Action operads and the free G-monoidal category on n invertible objects

Edward G. Prior

School of Mathematics and Statistics
University of Sheffield

September 2017

Abstract

Table of contents

Introduction

The central goal of this paper is to determine how one can construct free monoidal categories over invertible objects, for as many different kinds of monoidal category as possible. This will be achieved by framing the problem in terms of the theory of action operads, and then gradually exploring the features possessed by their algebras.

The motivation for this topic came from earlier work by the author which attempted to produce a classification theorem for 3-groups. In general, n-groups are a higher dimensional categorification of the standard notion of a group. While a group can be seen as a monoid in which all elements are invertible, a 2-group is a monoidal category in which all objects and morphisms are invertible in the appropriate sense, a 3-group is a monoidal 2-category with all data invertible, and so on. Much has already been written on the subject of 2-groups [1], including a theorem which classifies them completely in terms of group cohomology. The original intention of the author — which will hopefully still form the basis of a future paper — was to generalise this classification theorem to work for 3-groups, by taking each step in the proof and replacing it with a version using concepts from one dimension up. In particular, to replace the sections that involved group cohomology it would be necessary to develop a theory of braided 2-group cohomology. A cohomology of *symmetric* 2-groups already exists [18] [17], but proving that it is well-defined involves exploiting certain facts about symmetric monoidal categories, ones that do not immediately transfer to the braided case.

Thus the key to resolving the whole issue is to understand the behaviour of braided monoidal categories whose objects are all invertible. Indeed, it would suffice to know how to construct the free braided monoidal category on n invertible objects, for any value of $n \in \mathbb{N}$, but this in turn is fairly tricky. Over the course of the following chapters we shall see how to accomplish this task, as well as how to find the analagous free entry object for a large class of similar structures, what we will call the G-monoidal categories. These include the familiar symmetric monoidal categories, but also more unusual cases, such as ribbon braided monoidal categories.

First, we shall spend most of Chapter 1 covering definitions and results from the existing literature which will be relevant for reaching our objective. After beginning with a quick review of the concepts of monoidal categories and operads, we will introduce the main objects of study for this paper, the so-called 'action operads'. First appearing with extra restrictions as 'categorical operads' in the thesis of Nathalie Wahl [19], before being studied later in full generality by Alex Corner and Nick Gurski [4], action operads are kind of operad which generalise the notion of a group action upon a set. We will see how many common examples of operads-with-extra-structure — including the founding example of operad theory, the symmetric operads [14] — can be united in a single framework by viewing them as G-operads, ones that are acted on by some suitable action operad G. The translation operad $\mathrm{E}G$ will also be briefly introduced at this point, as a way to categorify certain aspects of a given action operad G. Following on from the discussion of G-operads will be a look into what the appropriate notion for the algebras of these operads should be. In particular, we will see how they differ slightly from the more typical definition of an operad algebra, due to an additional equivariance condition. During this we will see that a certain monoidal structure, present in all action operads G, will be inherited by the algebras of both G and $\mathrm{E}G$. Then at last all of the work in this chapter will come to a head in Theorem 1.28, a result of Gurski [8], where we learn that algebras of the G-operad $\mathrm{E}G$ are equivalent to kind of monoidal category, one equipped with extra permutative structure dictated by the nature of the action operad G. These are the G-monoidal categories, and thus by framing our questions about free braided monoidal categories in the language of action operad algebras, we will be able to produce results which are applicable to a much wider range of situations. Chapter 1 will close with a look at some of these result in the case of $\mathbb{G}_n$, the free algebra which is taken over some number $n \in \mathbb{N}$ of not-necessarily invertible objects.

Next, Chapter 2 will begin our investigation into the free $\mathrm{E}G$-algebra on n invertible objects, which we denote $L\mathbb{G}_n$. In it, we shall see how $L\mathbb{G}_n$ can be viewed as the initial object in a certain comma category of algebras, when paired with the obvious map between free algebras $\eta : \mathbb{G}_n \to L\mathbb{G}_n$. From this initial algebra prespective it will be possible for us to extract several important pieces of information about the structure of $L\mathbb{G}_n$, by using a technique where we exploit the properties of adjoint functors. First, by showing that the previously mentioned translation functor E forms an adjunction with the object monoid functor Ob, we will demonstrate that the objects of $L\mathbb{G}_n$ are the group completion of the objects of $\mathbb{G}_n$. Likewise, forming an adjunction between discrete category functor D and the connected component functor π_0 will let us prove that the components of $L\mathbb{G}_n$ are the group completion of $\pi_0(\mathbb{G}_n)$. However, a way of using this method to find the morphisms of $L\mathbb{G}_n$ will remain elusive. The closest we can get is by showing that the delooping functor B is right adjoint to a certain functor $\mathrm{M}(\_)^{\mathrm{ab}} : \mathrm{MonCat} \to \mathrm{CMon}$, which describes what we will call the 'collapsed morphisms' of a given monoidal category. In order to salvage this approach, we must therefore try to translate the defining property of $L\mathbb{G}_n$ into one that works solely within the category MonCat, and then also prove that both the algebra structure and the true morphisms of a given EG-algebra can be recovered from these new collapsed morphisms. This task will form the majority of the remaining three chapters.

?? will bring a couple of new ways for us to think about the algebra $L\mathbb{G}_n$. Instead of viewing it as part of an initial object like in Chapter 2, we will instead show that it forms the target of a coequaliser map $q : \mathbb{G}_{2n} \to L\mathbb{G}_n$, whose source now has twice as many generating objects as before. The simplest way to do this involves exhibiting q as the cokernel of an algebra map $\delta : \mathbb{G}_{2n} \to \mathbb{G}_{2n}$, which is designed in such a way that the additional n generators of $\mathbb{G}_{2n}$ will get sent by q onto the inverses of the n generators of $L\mathbb{G}_n$. Through this new perspective we will learn several important facts about the action α of $L\mathbb{G}_n$, including how we will eventually be able to recontruct it from $L\mathbb{G}_n$'s monoid of morphisms, once we finally understand them. This insight will then indicate how we can subtly change the coequaliser diagram for q, so that the preservation of the EG-action is now a consequence of the way that we have built it, rather than just an automatic feature of q being an algebra map. In other words, we will have demonstrated that the underlying monoidal functor of q is also a coequaliser, and thus have found a property which marks the free algebra $L\mathbb{G}_n$ as special within the world of monoidal categories. This is exactly what we need in order to leverage the left adjoint status of the functor $\mathrm{M}(\_)^{\mathrm{ab}}$, since it lives over the category MonCat and commutes with all colimits, like coequalisers. With a little work, our approach will then yield a description of the abelian group of collapsed morphisms $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$ as a quotient of the larger group of collapsed morphisms of $\mathbb{G}_{2n}$.

In **??**, we will see how to use the information that we've accumulated up to this point to build the morphisms of $L\mathbb{G}_n$. The idea is that the invertibility of the objects in this category will let us split the monoid $\mathrm{Mor}(L\mathbb{G}_n)$ into two relevant pieces. The first is a subgroup $(s \times t)(L\mathbb{G}_n)$, which encodes all of the ordered pairs of objects that appear as the source and target data of at least one morphism. The fact that there is such a subgroup — that we can choose a representative morphism for each source/target pair in a way which respects the tensor product of $L\mathbb{G}_n$ — is a consequence of the way that the morphisms of the free algebra $\mathbb{G}_n$ are structured. Specifically, the source and target monoid $(s \times t)(\mathbb{G}_n)$ is free, which lets us easily construct an inclusion $(s \times t)(\mathbb{G}_n) \to \mathrm{Mor}(L\mathbb{G}_n)$, whose image under the coequaliser q then forms the required inclusion for $(s \times t)(L\mathbb{G}_n)$. By comparison, the second subgroup that we need is much simpler, as it is just the homset of endomorphisms of the unit object, $L\mathbb{G}_n(I, I)$. Together, these two subgroups shape the whole of $\mathrm{Mor}(L\mathbb{G}_n)$, in the sense that the latter is a semidirect product of the former. Moreover, under certain circumstances which will include all of the motivating examples for this research, this semidirect product is actualy direct. This will allow us to easily perform abelianisations, group completions, and repeated quotients of $\mathrm{Mor}(L\mathbb{G}_n)$ until we arrive at the same the collapsed $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$ we had before, after which we will have successfully described a path from the morphisms of the free algebra $\mathbb{G}_{2n}$ to those of the invertible $L\mathbb{G}_n$. The rest of the chapter will then be concerned with simplifying this description, by carrying out some calculations that do not change for different instances of $L\mathbb{G}_n$. This will include an investigation into the way that action operads and the monoids we've built out of them will act under group completion and abelianisation.

Finally, in **??** we will compile all of the major results of the previous chapters into a single account of the free $\mathrm{E}G$-algebra on n invertible objects. The only piece of data still missing at this stage will be the action α, but a method for recovering it will have already been established back in **??**, so this will not present any further challenges. **????** are the focal point of the thesis, providing a step-by-step construction of the algebra $L\mathbb{G}_n$ for all values of $n \in \mathbb{N}$ and all action operads G. The remainder of the paper will then consist of applications of these theorems to calculate specific examples of free G-monoidal categories on invertible objects — the symmetric, the braided, and the ribbon braided.

Chapter 1

Operads and their algebras

Before we can talk about the main focus of this paper, the free EG-algebras on n invertible objects, we will need to work our way through several intermediate concepts. This chapter will cover the background material need to undertstand each of these structures in turn — monoidal categories, operads, action operads, G-operads, and operad algebras. Most of this content is due to other authors, and the reader is encouraged to refer to the given sources if they are interested in a more complete analysis of any of the featured topics. Towards the end of the chatper however we will start to see some novel results appear.

1.1 Monoidal categories

We shall start by reviewing some very basic definitions. Everything in this section can be found in any good introductory text on category theory, such as the foundational 'Categories for the Working Mathematician' [11] by Saunders Mac Lane.

Definition 1.1. A *monoidal category* is a category C equipped with

- a functor $\otimes : C \times C \to C$, called the *tensor product* of C
- an object $I \in C$, called the *unit*
- a natural isomorphism a, called the *associator*, with components

$$a_{x,y,z} \,:\, (x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z)$$

- two natural isomorphisms l and r, called the *left* and *right unitors*, with components

$$l_x : I \otimes x \longrightarrow x, \qquad r_x : x \otimes I \longrightarrow x$$

which satisfy two coherence conditions. The first of these, the pentagon identity, is best displayed as the commutative diagram

$$\begin{tikzcd}
 & (w \otimes x) \otimes (y \otimes z) \arrow[dr, "a_{w,x,y\otimes z}"] & \\
((w \otimes x) \otimes y) \otimes z \arrow[ur, "a_{w\otimes x,y,z}"] \arrow[d, "a_{w,x,y}\otimes \mathrm{id}_z"'] & & w \otimes (x \otimes (y \otimes z)) \\
(w \otimes (x \otimes y)) \otimes z \arrow[rr, "a_{w,x\otimes y,z}"] & & w \otimes ((x \otimes y) \otimes z) \arrow[u, "\mathrm{id}_z \otimes a_{x,y,z}"']
\end{tikzcd}$$

and shows that the operation $\otimes$ is weakly associative. Likewise the second condition, the triangle identity, corresponds to the diagram

$$\begin{tikzcd}
(x \otimes I) \otimes y \arrow[rr, "a_{x,I,y}"] \arrow[dr, "r_x \otimes \mathrm{id}_y"'] & & x \otimes (I \otimes y) \arrow[dl, "\mathrm{id}_y \otimes l_y"] \\
 & x \otimes y &
\end{tikzcd}$$

and represents the fact that I is a weak unit. A monoidal category in which the natural isomorphisms a, l, r are all identities — and thus the two coherence conditions old trivially — is said to be *strictly* monoidal. For contrast, we will therefore sometimes refer to the above kind of category as *weakly* monoidal.

While it isn't explicitly stated in Definition 1.1, notice that functoriality of $\otimes$ will induce the following relationship between the tensor product and composition in C:

$$(f' \circ f) \otimes (g' \circ g) \quad = \quad (f' \otimes g') \circ (f \otimes g)$$

This is known as the *interchange law* of C, and we will make heavy use of this fact in later chapters.

Monoidal categories are found everywhere thoughout mathematics. Commonly studied examples include the category of sets Set with the cartesian product $\times$, the category of abelian groups Ab under direct sum $\oplus$, and the category of K-vector spaces $\mathrm{K}-\mathrm{Vect}$ with its usual tensor product $\otimes_K$. Part of the reason for their ubiquity is that monoidal categories are, in some sense, really degenerate versions of a higher dimensional category, specifically a one-object bicategory. We will not be exploring the concept of higher categories in this paper (see for example [9] for a proper treatment), but suffice it to say that there are also other kinds of degenerate n-categories which appear to be common kinds of category-with-extra-structure.

Definition 1.2. A *braided* monoidal category is a monoidal category C equipped with an additional natural isomorphism,

$$\beta_{x,y} \,:\, x \otimes y \longrightarrow y \otimes x$$

called the *braiding*, which satisfies the hexagon identities,

$$\begin{tikzcd}
 & (x\otimes y)\otimes z \arrow[r, "\beta_{x\otimes y,z}"] \arrow[dl, "a_{x,y,z}"'] & z\otimes(x\otimes y) \arrow[dr, "a^{-1}_{z,x,y}"] & \\
x\otimes(y\otimes z) \arrow[dr, "\mathrm{id}_x\otimes\beta_{y,x}"'] & & & (z\otimes x)\otimes y \arrow[dl, "\beta_{z,x}\otimes\mathrm{id}_y"] \\
 & x\otimes(z\otimes y) \arrow[r, "a^{-1}_{x,z,y}"] & (x\otimes z)\otimes y &
\end{tikzcd}$$

$$\begin{tikzcd}
 & x\otimes(y\otimes z) \arrow[r, "\beta_{x,y\otimes z}"] \arrow[dl, "a^{-1}_{x,y,z}"'] & (y\otimes z)\otimes x \arrow[dr, "a_{y,z,x}"] & \\
(x\otimes y)\otimes z \arrow[dr, "\beta_{x,y}\otimes\mathrm{id}_z"'] & & & y\otimes(z\otimes x) \arrow[dl, "\mathrm{id}_y\otimes\beta_{z,x}"] \\
 & (y\otimes x)\otimes z \arrow[r, "a_{y,x,z}"] & y\otimes(x\otimes z) &
\end{tikzcd}$$

Again, though it isn't directly mentioned in, the above definition also implies another pair of coherence conditions for the unit in C, namely

$$\begin{tikzcd}
x\otimes I \arrow[rr, "\beta_{x,I}"] \arrow[dr, "r_x"'] & & I\otimes x \arrow[dl, "l_x"] \\
 & x &
\end{tikzcd}
\qquad
\begin{tikzcd}
I\otimes x \arrow[rr, "\beta_{I,x}"] \arrow[dr, "l_x"'] & & x\otimes I \arrow[dl, "r_x"] \\
 & x &
\end{tikzcd}$$

Definition 1.3. A *symmetric* monoidal category is a braided monoidal category C whose braiding satisfies an extra symmetry condition, $\beta^{-1}_{x,y} = \beta_{y,x}$.

Braided monoidal categories can be seen as the 'same' as doubly-degenerate tricategories, while symmetric monoidal categories 'are' triply-degenerate weak 4-categories.

For a more thorough explanation of this relationship, see [2] and [3] by Cheng and Gurski.

Strict symmetric monoidal categories are sometimes known as 'permutative categories', and it is not hard to see why. If we set $a, l, r = \mathrm{id}$, then in the symmetric case the diagrams from Definition 1.2 simplify to

$$\begin{aligned} \beta_{x \otimes y, z} &= (\beta_{z,x} \otimes \mathrm{id}_y) \circ (\mathrm{id}_x \otimes \beta_{y,x}), & \beta_{x,I} &= \mathrm{id}_x \\ \beta_{x, y \otimes z} &= (\mathrm{id}_y \otimes \beta_{z,x}) \circ (\beta_{y,x} \otimes \mathrm{id}_x), & \beta_{I,xI} &= \mathrm{id}_x \end{aligned}$$

Collectively, these identities represent the fact that for any collection of distinct objects $x_1, ..., x_n$ in a strict symmetric monoidal category X and any permutation $\sigma \in \mathrm{S}_n$, there exists a unique isomorphism

$$x_1 \otimes ... \otimes x_n \longrightarrow x_{\sigma^{-1}(1)} \otimes ... \otimes x_{\sigma^{-1}(n)}$$

built out of the symmetries β. In other words, elements of the symmetric groups S_n act like n-ary operations, which take in an appropriate number of objects and return some data for a strict symmmetric monoidal category. This is a fairly vague statement however; it would be nice if we could make it more rigourous.

1.2 Operads

What we need is the concept of an operad. These were first introduced by Peter May in the book 'The Geometry of Iterated Loop Spaces' [14], though our usage will be slightly different, for reasons discussed later.

Definition 1.4. An *operad* O in a symmetric monoidal category $(C, \otimes, I)$ is a structure consisting of

- a family of objects, $O(n)$ for $n \in \mathbb{N}$,
- a morphism $1 : I \to O(1)$, called the identity
- a family of morphisms,

  $$\mu_{n;k_1,...,k_n} : O(n) \otimes O(k_1) \otimes ... \otimes O(k_n) \longrightarrow O(k_1 + ... + k_n)$$

  called operadic multiplication.

This data is then subject to the unitality conditions

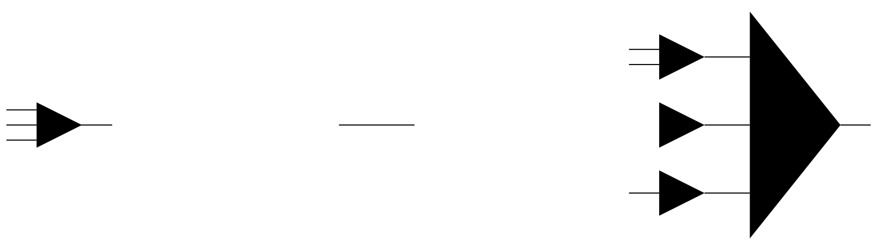
$$
\begin{tikzcd}
I \otimes O(n) \arrow[d, "1\otimes \mathrm{id}_{O(n)}"'] \arrow[dr, "l_x"] & \\
O(1) \otimes O(n) \arrow[r, "\mu_{1;n}"'] & O(n)
\end{tikzcd}
\qquad
\begin{tikzcd}
O(n) \otimes I \otimes ... \otimes I \arrow[d, "\mathrm{id}_{O(n)}\otimes 1 \otimes ... \otimes 1"'] \arrow[dr, "r_{x\otimes I \otimes ... \otimes I} \circ ... \circ r_x"] & \\
O(n) \otimes O(1) \otimes ... \otimes O(1) \arrow[r, "\mu_{n;1,...,1}"'] & O(n)
\end{tikzcd}
$$

for all $n \in \mathbb{N}$, and the associativity conditions

$$
\begin{tikzcd}
O(n) \otimes \prod O(m_i) \otimes \prod O(k_{1,j}) \otimes ... \otimes \prod O(k_{n,j}) \arrow[d, "\beta"'] \arrow[dr, "\mu \otimes \mathrm{id}"] & \\
O(n) \otimes \prod \Big(O(m_i) \otimes \prod O(k_{i,j}) \Big) \arrow[d, "\mathrm{id} \otimes \prod \mu"'] & O(m_1 + ... + m_n) \otimes \prod O(k_{i,j}) \arrow[d, "\mu"] \\
O(n) \otimes \prod O(k_{i,1} + ... + k_{i,m_i}) \arrow[r, "\mu"] & O(k_{1,1} + ... + k_{n,m_n})
\end{tikzcd}
$$

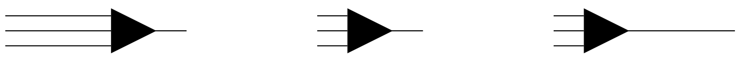
for all $n, m_1, ..., m_n, k_{1,1}, ..., k_{1,m_1}, ..., k_{n,1}, ..., k_{n,m_n} \in \mathbb{N}$.

The idea behind operads is that they are supposed to generalise the notion of ‘operations’. That is, objects $O(n)$ are to be thought of as somehow representing collections of n-ary operations, with the identity as a distinguished unary operation. Multipication in an operad is then motivated by the intuition that we can plug the outputs of n given operations into the inputs of an n-ary operation.



n-ary operation Identity Operadic multiplication

As an example, if we were to represent some operations pictorially as in the diagram above, then the figure on the right is what is meant by the multiplication $\mu : O(3) \times O(2) \times O(0) \times O(1) \rightarrow O(2 + 0 + 1)$. Under this interpretation, each of the coherence conditions for an operad represents some obvious fact about how generic n-ary operations should interact with one another. For instance, unitality of the identity is simply



$$\mu(\,x\,;\,1,1,1\,) \qquad = \qquad x \qquad = \qquad \mu(\,1\,;\,x\,)$$

As with most mathematical structures, operads naturally form a category, together with a suitable notion of morphisms between operads.

Definition 1.5. Given two operads O, O' in a symmetric monoidal category $(C, \otimes, I)$, a *map of operads* between them is a family of maps between their operations which preserve operadic composition. That is, any $f : O \to O'$ is composed of morphisms $f_n : O(n) \to O'(n)$, $n \in \mathbb{N}$ which satisfy

$$\begin{tikzcd} & I \arrow[dl, "1^O"'] \arrow[dr, "1^{O'}"] & \\ O(1) \arrow[rr, "f_1"] & & O'(1) \end{tikzcd} \qquad \begin{tikzcd} O(n) \otimes O(k_1) \otimes ... \otimes O(k_n) \arrow[r, "\mu^O"] \arrow[d, "f_n \otimes f_{k_1} \otimes ... \otimes f_{k_n}"'] & O(k_1 + ... + k_n) \arrow[d, "f_{k_1+...+k_n}"] \\ O'(n) \otimes O'(k_1) \otimes ... \otimes O'(k_n) \arrow[r, "\mu^{O'}"] & O'(k_1 + ... + k_n) \end{tikzcd}$$

for all $n, k_1, ..., k_n \in \mathbb{N}$. The category of operads and maps of operads in $(C, \otimes, I)$ is denoted $\mathrm{Op}(C)$, though in the case of Set we will just call it Op. Composition in this category is defined by termwise composition of families $f_n : O(n) \to O'(n)$, $g_n : O'(n) \to O''(n)$, and the identity morphisms $\mathrm{id}_O : O \to O$ are simply the families $\mathrm{id}_{O(n)}$ from C.

