11.5

Lines and Curves in Space

PEARSON

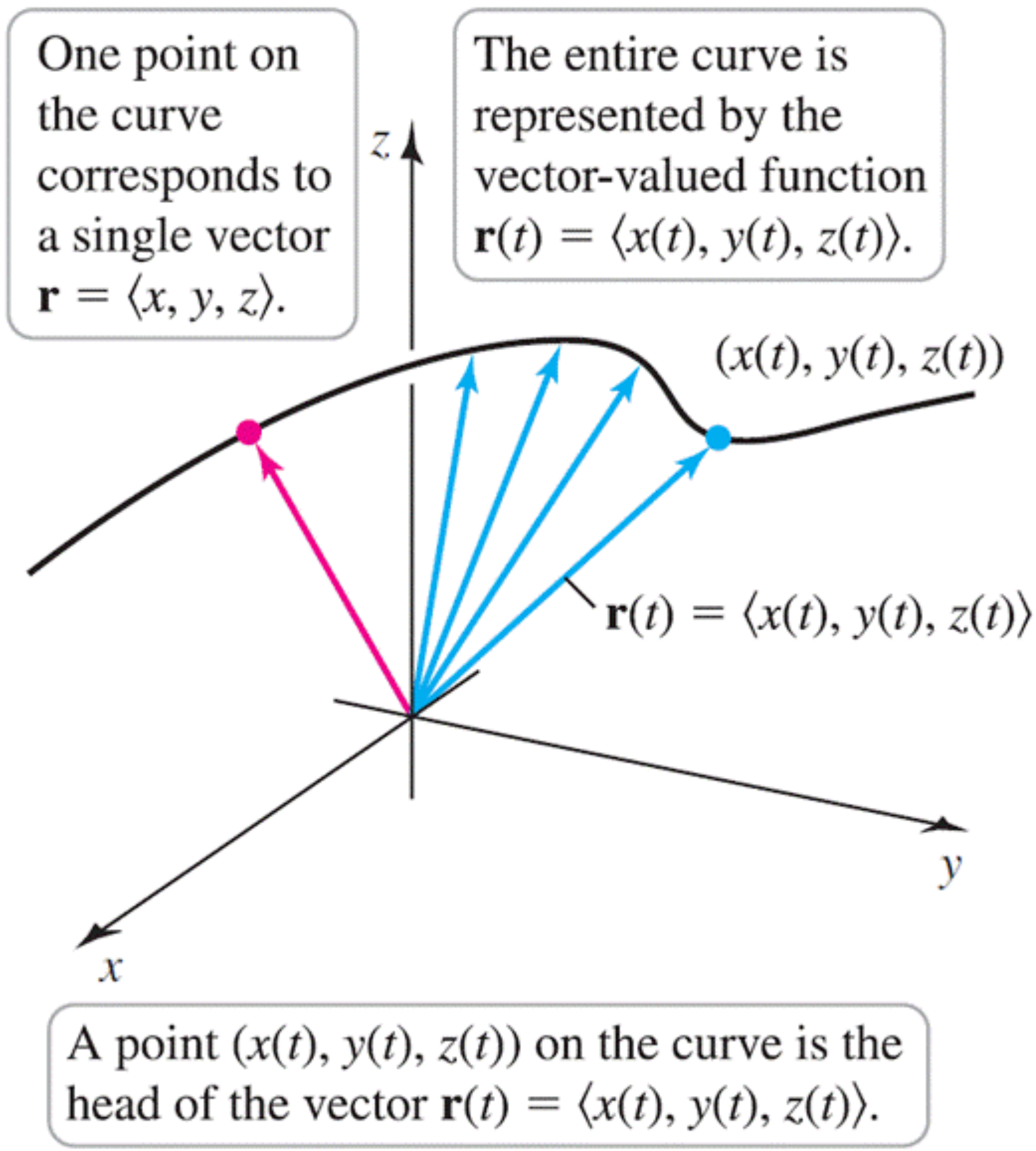

One point on the curve corresponds to a single vector $\mathbf{r} = \langle x, y, z \rangle$.
The entire curve is represented by the vector-valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.
z
$(x(t), y(t), z(t))$
$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$
y
x
A point $(x(t), y(t), z(t))$ on the curve is the head of the vector $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

