(1)}, ..., w^{(k)}$ and $u^{(1)}, ..., u^{(k)}$ are the sequences generated from w via Proposition 3.24, and likewise ${w'}^{(1)}, ..., {w'}^{(k')}$ and ${u'}^{(1)}, ..., {u'}^{(k')}$ from w', then $w^{(k)} = {w'}^{(k')}$ and $u^{(k)} = {u'}^{(k')}$.*

Proof. Consider the decomposition of the object $w^{(k)} \in \mathbb{N}^{*2n}$ as in Definition 3.23. Assume, for the sake of contradiction, that there exist $1 \leq j < |w^{(k)}|$ and $1 \leq m \leq n$ such that

$$d(w^{(k)}, j) = z_m, \qquad d(w^{(k)}, j+1) = z_{n+m}$$

Then we can use j and m to contruct a new element $u \in \mathbb{N}^{*4n}$, defined by

$$|u| = |w| - 1, \qquad d(u, i) = \begin{cases} \iota\Big(d(w^{(k)}, i)\Big) & \text{if} \quad 1 \leq i < j \\ z_{2n+m} & \text{if} \quad i = j \\ \iota\Big(d(w^{(k)}, i+1)\Big) & \text{if} \quad j < i \leq |u| \end{cases}$$

This u will then have the property that

$$\begin{aligned}
\tilde{\delta}(u) &= \tilde{\delta}\Big(\bigotimes_{i=1}^{|u|} d(u,i)\Big) \\
&= \bigotimes_{i=1}^{|u|} \tilde{\delta}\Big(\, d(u,i)\Big) \\
&= \bigotimes_{i=1}^{j-1} \tilde{\delta}\iota\Big(\, d(w^{(k)},i)\,\Big) \otimes \tilde{\delta}(z_{2n+m}) \otimes \bigotimes_{i=j+1}^{|u|} \tilde{\delta}\iota\Big(\, d(w^{(k)},i+1)\Big) \\
&= \bigotimes_{i=1}^{j-1} d(w^{(k)},i) \otimes z_m \otimes z_{n+m} \otimes \bigotimes_{i=j+2}^{|u|+1} d(w_k,i) \\
&= \bigotimes_{i=1}^{j-1} d(w^{(k)},i) \otimes d(w^{(k)},j) \otimes d(w^{(k)},j+1) \otimes \bigotimes_{i=j+2}^{|u|+1} d(w_k,i) \\
&= w^{(k)}
\end{aligned}$$

But this is impossible, since by Proposition 3.24 $u^{(k)}$ is the only object of $\mathbb{G}_{4n}$ whose image under $\tilde{\delta}$ is $w^{(k)}$, and this u we have constructed is manifestly not $w^{(k)}$. Thus we can conclude that there are no values of j and m for which

$$d(w^{(k)},j) \;=\; z_m, \qquad d(w^{(k)},j+1) \;=\; z_{n+m}$$

An analogous line of reasoning — using z_{3n+m} rather than z_{2n+m} in the definition of u — demonstrates that there are also no j,m with

$$d(w^{(k)},j) \;=\; z_{n+m}, \qquad d(w^{(k)},j+1) \;=\; z_m$$

As a result, for all $1 \leq i < |w^{(k)}|$

$$\tilde{c}\Big(\, d(w^{(k)},i+1)\Big) \quad \neq \quad \tilde{c}\Big(\, d(w^{(k)},i)\,\Big)^*$$

and this combined with the fact that

$$\bigotimes_{i=1}^{|w^{(k)}|} \tilde{c}\Big(\, d(w^{(k)},i)\Big) \quad = \quad \tilde{c}\Big(\bigotimes_{i=1}^{|w^{(k)}|} d(w^{(k)},i)\Big) \quad = \quad \tilde{c}(w^{(k)})$$

shows that the unique decomposition of $\tilde{c}(w^{(k)}) \in \mathbb{Z}^{*n}$ as in Definition 3.23 is given by

$$|\tilde{c}(w^{(k)})| \;=\; |w^{(k)}|, \qquad d\Big(\,\tilde{c}(w^{(k)}),i\Big) \quad = \quad \tilde{c}\Big(\, d(w^{(k)},i)\Big)$$

Next, let r be a function — not a homomorphism — defined by

$$\begin{array}{rcccl} r & : & \mathbb{Z}^{*n} & \to & \mathbb{N}^{*2n} \\ & : & z_i & \mapsto & z_i \\ & : & z_i^* & \mapsto & z_{n+i} \\ & : & x & \mapsto & \bigotimes_{i=1}^{|x|} r\Big(d(x,i) \Big) \end{array}$$

Then for $1 \leq i \leq n$,

$$r\tilde{c}(z_i) \;=\; r(z_i) \;=\; z_i, \qquad r\tilde{c}(z_{n+i}) \;=\; r(z_i^*) \;=\; z_{n+i}$$

and so it follows that

$$r\tilde{c}(w^{(k)}) \quad = \quad \bigotimes_{i=1}^{|w^{(k)}|} r\tilde{c}\Big(d(w^{(k)}, i) \Big) \quad = \quad \bigotimes_{i=1}^{|w^{(k)}|} d(w^{(k)}, i) \quad = \quad w^{(k)}$$

Finally, notice that the exact same logic as we've used above will work for $w'^{(k')}$ as well, so that $r\tilde{c}(w'^{(k')}) = w'^{(k')}$. Therefore,

$$\begin{array}{rcccccc} w_k & = & r\tilde{c}(w^{(k)}) & = & r\tilde{c}\tilde{I}(u^{(k-1)}) & = & r\tilde{c}\tilde{\delta}(u^{(k-1)}) \\ & = & r\tilde{c}(w^{(k-1)}) & = & \vdots & = & \vdots \\ & \vdots & & & & & \\ & = & r\tilde{c}(w^{(1)}) & & & & \\ & = & r\tilde{c}(w) & & & & \\ & = & r\tilde{c}(w') & & & & \\ & = & r\tilde{c}(w'^{(1)}) & = & r\tilde{c}\tilde{\delta}(u'^{(1)}) & = & r\tilde{c}\tilde{I}(u'^{(1)}) \\ & = & r\tilde{c}(w'^{(2)}) & = & \vdots & = & \vdots \\ & \vdots & & & & & \\ & = & r\tilde{c}(w'^{(k')}) & & & & \\ & = & w'^{(k')} & & & & \end{array}$$

as required. $\square$

It is this special property — shared by all w, w' for which $\tilde{c}(w) = \tilde{c}(w')$ — that will now let us prove that the coequaliser $\tilde{c}$ satisfies the condition which we couldn't prove about the cokernel c. In other words, with Propositions 3.24 and 3.25 we can now contruct a valid EG-action on the monoidal category $\tilde{C}$.

Proposition 3.26. *There is a unique action $\alpha_{\tilde{C}}$ making the category $\tilde{C}$ into* EG*-algebra and the functor $\tilde{c} : \mathbb{G}_{2n} \to \tilde{C}$ into a map of* EG*-algebras.*

Proof. We will try to affix an action to $\tilde{C}$ in the same way we thought about doing with the category C. In order for the functor $\tilde{c} : \mathbb{G}_{2n} \to \tilde{C}$ to be an EG-algebra map with respect to some $\alpha_{\tilde{C}}$, it must satisfy

$$\tilde{c}\Big(\alpha_{\mathbb{G}_{2n}}(g \,;\, f_1, ..., f_m) \Big) \quad = \quad \alpha_{\tilde{C}}(g \,;\, \tilde{c}(f_1), ..., \tilde{c}(f_m))$$

for all morphisms $f_1, ..., f_m$ in $\mathbb{G}_{2n}$, though given Lemma 1.27 it will be enough to have

$$\begin{aligned} \tilde{c}\Big(\alpha_{\mathbb{G}_{2n}}(g \,;\, \mathrm{id}_{w_1}, ..., \mathrm{id}_{w_m}) \Big) &= \alpha_{\tilde{C}}(g \,;\, \tilde{c}(\mathrm{id}_{w_1}), ..., \tilde{c}(\mathrm{id}_{w_m})) \\ &= \alpha_{\tilde{C}}(g \,;\, \mathrm{id}_{\tilde{c}(w_1)}, ..., \mathrm{id}_{\tilde{c}(w_m)}) \end{aligned}$$

But since we know from Corollary 3.21 that $\tilde{c}$ is surjective, this condition will actually suffice as a definition for $\alpha_{\tilde{C}}$, provided that we can prove it to be well-defined.

To that end, let $w_1, ..., w_m$ and $w'_1, ..., w'_m$ be any two sequences of objects in $\mathbb{G}_{2n}$ that have $\tilde{c}(w_i) = \tilde{c}(w'_i)$ for all $1 \leq i \leq m$. Then using Proposition 3.24, let $w_i^{(1)}, ..., w_i^{(k)}$ and $u_i^{(1)}, ..., u_i^{(k)}$ be the sequences we get from each w_i and $w_i'^{(1)}, ..., w_i'^{(k')}$, $u_i'^{(1)}, ..., u_i'^{(k')}$ those we get from w'_i. It follows that

$$\begin{aligned} \tilde{c}\Big(\alpha_{\mathbb{G}_{2n}}(g \,;\, \mathrm{id}_{w_1^{(i)}}, ..., \mathrm{id}_{w_m^{(i)}}) &= \tilde{c}\Big(\alpha_{\mathbb{G}_{2n}}(g \,;\, \mathrm{id}_{\tilde{\delta}(u_1^{(i)})}, ..., \mathrm{id}_{\tilde{\delta}(u_m^{(i)})}) \\ &= \tilde{c}\tilde{\delta}\Big(\alpha_{\mathbb{G}_{2n}}(g \,;\, \mathrm{id}_{u_1^{(i)}}, ..., \mathrm{id}_{u_m^{(i)}}) \\ &= \tilde{c}\tilde{I}\Big(\alpha_{\mathbb{G}_{2n}}(g \,;\, \mathrm{id}_{u_1^{(i)}}, ..., \mathrm{id}_{u_m^{(i)}}) \\ &= \tilde{c}\Big(\alpha_{\mathbb{G}_{2n}}(g \,;\, \mathrm{id}_{\tilde{I}(u_1^{(i)})}, ..., \mathrm{id}_{\tilde{I}(u_m^{(i)})}) \\ &= \tilde{c}\Big(\alpha_{\mathbb{G}_{2n}}(g \,;\, \mathrm{id}_{w_1^{(i+1)}}, ..., \mathrm{id}_{w_m^{(i+1)}}) \end{aligned}$$

and likewise for the w'. Thus from Proposition 3.25 we can conclude that

$$\begin{aligned} \tilde{c}\Big(\alpha_{\mathbb{G}_{2n}}(g \,;\, \mathrm{id}_{w_1}, ..., \mathrm{id}_{w_m}) &= \tilde{c}\Big(\alpha_{\mathbb{G}_{2n}}(g \,;\, \mathrm{id}_{w_1^{(1)}}, ..., \mathrm{id}_{w_m^{(1)}}) \\ &= \tilde{c}\Big(\alpha_{\mathbb{G}_{2n}}(g \,;\, \mathrm{id}_{w_1^{(2)}}, ..., \mathrm{id}_{w_m^{(2)}}) \\ &\;\vdots \\ &= \tilde{c}\Big(\alpha_{\mathbb{G}_{2n}}(g \,;\, \mathrm{id}_{w_1^{(k)}}, ..., \mathrm{id}_{w_m^{(k)}}) \\ &= \tilde{c}\Big(\alpha_{\mathbb{G}_{2n}}(g \,;\, \mathrm{id}_{w_1'^{(k')}}, ..., \mathrm{id}_{w_m'^{(k')}}) \\ &\;\vdots \\ &= \tilde{c}\Big(\alpha_{\mathbb{G}_{2n}}(g \,;\, \mathrm{id}_{w'_1}, ..., \mathrm{id}_{w'_m}) \end{aligned}$$

Thus the value of $\alpha_{\tilde{C}}(g; \mathrm{id}_{\tilde{c}(w_1)}, ..., \mathrm{id}_{\tilde{c}(w_m)})$ we gave earlier does not depend on our particular choice of w_i. Therefore $\alpha_{\tilde{C}}$ is indeed a well-defined EG-action on $\tilde{C}$, and the coequaliser $\tilde{c}$ from MonCat is a map of EG-algebras with respect to $\alpha_{\tilde{C}}$. ☐

3.5 Extracting $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$ from $\mathbb{G}_{2n}$

We are now finally ready to address problem 1 from the end of the previous chapter: how can we deal with the fact that our adjunction $\mathrm{M}(\_)^{\mathrm{gp,ab}} \dashv C$ involves monoidal categories rather than full EG-algebras? It turns out that this is all we really needed, as despite us originally conceiving of $L\mathbb{G}_n$ as a colimit in EGAlg$_S$ it can equally be viewed as a slightly more complicated colimit in MonCat.

Proposition 3.27. *The coequaliser functor $\tilde{c} : \mathbb{G}_{2n} \to \tilde{C}$ defined in Definition 3.19 is isomorphic as a map of* EG*-algebras to $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$, the cokernel of δ in* EGAlg$_S$*.*

Proof. First, consider what we know of the functor $\tilde{c}$. By definition it has the property for any $1 \leq i \leq 2n$

$$\tilde{c}\delta(z_i) \quad = \quad \tilde{c}\tilde{\delta}(z_{2n+i}) \quad = \quad \tilde{c}\tilde{I}(z_{2n+i}) \quad = \quad \tilde{c}(I) \quad = \quad I$$

so that $\tilde{c} \circ \delta$ is the constant functor on the unit object I. Moreover, given what we saw in **??** we know that $\tilde{c}$ is map of EG-algebras which has this property. But the cokernel map q is universal amongst maps like these, and so it follows that there must exist a unique map of EG-algebras $u : L\mathbb{G}_n \to \tilde{C}$ factoring $\tilde{c}$ through q. Conversely, the algebra map q is a monoidal functor for which $q \circ \delta = I$, while $\tilde{c}$ is the universal map in MonCat with this property. Thus there also exists a unique monoidal functor $v : \tilde{C} \to L\mathbb{G}_n$ which factors q through $\tilde{c}$.

Putting these facts together with the surjectivity of q and $\tilde{c}$ (from Corollaries 3.5 and 3.21 respectively), we can conclude that the maps u and u' form a isomorphism of monoidal categories:

$$\begin{aligned} u \circ v \circ \tilde{c} \;&=\; u \circ q \;=\; \tilde{c} \quad &\Longrightarrow \quad u \circ v \;&=\; \mathrm{id}_{\tilde{C}} \\ v \circ u \circ q \;&=\; v \circ \tilde{c} \;=\; q \quad &\Longrightarrow \quad v \circ u \;&=\; \mathrm{id}_{L\mathbb{G}_n} \end{aligned}$$

Furthermore, not only is u an algebra map, but v is one too. To see this, use the surjectivity of $\tilde{c}$ to find for any morphism f_i in $\tilde{C}$ a corresponding f_i' in $\mathbb{G}_{2n}$ with

$\tilde{c}(f_i') = f_i$. Then

$$\begin{aligned} v\Big(\alpha_{\tilde{C}}(\,g\,;\,f_1,...,f_m\,)\Big) &= v\Big(\alpha_{\tilde{C}}(\,g\,;\,\tilde{c}(f_1'),...,\tilde{c}(f_m')\,)\Big) \\ &= v\tilde{c}\Big(\alpha_{\mathbb{G}_{2n}}(\,g\,;\,f_1',...,f_m'\,)\Big) \\ &= q\Big(\alpha_{\mathbb{G}_{2n}}(\,g\,;\,f_1',...,f_m'\,)\Big) \\ &= \alpha_{L\mathbb{G}_n}\Big(\,g\,;\,q(f_1'),...,q(f_m')\Big) \\ &= \alpha_{L\mathbb{G}_n}\Big(\,g\,;\,v\tilde{c}(f_1'),...,v\tilde{c}(f_m')\Big) \\ &= \alpha_{L\mathbb{G}_n}\Big(\,g\,;\,v(f_1),...,v(f_m)\Big) \end{aligned}$$

Therefore (u, v) is also an isomorphism of EG-algebras $\tilde{C} \cong L\mathbb{G}_n$, and up to this isomorphism the algebra maps q and $\tilde{c}$ are the same. □

With our newfound ability to express the map $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$ as a colimit of monoidal categories, we can now set about using the adjunction from Proposition 2.24 to calculate $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$. The most obvious way to do this is to mimic what we did in Proposition 3.2 — apply the left adjoint functor to q and then commute it with the colimit to get a formula in terms of the known monoid $\mathrm{Mor}(\mathbb{G}_{2n})$.

Proposition 3.28. *Let Δ be the subgroup of $\mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}}$ generated by elements of the form*

$$\mathrm{M}(\tilde{\delta})^{\mathrm{gp,ab}}(f) \otimes \mathrm{M}(\tilde{I})^{\mathrm{gp,ab}}(f)^*, \qquad\qquad f \in \mathrm{M}(\mathbb{G}_{4n})^{\mathrm{gp,ab}}$$

Then the abelianisation of the group completion of the collapsed morphisms of $L\mathbb{G}_n$ is

$$\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \quad = \quad \mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \big/ \Delta$$

with $\mathrm{M}(q)^{\mathrm{gp,ab}}$ acting as the appropriate quotient map.

Proof. From Proposition 2.24, we know that $\mathrm{M}(\_)^{\mathrm{gp,ab}} : \mathrm{MonCat} \to \mathrm{Ab}$ is a left adjoint functor. This means that it preserves all colimits in MonCat, including the coequaliser use to define $\tilde{c}$, which from Proposition 3.27 we now know is really q. Thus

$$\mathrm{coeq}\Big(\mathrm{M}(\tilde{\delta})^{\mathrm{gp,ab}}, \mathrm{M}(\tilde{I})^{\mathrm{gp,ab}}\Big) \quad = \quad \mathrm{M}\Big(\mathrm{coeq}(\tilde{\delta}, \tilde{I})\Big)^{\mathrm{gp,ab}} \quad = \quad \mathrm{M}(q)^{\mathrm{gp,ab}}$$

or in other words, the following is a coequaliser diagram in the category of abelian groups:

$$\begin{tikzcd} \mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \arrow[r, shift left, "\mathrm{M}(\tilde{\delta})^{\mathrm{gp,ab}}"] \arrow[r, shift right, "\mathrm{M}(\tilde{I})^{\mathrm{gp,ab}}"'] & \mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \arrow[r, "\mathrm{M}(c)^{\mathrm{gp,ab}}"] & \mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \end{tikzcd}$$

But the coequaliser of two abelian group homomorphisms is just the quotient of their common target by the image of their difference. Hence in this case we have

$$\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \quad = \quad \mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \Big/ \mathrm{im}\Big(\,\mathrm{M}(\tilde{\delta})^{\mathrm{gp,ab}} - \mathrm{M}(\tilde{I})^{\mathrm{gp,ab}}\Big) \quad = \quad \mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \Big/ \Delta$$

□

Notice that the subgroup Δ contains all of $\mathrm{im}(\mathrm{M}(\delta)^{\mathrm{gp,ab}})$, but in general these two groups are not the same. This means that the effort we put into avoiding the naive mistake we could have made at the end of Section 3.2 was indeed worth it.

Now, at some point later on we will want to actually evaluate the quotient in for particular values of action operad G. This would be fairly tricky without an explicit desciption of the elements of Δ, so we need to take a moment to think about what we really mean by $\mathrm{M}(\tilde{\delta})^{\mathrm{gp,ab}}(f) \otimes \mathrm{M}(\tilde{I})^{\mathrm{gp,ab}}(f)^*$.

Lemma 3.29. *Δ is the subgroup of $\mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}}$ whose elements are tensor products of equivalence classes*

$$\begin{gathered}
\Big[\,\alpha_{\mathbb{G}_{2n}}\Big(\,\mu(\,g\,;\,e_{|\tilde{\delta}(x_1)|},\dots,e_{|\tilde{\delta}(x_m)|}\,)\,;\,\mathrm{id}_{x'_1},\dots,\mathrm{id}_{x'_{m'}}\,\Big)\,\Big] \\
\otimes \\
\Big[\,\alpha_{\mathbb{G}_{2n}}\Big(\,\mu(\,g\,;\,e_{|\tilde{I}(x_1)|},\dots,e_{|\tilde{I}(x_m)|}\,)\,;\,\mathrm{id}_{x''_1},\dots,\mathrm{id}_{x''_{m''}}\,\Big)\,\Big]^*
\end{gathered}$$

where $g \in G(m)$, the x_i are generators of $\mathbb{N}^{*4n}$, the x'_i, x''_i are generators of $\mathbb{N}^{*2n}$, and

$$\begin{aligned}
\tilde{\delta}(x_1 \otimes \dots \otimes x_m) \quad &= \quad x'_1 \otimes \dots \otimes x'_{m'} \\
\tilde{I}(x_1 \otimes \dots \otimes x_m) \quad &= \quad x''_1 \otimes \dots \otimes x''_{m''}
\end{aligned}$$

Proof. Let f be an element of $\mathrm{M}(\mathbb{G}_{4n})^{\mathrm{gp,ab}}$. By defintion this means that f is an equivalence class of morphisms from $\mathbb{G}_{4n}$, and so by Lemma 1.27 there must exist $g \in G(m)$ and $x_1, \dots, x_m \in \{z_1, \dots, z_{4n}\}$ for which

$$f \quad = \quad [\,\alpha_{\mathbb{G}_{4n}}(g; \mathrm{id}_{x_1}, \dots, \mathrm{id}_{x_m})\,]$$

Thus

$$\begin{aligned}
\mathrm{M}(\tilde{\delta})^{\mathrm{gp,ab}}(f) \quad &= \quad \mathrm{M}(\tilde{\delta})^{\mathrm{gp,ab}}\Big(\,[\,\alpha_{\mathbb{G}_{4n}}(g; \mathrm{id}_{x_1}, \dots, \mathrm{id}_{x_m})\,]\,\Big) \\
&= \quad \Big[\,\tilde{\delta}\Big(\,\alpha_{\mathbb{G}_{4n}}(g; \mathrm{id}_{x_1}, \dots, \mathrm{id}_{x_m})\,\Big)\,\Big] \\
&= \quad [\,\alpha_{\mathbb{G}_{2n}}(g; \mathrm{id}_{\tilde{\delta}(x_1)}, \dots, \mathrm{id}_{\tilde{\delta}(x_m)})\,]
\end{aligned}$$

But again using Lemma 1.27, we know it must be possible to express the action morphism $\alpha_{\mathbb{G}_{2n}}(g; \mathrm{id}_{\tilde{\delta}(x_1)}, \dots, \mathrm{id}_{\tilde{\delta}(x_m)})$ as an action morphism on the identities of generators.

