Chapter 11

Vectors and Vector-Valued Functions

11.1

Vectors in the Plane

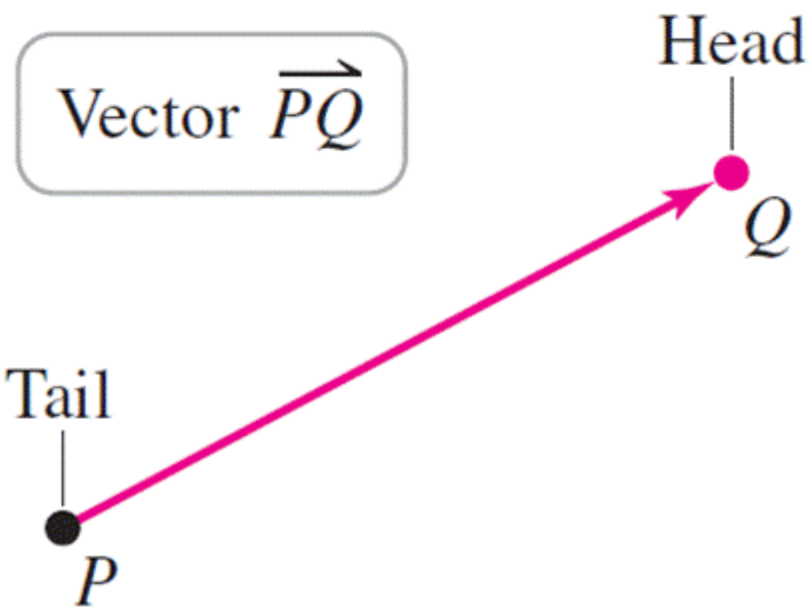


FIGURE 11.3

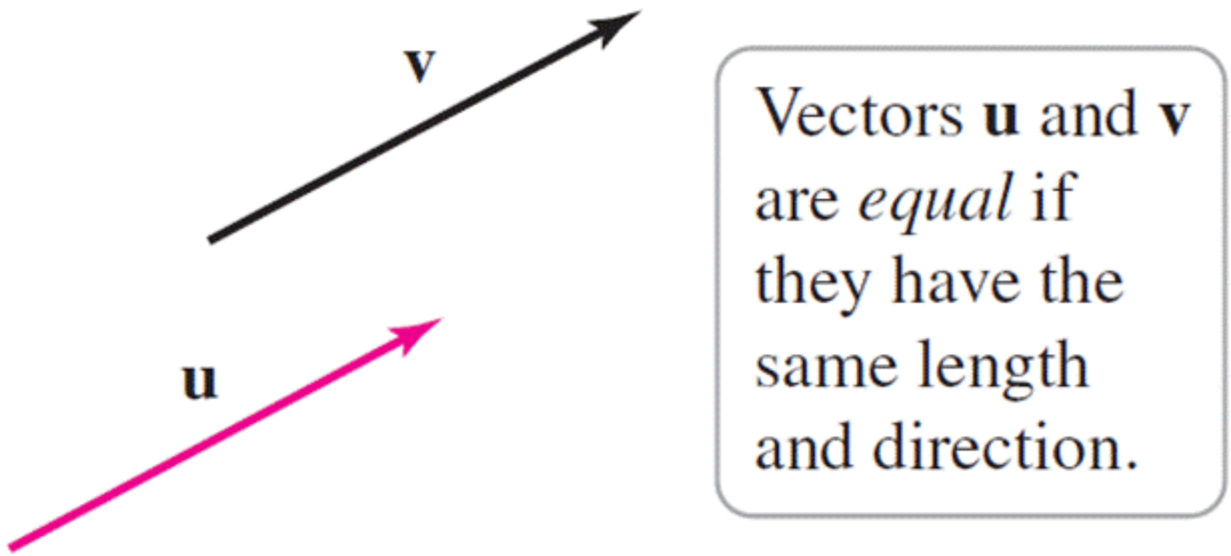

v
u
Vectors **u** and **v** are *equal* if they have the same length and direction.

FIGURE 11.4

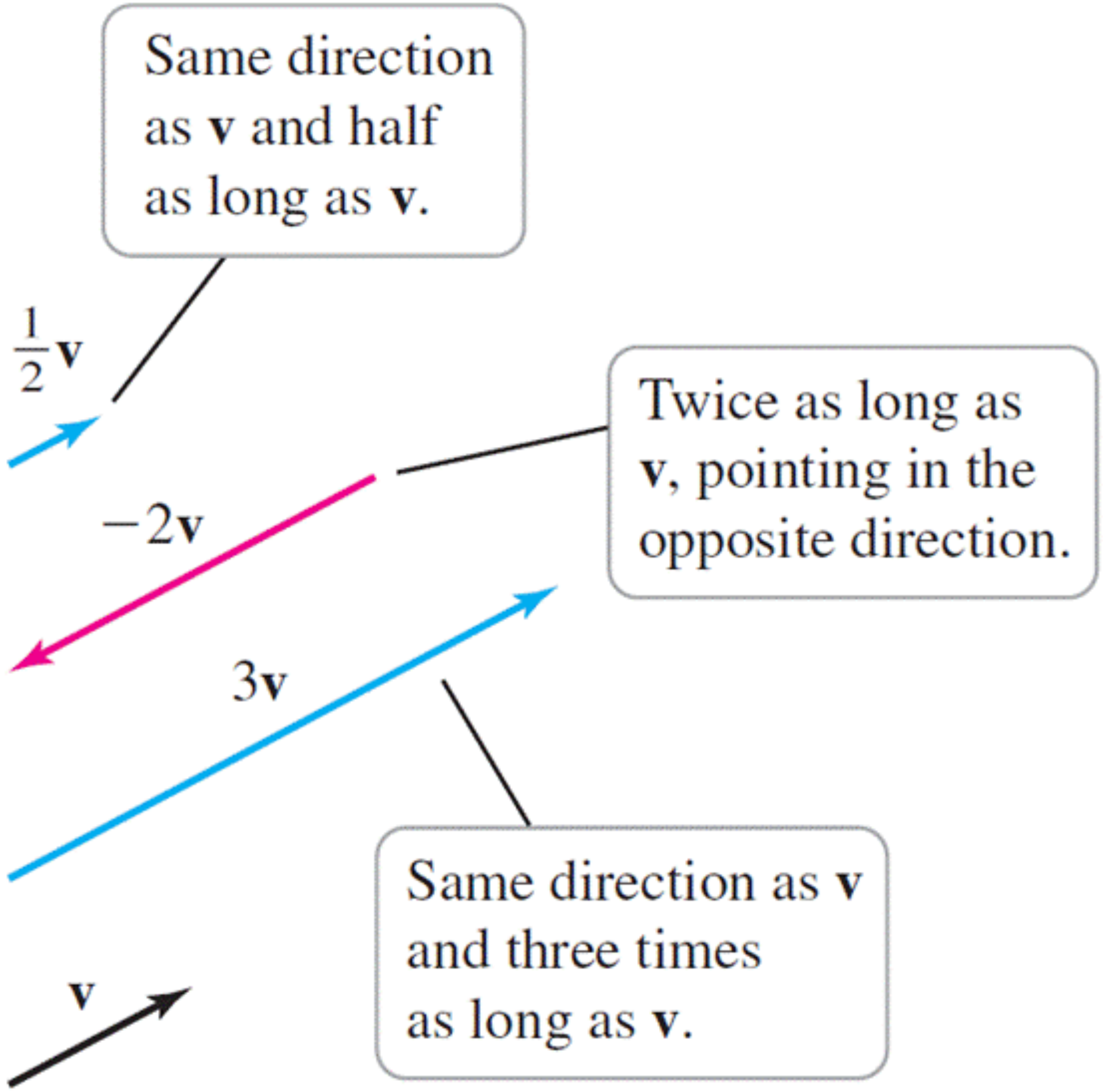
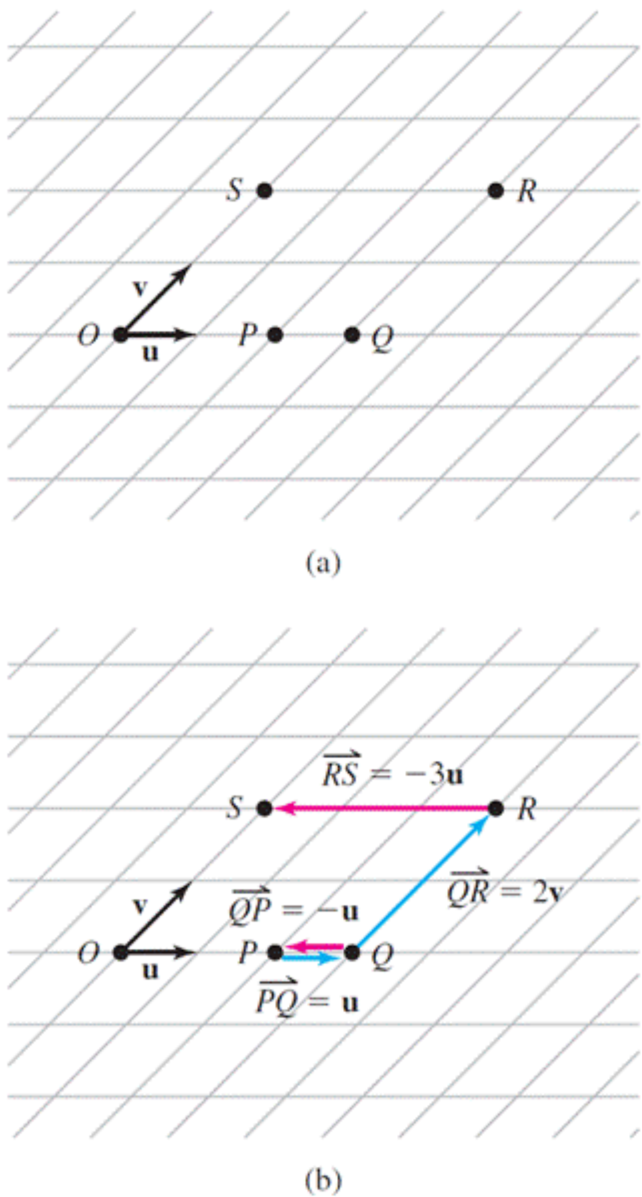