FIGURE 11.66

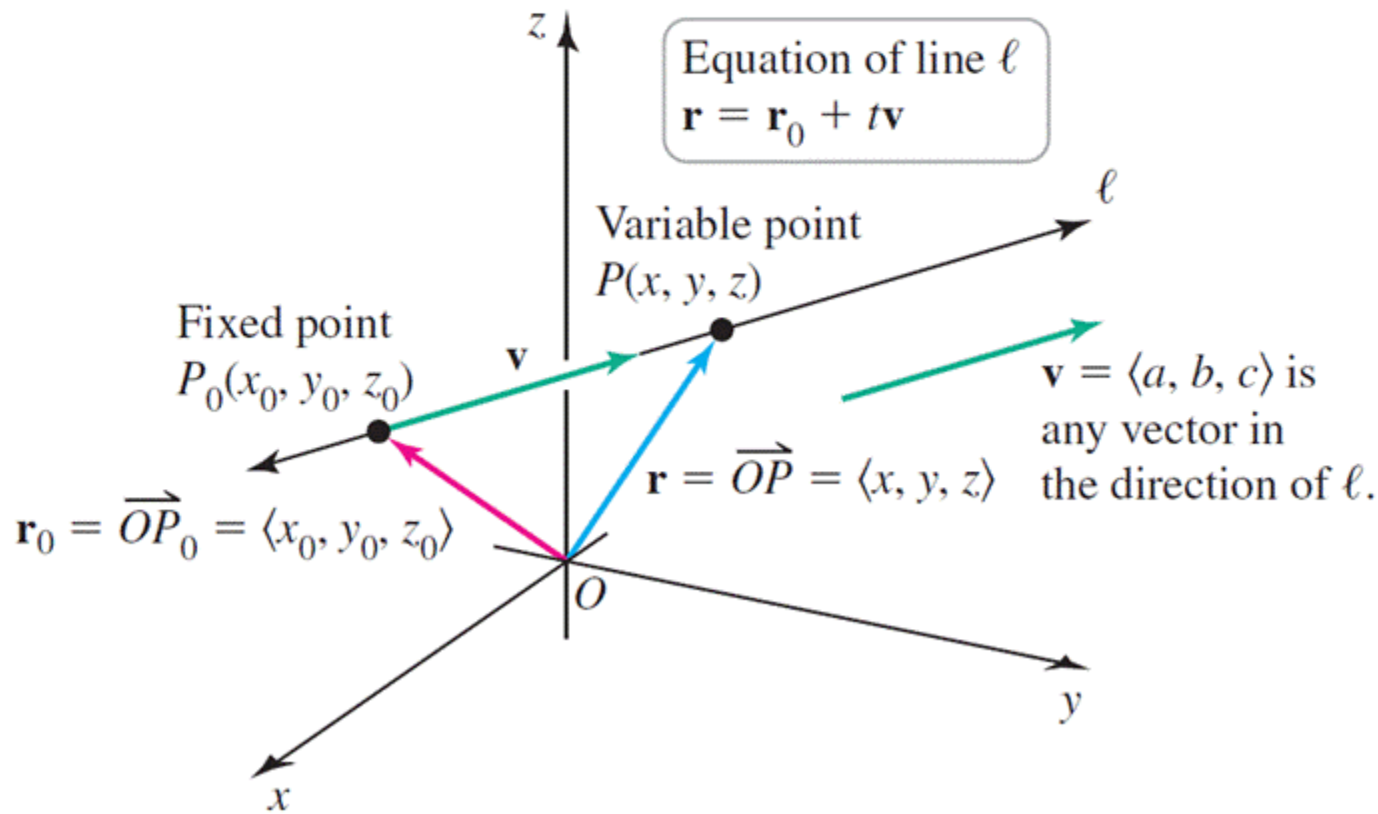
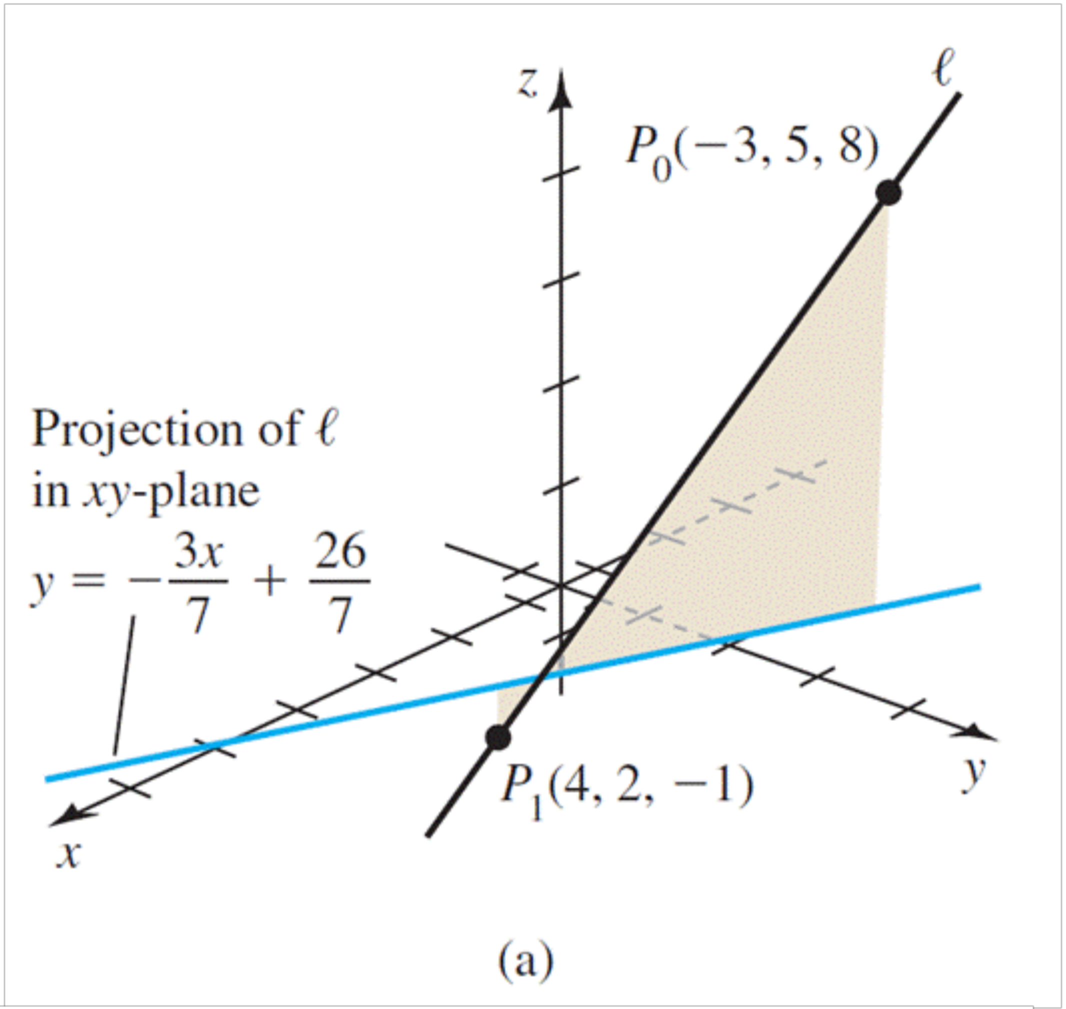

z
Equation of line ℓ
$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$
ℓ
Variable point
$P(x, y, z)$
Fixed point
$P_0(x_0, y_0, z_0)$
$\mathbf{v}$
$\mathbf{v} = \langle a, b, c \rangle$ is
any vector in
the direction of ℓ.
$\mathbf{r} = \overrightarrow{OP} = \langle x, y, z \rangle$
$\mathbf{r}_0 = \overrightarrow{OP_0} = \langle x_0, y_0, z_0 \rangle$
O
y
x