For a far more in depth explanation of operads and their intimate relationship with category theory, see the book ‘Higher Operads, Higher Categories’ [9] by Tom Leinster.

When we are working with operads in the category of sets, $(\mathrm{Set}, \times, 1)$, the objects $O(n)$ genuinely are collections of elements, with a distinguished identity $1 \in O(1)$. However, these elements still do not have to be operations in any way other than that they satisfy Definition 1.4, as we will see in the following examples.

Example 1.6 (The symmetric operad).
There is an operad in Set whose sets of operations $\mathrm{S}(n)$ for each $n \in \mathbb{N}$ are the underlying sets of the symmetric groups S_n. The identity element of this *symmetric operad* S is the identity permutation of a single object, $e_1 \in \mathrm{S}_1$, and the operadic multiplication is defined in the following way:

- First, there exist maps $\otimes : \mathrm{S}_m \times \mathrm{S}_n \to \mathrm{S}_{m+n}$ called the *direct sum* or *block sum* of permutations. For any $\sigma \in \mathrm{S}_m$ and $\tau \in \mathrm{S}_n$, these are given by

$$(\sigma \otimes \tau)(i) \quad = \quad \begin{cases} \sigma(i) & 1 \leq i \leq m \\ \tau(i-m)+m & m+1 \leq i \leq m+n \end{cases}$$

  As the name suggests, this direct sum is usually denote by the symbol $\oplus$, but we will stick with $\otimes$ so that our notation here matches all of the other tensor products we will see throughout this paper. Also, notice that the value of these direct sums in general are determined by those specific cases where one of the inputs is an identity permutation:

$$\sigma \otimes \tau \quad = \quad (\sigma \otimes e_n) \cdot (e_m \otimes \tau) \quad = \quad (e_m \otimes \tau) \cdot (\sigma \otimes e_n)$$

- Next, we'll define functions $(\_)_{(k_1,...,k_n)} : \mathrm{S}_n \to \mathrm{S}_{k_1+...+k_n}$ for all $n, k_1, ..., k_n \in \mathbb{N}$. These will act by sending each σ that permutes n individual objects to a corresponding $\sigma_{(k_1,...,k_n)}$ that permutes blocks of objects of size $k_1, ..., k_n$ in the same way. More concretely, if $k_1 + ... + k_{i-1} < j \leq k_1 + ... + k_i$ then

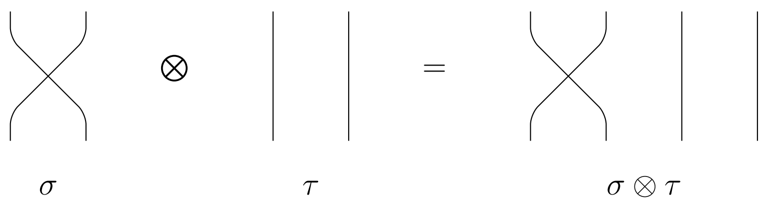
$$\sigma_{(k_1,...,k_n)}(j) \quad = \quad j - k_1 - ... - k_{i-1} + k_{\sigma^{-1}(1)} + ... + k_{\sigma^{-1}(\sigma(i)-1)}$$

- Finally, the multiplication maps $\mu : \mathrm{S}_n \times \mathrm{S}_{k_1} \times ... \times \mathrm{S}_{k_n} \to \mathrm{S}_{k_1+...+k_n}$ are given by

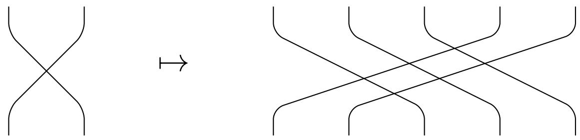
$$\begin{aligned} \mu(\sigma; \tau_1, ..., \tau_n) \; &:= \; \sigma_{(k_1,...,k_n)} \cdot (\tau_1 \otimes ... \otimes \tau_n) \\ &= \; (\tau_{\sigma^{-1}(1)} \otimes ... \otimes \tau_{\sigma^{-1}(n)}) \cdot \sigma_{(k_1,...,k_n)} \end{aligned}$$

  In other words, the operadic multiplication of permutations comes from both permutating objects within distinct blocks and also permuting the blocks themselves.

If we decide to represent elements of the symmetric operad pictorially — for example as strings which cross over another according to the appropriate permutation — then both $\sigma \otimes \tau$ and $\sigma_{(k_1,...,k_n)}$ have rather nice interpretations.



The direct sum of two permutations is just the result of placing two permutations ‘next to’ each other, as above, and block permutations are given by expanding string into some number of parallel strings:



σ $\sigma_{(3,2)}$

With a little work, we can actually replace the functions $(\_)_{(k_1,...,k_n)}$ with an explicit combination of group multiplication and tensor product. This is due to basic fact about the symmetric groups S_n, which is that they possess a presentation in terms of the *elementary transpositions* $(i\ \ i+1)$.

Lemma 1.7. *The group* S_n *is generated by the permutations* $(1\,2), ..., (n-1\ \ n)$*, subject to the relations*

$$\begin{array}{rcll} (i\ \ i+1)^2 & = & e & \\ (i-1\ \ i)(i\ \ i+1)(i-1\ \ i) & = & (i\ \ i+1)(i-1\ \ i)(i\ \ i+1) & \\ (i\ \ i+1)(j\ \ j+1) & = & (j\ \ j+1)(i\ \ i+1), & \qquad i+1<j \end{array}$$

Thus if $\sigma \in \mathrm{S}_n$ is a permutation with a decomposition $\sigma = \sigma_m \cdot ... \cdot \sigma_1$ in terms of elementary transpositons $\sigma_i \in \mathrm{S}_n$, we can break down the block permutation $\sigma_{(k_1,...,k_n)}$ into the m ‘elementary block transpositions’ $(\sigma_i)_{(k_1,...,k_n)}$:

$$\begin{array}{rcl} \sigma_{(k_1,...,k_n)}(j) & = & j - k_1 - ... - k_{i-1} + k_{\sigma^{-1}(1)} + ... + k_{\sigma^{-1}(\,\sigma(i)-1\,)} \\ & = & j - k_1 - ... - k_{i-1} \\ & & + k_{\sigma_1^{-1}(1)} + ... + k_{\sigma_1^{-1}(\,\sigma_1(i)-1\,)} \\ & & - k_{\sigma_1^{-1}(1)} - ... - k_{\sigma_1^{-1}(\,\sigma_1(i)-1\,)} \\ & & + k_{(\sigma_2\sigma_1)^{-1}(1)} + ... + k_{(\sigma_2\sigma_1)^{-1}(\,\sigma_2\sigma_1(i)-1\,)} \\ & & \vdots \\ & & - k_{(\sigma_{m-1}...\sigma_1^{-1}(1)} - ... - k_{(\sigma_{m-1}...\sigma_1)^{-1}(\,\sigma_{m-1}...\sigma_1(i)-1\,)} \\ & & + k_{(\sigma_m...\sigma_1)^{-1}(1)} + ... + k_{(\sigma_m...\sigma_1)^{-1}(\,\sigma_m...\sigma_1(i)-1\,)} \\ & = & \Big((\sigma_m)_{(k_1,...,k_n)} \cdot ... \cdot (\sigma_1)_{(k_1,...,k_n)} \Big)(j) \end{array}$$

However, since elementary transpositions only really permute two objects, they can be written as a block sum in the operad S involving the sole transposition of S_2, plus some number of identity permutations.

$$(i \;\; i+1) \quad = \quad e_{i-1} \otimes (1\,2) \otimes e_{n-i-1}$$

This means that the elementary block transpositions are

$$\begin{aligned}(i \;\; i+1)_{(k_1,\ldots,k_n)} \quad &= \quad (e_{i-1} \otimes (1\,2) \otimes e_{n-i-1})_{(k_1,\ldots,k_n)} \\ &= \quad e_{k_1+\ldots k_{i-1}} \otimes (1\,2)_{(k_i,k_{i+1})} \otimes e_{k_{i+1}+\ldots k_n}\end{aligned}$$

So all we need to know to fully understand the functions $(\_\_)_{(k_1,\ldots,k_n)}$ are the values they take on the transposition $(1\,2)$. These can be defined recursively, via

$$\begin{aligned}(1\,2)_{(0,n)} &= e_n, & (1\,2)_{(m+m',n)} &= \Big((1\,2)_{(m,n)} \otimes e_{m'} \Big) \cdot \Big(e_m \otimes (1\,2)_{(m',n)} \Big) \\ (1\,2)_{(m,0)} &= e_m, & (1\,2)_{(m,n+n')} &= \Big(e_n \otimes (1\,2)_{(m,n')} \Big) \cdot \Big((1\,2)_{(m,n)} \otimes e_{n'} \Big) \\ (1\,2)_{(1,1)} &= (1\,2) & & \end{aligned}$$

which all follow from the definition of $(\_\_)_{(k_1,\ldots,k_n)}$. Therefore all $\sigma_{(k_1,\ldots,k_n)}$ and hence all $\mu(\sigma;\tau_1,\ldots,\tau_n)$ can be expressed in terms of group multiplication $\cdot$ and direct sum $\otimes$, and the elementary permutations which constitute $\sigma, \tau_1, \ldots, \tau_n$.

Something very important to notice about the symmetric operad is that while its sets of operations S_n are groups, it is *not* an operad in the category of groups, because the operadic multiplication we have just outlined is not a group homomorphism. If it were, then it would obey

$$\mu(\sigma;\tau_1,\ldots,\tau_n) \cdot \mu(\sigma';\tau'_1,\ldots,\tau'_n) \quad = \quad \mu(\,\sigma\sigma'\,;\,\tau_1\tau'_1,\ldots,\tau_n\tau'_n\,)$$

for all $\sigma, \sigma' \in \mathrm{S}_n$, $\tau_i, \tau'_i \in \mathrm{S}_{k_i}$, but this is clearly false. As a counterexample, consider the fairly simple case

$$\begin{aligned}\mu\Big((1\,2)\,;\,e_2, e_1 \Big) \quad &= \quad (1\,2)_{(2,1)} \cdot (e_2 \otimes e_1) \quad &&= \quad (1\,2\,3) \cdot e_3 \quad &&= \quad (1\,2\,3) \\ \mu\Big(e_2\,;\,(1\,2), e_1 \Big) \quad &= \quad (e_2)_{(2,1)} \cdot \Big((1\,2) \otimes e_1 \Big) \quad &&= \quad e_3 \cdot (1\,2) \quad &&= \quad (1\,2) \\ \mu\Big((1\,2)\,;\,(1\,2), e_1 \Big) \quad &= \quad (1\,2)_{(2,1)} \cdot \Big((1\,2) \otimes e_1 \Big) \quad &&= \quad (1\,2\,3) \cdot (1\,2) && \end{aligned}$$

$$\begin{aligned}\Longrightarrow \; \mu\Big(e_2\,;\,(1\,2), e_1 \Big) \cdot \mu\Big((1\,2)\,;\,e_2, e_1 \Big) \quad &= \quad (1\,2) \cdot (1\,2\,3) \\ &\neq \quad (1\,2\,3) \cdot (1\,2) \\ &= \quad \mu\Big((1\,2)\,;\,(1\,2), e_1 \Big) \\ &= \quad \mu\Big(e_2 \cdot (1\,2)\,;\,(1\,2) \cdot e_2,\; e_1 \cdot e_1 \Big)\end{aligned}$$

This seems at first like pretty strange behaviour. After all, the symmetric groups play a central role in the theory of groups, so it would be reasonable to assume that their operad would be similarly crucial for the theory of group operads. But S is not the only family of groups whose operad is fundamentally set related.

Example 1.8 (The braid operad).
The *braid groups* B_n are the family of groups that result from taking the symmetric groups and removing the requirement that everything needs to be self-inverse. That is, the group B_n has a presentation on some *elementary braids* $b_1, ..., b_{n-1}$, given by the relations

$$b_i b_{i+1} b_i \;=\; b_{i+1} b_i b_{i+1}, \qquad\qquad b_i b_j \;=\; b_j b_i, \quad i+1 < j$$

As might be expected, the underlying sets of these groups also form an operad in Set known as the *braid operad* B, and they do so in a way directly analagous to the operad S. That is, the identity element of B is $e_1 \in B_1$, and the operadic multiplication is constructed as follows:

- Tensor products $\otimes : B_m \times B_n \to B_{m+n}$ are determined by setting

  $$x \otimes y \quad = \quad (x \otimes e_n) \cdot (e_m \otimes y) \quad = \quad (e_m \otimes x) \cdot (y \otimes e_n)$$

  for all $x \in B_m$, $y \in B_n$, and also

  $$b_i \quad = \quad e_{i-1} \otimes b \otimes e_{n-i-1}$$

  for any elementary braid $b_i \in B_n$, where b is the only elementary braid in B_2.

- The functions $(\_)_{(k_1,...,k_n)} : B_n \to B_{k_1+...+k_n}$ are first defined recursively on the elementary braid $b \in B_2$ by

  $$\begin{aligned} b_{(0,n)} &= e_n, & b_{(m+m',n)} &= (b_{(m,n)} \otimes e_{m'}) \cdot (e_m \otimes b_{(m',n)}) \\ b_{(m,0)} &= e_m, & b_{(m,n+n')} &= (e_n \otimes b_{(m,n')}) \cdot (b_{(m,n)} \otimes e_{n'}) \\ b_{(1,1)} &= b \end{aligned}$$

  then on arbitrary elementary braids $b_i \in B_n$ via

  $$(b_i)_{(k_1,...,k_n)} \quad = \quad e_{k_1+...k_{i-1}} \otimes b_{(k_i,k_{i+1})} \otimes e_{k_{i+1}+...k_n}$$

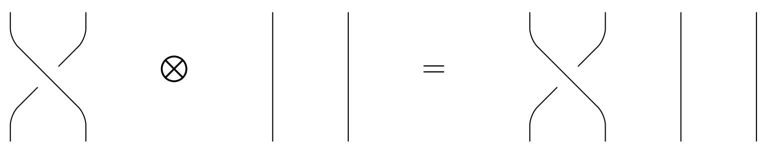
and finally on all elements of the braid groups by using their presentation in terms of the b_i,

$$\begin{aligned} x &= b_{i_m} \cdot ... \cdot b_{i_1} \\ \implies \quad x_{(k_1,...,k_n)} &= (b_{i_m})_{(k_1,...,k_n)} \cdot ... \cdot (b_{i_1})_{(k_1,...,k_n)} \end{aligned}$$

- Then as in the symmetric case, the multiplication maps $\mu : B_n \times B_{k_1} \times ... \times B_{k_n} \to B_{k_1+...+k_n}$ are just

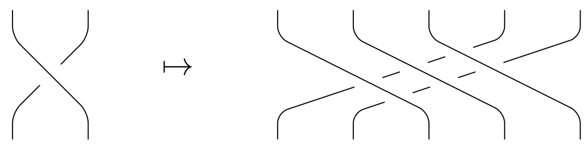
$$\mu(x; y_1, ..., y_n) \quad := \quad x_{(k_1,...,k_n)} \cdot (y_1 \otimes ... \otimes y_n)$$

These operations are exactly what they need to be in order for them to possess the same pictorial representations as the operations in S, but with actual braids replacing simple crossings. That is, the tensor product $x \otimes y$ is the braids x and y laid side-by-side,



$x \qquad y \qquad x \otimes y$

and the 'block braids' are multiple strings braided together in parallel,



$x \qquad x_{(3,2)}$

1.3 Action operads

It is not hard to see that the symmetric and braided operads both share certain features which are not otherwise common among operads of sets. This fact has been noticed by several different authors, each of whom proposed a slightly different definition and terminology for these sorts of structures. While older treatments exist — see for example [19] and [20] — in this paper we will be following the conventions laid out in [4], since they are the most general.

Definition 1.9. An *action operad* (G, π) consists of

- an operad G in the category of sets, whose $G(n)$ are also all groups

- a map of operads $\pi : G \to \mathrm{S}$ whose components $\pi_n : G(n) \to \mathrm{S}_n$ are also group homomorphisms

where the operadic multipliction of G and the group multiplication of the $G(n)$ are linked via the map π in the following way:

$$\mu(\, gg' \,;\, h_1h_1', ..., h_nh_n' \,) \quad = \quad \mu(g; h_{\pi(g')^{-1}(1)}, ..., h_{\pi(g')^{-1}(n)}) \cdot \mu(g'; h_1', ..., h_n')$$

The element $\pi(g)$ is called the *underlying permutation* of g, and as we can see the role it plays is to permute the inputs of an operadic multiplication when two of them are multiplied as group elements. This is exactly the behaviour we observed before with the symmetric operad; for instance, recalling our previous example we see that we should have had was

$$\begin{aligned} \mu\Big(\,(1\,2)\,;\, e_1, e_2\,\Big) &= (1\,2\,3) \\ \mu\Big(\,e_2\,;\, e_1, (1\,2)\,\Big) &= (e_2)_{(1,2)} \cdot \Big(\,e_1 \otimes (1\,2)\,\Big) \;=\; e_3 \cdot (2\,3) \;=\; (2\,3) \\ \mu\Big(\,(1\,2)\,;\,(1\,2), e_1\,\Big) &= (1\,2\,3) \cdot (1\,2) \end{aligned}$$

$$\begin{aligned} \Longrightarrow\; \mu\Big(\,e_2\,;\,(1\,2), e_1\,\Big) \cdot \mu\Big(\,(1\,2)\,;\, e_2, e_1\,\Big) &= (2\,3) \cdot (1\,2\,3) \\ &= (1\,2\,3) \cdot (1\,2) \\ &= \mu\Big(\,(1\,2)\,;\,(1\,2), e_1\,\Big) \\ &= \mu\Big(\,e_2 \cdot (1\,2)\,;\,(1\,2) \cdot e_2,\, e_1 \cdot e_1\,\Big) \end{aligned}$$

The effect that this has on the map μ also mirrors the way that we had to define operadic multiplication for S and B in stages. Specifically if for any action operad G we define

$$g_{(k_1,...,k_n)} := \mu(g; e_{k_1}, ..., e_{k_n}), \qquad\qquad g_1 \otimes ... \otimes g_n := \mu(e_n; g_1, ..., g_n)$$

then it follows from Definition 1.9 that

$$\begin{aligned} \mu(\,g\,;\, h_1, ..., h_n\,) &= \mu(\,g \cdot e_n\,;\, e_{k_1} \cdot h_1, ..., e_{k_n} \cdot h_n\,) \\ &= \mu(\,g\,;\, e_{\pi(e_n)^{-1}(k_1)} ..., e_{\pi(e_n)^{-1}(k_1)}\,) \cdot \mu(\,e_n\,;\, h_1, ..., h_n\,) \\ &= \mu(\,g\,;\, e_{k_1} ..., e_{k_1}\,) \cdot \mu(\,e_n\,;\, h_1, ..., h_n\,) \\ &= g_{(k_1,...,k_n)} \cdot (h_1 \otimes ... \otimes h_n) \end{aligned}$$

for all $g \in G(n)$, $h_i \in G(k_i)$, $n, k_1, ..., k_n \in \mathbb{N}$.

Now we can also see the reason why we chose the tensor product notation for the operation $\mu(e_n; \_, ..., \_)$ before. Just like the tensor product of a monoidal category, the definition of this $\otimes$ in G immediately implies an interchange law:

$$\begin{aligned}(g \cdot g') \otimes (h \cdot h') &= \mu(\, e_2\, ;\, gg',\, hh'\,) \\ &= \mu(e_2; g, h) \cdot \mu(e_2; g', h') \\ &= (g \otimes h) \cdot (g' \otimes h')\end{aligned}$$

This interaction between the operad and group structures of G places some restrictions on which groups we may choose to build action operads out of. One such consequence which we will need to refer to in later chapters is the following:

Lemma 1.10. *For any action operad G, the group $G(0)$ is abelian.*

Proof. This lemma is an example of the classic Eckmann-Hilton argument, first put forth in [6]. The idea is that if some set is equipped with two binary operations which obey some form of interchange, and both of them possess the same unit element e, then they are in reality a single, commutative operation.

In the case of $G(0)$, we know that it is closed under group multiplication, whose unit is the identity element e_0. But the operadic multiplication of G includes a map

$$\mu_{n;0,\dots,0} \;:\; G(n) \times G(0) \times \dots \times G(0) \;\longrightarrow\; G(0 + \dots + 0) \;=\; G(0)$$

which means that tensor products of elements in $G(0)$,

$$g_1 \otimes \dots \otimes g_n \quad = \quad \mu_{n;0,\dots,0}(e_n; g_1, \dots, g_n)$$

are also elements of $G(0)$. This operation has unit e_0 as well, since by associativity and unitality of operadic multipication,

$$\begin{aligned}g \otimes e_0 &= \mu(e_2; g, e_0) \\ &= \mu\Big(\, e_2\, ;\, \mu(e_1; g),\, \mu(e_0; -)\Big) \\ &= \mu\Big(\, \mu(e_2; e_1, e_0)\, ;\, g\, \Big) \\ &= \mu(e_1; g) \\ &= g\end{aligned}$$

and likewise for $e_0 \otimes g = g$. Moreover, we've just seen that the group multiplication and tensor product of G obey an interchange law. Therefore we can apply the Eckmann-Hilton argument: for any $g, h \in G(0)$,

$$g \otimes h \quad = \quad (g \cdot e_0) \otimes (e_0 \cdot h) \quad = \quad (g \otimes e_0) \cdot (e_0 \otimes h) \quad = \quad g \cdot h$$

and also

$$h \otimes g \quad = \quad (e_0 \cdot h) \otimes (g \cdot e_0) \quad = \quad (e_0 \otimes g) \cdot (h \otimes e_0) \quad = \quad g \cdot h$$

In other words, tensor product and group multiplication coincide on $G(0)$, and are commutative, so that $G(0)$ is an abelian group. $\square$

Much like standard operads, we can pair action operads with a natural notion of maps between them in order to form a category.