Same direction
as **v** and half
as long as **v**.
$\frac{1}{2}\mathbf{v}$
Twice as long as
v, pointing in the
opposite direction.
$-2\mathbf{v}$
$3\mathbf{v}$
Same direction as **v**
and three times
as long as **v**.
$\mathbf{v}$

FIGURE 11.5

DEFINITION **Scalar Multiples and Parallel Vectors**

Given a scalar c and a vector $\mathbf{v}$, the **scalar multiple** $c\mathbf{v}$ is a vector whose magnitude is $|c|$ multiplied by the magnitude of $\mathbf{v}$. If $c > 0$, then $c\mathbf{v}$ has the same direction as $\mathbf{v}$. If $c < 0$, then $c\mathbf{v}$ and $\mathbf{v}$ point in opposite directions. Two vectors are **parallel** if they are scalar multiples of each other.



S
R
v
O
u
P
Q
(a)
$\overrightarrow{RS} = -3\mathbf{u}$
S
R
$\overrightarrow{QR} = 2\mathbf{v}$
v
$\overrightarrow{QP} = -\mathbf{u}$
O
u
P
Q
$\overrightarrow{PQ} = \mathbf{u}$
(b)

FIGURE 11.6

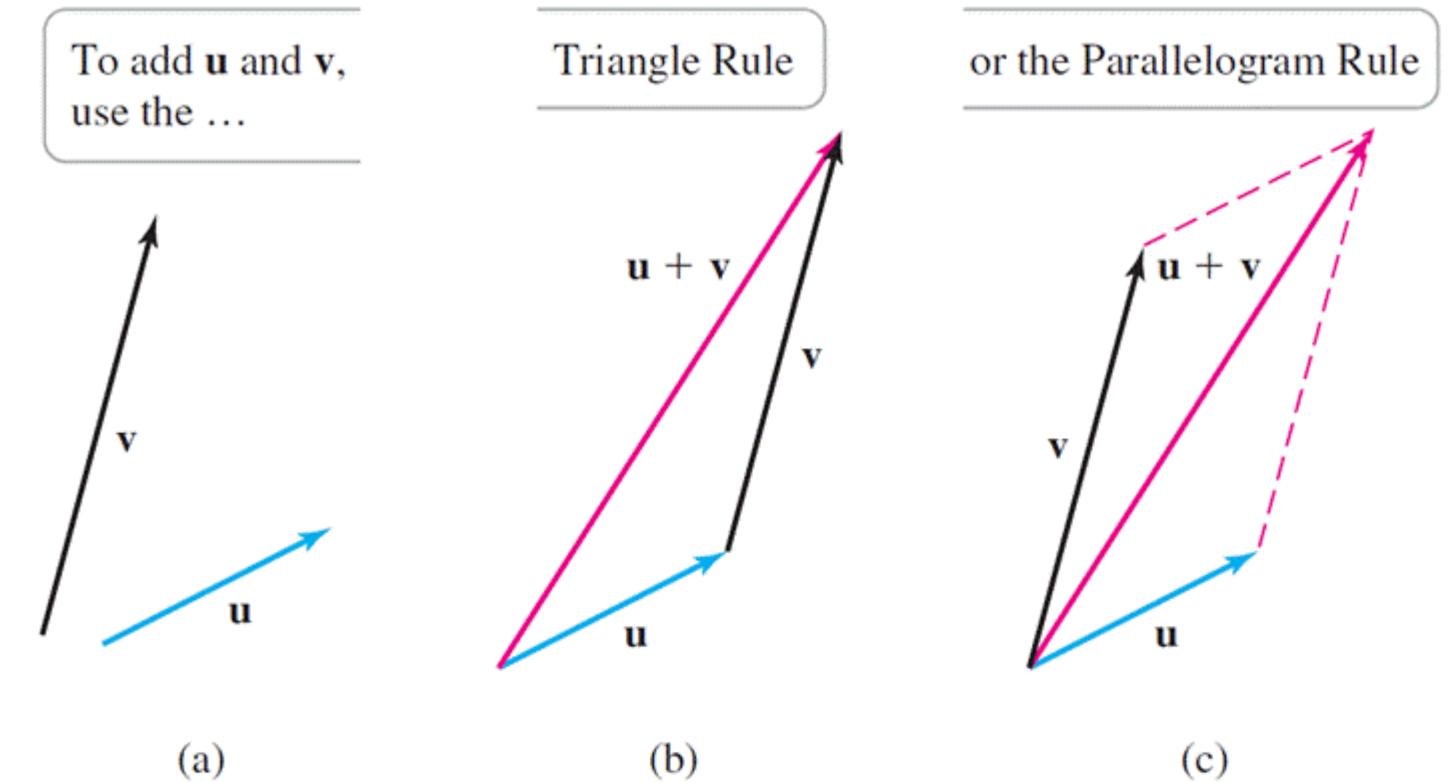

To add **u** and **v**, use the …
Triangle Rule
or the Parallelogram Rule
v
u
u + v
v
u
v
u + v
u
(a)
(b)
(c)

FIGURE 11.8

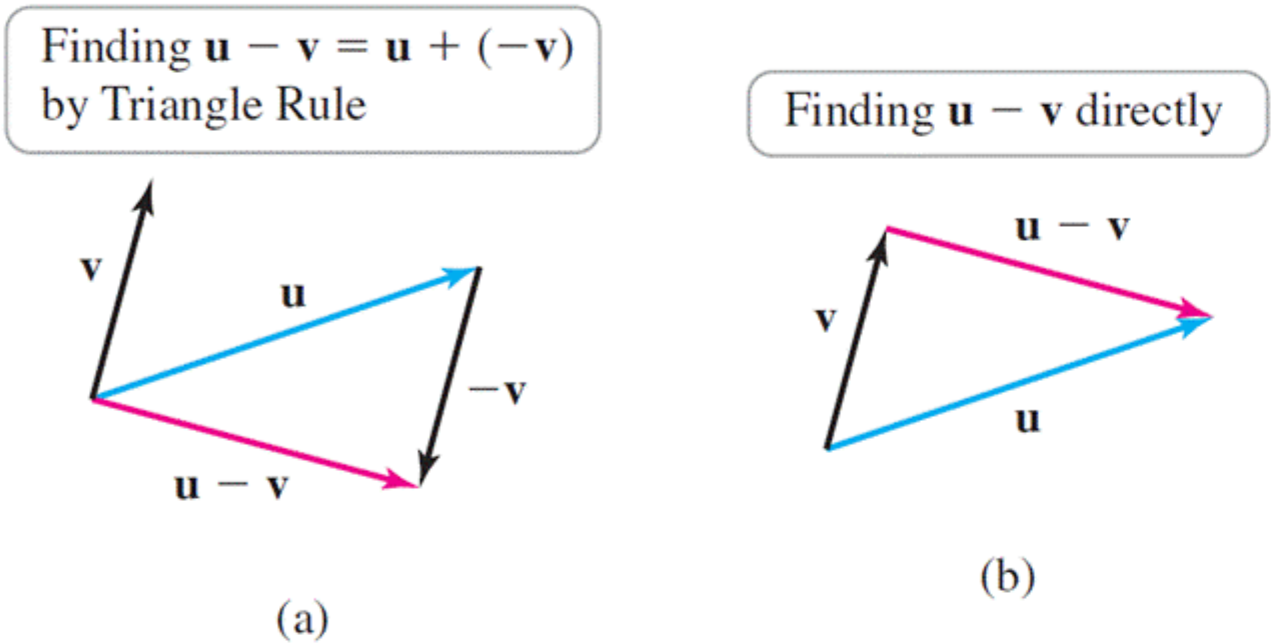

Finding $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$ by Triangle Rule
v
u
−v
u − v
(a)
Finding $\mathbf{u} - \mathbf{v}$ directly
u − v
v
u
(b)