FIGURE 11.67



z
ℓ
$P_0(-3, 5, 8)$
Projection of ℓ in xy-plane
$y = -\frac{3x}{7} + \frac{26}{7}$
$P_1(4, 2, -1)$
y
x
(a)

FIGURE 11.69

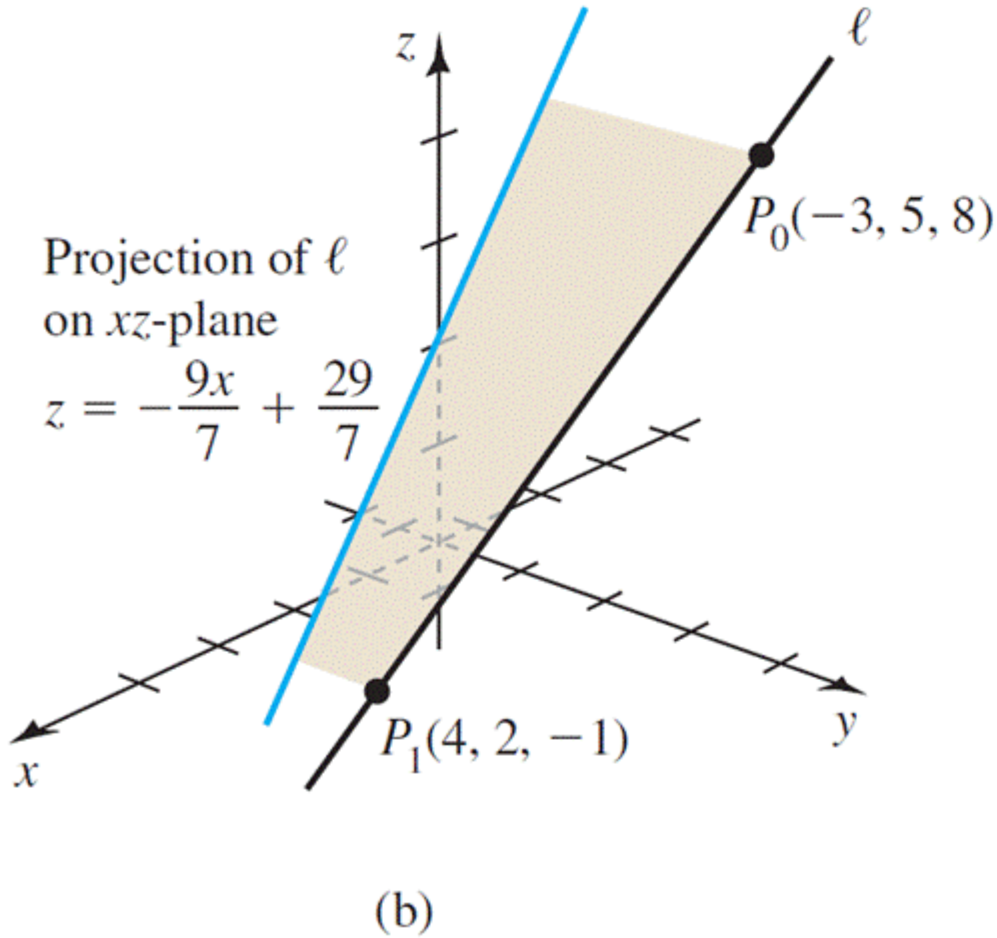
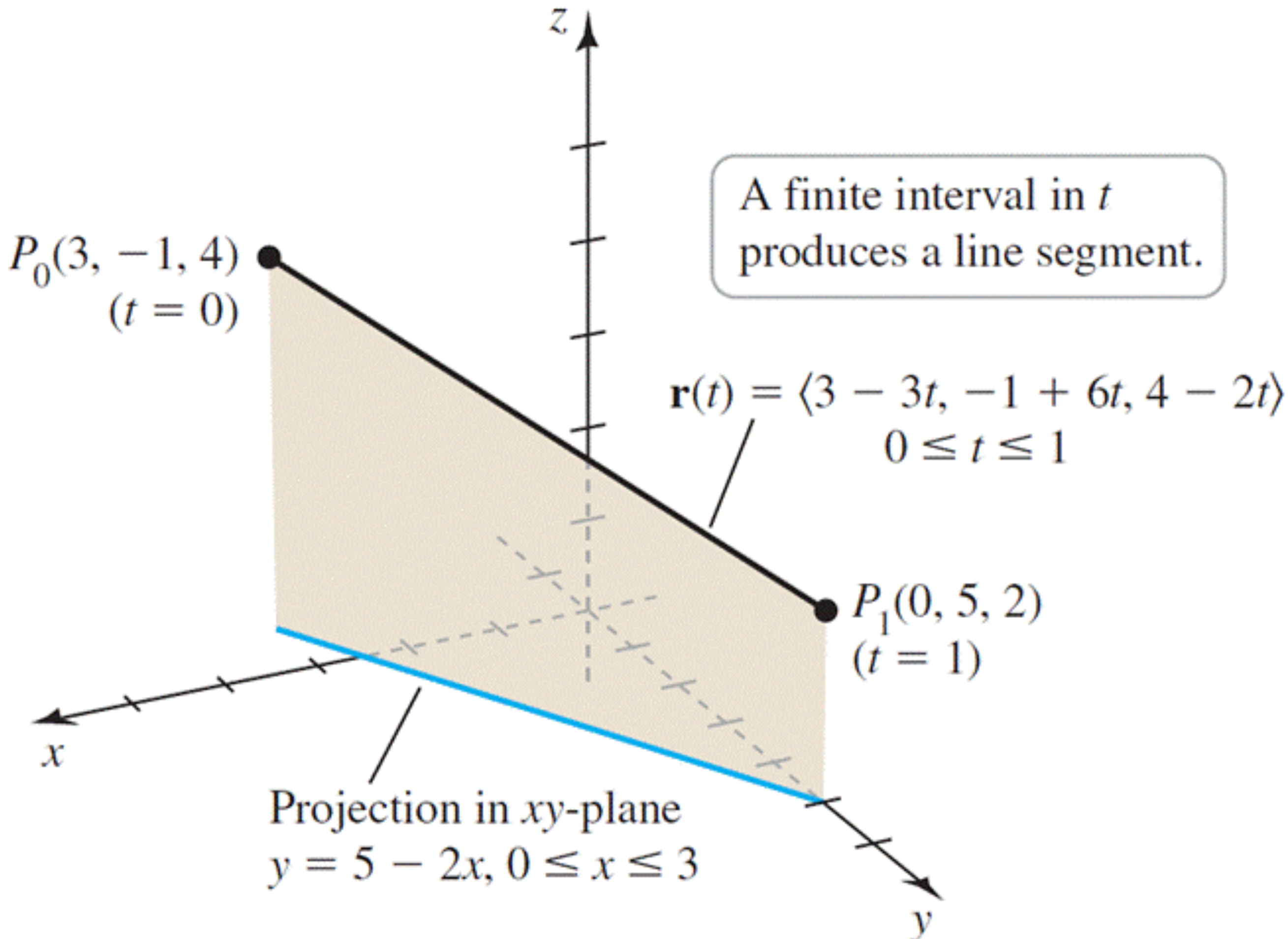

