Action operads, free algebras on invertible objects, and the classification of 3-groups

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Abstract

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Chapter 1

Free invertible algebras as initial objects

In this chapter we will start to consider how to construct free EG-algebras on some number of invertible objects. Specifically, we will begin by showing that such algebras are the initial objects of a particular comma category, in accordance with some well known properties of adjunctions and their units. Using this initial object prespective will allow us to recover all of the data associated with the objects of a given free invertible algebra — what those objects are, how they act under tensor product, and which pairs of objects form the source and target of at least one morphism. Unfortunately, a concrete decription of the morphisms themselves will ultimately remain elusive. We can get tantalisingly closer though, and an examiniation of the exact way that this method fails will provide the neccessary insight to motivate a more successful approach in **??**.

1.1 The free algebra on n invertible objects

We saw in **??** that the existence of a free EG-algebra on n objects can be proven by taking the left adjoint of a 2-functor which forgets about the algebra structure. Now we want to extend this idea into the realm of algebras on invertible objects. For the analogous approach, we will need to find a new 2-functor that lets us forget about non-invertible objects, and then hopefully we can find its left adjoint too, and use it to freely add inverses to $\mathbb{G}_n$. First though, we need to make this concept of 'forgetting non-invertible objects' a little more precise.

Definition 1.1.1. Given an EG-algebra X, we denote by X_{inv} the sub-EG-algebra containing all invertible objects in X and the isomorphisms between them.

Note that this is indeed a well-defined EG-algebra. If $x_1, ..., x_m$ are invertible objects with inverses $x_1^*, ..., x_m^*$, then $\alpha(g; x_1, ..., x_m)$ is an invertible object with inverse $\alpha(g; x_m^*, ..., x_1^*)$, since

$$\begin{aligned} & \alpha(g; x_1, ..., x_m) \otimes \alpha(g; x_m^*, ..., x_1^*) \\ = \; & \left(x_{\pi(g)^{-1}(1)} \otimes ... \otimes x_{\pi(g)^{-1}(m)} \right) \otimes \left(x^*_{\pi(g)^{-1}(m)} \otimes ... \otimes x^*_{\pi(g)^{-1}(1)} \right) \\ = \; & I \end{aligned}$$

$$\begin{aligned} & \alpha(g; x_m^*, ..., x_1^*) \otimes \alpha(g; x_1, ..., x_m) \\ = \; & \left(x^*_{\pi(g)^{-1}(m)} \otimes ... \otimes x^*_{\pi(g)^{-1}(1)} \right) \otimes \left(x_{\pi(g)^{-1}(1)} \otimes ... \otimes x_{\pi(g)^{-1}(m)} \right) \\ = \; & I \end{aligned}$$

Likewise, if $f_1, ..., f_m$ are isomorphisms from invertible objects $x_1, ..., x_m$ to invertible objects $y_1, ..., y_m$, then $\alpha(g; f_1, ..., f_m)$ is a map from the invertible object $\alpha(g; x_1, ..., x_m)$ to the invertible object $\alpha(g; y_1, ..., y_m)$, and it has an inverse $\alpha(g^{-1}; f^{-1}_{\pi(g)(1)}, ..., f^{-1}_{\pi(g)(m)})$, since

$$\begin{aligned} & \alpha\Big(g^{-1} ; f^{-1}_{\pi(g)(1)}, \dots, f^{-1}_{\pi(g)(m)} \Big) \circ \alpha(g ; f_1, ..., f_m) \\ = \; & \alpha\Big(g^{-1} g ; f_1^{-1} f_1, \dots, f_m^{-1} f_m \Big) \\ = \; & \mathrm{id}_{x_1 \otimes ... \otimes x_m} \end{aligned}$$

$$\begin{aligned} & \alpha(g ; f_1, ..., f_m) \circ \alpha\Big(g^{-1} ; f^{-1}_{\pi(g)(1)}, \dots, f^{-1}_{\pi(g)(m)} \Big) \\ = \; & \alpha\Big(g g^{-1} ; f_{\pi(g)(1)} f^{-1}_{\pi(g)(1)}, \dots, f_{\pi(g)(m)} f^{-1}_{\pi(g)(m)} \Big) \\ = \; & \mathrm{id}_{y_{\pi(g)(1)} \otimes ... \otimes y_{\pi(g)(m)}} \end{aligned}$$

Clearly then, X_{inv} is the correct algebra for our new forgetful 2-functor to send X to. Knowing this, we can contruct the rest of the functor fairly easily.