FIGURE 11.9

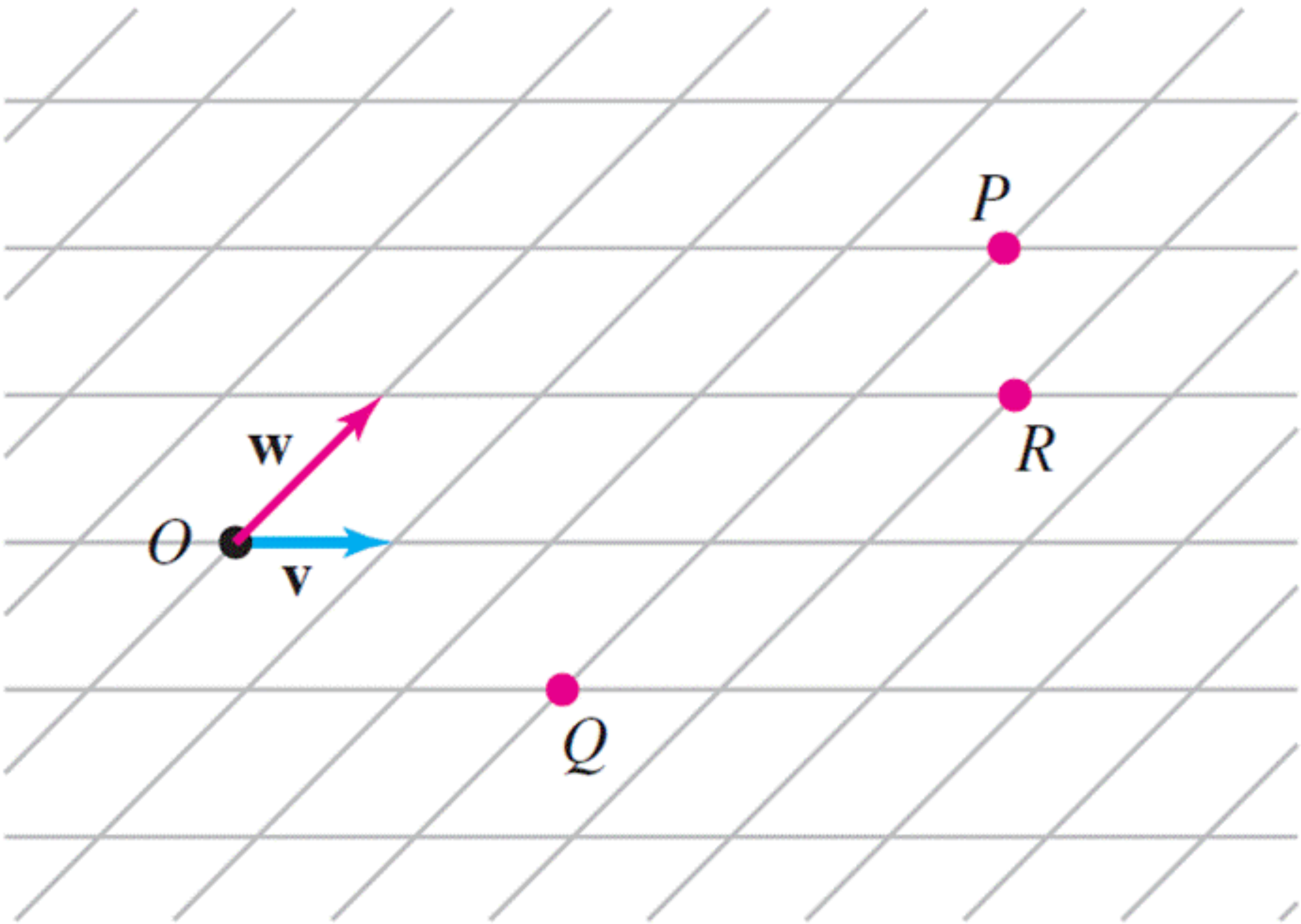


FIGURE 11.10

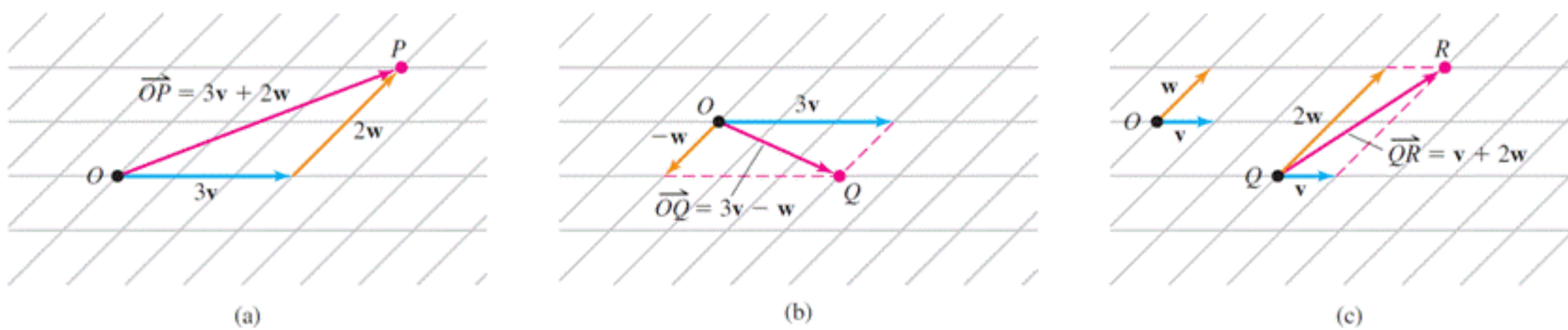
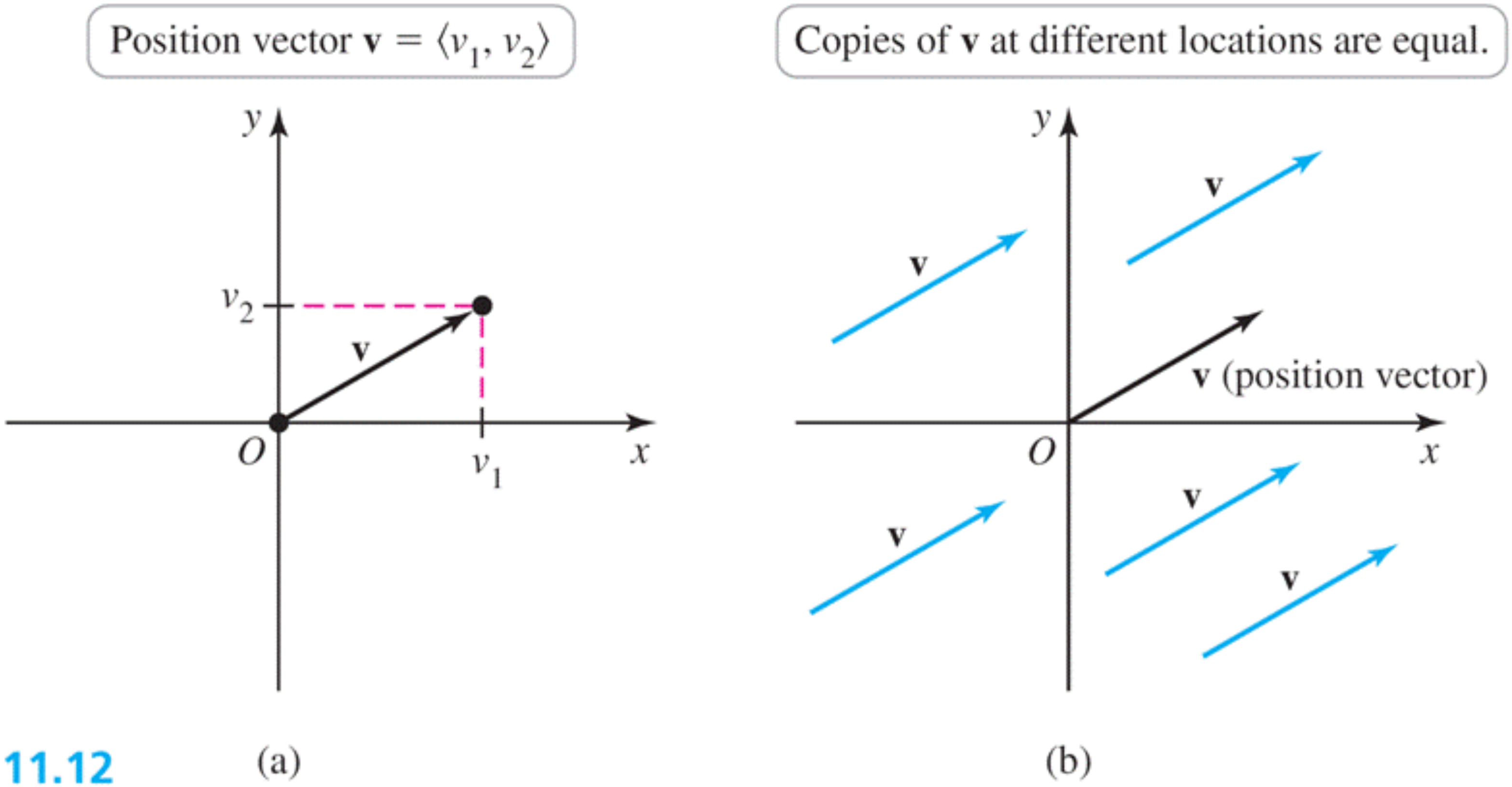