Since the source of this map is

$$\tilde{\delta}(x_1) \otimes ... \otimes \tilde{\delta}(x_m) \quad = \quad \tilde{\delta}(x_1 \otimes ... \otimes x_m) \quad = \quad x'_1 \otimes ... \otimes x'_{m'}$$

clearly the x'_i are the generators we want, and so by expanding the $\tilde{\delta}(x_i)$ as tensor products of these we find that

$$[\,\alpha_{\mathbb{G}_{2n}}(g; \mathrm{id}_{\tilde{\delta}(x_1)}, ..., \mathrm{id}_{\tilde{\delta}(x_m)})\,] \quad = \quad \Big[\,\alpha_{\mathbb{G}_{2n}}\Big(\,\mu(\,g\,;\,e_{|\tilde{\delta}(x_1)|}, ..., e_{|\tilde{\delta}(x_m)|}\,)\,;\,\mathrm{id}_{x'_1}, ..., \mathrm{id}_{x'_{m'}}\,\Big)\Big]$$

For analagous reasons we also get

$$\begin{aligned} \mathrm{M}(\tilde{I})^{\mathrm{gp,ab}}(f) &= [\,\alpha_{\mathbb{G}_{2n}}(g; \mathrm{id}_{\tilde{I}(x_1)}, ..., \mathrm{id}_{\tilde{I}(x_m)})\,] \\ &= \Big[\,\alpha_{\mathbb{G}_{2n}}\Big(\,\mu(\,g\,;\,e_{|\tilde{I}(x_1)|}, ..., e_{|\tilde{I}(x_m)|}\,)\,;\,\mathrm{id}_{x''_1}, ..., \mathrm{id}_{x''_{m''}}\,\Big)\Big] \end{aligned}$$

and using these equations the lemma follows immediately from the definition of Δ. $\square$

Chapter 4

Morphisms of free invertible algebras

The goal of this chapter will be to show that we can reconstruct all of the morphisms of $L\mathbb{G}_n$ from the abelian group $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$, and therefore that we can actually use the adjunction from Proposition 2.24 to help find a description of the free $\mathrm{E}G$-algebra on n invertible objects.

The first step towards this goal will involve splitting $\mathrm{Mor}(L\mathbb{G}_n)$ up as the product of two other monoids. The first of these will encode all of the possible combinations of source and target data for morphisms in $L\mathbb{G}_n$, while the second will just be the endomorphisms of the unit object, $L\mathbb{G}_n(I, I)$. In other words, we will see that the monoid $\mathrm{Mor}(L\mathbb{G}_n)$ can be broken down into a context where source and target are the only thing that matters, and another where they are irrelevant.

Once we have done this, we can then use the fact that $L\mathbb{G}_n(I, I)$ is always an abelian group to rewrite $\mathrm{Mor}(L\mathbb{G}_n)$ in terms of its abelian group completion, $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$. This is not quite the same thing as $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$, but they are close enough that we can find a simple equation linking the two, which will in turn allow us to frame the former in terms of the quotient of $\mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}}$ we described last chapter. All together, this will constitute an expression for $\mathrm{Mor}(L\mathbb{G}_n)$ that is built up of pieces which we know how to calculate.

4.1 Sources and targets in $L\mathbb{G}_n$

To get things started, we will spend this section considering the source and target information of morphisms in $L\mathbb{G}_n$.

Definition 4.1. For any EG-algebra X, denote by $s : \mathrm{Mor}(X) \to \mathrm{Ob}(X)$ and $t : \mathrm{Mor}(X) \to \mathrm{Ob}X)$ the monoid homomorphisms which send each morphism of X to its source and target, respectively. That is,

$$s(\, f : x \to y) \;=\; x, \qquad t(\, f : x \to y) \;=\; y$$

If we use the universal property of products, we can combine these source and target homomorphisms into a single map, $s \times t : \mathrm{Mor}(X) \to \mathrm{Ob}(X) \times \mathrm{Ob}(X)$. The monoid we are interested in finding is the image $L\mathbb{G}_n$ under its instance of this map.

Lemma 4.2. *Let X be an* EG*-algebra, and $s \times t : \mathrm{Mor}(X) \to \mathrm{Ob}(X)^2$ the map built from s and t using the universal property of products. Then the image of this map is*

$$(s \times t)(X) \;=\; \mathrm{Ob}(X) \times_{\pi_0(X)} \mathrm{Ob}(X)$$

where this pullback is taken over the canonical maps sending objects of X to their connected components:

$$\begin{tikzcd}
\mathrm{Ob}(X) \times_{\pi_0(X)} \mathrm{Ob}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ob}(X) \arrow[d, "{[_]}"] \\
\mathrm{Ob}(X) \arrow[r, "{[_]}"] & \pi_0(X)
\end{tikzcd}$$

Proof. By definition, there exists a morphism $f : x \to y$ between objects x, y of X if and only if they are in the same connected component, $[x] = [y]$. Thus

$$\begin{aligned}
(x, y) \;\in\; (s \times t)(X) \quad &\Longleftrightarrow \quad \exists\, f : \quad s(f) \;=\; x, \quad t(f) \;=\; y \\
&\Longleftrightarrow \quad [x] = [y] \\
&\Longleftrightarrow \quad (x, y) \;\in\; \mathrm{Ob}(X) \times_{\pi_0(X)} \mathrm{Ob}(X)
\end{aligned}$$

as required. $\square$

Recalling Lemma 1.26, Propositions 1.28 and 2.14, and Corollary 2.20, we can immediately conclude the following:

Corollary 4.3.

$$(s \times t)(\mathbb{G}_n) \quad = \quad \begin{cases} \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} & \textit{if } G \textit{ is crossed} \\ \mathbb{N}^{*n} & \textit{otherwise} \end{cases}$$

$$(s \times t)(L\mathbb{G}_n) \quad = \quad \begin{cases} \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} & \textit{if } G \textit{ is crossed} \\ \mathbb{Z}^{*n} & \textit{otherwise} \end{cases}$$

where the pullbacks are taken over the quotients of abelianisation for $(\mathbb{N}^{*n})^{\mathrm{ab}} = \mathbb{N}^n$ *and* $(\mathbb{Z}^{*n})^{\mathrm{ab}} = \mathbb{Z}^n$ *respectively.*

Next, we want to show that this $(s \times t)(L\mathbb{G}_n)$ we have described is in fact a submonoid of $\mathrm{Mor}(L\mathbb{G}_n)$. This is a little tricky though, since we don't currently know what the morphisms of $L\mathbb{G}_n$ even are. We will sidestep this problem by first proving the analogous statement for all $\mathbb{G}_n$, and then recovering the $L\mathbb{G}_n$ version from it later.

Now, by Lemma 1.31 we know that wanting $(s\times t)(\mathbb{G}_n)$ to be a submonoid of $\mathrm{Mor}(\mathbb{G}_n)$ is the same as asking if we can find an injective homomorphism $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \to G \times_{\mathbb{N}} \mathbb{N}^{*n}$, assuming G is crossed, or $\mathbb{N}^{*n} \to G \times_{\mathbb{N}} \mathbb{N}^{*n}$ if it is not. The latter case is pretty obvious, so we'll focus on crossed G for the moment. Creating a injective *function* $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \to G \times_{\mathbb{N}} \mathbb{N}^{*n}$ is not especially hard. For any pair $(w, w') \in \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$, the image of w and w' in the abelian group $\mathbb{N}^n$ is the same, which is to say that if $x_1, ..., x_m \in \{z_1, ..., z_n\}$ are the collection of generators for which

$$w \quad = \quad x_1 \otimes ... \otimes x_m$$

and there exists at least one permutation $\sigma \in S_m$ such that

$$w' \quad = \quad x_{\sigma(1)} \otimes ... \otimes x_{\sigma(m)}$$

Then since the underlying permutation maps $\pi : G(m) \to \mathrm{S}_m$ of a crossed action operad G are all surjective, we can always find an element of $g \in G(m)$ for which $\pi(g) = \sigma$. Thus in order to make our injective function all we need to do is make a choice $g =: \rho(w, w')$ like this to represent each (w, w'), and then set

$$\begin{aligned} \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} &\to G \times_{\mathbb{N}} \mathbb{N}^{*n} \\ (w, w') &\mapsto (\, \rho(w, w'), w \,) \end{aligned}$$

Injectivity follows because given a specific (g, w), the only element that could map onto it is $(w, \pi(g)(w))$.

So how do we know if we can choose these representatives $\rho(w, w')$ in such a way that the resulting function i is also a monoid homomorphism? If we could find a presentation of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ in terms of generators and relations then this would help a little, since we would only need to pick a $\rho(z, z')$ for each generator (z, z'), and then define all other $\rho(w, w')$ by way of tensor products:

$$\rho(v \otimes w, v' \otimes w') \quad = \quad \rho(v, v') \otimes \rho(w, w')$$

But then we would still need make sure that our choice of $\rho(z, z')$ obeyed the necessary relations on the generators of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$. Luckily for us though, this turns out to be no problem at all.

Proposition 4.4. $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ *is a free monoid.*

Proof. Given an element (w, w') of the monoid $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$, let $D(w, w')$ be the following set:

$$D(w, w') = \left\{ (u, u'), (v, v') \in \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \quad : \quad \begin{array}{rcl} (w, w') & = & (u, u') \otimes (v, v'), \\ (u, u') & \neq & (I, I), \\ (v, v') & \neq & (I, I) \end{array} \right\}$$

We can use these sets to recursively define a decomposition of any element (w, w') as a product of other elements of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$. Specifically, if $D(w, w')$ is empty then we say that the decomposition of (w, w') is just (w, w') itself, and otherwise we choose any $\Big((u, u'), (v, v') \Big) \in D(w, w')$ and say that the decomposition of (w, w') is the concatenation of the decomposition of (u, u') with the decomposition of (v, v'). Note that when we look at the lengths of these elements, as defined in Definition 1.30, $|u|$ and $|v|$ are always strictly smaller that $|w|$, and any strictly decreasing sequence of natural numbers is finite, so this process definitely terminates.,

Of course, we need to check that this decomposition of (w, w') is well-defined, which amounts to checking that the choice of $(u, u'), (v, v')$ we make at each stage won't change the eventual output. To that end, suppose for the sake of contradiction that $(u_1, u'_1), ..., (u_m, u'_m)$ and $(v_1, v'_1), ..., (v'_m, v'_{m'})$ are distinct decompositions of (w, w') we could arrive at using the above process. Notice that we can assume without loss of generality that $|u_1| < |v_1|$. If instead $|u_1| > |w_1|$, we can just swap the labels of the sequences, and if $|u_1| = |v_1|$ then we can just discard those elements and instead consider the decompositions $(u_2, u'_2), ..., (u_m, u'_m)$ and $(v_2, v'_2), ..., (v'_m, v'_{m'})$ of $(u_2, u'_2) \otimes ... \otimes$

$(u_m, u'_m) = (v_2, v'_2) \otimes ... \otimes (v'_m, v'_{m'})$. Since $(u_1, u'_1), ..., (u_m, u'_m)$ and $(v_1, v'_1), ..., (v'_m, v'_{m'})$ were distinct decompositions of (w, w'), in this way we will eventually reach some subsequences whose first elements are different; once we have, we can relabel them so that $|u_1| < |v_1|$.

Then by definition,

$$u_1 \otimes \Big(\bigotimes_{i=2}^{m} u_i \Big) \;=\; w \;=\; v_1 \otimes \Big(\bigotimes_{i=2}^{m'} v_i \Big)$$

But $w, u_1, v_1, \bigotimes_{i=2}^{m} u_i, \bigotimes_{i=2}^{m'} v_i$ are all elements of $\mathbb{N}^{*n}$, which is a free monoid, and so they each have a unique decomposition as products of the generators $\{z_1, ..., z_n\}$, and these all respect tensor products. Therefore, since $|u_1| < |v_1|$, there must exist some element a of $\mathbb{N}^{*n}$ such that

$$w \;=\; u_1 \otimes a \otimes \Big(\bigotimes_{i=2}^{m'} v_i \Big) \qquad \Longrightarrow \qquad v_1 \;=\; u_1 \otimes a$$

Since

$$|u'_1| \;=\; |u_1| \;<\; |v_1| \;=\; |v'_1|$$

we can also use exactly the same reasoning to find an a' in $\mathbb{N}^{*n}$ with $v'_1 = u'_1 \otimes a'$, and hence $(v_1, v'_1) = (u_1, u'_1) \otimes (a, a')$. Moreover, this (a, a') is an element of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$, because

$$\begin{array}{rcccl} & v_1 & = & u_1 \otimes a & \\ \Longrightarrow & [v_1] & = & [u_1 \otimes a] & = \; [u_1] + [a] \\ \\ & v'_1 & = & u'_1 \otimes a' & \\ \Longrightarrow & [v'_1] & = & [u'_1 \otimes a'] & = \; [u'_1] + [a'] \\ \\ \Longrightarrow & [a] & = & [v_1] - [u_1] & \\ & & & [v'_1] - [u'_1] & = \; [a'] \end{array}$$

In other words, we have shown that the pair $\Big((u_1, u'_1)(a, a') \Big)$ is an element of $D(v_1, v'_1)$. But by assumption $(v_1, v'_1), ..., (v'_m, v'_{m'})$ was a decomposition of (w, w'), and hence the $D(v_i, v'_i)$ were supposed to be empty for each i, since that is when the decomposition finding process terminates. This is a contradiction, and hence our assumption that $(u_1, u'_1), ..., (u_m, u'_m)$ and $(v_1, v'_1), ..., (v'_m, v'_{m'})$ were distinct decompositions of (w, w') is false. Therefore, each (w, w') in $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ has a unique decomposition in terms of elements (v_i, v_i') for which $D(v_i, v_i')$ is empty, and so $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ is the free monoid whose generators are all such elements. $\square$

It follows immediately from this that our earlier contruction of an injective function $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \to G \times_{\mathbb{N}} \mathbb{N}^{*n}$ can always be extended to be an inclusion of monoids.

Proposition 4.5. *$(s \times t)(\mathbb{G}_n)$ is (isomorphic to) a submonoid of* $\mathrm{Mor}(\mathbb{G}_n)$

Proof. First, assume that the action operad G is non-crossed. Then there exists an obvious injective monoid homomorphism

$$\begin{array}{rcrcl} i & : & (s \times t)(\mathbb{G}_n) & \to & \mathrm{Mor}(\mathbb{G}_n) \\ & : & \mathbb{N}^{*n} & \to & G \times_{\mathbb{N}} \mathbb{N}^{*n} \\ & : & w & \mapsto & (\, e_{|w|}, w\,) \end{array}$$

The homomorphism property follows from the fact that the length $|w|$ defined in Definition 1.30 is itself a homomorphism, so $|w \otimes w'| = |w| + |w'|$. Thus $(s \times t)(\mathbb{G}_n) \subseteq \mathrm{Mor}(\mathbb{G}_n)$ for non-crossed G.

Now assume that G is crossed. For each generator (z, z') of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$, the words $z, z' \in \mathbb{N}^{*n}$ are permuations of each other, and the map $\pi : G(|z|) \to \mathrm{S}_|z|$ is surjective, and so there must be some $g \in G(|z|)$ with the property that $\pi(g)(z) = z'$. Choose from among these a representative element, which we'll call $\rho(z, z')$. Then because $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ is a free monoid by Proposition 4.4, these choices will extend to a well-defined, monoid homomorphism

$$\rho : \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \longrightarrow G$$

which preserves underlying permutation. This map will now possess the property that

$$\pi(\rho(w, w'))(w) \quad = \quad w'$$

for any $(w, w') \in \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$, not just the generators. Then from ρ we'll define the homomorphism i to be

$$\begin{array}{rcrcl} i & : & (s \times t)(\mathbb{G}_n) & \to & \mathrm{Mor}(\mathbb{G}_n) \\ & : & \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} & \to & G \times_{\mathbb{N}} \mathbb{N}^{*n} \\ & : & (w, w') & \mapsto & (\, \rho(w, w'), w\,) \end{array}$$

Moreover, for any two elements (v, v'), (w, w') of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ we'll have

$$(\,\rho(v,v'),v\,) \quad = \quad (\,\rho(w,w'),w\,) \quad \Longrightarrow \quad \begin{array}{rcl} \rho(v,v') & = & \rho(w,w') \\ v & = & w \\ v' & = & \pi(\rho(v,v'))(v) \\ & = & \pi(\rho(w,w'))(w) \\ & = & w' \end{array}$$

and thus i is injective. Therefore the image of this i is a submonoid of $G \times_{\mathbb{N}} \mathbb{N}^{*n}$ which is isomorphic to $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$, so again $(s \times t)(\mathbb{G}_n) \subseteq \mathrm{Mor}(\mathbb{G}_n)$ as required. □

In other words, this result says that the source and target data of $\mathbb{G}_n$ is isomorphic to the monoid made up of action morphisms

$$\alpha\Big(\,\rho(x_1 \otimes ... \otimes x_m, x_{\sigma(1)} \otimes ... \otimes x_{\sigma(1)})\,;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,\Big)$$

when G is crossed, and

$$\alpha(e_m; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) \quad = \quad \mathrm{id}_{x_1 \otimes ... \otimes x_m}$$

otherwise, for $\sigma \in S_m$, $x_1, ..., x_m \in \{z_1, ..., z_n\}$. Now, in theory the map $\rho : \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \longrightarrow G$ that we use to choose representatives can be any valid homomorphism between those monoids for which

$$\pi(\rho(w,w'))(w) \quad = \quad w'$$

but later on we will be able to make things easier on ourselves by making a more systematic choice.

So now we have shown that $(s \times t)(\mathbb{G}_n)$ is a submonoid of $\mathrm{Mor}(\mathbb{G}_n)$, but what we were really interested in is whether or not $(s \times t)(\mathbb{G}_n)$ is a submonoid of $\mathrm{Mor}(\mathbb{G}_n)$. To recover the latter result from the former, we will use our cokernel map $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$. In particular, the surjectivity of q combined with the case $(s \times t)(\mathbb{G}_{2n}) \subseteq \mathrm{Mor}(\mathbb{G}_{2n})$ from Proposition 4.5, immediately gives us what we need.

Corollary 4.6. *$(s \times t)(L\mathbb{G}_n)$ is (isomorphic to) a submonoid of* $\mathrm{Mor}(L\mathbb{G}_n)$

Proof. Let $i : (s \times t)(\mathbb{G}_{2n}) \hookrightarrow \mathrm{Mor}(\mathbb{G}_{2n})$ be an inclusion which allows us to view $(s \times t)(\mathbb{G}_{2n})$ as a submonoid of $\mathrm{Mor}(\mathbb{G}_{2n})$, as in Proposition 4.5. Also, let $\mathrm{Mor}(q) : \mathrm{Mor}(\mathbb{G}_{2n}) \to \mathrm{Mor}(L\mathbb{G}_n)$ the restriction of the cokernel map $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$ onto morphisms. Then the image of the composite of these two homomorphisms,

$$\mathrm{im}\Big(\,\mathrm{Mor}(q)\circ i\,\Big) \quad = \quad q\Big(\,\mathrm{im}(i)\,\Big) \quad \cong \quad q\Big(\,(s\times t)(\mathbb{G}_{2n})\,\Big)$$

is clearly a submonoid of $\mathrm{Mor}(L\mathbb{G}_n)$.

But by Corollary 3.5 q is a surjective functor. This means that there can exist a morphism $w\to v$ in $L\mathbb{G}_n$ if and only if there exists at least one morphism $w'\to v'$ in $\mathbb{G}_{2n}$, for some w',v' which have $q(w')=w$ and $q(v')=v$. In other words,

$$q\Big(\,(s\times t)(\mathbb{G}_{2n})\,\Big) \;=\; (s\times t)(L\mathbb{G}_n)$$

and therefore the monoid $\mathrm{im}\Big(\,\mathrm{Mor}(q)\circ i\,\Big)$ that we saw above is really a submonoid of $\mathrm{Mor}(L\mathbb{G}_n)$ isomorphic to $(s\times t)(L\mathbb{G}_n)$, as required. □

4.2 Unit endomorphisms of $L\mathbb{G}_n$

To help us understand $\mathrm{Mor}(L\mathbb{G}_n)$, we decided to break it down into two smaller pieces. The first of these was the source/target data $(s\times t)(L\mathbb{G}_n)$, which we explored in the previous section. The other piece that we now have to consider is the monoid of unit endomorphisms, $L\mathbb{G}_n(I,I)$.

This is a particularly important submonoid of the morphisms $\mathrm{Mor}(L\mathbb{G}_n)$, since it is the only submonoid which is also a homset of the category $L\mathbb{G}_n$. Moreover, because the maps in $L\mathbb{G}_n(I,I)$ all share the same source and target, what we have is not just a monoid under tensor product but also under composition as well. This fact leads to a series of special properties for $L\mathbb{G}_n(I,I)$, the first of which is just another instance of the classic Eckmann-Hilton argument.

Lemma 4.7. *$L\mathbb{G}_n(I,I)$ is a commutative monoid under both tensor product and composition, with $f\otimes f'=f\circ f'$.*

Proof. Let f,f' be arbitrary elements of the monoid $L\mathbb{G}_n(I,I)$. Since both of these are morphisms in the monoidal category $L\mathbb{G}_n$, we can use the law of interchange to show that

$$\begin{aligned} f \otimes f' &= (f \circ \mathrm{id}_I) \otimes (\mathrm{id}_I \circ f') \\ &= (f \otimes \mathrm{id}_I) \circ (\mathrm{id}_I \otimes f') \\ &= f \circ f' \\ &= (\mathrm{id}_I \otimes f) \circ (f' \otimes \mathrm{id}_I) \\ &= (f' \circ \mathrm{id}_I) \otimes (\mathrm{id}_I \circ f) \\ &= f' \otimes f \end{aligned}$$

□

In fact, since we already proved that the morphisms of $L\mathbb{G}_n$ are all actions morphisms, we can take this one step further.