Definition 1.11. Given action operads G, G', a *map of action operads* $f : G \to G'$ is a map of operads in Set whose components $f_n : G(n) \to G'(n)$ are all group homomorphisms, and which preserves all underlying permutations:

$$\begin{tikzcd}
G(n) \arrow[rr, "f_n"] \arrow[dr, "\pi^G"'] & & G'(n) \arrow[dl, "\pi^{G'}"] \\
& \mathrm{S}_n &
\end{tikzcd}$$

The identity maps $\mathrm{id}_G : G \to G$ and the composites of action operad maps $g \circ f : G \to G' \to G''$ in Op are all well-defined maps of action operads themselves, and so together these constitute a category of action operads and their maps, called AOp.

There are a couple of operads which trivially have the structure of an action operad. First we have the *terminal operad* T, which has a single operation for each arity, so that $\mathrm{T}(n) = \{e_n\}$. Each of these sets can be seen as the trivial group, and it follows from this that the $\pi^{\mathrm{T}} : \mathrm{T}(n) \to \mathrm{S}_n$ must be the respective zero maps, the terminal homomorphisms in the category of groups. The action operad condition is then

$$\mu(e_n; e_{k_1}, ..., e_{k_n}) \cdot \mu(e_n; e_{k_1}, ..., e_{k_n}) \quad = \quad \mu(e_n; e_{k_1}, ..., e_{k_n})$$

which is really just

$$e_{k_1+...+k_n} \cdot e_{k_1+...+k_n} = e_{k_1+...+k_n}$$

and hence is trivially true. As its name suggests, the terminal operad is the terminal object in the category of operads, but it is also the *initial* object in the category of action operads. This is because for any other G in AOp the zero homomorphisms $\mathrm{T}(n) \to G(n)$ define the unique map of operads $f : \mathrm{T} \to G$.

On the other hand, the symmetric operad S itself functions as the terminal object in AOp. Its action operad structure is just given by the standard group multiplications on the S_n, with the identity maps $\mathrm{id}_{\mathrm{S}_n} : \mathrm{S}_n \to \mathrm{S}_n$ functioning as its π_n. To see terminality, notice that for any other action operad G, a valid morphism $f : G \to \mathrm{S}$ in AOp must obey

$$\pi^{\mathrm{S}} \circ f \;=\; \pi^G \qquad \Longrightarrow \qquad f \;=\; \pi^G$$

Thus there only one map of action operads $G \to \mathrm{S}$: the underlying permutation structure used to define G in the first place.

There are more interesting examples of action operads we can look at though. For instance, we know that the braid groups B_n have the same presentation as the symmetric groups, except without the relations $b_i^2 = e$. Thus if we take their quotients by these relations we will obtain a sequence of homomorphisms $B_n \to \mathrm{S}_n$, each sending $b_i \mapsto (i \;\; i+1)$. This provides a natural way to describe the underlying permutation of any braid, and indeed choosing these maps to form π^B gives a valid way of seeing the braid operad as an action operad. Another example can also be built with the so-called ribbon braid groups.

Definition 1.12. For each $n \in \mathbb{N}$, the *ribbon braid group* RB_n is the group whose presentation is the same as that of the braid group B_n, except with the addition of n new generators $t_1, ..., t_n$, known as the *twists*. These twists all commute with one other, and also commute with all braids except in the following cases:

$$b_i \cdot t_i \quad = \quad t_{i+1} \cdot b_i, \qquad\qquad b_i \cdot t_{i+1} \quad = \quad t_i \cdot b_i$$

The *ribbon braid operad* RB is then the operad made up of these groups in a way that extends the definition of the braid operad. In other words, the identity is still $e_1 \in RB_1$, and the operadic multiplication is built up in stages in exactly the same ways as in Example 1.8, but with some additional rules for dealing with twists. For the tensor product, we have that for any twist $t_i \in RB_n$,

$$t_i \quad = \quad e_{i-1} \otimes t \otimes e_{n-i}$$

where t is the sole twist in RB_1, and for the 'block twists' $t_{(m)}$ we again work recursively:

$$t_{(0)} \;=\; e_n, \qquad t_{(m+m')} \quad = \quad \big(t_{(m)} \otimes t_{(m')}\big) \cdot b_{(m',m)} \cdot b_{(m,m')}$$

Much as the symmetric groups can be represented by crossings of a collection of strings, and the braid groups by braidings of strings, the ribbon braid groups deal with the ways that one can braid together several flat ribbons, including the ability to twist a ribbon about its own axis by 360 degrees.

b t

This operad RB is also clearly an action operad, since we can just define $\pi^{RB} : RB_n \to \mathrm{S}_n$ to act like π^B on any braids, at which point the fact that $\pi(t) \in S_1 = \{e_1\}$ will automatically take care of the twists. To learn more about the ribbon braids and their operads, see Natalie Wahl's thesis [19] on the subject, or her subsequent paper with Paolo Salvatore [15].

The fact that the ribbon braid operad seems to contain the whole of the braid operad is the key to easily understanding its operadic structure. We can formalise this kind of relationship in the following way:

Definition 1.13. An action operad G is said to be a *sub action operad* of some other action operad G' if for all $n \in \mathbb{N}$ we have

$$G(n) \leq G'(n), \qquad \mu^G(g; h_1, ..., h_n) = \mu^{G'}(g; h_1, ..., h_n), \qquad \pi^G(g) = \pi^{G'}(g)$$

The most important example of sub action operads are those of the symmetric operad, S. This is because Definition 1.9 itself makes explicit reference to the symmetric groups, and so every action operad will end up being related to some sub-operad of S:

Definition 1.14. For any action operad G, the images of the underlying permutation maps $\pi_n^G : G(n) \to \mathrm{S}_n$ naturally form an action operad $\mathrm{im}(\pi^G)$, where

- the sets of operations are the images of G's sets of operations under the homomorphisms π^G:
$$\mathrm{im}(\pi^G)(n) \quad := \quad \mathrm{im}(\pi_n^G)$$
- the underlying permutation maps are the evident inclusions:
$$\pi_n^{\mathrm{im}(\pi^G)} : \mathrm{im}(\pi_n^G) \hookrightarrow \mathrm{S}_n$$
- the operad multiplication is the appropriate restriction of the multiplication of S:
$$\mu^{\mathrm{im}(\pi^G)}(\, g\, ;\, h_1, ..., h_n\,) \quad := \quad \mu^{\mathrm{S}}(\, g\, ;\, h_1, ..., h_n\,)$$

Clearly this $\mathrm{im}(\pi^G)$ is a sub action operad of the symmetric operad S, and so we will call it the *underlying permutation operad* of G.

For example, consider the action operad B we just saw in Example 1.8. For a given n, the braid group B_n is generated by $n-1$ elementary braids. But the underlying permutations of these braids are just the $n-1$ adjacent transpositions which generate the symmetric group S_n, and so the underlying permutation maps $\pi_n^B : B_n \to \mathrm{S}_n$ are all surjective. Thus the underlying permutation operad of B is just the whole symmetric action operad, $\mathrm{im}(\pi^B) = \mathrm{S}$.

It is even easier to see that S itself will have underlying permutations S, as the maps $\pi_n^{\mathrm{S}} = \mathrm{id} : \mathrm{S}_n \to \mathrm{S}_n$ are obviously surjective. Similarly, the trivial operad T is also its own underlying permutation action operad, as the image of the homomorphims $\pi_n^{\mathrm{T}} : \{e\} \to \mathrm{S}_n$ are trivial. Faced with rather dull examples like these, it might be tempting to try and construct some new action operads with more exotic underlying permutations, like maybe the alternating groups $\mathrm{A}_n \subset \mathrm{S}_n$. But it turns out that this is not possible; when it come to their underlying permutation operad, action operads come in exactly two flavours.

Definition 1.15. Let G be an action operad where $\mathrm{im}(\pi)(n)$ is the trivial group for each $n \in \mathbb{N}$. Then we say that G is *non-crossed*, since its operad multiplication will be a true group homomorphism:

$$\begin{aligned} \mu(\, gg' \,;\, h_1h_1', ..., h_nh_n' \,) &= \mu(\, g \,;\, h_{\pi(g')^{-1}(1)}, ..., h_{\pi(g')^{-1}(n)} \,)\mu(\, g' \,;\, h_1', ..., h_n' \,) \\ &= \mu(\, g \,;\, h_1, ..., h_n \,)\mu(\, g' \,;\, h_1', ..., h_n' \,) \end{aligned}$$

Likewise, a *crossed* action operad will refer to any that has a non-trivial underlying permutation operad.

Lemma 1.16. *An action operad G is crossed if and only if it has surjective underlying permutation maps $\pi_n : G(n) \to \mathrm{S}_n$. In other words, the underlying permutations operad of G must be either the trivial operad* T *or the symmetric operad* S.

Proof. Let $\mathrm{im}(\pi)$ be the underlying permutation operad of G, and let us assume that G is crossed, so that $\mathrm{im}(\pi)$ is not the trivial operad. This means that for some natural number n, the n-ary operations of $\mathrm{im}(\pi)$ include at least one permutation σ which is not the identity element of the relevant symmetric group S_n. Put another way, there must be some σ and some $1 \leq i \leq n$ for which $\sigma(i) \neq i$. But now consider evaluating the expression

$$\mu^{\mathrm{im}(\pi)}(\, \sigma \,;\, e_0, ..., e_0, e_1, e_0, ..., e_0, e_1, e_0, ..., e_0 \,)$$

where the e_1's above are appearing in the ith and $\sigma(i)$th coordinates, which we know are distinct. From the definitions of $\mathrm{im}(\pi)(n)$ and of operad multiplication in S, this permutation is really just

$$\mu^{\mathrm{S}}(\,\sigma\,;\,e_0,...,e_0,e_1,e_0,....,e_0,e_1,e_0,...,e_0\,) \quad = \quad (1\,2)$$

the only non-identity element of S_2. This proves that the map $\pi_2 : G(2) \to \mathrm{S}_2$ is indeed surjective, but more than that it shows that $\mathrm{im}(\pi)$ must contain every possible adjacent transposition, since for any $m \in \mathbb{N}$ we have

$$\begin{aligned} & \mu^{\mathrm{im}(\pi)}(\,e_n\,;\,e_1,...,e_1,(1\,2),e_1,....,e_1\,) \\ = \; & \mu^{\mathrm{S}}(\,e_n\,;\,e_1,...,e_1,(1\,2),e_1,....,e_1\,) \\ = \; & (m\;m+1) \in \mathrm{S}_n \end{aligned}$$

Then because adjacent transpositions generate the symmetric groups S_n, it follows that every permutation is actually an operation in $\mathrm{im}(\pi)$, so that it is really just the full symmetric operad S. Thus by only assuming that our action operad G was crossed, we have shown that all of the maps π_n must be surjective. □

1.4 G-Operads

The most important feature of action operads, and the reason for givning them that name in the first place, is that they are able to 'act' on other operads. The way that this is done for operads in the category of sets is a direct generalisation of the more familiar notion of group actions on sets. Before we begin then, we should recall what exactly is meant by an action of a group on a set.

Definition 1.17. For any set S and group H, a *(right) action* of H on S is a function $\cdot : S \times H \to S$ which respects the group multiplication of H. That is,

$$x \cdot e \,=\, x, \qquad\qquad x \cdot (hh') \,=\, (x \cdot h) \cdot h'$$

for any $x \in S$, $h, h' \in H$, and e the identity of H. The set S equipped with this action is known as an *H-set.*

In more categorical terms, an H-set is simply a functor $\mathrm{B}H \to \mathrm{Set}$. Here the notation $\mathrm{B}H$ refers to the category that has a single object $*$, and a homset $\mathrm{B}H(*,*)$ which is just isomorphic to the group H when viewed as a monoid under composition. The bridge between these two perspectives is that if the functor $\mathrm{B}H \to \mathrm{Set}$ sends $* \mapsto S$, then the rest of the functor constitutes a monoid homomorphism $H \to \mathrm{Set}(S, S)$. We can then see this as a special kind of function $S \times H \to S$, via the (right) tensor-hom adjunction for the category of sets:

$$\mathrm{Set}(A \times B, C) \quad \cong \quad \mathrm{Set}\Big(B,\, \mathrm{Set}(A, C)\Big)$$

Now we can take the idea of H-sets and apply it to the domain of operads and action operads.

Definition 1.18. Let G be an action operad. Then a G*-operad* in the category of sets is an operad O in Set, equipped with an action of the group $G(n)$ on the set $O(n)$ for each $n \in \mathbb{N}$, which respect the operadic multiplications of G and O in the following sense:

$$\mu^O(\, x \cdot g \,;\, y_1 \cdot h_1, ..., y_n \cdot h_n \,) \quad = \quad \mu^O(x; y_{\pi(g)^{-1}(1)}, ..., y_{\pi(g)^{-1}(n)}) \cdot \mu^G(g; h_1, ..., h_n)$$

Additionally, if a map of operads $f : O \to O'$ between two G-operads preserves all of the actions, so that the diagrams

$$\begin{tikzcd} O(n) \times G(n) \arrow[r, "f_n \times \mathrm{id}_{G(n)}"] \arrow[d] & O'(n) \times G(n) \arrow[d] \\ O(n) \arrow[r, "f_n"] & O'(n) \end{tikzcd}$$

commute for each $n \in \mathbb{N}$, then we say that f is a *map of* G*-operads* in Set. Together G-operads of sets and their maps form a category, which we shall call $G{-}\mathrm{Op}$.

It is a well-known fact that every group H can also be seen as an H-set, with the action $H \times H \to H$ being given by multiplication or the right. The equivariance axiom above has been chosen in such a way that we can immediately conclude an analogous result about operads. That is, any action operad G is also G-operad with actions $G(n) \times G(n) \to G(n)$ given by multiplication on the right, because under those conditions the defining equation of a G-operad simply becomes the defining equation for an action operad.

For certain specific G, the G-operads are already well-studied objects. If we take our action operad G to be the symmetric operad S, then since the map π^{S} is trivial we arrive at a rather straightforward variety of G-operads, those whose equivariance is given by

$$\mu^O(\, x \cdot \sigma \,;\, y_1 \cdot \tau_1, ..., y_n \cdot \tau_n \,) \quad = \quad \mu^O(x; y_{\sigma^{-1}(1)}, ..., y_{\sigma^{-1}(n)}) \cdot \mu^{\mathrm{S}}(\sigma; \tau_1, ..., \tau_n)$$

These S-operads are nowadays generally known as *symmetric operads*, or sometimes *permutative operads*. However, May's original definition [14] for 'operads' was actually this symmetric version, and so some authors prefer to reserve that term for these structures, instead calling the subject of Definition 1.4 'planar operads', or 'operads without permutation'. This should give an idea of just how important these symmetric operads really are. Prominent examples include the 'little cubes', 'little discs', and similar operads which helped motivate the development of operad theory. There are also *braided operads*, which are B-operads for the braid operad B — these appear in the work of Zbigniew Fiedorowicz [7].

As one might expect, the notion of G-operads can be extended from Set to work in other symmetric monoidal categories $(C, \otimes, I)$, by instead working with the operads within that category. Since we are aiming to connect action operads to symmetric and braided monoidal categories, the particular context we will be interested in is Cat, the category of (small) categories. Here the concept of a group action is particularly simple — it is just like a group action on sets, applied to both the objects and morphism of a category.

Definition 1.19. Let X be a category, and H a group which we will also think of as a discrete category. Then a *(right) action* of H on X is a functor $\cdot : X \times H \to X$ which respects the group multiplication of H. That is,

$$\begin{array}{rclcrcl} x \cdot e & = & x, & \qquad & x \cdot (hh') & = & (x \cdot h) \cdot h' \\ f \cdot \mathrm{id}_e & = & f, & \qquad & f \cdot \mathrm{id}_{hh'} & = & (f \cdot \mathrm{id}_h) \cdot \mathrm{id}_{h'} \end{array}$$

for any objects x and morphisms f of X, and elements $h, h' \in H$ with e the identity.

As before, we can view a group action like this as a functor $\mathrm{B}H \to \mathrm{Cat}$ where the sole object $*$ of $\mathrm{B}H$ is sent to the category X in question. This is because these are equivalent to monoid homomorphisms $H \to \mathrm{Cat}(X, X)$, which we can see as functors $X \times H \to X$ using the fact that Cat is copowered (on the right) over Set:

$$\mathrm{Cat}(X \times S, Y) \quad \cong \quad \mathrm{Set}\Big(\, S,\, \mathrm{Cat}(X, Y)\,\Big)$$

Here S is a set which again we identify with a discrete category.

Definition 1.20. Let G be an action operad. Then a *G-operad* in Cat is an operad O in $(\mathrm{Cat}, \times, 1)$, equipped with an action of the group $G(n)$ on the category $O(n)$ for each $n \in \mathbb{N}$, which respect the operadic multiplications of G and O via a higher dimensional version of the equation in Definition 1.18. Specifically, we require that the diagram

$$\begin{tikzcd}
O(n) \times G(n) \times \prod \Big(O(k_i) \times G(k_i) \Big) \arrow[r] \arrow[d, "\mathrm{id}\times(\pi,\mathrm{id})\times\mathrm{id}"'] & O(n) \times \prod O(k_i) \arrow[ddd, "\mu^O"] \\
O(n) \times \mathrm{S}_n \times G(n) \times \prod \Big(O(k_i) \times G(k_i) \Big) \arrow[d, "\beta"'] & \\
O(n) \times \mathrm{S}_n \times \prod O(k_i) \times G(n) \times \prod G(k_i) \arrow[d, "\tilde{\mu}^O \times \mu^G"'] & \\
O(k_1 + ... + k_n) \times G(k_1 + ... + k_n) \arrow[r] & O(k_1 + ... + k_n)
\end{tikzcd}$$

commutes for all $n, k_1, ..., k_n \in \mathbb{N}$. Here we are using $\tilde{\mu}^O$ to refer to the obvious functor $\mathrm{S}_n \times O(n) \times \prod O(k_i) \to O(k_1 + ... + k_n)$ which acts like μ^O but with suitably permuted inputs:

$$\tilde{\mu}^O(\,\sigma, x\,;\, y_1, ..., y_n\,) \quad := \quad \mu^O(\,x\,;\, y_{\sigma^{-1}(1)}, ..., y_{\sigma^{-1}(1)}\,)$$

The easiest way to produce examples of G-operads in Cat is to simply build them out of existing operads in the category of sets. In particular, if we design our categories of operations so that the morphisms are determined entirely by their source and target, then a single operad in Set will suffice to create one of these new higher dimensional operads.

Definition 1.21. For any monoid M, we will define its *translation category* $\mathrm{E}M$ to be the (monoidal) category whose objects are the elements of the monoid M, and whose morphisms consist of a unique isomorphism between each pair of objects. Also, for any monoid homomorphisms $h : M \to M'$ we can define a functor

$$\begin{array}{rcrcl}
\mathrm{E}h & : & \mathrm{E}M & \to & \mathrm{E}M' \\
 & : & m & \mapsto & h(m) \\
 & : & m \to m' & \mapsto & h(m) \to h(m')
\end{array}$$

This definition of $\mathrm{E}h$ obviously respects composition and identities, and so together with $\mathrm{E}M$ it describes a functor $\mathrm{E} : \mathrm{Mon} \to \mathrm{Cat}$.

Likewise, for any operad O in the category of sets we can define its *translation operad* $\mathrm{E}O$ to be the operad in Cat given by the data

$$(\mathrm{E}O)(n) := \mathrm{E}\big(O(n)\big), \qquad 1^{\mathrm{E}O} = \mathrm{E}(1^O), \qquad \mu^{\mathrm{E}O} = \mathrm{E}(\mu^O)$$

For each of the coherence conditions which $\mathrm{E}O$ must satisfy in order to be a well-defined operad in Cat, we can obtain them from the corresponding conditions that make O an operad in Set, by simply applying the functor E everywhere.

$\mathrm{E}O$ can be seen as a categorified version of the operad O. That is, it may live in Cat rather than Set, but in many other respects it behaves the same way that O does. Of particular interest to us is what this means in the case when O is really an action operad G. We saw earlier that any G is always a G-operad in the category of sets, with an action given by group multiplication. The categorified variant of this statement is the following:

Lemma 1.22. *For any action operad G, the translation operad $\mathrm{E}G$ is a G-operad in* Cat*, with actions*

$$\begin{array}{rcl} \mathrm{E}G(n) \times G(n) & \to & \mathrm{E}G(n) \\ (g, h) & \mapsto & gh \\ (g \to g', \mathrm{id}_h) & \mapsto & gh \to g'h \end{array}$$

The proof of this fact can be found in [8].

1.5 Operad algebras

As with many mathematical structures, we are not merely interested in operads for their own sake, but also for their algebras.