$\overrightarrow{OP} = 3\mathbf{v} + 2\mathbf{w}$
P
2w
O
3v
(a)
O
3v
−w
$\overrightarrow{OQ} = 3\mathbf{v} - \mathbf{w}$
Q
(b)
w
O
v
R
2w
$\overrightarrow{QR} = \mathbf{v} + 2\mathbf{w}$
Q
v
(c)

FIGURE 11.11

DEFINITION **Position Vectors and Vector Components**

A vector $\mathbf{v}$ with its tail at the origin and head at (v_1, v_2) is called a **position vector** (or is said to be in **standard position**) and is written $\langle v_1, v_2 \rangle$. The real numbers v_1 and v_2 are the x- and y-**components** of $\mathbf{v}$, respectively. The position vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are **equal** if and only if $u_1 = v_1$ and $u_2 = v_2$.



Position vector $\mathbf{v} = \langle v_1, v_2 \rangle$
y
v_2
v
O
v_1
x
(a)
Copies of **v** at different locations are equal.
y
v
v
v (position vector)
O
x
v
v
v
(b)

FIGURE 11.12

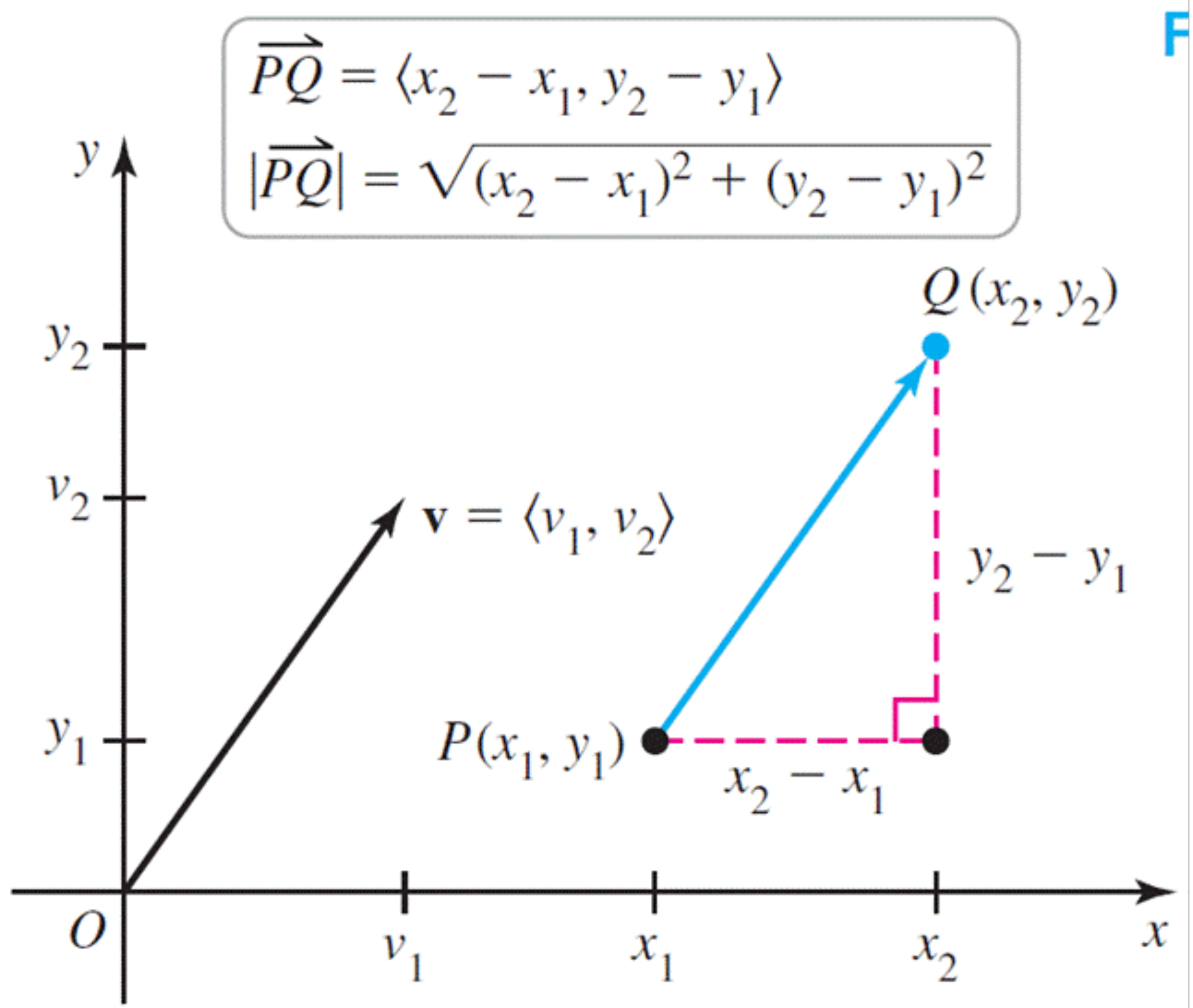

$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$
$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
y
y_2
v_2
y_1
O
v_1
x_1
x_2
x
$\mathbf{v} = \langle v_1, v_2 \rangle$
$Q(x_2, y_2)$
$P(x_1, y_1)$
$y_2 - y_1$
$x_2 - x_1$