Proposition 1.1.2. *The assignment $X \mapsto X_{\mathrm{inv}}$ can be extended to a 2-functor $(\_)_{\mathrm{inv}} : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$.*

Proposition 1.1.3. *The 2-functor $(\_)_{\mathrm{inv}} : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$ has a left adjoint, $L : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$.*

With this new 2-functor $L : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{E}G\mathrm{Alg}_S$, we now have the ability to 'freely add inverses to objects' in any EG-algebra we want. The algebra $L\mathbb{G}_n$ is then a clear candidate for our free algebra on n invertible objects, and indeed the proof of this is very simple.

Theorem 1.1.4. *There exists a free* EG*-algebra on* n *invertible objects. Specifically, the algebra* $L\mathbb{G}_n$ *is such that for any other* EG*-algebra* X*, we have an isomorphism of categories*

$$\mathrm{E}G\mathrm{Alg}_S(L\mathbb{G}_n, X) \quad \cong \quad (X_{\mathrm{inv}})^n$$

1.2 $L\mathbb{G}_n$ as an initial algebra

We have now proven that a free EG-algebra on n invertible objects indeed exists. But this fact on its own is not very helpful. To be able to actually use the free algebra $L\mathbb{G}_n$, we need to know how to contruct it explicitly, in terms of its objects and morphisms. We could do this by finding a detailed characterisation of the 2-functor L, and then applying this to our explicit description of $\mathbb{G}_n$ from **??**. However, this would probably take far more effort than is required, since it would involve determining the behaviour of L in many situtations that we aren't interested in. Also, we wouldn't be leveraging $\mathbb{G}_n$'s status as a free algebra to make the calculations any easier. We will try a different strategy instead, one that begins by noticing a special property of the functor L.

Proposition 1.2.1. *For any* EG*-algebra* X*, we have* $L(X)_{\mathrm{inv}} = L(X)$*.*

This result is not especially surprising. Intuitively, it just says that when you freely add inverses to an algebra, every object ends up with an inverse. But the upshot of this is that we now have another way of thinking about $L(X)$: as the target object of the unit of our adjunction, $\eta_X : X \to L(X)_{\mathrm{inv}}$. This means that we don't really need to know the entirety of L in order to determine the free algebra $L\mathbb{G}_n$, just its unit. To find this unit directly, we can turn to the following fact about adjunctions, for which a proof can be found in Lemma 2.3.5 of Leinster's *Basic Category Theory* [6].

Proposition 1.2.2. *Let* $F \dashv G : A \to B$ *be an adjunction with unit* η*. For any object* a *in* A*, let* $(a \downarrow G)$ *denote the comma category whose objects are pairs* (b, f) *consisting of an object* b *from* B *and a morphism* $f : a \to G(b)$ *from* A*, and whose morphisms* $h : (b, f) \to (b', f')$ *are morphisms* $f : b \to b'$ *from* B *such that* $G(f) \circ f = f'$*. Then the pair* $\big(F(a), \eta_a : a \to GF(a)\big)$ *is an initial object of* $(a \downarrow G)$*.*

Corollary 1.2.3. *If* $\phi : \mathbb{G}_n \to Z$ *is an initial object of* $(\mathbb{G}_n \downarrow \mathrm{inv})$*, then*

$$Z \quad \cong \quad (L\mathbb{G}_n)_{\mathrm{inv}} \quad = \quad L\mathbb{G}_n$$

Being able to view $L\mathbb{G}_n$ as the initial object in the comma category $(\mathbb{G}_n \downarrow \mathrm{inv})$ will prove immensely useful in the coming sections. This is because it lets us think about the properties of $L\mathbb{G}_n$ in terms of maps $\psi : \mathbb{G}_n \to X_{\mathrm{inv}}$, and this is exactly the context where we can exploit $\mathbb{G}_n$'s status as a free algebra.

Before moving on, we'll make a small change in notation. From now on, rather than writing objects in $(\mathbb{G}_n \downarrow \mathrm{inv})$ as maps $\psi : \mathbb{G}_n \to Y_{\mathrm{inv}}$, we will instead just let $X = Y_{\mathrm{inv}}$ and speak of maps $\psi : \mathbb{G}_n \to X$. This is purely to prevent the notation from becoming cluttered, and shouldn't be a problem so long as we always remember that the targets of these maps only ever contain invertible objects and morphisms.