Proposition 4.8. *$L\mathbb{G}_n(I, I)$ is an abelian group.*

Proof. From Lemma 3.6 we know that every morphism f in $L\mathbb{G}_n$ is of the form $\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$, for some $g \in G(m)$ and $x_i \in \mathbb{Z}^{*n}$. It follows immediately that

$$\begin{aligned} & \alpha(\, g\, ;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,) \circ \alpha(\, g^{-1}\, ;\, \mathrm{id}_{x_{\pi(g^{-1})(1)}}, ..., \mathrm{id}_{x_{\pi(g^{-1})(m)}}\,) \\ = \;& \alpha(\, gg^{-1}\, ;\, \mathrm{id}_{x_{\pi(g^{-1})(1)}}, ..., \mathrm{id}_{x_{\pi(g^{-1})(m)}}\,) \\ = \;& \alpha(\, e_m\, ;\, \mathrm{id}_{x_{\pi(g^{-1})(1)}}, ..., \mathrm{id}_{x_{\pi(g^{-1})(m)}}\,) \\ = \;& \mathrm{id}_{x_{\pi(g^{-1})(1)} \otimes ... \otimes x_{\pi(g^{-1})(m)}} \end{aligned}$$

$$\begin{aligned} & \alpha(\, g^{-1}\, ;\, \mathrm{id}_{x_{\pi(g^{-1})(1)}}, ..., \mathrm{id}_{x_{\pi(g^{-1})(m)}}\,) \circ \alpha(\, g\, ;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,) \\ = \;& \alpha(\, g^{-1}g\, ;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,) \\ = \;& \alpha(\, e_m\, ;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,) \\ = \;& \mathrm{id}_{x_1 \otimes ... \otimes x_m} \end{aligned}$$

In other words, every morphism $f : w \to v$ in $L\mathbb{G}_n$ has an inverse under composition,

$$f^{-1} \quad := \quad \alpha(g^{-1}; \mathrm{id}_{x_{\pi(g^{-1})(1)}}, ..., \mathrm{id}_{x_{\pi(g^{-1})(m)}})$$

But we know from Lemma 4.7 that tensor product and composition are the same for endomorphisms of the unit object of $L\mathbb{G}_n$. In particular this means that if some morphism $f : I \to I$ has a compositional inverse f^{-1}, then it will also be its monoidal inverse f^*. Thus every element of the commutative monoid $L\mathbb{G}_n(I, I)$ is invertible, or in other words $L\mathbb{G}_n(I, I)$ is an abelian group. □

Indeed, by using a slightly broader argument we can extend this result to every morphism of $L\mathbb{G}_n$.

Proposition 4.9. *Every morphism $f : w \to v$ in $L\mathbb{G}_n$ has an inverse under tensor product, $f^* : w^* \to v^*$. That is, the monoid* $\mathrm{Mor}(L\mathbb{G}_n)$ *is actually a group.*

Proof. For any $f : w \to v$ in $L\mathbb{G}_n$, consider the map $\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*}$, where f^{-1} is the compositional inverse of f, as in the proof of Proposition 4.8. This morphism has source $w^* \otimes v \otimes v^* = w^*$ and target $w^* \otimes w \otimes v^* = v^*$, which allows us to apply the law of interchange to get

$$\begin{aligned}
f \otimes (\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*}) &= \Big(f \circ \mathrm{id}_w \Big) \otimes \Big(\mathrm{id}_{v^*} \circ (\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*}) \Big) \\
&= \Big(f \otimes \mathrm{id}_{v^*} \Big) \circ \Big(\mathrm{id}_w \otimes (\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*}) \Big) \\
&= (f \otimes \mathrm{id}_{v^*}) \circ (f^{-1} \otimes \mathrm{id}_{v^*}) \\
&= (f \circ f^{-1}) \otimes (\mathrm{id}_{v^*} \circ \mathrm{id}_{v^*}) \\
&= \mathrm{id}_v \otimes \mathrm{id}_{v^*} \\
&= \mathrm{id}_I
\end{aligned}$$

and likewise

$$\begin{aligned}
(\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*}) \otimes f &= \Big((\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*}) \circ \mathrm{id}_{w^*} \Big) \otimes \Big(\mathrm{id}_v \circ f \Big) \\
&= \Big((\mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*}) \otimes \mathrm{id}_v \Big) \circ \Big(\mathrm{id}_{w^*} \otimes f \Big) \\
&= (\mathrm{id}_{w^*} \otimes f^{-1}) \circ (\mathrm{id}_{w^*} \otimes f) \\
&= (\mathrm{id}_{w^*} \circ \mathrm{id}_{w^*}) \otimes (f^{-1} \circ f) \\
&= \mathrm{id}_{w^*} \otimes \mathrm{id}_w \\
&= \mathrm{id}_I
\end{aligned}$$

In other words, $f^* := \mathrm{id}_{w^*} \otimes f^{-1} \otimes \mathrm{id}_{v^*}$ is the inverse of f in the monoid $\mathrm{Mor}(L\mathbb{G}_n)$, as required. ☐

So $\mathrm{Mor}(L\mathbb{G}_n)$ and $L\mathbb{G}_n(I, I)$ both turn out to be groups under tensor product. Obviously it follows from this that $L\mathbb{G}_n(I, I)$ is a not just a submonoid of $\mathrm{Mor}(L\mathbb{G}_n)$ but a subgroup — in particular an abelian subgroup, going by Proposition 4.8. But $L\mathbb{G}_n(I, I)$ is actually an even more special subgroup than this.

Proposition 4.10. $L\mathbb{G}_n(I, I)$ *is a normal subgroup of* $\mathrm{Mor}(L\mathbb{G}_n)$*. Moreover, if G is a crossed action operad, then* $L\mathbb{G}_n(I, I)$ *is a subgroup of the centre of* $\mathrm{Mor}(L\mathbb{G}_n)$*.*

Proof. From Propositions 4.8 and 4.9, we know that $L\mathbb{G}_n(I, I)$ is a subgroup of $\mathrm{Mor}(L\mathbb{G}_n)$. For normality, we need to again consider both crossed and non-crossed action operads separately.

If G is non-crossed, then by Corollary 2.20 we know that the map assigning objects of $L\mathbb{G}_n$ to their connected component is just the identity $\mathrm{id}_{\mathbb{Z}^{*n}}$. In other words, every object belongs to its own unique component, so that every morphism of $L\mathbb{G}_n$ is actually an endomorphism. It follows that the group $L\mathbb{G}_n(I,I)$ is the kernel of the source homomorphism s from Definition 4.1 — or equally the target homomorphism t.

$$L\mathbb{G}_n(I,I) \longrightarrow \mathrm{Mor}(L\mathbb{G}_n) \xrightarrow{s} \mathrm{Ob}(L\mathbb{G}_n)$$

The kernel of a group homomorphism is always a normal subgroup of that homomorphism's source, and so in our case we have $L\mathbb{G}_n(I,I) \leq \mathrm{Mor}(L\mathbb{G}_n)$.

For crossed G, recall from Lemma 1.22 that all crossed EG-algebras are spacial, and so in particular $L\mathbb{G}_n$ is. This means that for any $h \in L\mathbb{G}_n(I,I)$ and $w \in \mathrm{Ob}(L\mathbb{G}_n)$ we will always have $h \otimes \mathrm{id}_w = \mathrm{id}_w \otimes h$. Thus for any $f : w \to v$ in $\mathrm{Mor}(L\mathbb{G}_n)$, we get

$$\begin{aligned} h \otimes f &= (\mathrm{id}_I \circ h) \otimes (f \circ \mathrm{id}_w) \\ &= (\mathrm{id}_I \otimes f) \circ (h \otimes \mathrm{id}_w) \\ &= (f \otimes \mathrm{id}_I) \circ (\mathrm{id}_w \otimes h) \\ &= (f \circ \mathrm{id}_w) \otimes (\mathrm{id}_I \circ h) \\ &= f \otimes h \end{aligned}$$

That is, $L\mathbb{G}_n(I,I)$ is a subgroup of the centre of $\mathrm{Mor}(L\mathbb{G}_n)$. Then because

$$f \otimes h \otimes f^* \;=\; h \otimes f \otimes f^* \;=\; h \;\in L\mathbb{G}_n(I,I)$$

it follows that $L\mathbb{G}_n(I,I)$ is a normal subgroup of $\mathrm{Mor}(L\mathbb{G}_n)$. □

4.3 The morphisms of $L\mathbb{G}_n$

We have finally described all of the important properties of $(s \times t)(L\mathbb{G}_n)$ and $L\mathbb{G}_n(I,I)$ that we will need going forward. Putting all of these results together will allow us to characterize the morphisms of $L\mathbb{G}_n$ as a product of groups, as was promised at the beginning of this chapter. Before we do so though, it will be worth going over a few well-known pieces of group theory that we will be using in the proof of Proposition 4.14.

Definition 4.11. Let H, K and N be groups. Then we say that H is a *group extension* of K by N if there exists a short exact sequence

$$0 \longrightarrow N \overset{i}{\hookrightarrow} H \xrightarrow{p} K \longrightarrow 0$$

In other words, H is an extension of K by N whenever we have $K = H/N$. Moreover, if N is a subgroup of the centre of H, we say that this is a *central* extension, and if the map p has a right-inverse, $r : K \to H$, $p \circ r = \mathrm{id}_K$, then we say that it is a *split* extension.

Definition 4.12. Let H be a group with subgroup K and normal subgroup N. Then we say that H is a *semidirect product* $K \ltimes N$ if the underlying set of H is the same as underlying set of $K \times N$, but multiplication is given by

$$(k, n) \cdot (k', n') \quad = \quad (\, kk',\, nkn'k^{-1}\,)$$

Lemma 4.13. **??** *If H is a split extension of K by N then $H = K \ltimes N$, with $r : K \to H$ acting as the appropriate subgroup inclusion. Further, if H is a both split and central, then $H \cong K \times N$.*

Proof. Define a group homomorphism $f : H \to K \ltimes N$ by

$$f(h) \quad := \quad \Big(p(h), h \cdot rp(h)^{-1}\Big)$$

This is a well-defined homomorphism, since

$$\begin{aligned}
f(hh') &= \Big(p(hh'), hh' \cdot rp(hh')^{-1}\Big) \\
&= \Big(p(h) \cdot p(h'), h \cdot h' \cdot rp(h')^{-1} \cdot rp(h)^{-1}\Big) \\
&= \Big(p(h) \cdot p(h'), h \cdot rp(h)^{-1} \cdot rp(h) \cdot h' \cdot rp(h')^{-1} \cdot rp(h)^{-1}\Big) \\
&= \Big(p(h'), h \cdot rp(h)^{-1}, p(h)\Big) \cdot \Big(h' \cdot rp(h')^{-1}\Big) \\
&= f(h) \cdot f(h')
\end{aligned}$$

Next, define another map $f^{-1} : K \times N \to H$ by

$$f^{-1}(k, n) \quad := \quad n \cdot r(k)$$

f^{-1} is also well-defined, because

$$\begin{aligned}
f^{-1}\Big(\,(k, n) \cdot (k', n')\Big) &= f^{-1}\Big(\,kk',\, n \cdot r(k) \cdot n' \cdot r(k)^{-1}\Big) \\
&= \Big(n \cdot r(k) \cdot n' \cdot r(k)^{-1}\Big) \cdot r(kk') \\
&= n \cdot r(k) \cdot n' \cdot r(k)^{-1} \cdot r(k) \cdot r(k') \\
&= n \cdot r(k) \cdot n' \cdot r(k') \\
&= f^{-1}(k, n) \cdot f^{-1}(k', n')
\end{aligned}$$

and due to the fact that $p : N \hookrightarrow H \to K$ is the zero map, f and f^{-1} are inverses:

$$\begin{aligned} f^{-1}f(h) &= f^{-1}\Big(p(h),\, h\cdot rp(h)^{-1}\Big) \\ &= \Big(h\cdot rp(h)^{-1}\Big)\cdot r\Big(p(h)\Big) \\ &= h\cdot rp(h)^{-1}\cdot rp(h) \\ &= h \end{aligned}$$

$$\begin{aligned} ff^{-1}(k,n) &= f\Big(n\cdot r(k)\Big) \\ &= \Big(p\Big(n\cdot r(k)\Big),\, n\cdot r(k)\cdot rp\Big(n\cdot r(k)\Big)^{-1}\Big) \\ &= \Big(p(n)\cdot pr(k),\, n\cdot r(k)\cdot rpr(k)^{-1}\cdot rp(n)^{-1}\Big) \\ &= \Big(e\cdot k,\, n\cdot r(k)\cdot r(k)^{-1}\cdot e\Big) \\ &= (k,n) \end{aligned}$$

Thus f is an isomorphism of groups $H \cong K \ltimes N$. Also, if N is in the centre of H then the multiplication in $K \ltimes N$ becomes

$$\begin{aligned} (k,n)\cdot(k',n') &= (\,kk', nkn'k^{-1}\,) \\ &= (\,kk', nn'kk^{-1}\,) \\ &= (kk', nn') \end{aligned}$$

and so H really is the direct product of groups $K \times N$. □

With that out of the way, we can now produce an expression for the morphisms of $L\mathbb{G}_n$.

Proposition 4.14. *For any action operad G,*

$$\mathrm{Mor}(L\mathbb{G}_n) \quad\cong\quad (s\times t)(L\mathbb{G}_n) \ltimes L\mathbb{G}_n(I,I)$$

Moreover, if G is a crossed action operad, then

$$\mathrm{Mor}(L\mathbb{G}_n) \quad\cong\quad (s\times t)(L\mathbb{G}_n) \times L\mathbb{G}_n(I,I)$$

Proof. We just saw in Proposition 4.10 that $L\mathbb{G}_n(I,I)$ is a normal subgroup of $\mathrm{Mor}(L\mathbb{G}_n)$, so we can consider the quotient group

$$L\mathbb{G}_n(I,I) \hookrightarrow \mathrm{Mor}(L\mathbb{G}_n) \longrightarrow \mathrm{Mor}(L\mathbb{G}_n)\big/L\mathbb{G}_n(I,I)$$

By the universal property of quotients, the map $\mathrm{Mor}(L\mathbb{G}_n) \to \mathrm{Mor}(L\mathbb{G}_n)/L\mathbb{G}_n(I,I)$ will uniquely factor any homomorphism whose composite with the inclusion $L\mathbb{G}_n(I,I) \hookrightarrow$

$\mathrm{Mor}(L\mathbb{G}_n)$ is the zero map. But our source/target map $s\times t : \mathrm{Mor}(L\mathbb{G}_n) \to (s\times t)(L\mathbb{G}_n)$ is one such homomorphism, since for any $h : I \to I$ clearly $(s \times t)(h) = (I, I)$, which is the identity element in $(s\times t)(L\mathbb{G}_n)$. Therefore there must exist a unique homomorphism u making the triangle below commute:

$$\begin{tikzcd}
\mathrm{Mor}(L\mathbb{G}_n) \arrow[d] \arrow[dr, "s\times t"] & \\
\mathrm{Mor}(L\mathbb{G}_n)\big/L\mathbb{G}_n(I,I) \arrow[r, "u"] & (s\times t)(L\mathbb{G}_n)
\end{tikzcd}$$

This map u will be surjective — because $s \times t$ is — but in fact it will also be injective. This is because if two morphisms f, f' of $L\mathbb{G}_n$ have the same source and target, then the map $h = f^* \otimes f'$ is an element of $L\mathbb{G}_n(I, I)$ for which $f \otimes h = f'$, and so clearly f and f' are part of the same equivalence class in $\mathrm{Mor}(L\mathbb{G}_n)/L\mathbb{G}_n(I, I)$. More precisely,

$$\begin{aligned}
[f] \neq [f'] \quad &\Longrightarrow \quad [f]^* \otimes [f'] \neq [I] \\
&\Longrightarrow \quad [f^* \otimes f'] \neq [I] \\
&\Longrightarrow \quad f^* \otimes f' \notin L\mathbb{G}_n(I, I)
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \quad (s \times t)(f^* \otimes f') &\neq (I, I) \\
\Longrightarrow \quad (s \times t)(f)^* \otimes (s \times t)(f') &\neq (I, I) \\
\Longrightarrow \quad (s \times t)(f) &\neq (s \times t)(f')
\end{aligned}$$

Thus u is bijective, so that

$$\mathrm{Mor}(L\mathbb{G}_n)\big/L\mathbb{G}_n(I,I) \quad \cong \quad (s \times t)(L\mathbb{G}_n)$$

In other words, what have here is a group extension

$$0 \longrightarrow L\mathbb{G}_n(I, I) \hookrightarrow \mathrm{Mor}(L\mathbb{G}_n) \xrightarrow{s\times t} (s \times t)(L\mathbb{G}_n) \longrightarrow 0$$

But recall from Corollary 4.6 that $(s \times t)(L\mathbb{G}_n)$ is also a submonoid (and hence subgroup) of $\mathrm{Mor}(L\mathbb{G}_n)$, so that we have another map $i : (s \times t)(L\mathbb{G}_n) \to \mathrm{Mor}(L\mathbb{G}_n)$ for which $(s \times t) \circ i = \mathrm{id}_{(s\times t)(L\mathbb{G}_n)}$. That is, the above is a split extension of groups, or equivalently $\mathrm{Mor}(L\mathbb{G}_n)$ is a semi direct product $(s \times t)(L\mathbb{G}_n) \ltimes L\mathbb{G}_n(I, I)$. However, if G is crossed then we also saw in Proposition 4.10 that $L\mathbb{G}_n(I, I)$ is a subgroup of the centre of $\mathrm{Mor}(L\mathbb{G}_n)$, and so it will follow that $\mathrm{Mor}(L\mathbb{G}_n)$ is also a central extension of $(s \times t)(L\mathbb{G}_n)$. In that case $\mathrm{Mor}(L\mathbb{G}_n)$ is really just the direct product $(s \times t)(L\mathbb{G}_n) \times L\mathbb{G}_n(I, I)$, as required. □

In certain select cases, Proposition 4.14 will actually be sufficient to fully determine $\mathrm{Mor}(L\mathbb{G}_n)$ — specifically, whenever we know that the unit endomorphisms of $L\mathbb{G}_n$ are trivial. We already know of two examples like this, due to Proposition 3.9 and Lemma 3.11.

Corollary 4.15. *If G is a crossed action operad with $G(m) = G(0)$ for all $m \in \mathbb{N}$, then*

$$\mathrm{Mor}(L\mathbb{G}_n) \quad = \quad (s \times t)(L\mathbb{G}_n) \quad = \quad \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$$

Instead if G is a $G(1)$-generated action operad, then

$$\mathrm{Mor}(L\mathbb{G}_n) \quad = \quad (s \times t)(L\mathbb{G}_n) \quad = \quad \mathrm{Ob}(L\mathbb{G}_n) \quad = \quad \mathbb{Z}^{*n}$$

In the latter case, what this is saying is that every morphism in $L\mathbb{G}_n$ is just the identity element of some object.

But what about for more general $L\mathbb{G}_n$ with nontrivial unit endomorphisms? For crossed G, the key insight is that one half of the product in Proposition 4.14, $L\mathbb{G}_n(I, I)$, is always an abelian group. This means that it will remain untouched if we were to abelianise the entire product, thus providing a link between $\mathrm{Mor}(L\mathbb{G}_n)$ before and after abelianisation.

Proposition 4.16. *Let G be a crossed action operad. Then the endomorphisms of the unit object of $L\mathbb{G}_n$ are*

$$L\mathbb{G}_n(I, I) \quad = \quad \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}} \big/ (s \times t)(L\mathbb{G}_n)^{\mathrm{ab}}$$

and therefore

$$\mathrm{Mor}(L\mathbb{G}_n) \quad = \quad (s \times t)(L\mathbb{G}_n) \times \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}} \big/ (s \times t)(L\mathbb{G}_n)^{\mathrm{ab}}$$

Proof. From Proposition 4.14, we know that

$$\mathrm{Mor}(L\mathbb{G}_n) \quad = \quad (s \times t)(L\mathbb{G}_n) \times L\mathbb{G}_n(I, I)$$

Abelianising both sides of this equation, we get

$$\begin{aligned}\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}} &= \Big((s\times t)(L\mathbb{G}_n)\times L\mathbb{G}_n(I,I)\Big)^{\mathrm{ab}}\\ &= (s\times t)(L\mathbb{G}_n)^{\mathrm{ab}}\times L\mathbb{G}_n(I,I)^{\mathrm{ab}}\\ &= (s\times t)(L\mathbb{G}_n)^{\mathrm{ab}}\times L\mathbb{G}_n(I,I)\end{aligned}$$

since $L\mathbb{G}_n(I,I)$ is already abelian. Now, there is an obvious inclusion $(s\times t)(L\mathbb{G}_n)^{\mathrm{ab}} \hookrightarrow (s\times t)(L\mathbb{G}_n)^{\mathrm{ab}}\times L\mathbb{G}_n(I,I)$, and since everything here is abelain, all subgroups are normal subgroups. Thus we can take the quotient of the above equation by this map, to obtain

$$L\mathbb{G}_n(I,I) = \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}}\Big/(s\times t)(L\mathbb{G}_n)^{\mathrm{ab}}$$

Finally, we can now substitute this expression back into our original equation, giving

$$\mathrm{Mor}(L\mathbb{G}_n) = (s\times t)(L\mathbb{G}_n)\times \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}}\Big/(s\times t)(L\mathbb{G}_n)^{\mathrm{ab}}$$

as required. □

Unfortunately, there is no general version of Proposition 4.16 for when G is not crossed. If we try to abelianise the semideirect product from Proposition 4.14, we will arrive at a product of the relevant abelian group, but a new term will also appear indicating the degree to which $L\mathbb{G}_n(I,I)$ and $(s\times t)(L\mathbb{G}_n)$ fail to commute.

Lemma 4.17. *If H is semidirect product $K\ltimes N$, then its abelianisation is*

$$H^{\mathrm{ab}} = K^{\mathrm{ab}}\times N^{\mathrm{ab}}\Big/[N,K]$$

where $[N,K]$ is the commutator of N with K.

If we stick to working with crossed action operads however, we are now only one step away from our full expression for $\mathrm{Mor}(L\mathbb{G}_n)$. The last term whose value we do not know is $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}} = \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}}$, and as one might expect this is related to the value that the algebra takes under the collapsed morphism left adjoint, $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}} = \mathrm{M}(L\mathbb{G}_n)^{\mathrm{ab}}$

Proposition 4.18. *Let X be any monoidal category whose objects are morphisms are all invertible under tensor product. Then the group completion of the abelianisation of the collapsed morphisms of X are*

$$\mathrm{M}(X)^{\mathrm{ab}} \cong \mathrm{Mor}(X)^{\mathrm{ab}}\Big/\mathrm{Ob}(X)^{\mathrm{ab}}$$

where we are viewing $\mathrm{Ob}(X)$ *as a subgroup of* $\mathrm{Mor}(X)$ *under tensor product by using the inclusion*

$$\begin{array}{rcl} \mathrm{Ob}(X) & \to & \mathrm{Mor}(X) \\ x & \mapsto & \mathrm{id}_x \end{array}$$

Proof. Recall Lemma 2.21, which says that in any monoidal category with invertible objects,

$$f' \circ f \quad = \quad f' \otimes \mathrm{id}_{y*} \otimes f$$

We will proceed by checking what effect this relation in $\mathrm{Mor}(X)$ will have on the two quotients that we are comparing.