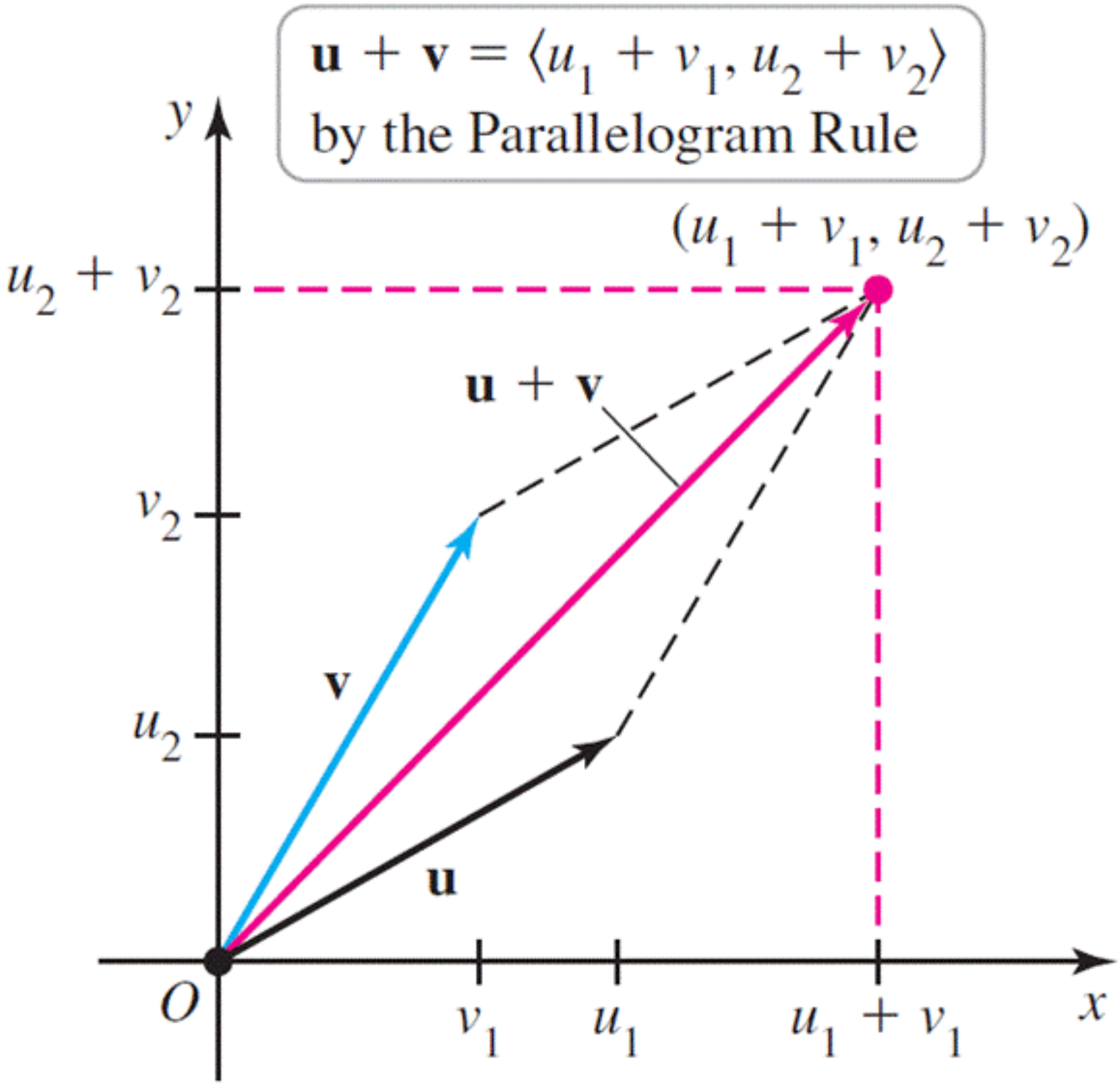
FIGURE 11.13

DEFINITION **Magnitude of a Vector**

Given the points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the **magnitude**, or **length**, of $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$, denoted $|\overrightarrow{PQ}|$, is the distance between P and Q:

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The magnitude of the position vector $\mathbf{v} = \langle v_1, v_2 \rangle$ is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$.



$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$
by the Parallelogram Rule
$(u_1 + v_1, u_2 + v_2)$
y
$u_2 + v_2$
v_2
u_2
$\mathbf{u} + \mathbf{v}$
$\mathbf{v}$
$\mathbf{u}$
O
v_1
u_1
$u_1 + v_1$
x

FIGURE 11.14

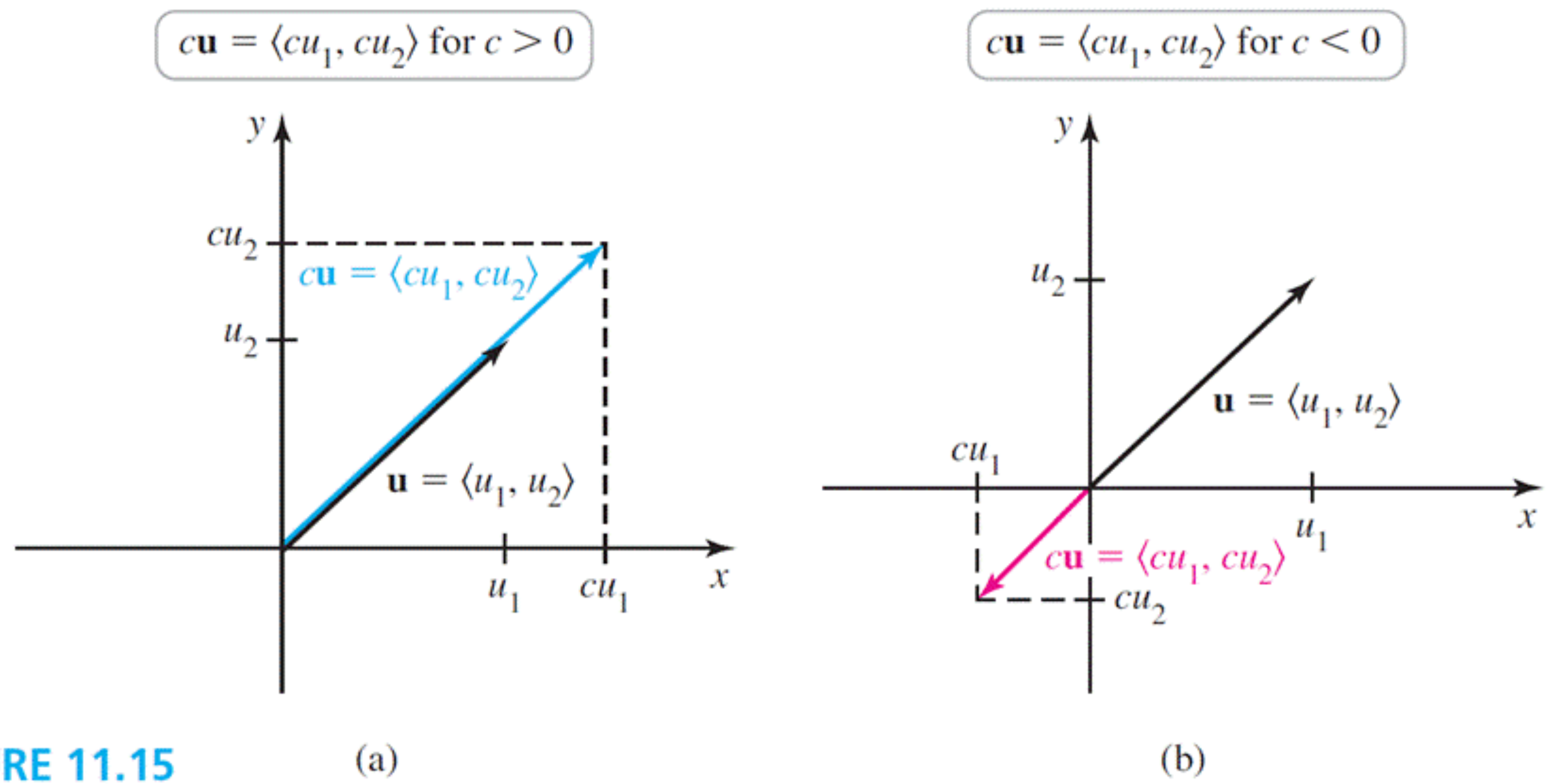

$c\mathbf{u} = \langle cu_1, cu_2 \rangle$ for $c > 0$
$c\mathbf{u} = \langle cu_1, cu_2 \rangle$ for $c < 0$
y
cu_2
u_2
$c\mathbf{u} = \langle cu_1, cu_2 \rangle$
$\mathbf{u} = \langle u_1, u_2 \rangle$
u_1
cu_1
x
(a)
y
u_2
$\mathbf{u} = \langle u_1, u_2 \rangle$
cu_1
u_1
x
$c\mathbf{u} = \langle cu_1, cu_2 \rangle$
cu_2
(b)

FIGURE 11.15

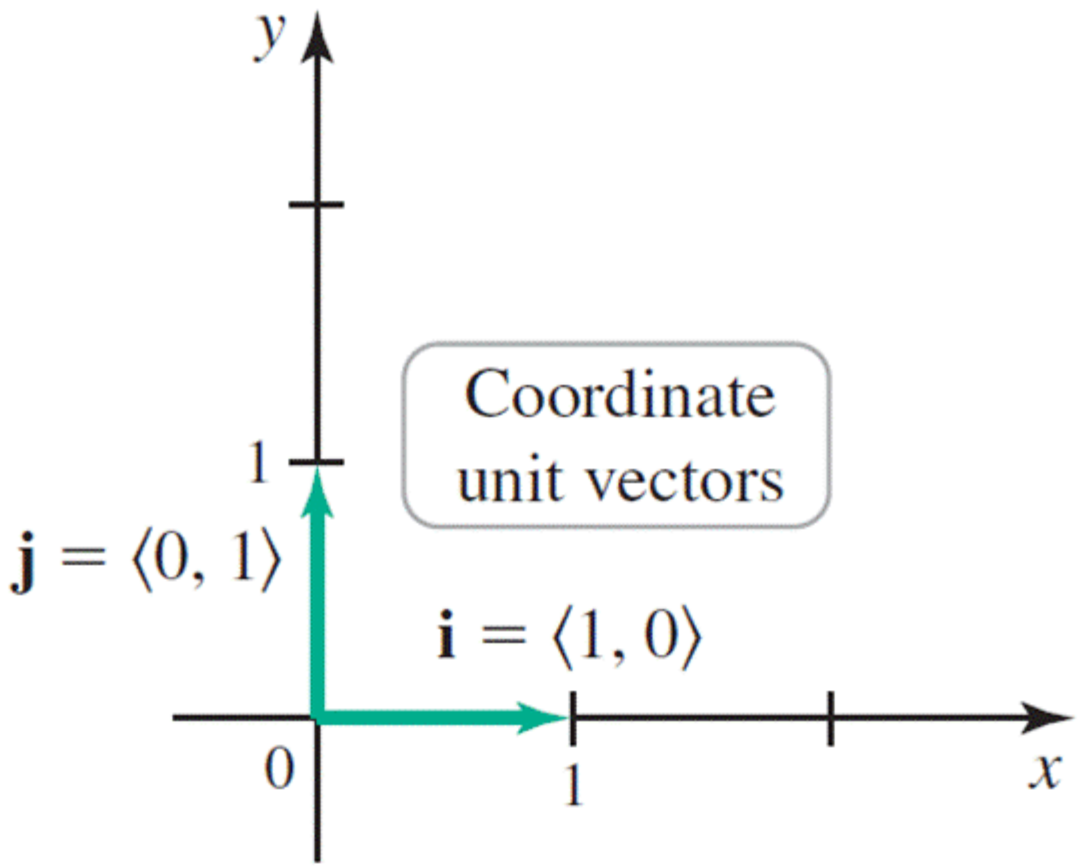

y
1
Coordinate
unit vectors
$\mathbf{j} = \langle 0, 1 \rangle$
$\mathbf{i} = \langle 1, 0 \rangle$
0
1
x