1.3 The objects of $L\mathbb{G}_n$

So now we know that $L\mathbb{G}_n$ is an initial object in the category $(\mathbb{G}_n \downarrow \mathrm{inv})$. But what does this actually tell us? After all, we do not currently have a method for finding initial objects in an arbitrary collection of EG-algebra maps. Because of this, we'll have to approach the problem step-by-step, using the initiality of η to extract different pieces of information about the algebra $L\mathbb{G}_n$ as we go. We'll begin by tring to find its objects.

Definition 1.3.1. Denote by $\mathrm{Ob} : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{Mon}$ be the functor that sends EG-algebras X to their monoid of objects $\mathrm{Ob}(X)$, and algebra maps $F : X \to Y$ to their underlying monoid homomorphism $\mathrm{Ob}(F) : \mathrm{Ob}(X) \to \mathrm{Ob}(Y)$.

In order to find $\mathrm{Ob}(L\mathbb{G}_n)$, we'll need to make use of an important result about the nature of Ob.

Definition 1.3.2. Recall that given a monoid M, the monoidal category EM is the one whose monoid of objects is M and which has a unique isomorphism between any two objects. We can view EM as not just a category but an EG-algebra, by letting the action on morphisms take the only possible values it can, given the required source and target. Similarly, for any monoid homomorphisms $h : M \to M'$ we can define a map of EG-algebras

$$\begin{array}{rcrcl} \mathrm{E}h & : & \mathrm{E}M & \to & \mathrm{E}M' \\ & : & m & \mapsto & h(m) \\ & : & m \to m' & \mapsto & h(m) \to h(m') \end{array}$$

This definition of Eh respects composition and identities, and so together with EM it describes a functor $\mathrm{E} : \mathrm{Mon} \to \mathrm{E}G\mathrm{Alg}_S$.

Proposition 1.3.3. *E is a right adjoint to the functor* Ob.

What Proposition 1.3.3 is essentially saying is that the functor Ob provides a way for us to move back and forth between the categories $\mathrm{E}G\mathrm{Alg}_S$ and Mon. By applying this reasoning to the universal property of the initial object η, we can then determine the value of $\mathrm{Ob}(L\mathbb{G}_n)$ in terms of a new universal property of $\mathrm{Ob}(\eta)$ in the category Mon. In particular, the algebras in $(\mathbb{G}_n \downarrow \mathrm{inv})$ are those whose objects are all invertible, and so the induced property of $\mathrm{Ob}(\eta)$ will end up saying something about the relationship between $\mathrm{Ob}(\mathbb{G}_n)$ and groups — those monoids whose elements are all invertible.

Definition 1.3.4. Let M be a monoid, M^{gp} a group, and $i : M \to M^{\mathrm{gp}}$ a monoid homomorphism between them. Then we say that M^{gp} is the *group completion* of M if for any other group H and homomorphism $h : M \to H$, there exists a unique homomorphism $u : M^{\mathrm{gp}} \to H$ such that $u \circ i = h$.

There are several different ways to actually calculate the group completion of a monoid. One is to use that fact that M^{gp} is the group whose group presentation is the same as the monoid presentation of M. That is, if M is the quotient of the free monoid on generators $\mathcal{G}$ by the relations $\mathcal{R}$, then M^{gp} is the quotient of the free *group* on generators $\mathcal{G}$ by relations $\mathcal{R}$. This makes finding the completion of free monoids particularly simple.

Proposition 1.3.5. *The object monoid of* $L\mathbb{G}_n$ *is* $\mathbb{Z}^{*n}$*, and the restriction of* η *to objects* $\mathrm{Ob}(\eta)$ *is the obvious inclusion* $\mathbb{N}^{*n} \hookrightarrow \mathbb{Z}^{*n}$*.*

This result makes concrete the sense in which the functor L represents 'freely adding inverses' to objects. Extending this same logic to connected components as well, it seems reasonable to expect that $\pi_0(L\mathbb{G}_n)$ should be $\mathbb{Z}^n$, the group completion of $\pi_0(\mathbb{G}_n) = \mathbb{N}^n$. This is indeed the case, and the proof proceeds in a way completely analagous to Proposition 1.3.5. First, we show that the process of taking connected components is part of an adjunction.

Definition 1.3.6. Denote by $\pi_0 : \mathrm{E}G\mathrm{Alg}_S \to \mathrm{CMon}$ be the functor that sends each algebra X to its commutaive monoid of connected components, $\pi_0(X)$, and sends each map of algebras $F : X \to Y$ to its restriction to connected components $\pi_0(F) : \pi_0(X) \to \pi_0(Y)$.