Definition 1.23. Let O be an operad in the symmetric monoidal category $(C, \otimes, I)$. Then an *algebra* of O is an object X in C, equipped with a family of morphisms $\alpha_n : O(n) \otimes X^{\otimes n} \to X$, $n \in \mathbb{N}$ called the action of O on X, which obey axioms that mirror those needed to define an operad. In other words, we a unitality condition

$$\begin{tikzcd} I \otimes X \arrow[d, "1 \otimes \mathrm{id}_X"'] \arrow[dr, "l_X"] & \\ O(1) \otimes X \arrow[r, "\alpha_1"'] & X \end{tikzcd}$$

and then for all $n, k_1, ..., k_n \in \mathbb{N}$ we have an associativity condition,

$$\begin{tikzcd}
O(n) \otimes \prod O(k_i) \otimes \prod X^{\otimes k_i} \arrow[d, "\beta"'] \arrow[dr, "\mu \otimes \mathrm{id}"] & \\
O(n) \otimes \prod \left(O(k_i) \otimes X^{\otimes k_i} \right) \arrow[d, "\mathrm{id} \otimes \prod \alpha"'] & O(k_1 + ... + k_n) \otimes X^{\otimes (k_1 + ... + k_n)} \arrow[d, "\alpha"] \\
O(n) \otimes X^{\otimes n} \arrow[r, "\alpha"] & X
\end{tikzcd}$$

As one might expect, a *map of algebras* $f : (X, \alpha^X) \to (Y, \alpha^Y)$ between two algebras of O is then simply a map between their underlying objects, $f : X \to Y$, which preserves this algebra structure:

$$\begin{tikzcd}
O(n) \times X^n \arrow[r, "\mathrm{id}^{O(n)} \times f^n"] \arrow[d, "\alpha^X"'] & O(n) \times Y^n \arrow[d, "\alpha^Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Together these form the category OAlg of all O-algebras and their maps.

When O is an operad in Set, an algebra of O is simply a realisation of the elements of the $O(n)$ as actual n-ary operations on some set. A similar statement is true in any concrete category, though with extra structure or restrictions depending on the nature of $(C, \otimes, I)$. When O is an operad in Cat we can also upgrade the category OAlg into a 2-category, by simply adding in monoidal natural transformations as the 2-morphisms between algebra maps. We will use the notation OAlg for both of these structures, since it is generally clear from the context whether we want to be working with categories or 2-categories.

As we've seen many times already, when the operad we are working with is actually an action operad, the presence of the additional group structure will cause something more interesting to happen. In this case, because the operadic multiplication $\mu^G(e_n; \_, ..., \_)$ in an action operad G can be interpreted as a tensor product, the operad algebras of G will end up inheriting a monoidal structure of their own.

Lemma 1.24. *Let G be an action operad, and X an algebra of G in the category of sets. Then X is a monoid with respect to the operation*

$$x_1 \otimes \ldots \otimes x_n \quad := \quad \alpha(e_n; x_1, \ldots, x_n)$$

and there exists a forgetful functor $G\mathrm{Alg} \to \mathrm{Mon}$ *sending the algebras of G to this underlying monoid structure.*

Similarly, let Y be an algebra of $\mathrm{E}G$ *in* Cat*. Then Y is a strict monoidal category with respect to the operation*

$$y_1 \otimes \ldots \otimes y_n \quad := \quad \alpha(e_n; y_1, \ldots, y_n), \qquad f_1 \otimes \ldots \otimes f_n \quad := \quad \alpha(e_n; f_1, \ldots, f_n)$$

and there is a forgetful 2-functor $\mathrm{E}G\mathrm{Alg} \to \mathrm{MonCat}_{\mathrm{st}}$ *sending these algebras to their underlying strict monoidal structure.*

Proof. We'll start by checking that above definition on X makes sense. Firstly, for any element $x \in X$ we want the one-fold tensor product of x with itself to just be x again. This is ensured by the unitality of the action α^X, which says that $\alpha(e_1; x) = x$. Next, we need to make sure that the tensor products for each arity are all compatible with each other, which follows from the associativity axiom for α^X:

$$\begin{aligned}
& (x_1 \otimes \ldots \otimes x_{k_1}) \otimes \ldots \otimes (x_{k_1+\ldots+k_{n-1}+1} \otimes \ldots \otimes x_{k_1+\ldots+k_n}) \\
= \; & \alpha\Big(e_n \,;\, \alpha(e_{k_1}; x_1, \ldots, x_{k_1}), \, \ldots, \, \alpha(e_{k_n}; x_{k_1+\ldots+k_{n-1}+1}, \ldots, x_{k_1+\ldots+k_n}) \Big) \\
= \; & \alpha\Big(\mu(e_n; e_{k_1}, \ldots, e_{k_n}) \,;\, x_1, \ldots, x_{k_1}, \ldots, x_{k_1+\ldots+k_{n-1}+1}, \ldots, x_{k_1+\ldots+k_n} \Big) \\
= \; & \alpha(e_{k_1+\ldots+k_n} \,;\, x_1, \ldots, x_{k_1+\ldots+k_n}) \\
= \; & x_1 \otimes \ldots \otimes x_{k_1+\ldots+k_n}
\end{aligned}$$

Perhaps unsurprising, this means that associativity axiom also forces the tensor product to be associative. Finally, a special case of the above — where $n = 2$ and the k_i are 0 and 1 — shows that the empty tensor product $\alpha(e_0; -)$ acts as the unit of $\otimes$. Thus X is indeed a well-defined monoid under the tensor product coming from its action. Moreover, since all algebra maps $f : X \to X'$ preserve actions they will also preserve this monoid structure,

$$f(x \otimes x') \quad = \quad f\Big(\alpha^X(e_2; x, x')\Big) \quad = \quad \alpha^{X'}\Big(e_2 \,;\, f(x), f(x')\Big) \quad = \quad f(x) \otimes f(x')$$

Therefore if we forget all of the features of our G-algebras other than the tensor product, what we are left with are monoids and monoid homomorphisms, and this defines an obvious functor $G\mathrm{Alg} \to \mathrm{Mon}$.

Turning now to the category Y, if we think of all of the functors in the unitality and associativity axioms for α^Y as acting just on objects, the exact same arguments we employed above will show that $(\mathrm{Ob}(Y), \otimes)$ is well-defined monoid. Likewise, restricting our view to morphisms will let us prove that $(\mathrm{Mor}(Y), \otimes)$ is a monoid, and then functoriality of α^Y tells us that we can stitch these two tensor products together into a single functor $\otimes : Y \times Y \to Y$.

$$\begin{aligned}(f : x \to y) \otimes (f' : x' \to y') &= \alpha(e_2; f, f') : \alpha(e_2; x, x') \to \alpha(e_2; y, y') \\ &= f \otimes f' : x \otimes x' \to y \otimes y'\end{aligned}$$

Thus Y as a whole has a tensor product, and because it comes from a monoid at both levels it will be *strictly* associative and unital. Therefore EG-algebras are strict monoidal categories, and since any algebra map $F : Y \to Y'$ will preserve this monoidal structure for the same reason we had before, there is an associated forgetful 2-functor EGAlg $\to$ $\mathrm{MonCat}_{\mathrm{st}}$ onto the 2-category of strict monoidal categories. □

In general, the algebras of G and EG will have a lot more structure to them than just this tensor product. For example, any algebra for the symmetric operad S will have an extra binary operation coming from the elementary permutation in S_2:

$$\begin{aligned} X \times X &\to X \\ (x, x') &\mapsto \alpha\Big((1\,2)\,;\, x, x' \Big)\end{aligned}$$

However, the rules that govern operad algebras do not put any extra restraints on this operation, which means that the resulting category SAlg ends up fairly boring due to its lack of specificity. The problem is that by using the concept of a standard operad algebra, we are ignoring the group multiplication of our action operads, since this is not something that every operad of sets can be expected to have.

What we need is a notion for algebras of a G-operad. Of course, as operads themselves any G-operad will already have algebras in the sense of Definition 1.23, but in general these won't respect the G-operadic actions, which anything worthy of the name 'G-operad algebra' should do. We can fix this by simply demanding that the maps α coequalise certain maps, chosen in a way which will force the equivariance to hold.

Definition 1.25. For any operad O in Set or Cat, a *G-operad algebra* X of O is just an operad algebra of O whose actions $\alpha_n : O(n) \times X^n \to X$ coequalise two maps $O(n) \times G(n) \times X^n \to O(n) \times X^n$, one coming from the action of $G(n)$ on $O(n)$, and the other from the reordering of X^n by the underlying permutations of $G(n)$.

More precisely, recall that the symmetric monoidal structures of $(\mathrm{Set}, \times, 1)$ and $(\mathrm{Cat}, \times, 1)$ provide us with several different isomorphisms $\beta : X^n \to X^n$. Indeed, there will be one for each permutation in S_n, and this gives rise to a natural embedding of monoids $\mathrm{S}_n \to \mathrm{Set}(X^n, X^n)$ or $\mathrm{S}_n \to \mathrm{Cat}(X^n, X^n)$. We can then use the (left) copower isomorphisms of the given categories to turn these embeddings into maps $\tilde{\beta} : \mathrm{S}_n \times X^n \to X^n$. With this notation, we define an algebra of O as a G-operad to be any algebra X of O as an operad, for which the following two composites are equal:

$$\begin{tikzcd}
O(n) \times G(n) \times X^n \arrow[rr, "\mathrm{id}_{O(n)} \times \pi \times \mathrm{id}_{X^n}"] \arrow[dr, "\cdot \times \mathrm{id}_{X^n}"'] & & O(n) \times \mathrm{S}_n \times X^n \arrow[dl, "\mathrm{id}_{O(n)} \times \tilde{\beta}"] \\
& O(n) \times X^n \arrow[d, "\alpha"] & \\
& X &
\end{tikzcd}$$

Maps between G-operad algebras are just maps between their underlying operad algebras, so the category of G-operad algebras for a given O is a subcategory of $O\mathrm{Alg}$, which we will denote by $O_G\mathrm{Alg}$. Generally, the objects of this category will just be referred to as O-algebras, with the G-operad perspective being implied.

So, what are the algebras of an action operad like in this context? Unfortunately, $G_G\mathrm{Alg}$ is even less interesting than $G\mathrm{Alg}$ was; it is simply the category of monoids, Mon. To see this, notice that if we view an action operad G as a G-operad with multiplication for its action, then the equivariance condition for an algebra X will become

$$\begin{aligned}
\alpha(g; x_1, ..., x_n) &= \alpha(\, e_n \cdot g\, ;\, x_1, ..., x_n\,) \\
&= \alpha(\, e_n\, ;\, x_{\pi(g)^{-1}(1)}, ..., x_{\pi(g)^{-1}(n)}\,) \\
&= x_{\pi(g)^{-1}(1)} \otimes ... \otimes x_{\pi(g)^{-1}(n)}
\end{aligned}$$

That is, the action α^X is entirely determined by the tensor product of X, and as we've already seen that this is unrestrained by the axioms for an operad algebra, so X is just an undecorated monoid.

However, the 2-category $\mathrm{E}G_G\mathrm{Alg}$ is far more exciting. Sure, the same argument we've just used for G will ensure that the action reduces to the tensor product on objects, but on morphisms the underlying permutative structure π will finally come into play. As an example, for the symmetric operad S we know that any ES-algebra X

must contain action morphisms of the form

$$\begin{array}{rcl} \alpha\Big(\, e_2 \to (1\,2)\,;\, \mathrm{id}_x, \mathrm{id}_y \Big) & : & \alpha\Big(\, e_2\,;\, x, y \Big) \;\; \to \;\; \alpha\Big(\, (1\,2)\,;\, x, y \Big) \\ & : & \qquad x \otimes y \;\; \to \;\; y \otimes x \end{array}$$

for all objects x,y. Indeed, it is not to difficult to see that these morphisms are really the symmetries $\beta_{x,y}$ for a strict symmetric monoidal category. For instance, the relation $\beta_{y,x} \circ \beta_{x,y} = \mathrm{id}_{x \otimes y}$ comes from the S-operad algebra equivariance axiom, and the fact that the functor α preserves composition:

$$\begin{array}{rl} & \alpha\Big(\, e_2 \to (1\,2)\,;\, \mathrm{id}_y, \mathrm{id}_x \Big) \circ \alpha\Big(\, e_2 \to (1\,2)\,;\, \mathrm{id}_x, \mathrm{id}_y \Big) \\ = & \alpha\Big(\, (1\,2)\cdot(1\,2) \to e_2 \cdot (1\,2)\,;\, \mathrm{id}_y, \mathrm{id}_x \Big) \circ \alpha\Big(\, e_2 \to (1\,2)\,;\, \mathrm{id}_x, \mathrm{id}_y \Big) \\ = & \alpha\Big(\, (1\,2) \to e_2\,;\, \mathrm{id}_x, \mathrm{id}_y \Big) \circ \alpha\Big(\, e_2 \to (1\,2)\,;\, \mathrm{id}_x, \mathrm{id}_y \Big) \\ = & \alpha\Big(\, e_2 \to (1\,2) \to e_2\,;\, \mathrm{id}_x \circ \mathrm{id}_x, \mathrm{id}_y \circ \mathrm{id}_y \Big) \\ = & \alpha(\mathrm{id}_{e_2}; \mathrm{id}_x, \mathrm{id}_y) \\ = & \mathrm{id}_{\alpha(e_2;x,y)} \\ = & \mathrm{id}_{x \otimes y} \end{array}$$

The questions that should follow from this observation are obvious. What about the braid operad B? Are the objects of EB_BAlg braided monoidal categories, in the same way that those of ES$_{\mathrm{S}}$Alg are symmetric monoidal? What about the algebras of the ribbon braid operad, what sort of monoidal category are they? And do the S-operad algebras of ES have any additional structure, other than their symmetries?

It turns out that there is a theorem which answers all of these questions at once, for all possible G. To properly state it though, we'll need some new terminology.

Definition 1.26. Let a $(\mathcal{G}, \mathcal{R})$-*monoidal category* mean a strict monoidal category X, equipped with a set of natural isomorphisms

$$\mathcal{G} \;=\; \Big\{ \, (f; \pi_f) \;:\; x_1 \otimes ... \otimes x_n \xrightarrow{f} x_{\pi_f^{-1}(1)} \otimes ... \otimes x_{\pi_f^{-1}(n)} \Big\}$$

which are subject to relations of the form

$$r \;:\; (f_{1,1} \otimes ... \otimes f_{1,k_1}) \circ ... \circ (f_{n,1} \otimes ... \otimes f_{n,k_n}) \;=\; (f'_{1,1} \otimes ... \otimes f'_{1,k'_1}) \circ ... \circ (f'_{n',1} \otimes ... \otimes f'_{n',k'_{n'}})$$

for each element in the set

$$\mathcal{R} \;=\; \Big\{ \, (r; n, n', \underline{k}, \underline{k'}, \underline{f}, \underline{f'}) \;:\; n, n' \in \mathbb{N},\, \underline{k} \in \mathbb{N}^n,\, \underline{k'} \in \mathbb{N}^{n'},\, \underline{f} \in \mathcal{G}^{\Sigma \underline{k}},\, \underline{f'} \in \mathcal{G}^{\Sigma \underline{k'}} \Big\}$$

Definition 1.27. Let G be an action operad. Then a *G-monoidal category* will refer to the notion of $(\mathcal{G},\mathcal{R})$-monoidal category we get from G by setting

$$\mathcal{G} = \Big\{ \Big(g\,;\, \pi(g) \Big) \;:\; \forall\, g \in G \Big\}$$

and having $\mathcal{R}$ contain one element $(r; n, n', \underline{k}, \underline{k'}, \underline{g}, \underline{g'})$ for each relation

$$r \;:\; (g_{1,1} \otimes \ldots \otimes g_{1,k_1}) \cdot \ldots \cdot (g_{n,1} \otimes \ldots \otimes g_{n,k_n}) \;=\; (g'_{1,1} \otimes \ldots \otimes g'_{1,k'_1}) \cdot \ldots \cdot (g'_{n',1} \otimes \ldots \otimes g'_{n',k'_{n'}})$$

satisfied by the action operad G.

Theorem 1.28. *For any action operad G, the algebras of $\mathrm{E}G$ are precisely the G-monoidal categories. Furthermore, for every notion of $(\mathcal{G},\mathcal{R})$-monoidal categories, there exists some action operad G for which they are equivalent to G-monoidal categories, and thus $\mathrm{E}G$-algebras.*

Proof. See [8], Theorem 3.11 and Corollary 3.12. □

This powerful result lets us to freely move back and forth between the worlds of action operads and strict monoidal categories, allowing us to reframe our questions about the latter into ones concerning the former. For instance, it is not difficult to see that the action operad corresponding to braided monoidal categories is the braid operad B. Thus if we want to describe certain kinds of free braided monoidal category, we can instead choose to look for the same sorts of free $\mathrm{E}B$-algebra. Moreover, this equivalence reveals a way to generate new examples of either structures. Using an eariler example, we know that the ribbon braid groups form an action operad RB, and so we can immediately conclude that there exists some notion of ribbon braided monoidal category [19], sometimes also known as balanced monoidal categories [16]. Conversely, if we had already known about these ribbon categories then we could have surmised from their strict versions that the ribbon braid groups formed an action operad.

Also, Theorem 1.28 will lead to a simplification for how we describe the action α of an $\mathrm{E}G$-algebra X. First, from now on we will generally only speak of the action as an operation that can be applied to the morphisms of X, because while α is really a functor its effect on objects is already covered by any discussion of the tensor product $\otimes$. Secondly, when the $\mathrm{E}G$ coordinate of α contains the unique morphism $g \to h$, we can always use the action of G on $\mathrm{E}G$ to rewrite things so that we have the morphism $e \to hg^{-1}$ instead. We saw this briefly in the symmetric example we looked at, but the definition of $\mathcal{G}$ in Definition 1.27 along with Theorem 1.28 shows that this shift to a single variable will never cause any problems or additional considerations. Thus from now on we will freely identify the morphism $g \to h$ in EG with the element hg^{-1}.

1.6 The free EG-algebra on n objects

From here on out, everything we do in this paper will be geared towards goal of describing the free EG-algebras on n invertible objects, for each action operad G. By Theorem 1.28, this will then tell us how to construct the equivalent free structure for whole host of strict monoidal categories. Before we attempt this however, it is crucial that we understand a much simpler case, where we do not require that our objects be invertible.

Definition 1.29. Let G be an action operad. Then for any category X and $k \in \mathbb{N}$, we will denote by $\mathrm{E}G(k) \times_{G(k)} X^k$ the coequaliser of the two functors $\mathrm{E}G(k) \times G(k) \times X^k \to \mathrm{E}G(k) \times X^k$ from Definition 1.25:

$$\begin{tikzcd}
\mathrm{E}G(k) \times G(k) \times X^k \arrow[rr, "\mathrm{id}_{\mathrm{E}G(k)} \times \pi \times \mathrm{id}_{X^k}"] \arrow[dr, "\cdot \times \mathrm{id}_{X^k}"'] & & \mathrm{E}G(k) \times \mathrm{S}_k \times X^k \arrow[dl, "\mathrm{id}_{\mathrm{E}G(k)} \times \tilde{\beta}"] \\
& \mathrm{E}G(k) \times X^k \arrow[d] & \\
& \mathrm{E}G(k) \times_{G(k)} X^k &
\end{tikzcd}$$

Proposition 1.30. *Let $\{z_1, ..., z_n\}$ be an n-object set, which can also be considered as a discrete category. Then the free* EG*-algebra on n objects is the algebra $\mathbb{G}_n$ whose underlying category is*

$$\mathbb{G}_n \quad := \quad \coprod_{k \in \mathbb{N}} \mathrm{E}G(k) \times_{G(k)} \{z_1, ..., z_n\}^k$$

where for all $m, k_1, ..., k_m \in \mathbb{N}$, $g \in G(m)$, $x_i \in \{z_1, ..., z_n\}$ the action is given by

$$\alpha\Big(\, g\,;\, (h_1; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_{k_1}}),\, ...,\, (h_m; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_{k_m}})\,\Big) \;=\; \Big(\, \mu(g; h_1, .., h_m)\,;\, \mathrm{id}_{x_1},\, ...,\, \mathrm{id}_{x_{k_m}}\,\Big)$$

In other words, for any EG*-algebra* X*,*

$$\mathrm{E}G_G\mathrm{Alg}(\mathbb{G}_n, X) \quad\cong\quad \mathrm{Cat}(\{z_1, ..., z_n\}, X) \quad\cong\quad X^n$$

Again, this is something already covered by the work of Gurski [4], so we won't go through all of the details here. The basic idea is that since the actions α_m : $\mathrm{E}G(m) \times X^m \to X$ of any EG-algebra coequalise the diagram from Definition 1.25, the universal property of $\mathrm{E}G(k) \times_{G(k)} X^k$ will allow us to factor them uniquely though some α',

$$\begin{tikzcd}
\mathrm{E}G(k) \times X^k \arrow[d] \arrow[dr, "\alpha_k"] & \\
\mathrm{E}G(k) \times_{G(k)} X^k \arrow[r, dashed, "\alpha'_k"'] & X
\end{tikzcd}$$

This then lets us upgrade any choice $f : \{z_1, ..., z_n\} \to X$ of n objects from X into an algebra map $\mathbb{G}_n \to X$:

$$\textstyle\coprod_{k\in\mathbb{N}} \mathrm{E}G(k) \times_{G(k)} \{z_1, ..., z_n\}^k \xrightarrow{\coprod \mathrm{id}\times f^k} \coprod_{k\in\mathbb{N}} \mathrm{E}G(k) \times_{G(k)} X^k \xrightarrow{\coprod \alpha'_k} X$$

Proposition 1.30 serves as a fairly opaque definition of $\mathbb{G}_n$ at first, so we'll spend a little time now unpacking it. Recall that $\coprod_{k\in\mathbb{N}} \mathrm{E}G(k) \times_{G(k)} \{z_1, ..., z_n\}^k$ is the coequalizer of the maps

$$\textstyle\coprod_{k\in\mathbb{N}} \mathrm{E}G(k) \times G(k) \times \{z_1, ..., z_n\}^k \rightrightarrows \coprod_{k\in\mathbb{N}} \mathrm{E}G(k) \times \{z_1, ..., z_n\}^k$$

that come from the action of $G(k)$ on $\mathrm{E}G(k)$ by multiplication on the right,

$$\begin{array}{rcl}
\mathrm{E}G(k) \times G(k) & \to & \mathrm{E}G(k) \\
(g, h) & \mapsto & gh \\
(\,! : g \to g', \mathrm{id}_h\,) & \mapsto & ! : gh \to g'h
\end{array}$$

and the action of $G(k)$ on $\{z_1, ..., z_n\}^k$ by underlying permutations,

$$\begin{array}{rcl}
G(k) \times \{z_1, ..., z_n\}^k & \to & \{z_1, ..., z_n\}^k \\
(\,h\,;\, x_1, ..., x_k\,) & \mapsto & (x_{\pi(h^{-1})(1)}, ..., x_{\pi(h^{-1})(k)}) \\
(\mathrm{id}_h\,;\, \mathrm{id}_{(x_1,...,x_k)}\,) & \mapsto & \mathrm{id}_{(x_{\pi(h^{-1})(1)},...,x_{\pi(h^{-1})(k)})}
\end{array}$$

Thus objects in this algebra are equivalence classes of tuples $(g; x_1, ..., x_m)$, for some $g \in G(m)$ and $x_i \in \{z_1, ..., z_n\}$, under the relation

$$(gh \,;\, x_1,\, ...,\, x_m) \sim (g \,;\, x_{\pi(h)^{-1}(1)},\, ...,\, x_{\pi(h)^{-1}(m)})$$

But we can use this relation to rewrite any $(g; x_1, ..., x_m)$ uniquely in the form $(e_m; x'_1, ..., x'_m) = x'_1 \otimes ... \otimes x'_m$ where $x'_i = x_{\pi(g)(i)}$, and this means that each such equivalence class is just a tensor product for some unique sequence of generators z_i. More concretely, we have:

Lemma 1.31. $\mathrm{Ob}(\mathbb{G}_n)$ *is the free monoid on n generators, which is $\mathbb{N}^{*n}$, the free product of n copies of* $\mathbb{N}$.