z
ℓ
Projection of ℓ
on xz-plane
$z = -\frac{9x}{7} + \frac{29}{7}$
$P_0(-3, 5, 8)$
$P_1(4, 2, -1)$
x
y
(b)

FIGURE 11.69



z
A finite interval in t produces a line segment.
P0(3, −1, 4)
(t = 0)
r(t) = ⟨3 − 3t, −1 + 6t, 4 − 2t⟩
0 ≤ t ≤ 1
P1(0, 5, 2)
(t = 1)
x
Projection in xy-plane
y = 5 − 2x, 0 ≤ x ≤ 3
y

FIGURE 11.70

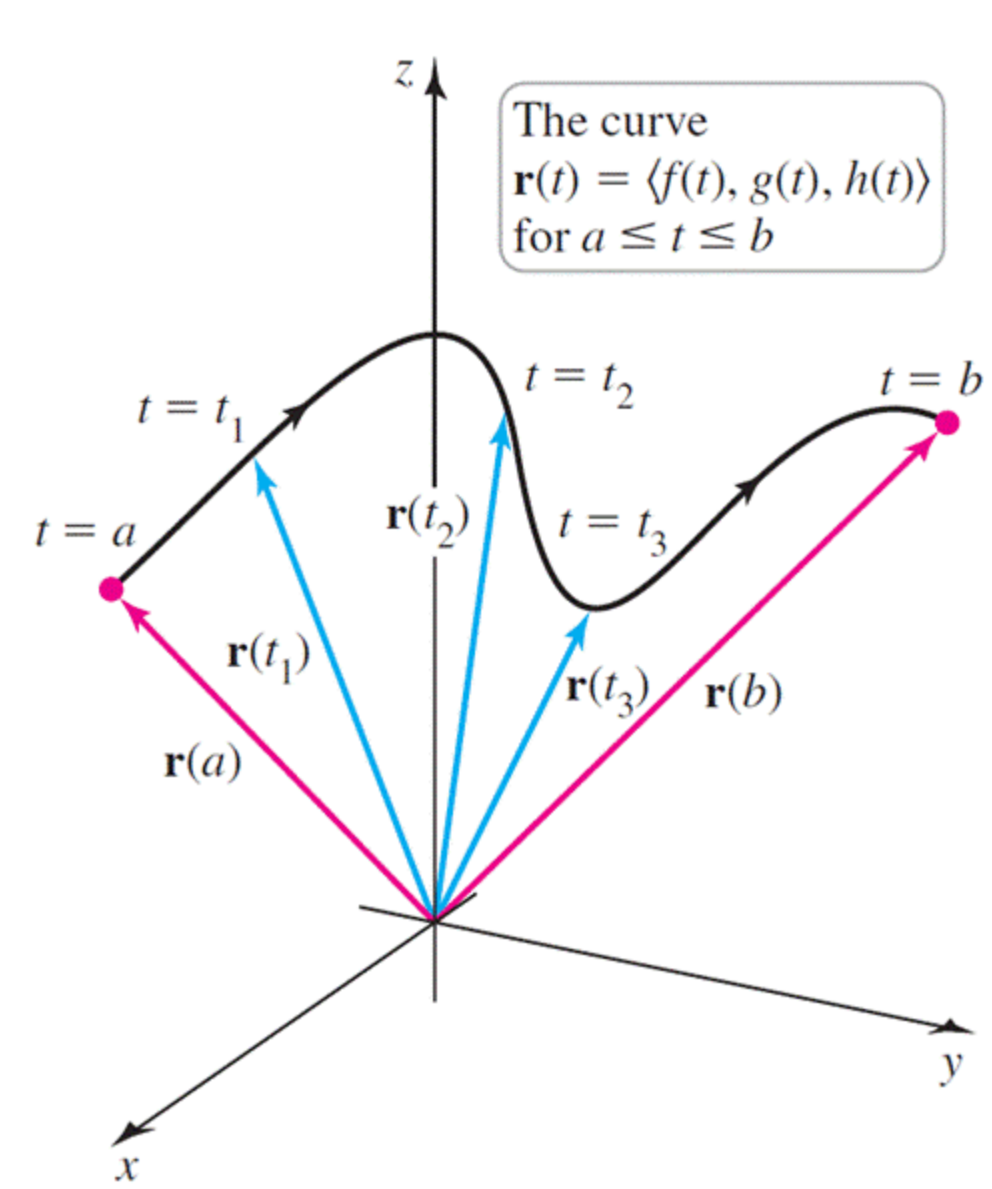
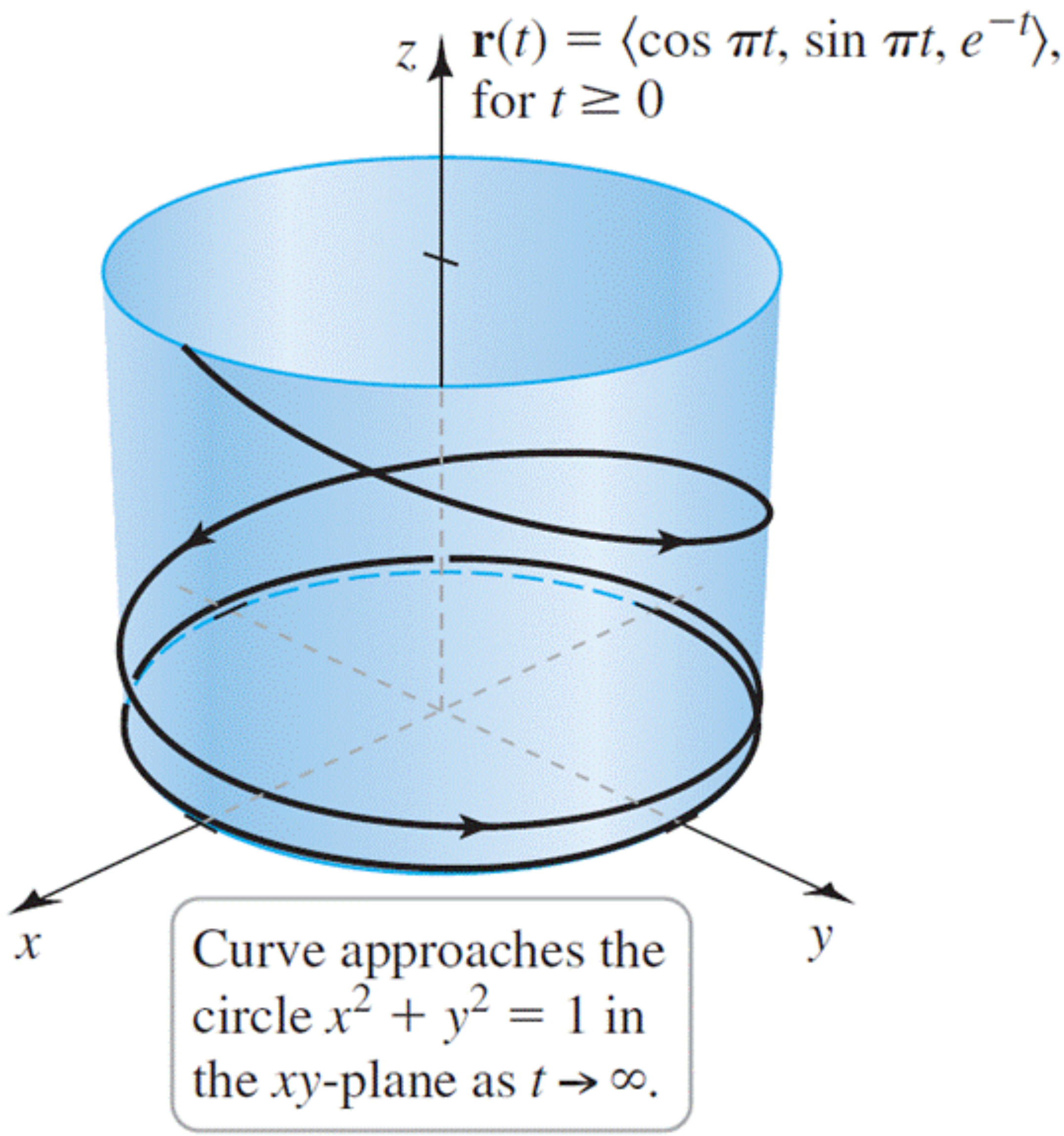

z
The curve
$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$
for $a \le t \le b$
$t = t_1$
$t = t_2$
$t = b$
$t = a$
$\mathbf{r}(t_2)$
$t = t_3$
$\mathbf{r}(t_1)$
$\mathbf{r}(t_3)$
$\mathbf{r}(b)$
$\mathbf{r}(a)$
y
x

FIGURE 11.71

DEFINITION **Limit of a Vector-Valued Function**

A vector-valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as t approaches a, written $\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}$, provided $\lim_{t \to a} |\mathbf{r}(t) - \mathbf{L}| = 0$.