Definition 1.3.7. There exists an inclusion of 2-categories Set $\hookrightarrow$ Cat which allows us to view any set S as a discrete category, one whose objects are just the elements of S and whose morphisms are all identities. If the given set also happens to be a monoid then there is an obvious way to see this discete category as a monoidal category, and so we have a similar inclusion Mon $\hookrightarrow$ MonCat. Finally, if a given monoid happens to be commutative then there is a unique way to assign an EG-action to its discete category. This works because for any elements c_i of a commutive monoid C, the morphism $\alpha(g; \mathrm{id}_{c_1}, ..., \mathrm{id}_{c_m})$ must have source and target $c_1 \otimes ... \otimes c_m = c_{\pi(g^{-1})(1)} \otimes ... \otimes c_{\pi(g^{-1})(m)}$, and therefore it can only be $\mathrm{id}_{c_1 \otimes ... \otimes c_m}$. This choice of action yields one last inclusion CMon $\hookrightarrow$ EGAlg$_S$, which we shall call D.

Proposition 1.3.8. *D is a right adjoint to the functor π_0.*

Now we can utilise Proposition 1.3.8 to draw out a universal property of $\pi_0(L\mathbb{G}_n)$, just as we did with Proposition 1.3.3 and $\mathrm{Ob}(L\mathbb{G}_n)$. This time, since we are dealing with commutative monoids, the requirement that everything be invertible will lead us to consider abelian groups.

Proposition 1.3.9. *The connected components of $L\mathbb{G}_n$ are $\mathbb{Z}^n$, with the restriction of η to components $\pi_0(\eta)$ being the obvious inclusion $\mathbb{N}^n \hookrightarrow \mathbb{Z}^n$, and the assignment of objects to their component $[\_] : \mathrm{Ob}(L\mathbb{G}_n) \to \pi_0(L\mathbb{G}_n)$ being the quotient map of abelianisation $\mathbb{Z}^{*n} \to \mathbb{Z}^n$.*

1.4 The morphisms of $L\mathbb{G}_n$

Now that we understand the objects of the algebra $L\mathbb{G}_n$, the next most obvious thing to look for are its morphisms, $\mathrm{Mor}(L\mathbb{G}_n)$. It would be nice to construct this collection in the same way we constructed $\mathrm{Ob}(L\mathbb{G}_n)$ and $\pi_0(L\mathbb{G}_n)$, by applying the left adjoint of some adjunction to the initial map η. Before we can do this however, we need to ask ourselves a question. What sort of mathematical object is $\mathrm{Mor}(L\mathbb{G}_n)$, exactly?

Given a pair of morphisms $f : x \to y, f' : x' \to y'$ in an EG-algebra X, there are two basic binary operations we can perform. First, we can take their tensor product $f \otimes f'$, and this together with the unit map id_I imbues $\mathrm{Mor}(X)$ with the structure of a monoid. Second, if we have $y = x'$ then we can form the composite morphism $f' \circ f$. However, consider the following fact: if y is an invertible object, then

$$\begin{aligned} f' \circ f &= (f' \otimes \mathrm{id}_I) \circ (\mathrm{id}_I \otimes f) \\ &= (f' \otimes \mathrm{id}_{y*} \otimes \mathrm{id}_y) \circ (\mathrm{id}_y \otimes \mathrm{id}_{y*} \otimes f) \\ &= (f' \circ \mathrm{id}_y) \otimes (\mathrm{id}_{y*} \circ \mathrm{id}_{y*}) \otimes (\mathrm{id}_y \circ f) \\ &= f' \otimes \mathrm{id}_{y*} \otimes f \end{aligned}$$

In other words, composition along invertible objects in X is determined completely by the monoidal structure of X. In the case of $L\mathbb{G}_n$, where every object is invertible, this means that if we understand $\mathrm{Mor}(L\mathbb{G}_n)$ as a monoid then we will be able to recover the operation $\circ$ in its entirety. For that reason, we will choose to ignore composition of elements of $\mathrm{Mor}(X)$ for the time being, and focus on its status as a monoid.

Now we try to proceed as we did before, by showing that $\mathrm{Mor}(X)$ is part of an adjunction.