Similarly, the morphisms of $\mathbb{G}_n$ are all of the form

$$(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) \,:\, x_1 \otimes ... \otimes x_m \;\to\; x_{\pi(g^{-1})(1)} \otimes ... \otimes x_{\pi(g^{-1})(m)}$$

for some $g \in G(m)$ and $x_i \in \{z_1, ..., z_n\}$. However, notice the definition of the action α of $\mathbb{G}_n$, we can rewrite these as

$$\begin{aligned}(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) &= \Big(\mu(g; e_1, ..., e_1)\,;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m} \Big) \\ &= \alpha\Big(g \,;\, (e_1; \mathrm{id}_{x_1}),\, ...,\, (e_1; \mathrm{id}_{x_m}) \Big) \\ &= \alpha(g;\, \mathrm{id}_{(e_1;x_1)}, ..., \mathrm{id}_{(e_1;x_m)}) \\ &= \alpha(g;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})\end{aligned}$$

That is, the free EG-algebra $\mathbb{G}_n$ does not have any objects or morphisms that do not arise straightforwardly from the tensor product and action.

Lemma 1.32. *Every morphism of $\mathbb{G}_n$ can be expressed uniquely as an action morphism*

$$\alpha(g \,;\, \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) \,:\, x_1 \otimes ... \otimes x_m \;\to\; x_{\pi(g)^{-1}(1)} \otimes ... \otimes x_{\pi(g)^{-1}(m)}$$

for some $g, g' \in G(m)$ and $x_i \in \{z_1, ..., z_n\}$.

As an immediate consequence of this, the source and target of any given morphism in $\mathbb{G}_n$ must be related to one another via some permutation of the form $\pi(g)$. In other words, the connected components of $\mathbb{G}_n$ will depend upon the underlying permutation operad of G, in the following way:

Proposition 1.33. *Considered as a monoid under tensor product,*

$$\pi_0(\mathbb{G}_n) \quad = \quad \begin{cases} \mathbb{N}^n & \text{if } G \text{ is crossed} \\ \mathbb{N}^{*n} & \text{otherwise} \end{cases}$$

Also, the canonical homomorphism sending objects in $\mathbb{G}_n$ to their connected component,

$$[\_] \,:\, \mathrm{Ob}(\mathbb{G}_n) \to \pi_0(\mathbb{G}_n)$$

is the quotient map of abelianisation

$$\mathrm{ab} \,:\, \mathbb{N}^{*n} \to (\mathbb{N}^{*n})^{\mathrm{ab}} \,=\, \mathbb{N}^n$$

when G is crossed, and the identity map $\mathrm{id}_{\mathbb{N}^{*n}}$ otherwise.

Proof. By Lemma 1.32, all morphisms in $\mathbb{G}_n$ can be written uniquely as $\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$, for some $g \in G(m)$ and $x_i \in \{z_1, ..., z_n\}$. Since maps of this form have source $x_1 \otimes ... \otimes x_m$ and target $x_{\pi(g^{-1})(1)} \otimes ... \otimes x_{\pi(g^{-1})(m)}$, we see that the only pairs of object which might have a morphism between them are those that can be expanded as tensor products that differ by some permutation.

If our action operad G is crossed, then for any two objects like this — say source $x_1 \otimes ... \otimes x_m$ and target $x_{\sigma^{-1}(1)} \otimes ... \otimes x_{\sigma^{-1}(m)}$ for an arbitrary $\sigma \in \mathrm{S}_m$ — we can always find a map $\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$ between them, because by Lemma 1.16 the underlying permutations maps $\pi_m : G(m) \to S_m$ are all surjective and so there must exist at least one g with $\pi(g) = \sigma$. In particular, for any two generating objects z_i and $z_j$of $\mathbb{G}_n$ there must exist at least morphism between $z_i \otimes z_j$ and $z_j \otimes z_i$, and therefore

$$[z_i] \otimes [z_j] \quad = \quad [z_i \otimes z_j] \quad = \quad [z_j \otimes z_i] \quad = \quad [z_j] \otimes [z_i]$$

Thus the canonical map $[\_] : \mathrm{Ob}(\mathbb{G}_n) \to \pi_0(\mathbb{G}_n)$ is the one that makes the free product of $\mathbb{N}^{*n}$ commutative, that is, the quotient map for the abelianisation $\mathrm{ab} : \mathbb{N}^{*n} \to (\mathbb{N}^{*n})^{\mathrm{ab}}$, and so $\pi_0(\mathbb{G}_n) = \mathbb{N}^n$.

Conversely, if G is non-crossed then its underlying permutation operad $\mathrm{im}(\pi)$ is trivial, and so the only morphisms we have in $\mathbb{G}_n$ will be those of the form

$$\alpha(\, e_m \,; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m} \,) \quad = \quad \mathrm{id}_{x_1} \otimes ... \otimes \mathrm{id}_{x_m} \quad = \quad \mathrm{id}_{x_1 \otimes ... \otimes x_m}$$

Therefore the map $[\_]$ just sends each object to its identity morphism, and since that function is one-to-one and onto it follows that

$$\pi_0(\mathbb{G}_n) \quad = \quad \mathrm{Ob}(\mathbb{G}_n) \quad = \quad \mathbb{N}^{*n}, \qquad\qquad [\_] \quad = \quad \mathrm{id}_{\mathbb{N}^{*n}}$$

by Lemma 1.31. □

?? is not the only way that the behaviour of tthe algebra $\mathbb{G}_n$ is contigent on whether or not G is crossed. A strict monoidal category X is said to be *spacial* if all of its identity morphisms commute with the endomorphisms of the unit object:

$$f \otimes \mathrm{id}_x = \mathrm{id}_x \otimes f, \qquad x \in \mathrm{Ob}(X), f \in X(I, I)$$

The motivation for the name 'spacial' comes from the context of string diagrams [16]. In a string diagram, the act of tensoring two strings together is represented by placing those strings side by side. Since the defining feature of the unit object is that tensoring it with other objects should have no effect, the unit object is therefore represented diagrammatically by the absense of a string. An endomorphism of the unit thus appears as an entity with no input or output strings, detached from the rest of the diagram. In a real-world version of these diagrams, made out of physical strings arranged in real space, we could use this detachedness to grab these endomorphisms and slide them over or under any strings we please, without affecting anything else in the diagram. This ability is embodied algebraically by the equation above, and hence categories which obey it are called 'spacial'.

It turns out that the crossedness of an action operad has a direct effect on the spaciality of algebras.

Lemma 1.34. *If G is a crossed action operad, then all* EG*-algebras are spacial.*

Proof. Let G be a crossed action operad, let X be a EG-algebra, and fix $x \in \mathrm{Ob}(X)$ and $f : I \to I$. From Lemma 1.16 we know that $\pi : G(2) \to S_2$ is surjective, so that the set $\pi^{-1}(\,(1\,2)\,)$ is non-empty, and from the rules for composition of action morphisms we see that for any such $g \in \pi^{-1}(\,(1\,2)\,)$,

$$\begin{aligned} \alpha(\,g\,;\,\mathrm{id}_x,\,\mathrm{id}_I\,) \circ \alpha(\,e_2\,;\,\mathrm{id}_x,\,f\,) \quad &= \quad \alpha(\,g\,;\,\mathrm{id}_x,\,f\,) \\ &= \quad \alpha(\,e_2\,;\,f,\,\mathrm{id}_x\,) \circ \alpha(\,g\,;\,\mathrm{id}_x,\,\mathrm{id}_I\,) \end{aligned}$$

Thus in order to obtain the result we're after, it will suffice to find a particular $g \in \pi^{-1}(\,(1\,2)\,)$ for which

$$\alpha(\,g\,;\,\mathrm{id}_x,\,\mathrm{id}_I\,) = \mathrm{id}_x$$

However, since

$$\begin{aligned}\alpha(\,g\,;\,\mathrm{id}_x,\,\mathrm{id}_I\,) &= \alpha(\,g\,;\,\mathrm{id}_x,\,\alpha(e_0;-)\,)\\ &= \alpha(\,\mu(g;e_1,e_0)\,;\,\mathrm{id}_x\,)\end{aligned}$$

all we really need is to find a $g \in \pi^{-1}(\,(1\,2)\,)$ for which

$$\mu(g;e_1,e_0) = e_1$$

To this end, choose an arbitrary element $h \in \pi^{-1}(\,(1\,2)\,)$. This h probably won't obey the above equation, but we can use it to construct a new element g which does. Specifically, define

$$k \;:=\; \mu(\,h\;;\,e_1,\,e_0\,)$$

and then consider

$$g \;:=\; h \cdot \mu(e_2;k^{-1},e_1)$$

To see that this is the correct choice of g, first note that we must have $\pi(k) = e_1$, since this is the only element of S_1. Following from that, we have

$$\begin{aligned}\pi\Big(\,\mu(e_2;k^{-1},e_1)\,\Big) &= \mu\Big(\,\pi(e_2)\;;\,\pi(k^{-1}),\,\pi(e_1)\,\Big)\\ &= \mu\Big(\,e_2\;;\,e_1,\,e_1\,\Big)\\ &= e_2\end{aligned}$$

and hence

$$\begin{aligned}\pi(g) &= \pi\Big(h\cdot\mu(e_2;k^{-1},e_1)\Big)\\ &= \pi(h)\cdot\pi\Big(\mu(e_2;k^{-1},e_1)\Big)\\ &= (1\,2)\cdot e_2\\ &= (1\,2).\end{aligned}$$

So g is indeed in $\pi^{-1}(\,(1\,2)\,)$, and furthermore

$$\begin{aligned}\mu(g;e_1,e_0) &= \mu\Big(\,h\cdot\mu(e_2;k^{-1},e_1)\;;\,e_1,\,e_0\,\Big)\\ &= \mu(\,h\;;\,e_1,\,e_0\,)\cdot\mu\Big(\,\mu(e_2;k^{-1},e_1)\;;\,e_1,\,e_0\,\Big)\\ &= \mu(\,h\;;\,e_1,\,e_0\,)\cdot\mu\Big(\,e_2\;;\,\mu(k^{-1};e_1),\,\mu(e_1;e_0)\,\Big)\\ &= \mu(\,h\;;\,e_1,\,e_0\,)\cdot\mu(\,e_2\;;\,k^{-1},e_0\,)\\ &= k\cdot k^{-1}\\ &= e_1\end{aligned}$$

Therefore, $h \cdot \mu(e_2; k^{-1}, e_1)$ is exactly the g we were looking for, and so working backwards through the proof we obtain the required result:

$$\begin{array}{rrcl} & \mu(g; e_1, e_0) & = & e_1 \\ \Longrightarrow & \alpha(\, g\,;\, \mathrm{id}_x,\, \mathrm{id}_I\,) & = & \mathrm{id}_x \end{array}$$

$$\begin{array}{rrcl} \alpha(\, g\,;\, \mathrm{id}_x,\, \mathrm{id}_I\,) \circ & \alpha(\, e_2\,;\, \mathrm{id}_x,\, f\,) & = & \alpha(\, e_2\,;\, f,\, \mathrm{id}_x\,) \circ \alpha(\, g\,;\, \mathrm{id}_x,\, \mathrm{id}_I\,) \\ \Longrightarrow & \alpha(\, e_2\,;\, \mathrm{id}_x,\, f\,) & = & \alpha(\, e_2\,;\, f,\, \mathrm{id}_I\,) \end{array}$$

□

Finally, Lemma 1.32 also gives a complete description of how the morphisms of $\mathbb{G}_n$ interact as a monoid under tensor product, though to best express this we need a bit of new terminology.

Definition 1.35. Let G be an action operad. Then we will also the notation G to denote the *underlying monoid* of this action operad. This is the natural way to consider G as a monoid, with its element set being all of its elements together, $\bigsqcup_m G(m)$, and with tensor product as its binary operation, $g \otimes h = \mu(e_2; g, h)$.

Also, note that this monoid comes equipped with a homomorphism $|\_| : G \to \mathbb{N}$, sending each $g \in G$ to the natural number m if and only if g is an element of the group $G(m)$. We'll call this number $|g|$ the *length* of g.

Definition 1.36. Let S be a set and $F(S)$ the free monoid on S, the monoid whose elements are strings of elements of S and whose binary operation is concatenation. Then we will denote by

$$|\_| : F(S) \to \mathbb{N}$$

the monoid homomorphism defined by sending each element of $S \subseteq F(S)$ to 1, and therefore also each concatenation of n elements of S to the natural number n. Again, we will call $|x|$ the *length* of $x \in F(S)$.

Lemma 1.37. *The monoid of morphisms of the algebra* $\mathbb{G}_n$ *is*

$$\mathrm{Mor}(\mathbb{G}_n) \;\cong\; G \times_{\mathbb{N}} \mathbb{N}^{*n}$$

where this pullback is taken over the respective length homomorphisms,

$$\begin{tikzcd} G \times_{\mathbb{N}} \mathbb{N}^{*n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{N}^{*n} \arrow[d, "|\_|"] \\ G \arrow[r, "|\_|"] & \mathbb{N} \end{tikzcd}$$

using the fact that $\mathbb{N}^{*n}$ *is the free monoid* $F\Big(\{z_1, ..., z_n\}\Big)$.

Proof. An element of $G \times_{\mathbb{N}} F\Big(\{z_1, ..., z_n\}\Big)$ is just an element $g \in G(m)$ for some m, together with an m-tuple of objects $(x_1, ..., x_m)$ from the set of generators $\{z_1, ..., z_n\}$. Thus the action on $\mathbb{G}_n$ defines an obvious function

$$\begin{array}{rcrcl} \alpha & : & G \times_{\mathbb{N}} F\Big(\{z_1, ..., z_n\}\Big) & \to & \mathrm{Mor}(\mathbb{G}_n) \\ & : & (g; x_1, ..., x_m) & \mapsto & \alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m}) \end{array}$$

But by Lemma 1.32, each element of $\mathrm{Mor}(\mathbb{G}_n)$ can be expressed in the form $\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_m})$ for a unique collection $(g; x_1, ..., x_m)$, and so this function α is actually a bijection of sets. Furthermore, this function preserves tensor product, since

$$\begin{array}{rcl} \alpha\Big((g; f_1, ..., f_m) \otimes (g'; f'_1, ..., f'_m)\Big) & = & \alpha(\, g \otimes g' \,;\, f_1, ..., f_m, f'_1, ..., f'_m \,) \\ & = & \alpha(\, g \,;\, f_1, ..., f_m \,) \otimes \alpha(\, g' \,;\, f'_1, ..., f'_m \,) \end{array}$$

and hence it is a monoid isomorphism, as required. □

Chapter 2

Free invertible algebras as initial objects

In this chapter we will start to consider how to construct free EG-algebras on some number of invertible objects. Specifically, we will begin by showing that such algebras are the initial objects of a particular comma category, in accordance with some well known properties of adjunctions and their units. Using this initial object prespective will allow us to recover all of the data associated with the objects of a given free invertible algebra — what those objects are, how they act under tensor product, and which pairs of objects form the source and target of at least one morphism. Unfortunately, a concrete description of the morphisms themselves will ultimately remain elusive. We can get tantalisingly closer though, and an examiniation of the exact way that this method fails will provide the neccessary insight to motivate a more successful approach in **??**.

2.1 The free algebra on n invertible objects

We saw in **??** that the existence of a free EG-algebra on n objects can be proven by taking the left adjoint of a 2-functor which forgets about the algebra structure. Now we want to extend this idea into the realm of algebras on invertible objects. For the analogous approach, we will need to find a new 2-functor that lets us forget about non-invertible objects, and then hopefully we can find its left adjoint too, and use it to freely add inverses to $\mathbb{G}_n$. First though, we need to make this concept of 'forgetting non-invertible objects' a little more precise.

Definition 2.1. Given an EG-algebra X, we'll denote by X_{inv} the sub-EG-algebra containing all invertible objects in X and the isomorphisms between them.

Note that this is indeed a well-defined EG-algebra. If $f_1, ..., f_m$ are isomorphisms from invertible objects $x_1, ..., x_m$ to invertible objects $y_1, ..., y_m$, then $\alpha(g; f_1, ..., f_m)$ is a map from the invertible object $\alpha(g; x_1, ..., x_m)$ to the invertible object $\alpha(g; y_1, ..., y_m)$, and it has an inverse $\alpha(g^{-1}; f^{-1}_{\pi(g)(1)}, ..., f^{-1}_{\pi(g)(m)})$, since

$$\begin{aligned} & \alpha\Big(g^{-1} \,;\, f^{-1}_{\pi(g)(1)},\, ...,\, f^{-1}_{\pi(g)(m)} \Big) \circ \alpha(\, g \,;\, f_1, ..., f_m \,) \\ = \;& \alpha\Big(g^{-1}g \,;\, f_1^{-1}f_1,\, ...,\, f_m^{-1}f_m \Big) \\ = \;& \mathrm{id}_{x_1 \otimes ... \otimes x_m} \end{aligned}$$

$$\begin{aligned} & \alpha(\, g \,;\, f_1, ..., f_m \,) \circ \alpha\Big(g^{-1} \,;\, f^{-1}_{\pi(g)(1)},\, ...,\, f^{-1}_{\pi(g)(m)} \Big) \\ = \;& \alpha\Big(gg^{-1} \,;\, f_{\pi(g)(1)}f^{-1}_{\pi(g)(1)},\, ...,\, f_{\pi(g)(m)}f^{-1}_{\pi(g)(m)} \Big) \\ = \;& \mathrm{id}_{y_{\pi(g)(1)} \otimes ... \otimes y_{\pi(g)(m)}} \end{aligned}$$

Clearly then, X_{inv} is the correct algebra for our new forgetful 2-functor to send X to. Knowing this, we can contruct the rest of the functor fairly easily.

Proposition 2.2. *The assignment $X \mapsto X_{\text{inv}}$ can be extended to a 2-functor $(\_)_{\text{inv}}$: $\mathrm{E}G_G\mathrm{Alg} \to \mathrm{E}G_G\mathrm{Alg}$.*

Proof. Let $F : X \to Y$ be a (strict) map of EG-algebras. If x is an invertible object in X with inverse x^*, then $F(x)$ is an invertible object in Y with inverse $F(x^*)$, by

$$\begin{aligned} F(x) \otimes F(x^*) &= F(x \otimes x^*) = F(I) = I \\ F(x^*) \otimes F(x) &= F(x^* \otimes x) = F(I) = I \end{aligned}$$

Since F sends invertible objects to invertible objects, it will also send isomorphisms of invertible objects to isomorphisms of invertible objects. In other words, the map $F : X \to Y$ can be restricted to a map $F_{\text{inv}} : X_{\text{inv}} \to Y_{\text{inv}}$. Moreover, we have that

$$(F \circ G)_{\text{inv}}(x) = F \circ G(x) = F_{\text{inv}} \circ G_{\text{inv}}(x)$$

$$(F \circ G)_{\text{inv}}(f) = F \circ G(f) = F_{\text{inv}} \circ G_{\text{inv}}(f)$$

and so the assignment $F \mapsto F_{\text{inv}}$ is clearly functorial. Next, let $\theta : F \Rightarrow G$ be a monoidal natural transformation. Choose an invertible object x from X, and consider the component map of its inverse, $\theta_{x^*} : F(x^*) \to G(x^*)$. Since θ is monoidal, we have $\theta_{x^*} \otimes \theta_x = \theta_I = I$ and $\theta_x \otimes \theta_{x^*} = I$, or in other words that θ_{x^*} is the monoidal inverse of θ_x. We can use this fact to construct a compositional inverse as well, namely $\mathrm{id}_{F(x)} \otimes \theta_{x^*} \otimes \mathrm{id}_{G(x)}$, which can be seen as follows:

$$\begin{aligned}\left(\mathrm{id}_{F(x)} \otimes \theta_{x^*} \otimes \mathrm{id}_{G(x)}\right) \circ \theta_x &= \theta_x \otimes \theta_{x^*} \otimes \mathrm{id}_{G(x)} \\ &= \mathrm{id}_{G(x)}\end{aligned}$$

$$\begin{aligned}\theta_x \circ \left(\mathrm{id}_{F(x)} \otimes \theta_{x^*} \otimes \mathrm{id}_{G(x)}\right) &= \mathrm{id}_{F(x)} \otimes \theta_{x^*} \otimes \theta_x \\ &= \mathrm{id}_{F(x)}\end{aligned}$$

Therefore, we see that all the components of our transformation on invertible objects are isomorphisms, and hence we can define a new transformation $\theta_{\mathrm{inv}} : F_{\mathrm{inv}} \Rightarrow G_{\mathrm{inv}}$ whose components are just $(\theta_{\mathrm{inv}})_x = \theta_x$. The assignment $\theta \mapsto \theta_{\mathrm{inv}}$ is also clearly functorial, and thus we have a complete 2-functor $(\_)_{\mathrm{inv}} : \mathrm{E}G_G\mathrm{Alg} \to \mathrm{E}G_G\mathrm{Alg}$. $\square$

Proposition 2.3. *The 2-functor* $(\_)_{\mathrm{inv}} : \mathrm{E}G_G\mathrm{Alg} \to \mathrm{E}G_G\mathrm{Alg}$ *has a left adjoint,* $L : \mathrm{E}G_G\mathrm{Alg} \to \mathrm{E}G_G\mathrm{Alg}$.