FIGURE 11.16

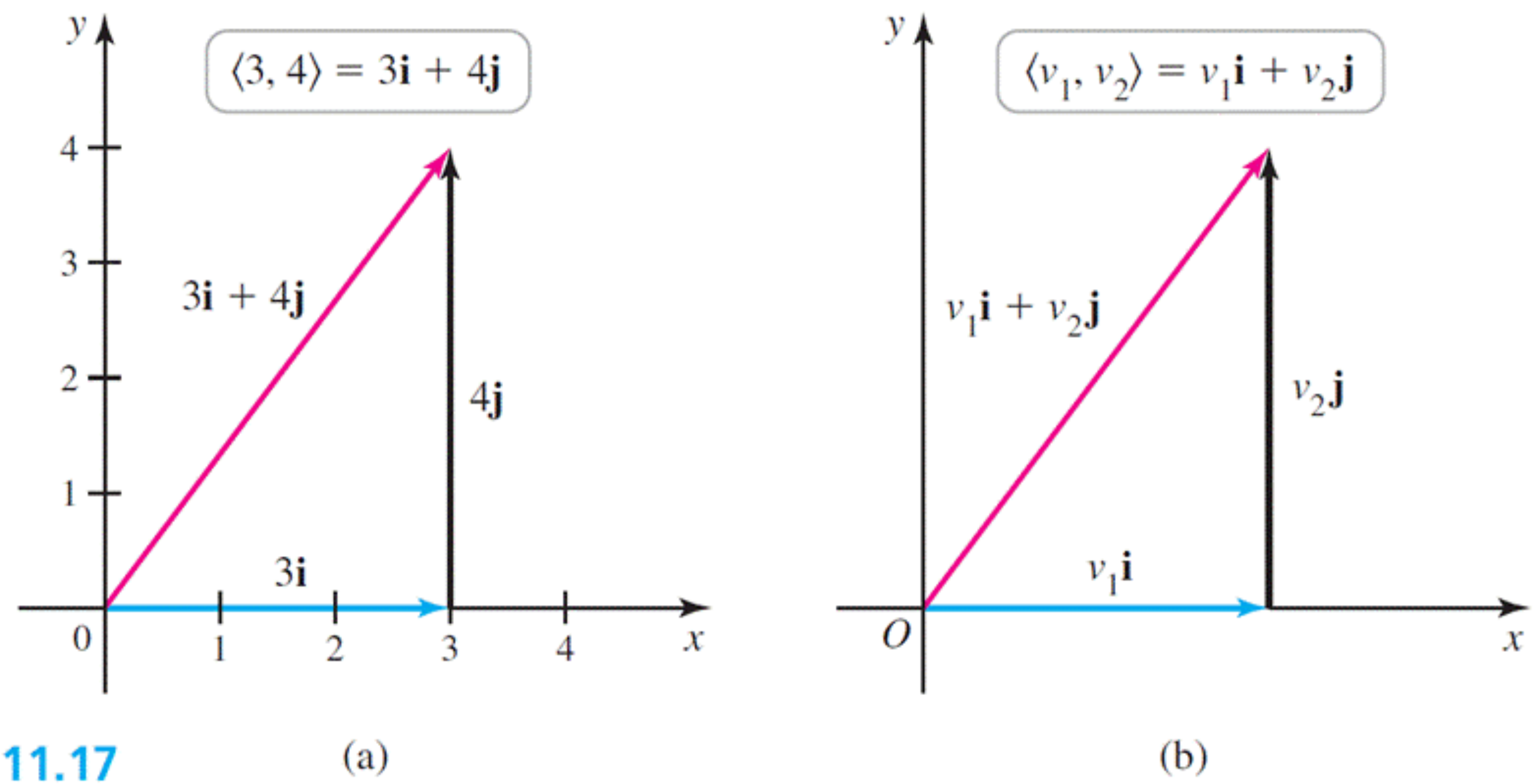

$\langle 3, 4\rangle = 3\mathbf{i} + 4\mathbf{j}$
$3\mathbf{i} + 4\mathbf{j}$
$4\mathbf{j}$
$3\mathbf{i}$
y
4
3
2
1
0
1
2
3
4
x
(a)
$\langle v_1, v_2\rangle = v_1\mathbf{i} + v_2\mathbf{j}$
$v_1\mathbf{i} + v_2\mathbf{j}$
$v_2\mathbf{j}$
$v_1\mathbf{i}$
y
O
x
(b)

FIGURE 11.17

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} \text{ and } -\mathbf{u} = -\frac{\mathbf{v}}{|\mathbf{v}|} \text{ have length 1.}$$

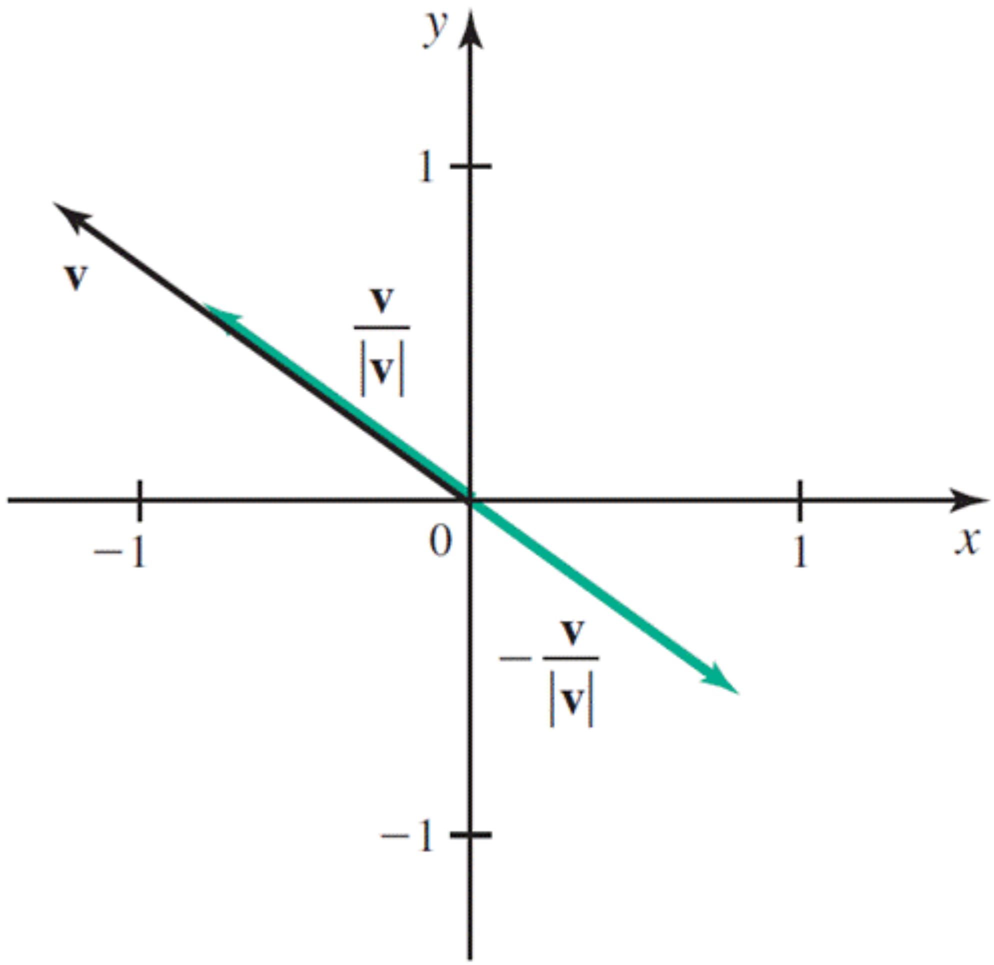


FIGURE 11.18

DEFINITION **Unit Vectors and Vectors of a Specified Length**

A **unit vector** is any vector with length 1. Given a nonzero vector $\mathbf{v}$, $\pm\dfrac{\mathbf{v}}{|\mathbf{v}|}$ are unit vectors parallel to $\mathbf{v}$. For a scalar $c > 0$, the vectors $\pm\dfrac{c\mathbf{v}}{|\mathbf{v}|}$ are vectors of length c parallel to $\mathbf{v}$.