Definition 1.4.1. Let $\mathrm{Mor} : \mathrm{MonCat} \to \mathrm{Mon}$ be the functor which sends algebras X to their monoid of morphisms $\mathrm{Mor}(X)$, and sends algebra maps $F : X \to Y$ to the monoid homomorphism

$$\begin{array}{rcrcl} \mathrm{Mor}(F) & : & \mathrm{Mor}(X) & \to & \mathrm{Mor}(Y) \\ & : & f : x \to x' & \mapsto & F(f) : F(x) \to F(x') \end{array}$$

Definition 1.4.2. For a given abelian group A, let $C(A)$ represent the monoidal category defines as follows:

- The objects of $C(A)$ are the monoid A, with the monoid multiplication as the tensor product and the identity element e as the monoidal unit.
- For any two objects $a, a' \in A$, the homset $C(A)(a, a')$ is isomorphic to the underlying set of A.
- From the above, the morphisms of $C(A)$ will clearly be

  $$\mathrm{Mor}\Big(C(A)\Big) \;=\; A \times A^2 \;=\; A^3$$

  when viewed as a set, but this equality also holds as monoids, so that the tensor product is defined componentwise using the monoid multiplication of A.
- For any two composable morphisms (a, b, b'), (a', b', b'') of $C(A)$, their composite is the morphism

  $$(a', b', b'') \circ (a, b, b') \;=\; \Big(a(b')^* a',\; b,\; b''\Big)$$

Likewise, for any group homomorphism $h : A \to A'$ between abelian groups, denote by $C(h) : C(A) \to C(A')$ the obvious monoidal functor which acts like h on objects and h^3 on morphisms. This defines a functor $C : \mathrm{Ab} \to \mathrm{MonCat}$ from the category of abelian groups onto the category of monoidal categories.

Intuitively, $C(A)$ is the the monoidal category that we can build out of A by using the trick we discussed before for extracting composition from the tensor product, $f' \circ f = f' \otimes \mathrm{id}_{y*} \otimes f$. This is why we had to choose A to be a group, as this can only work when all of the objects of $C(A)$ are invertible. Notice also that commutativity is required in order for $C(A)$ to be a well-defined monoidal category, since we need its operations to obey an interchange law, and thus

$$\begin{aligned}
(aa', e, e) &= (a, e, e) \otimes (a', e, e) \\
&= \Big(\mathrm{id}_e \circ (a, e, e)\Big) \otimes \Big((a', e, e) \circ \mathrm{id}_e\Big) \\
&= \Big((a', e, e) \otimes \mathrm{id}_e\Big) \circ \Big(\mathrm{id}_e \otimes (a, e, e)\Big) \\
&= (a', e, e) \circ (a, e, e) \\
&= (a'a, e, e)
\end{aligned}$$

This is the classic Eckmann-Hilton argument.

Proposition 1.4.3. *The functor C is a right adjoint to the functor* $\mathrm{Mor}(\_)^{\mathrm{gp,ab}} : \mathrm{MonCat} \to \mathrm{Ab}$.

Proposition 1.4.3 is very similar to Propositions 1.3.3 and 1.3.8, but it falls short in a few very important ways. First, if we want to use an adjunction to find a relationship between the morphisms of $\mathbb{G}_n$ and $L\mathbb{G}_n$, like what we did in Propositions 1.3.5 and 1.3.9, then what we need is an adjunction involving $\mathrm{E}G\mathrm{Alg}_S$, not MonCat. This is because η can only be used to factor algebra maps $\mathbb{G}_n \to X_{\mathrm{inv}}$ through $L\mathbb{G}_n$, and not arbitrary monoidal functors. Likewise, we would rather have the other side of the adjunction be Mon instead of Ab so that we could work with the functor Mor directly, and not its group completed, abelianised version.

Unfortunately this adjunction seems to be the best we can do. We already saw that we need A to be an abelian group for $C(A)$ to have composition and interchange, and given an arbitrary abelian group that we want to be the morphisms of an algebra, there is no general method for assigning it an $\mathrm{E}G$-action. As this is what we are stuck with, we will not be able to use the η method to extract the information we need about the morphisms of $L\mathbb{G}_n$, and so we must try a less straightforward approach.

1.5 Sources and targets in $L\mathbb{G}_n$

The goal of these next couple of sections will be to show that we can reconstruct the all of morphisms of $L\mathbb{G}_n$ from just the abelian group $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$, and therefore that we can actually use the adjunction from **??** to help find a description of $L\mathbb{G}_n$. The way we will do this is by splitting $\mathrm{Mor}(L\mathbb{G}_n)$ up as the product of two other monoids. The first of these will encode all of the possible combinations of source and target data for morphisms in $L\mathbb{G}_n$, while the second will just be the endomorphisms of the unit object, $L\mathbb{G}_n(I, I)$. In other words, we will see that the monoid $\mathrm{Mor}(L\mathbb{G}_n)$ can be broken down into a context where source and target are the only thing that matters, and another where they are irrelevant. Once we have done this, we can then use the fact that $L\mathbb{G}_n(I, I)$ is always an abelian group to rewrite $\mathrm{Mor}(L\mathbb{G}_n)$ in terms of $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$.