Proof. To begin, consider the 2-monad $\mathrm{E}G(\_)$. This is a finitary monad — that is it preserves all filtered colimits — and it is a 2-monad over Cat, which is locally finitely presentable. It follows from this that $\mathrm{E}G_G\mathrm{Alg}$ is itself locally finitely presentable. Thus if we want to prove $(\_)_{\mathrm{inv}}$ has a left adjoint, we can use the Adjoint Functor Theorem for locally finitely presentable categories, which amounts to showing that $(\_)_{\mathrm{inv}}$ preserves both limits and filtered colimits.

- Given an indexed collection of $\mathrm{E}G$-algebras X_i, the $\mathrm{E}G$-action of their product $\prod X_i$ is defined componentwise. In particular, this means that the tensor product of two objects in $\prod X_i$ is just the collection of the tensor products of their components in each of the X_i. An invertible object in $\prod X_i$ is thus simply a family of invertible objects from the X_i — in other words, $(\prod X_i)_{\mathrm{inv}} = \prod (X_i)_{\mathrm{inv}}$.

- Given maps of $\mathrm{E}G$-algebras $F : X \to Z$, $G : Y \to Z$, the $\mathrm{E}G$-action of their pullback $X \times_Z Y$ is also defined componentwise. It follows that an invertible object in $X \times_Z Y$ is just a pair of invertible objects (x, y) from X and Y, such that $F(x) = G(y)$. But this is the same as asking for a pair of objects (x, y) from X_{inv} and Y_{inv} such that $F_{\mathrm{inv}}(x) = G_{\mathrm{inv}}(y)$, and hence $(X \times_Z Y)_{\mathrm{inv}} = X_{\mathrm{inv}} \times_{Z_{\mathrm{inv}}} Y_{\mathrm{inv}}$.

- Given a filtered diagram D of $\mathrm{E}G$-algebras, the $\mathrm{E}G$-action of their colimit $\mathrm{colim}(D_n)$ is defined in the following way: use filteredness to find an algebra which contains (representatives of the classes of) all the things you want to act on, then apply the action of that algebra. In the case of tensor products this means that $[x] \otimes [y] = [x \otimes y]$, and thus an invertible object in $\mathrm{colim}(D_n)$ is just (the class of) an invertible object in one of the algebras of D. In other words, $\mathrm{colim}(D_n)_{\mathrm{inv}} = \mathrm{colim}(D_{\mathrm{inv}})$.

Preservation of products and pullbacks gives preservation of limits, and preservation of limits and filtered colimits gives the result. □

With this new 2-functor $L : \mathrm{E}G_G\mathrm{Alg} \to \mathrm{E}G_G\mathrm{Alg}$, we now have the ability to 'freely add inverses to objects' in any EG-algebra we want. The algebra $L\mathbb{G}_n$ is then a clear candidate for our free algebra on n invertible objects, and indeed the proof of this is very simple.

Theorem 2.4. *There exists a free* EG*-algebra on n invertible objects. Specifically, the algebra $L\mathbb{G}_n$ is such that for any other* EG*-algebra X, we have an isomorphism of categories*

$$\mathrm{E}G_G\mathrm{Alg}(L\mathbb{G}_n, X) \quad \cong \quad (X_{\mathrm{inv}})^n$$

Proof. Using the adjunction from Proposition 2.3 along with the one from **??**, we see that

$$\begin{aligned} U(X_{\mathrm{inv}})^n &= \mathrm{Cat}(\{z_1, ..., z_n\}, U(X_{\mathrm{inv}})) \\ &\cong \mathrm{E}G_G\mathrm{Alg}(F(\{z_1, ..., z_n\}), X_{\mathrm{inv}}) \\ &\cong \mathrm{E}G_G\mathrm{Alg}(LF(\{z_1, ..., z_n\}), X) \end{aligned}$$

As before, X_{inv} and $U(X_{\mathrm{inv}})$ are obviously isomorphic as categories, and so $LF(\{z_1, ..., z_n\}) = L\mathbb{G}_n$ satisfies the requirements for the free algebra on n invertible objects. □

2.2 $L\mathbb{G}_n$ as an initial algebra

We have now proven that a free EG-algebra on n invertible objects indeed exists. But this fact on its own is not very helpful. To be able to actually use the free algebra $L\mathbb{G}_n$, we need to know how to contruct it explicitly, in terms of its objects and morphisms. We could do this by finding a detailed characterisation of the 2-functor L, and then applying this to our explicit description of $\mathbb{G}_n$ from Proposition 1.30. However, this would probably take far more effort than is required, since it would involve determining the behaviour of L in many situtations that we aren't interested in. Also, we wouldn't be leveraging $\mathbb{G}_n$'s status as a free algebra to make the calculations any easier. We will try a different strategy instead, one that begins by noticing a special property of the functor L.

Proposition 2.5. *For any* EG*-algebra* X*, we have* $L(X)_{\text{inv}} = L(X)$*.*

Proof. From the definition of adjunctions, the isomorphisms

$$\mathrm{E}G_G\mathrm{Alg}(LX, Y) \quad \cong \quad \mathrm{E}G_G\mathrm{Alg}(X, Y_{\text{inv}})$$

are subject to certain naturality conditions. Specifically, given $F : X' \to X$ and $G : Y \to Y'$ we get a commutative diagram

$$\begin{tikzcd}
\mathrm{E}G_G\mathrm{Alg}(LX, Y) \arrow[r, "\sim"] \arrow[d, "G\circ_\circ LF"'] & \mathrm{E}G_G\mathrm{Alg}(X, Y_{\text{inv}}) \arrow[d, "G_{\text{inv}}\circ_\circ F"] \\
\mathrm{E}G_G\mathrm{Alg}(LX', Y') \arrow[r, "\sim"] & \mathrm{E}G_G\mathrm{Alg}(X', Y'_{\text{inv}})
\end{tikzcd}$$

Consider the case where F is the identity map $\mathrm{id}_X : X \to X$ and G is the inclusion $j : L(X)_{\text{inv}} \to L(X)$. Note that because j is an inclusion, the restriction $j_{\text{inv}} : (L(X)_{\text{inv}})_{\text{inv}} \to L(X)_{\text{inv}}$ is also an inclusion, but since $((\_)_{\text{inv}})_{\text{inv}} = (\_)_{\text{inv}}$, we have that $j_{\text{inv}} = \mathrm{id}$. It follows that

$$\begin{tikzcd}
\mathrm{E}G_G\mathrm{Alg}(LX, LX_{\text{inv}}) \arrow[r, "\sim"] \arrow[d, "j\circ_"'] & \mathrm{E}G_G\mathrm{Alg}(X, LX_{\text{inv}}) \arrow[d, equal] \\
\mathrm{E}G_G\mathrm{Alg}(LX, LX) \arrow[r, "\sim"] & \mathrm{E}G_G\mathrm{Alg}(X, LX_{\text{inv}})
\end{tikzcd}$$

Therefore, for any map $f : LX \to LX$ there exists a unique $g : LX \to LX_{\text{inv}}$ such that $j \circ g = f$. But this means that for any such f, we must have $\mathrm{im}(f) \subseteq L(X)_{\text{inv}}$, and so in particular $L(X) = \mathrm{im}(\mathrm{id}_{LX}) \subseteq L(X)_{\text{inv}}$. Since $L(X)_{\text{inv}} \subseteq L(X)$ by definition, we obtain the result. ☐

This result is not especially surprising. Intuitively, it just says that when you freely add inverses to an algebra, every object ends up with an inverse. But the upshot of this is that we now have another way of thinking about $L(X)$: as the target object of the unit of our adjunction, $\eta_X : X \to L(X)_{\text{inv}}$. This means that we don't really need to know the entirety of L in order to determine the free algebra $L\mathbb{G}_n$, just its unit. To find this unit directly, we can turn to the following fact about adjunctions, for which a proof can be found in Lemma 2.3.5 of Leinster's *Basic Category Theory* [10].

Proposition 2.6. *Let $F \dashv G : A \to B$ be an adjunction with unit η. For any object a in A, let $(a \downarrow G)$ denote the comma category whose objects are pairs (b, f) consisting of an object b from B and a morphism $f : a \to G(b)$ from A, and whose morphisms $h : (b, f) \to (b', f')$ are morphisms $f : b \to b'$ from B such that $G(f) \circ f = f'$. Then the pair $\Big(F(a), \eta_a : a \to GF(a)\Big)$ is an initial object of $(a \downarrow G)$.*

Corollary 2.7. *$\eta_{\mathbb{G}_n} : \mathbb{G}_n \to (L\mathbb{G}_n)_{\text{inv}} = L\mathbb{G}_n$ is an initial object of $(\mathbb{G}_n \downarrow \text{inv})$.*

Being able to view $L\mathbb{G}_n$ as the initial object in the comma category $(\mathbb{G}_n \downarrow \text{inv})$ will prove immensely useful in the coming sections. This is because it lets us think about the properties of $L\mathbb{G}_n$ in terms of maps $\psi : \mathbb{G}_n \to X_{\text{inv}}$, and this is exactly the context where we can exploit $\mathbb{G}_n$'s status as a free algebra. As a result, its worth taking some time to think about what exactly this map $\eta_{\mathbb{G}_n}$ is.

Lemma 2.8. *The initial object $\eta_{\mathbb{G}_n} : \mathbb{G}_n \to L\mathbb{G}_n$ is the obvious map from the free* EG*-algebra on n objects into the free* EG*-algebra on n* invertible *objects. That is, $\eta_{\mathbb{G}_n}$ is the algebra map defined by*

$$\begin{array}{rcrcl} \eta_{\mathbb{G}_n} & : & \mathbb{G}_n & \to & L\mathbb{G}_n \\ & : & F(\{z_1, ..., z_n\}) & \to & LF(\{z_1, ..., z_n\}) \\ & : & z_i & \mapsto & z_i \end{array}$$

Proof. Consider the n-tuple $(z_1, ..., z_n)$ in $(\mathbb{G}_n)^n$. Clearly the image of $(z_1, ..., z_n)$ under the functor L is just the object $(z_1, ..., z_n)$ in the algebra

$$L\Big((\mathbb{G}_n)^n\Big) \quad = \quad (L\mathbb{G}_n)^n \quad = \quad LF(\{z_1, ..., z_n\})^n$$

But the image of $(z_1, ..., z_n) \in (\mathbb{G}_n)^n$ under the isomorphism

$$\mathrm{E}G_G\mathrm{Alg}(\,\mathbb{G}_n, \mathbb{G}_n\,) \quad \cong \quad (\mathbb{G}_n)^n$$

is just the identity map $\mathrm{id}_{\mathbb{G}_n}$. Thus by functoriality of L, the map $L(\mathrm{id}_{\mathbb{G}_n}) = \mathrm{id}_{L\mathbb{G}_n}$ must be the one which corresponds to the n-tuple $(z_1, ..., z_n) \in (L\mathbb{G}_n)^n$ image via the isomorphism

$$\mathrm{E}G_G\mathrm{Alg}(\,L\mathbb{G}_n, L\mathbb{G}_n\,) \quad \cong \quad (L\mathbb{G}_n)^n$$

Furthermore, the $\mathbb{G}_n$ component of the unit η is by definition the image of the identity map $\mathrm{id}_{L\mathbb{G}_n}$ under the isomorphism

$$\mathrm{E}G_G\mathrm{Alg}(\,L\mathbb{G}_n, L\mathbb{G}_n\,) \quad \cong \quad \mathrm{E}G_G\mathrm{Alg}(\,\mathbb{G}_n, L\mathbb{G}_n\,)$$

Hence it follows that $\eta_{\mathbb{G}_n}$ is the map that corresponds to $(z_1, ..., z_n)$ via

$$\mathrm{E}G_G\mathrm{Alg}(\,\mathbb{G}_n, L\mathbb{G}_n\,) \quad \cong \quad (L\mathbb{G}_n)^n$$

which is exactly the definition given in the statement of the lemma. □

This increadibly simple description makes the map η very easy to work with. For example, we immediately obtain the following property, one which we will use frequently throughout the rest of the paper:

Corollary 2.9. *η is an epimorphism in* $\mathrm{E}G_G\mathrm{Alg}$.

Proof. Let $\phi, \psi : L\mathbb{G}_n \to X$ be a pair of algebra maps for which $\phi \circ \eta = \psi \circ \eta$. Then on the generators of $L\mathbb{G}_n$ we have

$$\begin{array}{rcccccc} \phi(z_i) & = & \phi\eta(z_i) & = & \psi\eta(z_i) & = & \psi(z_i) \\ \Longrightarrow & & \phi_{\mathrm{inv}}(z_i) & = & \psi_{\mathrm{inv}}(z_i) & & \end{array}$$

But $L\mathbb{G}_n$ is the free EG-algebra on n invertible objects, so maps $L\mathbb{G}_n \to X_{\mathrm{inv}}$ are determined uniquely by where they those generating objects. It follows that $\phi_{\mathrm{inv}} = \psi_{\mathrm{inv}}$, and if $i : X_{\mathrm{inv}} \to X$ is the obvious inclusion,

$$\phi \quad = \quad i\phi_{\mathrm{inv}} \quad = \quad i\psi_{\mathrm{inv}} \quad = \quad \psi$$

□

Before moving on, we'll make a small change in notation. From now on, rather than writing objects in $(\mathbb{G}_n \downarrow \mathrm{inv})$ as maps $\psi : \mathbb{G}_n \to Y_{\mathrm{inv}}$, we will instead just let $X = Y_{\mathrm{inv}}$ and speak of maps $\psi : \mathbb{G}_n \to X$. This is purely to prevent the notation from becoming cluttered, and shouldn't be a problem so long as we always remember that the targets of these maps only ever contain invertible objects and morphisms. We'll also drop the subscript from $\eta_{\mathbb{G}_n}$, since it is the only component of the unit we'll ever use.

2.3 The objects of $L\mathbb{G}_n$

So now we know that $L\mathbb{G}_n$ is an initial object in the category $(\mathbb{G}_n \downarrow \mathrm{inv})$. But what does this actually tell us? After all, we do not currently have a method for finding initial objects in an arbitrary collection of EG-algebra maps. Because of this, we'll have to approach the problem step-by-step, using the initiality of η to extract different pieces of information about the algebra $L\mathbb{G}_n$ as we go. We'll begin by tring to find its objects.

Definition 2.10. Denote by $\mathrm{Ob} : \mathrm{E}G_G\mathrm{Alg} \to \mathrm{Mon}$ be the functor that sends EG-algebras X to their monoid of objects $\mathrm{Ob}(X)$, and algebra maps $F : X \to Y$ to their underlying monoid homomorphism $\mathrm{Ob}(F) : \mathrm{Ob}(X) \to \mathrm{Ob}(Y)$.

In order to find $\mathrm{Ob}(L\mathbb{G}_n)$, we'll need to make use of an important result about the nature of Ob.

Definition 2.11. Recall from Definition 1.21 that given a monoid M, the monoidal category EM is the one whose monoid of objects is M and which has a unique isomorphism between any two objects. We can view EM as not just a category but an EG-algebra, by letting the action on morphisms take the only possible values it can, given the required source and target. Then for any monoid homomorphisms $h : M \to M'$, the definition of $\mathrm{E}h : \mathrm{E}M \to \mathrm{E}M'$ given in Definition 1.21 must be a well-defined map of EG-algebras, by functoriality. Thus we can also view E as a functor $\mathrm{Mon} \to \mathrm{E}G_G\mathrm{Alg}$.

Proposition 2.12. E *is a right adjoint to the functor* Ob*.*

Proof. For any EG-algebra X, a map $F : X \to \mathrm{E}M$ is determined entirely by its restriction to objects, the monoid homomorphism $\mathrm{Ob}(F) : \mathrm{Ob}(X) \to M$. This is because functorality of F ensures that any map $x \to x'$ in X must be sent to a map $F(x) \to F(x')$ in EM, and by the definition of E there is always exactly one of these to choose from. In other words, we have an isomorphism between the homsets

$$\mathrm{E}G_G\mathrm{Alg}(\,X,\,\mathrm{E}M\,) \quad \cong \quad \mathrm{Mon}(\,\mathrm{Ob}(X),\,M\,)$$

Additionally, this isomorphism is natural in both coordinates. That is, for any $G : X \to X'$ in $\mathrm{E}G_G\mathrm{Alg}$ and $h : M \to M'$ in Mon, the diagram

$$\begin{tikzcd}
\mathrm{E}G_G\mathrm{Alg}(X, \mathrm{E}M) \arrow[r, "\sim"] \arrow[d, "\mathrm{E}h\circ_\circ G"'] & \mathrm{Mon}(\mathrm{Ob}(X), M) \arrow[d, "h\circ_\circ\mathrm{Ob}(G)"] \\
\mathrm{E}G_G\mathrm{Alg}(X', \mathrm{E}M') \arrow[r, "\sim"] & \mathrm{Mon}(\mathrm{Ob}(X'), M')
\end{tikzcd}$$

commutes, because

$$\mathrm{Ob}(\,\mathrm{E}h \circ F \circ G\,) \quad = \quad \mathrm{Ob}(Eh) \circ \mathrm{Ob}(F) \circ \mathrm{Ob}(G) \quad = \quad h \circ \mathrm{Ob}(F) \circ \mathrm{Ob}(G)$$

Therefore, $\mathrm{Ob} \dashv \mathrm{E}$. $\square$

What Proposition 2.12 is essentially saying is that the functor Ob provides a way for us to move back and forth between the categories $\mathrm{E}G_G\mathrm{Alg}$ and Mon. By applying this reasoning to the universal property of the initial object η, we can then determine the value of $\mathrm{Ob}(L\mathbb{G}_n)$ in terms of a new universal property of $\mathrm{Ob}(\eta)$ in the category Mon. In particular, the algebras in $(\mathbb{G}_n \downarrow \mathrm{inv})$ are those whose objects are all invertible, and so the induced property of $\mathrm{Ob}(\eta)$ will end up saying something about the relationship between $\mathrm{Ob}(\mathbb{G}_n)$ and groups — those monoids whose elements are all invertible.

Definition 2.13. Let M be a monoid, M^{gp} a group, and $i : M \to M^{\mathrm{gp}}$ a monoid homomorphism between them. Then we say that M^{gp} is the *group completion* of M if for any other group H and homomorphism $h : M \to H$, there exists a unique homomorphism $u : M^{\mathrm{gp}} \to H$ such that $u \circ i = h$.

There are several different ways to actually calculate the group completion of a monoid. One is to use that fact that M^{gp} is the group whose group presentation is the same as the monoid presentation of M. That is, if M is the quotient of the free monoid on generators $\mathcal{G}$ by the relations $\mathcal{R}$, then M^{gp} is the quotient of the free *group* on generators $\mathcal{G}$ by relations $\mathcal{R}$. This makes finding the completion of free monoids particularly simple.

Proposition 2.14. *The object monoid of $L\mathbb{G}_n$ is $\mathbb{Z}^{*n}$, the group completion of the object monoid of $\mathbb{G}_n$. The restriction of η on objects, $\mathrm{Ob}(\eta)$, is then the obvious inclusion $\mathbb{N}^{*n} \hookrightarrow \mathbb{Z}^{*n}$.*

Proof. Let H be a group, and $h : \mathrm{Ob}(\mathbb{G}_n) \to H$ a monoid homomorphism. By Proposition 2.12 we have an isomorphism of homsets

$$\mathrm{E}G_G\mathrm{Alg}(\,\mathbb{G}_n,\, \mathrm{E}H\,) \quad \cong \quad \mathrm{Mon}(\,\mathrm{Ob}(\mathbb{G}_n),\, H\,)$$

Denote by $h' : \mathbb{G}_n \to \mathrm{E}H$ the map of $\mathrm{E}G$-algebras corresponding to h under this isomorphism. Since H is a group, every object in $\mathrm{E}H$ is invertible, and so h' is an object of $(\mathbb{G}_n \downarrow \mathrm{inv})$. Thus, by initiality of η, there must exist a unique map $u : L\mathbb{G}_n \to \mathrm{E}G$ making the lefthand traingle below commute:

$$\begin{tikzcd}
\mathbb{G}_n \arrow[d, "\eta"'] \arrow[dr, "h'"] & \\
L\mathbb{G}_n \arrow[r, "u"'] & \mathrm{E}H
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathrm{Ob}(\mathbb{G}_n) \arrow[d, "\mathrm{Ob}(\eta)"'] \arrow[dr, "h"] & \\
\mathrm{Ob}(L\mathbb{G}_n) \arrow[r, "\mathrm{Ob}(u)"'] & H
\end{tikzcd}$$

It follows that the righthand triangle — which is the image of the first under Ob — also commutes. Hence for any group H and homomorphism $h : \mathrm{Ob}(\mathbb{G}_n) \to H$, there is at least one map which factors h through $\mathrm{Ob}(\eta)$.