11.2

Vectors in Three Dimensions

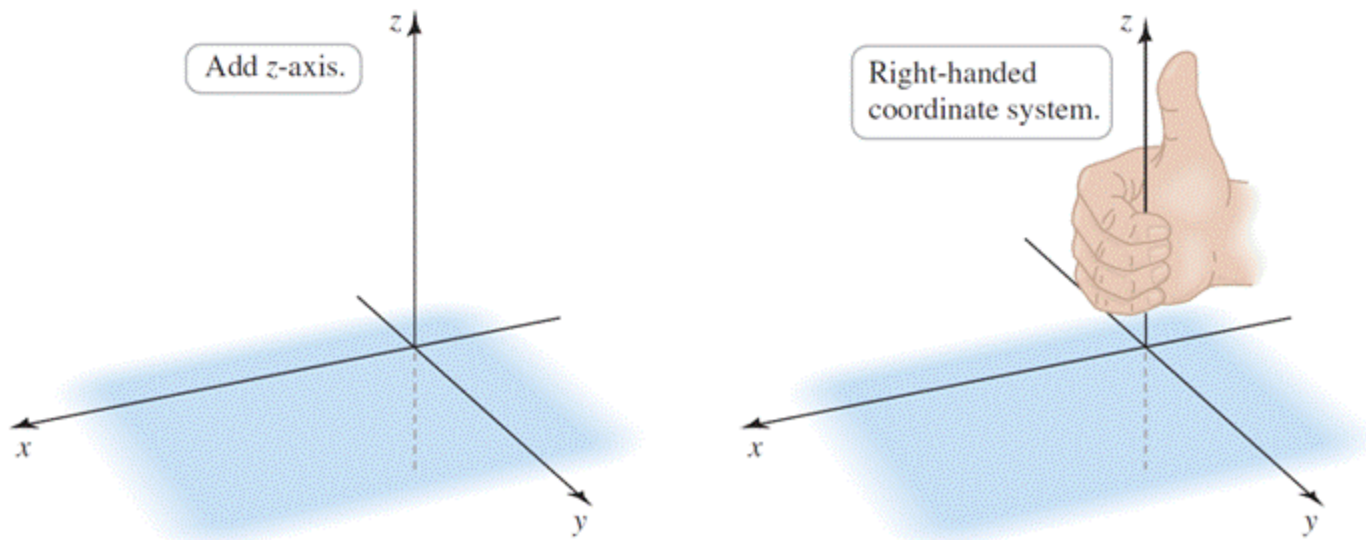


FIGURE 11.25

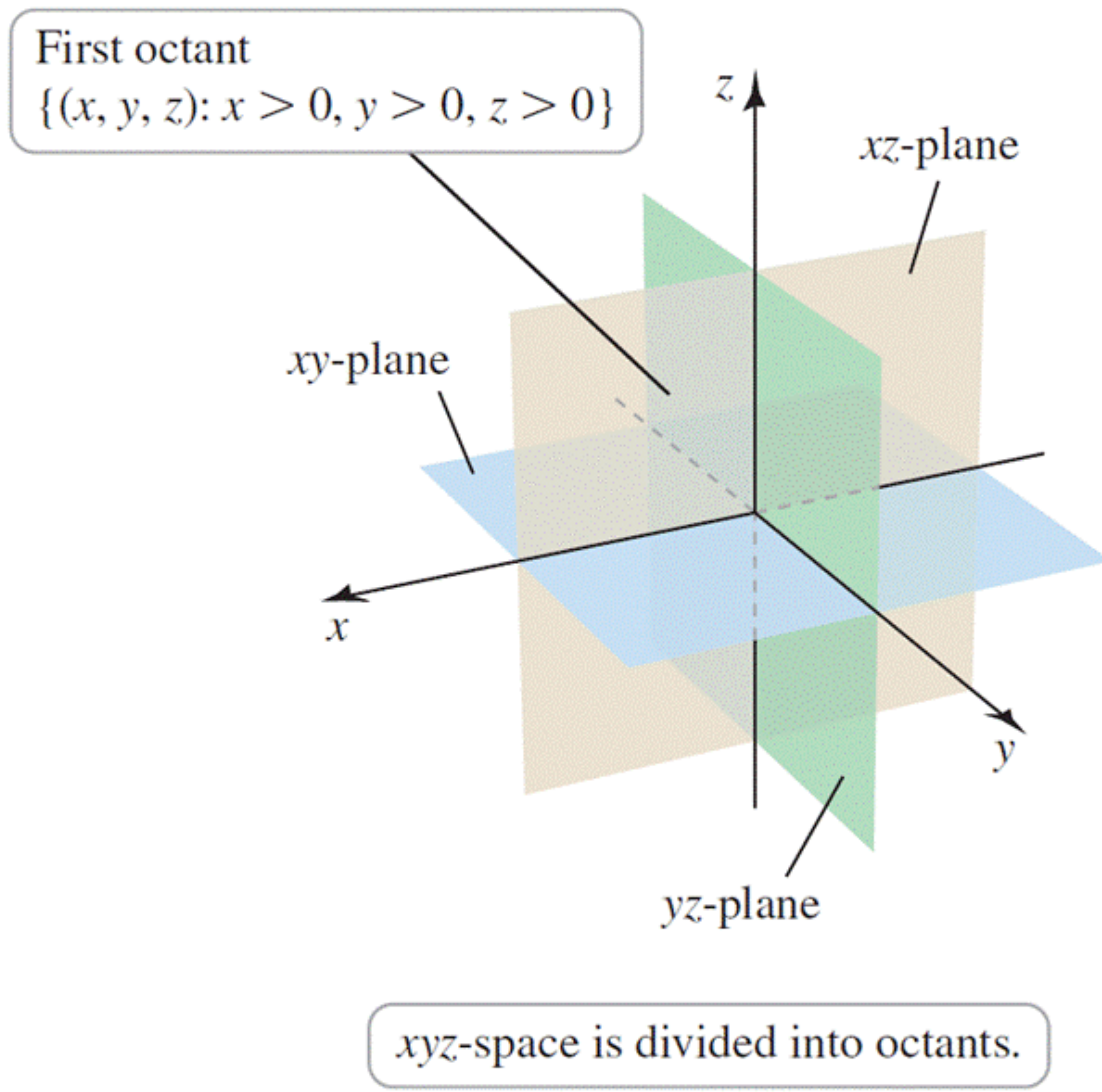

First octant
$\{(x, y, z): x > 0, y > 0, z > 0\}$
z
xz-plane
xy-plane
x
y
yz-plane
xyz-space is divided into octants.