To get things started, we will spend this section considering the source and target information of morphisms in $L\mathbb{G}_n$.

Definition 1.5.1. For any $\mathrm{E}G$-algebra X, denote by $s : \mathrm{Mor}(X) \to \mathrm{Ob}(X)$ and $t : \mathrm{Mor}(X) \to \mathrm{Ob}X)$ the monoid homomorphisms which send each morphism of X to its source and target, respectively. That is,

$$s(\,f : x \to y) \;=\; x, \qquad t(\,f : x \to y) \;=\; y$$

If we use the universal property of products, we can combine these two homomorphisms into a single map, $s \times t : \mathrm{Mor}(X) \to \mathrm{Ob}(X) \times \mathrm{Ob}(X)$. The monoid we are interested in finding is the image $L\mathbb{G}_n$ under its instance of this map.

Lemma 1.5.2. *Let X be an $\mathrm{E}G$-algebra, and $s \times t : \mathrm{Mor}(X) \to \mathrm{Ob}(X)^2$ the map built from s and t using the universal property of products. Then the image of this map is*

$$(s \times t)(X) \;=\; \mathrm{Ob}(X) \times_{\pi_0(X)} \mathrm{Ob}(X)$$

where this pullback is taken over the canonical maps sending objects of X to their connected components:

$$\begin{tikzcd}
\mathrm{Ob}(X) \times_{\pi_0(X)} \mathrm{Ob}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ob}(X) \arrow[d, "{[_]}"] \\
\mathrm{Ob}(X) \arrow[r, "{[_]}"] & \pi_0(X)
\end{tikzcd}$$

Recalling **????** and Propositions 1.3.5 and 1.3.9, we can immediately conclude the following:

Corollary 1.5.3.

$$\begin{aligned}
(s \times t)(\mathbb{G}_n) &= \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \\
(s \times t)(L\mathbb{G}_n) &= \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n}
\end{aligned}$$

where these pullbacks are taken over the quotients of abelianisation for $(\mathbb{N}^{*n})^{\text{ab}} = \mathbb{N}^n$ *and* $(\mathbb{Z}^{*n})^{\text{ab}} = \mathbb{Z}^n$ *respectively.*

Next, we want to show that this $(s \times t)(L\mathbb{G}_n)$ we have described is in fact a submonoid of $\text{Mor}(L\mathbb{G}_n)$. This is a little tricky though, since we don't currently know what the morphisms of $L\mathbb{G}_n$ even are. We will sidestep this problem by first proving the analogous statement for all $\mathbb{G}_n$, and then recovering the $L\mathbb{G}_n$ version from it later.

To that end we need to find a description of the monoid $\text{Mor}(\mathbb{G}_n)$. **??** already tells us everything we need to know about these morphisms, but it will be helpful for us to give a nice compact description.

Definition 1.5.4. Let S be a set and $F(S)$ the free monoid on S, the monoid whose elements are strings of elements of S and whose binary operation is concatenation. Then we will denote by

$$|\_| : F(S) \to \mathbb{N}$$

the monoid homomorphism defined by sending each element of $S \subseteq F(S)$ to 1, and therefore also each concatenation of n elements of S to the natural number n. We will call $|x|$ the *length* of $x \in F(S)$.

Definition 1.5.5. Let G be an action operad. Then we will also the notation G to denote the *underlying monoid* of this action operad. This is the natural way to consider G as a monoid, with its element set being all of its elements together, $\coprod_m G(m)$, and with tensor product as its binary operation, $g \otimes h = \mu(e_2; g, h)$.

Also, note that this monoid comes equipped with a homomorphism $|\_| : G \to \mathbb{N}$, sending each $g \in G$ to the natural number m if and only if g is an element of the group $G(m)$. Again, we'll call this number $|g|$ the *length* of g.

Lemma 1.5.6. *The monoid of morphisms of the algebra* $\mathbb{G}_n$ *is*

$$\text{Mor}(\mathbb{G}_n) \cong G \times_{\mathbb{N}} \mathbb{N}^{*n}$$

where this pullback is taken over the respective length homomorphisms,

$$\begin{tikzcd} G \times_{\mathbb{N}} \mathbb{N}^{*n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{N}^{*n} \arrow[d, "|\_|"] \\ G \arrow[r, "|\_|"] & \mathbb{N} \end{tikzcd}$$

using the fact that $\mathbb{N}^{*n}$ *is the free monoid* $F\Big(\{z_1, ..., z_n\}\Big)$.