First, consider the canonical homomorphism $\psi : \mathrm{Mor}(X) \to \mathrm{M}(X) \to \mathrm{M}(X)^{\mathrm{ab}}$, where $\mathrm{Mor}(X)$ is being considered as a group under $\otimes$. Also $\mathrm{M}(X)$ is a group rather than just a monoid, since if f^* is the inverse of f under tensor product in $\mathrm{Mor}(X)$, then the equivalence class $\mathrm{M}(f^*)$ is an inverse of $\mathrm{M}(f)$ under the collapsed product of $\mathrm{M}(X)$. Clearly this map obeys the relation $\psi(f' \circ f) = \psi(f' \otimes f)$ for any $f : x \to y$, $f' : y \to z$ in X, because it passes through $\mathrm{M}(X)$, and so we also have

$$\begin{array}{rcl} \psi(f' \otimes f) & = & \psi(f' \circ f) \\ & = & \psi(f' \otimes \mathrm{id}_{y*} \otimes f) \\ & = & \psi(f') \otimes \psi(\mathrm{id}_{y*}) \otimes \psi(f) \\ & = & \psi(f') \otimes \psi(f) \otimes \psi(\mathrm{id}_{y*}) \\ & = & \psi(f' \otimes f) \otimes \psi(\mathrm{id}_{y*}) \\ & & \\ \Longrightarrow \quad \psi(\mathrm{id}_{y*}) & = & e \end{array}$$

But since ψ is also a map from $\mathrm{Mor}(X)$ onto an abelian group, we know that it must factor uniquely though some other homomorphism $\mathrm{Mor}(X)^{\mathrm{ab}} \to \mathrm{M}(X)^{\mathrm{ab}}$, which we will call ψ'. This map will inherit from ψ the property that

$$\psi'\Big(\mathrm{ab}(\mathrm{id}_x)\Big) \quad = \quad \psi(\mathrm{id}_x) \quad = \quad e$$

for all $x \in \mathrm{Ob}(X)$.

Now let A be an abelian group and $\phi : \mathrm{Mor}(X)^{\mathrm{ab}} \to A$ any homomorphism of groups which satisfies the condition $\phi(\mathrm{ab}(\mathrm{id}_x)) = e$ for all objects x. Then

$$\begin{aligned}
\phi\Big(\mathrm{ab}(f' \circ f)\Big) &= \phi\Big(\mathrm{ab}(f' \otimes \mathrm{id}_{y*} \otimes f)\Big) \\
&= \phi\Big(\mathrm{ab}(f')\Big) \otimes \phi\Big(\mathrm{ab}(\mathrm{id}_{y*})\Big) \otimes \phi\Big(\mathrm{ab}(f)\Big) \\
&= \phi\Big(\mathrm{ab}(f')\Big) \otimes \phi\Big(\mathrm{ab}(f)\Big) \\
&= \phi\Big(\mathrm{ab}(f' \otimes f)\Big)
\end{aligned}$$

By Lemma 2.25 this is the defining relation for the group $\mathrm{M}(X)^{\mathrm{ab}}$. It follows that for any ϕ with $\phi(\mathrm{ab}(\mathrm{id}_x)) = e$, there must exist a unique homomorphism $\mathrm{M}(X)^{\mathrm{ab}} \to A$ which factors ϕ through ψ'. But this in turn is just the universal property of the quotient $\mathrm{Mor}(X)^{\mathrm{ab}}/\mathrm{Ob}(X)^{\mathrm{ab}}$ in Ab. Since colimits like quotient groups are unique up to isomorphism, we can therefore conclude that

$$\mathrm{M}(X)^{\mathrm{ab}} \quad \cong \quad \mathrm{Mor}(X)^{\mathrm{ab}} \big/ \mathrm{Ob}(X)^{\mathrm{ab}}$$

□

Now it just remains to chain together all of our previous results.

Proposition 4.19. *For crossed action operads G, the morphism monoid of $L\mathbb{G}_n$ is equal to*

$$\mathrm{Mor}(L\mathbb{G}_n) \quad = \quad \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \times \frac{\left(\mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \big/ \Delta\right)}{\left((\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}} \big/ \mathbb{Z}^n\right)}$$

Proof. Consider the quotient group

$$L\mathbb{G}_n(I, I) \quad = \quad \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}} \big/ (s \times t)(L\mathbb{G}_n)^{\mathrm{ab}}$$

This quotient clearly depends on the way that have chosen to see $(s \times t)(L\mathbb{G}_n)$ as a subgroup of of the morphisms $L\mathbb{G}_n$. Recall that back in the proof of Proposition 4.5, we used the freeness of the monoid $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ to define a subgroup by choosing values for some function ρ on generators. Since these $\rho(z, z')$ can be whichever element of the appropriate $G(m)$ we want, we can retroactively pick them in a way that makes our current calculations easier. Specifically, if we let $\rho(z_i, z_i) = e_1$ for each generator z_i of $\mathbb{N}^{*n}$, then the corresponding element of the subgroup $(s \times t)(L\mathbb{G}_n)$ will be

$$\alpha_{L\mathbb{G}_n}(e_1; z_i) \quad = \quad \mathrm{id}_{z_i}$$

Given this choice, clearly the group

$$\mathrm{Ob}(L\mathbb{G}_n) \quad \cong \quad \{\, \mathrm{id}_x \,;\, x \in \mathrm{Ob}(L\mathbb{G}_n)\}$$

will be a subgroup of $(s \times t)(L\mathbb{G}_n)$, and thus $\mathrm{Ob}(L\mathbb{G}_n)^{\mathrm{ab}}$ a normal subgroup of $(s \times t)(L\mathbb{G}_n)^{\mathrm{ab}}$. It follows that

$$\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}} \big/ (s \times t)(L\mathbb{G}_n)^{\mathrm{ab}} \quad = \quad \frac{\left(\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}} \big/ \mathrm{Ob}(L\mathbb{G}_n)^{\mathrm{ab}}\right)}{\left((s \times t)(L\mathbb{G}_n)^{\mathrm{ab}} \big/ \mathrm{Ob}(L\mathbb{G}_n)^{\mathrm{ab}}\right)}$$

Using Proposition 4.18 to change the numerator and Proposition 2.14 and Corollary 4.3 to simplify the denominator, this quotient becomes

$$\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}} \big/ (s \times t)(L\mathbb{G}_n)^{\mathrm{ab}} \quad = \quad \frac{\left(\mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \big/ \Delta\right)}{\left((\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}} \big/ \mathbb{Z}^n\right)}$$

But from Proposition 4.16 we know that

$$\mathrm{Mor}(L\mathbb{G}_n) \quad = \quad (s \times t)(L\mathbb{G}_n) \times \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}} \big/ (s \times t)(L\mathbb{G}_n)^{\mathrm{ab}}$$

and together these give the required description of the morphisms of $(L\mathbb{G}_n$. □

4.4 Abelianising sources and targets

To say that the expression for $\mathrm{Mor}(L\mathbb{G}_n)$ we just found is 'complicated' would probably be an understament. If we are to have any hope of eventually being able to use Proposition 4.19, we need to work out a more explicit presentation for its quotient part. We'll start by trying to find the value of $(s \times t)(L\mathbb{G}_n)^{\mathrm{ab}}$ for crossed G, the abelian group $(\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}}$. This will require some careful consideration, since in general limits such as the pullback do not interact with abelianisation in a simple way. What would help is a suitable presentation of $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$ in terms of generators and relations.

Proposition 4.20. *The group $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$ is generated by the elements*

$$\langle x \rangle \quad := \quad (x, x) \qquad \textit{and} \qquad \langle xy \rangle \quad := \quad (xy, yx)$$

where $x, y \in \{z_1, ..., z_n, z_1^*, ..., z_n^*\}$ *are generators of the free group* $\mathbb{Z}^{*n}$ *or their inverses. These are subject to the relations*

$$\langle x\rangle^{-1} = \langle x^*\rangle, \qquad \langle xy\rangle^{-1} = \langle y^*x^*\rangle$$

$$\langle xx^*\rangle = e = \langle x^*x\rangle, \qquad \langle xx\rangle = \langle x\rangle^2$$

$$\langle xy\rangle\langle x^*\rangle\langle xy^*\rangle = \langle x\rangle$$

$$\langle xy\rangle\langle x^*\rangle\langle y^*\rangle\langle yx\rangle = \langle x\rangle\langle y\rangle = \langle yx\rangle\langle x^*\rangle\langle y^*\rangle\langle xy\rangle$$

$$\langle xy\rangle\langle x^*\rangle\langle xz\rangle\langle x^*\rangle\langle z^*\rangle\langle y^*\rangle\langle yz\rangle\langle y^*\rangle\langle yx\rangle\langle y\rangle\langle x^*\rangle\langle z^*\rangle^{-1}\langle zx\rangle\langle z^*\rangle\langle zy\rangle = \langle x\rangle\langle y\rangle\langle z\rangle$$

Proof. We'll begin by constructing a certain monoidal category, which we'll call Z.

- The objects of Z are the elements of the group $\mathbb{Z}^{*n}$, with the usual multiplication as the tensor product.

- There is a unique morphisms between any two objects x and y for which $\mathrm{ab}(x) = \mathrm{ab}(y)$, where $\mathrm{ab} : \mathbb{Z}^{*n} \to \mathbb{Z}^n$ is the quotient map of abelianisation. In other words, the morphisms of Z are the elements of $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$, with multiplication as the tensor product and composition given by

$$(x, y) \circ (y, z) = (x, z)$$

- The identity map on an object x is then the unique map $(x, x) : x \to x$.

Z is almost the subcategory of $L\mathbb{G}_n$ whose morphisms are the subgroup isomorphic to $(s \times t)(L\mathbb{G}_n)$ that we chose in Corollary 4.6. However, we never required those representatives to be closed under composition, so Z is a strictly formal version of the subcategory on $(s \times t)(L\mathbb{G}_n)$, one that doesn't involve any specific choice of the map ρ. It is a well-defined monoidal category; the only thing that might not be immediately clear is the law of interchange, which is just given by

$$\begin{aligned}
\Big((x, y) \circ (y, z)\Big) \otimes \Big((x', y') \circ (y', z')\Big) &= (x, z) \otimes (x', z') \\
&= (xx', zz') \\
&= (xx', yy') \circ (yy', zz') \\
&= \Big((x, y) \otimes (x', y')\Big) \circ \Big((y, z) \otimes (y', z')\Big)
\end{aligned}$$

But now recall from Lemma 2.21 that in any monoidal category the composition of morphisms along an intertible object can be rewritten in terms of only the tensor product. In the case of Z, where all of the objects have inverses, we will have

$$(x, y) \circ (y, z) \quad = \quad (x, y) \otimes (y^*, y^*) \otimes (y, z)$$

Using this composition operation will make it easier to understand the structure of the group $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$.

Next, let $\mathbb{S}_{2n}$ be the free ES-algebra on $2n$ objects, where S is the symmetric action operad. Then there is an obvious monoidal functor $\psi : \mathbb{S}_{2n} \to Z$, given by

$$\begin{array}{cccc} \psi & : & \mathbb{S}_{2n} & \to & Z \\ & : & z_i & \mapsto & z_i \\ & : & z_{n+i} & \mapsto & z_i^* \\ & : & \alpha(\sigma; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) & \mapsto & (x_1 \otimes ... \otimes x_m, x_{\sigma(1)} \otimes ... \otimes x_{\sigma(m)}) \end{array}$$

A neccessary condition for (y, y') to be an element of $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$ is that there exists some sequence of generators and their inverses $x_1, ..., x_m \in \{z_1, ..., z_n, z_1^*, ..., z_n^*\}$ and some permutation $\sigma \in S_m$ for which

$$\begin{array}{rcl} y & = & x_1 \otimes ... \otimes x_m \\ y' & = & x_{\sigma(1)} \otimes ... \otimes x_{\sigma(m)} \end{array}$$

Hence the functor ψ is clearly surjective. It follows from this that if we can find a collection of morphisms which generate $\mathrm{Mor}(\mathbb{S}_{2n})$ under composition and tensor product, their images under ψ will also generate $\mathrm{Mor}(Z) = \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$ under composition and tensor product, and hence under just tensor product. To begin, we know that any permutation $\sigma \in S_m$ can be written as a product $\sigma_{i_k} \cdot ... \cdot \sigma_{i_1}$ of elementary transpositions — elements of S_m which only swap two adjacent positions. This means that

$$\begin{array}{rcl} \alpha(\,\sigma\,;\,\mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,) & = & \alpha(\,\sigma_{i_k} \cdot ... \cdot \sigma_{i_1}\,;\,\mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,) \\ & = & \alpha(\,\sigma_{i_1}\,;\,\mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,) \circ \\ & & \alpha(\,\sigma_{i_2}\,;\,\mathrm{id}_{x_{\sigma_{i_1}(1)}}, ..., \mathrm{id}_{x_{\sigma_{i_1}(m)}}\,) \circ ... \circ \\ & & \alpha(\,\sigma_{i_k}\,;\,\mathrm{id}_{x_{\sigma_{i_{k-1}} \cdot ... \cdot \sigma_{i_1}(1)}}, ..., \mathrm{id}_{x_{\sigma_{i_{k-1}} \cdot ... \cdot \sigma_{i_1}(m)}}\,) \end{array}$$

Then for any such elementary transposition $\sigma_i = (i\ i+1) \in S_m$ we will have

$$\begin{aligned} \alpha(\,(i\ i+1)\,;\,\mathrm{id}_{x_1},...,\mathrm{id}_{x_m}\,) &= \alpha(\,e_{i-1}\otimes(12)\otimes e_{m-i-1}\,;\,\mathrm{id}_{x_1},...,\mathrm{id}_{x_m}\,) \\ &= \mathrm{id}_{x_1\otimes...\otimes x_{i-1}}\otimes\alpha(\,(12)\,;\,\mathrm{id}_{x_i},\mathrm{id}_{x_{i+1}}\,)\otimes\mathrm{id}_{x_{i+2}\otimes...\otimes x_m} \end{aligned}$$

Therefore all of the morphisms of $\mathbb{S}_{2n}$ are generated by just the identities and the action maps $\alpha(\,(12)\,;\,\mathrm{id}_{x_1},\mathrm{id}_{x_2}\,)$ for all pairs $x_1, x_2 \in \{z_1,...,z_{2n}\}$. Passing through ψ, this means that elements of $\mathbb{Z}^{*n}\times_{\mathbb{Z}^n}\mathbb{Z}^{*n}$ can always be expressed as a tensor product of elements of the form

$$(x,x) \qquad \text{or} \qquad (xy,yx), \qquad\qquad x,y\in\{z_1,...,z_n,z_1^*,...,z_n^*\}$$

These are exactly the $\langle x\rangle$ and $\langle xy\rangle$ given in the statement of the proposition.

Now we need to consider what relations these generators will obey. Firstly, their definitions overlap in the following case:

$$\langle xx\rangle \quad = \quad (xx,xx) \quad = \quad (x,x)\otimes(x,x) \quad = \quad \langle x\rangle\langle x\rangle$$

Then we have the law of interchange we discussed earlier. By Lemma 2.21, we'll get

$$\begin{aligned} \langle xy\rangle\langle x^*\rangle\langle y^*\rangle\langle yx\rangle &= (xy,yx)\otimes(x^*,x^*)\otimes(y^*,y^*)\otimes(yx,xy) \\ &= (xy,yx)\otimes(yx,yx)^*\otimes(yx,xy) \\ &= (xy,yx)\circ(yx,xy) \\ &= (yx,xy)\otimes(yx,yx)^*\otimes(yx,xy) \\ &= (yx,xy)\otimes(x^*,x^*)\otimes(y^*,y^*)\otimes(xy,yx) \\ &= \langle yx\rangle\langle x^*\rangle\langle y^*\rangle\langle xy\rangle \end{aligned}$$

Also, by functoriality these generators will inherit any relations on the corresponding morphisms of $\mathbb{S}_{2n}$, which in turn are just relations among different elementary transpositions. Each symmetric group S_m is subject to three families of these, namely

$$\begin{aligned} (\sigma_i)^2 &= e \\ \sigma_i\sigma_j &= \sigma_j\sigma_i, \qquad j\neq i\pm 1 \\ (\sigma_i\sigma_{i+1})^3 &= e \end{aligned}$$

The first corresponds to

$$\begin{array}{rrcl} & (xy, yx) \circ (yx, xy) & = & (xy, xy) \\ \Longrightarrow & (xy, yx) \otimes (yx, yx)^* \otimes (yx, xy) & = & (x, x) \otimes (y, y) \\ \Longrightarrow & (xy, yx) \otimes (x^*, x^*) \otimes (y^*, y^*) \otimes (yx, xy) & = & (x, x) \otimes (y, y) \\ \Longrightarrow & \langle xy \rangle \langle x^* \rangle \langle y^* \rangle \langle yx \rangle & = & \langle x \rangle \langle y \rangle \end{array}$$

The second relation is just an example of interchange, which we have already looked at. The third yields

$$\begin{array}{rcl} (xy, yx)(x^*, x^*)(xz, zx)(x^*, x^*)(z^*, z^*)(y^*, y^*)(yz, zy) & & \\ (y^*, y^*)(yx, xy)(y^*, y^*)(x^*, x^*)(z^*, z^*)(zx, xz)(z^*, z^*)(zy, yz) & = & (x, x)(y, y)(z, z) \end{array}$$

$$\Longrightarrow \quad \langle xy \rangle \langle x^* \rangle \langle xz \rangle \langle x^* \rangle \langle z^* \rangle \langle y^* \rangle \langle yz \rangle \langle y^* \rangle \langle yx \rangle \langle y^* \rangle \langle x^* \rangle \langle z^* \rangle \langle zx \rangle \langle z^* \rangle \langle zy \rangle \quad = \quad \langle x \rangle \langle y \rangle \langle z \rangle$$

Finally, we need to check how the invertibility of the objects of Z interacts with these generators. Most obviously, we have

$$\begin{array}{rcccccl} \langle x \rangle^{-1} & = & (x, x)^* & = & (x^*, x^*) & = & \langle x^* \rangle \\ \langle xy \rangle^{-1} & = & (xy, yx)^* & = & (y^* x^*, x^* y^*) & = & \langle y^* x^* \rangle \\ & & & & & & \\ \langle xx^* \rangle & = & (xx^*, x^* x) & = & (I, I) & = & e \\ \langle x^* x \rangle & = & (x^* x, xx^*) & = & (I, I) & = & e \end{array}$$

But we can also insert an element and its inverse into different points of the source and target:

$$\begin{array}{rcl} \langle x \rangle & = & (x, x) \\ & = & (xyy^*, yy^* x) \\ & = & (xyy^*, yxy^*) \circ (yxy^*, yy^* x) \\ & = & (xyy^*, yxy^*) \otimes (yxy^*, yxy^*)^* \otimes (yxy^*, yy^* x) \\ & = & (xy, yx) \otimes (y^*, y^*) \otimes (y, y) \otimes (x, x)^* (y^*, y^*) \otimes (y, y) \otimes (xy^*, y^* x) \\ & = & \langle xy \rangle \langle x^* \rangle \langle xy^* \rangle \end{array}$$

The relations $(xy, yx) = (zz^* xy, yzz^* x)$ and so forth are all composed of successive instance of the above, so these are all of the relations on our generators $\langle x \rangle$ and $\langle xy \rangle$. $\square$

Of course, the collection of relations we just gave in Proposition 4.20 are nowhere near minimal. Many of them clearly interact with each other in ways that would let us simplify or cancel some relations, or even generators. However, we will not expend any effort trying to do this, because we do not need to. With this inefficient presentation of $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$ in hand, we have in a sense already found its abelianisation. After all, for any presentation of some group H, the group H^{ab} possesses a presentation consisting of the exact same generators, subject to the same relations, plus a commutativity condition. This too will not normally be the most efficient description of the new group, but that remains true even if the presentation of H we started with was minimal, and so any time spent finding one will just be wasted. Instead, we'll suppress the urge to simplify Proposition 4.20 and move straight on to tackling $(\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}}$.