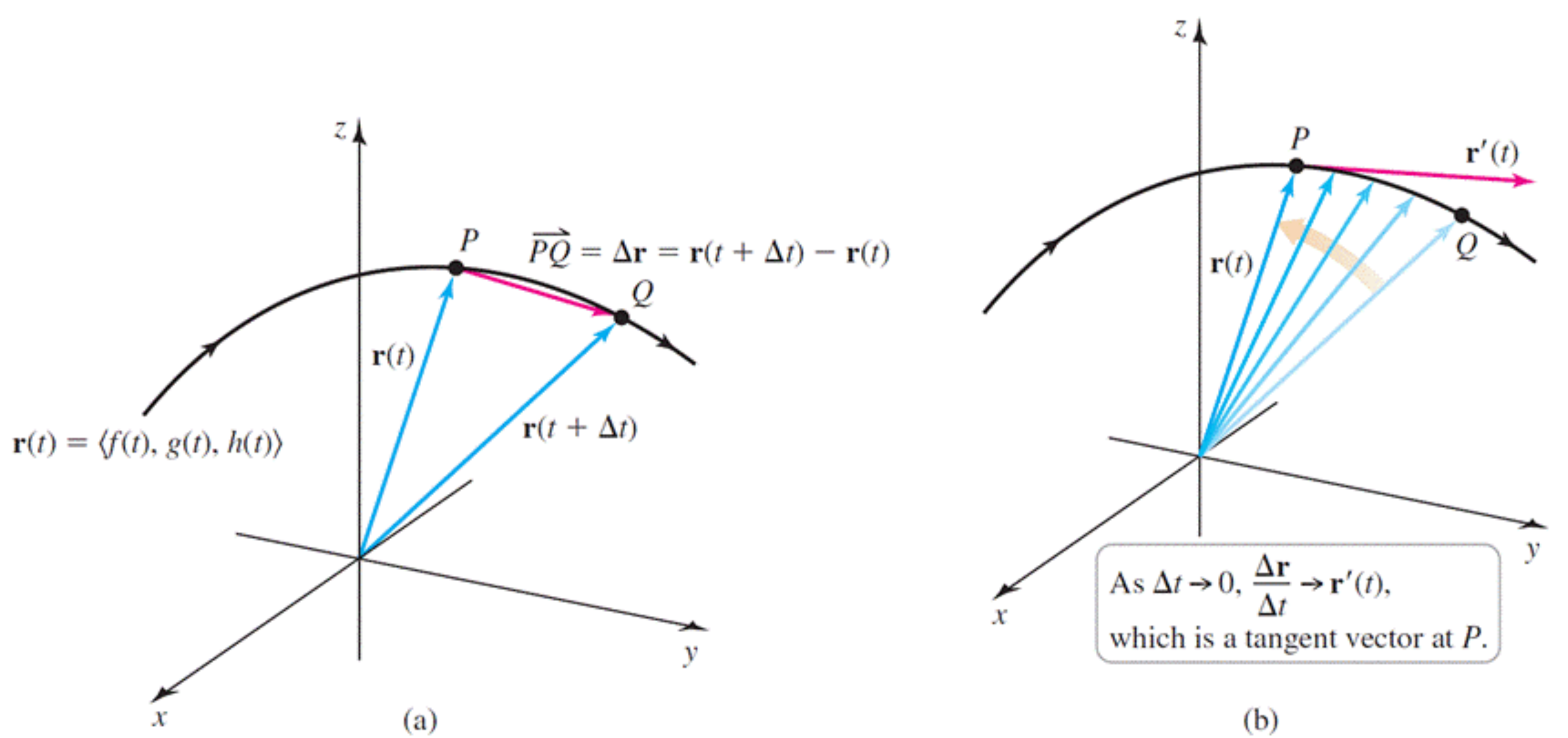


$\mathbf{r}(t) = \langle \cos \pi t, \sin \pi t, e^{-t} \rangle$,
for $t \geq 0$
z
x
y
Curve approaches the circle $x^2 + y^2 = 1$ in the xy-plane as $t \to \infty$.

FIGURE 11.76

11.6

Calculus of Vector-Valued Functions



z
P
$\overrightarrow{PQ} = \Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$
Q
$\mathbf{r}(t)$
$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$
$\mathbf{r}(t + \Delta t)$
y
x
(a)
z
P
$\mathbf{r}'(t)$
Q
$\mathbf{r}(t)$
y
x
As $\Delta t \to 0$, $\dfrac{\Delta \mathbf{r}}{\Delta t} \to \mathbf{r}'(t)$,
which is a tangent vector at P.
(b)