Now, we want to know if $(s \times t)(\mathbb{G}_n)$ can be seen as a submonoid of $\mathrm{Mor}(\mathbb{G}_n)$, which we now see is the same as asking if we can find an injective homomorphism $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \to G \times_{\mathbb{N}} \mathbb{N}^{*n}$. Creating a *function* like this would not be especially hard. For any pair $(w, w') \in \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$, the image of w and w' in the abelian group $\mathbb{N}^n$ is the same, which is to say that the words $w, w' \in \mathbb{N}^{*n}$ are permuations of each other. Since the underlying permutation maps $\pi : G(m) \to \mathrm{S}_m$ in the definition of the action operad G are all surjective, we can always find an element of $g \in G(|w|)$ for which $\pi(g)(w) = w'$. Thus in order to make an injective function $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \to G \times_{\mathbb{N}} \mathbb{N}^{*n}$, all we need to do is make a choice $g_{(w,w')}$ like this for each (w, w'), and then set

$$\begin{array}{rcl} \mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} & \to & G \times_{\mathbb{N}} \mathbb{N}^{*n} \\ (w, w') & \mapsto & (\, g_{(w,w')}, w \,) \end{array}$$

Injectivity follows from

$$(\, g_{(w,w')}, w \,) \;=\; (\, g_{(v,v')}, v \,) \quad\Longrightarrow\quad \begin{array}{rcl} g_{(w,w')} & = & g_{(v,v')} \\ w & = & v \\ w' & = & \pi(g_{(w,w')})(w) \\ & = & \pi(g_{(v,v')})(v) \\ & = & v' \end{array}$$

So how do we know if we can choose these $g_{(w,w')}$ in such a way that the resulting function is also a monoid homomorphism? If we could find a presentation of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ in terms of generators and relations then this would help a little, since we would only need to pick a $g_{(z,z')}$ for each generator (z, z'), and then define all other g by way of products.

$$g_{(vw,v'w')} = g_{(v,v')} g_{(w,w')}$$

But we would still need to know if our choice of $g_{(z,z')}$ obeyed the necessary relations on the generators of $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$. Luckily for us though, this turns out to be no problem at all.

Proposition 1.5.7. $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n}$ *is a free monoid.*

It follows immediately from this that our contruction of an injective function $\mathbb{N}^{*n} \times_{\mathbb{N}^n} \mathbb{N}^{*n} \to G \times_{\mathbb{N}} \mathbb{N}^{*n}$ can be extended to be an inclusion of monoids.

Proposition 1.5.8. $(s \times t)(\mathbb{G}_n)$ *is (isomorphic to) a submonoid of* $\mathrm{Mor}(\mathbb{G}_n)$

So, now we know that $(s \times t)(\mathbb{G}_n) \subseteq \mathrm{Mor}(\mathbb{G}_n)$. But what we are really interested in is whether $(s \times t)(\mathbb{G}_n) \subseteq \mathrm{Mor}(\mathbb{G}_n)$, and it is not really clear how we can recover this result from the former. In order to do so, we need to employ the following result:

Proposition. *There exists a surjective map of* $\mathrm{E}G$*-algebras* $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$

The proof of this fact is a long process that will require many subproofs, and so to allow us to remain focused on our current task we will gloss over the details for the time being. Instead, the question of the existence and surjectivity of q will be covered at length in **??**.

Now, consider again Proposition 1.5.8, in particular the case $(s \times t)(\mathbb{G}_{2n}) \subseteq \mathrm{Mor}(\mathbb{G}_{2n})$. From this we can use the map q to immediately conclude that $(s \times t)(L\mathbb{G}_n) \subseteq \mathrm{Mor}(L\mathbb{G}_n)$ as well, since by surjectivity of q this statement is just equivalent to saying $q\Big((s \times t)(\mathbb{G}_{2n}) \Big) \subseteq q\Big(\mathrm{Mor}(\mathbb{G}_{2n}) \Big)$. More precisely, we have the following:

Corollary 1.5.9. $(s \times t)(L\mathbb{G}_n)$ *is (isomorphic to) a submonoid of* $\mathrm{Mor}(L\mathbb{G}_n)$

1.6 Unit endomorphisms of $L\mathbb{G}_n$

To help us understand $\mathrm{Mor}(L\mathbb{G}_n)$, we decided to break it down into two smaller pieces. The first of these was the source/target data $(s \times t)(L\mathbb{G}_n)$, which we explored in the previous section. The other piece that we now have to consider is the monoid of unit endomorphisms, $L\mathbb{G}_n(I, I)$.