Proposition 4.21.

$$(\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}} \quad = \quad \mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}}$$

Proof. It follows immediately from Proposition 4.20 that the group $(\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}}$ has a presentation on generators

$$\langle x \rangle, \quad \langle xy \rangle, \quad x, y \in \{z_1, ..., z_n, z_1^*, ..., z_n^*\}$$

subject to the relations

$$\langle x \rangle^{-1} \quad = \quad \langle x^* \rangle, \qquad \langle xy \rangle^{-1} \quad = \quad \langle y^* x^* \rangle$$

$$\langle xx^* \rangle \quad = \quad e \quad = \quad \langle x^* x \rangle, \qquad \langle xx \rangle \quad = \quad \langle x \rangle^2$$

$$\langle xy \rangle \langle x^* \rangle \langle xy^* \rangle \quad = \quad \langle x \rangle$$

$$\langle xy \rangle \langle x^* \rangle \langle y^* \rangle \langle yx \rangle \quad = \quad \langle x \rangle \langle y \rangle \quad = \quad \langle yx \rangle \langle x^* \rangle \langle y^* \rangle \langle xy \rangle$$

$$\langle xy \rangle \langle x^* \rangle \langle xz \rangle \langle x^* \rangle \langle z^* \rangle \langle y^* \rangle \langle yz \rangle \langle y^* \rangle \langle yx \rangle \langle y^* \rangle \langle x^* \rangle \langle z^* \rangle \langle zx \rangle \langle z^* \rangle \langle zy \rangle \quad = \quad \langle x \rangle \langle y \rangle \langle z \rangle$$

but then also the commutativity conditions

$$\begin{aligned} \langle x \rangle \langle y \rangle &= \langle y \rangle \langle x \rangle \\ \langle x \rangle \langle yz \rangle &= \langle z \rangle \langle xy \rangle \\ \langle wx \rangle \langle yz \rangle &= \langle yz \rangle \langle wx \rangle \end{aligned}$$

Rearranging all of the former equations with the latter in mind, we get

$$\langle x\rangle^{-1} = \langle x^*\rangle, \qquad \langle xy\rangle^{-1} = \langle y^*x^*\rangle$$

$$\langle xx^*\rangle = e = \langle x^*x\rangle, \qquad \langle xx\rangle = \langle x\rangle^2 = \langle xy\rangle\langle xy^*\rangle$$

$$\langle xy\rangle\langle yx\rangle = \langle x\rangle^2\langle y\rangle^2$$

$$\langle xy\rangle\langle yx\rangle\langle xz\rangle\langle zx\rangle\langle yz\rangle\langle zy\rangle = \langle x\rangle^4\langle y\rangle^4\langle z\rangle^4$$

The last of these relations is just a consequence of the one above that,

$$\begin{aligned}\langle xy\rangle\langle yx\rangle\langle xz\rangle\langle zx\rangle\langle yz\rangle\langle zy\rangle &= \Big(\langle x\rangle^2\langle y\rangle^2\Big)\Big(\langle x\rangle^2\langle z\rangle^2\Big)\Big(\langle y\rangle^2\langle y\rangle^2\Big) \\ &= \langle x\rangle^4\langle y\rangle^4\langle z\rangle^4\end{aligned}$$

and in turn, the second-to-last follows from the relation above it,

$$\begin{aligned}\langle x\rangle^2\langle y\rangle^2 &= \Big(\langle xy\rangle\langle xy^*\rangle\Big)\Big(\langle yx\rangle\langle yx^*\rangle\Big) \\ &= \langle xy\rangle\langle yx\rangle\langle xy^*\rangle\langle xy^*\rangle^{-1} \\ &= \langle xy\rangle\langle yx\rangle\end{aligned}$$

Without these, we are just left with equations in two or fewer variables. Then for any two $z_i, z_j \in \mathbb{Z}^{*n}$, $i < j$, the first three relations tell us that we only need to consider generators of the form

$$\langle z_i\rangle, \quad \langle z_j\rangle, \quad \langle z_iz_j\rangle, \quad \langle z_i^*z_j\rangle, \quad \langle z_iz_j^*\rangle, \quad \langle z_i^*z_j^*\rangle$$

Finally, the remaining relation $\langle x\rangle^2 = \langle xy\rangle\langle xy^*\rangle$ induces a system of four linear equations on these six generators, which can be solved to give

$$\begin{aligned}\langle z_i^*z_j\rangle &= \langle z_j\rangle^2\langle z_iz_j\rangle^{-1} \\ \langle z_iz_j^*\rangle &= \langle z_i\rangle^2\langle z_iz_j\rangle^{-1} \\ \langle z_i^*z_j^*\rangle &= \langle z_i\rangle^{-2}\langle z_j\rangle^{-2}\langle z_iz_j\rangle\end{aligned}$$

and three independent variables, $\langle z_i\rangle$, $\langle z_j\rangle$, and $\langle z_iz_j\rangle$. In other words, $(\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}}$ is the free abelian group generated by elements of this form, for $1 \le i < j \le n$, which means that

$$(\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}} = \mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}}$$

□

From this presentation, it should be immediately obvious how to calculate the denominator from Proposition 4.19.

Corollary 4.22.

$$(\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}} \Big/ \mathbb{Z}^n \quad = \quad \mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}} \Big/ \mathbb{Z}^n \quad = \quad \mathbb{Z}^{\binom{n}{2}}$$

Proof. The $\mathbb{Z}^n$ term in the product of Proposition 4.21 represents the free abelian group generated by the morphisms

$$\langle x \rangle \quad := \quad (x, x) \quad = \quad \mathrm{id}_x$$

But this is exactly the same $\mathbb{Z}^n$ group that appears in the denominator of our quotient, $\mathrm{Ob}(L\mathbb{G}_n)^{\mathrm{ab}}$, so they cancel straightforwardly. □

Before moving on, we should be clear about exactly which $\mathbb{Z}^{\binom{n}{2}}$ subgroup of $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{ab}}$ we have just identified — after all, we will eventually need to perform a quotient involving it. In Proposition 4.20 we defined the generators $\langle z_i z_j \rangle$ to be the elements $(z_i \otimes z_j, z_j \otimes z_i)$ of the monoid $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$, which are the source/target combinations of morphisms of $L\mathbb{G}_n$. Using Corollary 4.6 we can identify this with a particular submonoid of the morphisms of $L\mathbb{G}_n$, specifically the image under q of the submonoid $\mathbb{N}^{*2n} \times_{\mathbb{N}^{2n}} \mathbb{N}^{*2n} = (s \times t)(\mathbb{G}_{2n}) \subseteq \mathrm{Mor}(\mathbb{G}_{2n})$ we chose in Proposition 4.5. In particular, since on objects we have $q(z_i) = z_i$ for all $1 \leq i \leq n$, the generators $(z_i \otimes z_j, z_j \otimes z_i)$ of $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$ are clearly the image of the generators $(z_i \otimes z_j, z_j \otimes z_i)$ of $\mathbb{N}^{*2n} \times_{\mathbb{N}^{2n}} \mathbb{N}^{*2n}$.

Thus, consider the following commutative diagram, whose top-left region comes from Corollary 4.6, bottom-left from the naturality of the adjoint functor $\mathrm{M}(\_)^{\mathrm{gp,ab}}$, and right-hand square from Proposition 4.18.

$$\begin{tikzcd}
 & (s \times t)(\mathbb{G}_{2n}) \arrow[rr, "q"] \arrow[dl, hook] & & (s \times t)(L\mathbb{G}_n) \arrow[dl, hook] \arrow[dr] & \\
\mathrm{Mor}(\mathbb{G}_{2n}) \arrow[rr, "q"] \arrow[dr] & & \mathrm{Mor}(L\mathbb{G}_n) \arrow[dr] & & \dfrac{(s \times t)(L\mathbb{G}_n)^{\mathrm{ab}}}{\mathrm{Ob}(L\mathbb{G}_n)^{\mathrm{ab}}} \arrow[dl, hook] \\
 & \mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \arrow[rr, "\mathrm{M}(q)^{\mathrm{gp,ab}}"] & & \mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}} &
\end{tikzcd}$$

What we've just said that if we start with the element $(z_i \otimes z_j, z_j \otimes z_i)$ of $(s \times t)(\mathbb{G}_{2n})$, moving clockwise around the diagram will send it to the generator $\langle z_i z_j \rangle$ in $(s \times t)(L\mathbb{G}_n)^{\mathrm{ab}}/\mathrm{Ob}(L\mathbb{G}_n)^{\mathrm{ab}} = \mathbb{Z}^{\binom{n}{2}}$. If we instead move anticlockwise, then we will first pass to our chosen representative $\alpha_{\mathbb{G}_{2n}}(\rho(z_i \otimes z_j, z_j \otimes z_i); \mathrm{id}_{z_i}, \mathrm{id}_{z_j})$ in $\mathrm{Mor}(\mathbb{G}_{2n})$, then its equivalence class in $\mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}}$, then its equivalence class in $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$, using the fact that $\mathrm{M}(q)^{\mathrm{gp,ab}}$ is the canonical map associated with the quotient

$$\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \quad = \quad \mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \big/ \Delta$$

which we proved back in Section 3.5. Since the bottom-right inclusion completes this circuit, we see that the specific subgroup we are talking about in **??** is

$$\mathbb{Z}^{\binom{n}{2}} = \Big\{ \Big[\alpha_{\mathbb{G}_{2n}} \Big(\rho(z_i \otimes z_j, z_j \otimes z_i) \, ; \, \mathrm{id}_{z_i}, \mathrm{id}_{z_j} \Big) \Big] \; : \; 1 \leq i < j \leq n \Big\} \subseteq \mathrm{M}(L\mathbb{G}_n)^{\mathrm{ab}}$$

Of course, ρ was an arbitrary permutation-preserving map $\mathbb{N}^{*n} \times_{\mathbb{N}} \mathbb{N}^{*n} \to G$, chosen using the freeness of its source monoid. Thus if we wanted to we could just pick the same element $\rho(2) \in \pi^{-1}((1\,2))$ to act as $\rho(z_i \otimes z_j, z_j \otimes z_i)$ for all i, j, and for simplicity's sake we will indeed assume this from now on.

4.5 Group completion of action operads

The next group we are interested in understanding a little better is $\mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}}$. Per **??**, the operations needed to produce this group out of $\mathrm{Mor}(\mathbb{G}_{2n}) = G \times_{\mathbb{N}} \mathbb{N}^{*2n}$ can be done in any order we choose, and so we will save the identification of $\otimes$ and $\circ$ until last. This will let us keep the tensor product as simple as possible whilst we are in the process of group completing and abelianising it.

So the obvious place to start is to ask how to simplify the expression $(G \times_{\mathbb{N}} \mathbb{N}^{*2n})^{\mathrm{gp}}$. In principle we might not be able to, since for generic G we lack any sort of a presentation by generators and relations. It would help if we at least knew that the group completion map $\mathrm{gp} : G \to G^{\mathrm{gp}}$ was injective — or equivalently, that there exists any group H and injective homomorphism $G \to H$ — but proving this kind of statement is notoriously tricky. In 1935, a paper by Anton Sushkevich 'proved' that a semigroup, and thus a monoid, could be embedded into a group if and only if it was cancellative.

Definition 4.23. We say that a monoid M is *left-cancellative* if for any $a, b, c \in M$, we have

$$ab = ac \quad \Longrightarrow \quad b = c$$

That is, common factors may be cancelled out on the left. Similarly, we call M *right-cancellative* if common factors can be cancelled on the right:

$$ac = bc \quad \Longrightarrow \quad a = b$$

A monoid that is both left- and right-cancellative is simply referred to as *cancellative*.

However, just two years later Anatoly Malcev published a simple counterexample [11] to Sushkevich's proposition. To make matters worse, in 1939 Malcev would go on to show that the actual set of neccessary and sufficient conditions for a semigroup to be embeddible in a group consisted of an infinite collection of independent relations [12]. Thus the requirement that the group completion of monoid be injective is a deceptively complicated one.

Luckily for us though, there does exist a much simpler set of sufficient-but-not-neccessary conditions for embeddibility which all action operads G happen to satisfy. These come from a 1948 paper by Raouf Doss [10], and in addition to cancellativity they depend on the way that a monoid deals with multiples of different elements being equal.

Definition 4.24. An element a of a monoid M is said to be *regular on the left* if it shares a common left-multiple with every other element of M. That is,

$$\forall\, b \in M, \quad \exists\, c, d \in M \quad : \quad ca = db$$

The monoid as a whole is said to be *regular on the left* if all of its elements are, but we can also define a notion of M being *quasi-regular on the left*. This means that any two elements a, b of M will share a common left-multiple if and only if

$$\exists\, c, d \in M \quad : \quad ca = db, \qquad c \text{ or } d \text{ is regular in } M$$

Again, we can define a similar condition for being quasi-regular on the right, and we say that a monoid is *quasi-regular* when it is both.

Proposition 4.25. *If a monoid M is cancellative and quasi-regular on the left, then it can be embedded into a group.*

For a given action operad, both of these conditions will follow from the way that operadic multiplication interacts with the elements of the abelian group $G(0)$.

Proposition 4.26. *Every action operad G is both cancellative and quasi-regular as a monoid under tensor product.*

Proof. Let g and g' be any elements of G which share a left-multiple, so that there exists at least one pair h, h' in G for which

$$h \otimes g \quad = \quad h' \otimes g'$$

and without loss of generality assume that $|g| \geq |g'|$, so also $|h| \leq |h'|$. The operadic product $\mu(h; e_0, ..., e_0)$ is clearly an element of the group $G(0)$, and we know from **??** that this is an abelian group under tensor product, so also let $\mu(h; e_0, ..., e_0)^*$ be its inverse. Then

$$\begin{aligned}
g \quad &= \quad \mu(h; e_0, ..., e_0)^* \otimes \mu(h; e_0, ..., e_0) \otimes \mu(g; e_1, ..., e_1) \\
&= \quad \mu(h; e_0, ..., e_0)^* \otimes \mu\Big(\, e_2\, ;\, \mu(h; e_0, ..., e_0), \mu(g; e_1, ..., e_1)\Big) \\
&= \quad \mu(h; e_0, ..., e_0)^* \otimes \mu\Big(\, \mu(e_2; h, g)\, ;\, e_0, ..., e_0, e_1, ..., e_1\Big) \\
&= \quad \mu(h; e_0, ..., e_0)^* \otimes \mu\Big(\, h \otimes g\, ;\, e_0, ..., e_0, e_1, ..., e_1\Big) \\
&= \quad \mu(h; e_0, ..., e_0)^* \otimes \mu\Big(\, h' \otimes g'\, ;\, e_0, ..., e_0, e_1, ..., e_1\Big) \\
&= \quad \mu(h; e_0, ..., e_0)^* \otimes \mu\Big(\, \mu(e_2; h', g')\, ;\, e_0, ..., e_0, e_1, ..., e_1\Big) \\
&= \quad \mu(h; e_0, ..., e_0)^* \otimes \mu\Big(\, e_2\, ;\, \mu(h'; e_0, ..., e_0, e_1, ..., e_1), \mu(g'; e_1, ..., e_1)\Big) \\
&= \quad \mu(h; e_0, ..., e_0)^* \otimes \mu(h'; e_0, ..., e_0, e_1, ..., e_1) \otimes \mu(g'; e_1, ..., e_1) \\
&= \quad \Big(\mu(h; e_0, ..., e_0)^* \otimes \mu(h'; e_0, ..., e_0, e_1, ..., e_1)\Big) \otimes g' \\
&=: \quad k \otimes g'
\end{aligned}$$

Put another way,

$$\exists\, e_0, k \in G \quad : \quad e_0 \otimes g \quad = \quad g \quad = \quad k \otimes g'$$

and e_0 obviously regular, since it is the unit I in G. Thus G is quasi-regular on the left. For quasi-regularaity on the right, there is an argument which is completely analagous to what we have done already, but which lets us rewrite h' as $h \otimes k'$ for some $k' \in G$.

Moreover, if we set $h = h'$ then we see that

$$k \quad = \quad \mu(h; e_0, ..., e_0)^* \otimes \mu(h; e_0, ..., e_0) \quad = \quad I$$

and so

$$h \otimes g = h \otimes g' \quad \Longrightarrow \quad g = g'$$

which is left-cancellativity. Right-cancellativity follows from quasi-regularaity on the right in the same way. □

Corollary 4.27. *The canonical map* $\mathrm{gp} : G \to G^{\mathrm{gp}}$ *associated with the group completion of* G *is an inclusion.*

From now on we'll just write g for $\mathrm{gp}(g)$ and g^* for $\mathrm{gp}(g)^*$, in order to save on space.

4.6 Freely generated action operads

Knowing that the monoid $G \times_{\mathbb{N}} \mathbb{N}^{*n}$ always has a particularly well-behaved group completion is a good first step towards finding a description for said completion. However, it is worth noting that **??** is true for all action operads G, which is more than we really need. After all, the only reason we care about $\mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}}$ is that we know from Proposition 4.19 that it is crucial to understanding the morphisms of *crossed* action operads. Thus it would be nice if we could use some consequence of crossed-ness to tell us even more about the inclusion map $\mathrm{gp} : G \times_{\mathbb{N}} \mathbb{N}^{*n} \to (G \times_{\mathbb{N}} \mathbb{N}^{*n})^{\mathrm{gp}}$.

One such consequence was given back in Proposition 3.9. If G is a crossed action operad, then the action operad G' defined by $G'(m) = G(m)/G(0)$ possesses the same free algebra on invertible algebra that G does. In other words, we don't even need to worry about finding $\mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}}$ for all crossed G, merely those which have a trivial $G(0)$. As it turns out, this fact is hugely relevant to our search for group completions, since elements of $G(0)$ are the only ones in G which might already have an inverse under tensor product. This follows from additivity of lengths:

$$\begin{aligned} g \otimes h = e_0 \quad &\Longrightarrow \quad |g| + |h| = |e_0| = 0 \\ &\Longrightarrow \quad |g| = -|h|, \qquad |g|, |h| \in \mathbb{N} \\ &\Longrightarrow \quad |g| = |h| = 0 \end{aligned}$$

Cancellativity, quasi-regularity, and lack of invertible objects then combine to give something much stronger than mere injectivity of the group completion map.

Proposition 4.28. *If* G *is an action operad with trivial* $G(0)$*, then* G *is a free monoid under tensor product.*

Proof. Let $\mathcal{G}$ be a subset of the monoid G, and $\mathcal{R}$ a collection of relations on the elements of $\mathcal{G}$, such that $(\mathcal{G}, \mathcal{R})$ is a presentation of G. Notice that every relation in $\mathcal{R}$ can be written in the form $h \otimes g = h' \otimes g'$, where $g, g' \in \mathcal{G}$ are generators and $h, h' \in G$ some other elements. This is because the only other kind of relations are one like $h \otimes g = e_0$, and as we've seen this is not possible if $G(0)$ is trivial. We'll assume that in this case $|g| \geq |g'|$ and hence $|h| \leq |h'|$. Using the reasoning from the proof of Proposition 4.26, we can then find $k, k' \in G$ for which

$$g = k \otimes g', \qquad h' = h \otimes k'$$

It follows that

$$h \otimes k \otimes g' \quad = \quad h \otimes g \quad = \quad h' \otimes g' \quad = \quad h \otimes k \otimes g'$$

and thus by left- and right-cancellativity, $k = k'$. In other words, the relation $h \otimes g = h' \otimes g'$ implies and is implied by a pair of relations $g = k \otimes g'$, $h' = h \otimes k$.

There are a few scenarios to consider here.

- $|k| = |g|$. This is actually not possible, as it would follow from additivity of length that $|g'| = 0$, and thus by assumption $g' = e_0$, which is not a generator of G.

- $|k| = 0$. This would mean that $k = e_0$, and so we'd also get $g = g'$ and $h = h'$. Thus we could simplify the presentation of G by replacing the relation $h \otimes g = h' \otimes g'$ in the set $\mathcal{R}$ with $h' = h$.

- $0 < |k| < |g|$. In this case $|g| > |g'|$ and thus $g \neq g'$, and so we could change our presentation of G by replacing g with k in the generator set $\mathcal{G}$, and also $h \otimes g = h' \otimes g'$ by $h' = h \otimes k$ in the relations $\mathcal{R}$.

Notice that in the latter two cases, we are always changing generators for ones that have strictly smaller lengths, and replacing relations with ones whose left- and right-hand side have strictly smaller total length. But lengths are natural numbers, and therefore if we choose any relation in $\mathcal{R}$ and repeatedly apply this process to it, after a finite number of steps we will find that we have replaced it with $e_0 = e_0$, the only relation whose sides have total length 0. Proceeding like this will let us eliminated all of the relations in $\mathcal{R}$, leaving us with a set $\mathcal{G}$ that freely generates the action operad G under tensor product. $\square$

Whenever we can be sure of that G is a free monoid — whether by using Proposition 4.28 or some other method — this freeness will carry over directly to the algebras $\mathbb{G}_n$, giving us a new way to represent their morphisms.

Proposition 4.29. *Let $\mathcal{G}$ be a set that freely generates the action operad G under tensor product, and for each $m \in \mathbb{N}$ define $\mathcal{G}_m := \mathcal{G} \cap G(m)$, the subset of $\mathcal{G}$ containing all elements of length m. Then the monoid* $\mathrm{Mor}(\mathbb{G}_n)$ *is*

$$G \times_{\mathbb{N}} \mathbb{N}^{*n} \quad = \quad \mathbb{N}^{*(\,|\mathcal{G}_0|+n|\mathcal{G}_1|+n^2|\mathcal{G}_2|+\dots\,)}$$

Proof. Let (g, w) be an arbitrary element of $G \times_{\mathbb{N}} \mathbb{N}^{*n}$. The monoid G is free of the generators $\mathcal{G}$, and $\mathbb{N}^{*n}$ is free on $\{z_1, ..., z_n\}$, so we can find unique expansions of g and w as tensor products

$$\begin{array}{rclrcl} g & = & g_1 \otimes ... \otimes g_k, & g_1, ..., g_k & \in & \mathcal{G} \\ w & = & x_1 \otimes ... \otimes x_m, & x_1, ..., x_m & \in & \{z_1, ..., z_n\} \end{array}$$

But each of the generators $z_1, ..., z_n$ has length 1, so the index m here is really just the length $|w|$, which by the definition of $G \times_{\mathbb{N}} \mathbb{N}^{*n}$ is also the length $|g| = |g_1| + ... + |g_k|$. Therefore we may write

$$\begin{array}{rcl} (g, w) & = & (g_1 \otimes ... \otimes g_k, x_1 \otimes ... \otimes x_{|w|}) \\ & = & (g_1, x_1 \otimes ... \otimes x_{|g_1|}) \otimes (g_2, x_{|g_1|+1} \otimes ... \otimes x_{|g_1|+|g_2|}) \otimes ... \\ & & \otimes (g_k, x_{|g_1|+...+|g_{k-1}|+1} \otimes ... \otimes x_{|g_1|+...+|g_k|}) \end{array}$$

That is, every element in $G \times_{\mathbb{N}} \mathbb{N}^{*n}$ may be expressed as a product of elements from the subset $\mathcal{G} \times_{\mathbb{N}} \mathbb{N}^{*n}$. Furthermore, the freeness of G and $\mathbb{N}^{*n}$ make sure that this expansion is unique, since

$$\begin{array}{rl} & (g_1, x_1 \otimes ... x_{|g_1|}) \otimes ... \otimes (g_k, x_{|g_1|+...+|g_{k-1}|+1} \otimes ... \otimes x_{|g_1|+...+|g_k|}) \\ = & (g'_1, x'_1 \otimes ... \otimes x'_{|g'_1|}) \otimes ... \otimes (g'_{k'}, x'_{|g'_1|+...+|g'_{k'-1}|+1} \otimes ... \otimes x'_{|g'_1|+...+|g'_{k'}|}) \end{array}$$

$$\Longrightarrow \qquad g_1 \otimes ... \otimes g_k \;=\; g'_1 \otimes ... \otimes g'_{k'}, \qquad x_1 \otimes ... \otimes x_m \;=\; x'_1 \otimes ... \otimes x'_{m'}$$

$$\Longrightarrow \qquad g_i \;=\; g'_i, \qquad 1 \le i \le k = k', \qquad x_j \;=\; x'_j, \qquad 1 \le j \le m = m'$$

Thus $G \times_{\mathbb{N}} \mathbb{N}^{*n}$ is the free monoid on the set

$$\mathcal{G} \times_{\mathbb{N}} \mathbb{N}^{*n} \quad = \quad \mathcal{G}_0 \times \{z_1, ..., z_n\}^0 \cup \mathcal{G}_1 \times \{z_1, ..., z_n\}^1 \cup \mathcal{G}_2 \times \{z_1, ..., z_n\}^2 \cup ...$$

which is just the free product of $\mathbb{N}$ with itself equal to the number of generators

$$\begin{aligned} |\mathcal{G} \times_{\mathbb{N}} \mathbb{N}^{*n}| &= |\mathcal{G}_0| \cdot |\{z_1, ..., z_n\}^0| + |\mathcal{G}_1| \cdot |\{z_1, ..., z_n\}^1| + |\mathcal{G}_2| \cdot |\{z_1, ..., z_n\}^2| + ... \\ &= |\mathcal{G}_0| + n|\mathcal{G}_1| + n^2|\mathcal{G}_2| + ... \end{aligned}$$

□

This obviously makes the group completion and abelianisation which we want to do trivial.