But now recall from Corollary 2.9 that η is an epimorphism. Left adjoint functors preseve epimorphisms, which means that $\mathrm{Ob}(\eta)$ is one too, and so for any $v : \mathrm{Ob}(L\mathbb{G}_n) \to H$,

$$\begin{aligned} v \circ \mathrm{Ob}(\eta) = h \quad &\Longrightarrow \quad v \circ \mathrm{Ob}(\eta) = \mathrm{Ob}(u) \circ \mathrm{Ob}(\eta) \\ &\Longrightarrow \quad v = \mathrm{Ob}(u) \end{aligned}$$

Thus there is actually only one possible map which factors h through $\mathrm{Ob}(\eta)$, and therefore every homomorphism from $\mathrm{Ob}(\mathbb{G}_n)$ onto a group factors uniquely through the group $\mathrm{Ob}(L\mathbb{G}_n)$. In other words, $\mathrm{Ob}(L\mathbb{G}_n)$ is the group completion $\mathrm{Ob}(\mathbb{G}_n)^{\mathrm{gp}}$. Since by Lemma 1.31 the object monoid of $\mathbb{G}_n$ is $\mathbb{N}^{*n}$, the free monoid on n generators, we can conclude that

$$\mathrm{Ob}(L\mathbb{G}_n) \quad = \quad \mathrm{Ob}(\mathbb{G}_n)^{\mathrm{gp}} \quad = \quad (\mathbb{N}^{*n})^{\mathrm{gp}} \quad = \quad \mathbb{Z}^{*n}$$

the free group on n generators. Moreover, the map $\mathrm{Ob}(\eta)$ is then the inclusion of $\mathrm{Ob}(\mathbb{G}_n)$ into its completion, which is just $\mathbb{N}^{*n} \hookrightarrow \mathbb{Z}^{*n}$. □

2.4 The connected components of $L\mathbb{G}_n$

The core result of Proposition 2.14 — that $\mathrm{Ob}(L\mathbb{G}_n)$ is the group completion of $\mathrm{Ob}(\mathbb{G}_n)$ — makes concrete the sense in which the functor L represents 'freely adding inverses' to objects. Extending this same logic to connected components as well, it would seem reasonable to expect that $\pi_0(L\mathbb{G}_n)$ is the group completion of $\pi_0(\mathbb{G}_n)$ as well. This is indeed the case, and the proof proceeds in a way completely analagous to Proposition 2.14.

First, we want to show that the process of taking connected components forms part of an adjunction. To do this we are going to need a category from which we can draw the kind of structures that can act as the components of an EG-algebra. Exactly which category this should be will depend on our choice of action operad G, or more precisely its underlying permutations.

Definition 2.15. For a given action operad G, denote by $\mathrm{im}(\pi)-\mathrm{Mon}$ the full subcategory of Mon on those monoids whose multiplication is invariant under the permutations in $\mathrm{im}(\pi)$. That is, a monoid M is in $\mathrm{im}(\pi)-\mathrm{Mon}$ if and only if

$$m_1, ..., m_n \in M,\ g \in G(n) \quad \Longrightarrow \quad m_1...m_n \ = \ m_{\pi(g)^{-1}(1)}...m_{\pi(g)^{-1}(n)}$$

Of course, by Lemma 1.16 there are really only two examples of such a $\mathrm{im}(\pi)-\mathrm{Mon}$. If the underlying permutations of G are trivial, then $\mathrm{im}(\pi)-\mathrm{Mon}$ is just the whole of the category Mon; if instead G is crossed then we are asking for monoids whose multiplication is invariant under arbitrary permutations from S, and so $\mathrm{im}(\pi)-\mathrm{Mon}$ is just the category of *commutative* monoids, CMon. Regardless, when we are working with an arbitrary action operad G, the category $\mathrm{im}(\pi)-\mathrm{Mon}$ is exactly the collection of possible connected components that we were looking for.

Lemma 2.16. *Let G be an action operad and $\mathrm{im}(\pi)$ its underlying permutation action operad. Then there is a functor*

$$\pi_0 : \mathrm{E}G_G\mathrm{Alg} \to \mathrm{im}(\pi)-\mathrm{Mon}$$

which sends each algebra X to its monoid of connected components $\pi_0(X)$, and sends each map of algebras $F : X \to Y$ to its restriction to connected components $\pi_0(F) : \pi_0(X) \to \pi_0(Y)$.

Proof. Let $x_1, ..., x_n$ be an arbitrary collection of objects from the algebra X, and g an element of the group $G(n)$. Then the action of G guarantees the existence of a morphism

$$\alpha(g; \mathrm{id}_{x_1}, ..., \mathrm{id}_{x_n}) \ : \ x_1 \otimes ... \otimes x_n \to x_{\pi(g^{-1})(1)} \otimes ... \otimes x_{\pi(g^{-1})(n)}$$

By definition the source and target of this morphism belong to the same connected component, and hence

$$\begin{array}{rrcl} & [\, x_1 \otimes ... \otimes x_n \,] & = & [\, x_{\pi(g^{-1})(1)} \otimes ... \otimes x_{\pi(g^{-1})(n)} \,] \\ \Longrightarrow & [x_1] \otimes ... \otimes [x_n] & = & [x_{\pi(g^{-1})(1)}] \otimes ... \otimes [x_{\pi(g^{-1})(n)}] \end{array}$$

But since the x_i are just arbitrary objects of X, the components $[x_i]$ are an arbitrary collection of elements from $\pi_0(X)$, and likewise for the group element g and the permutation $\pi(g)$. Therefore multiplication in the monoid $\pi_0(X)$ is invariant under all permutations in the images of the homomorphisms $\pi_n : G(n) \to S_n$, and thus $\pi_0(X)$ is an object of $\mathrm{im}(\pi)-\mathrm{Mon}$, as required. Well-definedness of the functor π_0 on morphisms then follows immediately from the fullness of $\mathrm{im}(\pi)-\mathrm{Mon}$. $\square$

Now that we have a functor which represents the act of finding the connected component monoid of an algebra, we need another functor heading in the opposite direction, so that we can construct an adjunction between them.

Definition 2.17. There exists an inclusion of 2-categories $\mathrm{D} : \mathrm{Set} \hookrightarrow \mathrm{Cat}$ which allows us to view any set S as a *discrete category*, one whose objects are just the elements of S and whose morphisms are all identities. If the given set also happens to be a monoid M, then there is an obvious way to see the discete category $\mathrm{D}M$ as a monoidal category, and so we have a similar inclusion $\mathrm{D} : \mathrm{Mon} \hookrightarrow \mathrm{MonCat}$. Finally, for any action operad G and object M of the category $\mathrm{im}(\pi)-\mathrm{Mon}$, there is a unique way to assign an $\mathrm{E}G$-action to the discete category $\mathrm{D}M$. This works because for any elements $m_1, ..., m_n \in M$ and $g \in G(n)$, the morphism $\alpha(g; \mathrm{id}_{m_1}, ..., \mathrm{id}_{m_n})$ must have source and target

$$m_1 \otimes ... \otimes m_n \quad = \quad m_{\pi(g^{-1})(1)} \otimes ... \otimes m_{\pi(g^{-1})(m)}$$

and therefore it can only be the morphism $\mathrm{id}_{m_1 \otimes ... \otimes m_n}$. This choice of action yields one last inclusion $\mathrm{CMon} \hookrightarrow \mathrm{E}G_G\mathrm{Alg}$, which we shall also call D.

Proposition 2.18. D *is a right adjoint to the functor* π_0*.*

Proof. Consider a map of $F : X \to \mathrm{D}C$ from some $\mathrm{E}G$-algebra X onto the discrete $\mathrm{E}G$-algebra for a monoid M in $\mathrm{im}(\pi)-\mathrm{Mon}$. For any $f : x \to x'$ in X, the morphism $F(f)$ must be an identity map in $\mathrm{D}M$, since these are the only morphisms that $\mathrm{D}M$ has. It follows that x and x' being in the same connected component will imply $F(x) = F(x')$, and so F is determined entirely by its restriction to connected components, the monoid homomorphism $\pi_0(F) : \pi_0(X) \to M$. In other words, we have an isomorphism between the homsets

$$\mathrm{E}G_G\mathrm{Alg}(\, X, \mathrm{D}M\,) \quad \cong \quad \mathrm{im}(\pi)-\mathrm{Mon}(\, \pi_0(X), M\,)$$

This isomorphism is natural in both coordinates, since for any $G : X \to X'$ in $\mathrm{E}G_G\mathrm{Alg}$ and $h : M \to M'$ in $\mathrm{im}(\pi)-\mathrm{Mon}$,

$$\pi_0(\,\mathrm{D}h \circ F \circ G\,) \quad = \quad \pi_0(\mathrm{D}h) \circ \pi_0(F) \circ \pi_0(G) \quad = \quad h \circ \pi_0(F) \circ \pi_0(G)$$

and so the diagram

$$\begin{tikzcd}
\mathrm{E}G_G\mathrm{Alg}(X, \mathrm{D}M) \arrow[r, "\sim"] \arrow[d, "\mathrm{D}h\circ_\circ G"'] & \mathrm{im}(\pi)\text{−}\mathrm{Mon}\Big(\pi_0(X), M\Big) \arrow[d, "h\circ_\circ\pi_0(G)"] \\
\mathrm{E}G_G\mathrm{Alg}(X', \mathrm{D}M') \arrow[r, "\sim"] & \mathrm{im}(\pi)\text{−}\mathrm{Mon}\Big(\pi_0(X'), M'\Big)
\end{tikzcd}$$

commutes. Therefore, $\pi_0 \dashv \mathrm{D}$. □

Now we can utilise Proposition 2.18 to draw out a universal property of $\pi_0(L\mathbb{G}_n)$, just as we did with $\mathrm{Ob}(L\mathbb{G}_n)$ in Proposition 2.12.

Proposition 2.19. *The connected components of $L\mathbb{G}_n$ are the group completion of the connected components of $\mathbb{G}_n$. Also, the restriction of η onto connected components, $\pi_0(\eta)$, is the canonical map $\pi_0(\mathbb{G}_n) \to \pi_0(\mathbb{G}_n)^{\mathrm{gp}}$ associated with that group completion.*

Proof. Let H be a group which is also an object of $\mathrm{im}(\pi)\text{−}\mathrm{Mon}$, and let $h : \pi_0(\mathbb{G}_n) \to H$ be a monoid homomorphism. By Proposition 2.18 there is a homset isomorphism

$$\mathrm{E}G_G\mathrm{Alg}(\,\mathbb{G}_n,\, \mathrm{D}H\,) \quad\cong\quad \mathrm{im}(\pi)\text{−}\mathrm{Mon}(\,\pi_0(\mathbb{G}_n),\, H\,)$$

and thus some $\mathrm{E}G$-algebra map $h' : \mathbb{G}_n \to \mathrm{D}H$ corresponding to h. As H is a group, every object of $\mathrm{D}H$ is invertible, and so h' is an object of $(\mathbb{G}_n \downarrow \mathrm{inv})$. It follows that there exists a unique map $u : L\mathbb{G}_n \to \mathrm{D}M$ which factors h' through the initial object η:

$$\begin{tikzcd}
\mathbb{G}_n \arrow[d, "\eta"'] \arrow[dr, "h'"] & \\
L\mathbb{G}_n \arrow[r, "u"'] & \mathrm{D}H
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\pi_0(\mathbb{G}_n) \arrow[d, "\pi_0(\eta)"'] \arrow[dr, "h"] & \\
\pi_0(L\mathbb{G}_n) \arrow[r, "\pi_0(u)"'] & H
\end{tikzcd}$$

Applying the functor π_0 everywhere, we see that $\pi_0(u)$ must also factor h through the homomorphism $\pi_0(\eta)$. Moreover, since η is an epimorphism and π_0 a left adjoint functor, $\pi_0(\eta)$ is an epimorphism too, and so $\pi_0(u)$ is the only map with this property. Therefore, any monoid homomorphism $\pi_0(\mathbb{G}_n) \to H$ will factor uniquely through $\pi_0(L\mathbb{G}_n)$, so long as H is in $\mathrm{im}(\pi)\text{−}\mathrm{Mon}$.

Now consider another monoid homomorphism $k : \pi_0(\mathbb{G}_n) \to K$, where this time K is still a group but not neccessarily in $\mathrm{im}(\pi)\text{−}\mathrm{Mon}$. From Lemma 2.16, we know that

$\pi_0(\mathbb{G}_n)$ is still an object of $\mathrm{im}(\pi)-\mathrm{Mon}$, and from this we can conclude that the image $\mathrm{im}(k)$ will be too:

$$\begin{array}{rrcl} x_1, ..., x_m \in \pi_0(\mathbb{G}_n),\ g \in G(n) & \implies \quad x_1 \otimes ... \otimes x_m & = & x_{\pi(g)(1)} \otimes ... \otimes x_{\pi(g)(m)} \\ & \implies \quad k(\, x_1 \otimes ... \otimes x_m\,) & = & k(\, x_{\pi(g)(1)} \otimes ... \otimes x_{\pi(g)(m)}\,) \\ & \implies \quad k(x_1) \otimes ... \otimes k(x_m) & = & k(x_{\pi(g)(1)}) \otimes ... \otimes k(x_{\pi(g)(m)}) \end{array}$$

Also, since $\mathrm{im}(k)$ is a submonoid of the group K, it is a group as well. Thus if we denote by $k_{\mathrm{im}} : \mathrm{Ob}(\mathbb{G}_n) \to \mathrm{im}(k)$ the restriction of k to it image, then k_{im} is a map in $\mathrm{im}(\pi)-\mathrm{Mon}$ out of $\mathrm{Ob}(\mathbb{G}_n)$ and onto a group, and therefore by what we showed earlier there exists a unique homomorphism $v : \mathrm{Ob}(L\mathbb{G}_n) \to \mathrm{im}(k)$ with the property $v \circ \pi_0(\eta) = k_{\mathrm{im}}$. Composing this v with the inclusion $i : \mathrm{im}(k) \hookrightarrow K$, we see that

$$i \circ v \circ \pi_0(\eta) \ = \ i \circ k_{\mathrm{im}} \ = \ k$$

and $i \circ v$ must be the only map for which this is true, for restricting this equation back on $\mathrm{im}(k)$ yields the unique property of v again. Thus $\pi_0(\eta)$ will actually take any homomorphism from $\mathrm{Ob}(\mathbb{G}_n)$ onto a group and factor it through $\pi_0(L\mathbb{G}_n)$ in a unique way, not just those homomorphisms in $\mathrm{im}(\pi)-\mathrm{Mon}$. In other words,

$$\pi_0(L\mathbb{G}_n) \quad = \quad \pi_0(\mathbb{G}_n)^{\mathrm{gp}}$$

and $\pi_0(\eta)$ is the canonical map of this group completion. □

As we've said before, this result is a reflection of the fact that the functor L is trying to add inverses the objects of $\mathbb{G}_n$ freely, that is, with as little effect on the rest of the algebra as possible. Indeed, if we happen to know whether or not our action operad G is crossed then we can now calculate exactly what the effect on the components will be.

Corollary 2.20. *If G is a crossed action algebra then*

- *the connected components of $L\mathbb{G}_n$ are the monoid $\mathbb{Z}^n$*
- *the restriction of η to components is the obvious inclusion $\mathbb{N}^n \hookrightarrow \mathbb{Z}^n$*
- *the assignment of objects to their component is given by the quotient map of abelianisation* $\mathrm{ab} : \mathbb{Z}^{*n} \to \mathbb{Z}^n$

If instead G is non-crossed, then

- *the connected components of $L\mathbb{G}_n$ are the monoid $\mathbb{Z}^{*n}$*

- *the restriction of* η *to components is the obvious inclusion* $\mathbb{N}^{*n} \hookrightarrow \mathbb{Z}^{*n}$
- *the assignment of objects to their component is* $\mathrm{id}_{\mathbb{Z}^{*n}}$

Proof. Combining Propositions 1.33 and 2.19, we see that

$$\pi_0(L\mathbb{G}_n) \quad = \quad \pi_0(\mathbb{G}_n)^{\mathrm{gp}} \quad = \quad \begin{cases} (\mathbb{N}^n)^{\mathrm{gp}} \quad = \quad \mathbb{Z}^n & \text{if } G \text{ is crossed} \\ (\mathbb{N}^{*n})^{\mathrm{gp}} \quad = \quad \mathbb{Z}^{*n} & \text{otherwise} \end{cases}$$

Moreover, Proposition 2.19 says that restriction of η to connected components, $\pi_0(\eta)$, will be the homomorphism associated with these group completion, which means the inclusion $\mathbb{N}^n \hookrightarrow \mathbb{Z}^n$ when G is crossed and $\mathbb{N}^{*n} \hookrightarrow \mathbb{Z}^{*n}$ when it is not.

Next, by Proposition 1.33 we know that the map $[\_] : \mathrm{Ob}(\mathbb{G}_n) \to \pi_0(\mathbb{G}_n)$ sending objects of $\mathbb{G}_n$ to their connected component is either the quotient map of abelianisation $\mathbb{N}^{*n} \to \mathbb{N}^n$ or the identity on $\mathbb{N}^{*n}$, depending on whether or not it is crossed. If we also use $[\_]$ to denote the map sending objects of $L\mathbb{G}_n$ to their components, it then follows from functoriality of η that the corresponding choice of the followings two diagrams will commute:

$$\begin{tikzcd} \mathbb{N}^{*n} \arrow[r, hook, "\mathrm{Ob}(\eta)"] \arrow[d, two heads, "{[\_]}"'] & \mathbb{Z}^{*n} \arrow[d, "{[\_]}"] \\ \mathbb{N}^n \arrow[r, hook, "\pi_0(\eta)"'] & \mathbb{Z}^n \end{tikzcd} \qquad \begin{tikzcd} \mathbb{N}^{*n} \arrow[r, hook, "\mathrm{Ob}(\eta)"] \arrow[d, equal, "{[\_]}"'] & \mathbb{Z}^{*n} \arrow[d, "{[\_]}"] \\ \mathbb{N}^{*n} \arrow[r, hook, "\pi_0(\eta)"'] & \mathbb{Z}^{*n} \end{tikzcd}$$

Using the values of $[\_]$ from Proposition 1.33, $\mathrm{Ob}(\eta)$ from Proposition 2.14, and $\pi_0(\eta)$ from earlier in this proof, it follows that for any generator z_i of $\mathbb{Z}^{*n}$,

$$[z_i] \quad = \quad [\mathrm{Ob}(\eta)(z_i)] \quad = \quad \pi_0(\eta)([z_i]) \quad = \quad \pi_0(\eta)(z_i) \quad = \quad z_i$$

But this description of $[\_] : \mathrm{Ob}(L\mathbb{G}_n) \to \pi_0(L\mathbb{G}_n)$ on generators is either the definition of the quotient map $\mathrm{ab} : \mathbb{Z}^{*n} \to (\mathbb{Z}^{*n})^{\mathrm{ab}}$ or the identity $\mathrm{id} : \mathbb{Z}^{*n} \to \mathbb{Z}^{*n}$, depending on the value of target monoid, as required. ☐

2.5 The collapsed morphisms of $L\mathbb{G}_n$

Now that we understand the objects and connected components of the algebra $L\mathbb{G}_n$, the next most obvious thing to look for are its morphisms, $\mathrm{Mor}(L\mathbb{G}_n)$. It would be nice to construct this collection in the same way we constructed $\mathrm{Ob}(L\mathbb{G}_n)$ and $\pi_0(L\mathbb{G}_n)$, by applying the left adjoint of some adjunction to the initial map η. Before we can do this however, we need to ask ourselves a question. What sort of mathematical object is $\mathrm{Mor}(L\mathbb{G}_n)$, exactly?

Given a pair of morphisms $f : x \to y, f' : y' \to z$ in an EG-algebra X, there are two basic binary operations we can perform. First, we can take their tensor product $f \otimes f'$, and this together with the unit map id_I imbues $\mathrm{Mor}(X)$ with the structure of a monoid. Second, if we have $y = y'$ then we can form the composite morphism $f' \circ f$. However, these two operations are not as different as they first appear.

Lemma 2.21. *Let $f : x \to y$ and $f' : y \to z$ be morphisms in some monoidal category, and y is an invertible object of that category. Then*

$$f' \circ f \quad = \quad f' \otimes \mathrm{id}_{y*} \otimes f$$

Proof. By the interchange law for monoidal categories,

$$\begin{aligned} f' \circ f &= (f' \otimes \mathrm{id}_I) \circ (\mathrm{id}_I \otimes f) \\ &= (f' \otimes \mathrm{id}_{y*} \otimes \mathrm{id}_y) \circ (\mathrm{id}_y \otimes \mathrm{id}_{y*} \otimes f) \\ &= (f' \circ \mathrm{id}_y) \otimes (\mathrm{id}_{y*} \circ \mathrm{id}_{y*}) \otimes (\mathrm{id}_y \circ f) \\ &= f' \otimes \mathrm{id}_{y*} \otimes f \end{aligned}$$

□

In other words, composition along invertible objects in X can always be restated in terms of the tensor product. Thus in cases where every object of X is invertible, the monoidal structure together with knowledge of each morphisms source and target will be enough to determine X uniquely. Since all objects in $L\mathbb{G}_n$ are invertible, this means that we could choose to ignore composition of elements of $\mathrm{Mor}(L\mathbb{G}_n)$ for the time being, and focus on its status as a monoid under tensor product.

However, we are trying to extract information about the morphisms of $L\mathbb{G}_n$ by building some sort of left adjoint functor. Presumably we will also be able to apply it to other EG-algebras, some of which won't have all of their objects invertible, and so we can't just use $\mathrm{Mor}(-) : \mathrm{E}G_G\mathrm{Alg} \to \mathrm{Mon}$. What we need is a way to modify the morphism monoid of a category so that both composition and tensor product are recoverable from a single operation. Of course, there is one very easy method for achieving this — simply force $\otimes$ and $\circ$ to be equal.