This is a particularly important submonoid of the morphisms $\mathrm{Mor}(L\mathbb{G}_n)$, since it is the only submonoid which is also a homset of the category $L\mathbb{G}_n$. Moreover, because the maps in $L\mathbb{G}_n(I, I)$ all share the same source and target, what we have is not just a monoid under tensor product but also under composition as well. This fact leads to a series of special properties for $L\mathbb{G}_n(I, I)$, the first of which is just another instance of the classic Eckmann-Hilton argument.

Lemma 1.6.1. $L\mathbb{G}_n(I, I)$ *is a commutative monoid under both tensor product and composition, with* $f \otimes f' = f \circ f'$.

Knowing that the operations of composition and tensor product happen to coincide on $L\mathbb{G}_n(I, I)$, we can now translate categorical information about the morphisms of $L\mathbb{G}_n$ into monoidal information for its unit endomorphisms. Of course, we still do not know much about $\mathrm{Mor}(L\mathbb{G}_n)$, and so it might not be immediately obvious what facts we could use in this way. However, in order to prove Corollary 1.5.9 we just recently made the assumption that there exists some surjective map of $\mathrm{E}G$-algebras

$q : \mathbb{G}_{2n} \to L\mathbb{G}_n$. This functor will let us take **??**, a statement about the morphisms $\mathbb{G}_{2n}$, and extend the result onto $L\mathbb{G}_n$:

Lemma 1.6.2. *Every morphism in $L\mathbb{G}_n$ can be expressed as $\alpha_{L\mathbb{G}_n}(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$, for some $g \in G(m)$ and $x_i \in \mathbb{Z}^{*n}$.*

It is worth noting that, unlike with $\mathbb{G}_n$, this ability to express a morphism from $L\mathbb{G}_n$ as an action morphism is *not* necessarily unique. Regardless, Lemma 1.6.2 now gives us the next important property of $L\mathbb{G}_n(I, I)$, namely:

Proposition 1.6.3. *$L\mathbb{G}_n(I, I)$ is an abelian group.*

In fact, with a slightly broader argument we can extend this result to every morphism of $L\mathbb{G}_n$.

Proposition 1.6.4. *Every morphism $f : w \to v$ in $L\mathbb{G}_n$ has an inverse under tensor product, $f^* : w^* \to v^*$. That is, the monoid $\mathrm{Mor}(L\mathbb{G}_n)$ is actually a group.*

So $\mathrm{Mor}(L\mathbb{G}_n)$ and $L\mathbb{G}_n(I, I)$ both turn out to be groups under tensor product. Obviously it follows from this that $L\mathbb{G}_n(I, I)$ is a subgroup of $\mathrm{Mor}(L\mathbb{G}_n)$ — in particular an abelian subgroup, going by Proposition 1.6.3. But $L\mathbb{G}_n(I, I)$ is actually an even more special subgroup than this:

Proposition 1.6.5. *$L\mathbb{G}_n(I, I)$ is a normal subgroup of $\mathrm{Mor}(L\mathbb{G}_n)$.*

This is the last important property of $L\mathbb{G}_n(I, I)$ that we need. Now we finally have enough information to show that the morphism monoid of $L\mathbb{G}_n$ really does split apart into the smaller pieces that we claimed it did.

Proposition 1.6.6.

$$\mathrm{Mor}(L\mathbb{G}_n) \quad = \quad (s \times t)(L\mathbb{G}_n) \times L\mathbb{G}_n(I, I)$$

Proposition 1.6.7. *The endomorphisms of the unit object of $L\mathbb{G}_n$ are*

$$L\mathbb{G}_n(I, I) \quad = \quad \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{ab}} \big/ \mathbb{Z}^n$$

and therefore

$$\mathrm{Mor}(L\mathbb{G}_n) \quad = \quad \mathbb{Z}^{*n} \times_{\mathbb{Z}^n} \mathbb{Z}^{*n} \times \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}} \big/ \mathbb{Z}^n$$

Note that its not entirely clear here exactly which $\mathbb{Z}^n$ subgroup of $\mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$ is being referenced in the statement of Proposition 1.6.7. This is because the existence of such a quotient relied on our assumption that the algebra map $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$ exists, and so we will not be able to actually perform this quotient until we understand where q comes from.

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