Corollary 4.30. *If $\mathcal{G}$ is a set that freely generates G under tensor product, and $\mathcal{G}_m := \mathcal{G} \cap G(m)$, then the abelian group $\mathrm{Mor}(\mathbb{G}_n)^{\mathrm{gp,ab}}$ is*

$$(G \times_{\mathbb{N}} \mathbb{N}^{*n})^{\mathrm{gp,ab}} = \mathbb{Z}^{|\mathcal{G}_0| + n|\mathcal{G}_1| + n^2|\mathcal{G}_2| + ...}$$

Now all that remains is to account for what happens when we collapse the morphisms of $\mathbb{G}_n$ — that is, evaluate the quotient

$$\mathrm{M}(\mathbb{G}_n)^{\mathrm{gp,ab}} = \mathbb{Z}^{|\mathcal{G}_0| + n|\mathcal{G}_1| + n^2|\mathcal{G}_2| + ...} \big/ \otimes \sim \circ$$

Unfortunately, because this will depend on the exact multiplicative structure of the operad groups $G(m)$, there is no way to carry out this computation in general. The best we can say is that as composition in $\mathrm{Mor}(\mathbb{G}_n)$ is partly determined by the group multiplication of the $G(m)$, then in place of $\mathcal{G}$ in the quotient in Corollary 4.30 it would suffice to have some collection of elements which generate G when using multiplication as well as tensor product.

Lemma 4.31. *Let $\mathcal{G}$ be a subset of the action operad G that freely generates it under tensor product, and let $\mathcal{G}'$ be a subset of $\mathcal{G}$ which generates G under a combination of tensor product and group multiplication. Also let $\mathcal{G}_m := \mathcal{G} \cap G(m)$ and $\mathcal{G}'_m := \mathcal{G}' \cap G(m)$. Then*

$$\mathbb{Z}^{|\mathcal{G}_0| + n|\mathcal{G}_1| + n^2|\mathcal{G}_2| + ...} \big/ \otimes \sim \circ = \mathbb{Z}^{|\mathcal{G}'_0| + n|\mathcal{G}'_1| + n^2|\mathcal{G}'_2| + ...} \big/ \otimes \sim \circ$$

Proof. Compostion in $\mathrm{Mor}(\mathbb{G}_n)$ is given by

$$\alpha(g'; \mathrm{id}_{x_{\pi(g^{-1})(1)}}, ..., \mathrm{id}_{x_{\pi(g^{-1})(m)}}) \circ \alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) = \alpha(g'g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$$

which in $G \times_{\mathbb{N}} \mathbb{N}^{*n}$ terms is

$$\Big(g', \pi(g^{-1})(w)\Big) \circ (g, w) = (g'g, w)$$

Thus any element (g, w) of $\mathcal{G} \times_{\mathbb{N}} \mathbb{N}^{*n}$ can be expressed in terms of elements of $\mathcal{G}' \times_{\mathbb{N}} \mathbb{N}^{*n}$ by way of tensor product and compostion. All we need to do is find and expansion for g using $\mathcal{G}'$, and then pull all of the multiplication and tensors outside of the brackets via the equation above and those we employed back in Proposition 4.4. This means that when we take the quotient by the relation $\otimes \sim \circ$, the equivalence class for (g, w) will be a tensor product of equivalence classes of elements from $\mathcal{G}' \times_{\mathbb{N}} \mathbb{N}^{*n}$. In other words, every generator of $\mathbb{Z}^{|\mathcal{G}_0|+n|\mathcal{G}_1|+n^2|\mathcal{G}_2|+\dots}/\otimes \sim \circ$ is contained within the subgroup coming from $\mathcal{G}'$, and therefore so is the whole of the group. That is,

$$\begin{aligned}\mathbb{Z}^{|\mathcal{G}_0|+n|\mathcal{G}_1|+n^2|\mathcal{G}_2|+\dots} \big/ \otimes \sim \circ &= \mathbb{Z}^{|\mathcal{G}' \cap \mathcal{G}_0|+n|\mathcal{G}' \cap \mathcal{G}_1|+n^2|\mathcal{G}' \cap \mathcal{G}_2|+\dots} \big/ \otimes \sim \circ \\ &= \mathbb{Z}^{|\mathcal{G}'_0|+n|\mathcal{G}'_1|+n^2|\mathcal{G}'_2|+\dots} \big/ \otimes \sim \circ\end{aligned}$$

$\square$

Beyond this, the value of this quotient will have to be found seperately for each individual action operad.

Chapter 5

Complete descriptions of free invertible algebras

At last, we finally have an expression for the morphisms of $L\mathbb{G}_n$, one built out of smaller parts which we know how to calculate. This means that it is almost time to draw together everything we have done over the past three chapters into a single, complete description of free invertible EG-algebras — at least, in cases where G is crossed or $G(1)$-generated.

5.1 The action of $L\mathbb{G}_n$

At this stage, there is only one part of this EG-algebra that we have yet to find — its action, $\alpha_{L\mathbb{G}_n}$. When our action operad G is $G(1)$-generated, everything is so simple that there is really only one thing the action could be.

Lemma 5.1. *Let G be a $G(1)$-generated action operad, g an element of $G(m)$ for some $m \in \mathbb{N}$, and $x_1, ..., x_m$ elements of $\mathbb{Z}^{*n}$. Then the action of $L\mathbb{G}_n$ is given by*

$$\alpha_{L\mathbb{G}_n}(\, g\, ;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}\,) \quad = \quad \mathrm{id}_{x_1 \otimes ... \otimes x_m}$$

Proof. In order for $\alpha_{L\mathbb{G}_n}$ to be a well-defined EG-action, the map $\alpha_{L\mathbb{G}_n}(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$ needs to have source $x_1 \otimes ... \otimes x_m$ and target $x_{\pi(g^{-1})(1)} \otimes ... \otimes x_{\pi(g^{-1})(m)}$, where by noncrossedness of G the latter is also $x_1 \otimes ... \otimes x_m$. But we know from Corollary 4.15 that all morphisms in this $L\mathbb{G}_n$ are identities, and hence we get the result. $\square$

For crossed G, things are more complicated. What we need to do is employ the trick that was previously mentioned in Section 3.3, where we exploit the surjectivity of the algebra map $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$. This will allow us to express $\alpha_{L\mathbb{G}_n}$ in terms of the action $\alpha_{\mathbb{G}_{2n}}$.

Proposition 5.2. *Let G be a crossed action operad, and for some $m \in \mathbb{N}$ choose an element $g \in G(m)$ and morphisms $(x_1, y_1, h_1), ..., (x_m, y_m, h_m)$ in $L\mathbb{G}_n$. That is, the (x_i, y_i) are pairs of objects from $(s \times t)(L\mathbb{G}_n)$, and the h_i are morphisms in $L\mathbb{G}_n(I, I)$. Then the action of $L\mathbb{G}_n$ is given by*

$$\alpha_{L\mathbb{G}_n}\Big(\, g\,;\, (x_1, y_1, h_1), ..., (x_m, y_m, h_m)\Big)$$
$$=$$
$$\Big(\, \textstyle\bigotimes_i x_i, \quad \bigotimes_i y_{\pi(g^{-1})(i)}, \quad \Psi\alpha_{\mathbb{G}_{2n}}(\, g\,;\, \mathrm{id}_{q^{-1}(y_1)}, ..., \mathrm{id}_{q^{-1}(y_m)}\,) \otimes (\bigotimes_i h_i)\Big)$$

Here $q^{-1} : \mathrm{Ob}(L\mathbb{G}_n) \to \mathrm{Ob}(\mathbb{G}_{2n})$ is the function

$$\begin{array}{rcrcl} q^{-1} & : & \mathbb{Z}^{*n} & \to & \mathbb{N}^{*2n} \\ & : & z_i & \mapsto & z_i \\ & : & z_i^* & \mapsto & z_{n+1} \\ & : & w & \mapsto & \bigotimes_{i=1}^{|w|} q^{-1}\Big(d(w, i)\Big) \end{array}$$

with $\bigotimes_{i=1}^{|w|} d(w, i)$ the decomposition of w given in Definition 3.23, and $\Psi : \mathrm{Mor}(\mathbb{G}_{2n}) \to L\mathbb{G}_n(I, I)$ is the canonical map associated with the repeated quotient

$$\begin{tikzcd} \mathrm{Mor}(\mathbb{G}_{2n}) \arrow[r] & \mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \arrow[r] & \mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \big/ \Delta \arrow[d, equal] & \\ & & \mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \arrow[r] & \mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \big/ \mathbb{Z}^{\binom{n}{2}} \arrow[d, equal] \\ & & & L\mathbb{G}_n(I, I) \end{tikzcd}$$

Proof. Firstly, by the rules governing EG-actions and Lemma 2.21, we know that

$$\begin{aligned} & \alpha_{L\mathbb{G}_n}\Big(\, g\,;\, (x_1, y_1, h_1), ..., (x_m, y_m, h_m)\Big) \\ = \;& \alpha_{L\mathbb{G}_n}(\, g\,;\, \mathrm{id}_{y_1}, ..., \mathrm{id}_{y_m}\,) \circ \Big(\, (x_1, y_1, h_1) \otimes ... \otimes (x_m, y_m, h_m)\Big) \\ = \;& \alpha_{L\mathbb{G}_n}(\, g\,;\, \mathrm{id}_{y_1}, ..., \mathrm{id}_{y_m}\,) \circ (\, x_1 \otimes ... \otimes x_m,\; y_1 \otimes ... \otimes y_m,\; h_1 \otimes ... \otimes h_m\,) \\ = \;& \alpha_{L\mathbb{G}_n}(\, g\,;\, \mathrm{id}_{y_1}, ..., \mathrm{id}_{y_m}\,) \otimes \mathrm{id}^*_{y_1 \otimes ... \otimes y_m} \otimes (\, x_1 \otimes ... \otimes x_m,\; y_1 \otimes ... \otimes y_m,\; h_1 \otimes ... \otimes h_m\,) \end{aligned}$$

Since we already understand tensor products of objects and unit endomorphisms, we now only need to find the action morphisms on identities. Moreover, we know that the source and target of $\alpha_{L\mathbb{G}_n}(g; \mathrm{id}_{y_1}, ..., \mathrm{id}_{y_m})$ have to be $y_1 \otimes ... \otimes y_m$ and $y_{\pi(g^{-1})(1)} \otimes ... \otimes y_{\pi(g^{-1})(m)}$ respectively, so to see this morphism as an element of the monoid

$$\mathrm{Mor}(L\mathbb{G}_n) \quad \cong \quad (s \times t)(L\mathbb{G}_n) \times L\mathbb{G}_n(I, I)$$

all that is left to understand is its projection onto the unit endormorphisms.

Now, recall that $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$ is a surjective map of EG-algebras, so that for any $f_i \in \mathrm{Mor}(L\mathbb{G}_n)$ there exist $f'_i \in \mathrm{Mor}(\mathbb{G}_{2n})$ with $q(f'_i) = f_i$, and hence

$$q\Big(\alpha_{\mathbb{G}_{2n}}(\, g \,;\, f'_1, ..., f'_m \,) \Big) \quad = \quad \alpha_{L\mathbb{G}_n}(\, g \,;\, f_1, ..., f_m \,)$$

In particular, for the identities $\mathrm{id}_{y_i} \in \mathrm{Mor}(L\mathbb{G}_n)$ we can choose $\mathrm{id}_{q^{-1}(y_i)} \in \mathrm{Mor}(\mathbb{G}_{2n})$, as by design $q(\mathrm{id}_{q^{-1}(y_i)}) = \mathrm{id}_{qq^{-1}(y_i)} = \mathrm{id}_{y_i}$. This means that if we denote by $p_I : \mathrm{Mor}(L\mathbb{G}_n) \to L\mathbb{G}_n(I, I)$ the projection onto unit endomorphisms, we will have

$$p_I\Big(\alpha_{L\mathbb{G}_n}(\, g \,;\, \mathrm{id}_{y_1}, ..., \mathrm{id}_{y_m} \,) \Big) \quad = \quad p_I q\Big(\alpha_{\mathbb{G}_{2n}}(\, g \,;\, \mathrm{id}_{q^{-1}(y_1)}, ..., \mathrm{id}_{q^{-1}(y_m)} \,) \Big)$$

But $p_I \circ q$ is a map that can be described in a different way. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Mor}(\mathbb{G}_{2n}) \arrow[r, "q"] \arrow[dd] & \mathrm{Mor}(L\mathbb{G}_n) \arrow[r, "p_I"] \arrow[d, "\mathrm{ab}"] & L\mathbb{G}_n(I, I) \arrow[d, equal] \\
 & \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}} \arrow[r] \arrow[d] & \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}} \big/ (s \times t)(L\mathbb{G}_n)^{\mathrm{ab}} \arrow[d, equal] \\
\mathrm{M}(\mathbb{G}_{2n})^{\mathrm{gp,ab}} \arrow[r, "\mathrm{M}(q)^{\mathrm{gp,ab}}"] & \mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \arrow[r] & \mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \big/ \mathbb{Z}^{\binom{n}{2}}
\end{tikzcd}$$

where all unlabelled arrows are the appriate quotient maps. The region on the left commutes by naturality of the adjoint functor $\mathrm{M}(\_)^{\mathrm{gp,ab}}$, and the bottom-right square uses the fact that

$$\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}} \big/ (s \times t)(L\mathbb{G}_n)^{\mathrm{ab}} \quad = \quad \frac{\Big(\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}} / \mathrm{Ob}(L\mathbb{G}_n)^{\mathrm{ab}}\Big)}{\Big((s \times t)(L\mathbb{G}_n)^{\mathrm{ab}} / \mathrm{Ob}(L\mathbb{G}_n)^{\mathrm{ab}}\Big)} \quad = \quad \mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \big/ \mathbb{Z}^{\binom{n}{2}}$$

As for the square on the top-right, remember that our product description of morphisms of $L\mathbb{G}_n$ came from a split extension of groups

$$L\mathbb{G}_n(I,I) \hookrightarrow \mathrm{Mor}(L\mathbb{G}_n) \underset{\longleftarrow}{\xrightarrow{s\times t}} (s\times t)(L\mathbb{G}_n)$$

Thus by the proof of **??**, the specific isomorphism we are using is

$$\begin{aligned} \mathrm{Mor}(L\mathbb{G}_n) &\cong (s\times t)(L\mathbb{G}_n)\times L\mathbb{G}_n(I,I) \\ f &\mapsto \Big(s(f),\, t(f),\, f\otimes i\Big(s(f), t(f) \Big)^* \Big) \end{aligned}$$

and so the projection p_I is given by tensoring a morphism with the inverse of the representative of its source and target under the inclusion $(s\times t)(L\mathbb{G}_n) \hookrightarrow \mathrm{Mor}(L\mathbb{G}_n)$. But the monoid $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}}/(s\times t)(L\mathbb{G}_n)^{\mathrm{ab}}$ is exactly what we get when we quotient out by those representatives, so we see that

$$\begin{aligned} [\mathrm{ab}(f)] &= [\mathrm{ab}(f)]\otimes \Big[\mathrm{ab}\Big(i\Big(s(f), t(f) \Big)^* \Big)\Big] \\ &= \Big[\mathrm{ab}\Big(f\otimes i\Big(s(f), t(f) \Big)^* \Big)\Big] \\ &= \mathrm{ab}\Big(p_I(f) \Big) \\ &= p_I(f) \end{aligned}$$

Here we've used that fact that the equivalence classe of a unit endomorphism under the quotient map $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \to \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}}/(s\times t)(L\mathbb{G}_n)^{\mathrm{ab}} = L\mathbb{G}_n(I,I)$ is just the same endomorphism again, and also that $L\mathbb{G}_n(I,I)^{\mathrm{ab}} = L\mathbb{G}_n(I,I)$.

Thus all of the regions within the diagram do commute, and hence so will the outside. That is, $p_I\circ q$ is equal to the composite running along the left and bottom edges, which is what we called Ψ. This means that the projection onto $L\mathbb{G}_n(I,I)$ of our action on identities is

$$\begin{aligned} p_I\Big(\alpha_{L\mathbb{G}_n}(\, g\,;\, \mathrm{id}_{y_1}, ..., \mathrm{id}_{y_m}\,) \Big) &= p_I q\Big(\alpha_{\mathbb{G}_{2n}}(\, g\,;\, \mathrm{id}_{q^{-1}(y_1)}, ..., \mathrm{id}_{q^{-1}(y_m)}\,) \Big) \\ &= \Psi\Big(\alpha_{\mathbb{G}_{2n}}(\, g\,;\, \mathrm{id}_{q^{-1}(y_1)}, ..., \mathrm{id}_{q^{-1}(y_m)}\,) \Big) \end{aligned}$$

and so the whole action on identities is given by

$$\alpha_{L\mathbb{G}_n}(\, g\,;\, \mathrm{id}_{y_1}, ..., \mathrm{id}_{y_m}\,)$$
$$=$$
$$\Big(\, y_1\otimes ...\otimes y_m,\quad y_{\pi(g^{-1})(1)}\otimes ...\otimes y_{\pi(g^{-1})(m)},\quad \Psi\alpha_{\mathbb{G}_{2n}}(\, g\,;\, \mathrm{id}_{q^{-1}(y_1)}, ..., \mathrm{id}_{q^{-1}(y_m)}\,) \,\Big)$$

Therefore in the more general case, the action of $L\mathbb{G}_n$ is given by

$$\alpha_{L\mathbb{G}_n}\Big(g\,;\,(x_1,y_1,h_1),...,(x_m,y_m,h_m)\Big)$$
$$=$$
$$\Big(\bigotimes_i y_i,\, \bigotimes_i y_{\pi(g^{-1})(i)},\, \Psi\alpha_{\mathbb{G}_{2n}}(\, g\,;\, \mathrm{id}_{q^{-1}(y_1)},...,\mathrm{id}_{q^{-1}(y_m)}\,)\Big) \otimes\, \mathrm{id}^*_{\otimes_i y_i} \otimes (\, \bigotimes_i x_i,\, \bigotimes_i y_i,\, \bigotimes_i h_i\,)$$
$$=$$
$$\Big(\bigotimes_i y_i, \quad \bigotimes_i y_{\pi(g^{-1})(i)}, \quad \Psi\alpha_{\mathbb{G}_{2n}}(\, g\,;\, \mathrm{id}_{q^{-1}(y_1)},...,\mathrm{id}_{q^{-1}(y_m)}\,)\Big)$$
$$\otimes\, (\, \bigotimes_i y_i,\, \bigotimes_i y_i,\, \mathrm{id}_I\,)^* \,\otimes\, (\, \bigotimes_i x_i,\, \bigotimes_i y_i,\, \bigotimes_i h_i\,)$$
$$=$$
$$\Big(\bigotimes_i x_i, \quad \bigotimes_i y_{\pi(g^{-1})(i)}, \quad \Psi\alpha_{\mathbb{G}_{2n}}(\, g\,;\, \mathrm{id}_{q^{-1}(y_1)},...,\mathrm{id}_{q^{-1}(y_m)}\,) \otimes (\bigotimes_i h_i)\Big)$$

□

5.2 A full description of $L\mathbb{G}_n$

With this last proposition proven, the results in this paper now collectively describe how to construct the free EG-algebras on n invertible objects for most values of G. However, since this characterization was discovered by us in such a piecemeal fashion, we will now restate everything in one place, for ease of reading. We'll begin with the uncrossed case, or as much of it as we were able to draw a complete conclusion about.

Theorem 5.3. *Let G be a $G(1)$-generated action operad. Then the free* EG*-algebra on n invertible objects is just the discrete category*

$$L\mathbb{G}_n \quad = \quad \mathbb{Z}^{*n}$$

equipped with a tensor product which is the usual monoid multiplication, and an EG*-action given by*

$$\alpha_{L\mathbb{G}_n}(\, g\,;\, \mathrm{id}_{x_1},...,\mathrm{id}_{x_m}\,) \quad = \quad \mathrm{id}_{x_1\otimes ...\otimes x_m}$$

Proof. The object monoid is from Proposition 2.14, the fact that $L\mathbb{G}_n$ is discrete follows from Corollary 4.15, and the action is given by Lemma 5.1. □

It is a shame that we were not able to find a formulation for uncrossed $L\mathbb{G}_n$ in full generality; this will have to be the subject of future research. For crossed action operads however, we were able to achieve this.