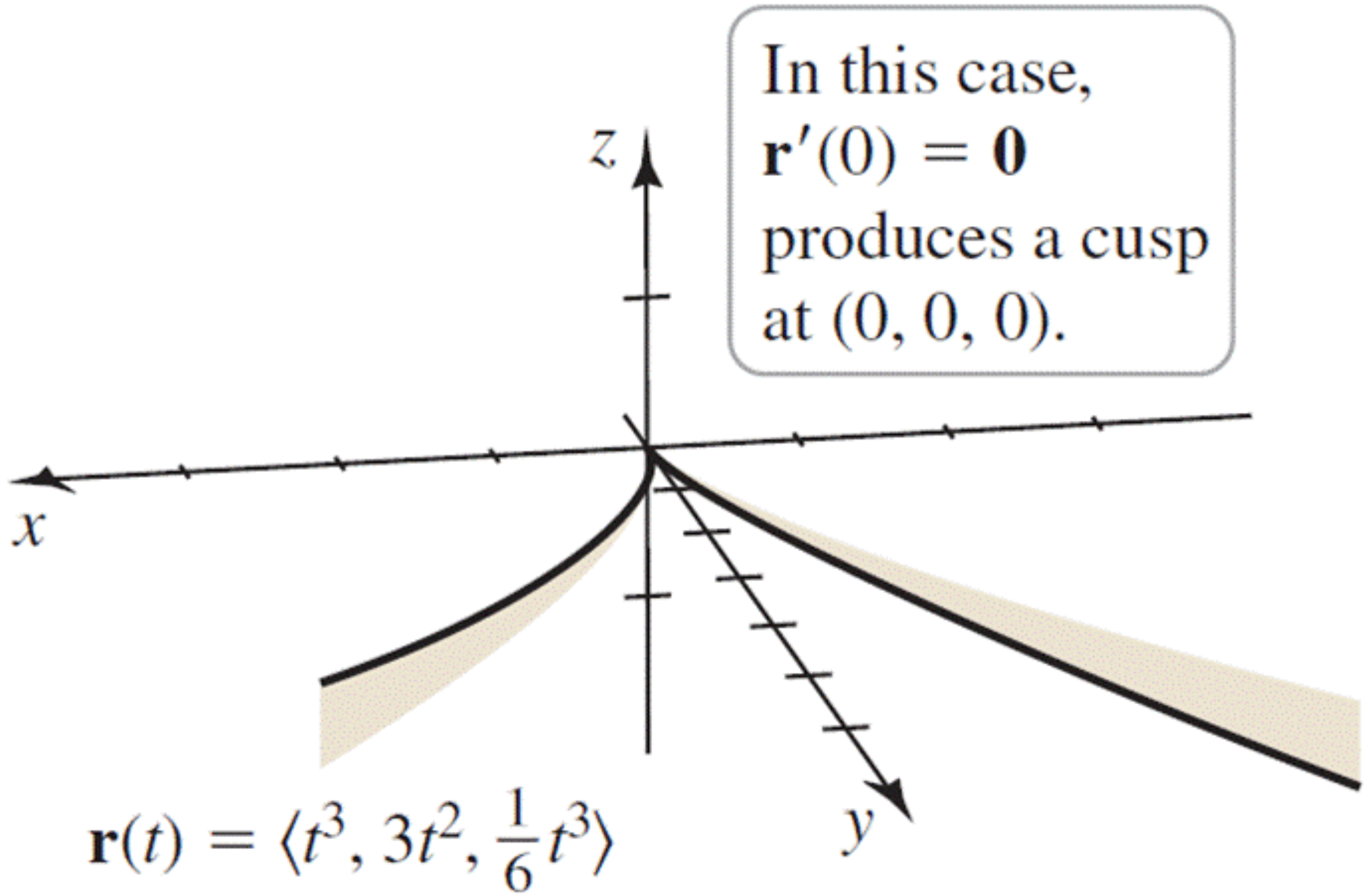
FIGURE 11.77

DEFINITION **Derivative and Tangent Vector**

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions on (a, b). Then $\mathbf{r}$ has a **derivative** (or is **differentiable**) on (a, b) and

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

Provided $\mathbf{r}'(t) \neq \mathbf{0}$, $\mathbf{r}'(t)$ is a **tangent vector** (or velocity vector) at the point corresponding to $\mathbf{r}(t)$.



In this case,
$\mathbf{r}'(0) = \mathbf{0}$
produces a cusp
at (0, 0, 0).
z
x
y
$\mathbf{r}(t) = \langle t^3, 3t^2, \frac{1}{6}t^3 \rangle$

FIGURE 11.78

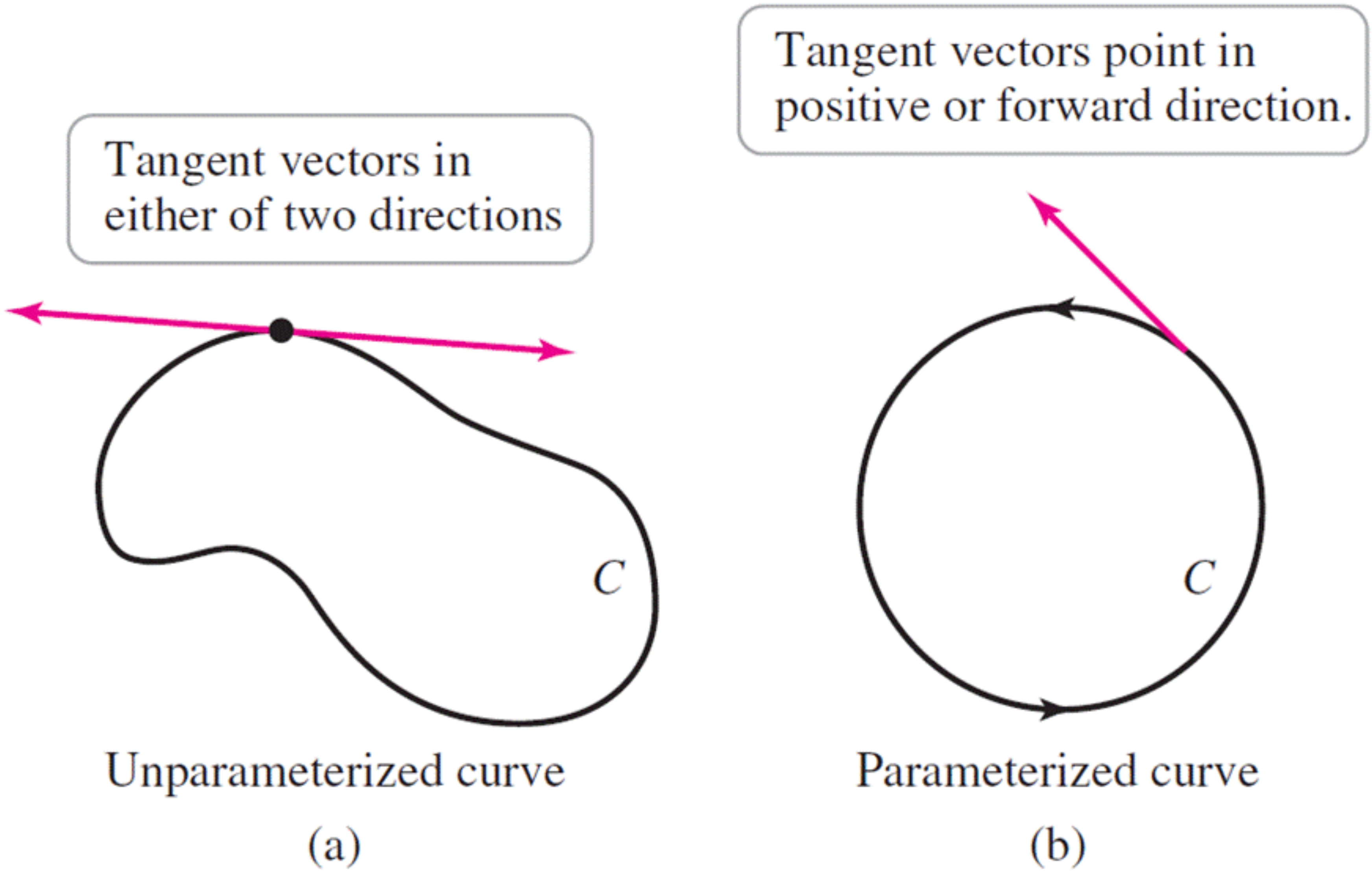

Tangent vectors in
either of two directions
C
Unparameterized curve
(a)
Tangent vectors point in
positive or forward direction.
C
Parameterized curve
(b)