Definition 2.22. Let $\mathrm{M} : \mathrm{MonCat} \to \mathrm{Mon}$ be the functor which sends monoidal categories X to the quotient of their monoid of morphisms by the relation that sets

$\otimes = \circ$.

$$\mathrm{M}X \quad = \quad {\mathrm{Mor}(X)}\big/{f' \circ f \sim f' \otimes f}$$

Each monoidal functors $F : X \to Y$ is then sent to the monoid homomorphism

$$\begin{array}{rccc} \mathrm{M}(F) & : & \mathrm{M}X & \to & \mathrm{M}Y \\ & : & \mathrm{M}(f) & \mapsto & \mathrm{M}\Big(F(f) \Big) \end{array}$$

where $\mathrm{M}(f)$ refers to the equivalence class of the map f under the quotient $\mathrm{Mor}(X) \to \mathrm{M}(X)$. This homomorphism is well-defined, since it respects the relation $\otimes = \circ$:

$$\begin{array}{rcl} \mathrm{M}(F)(\, f' \circ f \,) & = & \mathrm{M}\Big(F(f' \circ f) \Big) \\ & = & \mathrm{M}\Big(F(f') \circ F(f) \Big) \\ & = & \mathrm{M}\Big(F(f') \Big) \circ \mathrm{M}\Big(F(f) \Big) \\ & = & \mathrm{M}\Big(F(f') \Big) \otimes \mathrm{M}\Big(F(f) \Big) \\ & = & \mathrm{M}\Big(F(f') \otimes F(f) \Big) \\ & = & \mathrm{M}\Big(F(f' \otimes f) \Big) \\ & = & \mathrm{M}(F)(\, f' \otimes f \,) \end{array}$$

We will call $\mathrm{M}X$ the *collapsed* morphisms of the X.

From now on we will generally refer to the single operation in $\mathrm{M}X$ as $\otimes$ rather than $\circ$, unless we are focusing on some aspect best understood using compositon. This convention makes it easier to remember that because the tensor product is defined between all pairs of morphisms in X, the equivalence class $\mathrm{M}(f') \otimes \mathrm{M}(f)$ will always contain the morphism $f' \otimes f$, but not neccessarily $f' \circ f$, as it might fail to exist.

Now we need a candidate for the right adjoint to the functor M.

Definition 2.23. For a given monoid M, let $\mathrm{B}M$ represent the one-object category whose morphisms are the elements of M, with monoid multiplication as composition. This is known as the *delooping* of M, for reasons that come from homotopy theory. Likewise, for any monoid homomorphism $h : M \to M'$ between abelian groups, denote by $\mathrm{B}h : \mathrm{B}M \to \mathrm{B}M'$ the obvious monoidal functor which acts like h on morphisms. This defines a delooping functor $\mathrm{B} : \mathrm{Mon} \to \mathrm{Cat}$ from the category of monoids onto the category of small categories.

Moreover, let C be a commutative monoid. Then we can view $\mathrm{B}C$ as a monoidal category, with the tensor product also given by the multiplication in C, and the sole object as the unit I. Clearly for any homomorphism between commutative monoids $h : C \to C'$ the corresponding functor $\mathrm{B}h : \mathrm{B}C \to \mathrm{B}C'$ will preserve this monoidal structure, as it is already preserving it as compositon. Thus the restriction of B to commutative monoids also gives a functor CMon $\to$ MonCat, which we will still call B.

Commutativity is required in order for BC to be a well-defined monoidal category because we need its operations $\circ$ and $\otimes$ to obey the interchange law for monoidal categories:

$$\begin{array}{rrcl} & (\mathrm{id}_I \circ f) \otimes (f' \otimes \mathrm{id}_I) & = & (\mathrm{id}_I \otimes f') \circ (f \otimes \mathrm{id}_I) \\ \Longrightarrow & \mathrm{id}_I \cdot f \cdot f' \cdot \mathrm{id}_I & = & \mathrm{id}_I \cdot f' \cdot f \cdot \mathrm{id}_I \\ \Longrightarrow & f \cdot f' & = & f' \cdot f \end{array}$$

Proposition 2.24. B *is a right adjoint to the functor* $\mathrm{M}(\_)^{\mathrm{ab}} : \mathrm{MonCat} \to \mathrm{CMon}$.

Proof. Let X be a monoidal category, C a commutative monoid, and $F : X \to \mathrm{B}C$ a monoidal functor. For any $f : x \to x'$ in X, the morphism $F(f)$ is just an element of the monoid C, and so F can be used to define a function

$$\begin{array}{rrcl} F' : & \mathrm{M}(X)^{\mathrm{ab}} & \to & C \\ : & \mathrm{ab} \circ \mathrm{M}(f) & \mapsto & F(f) \end{array}$$

where ab is the quotient map of abelianisation $\mathrm{M}(X) \to \mathrm{M}(X)^{\mathrm{ab}}$. This F' is a well-defined monoid homomorphism; it preserves multiplication and respects the relation $\otimes = \circ$ because the monoid multiplication of C is acts as both tensor product and composition in BC.

$$\begin{array}{rcl} F'\Big(\mathrm{abM}(f' \circ f)\Big) & = & F(f' \circ f) \\ & = & F(f') \circ F(f) \\ & = & F(f') \cdot F(f) \\ & = & F(f') \otimes F(f) \\ & = & F(f' \otimes f) \\ & = & F'\Big(\mathrm{abM}(f' \otimes f)\Big) \end{array}$$

Conversely, if $h : \mathrm{M}(X)^{\mathrm{ab}} \to C$ is a monoid homomorphism, we can define from it a monoidal functor

$$\begin{array}{rrcl} h' : & X & \mapsto & \mathrm{B}C \\ : & x & \mapsto & I \\ : & f : x \to y & \mapsto & h\Big(\mathrm{abM}(f)\Big) : I \to I \end{array}$$

Yet again, the monoidal functor h' is well-defined because the fact that $\otimes = \circ$ in $\mathrm{B}C$ forces h' to respect that relation.

$$\begin{aligned}
h'(f' \circ f) &= h\Big(\mathrm{abM}(f' \circ f)\Big) \\
&= h\Big(\mathrm{abM}(f') \circ \mathrm{M}(f')\Big) \\
&= h\Big(\mathrm{abM}(f')\Big) \circ h\Big(\mathrm{abM}(f')\Big) \\
&= h\Big(\mathrm{abM}(f')\Big) \cdot h\Big(\mathrm{abM}(f')\Big) \\
&= h\Big(\mathrm{abM}(f')\Big) \otimes h\Big(\mathrm{abM}(f')\Big) \\
&= h\Big(\mathrm{abM}(f') \otimes \mathrm{abM}(f')\Big) \\
&= h\Big(\mathrm{abM}(f' \otimes f')\Big) \\
&= h'(f' \otimes f)
\end{aligned}$$

But these assignments $F \mapsto F'$ and $h \mapsto h'$ are clearly inverse to one another. For any $F : X \to \mathrm{B}C$ applying them twice gives

$$\begin{array}{rrrcl}
F'' & : & X & \to & \mathrm{B}C \\
 & : & x & \mapsto & I \\
 & : & f : x \to y & \mapsto & F'\Big(\mathrm{abM}(f)\Big) : I \to I \quad = \quad F(f)
\end{array}$$

and similarly for $h : \mathrm{M}X \to C$ we get

$$\begin{array}{rrrcl}
h'' & : & \mathrm{M}(X)^{\mathrm{ab}} & \to & C \\
 & : & \mathrm{abM}(f) & \mapsto & h'(f) \quad = \quad h\Big(\mathrm{abM}(f)\Big)
\end{array}$$

In other words, we have an isomorphism between the homsets

$$\mathrm{MonCat}(\,X, \mathrm{B}C\,) \quad \cong \quad \mathrm{CMon}(\,\mathrm{M}(X)^{\mathrm{ab}}, C\,)$$

This isomorphism is natural in both coordinates, as for any monoidal functor $G : X \to X'$ and homomorphism $h : C \to C'$ between commutative monoids,

$$\mathrm{abM}(\,\mathrm{B}h \circ F \circ G\,) \quad = \quad \mathrm{abM}(\mathrm{B}h) \circ \mathrm{abM}(F) \circ \mathrm{abM}(G) \quad = \quad h \circ \mathrm{abM}(F) \circ \mathrm{abM}(G)$$

and so the diagram

$$\begin{tikzcd}
\mathrm{MonCat}(X, \mathrm{B}C) \arrow[r, "\sim"] \arrow[d, "\mathrm{B}h \circ _ \circ G"'] & \mathrm{CMon}\Big(\mathrm{M}(X)^{\mathrm{ab}}, C\Big) \arrow[d, "h \circ _ \circ \mathrm{abM}G"] \\
\mathrm{MonCat}(X', \mathrm{B}C') \arrow[r, "\sim"] & \mathrm{CMon}\Big(\mathrm{M}(X')^{\mathrm{ab}}, M'\Big)
\end{tikzcd}$$

commutes. Therefore, $\mathrm{M}(\_)^{\mathrm{ab}} \dashv \mathrm{B}$. □

Proposition 2.24 seems at first glance very similar to Propositions 2.12 and 2.18. However, our goal was to discover the relationship between the morphisms of $\mathbb{G}_n$ and $L\mathbb{G}_n$, paralleling what we did in Propositions 2.14 and 2.19, and in that regard M falls short in two very important ways.

1. What we really wanted to have was an adjunction involving $\mathrm{E}G_G\mathrm{Alg}_S$, not MonCat. This is because our previous methodology involved applying our left adjoint functors to η and then using its initial property to factor various maps through $L\mathbb{G}_n$. But η is an initial object in $(\mathbb{G}_n \downarrow \mathrm{inv})$, and so we only know how to use it to factor *algebra* maps $\mathbb{G}_n \to X_{\mathrm{inv}}$, and not general monoidal functors.

2. Even if we do find a way to use this adjunction to extract information about $L\mathbb{G}_n$, it will not be the monoid $\mathrm{Mor}(L\mathbb{G}_n)$ we were originally after, only a strange abelianised version where tensor product and composition coincide.

Unfortunately, this adjunction seems to be the best we can do. The only general method for assigning an $\mathrm{E}G$-action to the monoidal category $\mathrm{B}C$ for all C is to set all of its action morphisms $\alpha(g; \mathrm{id}_I, ..., \mathrm{id}_I)$ to be id_I. This would then cause the homomorphism $\mathrm{M}X \to C$ corresponding to any algebra map $X \to \mathrm{B}C$ to be the zero map if X has only action morphisms. Given Lemma 1.32, this is clearly no use. However, it turns out that this approach is fixable. To that end, we will spend the bulk of the next two chapters directly addressing problems 1 and 2.

For now though, we will make one last small alteration to our plan going forward. Instead of working directly with the functor $\mathrm{M}(\_)^{\mathrm{ab}} : \mathrm{MonCat} \to \mathrm{CMon}$, we will instead focus on its composite with the group completion functor, $(\_)^{\mathrm{gp}} : \mathrm{CMon} \to \mathrm{Ab}$. It may not be clear yet why we would choose to do this, but over the next couple of chapters we will frequently find ourselves having to forming quotients of certain algebraic objects. If we were to stick with the functor M these would all be commutative monoid quotients, whereas by making the switch to $\mathrm{M}(\_)^{\mathrm{gp,ab}}$ they will be abelian groups instead, which are far easier to work with. Also, notice that since the process of group completion is left adjoint to the forgetful functor $\mathrm{Ab} \to \mathrm{CMon}$, its composite with the left adjoint $\mathrm{M}(\_)^{\mathrm{ab}}$ will be a left adjoint functor too. Thus with this new functor we will be able use all of the same important properties that we would have done with $\mathrm{M}(\_)^{\mathrm{ab}}$, such as the preservation of colimits. Moreover, while we won't prove this for some time, it turns out that the morphisms of $L\mathbb{G}_n$ actually form a group under tensor product. This means that whatever method we would have used to recover $\mathrm{Mor}(L\mathbb{G}_n)$ from $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{ab}}$ will still let us recover $\mathrm{Mor}(L\mathbb{G}_n) = \mathrm{Mor}(L\mathbb{G}_n)^{\mathrm{gp}}$ from $\mathrm{M}(L\mathbb{G}_n)^{\mathrm{gp,ab}}$.

Before we move on, we should spend a little time thinking about this new functor $\mathrm{M}(\_)^{\mathrm{gp,ab}}$. Specifically, we might ask in what order we have to carry out its constituent parts: the collapsing of $\circ$ and $\otimes$ into a single operation, group completion, and abelianisation. It is a well known fact that group completion and abelianisation commute:

$$\begin{tikzcd} \mathrm{Mon} \arrow[r, "(\_)^{\mathrm{gp}}"] \arrow[d, "(\_)^{\mathrm{ab}}"'] & \mathrm{Grp} \arrow[d, "(\_)^{\mathrm{ab}}"] \\ \mathrm{CMon} \arrow[r, "(\_)^{\mathrm{gp}}"] & \mathrm{Ab} \end{tikzcd}$$

Indeed, we already assume this when talking of 'the' canonical map $\mathrm{M}(X)^{\mathrm{gp,ab}}$. But a more interesting question is whether it matters if we choose to group complete or abelianise the tensor product of a monoidal category before or after we collapse its morphisms.

Lemma 2.25. *For any monoidal category X, define*

$$\mathrm{M}_{\mathrm{gp}}(X) \cong \mathrm{Mor}(X)^{\mathrm{gp}} \Big/ \mathrm{gp}(f' \circ f) \sim \mathrm{gp}(f' \otimes f)$$

$$\mathrm{M}_{\mathrm{ab}}(X) \cong \mathrm{Mor}(X)^{\mathrm{ab}} \Big/ \mathrm{ab}(f' \circ f) \sim \mathrm{ab}(f' \otimes f)$$

Then

$$\mathrm{M}_{\mathrm{gp}}(X) = \mathrm{M}(X)^{\mathrm{gp}}, \qquad \mathrm{M}_{\mathrm{ab}}(X) = \mathrm{M}(X)^{\mathrm{ab}}$$

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
 & \mathrm{M}(X) \arrow[r, "\mathrm{gp}"] \arrow[ddr, dashed, "v"] & \mathrm{M}(X)^{\mathrm{gp}} \arrow[dd, dashed, shift left, "u'"] \\
\mathrm{Mor}(X) \arrow[ur, "\mathrm{M}"] \arrow[dr, "\mathrm{gp}"'] & & \\
 & \mathrm{Mor}(X)^{\mathrm{gp}} \arrow[r, "\mathrm{M}"'] \arrow[uur, dashed, "u"'] & \mathrm{M}_{\mathrm{gp}}(X) \arrow[uu, dashed, shift left, "v'"]
\end{tikzcd}$$

Here all of the solid arrows are the respective canonical homomorphisms.

Starting from the left, the top edge of the diagram is a map coming out of $\mathrm{Mor}(X)$ and going into a group, and so by the universal property of the group completion there is a unique homomorphism u factoring it through $\mathrm{Mor}(X)^{\mathrm{gp}}$. But now this u is a map out of $\mathrm{Mor}(X)^{\mathrm{gp}}$ and into group where tensor product and compositon are equal, and so by the universal property of the quotient this factors once more through the map u'. On the other hand, the bottom edge of the diagram will factor through the map v because of the collapsed morphisms property, and then through the map v' due to the group completion property. Then this diagram says that

$$\begin{aligned}
v' \circ u' \circ \mathrm{gp} \circ \mathrm{M} &= v' \circ u' \circ u \circ \mathrm{gp} \\
&= v' \circ \mathrm{M} \circ \mathrm{gp} \\
&= u \circ \mathrm{gp} \\
&= \mathrm{gp} \circ \mathrm{M}
\end{aligned}$$

But $\mathrm{M} : \mathrm{Mor}(X) \to \mathrm{M}(X)$ is the map associated with a quotient, and so it is an epimorphism. Thus we can cancel it out on the right, leaving just

$$v' \circ u' \circ \mathrm{gp} \quad = \quad \mathrm{gp}$$

Then from this we can conclude that for any $\mathrm{M}(f) \in \mathrm{M}(X)$,

$$\begin{aligned}
v'u'\Big(\mathrm{gpM}(f)\Big) &= \mathrm{gpM}(f) \\
v'u'\Big(\mathrm{gpM}(f)^*\Big) &= v'u'\Big(\mathrm{gpM}(f)\Big)^* = \mathrm{gpM}(f)^*
\end{aligned}$$

All elements of $\mathrm{M}(X)^{\mathrm{gp}}$ can be written as $\mathrm{gpM}(f)$ or $\mathrm{gpM}(f)^*$ for at least one f, so this really says that $v' \circ u'$ is the identity homomorphisms on $\mathrm{M}(X)^{\mathrm{gp}}$.

A completely analagous argument can also be by made starting from the bottom edge of the diagram instead, and then concluding that $u' \circ v' = \mathrm{id}_{\mathrm{M}_{\mathrm{gp}}(X)}$. Furthermore, we can construct another diagram using the universal property of the abelianisation,

$$\begin{tikzcd}
 & \mathrm{M}(X) \arrow[r, "\mathrm{ab}"] \arrow[dr, dashed, "v''"] & \mathrm{M}(X)^{\mathrm{ab}} \arrow[d, dashed, shift left, "u'''"] \\
\mathrm{Mor}(X) \arrow[ur, "\mathrm{M}"] \arrow[dr, "\mathrm{ab}"'] & & \\
 & \mathrm{Mor}(X)^{\mathrm{ab}} \arrow[r, "\mathrm{M}"'] \arrow[ur, dashed, "u''"'] & \mathrm{M}_{\mathrm{ab}}(X) \arrow[u, dashed, shift left, "v'''"]
\end{tikzcd}$$

and then through a series of analagous arguments conclude that $v''' \circ u''' = \mathrm{id}_{\mathrm{M}(X)^{\mathrm{ab}}}$ and $u''' \circ v''' = \mathrm{id}_{\mathrm{M}_{\mathrm{ab}}(X)}$. All together, these yield the two isomorphisms given in the statement of the proposition. $\square$

In other words, we do not need to worry about order of operations when using the left adjoint functor $\mathrm{M}(\_)^{\mathrm{gp,ab}}$. This is very convenient, and later on when we actually need to evalute particular $\mathrm{M}(X)^{\mathrm{gp,ab}}$, we will use this fact to carry out the calculation in whichever order proves easiest.

Bibliography

[1] John C. Baez; Aaron D. Lauda. *Higher-Dimensional Algebra V: 2-Groups.* https://arxiv.org/pdf/math/0307200.pdf

[2] Eugenia Cheng; Nick Gurski. *The periodic table of n-categories for low dimensions I: degenerate categories and degenerate bicategories* Categories in Algebra, Geometry and Mathematical Physics, Contemp. Math. 431, 2005, 143-164 https://arxiv.org/pdf/0706.2307.pdf

[3] Eugenia Cheng; Nick Gurski. *The periodic table of n-categories for low dimensions II: degenerate tricategories* arXiv:0706.2307

[4] Alexander S. Corner; Nick Gurski. *Operads with general groups of equivariance, and some 2-categorical aspects of operads in Cat* arXiv:1312.5910 [math.CT]

[5] Raouf Doss. *Sur l'immersion d'un semi-groupe dans un groupe* Bulletin des Sciences Mathématiques, (2) 72, (1948). 139–150.

[6] Eckmann, B.; Hilton, P. J. *Group-like structures in general categories. I. Multiplications and comultiplications* Mathematische Annalen, 145 (3), pp. 227–255

[7] Z. Fiedorowicz; M. Stelzer; R.M. Vogt. *Homotopy Colimits of Algebras Over Cat-Operads and Iterated Loop Spaces* Advances in Mathematics 248, 2013, 1089-1155

[8] Nick Gurski. *Operads, tensor products, and the categorical Borel construction.* arXiv:1508.04050 [math.CT].

[9] Tom Leinster. *Higher Operads, Higher Categories* London Mathematical Society Lecture Note Series 298, Cambridge University Press, Cambridge, 2004.

[10] Tom Leinster. *Basic Category Theory* Cambridge Studies in Advanced Mathematics, Vol. 143, Cambridge University Press, Cambridge, 2014. https://arxiv.org/pdf/1612.09375v1.pdf

[11] Saunders Mac Lane. *Categories for the Working Mathematician* Springer Science+Business Media, 1971

[12] Malcev, A. *On the Immersion of an Algebraic Ring into a Field* Mathematische Annalen 113 (1937): DCLXXXVI-DCXCI.

[13] Malcev, A. *On the immersion of associative systems into groups* Matematicheskii Sbornik 6 (1939), no.2, 331-336

[14] J. P. May. *The Geometry of Iterated Loop Spaces* Springer-Verlag Berlin Heidelberg 1972 http://www.math.uchicago.edu/ may/BOOKS/gils.pdf

[15] Paolo Salvatore; Nathalie Wahl. *Framed discs operads and Batalin-Vilkovisky algebras* Q. J. Math., 54(2):213–231, 2003.

[16] Peter Selinger. *A survey of graphical languages for monoidal categories.* http://www.mscs.dal.ca/ selinger/papers/graphical.pdf

[17] K. H. Ulbrich. *Group cohomology for Picard categories* Journal of Algebra, 91 (1984), pp. 464-498 http://dx.doi.org/10.1016/0021-8693(84)90114-5

[18] K. H. Ulbrich. *Kohärenz in Kategorien mit Gruppenstruktur* Journal of Algebra, 72 (1981), pp. 279-295 http://dx.doi.org/10.1016/0021-8693(81)90295-7

[19] Nathalie Wahl. *Ribbon braids and related operads* Thesis (Ph.D.)–University of Oxford (2001)

[20] Wenbin Zhang. *Group Operads and Homotopy Theory* arXiv:1111.7090 [math.AT]