Theorem 5.4. *Let G be a crossed action algebra, and let G' be the action operad defined by $G'(m) := G(m)/G(0)$. Choose a subset $\mathcal{G}$ that generates G' under a combination of tensor product and group multiplication, which itself has subsets $\mathcal{G}_m := \mathcal{G} \cap G'(m)$. Then denote by A the abelian group obtained from the free abelian group*

$$\mathrm{F}(\mathcal{G} \times_{\mathbb{N}} \mathbb{N}^{*2n}) \quad = \quad \mathbb{Z}^{2n|\mathcal{G}_1|+(2n)^2|\mathcal{G}_2|+\dots}$$

via the following steps:

1. *For all $g, g' \in G(m)$ and $w \in \mathbb{N}^{*2n}$ with $|w| = m$, quotient out by the relation*

$$(g, w) \otimes \Big(g', \pi(g^{-1})(w)\Big) \quad \sim \quad (g \cdot g', w)$$

2. *Quotient out by the subgroup Δ, which is generated by the equivalence classes of elements of the form*

$$\begin{gathered}\Big(\mu(\,g\,;\, e_{|\tilde{\delta}(x_1)|}, \dots, e_{|\tilde{\delta}(x_m)|}\,),\, \tilde{\delta}(\,x_1 \otimes \dots \otimes x_m\,)\Big)\\ \otimes \\ \Big(\mu(\,g\,;\, e_{|\tilde{I}(x_1)|}, \dots, e_{|\tilde{I}(x_m)|}\,),\, \tilde{I}(\,x_1 \otimes \dots \otimes x_m\,)\Big)^*\end{gathered}$$

   where $g \in G(m)$, the x_i are generators of $\mathbb{N}^{*4n}$, and for all $1 \leq i \leq n$,

$$\begin{aligned}\tilde{\delta}(z_i) &= z_i, & \tilde{I}(z_i) &= z_i,\\ \tilde{\delta}(z_{n+i}) &= z_{n+i}, & \tilde{I}(z_{n+i}) &= z_{n+i},\\ \tilde{\delta}(z_{2n+i}) &= z_i \otimes z_{n+i}, & \tilde{I}(z_{2n+i}) &= I,\\ \tilde{\delta}(z_{3n+i}) &= z_{n+i} \otimes z_i & \tilde{I}(z_{3n+i}) &= I\end{aligned}$$

3. *Choose any $\rho(2) \in \pi^{-1}((1\,2))$, and then quotient out by the $\mathbb{Z}^{\binom{n}{2}}$ subgroup generated by the equivalence classes of the elements*

$$\Big(\rho(2)\,;\, z_i, z_j\Big), \qquad 1 \leq i < j \leq n$$

Also, denote by $\Psi : G \times_{\mathbb{N}} \mathbb{N}^{*2n} \to A$ the corresponding quotient map. Then the free EG*-algebra on n invertible objects is the category*

$$L\mathbb{G}_n \quad = \quad \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \,\times\, \mathrm{B}A$$

equipped with a tensor product given by componentwise monoid multiplication,

$$(x', y', h') \otimes (x, y, h) \;=\; (\, x' \otimes x,\; y' \otimes y,\; h'h \,)$$

and an EG*-action given by*

$$\begin{array}{c} \alpha_{L\mathbb{G}_n}\Big(\, g\,;\, (x_1, y_1, h_1), ..., (x_m, y_m, h_m) \,\Big) \\ = \\ \Big(\; \bigotimes_i x_i, \quad \bigotimes_i y_{\pi(g^{-1})(i)}, \quad \Psi\alpha_{\mathbb{G}_{2n}}(\, g\,;\, \mathrm{id}_{q^{-1}(y_1)}, ..., \mathrm{id}_{q^{-1}(y_m)} \,) \otimes (\bigotimes_i h_i) \Big) \end{array}$$

where q^{-1} *is the function*

$$\begin{array}{rcrcl} q^{-1} & : & \mathbb{Z}^{*n} & \to & \mathbb{N}^{*2n} \\ & : & z_i & \mapsto & z_i \\ & : & z_i^* & \mapsto & z_{n+1} \\ & : & w & \mapsto & \bigotimes_{i=1}^{|w|} q^{-1}\Big(\, d(w, i) \,\Big) \end{array}$$

Proof. First, notice that we are allowed to quotient out by a factor of $G(0)$ because of Proposition 3.9. Then the category $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \times \mathrm{B}A$ is just the one which has objects $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}$, morphisms $\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \times A$, and composition

$$(y, z, h') \circ (x, y, h) \quad = \quad (x, z, h'h)$$

We know that the objects and morphisms are correct by Propositions 2.14, 4.19 and 4.29 and Corollary 4.22, and since these deal with their monoidal structure too we can also see that the given tensor product is correct. Then for composition, it follows from Lemma 2.21 that

$$\begin{aligned} (y, z, h') \circ (x, y, h) &= (y, z, h') \otimes \mathrm{id}_{y^*} \otimes (x, y, h) \\ &= (y, z, h') \otimes (y^*, y^*, \mathrm{id}_I) \otimes (x, y, h) \\ &= (\, y \otimes y^* \otimes x,\; z \otimes y^* \otimes y,\; h' \otimes \mathrm{id}_I \otimes h \,) \\ &= (x, z, h'h) \end{aligned}$$

The action we just found in Proposition 5.2 then completes this description of $L\mathbb{G}_n$. □

5.3 Free symmetric monoidal categories on invertible objects

Even collected all together, Theorem 5.4 is still a fairly opaque result. In the next couple of sections we will work through some specific applications of the theorem, which will hopefully prove enlightening in this regard. A good place to start will be with the simplest of all the crossed action operads, the symmetric operad S. As one might expect, the free invertible algebras $L\mathbb{S}_n$ have a particularly straightforward form when viewed as monoidal categories.

Proposition 5.5. *The underlying monoidal category of the free* ES*-algebra on n invertible objects is*

$$L\mathbb{S}_n \quad = \quad \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \times \mathrm{B}\mathbb{Z}_2^n$$

with componentwise tensor product.

Proof. The symmetric operad has only one nullary operation, e_0, the identity permutation on 0 objects, and so the quotient operad S/S_0 is still just S. Moreover, we saw back in Section 1.1 that the symmetric groups S_m are generated by the elementary transpositions $(i\ \ i+1)$, which in turn are tensor products

$$\begin{aligned} (i\ \ i+1) &= e_{i-1} \otimes (1\,2) \otimes e_{m-i-1} \\ &= (e_1)^{\otimes(i-1)} \otimes (1\,2) \otimes (e_1)^{\otimes(m-i-1)} \end{aligned}$$

in the operad S. Therefore the set $\mathcal{S} = \{e_1, (1\,2)\}$ will suffice to generate S under multiplication and tensor product, and so our search for the unit endomorphisms of $L\mathbb{S}_n$ can begin with the group

$$\mathbb{Z}^{2n|\mathcal{S}_1|+(2n)^2|\mathcal{S}_2|+\ldots} \quad = \quad \mathbb{Z}^{2n+(2n)^2}$$

First of all, we need to collapse the composition and tensor product inherited from $\mathbb{S}_{2n}$ into the same operation. For the generators with permutation part e_1, we have

$$\begin{aligned} (e_1; z_i) \otimes (e_1; z_i) &\sim (e_1 \cdot e_1; z_i) = (e_1; z_i) \\ \Longrightarrow \quad (e_1; z_i) &\sim I \end{aligned}$$

and this will allow us to immediately eliminate those elements, leaving the group $\mathbb{Z}^{(2n)^2}$ coming from the $(1\,2)$ generators. The effect that collapsing compositon has on these elements will depend on how elementary transpositions interact under group multiplication. This comes down two three conditions from Lemma 1.3,

$$\begin{aligned}(i\ \ i+1)^2 &= e\\(i-1\ \ i)(i\ \ i+1)(i-1\ \ i) &= (i\ \ i+1)(i-1\ \ i)(i\ \ i+1)\\(i\ \ i+1)(j\ \ j+1) &= (j\ \ j+1)(i\ \ i+1), \qquad\qquad i+1<j\end{aligned}$$

The last of these will not induce any new relation on our generators, since they all already commute. Likewise, we know that

$$(i\ \ i+1) = e_{i-1}\otimes(1\,2)\otimes e_{n-i-1}, \qquad (e_1;z_1)\sim I$$

for any i, and so the second condition is implied by the specific case

$$(1\,2)(2\,3)(1\,2) = (2\,3)(1\,2)(2\,3)$$

which only produces a commutativity condition on our generators:

$$\begin{aligned}&\Big((1\,2)\,;\,z_i,z_j\Big)\otimes\Big((1\,2)\,;\,z_i,z_k\Big)\otimes\Big((1\,2)\,;\,z_j,z_k\Big)\\ \sim\ &(e_1;z_k)\otimes\Big((1\,2)\,;\,z_i,z_j\Big)\otimes\Big((1\,2)\,;\,z_i,z_k\Big)\otimes(e_1;z_j)\otimes(e_1;z_i)\otimes\Big((1\,2)\,;\,z_j,z_k\Big)\\ \sim\ &\Big((e_1\otimes(1\,2))\cdot((1\,2)\otimes e_1)\cdot(e_1\otimes(1\,2))\,;\,z_i,z_j,z_k\Big)\\ =\ &\Big((2\,3)(1\,2)(2\,3)\,;\,z_i,z_j,z_k\Big)\\ =\ &\Big((1\,2)(2\,3)(1\,2)\,;\,z_i,z_j,z_k\Big)\\ =\ &\Big(((1\,2)\otimes e_1)\cdot(e_1\otimes(1\,2))\cdot((1\,2)\otimes e_1)\,;\,z_i,z_j,z_k\Big)\\ \sim\ &\Big((1\,2)\,;\,z_j,z_k\Big)\otimes(e_1;z_i)\otimes(e_1;z_j)\otimes\Big((1\,2)\,;\,z_i,z_k\Big)\otimes\Big((1\,2)\,;\,z_i,z_j\Big)\otimes(e_1;z_k)\\ \sim\ &\Big((1\,2)\,;\,z_j,z_k\Big)\otimes\Big((1\,2)\,;\,z_i,z_k\Big)\otimes\Big((1\,2)\,;\,z_i,z_j\Big)\end{aligned}$$

Thus the only restraint we need to impose on our remaining generators is

$$\begin{aligned}\Big((1\,2)\,;\,z_i,z_j\Big)\otimes\Big((1\,2)\,;\,z_j,z_i\Big) &\sim \Big((1\,2)\cdot(1\,2)\,;\,z_i,z_j\Big)\\ &= (e_2;z_i,z_j)\\ &= (e_1;z_i)\otimes(e_1;z_j)\\ &= I\end{aligned}$$

which can be treated as two different cases depending on the values of the indices. From $i\neq j$ we will get $\binom{2n}{2}$ pairs of distinct generators $((1\,2);z_i,z_j)$, $((1\,2);z_j,z_i)$ whose equivalence classes are inverses of one other, and from $i=j$ we see that the classes of the $2n$ generators $((1\,2); z_i, z_i)$ are all self-inverse. In other words,

$$\mathbb{Z}^{2n+(2n)^2} \big/ \otimes \sim \circ \quad = \quad \mathbb{Z}_2^{2n} \times \mathbb{Z}^{\binom{2n}{2}}$$

where Z_2 is the cyclic group of order 2.

Next, we need to consider the subgroup Δ, which comes from the equivalence classes of elements of the form

$$\begin{gathered} \Big(\mu(\, g\,;\, e_{|\tilde{\delta}(x_1)|}, ..., e_{|\tilde{\delta}(x_m)|}\,),\; \tilde{\delta}(\, x_1 \otimes ... \otimes x_m\,) \Big) \\ \otimes \\ \Big(\mu(\, g\,;\, e_{|\tilde{I}(x_1)|}, ..., e_{|\tilde{I}(x_m)|}\,),\; \tilde{I}(\, x_1 \otimes ... \otimes x_m\,) \Big)^* \end{gathered}$$

for $x_i \in \{z_1, ..., z_{4n}\}$. At this point we are only interested in cases where g is $(1\,2)$, and thus $m = 2$, so pick any $1 \leq i, j \leq n$ and then suppose that $x_1 = z_i$ and $x_2 = z_j$. The corresponding element will just be

$$\Big(\mu(\,(1\,2)\,;\, e_1, e_1\,)\,;\, z_i, z_j \Big) \otimes \Big(\mu(\,(1\,2)\,;\, e_1, e_1\,)\,;\, z_i, z_j \Big)^* \quad = \quad I$$

which contributes nothing to the group Δ; the same is also true when instead either $x_1 = z_{n+i}$ or $x_2 = z_{n+j}$, or both. A more interesting result is what happens when $x_1 = z_i$ and $x_2 = z_{2n+j}$:

$$\begin{array}{cl} & \Big(\mu(\,(1\,2)\,;\, e_1, e_2\,)\,;\, z_i, z_j, z_{n+j} \Big) \otimes \Big(\mu(\,(1\,2)\,;\, e_1, e_0\,)\,;\, z_i \Big)^* \\ = & \Big((\, e_1 \otimes (1\,2)\,) \cdot (\,(1\,2) \otimes e_1\,)\,;\, z_i, z_j, z_{n+j} \Big) \otimes \Big(e_1\,;\, z_i \Big)^* \\ \sim & \Big((\, e_1 \otimes (1\,2)\,) \cdot (\,(1\,2) \otimes e_1\,)\,;\, z_i, z_j, z_{n+j} \Big) \\ = & (e_1; z_j) \otimes (\,(1\,2)\,; z_i, z_{n+j}\,) \otimes (\,(1\,2)\,; z_i, z_j\,) \otimes (e_1; z_{n+j}) \\ \sim & (\,(1\,2)\,; z_i, z_{n+j}\,) \otimes (\,(1\,2)\,; z_i, z_j\,) \end{array}$$

The presence of elements like the above will mean that when we quotient out by Δ, we will force equivalence classes of the generators $((1\,2); z_i, z_j)$ and $((1\,2); z_i, z_{n+j})$ to become inverses of one another. In an analogous way, setting $x_1 = z_{2n+j}$ and $x_2 = z_j$ shows that $((1\,2); z_{n+i}, z_j)$ will also become an inverse of $((1\,2); z_i, z_j)$, which means that $((1\,2); z_{n+i}, z_j) \sim ((1\,2); z_i, z_{n+j})$, whilst the choices $x_1 = z_{n+i}$ and $x_2 = z_{2n+j}$ will yield $((1\,2); z_{n+i}, z_j)^* \sim ((1\,2); z_{n+i}, z_{n+j})$, and hence $((1\,2); z_{n+i}, z_{n+j}) \sim ((1\,2); z_i, z_j)$. All other combinations of x_1, x_2 will end up repeating one of these relations, and so when we are done all that is left are the $n^2 = n + \binom{n}{2}$ generators of the form $((1\,2); z_i, z_j)$.

That is,

$$\mathbb{Z}_2^{2n} \times \mathbb{Z}^{\binom{2n}{2}} \big/ \Delta \quad = \quad \mathbb{Z}_2^n \times \mathbb{Z}^{\binom{n}{2}}$$

The last step needed in order to find the group A is to quotient out by a $\mathbb{Z}^{\binom{n}{2}}$ subgroup, the one generated by equivalence classes of elements $(\rho(2); z_i, z_j)$ for given $\rho(2) \in \pi^{-1}((1\,2))$ and $1 \leq i < j \leq n$. Of course, the underlying permutation map of permutations π^{S} is the identity, so $\rho(2)$ must be $(1\,2)$ itself. This gives a nice easy final quotient,

$$\mathbb{Z}_2^n \times \mathbb{Z}^{\binom{n}{2}} \big/ \mathbb{Z}^{\binom{n}{2}} \quad = \quad \mathbb{Z}_2^n$$

and so the underlying monoidal category we are looking for is

$$L\mathbb{S}_n \quad = \quad \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \; \times \; \mathrm{B}\mathbb{Z}_2^n$$

□

If we are to understand $L\mathbb{S}_n$'s role as a *symmetric* monoidal category, we now just need to use the rest of Theorem 5.4 to find its ES-action. This operation too is incredibly simple.

Proposition 5.6. *The action of $L\mathbb{S}_n$ is fully determined by the values*

$$\alpha\Big((1\,2)\,;\, \mathrm{id}_{z_i}, \mathrm{id}_{z_j} \Big) \quad = \begin{cases} \Big(z_i \otimes z_j,\ z_j \otimes z_i,\ (0, ..., 0) \Big) & \textit{if} \quad i \neq j \\ \Big(z_i \otimes z_i,\ z_i \otimes z_i,\ (0, ..., 0, 1, 0, ..., 0) \Big) & \textit{if} \quad i = j \end{cases}$$

where the 1 *appears in the ith coordinate of $\mathbb{Z}_2^n$, along with the identities*

$$\begin{aligned} \alpha\Big((1\,2)\,;\, \mathrm{id}_{z_i}, \mathrm{id}_{z_j} \Big) \quad &= \quad \alpha\Big((1\,2)\,;\, \mathrm{id}_{z_i^*}, \mathrm{id}_{z_j} \Big) \\ &= \quad \alpha\Big((1\,2)\,;\, \mathrm{id}_{z_i}, \mathrm{id}_{z_j^*} \Big) \\ &= \quad \alpha\Big((1\,2)\,;\, \mathrm{id}_{z_i^*}, \mathrm{id}_{z_j^*} \Big) \end{aligned}$$

Proof. We know that all EG-actions obey the conditions

$$\alpha(g; f_1, ..., f_m) \quad = \quad \alpha(g; \mathrm{id}_{y_1}, ..., \mathrm{id}_{y_m}) \circ (f_1 \otimes ... \otimes f_m)$$

for all morphisms $f_i : x_i \to y_i$, and

$$\begin{aligned} & \alpha(\, g\, ;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_{i-1}}, \mathrm{id}_{x_i \otimes x_i'}, \mathrm{id}_{x_{i+1}}, ...\mathrm{id}_{x_m}\,) \\ = \; & \alpha\Big(\, g\, ;\, \alpha(e_1; \mathrm{id}_{x_1}), ..., \alpha(e_1; \mathrm{id}_{x_{i-1}}), \alpha(e_2; \mathrm{id}_{x_i}, \mathrm{id}_{x_i'}), \alpha(e_1; \mathrm{id}_{x_{i+1}}), ..., \alpha(e_1; \mathrm{id}_{x_m})\,\Big) \\ = \; & \alpha\Big(\, \mu(g; e_1, ..., e_1, e_2, e_1, ..., e_1)\, ;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_{i-1}}, \mathrm{id}_{x_i}, \mathrm{id}_{x_i'}, \mathrm{id}_{x_{i+1}}, ..., \mathrm{id}_{x_m}\,\Big) \end{aligned}$$

for all elements $g \in G$ and objects $x_1, ..., x_m, x_i'$. Hence we can definitely recover all values of $\alpha_{\mathbb{S}_{2n}}$ from just those on identities morphisms, and more specifically identities of generators and their inverses. Further, the fact that we can express any permutation $\sigma \in \mathrm{S}$ in terms of e_1 and $(1\,2)$ via tensor product and group multiplication tells us that the action will also be determined just by its values on the transposition $(1\,2)$. Thus the equations in the statement of the proposition really would suffice to fix $\alpha_{L\mathbb{S}_n}$; all we need to do now is prove that they hold. The sources and targets are easy enough to verify, so we'll focus on the $\mathbb{Z}_2^n$ coordinate.

Per Theorem 5.4, we will start by forming the action morphisms

$$\alpha_{\mathbb{S}_{2n}}\Big(\,(1\,2)\,;\, \mathrm{id}_{q^{-1}(z_i)}, \mathrm{id}_{q^{-1}(z_j)}\,\Big) \quad = \quad \alpha_{\mathbb{S}_{2n}}\Big(\,(1\,2)\,;\, \mathrm{id}_{z_i}, \mathrm{id}_{z_j}\,\Big)$$

and then find their images under the map $\Psi : \mathrm{S} \times_{\mathbb{N}} \mathbb{N}^{*2n} \to \mathbb{Z}_2^n$. However, we just saw in the proof of Proposition 5.5 how this monoid homomorphism is built up as a composite

$$\mathrm{S} \times_{\mathbb{N}} \mathbb{N}^{*2n} \longrightarrow \mathbb{Z}^{2n+(2n)^2} \longrightarrow \mathbb{Z}_2^{2n} \times \mathbb{Z}^{\binom{2n}{2}} \longrightarrow \mathbb{Z}_2^n \times \mathbb{Z}^{\binom{n}{2}} \longrightarrow \mathbb{Z}_2^n$$

When $i \neq j$, the equivalence classes of the morphisms $\alpha((1\,2); \mathrm{id}_{z_i}, \mathrm{id}_{z_j})$ get sent to zero by the rightmost arrow, whereas the $\alpha((1\,2); \mathrm{id}_{z_i}, \mathrm{id}_{z_i})$ are each sent to a different generator of $\mathbb{Z}_2^n$, which is denoted by the appropriate n-tuple $(0, ..., 0, 1, 0, ..., 0)$.

So now we just need to check the morphisms involving the inverses of generators as well. The $\mathbb{S}_{2n}$ versions of these are

$$\begin{aligned} \alpha_{\mathbb{S}_{2n}}\Big(\,(1\,2)\,;\, \mathrm{id}_{q^{-1}(z_i^*)}, \mathrm{id}_{q^{-1}(z_j)}\,\Big) &= \alpha_{\mathbb{S}_{2n}}\Big(\,(1\,2)\,;\, \mathrm{id}_{z_{n+i}}, \mathrm{id}_{z_j}\,\Big) \\ \alpha_{\mathbb{S}_{2n}}\Big(\,(1\,2)\,;\, \mathrm{id}_{q^{-1}(z_i)}, \mathrm{id}_{q^{-1}(z_j^*)}\,\Big) &= \alpha_{\mathbb{S}_{2n}}\Big(\,(1\,2)\,;\, \mathrm{id}_{z_i}, \mathrm{id}_{z_{n+j}}\,\Big) \\ \alpha_{\mathbb{S}_{2n}}\Big(\,(1\,2)\,;\, \mathrm{id}_{q^{-1}(z_i^*)}, \mathrm{id}_{q^{-1}(z_j^*)}\,\Big) &= \alpha_{\mathbb{S}_{2n}}\Big(\,(1\,2)\,;\, \mathrm{id}_{z_{n+i}}, \mathrm{id}_{z_{n+j}}\,\Big) \end{aligned}$$

But again, we saw in the proof of Proposition 5.5 that the second-to-last arrow in the above diagram — the one representing the quotient by Δ — will make the equivalence class of $\alpha((1\,2); \mathrm{id}_{z_i}, \mathrm{id}_{z_j})$ equal to that of $\alpha((1\,2); \mathrm{id}_{z_{n+i}}, \mathrm{id}_{z_{n+j}})$, and inverse to the class containing both $\alpha((1\,2); \mathrm{id}_{z_{n+i}}, \mathrm{id}_{z_j})$ and $\alpha((1\,2); \mathrm{id}_{z_{n+i}}, \mathrm{id}_{z_j})$. Since every element of the group $\mathbb{Z}_2^n$ is self-inverse, this amounts to saying that all of these morphisms are equivalent under Ψ, which completes the proof. $\square$

Thus we see that in the free symmetric monoidal category on n invertible objects, every morphism can be expressed as a composite of tensor products of identities and symmetries maps

$$\beta_{z_i,z_j} \quad = \quad \alpha\Big(\,(1\,2)\,;\,\mathrm{id}_{z_i},\mathrm{id}_{z_j}\,\Big)$$

Moreover, two parallel morphisms in $L\mathbb{S}_n$ are equal if and only if the number of symmetries from

$$\Big\{\,\beta_{z_i,z_i},\beta_{z_i^*,z_i},\beta_{z_i,z_i^*},\beta_{z_i^*,z_i^*}\,\Big\}$$

appearing in these two expressions has the same parity, for each $1 \leq i \leq n$.