FIGURE 11.79

DEFINITION **Unit Tangent Vector**

Let $\mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a smooth parameterized curve for $a \leq t \leq b$. The **unit tangent vector** for a particular value of t is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

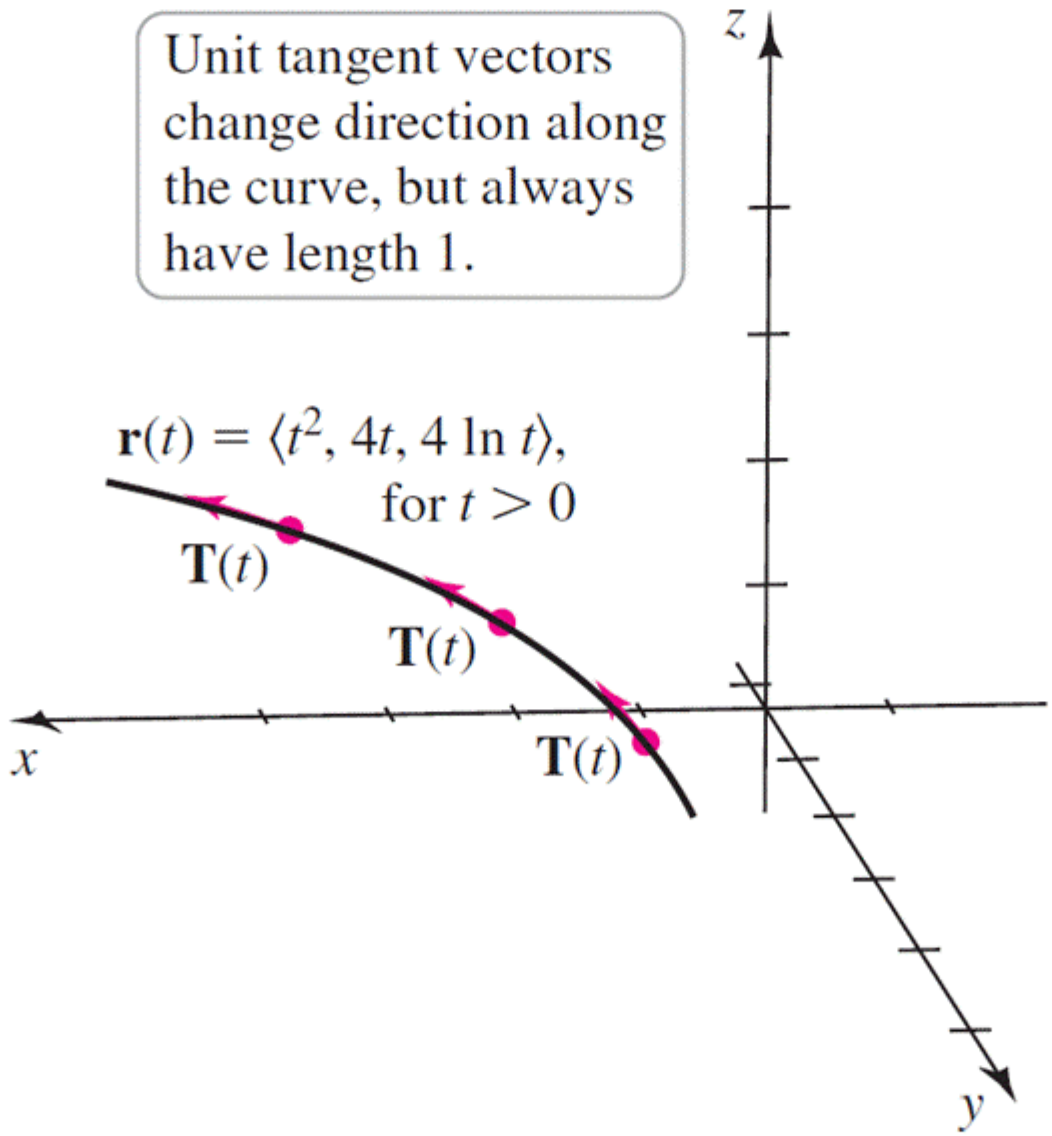

Unit tangent vectors change direction along the curve, but always have length 1.
$\mathbf{r}(t) = \langle t^2, 4t, 4 \ln t \rangle$,
for $t > 0$
$\mathbf{T}(t)$
$\mathbf{T}(t)$
$\mathbf{T}(t)$
x
y
z

FIGURE 11.80

THEOREM 11.7 Derivative Rules

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector-valued functions and let f be a differentiable scalar-valued function, all at a point t. Let $\mathbf{c}$ be a constant vector. The following rules apply.

1. $\dfrac{d}{dt}(\mathbf{c}) = \mathbf{0}$ Constant Rule
2. $\dfrac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$ Sum Rule
3. $\dfrac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ Product Rule
4. $\dfrac{d}{dt}(\mathbf{u}(f(t))) = \mathbf{u}'(f(t))f'(t)$ Chain Rule
5. $\dfrac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ Dot Product Rule
6. $\dfrac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ Cross Product Rule

DEFINITION **Indefinite Integral of a Vector-Valued Function**

Let $\mathbf{r} = f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k}$ be a vector function and let $\mathbf{R} = F\,\mathbf{i} + G\,\mathbf{j} + H\,\mathbf{k}$, where F, G, and H are antiderivatives of f, g, and h, respectively. The indefinite integral of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C},$$

where $\mathbf{C}$ is an arbitrary constant vector.

DEFINITION **Definite Integral of a Vector-Valued Function**

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are integrable on the interval $[a, b]$.

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\mathbf{k}$$