5.4 Free braided monoidal categories on invertible objects

Having successfully understood the symmetric monoidal case, we should now be ready to tackle the very similar world of braided monoidal categories. Indeed, since the only difference between the braid groups B_n and the symmetry groups S_n is the presense or absense of a self-invertibility condition, the abelian group $L\mathbb{B}_n(I,I)$ is simply the value we would gotten for $L\mathbb{S}_n(I,I)$ if we had never set $((1\,2);z_i,z_j)\otimes((1\,2);z_i,z_j)\sim I$.

Proposition 5.7. *The underlying monoidal category of the free* $\mathrm{E}B$*-algebra on* n *invertible objects is*

$$L\mathbb{B}_n \quad = \quad \mathbb{Z}^{*n}\times_{\mathbb{Z}^n}\mathbb{Z}^{*n}\;\times\;\mathrm{B}(\mathbb{Z}^n\times\mathbb{Z}^{\binom{n}{2}})$$

with componentwise tensor product.

Proof. The beginning of this proof is identical to that of Proposition 5.5. First, the braid operad B has $B_0=\{e_0\}$, so we don't need to take a quotient of our action operad. Next, we know from Example 1.4 that the braid groups B_m are generated by the elementary braids b_i, and these are just tensor products

$$b_i \quad = \quad (e_1)^{\otimes(i-1)}\otimes b\otimes(e_1)^{\otimes(m-i-1)}$$

where b is the elementary braid of B_2. Thus we can generate B under multiplication and tensor product from the set $\mathcal{B} = \{e_1, b\}$, and so as before we get

$$\mathbb{Z}^{2n|\mathcal{B}_1|+(2n)^2|\mathcal{B}_2|+\ldots} \quad = \quad \mathbb{Z}^{2n+(2n)^2}$$

Collapsing the composition of $\mathbb{B}_{2n}$ will then let us eliminate any generators involving e_1, since

$$\begin{aligned} (e_1; z_i) \otimes (e_1; z_i) \;&\sim\; (e_1 \cdot e_1; z_i) \;=\; (e_1; z_i) \\ \Longrightarrow \quad (e_1; z_i) \;&\sim\; I \end{aligned}$$

Moreover, the rules governing the elementary braids only state that

$$b_i b_{i+1} b_i \;=\; b_{i+1} b_i b_{i+1}, \qquad\qquad b_i b_j \;=\; b_j b_i, \quad i+1 < j$$

both of which just produce commutativity conditions on the remaining generators. In the latter case this should be obvious, and in the former it follows from the fact that

$$\begin{aligned} & (\,b\,;\, z_i, z_j\,) \,\otimes\, (\,b\,;\, z_i, z_k\,) \,\otimes\, (\,b\,;\, z_j, z_k\,) \\ \sim\;& (e_1; z_k) \otimes (\,b\,;\, z_i, z_j\,) \otimes (\,b\,;\, z_i, z_k\,) \otimes (e_1; z_j) \otimes (e_1; z_i) \otimes (\,b\,;\, z_j, z_k\,) \\ \sim\;& \Big(\, (e_1 \otimes b) \cdot (b \otimes e_1) \cdot (e_1 \otimes b)\,;\, z_i, z_j, z_k \Big) \\ =\;& (\, b_2 b_1 b_2\,;\, z_i, z_j, z_k\,) \\ =\;& (\, b_1 b_2 b_1\,;\, z_i, z_j, z_k\,) \\ =\;& \Big((b \otimes e_1) \cdot (e_1 \otimes b) \cdot (b \otimes e_1)\,;\, z_i, z_j, z_k \Big) \\ \sim\;& (\,b\,;\, z_j, z_k\,) \otimes (e_1; z_i) \otimes (e_1; z_j) \otimes (\,b\,;\, z_i, z_k\,) \otimes (\,b\,;\, z_i, z_j\,) \otimes (e_1; z_k) \\ \sim\;& (\,b\,;\, z_j, z_k\,) \otimes (\,b\,;\, z_i, z_k\,) \otimes (\,b\,;\, z_i, z_j\,) \end{aligned}$$

Thus we again arrive at a group $\mathbb{Z}^{(2n)^2}$, whose generators all have the form $(b; z_i, z_j)$. But without the self-invertibility that we had in the symmetric case we are already done with step 1 of Theorem 5.4, so that

$$\mathbb{Z}^{2n+(2n)^2} \big/ \otimes \sim \circ \quad = \quad \mathbb{Z}^{(2n)^2}$$

For step 2, we need quotient out by the subgroup Δ. For exactly the same reasons as in Proposition 5.5, we see that it contains the equivalence classes of the elements

$$\begin{aligned} & \Big(\mu(\,b\,;\,e_1,e_2\,)\,;\,z_i,z_j,z_{n+j}\Big) \otimes \Big(\mu(\,b\,;\,e_1,e_0\,)\,;\,z_i\Big)^* \\ = \;& \Big(\,(\,e_1\otimes b\,)\cdot(\,b\otimes e_1\,)\,;\,z_i,z_j,z_{n+j}\Big) \otimes (\,e_1\,;\,z_i\,)^* \\ \sim \;& \Big(\,(\,e_1\otimes b\,)\cdot(\,b\otimes e_1\,)\,;\,z_i,z_j,z_{n+j}\Big) \\ \sim \;& (e_1;z_j)\otimes(b;z_i,z_{n+j})\otimes(b;z_i,z_j)\otimes(e_1;z_{n+j}) \\ \sim \;& (b;z_i,z_{n+j})\otimes(b;z_i,z_j) \end{aligned}$$

for $1 \leq i,j \leq n$, as well as ones like

$$(b;z_{n+i},z_j)\otimes(b;z_i,z_j), \qquad (b;z_{n+i},z_{n+j})\otimes(b;z_{n+i},z_j)$$

and so forth. This means that our quotient group will be

$$\mathbb{Z}^{(2n)^2} \big/ \Delta \quad = \quad \mathbb{Z}^{n^2}$$

whose generators are the classes $[(b;z_i,z_j)] = [(b;z_{n+i},z_{n+j})]$, with inverses $[(b;z_{n+i},z_j)] = [(b;z_i,z_{n+j})]$. Moreover, this group clearly has a $\mathbb{Z}^{\binom{n}{2}}$ subgroup coming from those classes $[(b;z_i,z_j)]$ which have $1 \leq i < j \leq n$. Thus if we choose $\rho(2) \in \pi^{-1}((1\,2))$ to be the elementary braid b, the third and final quotient will give

$$\mathbb{Z}^{n^2} \big/ \mathbb{Z}^{\binom{n}{2}} \quad = \quad \mathbb{Z}^{n^2-\binom{n}{2}} \quad = \quad \mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}}$$

and therefore

$$L\mathbb{B}_n \quad = \quad \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \;\times\; \mathrm{B}(\mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}})$$

as a monoidal category. $\square$

Just to be clear, the first n generators of this group $\mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}}$ are the images under $q : \mathbb{B}_{2n} \to L\mathbb{B}_n$ of the action morphisms $\alpha_{\mathbb{B}_{2n}}(b;\mathrm{id}_{z_i},\mathrm{id}_{z_i})$, and the other $\binom{n}{2}$ come from the $\alpha_{\mathbb{B}_{2n}}(b;\mathrm{id}_{z_i},\mathrm{id}_{z_j})$ for $i > j$. This seems a little strange at first — why would $L\mathbb{B}_n$ have this kind of directionality to it, where the $i < j$ generators have been cancelled out but the $i > j$ remain? The important thing to realise is this group is representing the unit endomorphisms $L\mathbb{B}_n(I,I)$, which have the same source and target. By contrast, if $i \neq j$ then $\alpha_{\mathbb{B}_{2n}}(b;\mathrm{id}_{z_i},\mathrm{id}_{z_j})$ will have distinct source and target $z_i \otimes z_j \neq z_j \otimes z_i$, and thus the only way we can add it onto a composite without changing the source and target is to also add in the corresponding $\alpha_{\mathbb{B}_{2n}}(b;\mathrm{id}_{z_j},\mathrm{id}_{z_i})$ somewhere. Therefore we really only need to keep track of one of these two kinds of morphisms, such as all of the ones where $i > j$. This is also reflected in the action of this algebra.

Proposition 5.8. *The action of $L\mathbb{B}_n$ is fully determined by the values*

$$\alpha(\,b\,;\,\mathrm{id}_{z_i},\mathrm{id}_{z_j}\,) \quad = \begin{cases} \Big(z_i \otimes z_j,\ z_j \otimes z_i,\ (0,...,0)\Big) & \text{if} \quad i < j \\ \Big(z_i \otimes z_j,\ z_j \otimes z_i,\ (0,...,0,1,0,...,0)\Big) & \text{if} \quad i \geq j \end{cases}$$

where the 1 *appears in the ith coordinate of $\mathbb{Z}^n$ when $i = j$, and the (i,j)th coordinate of $\mathbb{Z}^{\binom{n}{2}}$ when $i > j$, and also*

$$\begin{aligned} \alpha(\,b\,;\,\mathrm{id}_{z_i},\mathrm{id}_{z_j}\,) &= \alpha(\,b\,;\,\mathrm{id}_{z_i^*},\mathrm{id}_{z_j}\,)^* \\ &= \alpha(\,b\,;\,\mathrm{id}_{z_i},\mathrm{id}_{z_j^*}\,)^* \\ &= \alpha(\,b\,;\,\mathrm{id}_{z_i^*},\mathrm{id}_{z_j^*}\,) \end{aligned}$$

Proof. Similarly to the symmetric case, the fact that any braid $x \in B_m$ can be written as tensor product and group multiple of e_1 and b will let us recover all of the values of $\alpha_{L\mathbb{S}_n}$ from just those four families of action morphisms which appear in the proposition. Their sources and targets are clearly correct, so all we need to do examine their $\mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}}$ coordinates.

We saw in the the proof of Proposition 5.7 that under the map

$$B \times_{\mathbb{N}} \mathbb{N}^{*2n} \longrightarrow \mathbb{Z}^{2n+(2n)^2} \longrightarrow \mathbb{Z}^{(2n)^2} \longrightarrow \mathbb{Z}^{n^2} \longrightarrow \mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}}$$

the action morphisms

$$\alpha_{\mathbb{S}_{2n}}(\,b\,;\,\mathrm{id}_{q^{-1}(z_i)},\mathrm{id}_{q^{-1}(z_j)}\,) \quad = \quad \alpha_{\mathbb{S}_{2n}}(\,b\,;\,\mathrm{id}_{z_i},\mathrm{id}_{z_j}\,)$$

are sent to one of the generators of $\mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}}$ when $i \geq j$, and are sent to zero otherwise. Moreover, we also proved that the morphisms

$$\alpha_{\mathbb{S}_{2n}}(\,b\,;\,\mathrm{id}_{q^{-1}(z_i^*)},\mathrm{id}_{q^{-1}(z_j^*)}\Big) \quad = \quad \alpha_{\mathbb{S}_{2n}}(\,b\,;\,\mathrm{id}_{z_{n+i}},\mathrm{id}_{z_{n+j}}\,)$$

are sent to the exact same generators as the $\alpha_{\mathbb{S}_{2n}}(b; \mathrm{id}_{z_i}, \mathrm{id}_{z_j})$, whilst the corresponding

$$\begin{aligned} \alpha_{\mathbb{S}_{2n}}(\,b\,;\,\mathrm{id}_{q^{-1}(z_i^*)},\mathrm{id}_{q^{-1}(z_j)}\,) &= \alpha_{\mathbb{S}_{2n}}(\,b\,;\,\mathrm{id}_{z_{n+i}},\mathrm{id}_{z_j}\,) \\ \alpha_{\mathbb{S}_{2n}}(\,b\,;\,\mathrm{id}_{q^{-1}(z_i)},\mathrm{id}_{q^{-1}(z_j^*)}\,) &= \alpha_{\mathbb{S}_{2n}}(\,b\,;\,\mathrm{id}_{z_i},\mathrm{id}_{z_{n+j}}\,) \end{aligned}$$

are sent to that generator's inverse. Thus by Theorem 5.4, we obtain the required relations for the action $\alpha_{L\mathbb{S}_n}$. □

To put this in a more categorical perspective, suppose that we decide to call the following kinds of braiding isomorphisms 'positive',

$$\begin{aligned}
\beta_{z_i,z_j} &= \alpha(\, b\,;\, \mathrm{id}_{z_i}, \mathrm{id}_{z_j}\,), & \beta^{-1}_{z_i^*,z_j} &= \alpha(\, b\,;\, \mathrm{id}_{z_i^*}, \mathrm{id}_{z_j}\,)^{-1}, \\
\beta^{-1}_{z_i,z_j^*} &= \alpha(\, b\,;\, \mathrm{id}_{z_i}, \mathrm{id}_{z_j^*}\,)^{-1}, & \beta_{z_i^*,z_j^*} &= \alpha(\, b\,;\, \mathrm{id}_{z_i^*}, \mathrm{id}_{z_j^*}\,)
\end{aligned}$$

and likewise call their inverses 'negative'. Then what Proposition 5.8 is saying is that in the free braided monoidal category on n invertible objects, parallel morphisms coincide only when the number of positive braidings minus the number of negative braidings they contain is the same.

Something else to notice about $L\mathbb{B}_n$ is that we've actually seen its unit endomorphism group before. Back in Proposition 4.21 we proved that for any crossed action operad G,

$$(s \times t)(L\mathbb{G}_n)^{\mathrm{ab}} \quad = \quad (\mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n})^{\mathrm{ab}} \quad = \quad \mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}}$$

This means that in the case of the braid operad, we have the unusual identity

$$(s \times t)(L\mathbb{B}_n)^{\mathrm{ab}} \quad \cong \quad L\mathbb{B}_n(I, I)$$

What is the significance of this fact? It is not entirely clear, though certainly the isomorphism involved is highly nontrivial. For example the $\mathbb{Z}^n$ subgroup of $(s\times t)(L\mathbb{B}_n)^{\mathrm{ab}}$ has generators representing maps with source and target $z_i \to z_i$, $1 \leq i \leq n$, while the same generators of $\mathbb{Z}^n \subseteq L\mathbb{B}_n(I, I)$ represent the braidings $\beta_{z_i,z_i} = \alpha(b; \mathrm{id}_{z_i}, \mathrm{id}_{z_i})$. Of course, it is possible that this connection between the groups that make up $\mathrm{Mor}(L\mathbb{B}_n)$ could simply be a conincidence. It would help if we could compare B to another action operad which shares this property — either another crossed G whose algebra has the same underlying category as the $L\mathbb{B}_n$, or an uncrossed G whose algebra has $L\mathbb{G}_n(I, I) = (\mathbb{Z}^{*n})^{\mathrm{ab}} = \mathbb{Z}^n$ — but none of these are currently known to the author.

5.5 Free ribbon braided monoidal categories on invertible objects

The last action operad whose invertible algebras we will calculate explicitly is the ribbon braid operad, RB. The details will prove largely similar to those we saw for the braided case in Proposition 5.7, much as the braided case itself was built upon the symmatric case with a few small changes.

Proposition 5.9. *The underlying monoidal category of the free* ERB*-algebra on* n *invertible objects is*

$$L\mathbb{RB}_n \quad = \quad \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \;\times\; \mathrm{B}(\mathbb{Z}^n \times \mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}})$$

with componentwise tensor product. Moreover, the action of $L\mathbb{RB}_n$ *is determined by its restriction to the subcategory* $L\mathbb{B}_n \subseteq L\mathbb{RB}_n$*, plus the values*

$$\alpha(\,t\,;\,\mathrm{id}_{z_i}\,) \quad = \quad \Big(\,z_i,\, z_i,\, (0,...,0,1,0,...,0)\,\Big)$$

where the 1 *appears in the* i*th coordinate of the copy of* $\mathbb{Z}^n$ *which is not shared with* $L\mathbb{B}_n$*, and*

$$\alpha(\,t\,;\,\mathrm{id}_{z_i^*}\,) \quad = \quad \alpha(\,t\,;\,\mathrm{id}_{z_i}\,)^* \otimes \alpha(\,b\,;\,\mathrm{id}_{z_i}, \mathrm{id}_{z_i}\,)^{\otimes 2}$$

Proof. The ribbon braid operad has $RB_0 = \{e_0\}$ and is generated under $\otimes$ and $\cdot$ by the set $\mathcal{RB} = \{e_1, b, t\}$. Thus our starting point will be the group

$$\mathbb{Z}^{2n|\mathcal{RB}_1|+(2n)^2|\mathcal{RB}_2|+\dots} \quad = \quad \mathbb{Z}^{4n+(2n)^2}$$

Since the free EB-algebra $\mathbb{B}_{2n}$ is clearly a subcategory of $\mathbb{RB}_{2n}$, when we collapse its composition we will at the least have to quotient out by all of the same relations we did in Proposition 5.7. This will amount to eliminating all of the e_1 generators, which will get us down to $\mathbb{Z}^{2n+(2n)^2}$. We also have to collapse our morphisms according to the rules which govern multiplication by twists, but just as with the braids it turns out that these are already implicit in commutativity. For example, in RB_2 we have

$$\begin{aligned}
(\,b\,;\,z_i, z_j\,) \otimes (\,t\,;\,z_i\,) \;&\sim\; (\,b\,;\,z_i, z_j\,) \otimes (\,t\,;\,z_i\,) \otimes (e_1; z_j) \\
&\sim\; \Big(\,b \cdot (t \otimes e_1)\,;\,z_i, z_j\,\Big) \\
&=\; (\,b_1 t_1\,;\,z_i, z_j\,) \\
&=\; (\,t_2 b_1\,;\,z_i, z_j\,) \\
&=\; \Big(\,(e_1 \otimes t) \cdot b\,;\,z_i, z_j\,\Big) \\
&\sim\; (e_1; z_j) \otimes (t; z_i) \otimes (b; z_i, z_j) \\
&\sim\; (t; z_i) \otimes (b; z_i, z_j)
\end{aligned}$$

Therefore,

$$\mathbb{Z}^{4n+(2n)^2} \Big/ \otimes \sim \circ \quad = \quad \mathbb{Z}^{2n+(2n)^2}$$

The next step is to quotient out by Δ, and again this will at the very least end up imposing all of the same constraints that we had in the braided case, namely

$$[\,(\,b\,;z_i,z_j\,)\,] \quad = \quad [\,(\,b\,;\,z_{n+i},z_j\,)\,]^* \quad = \quad [\,(\,b\,;\,z_i,z_{n+j}\,)\,]^* \quad = \quad [\,(\,b\,;\,z_{n+i},z_{n+j}\,)\,]$$

But we also have those elements of Δ which come from the twist t:

$$\begin{aligned}
& \Big(\mu(\,t\,;\,e_2\,)\,;\,z_i,z_{n+i}\Big) \otimes \Big(\mu(\,t\,;\,e_0\,)\,;\,-\Big)^* \\
= \;& \Big(\,(\,t\otimes t\,)\cdot b\cdot b\,;\,z_i,z_{n+i}\Big) \otimes (e_0;-)^* \\
= \;& \Big(\,(\,t\otimes t\,)\cdot b\cdot b\,;\,z_i,z_{n+i}\Big) \\
\sim \;& (t;z_i)\otimes(t;z_{n+i})\otimes(b;z_{n+i},z_i)\otimes(b;z_i,z_{n+i}) \\
\sim \;& (t;z_i)\otimes(t;z_{n+i})\otimes(b;z_i,z_i)^*\otimes(b;z_i,z_i)^*
\end{aligned}$$

Quotienting out by these will allow us to express twists on objects with index greater than n in terms of the other generators,

$$[\,(\,t\,;\,z_{n+i}\,)\,] \quad = \quad [\,(\,t\,;\,z_i\,)\,]^*\otimes[\,(\,b\,;\,z_i,z_i\,)\,]^{\otimes 2}$$

and so overall we will get

$$\mathbb{Z}^{2n+(2n)^2}\big/\Delta \quad = \quad \mathbb{Z}^{n+n^2}$$

Then for the $\mathbb{Z}^{\binom{n}{2}}$ subgroup coming from the representatives $\rho(2)$ we can just choose the same one as in the braided case, so that

$$\mathbb{Z}^{n+n^2}\big/\mathbb{Z}^{\binom{n}{2}} \quad = \quad \mathbb{Z}^{n+n^2-\binom{n}{2}} \quad = \quad \mathbb{Z}^n\times\mathbb{Z}^n\times\mathbb{Z}^{\binom{n}{2}}$$

and therefore

$$L\mathbb{RB}_n \quad = \quad \mathbb{Z}^{*n}\times_{\mathbb{Z}^n}\mathbb{Z}^{*n}\;\times\;\mathrm{B}(\mathbb{Z}^n\times\mathbb{Z}^n\times\mathbb{Z}^{\binom{n}{2}})$$

Finally, the same reasoning we have used previously tells us that we can recover the whole action of $L\mathbb{RB}_n$ from just the values

$$\alpha(\,b\,;\,\mathrm{id}_{z_i},\mathrm{id}_{z_j}\,),\qquad \alpha(\,b\,;\,\mathrm{id}_{z_i^*},\mathrm{id}_{z_j}\,)\qquad \alpha(\,b\,;\,\mathrm{id}_{z_i},\mathrm{id}_{z_j^*}\,),\qquad \alpha(\,b\,;\,\mathrm{id}_{z_i^*},\mathrm{id}_{z_j^*}\,)$$

$$\alpha(\,t\,;\,\mathrm{id}_{z_i}\,)\qquad \alpha(\,t\,;\,\mathrm{id}_{z_i^*}\,)$$

The process for working out the first four is no different than before, which means that $\alpha_{L\mathbb{RB}_n}$ acts on the braids in the exact same ways that $\alpha_{L\mathbb{B}_n}$ does. Furthermore, it is not hard to see that

$$\alpha(\, t\,;\, \mathrm{id}_{z_i}\,) \quad = \quad \Big(\, z_i,\; z_i,\; (0, ..., 0, 1, 0, ..., 0)\,\Big)$$

where the 1 corresponds to the $(t; z_i)$ generator of $\mathbb{Z}^n \times \mathbb{Z}^n \times \mathbb{Z}^{\binom{n}{2}}$, and also that the process of quotienting by Δ will translate to

$$\alpha(\, t\,;\, \mathrm{id}_{z_i^*}\,) \quad = \quad \alpha(\, t\,;\, \mathrm{id}_{z_i}\,)^* \otimes \alpha(\, b\,;\, \mathrm{id}_{z_i}, \mathrm{id}_{z_i}\,)^{\otimes 2}$$

as required. □

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