FIGURE 11.26

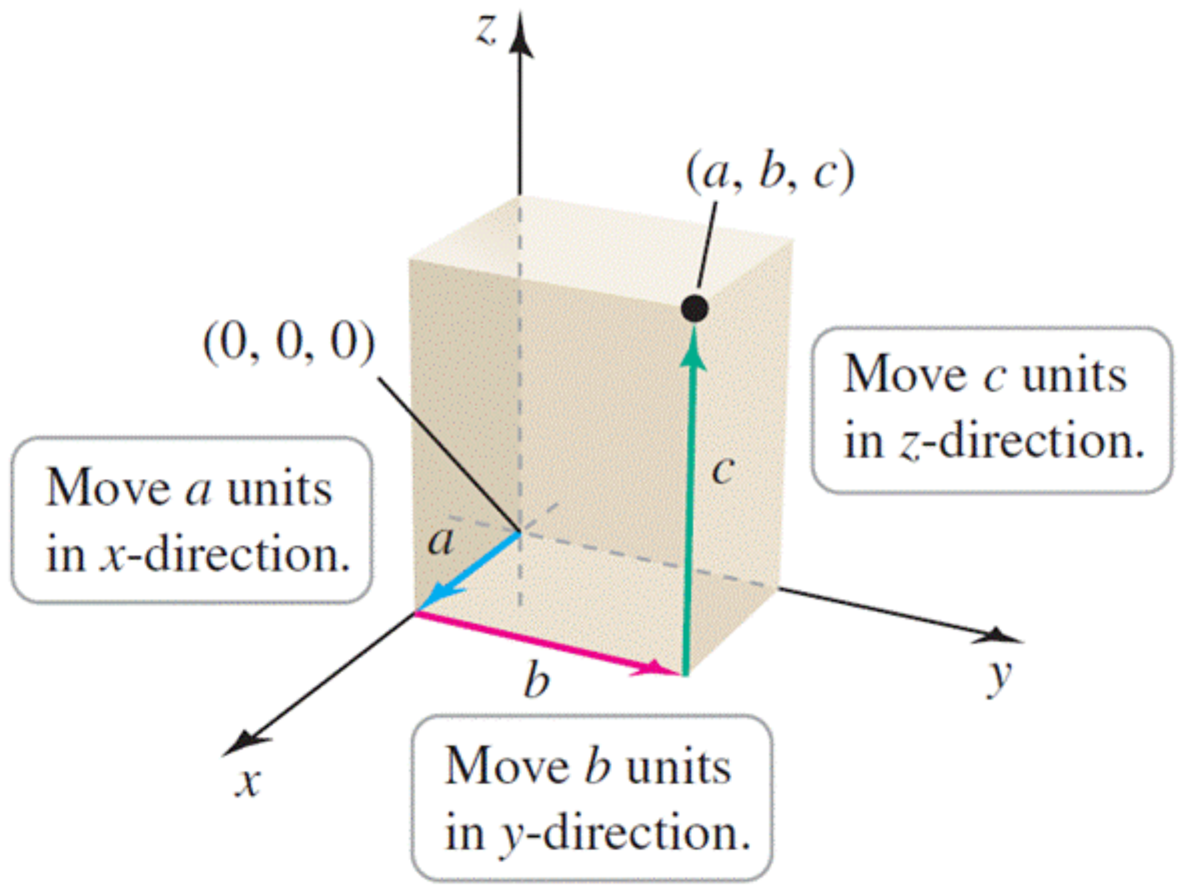

z
(a, b, c)
(0, 0, 0)
Move c units
in z-direction.
Move a units
in x-direction.
a
c
b
y
x
Move b units
in y-direction.

FIGURE 11.27

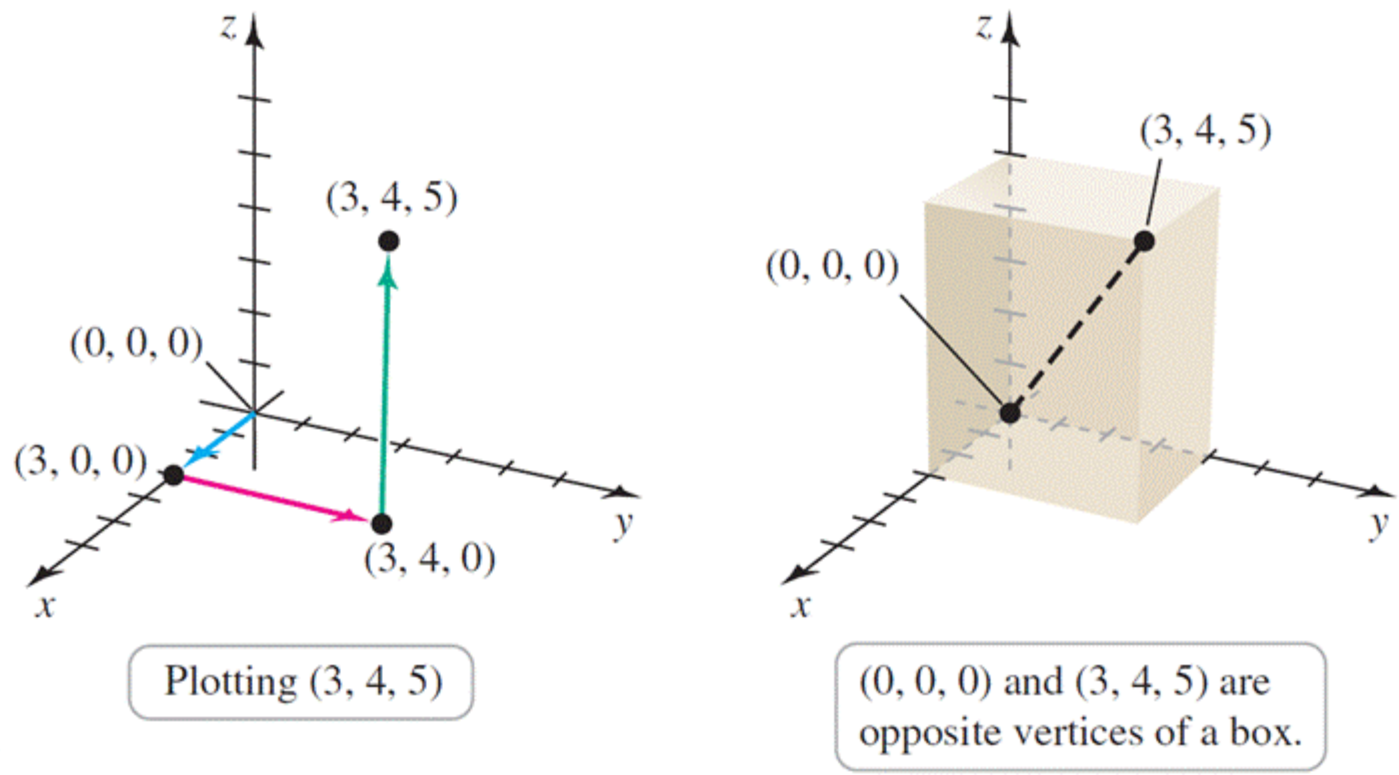

z
(3, 4, 5)
(0, 0, 0)
(3, 0, 0)
(3, 4, 0)
y
x
Plotting (3, 4, 5)
z
(3, 4, 5)
(0, 0, 0)
y
x
(0, 0, 0) and (3, 4, 5) are opposite vertices of a box.

FIGURE 11.28

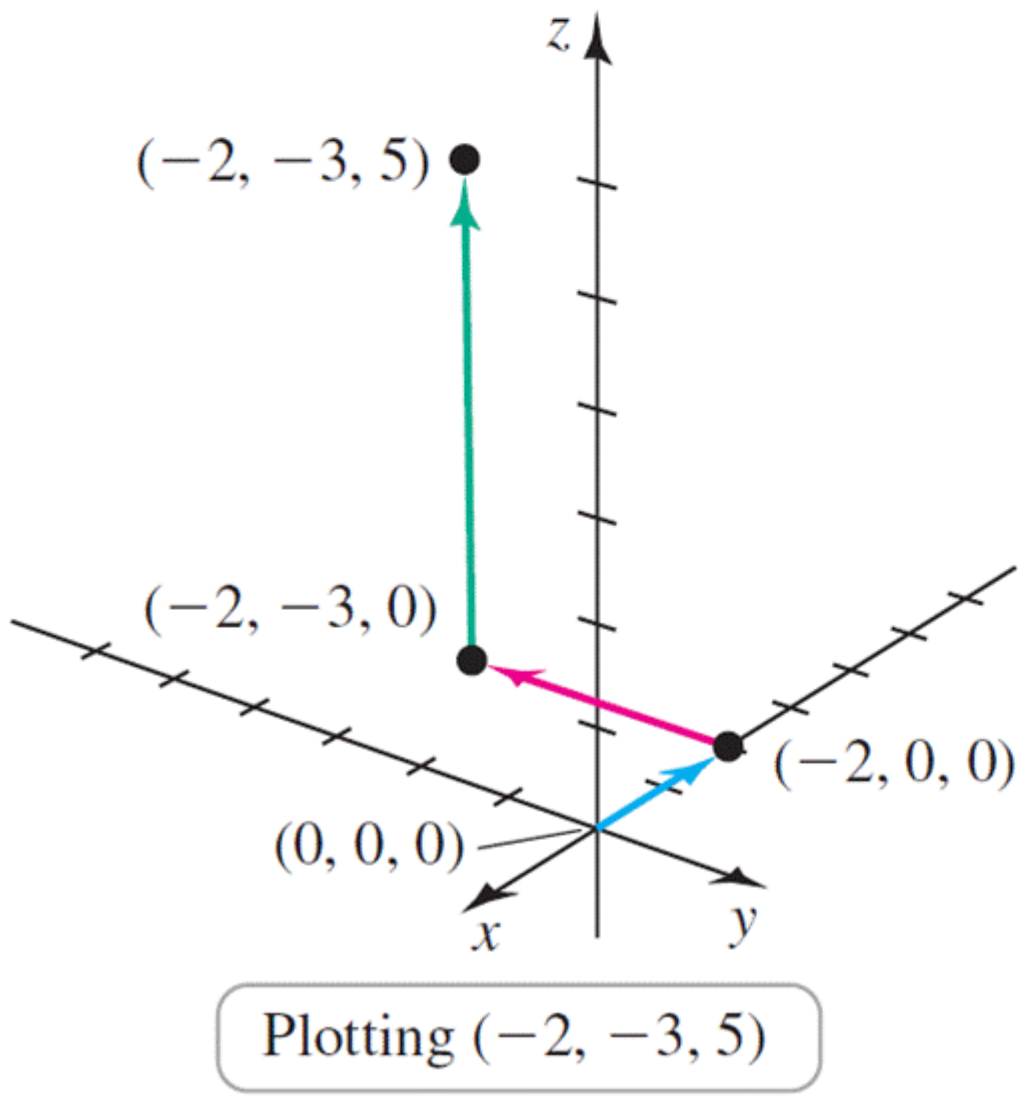


FIGURE 11.29

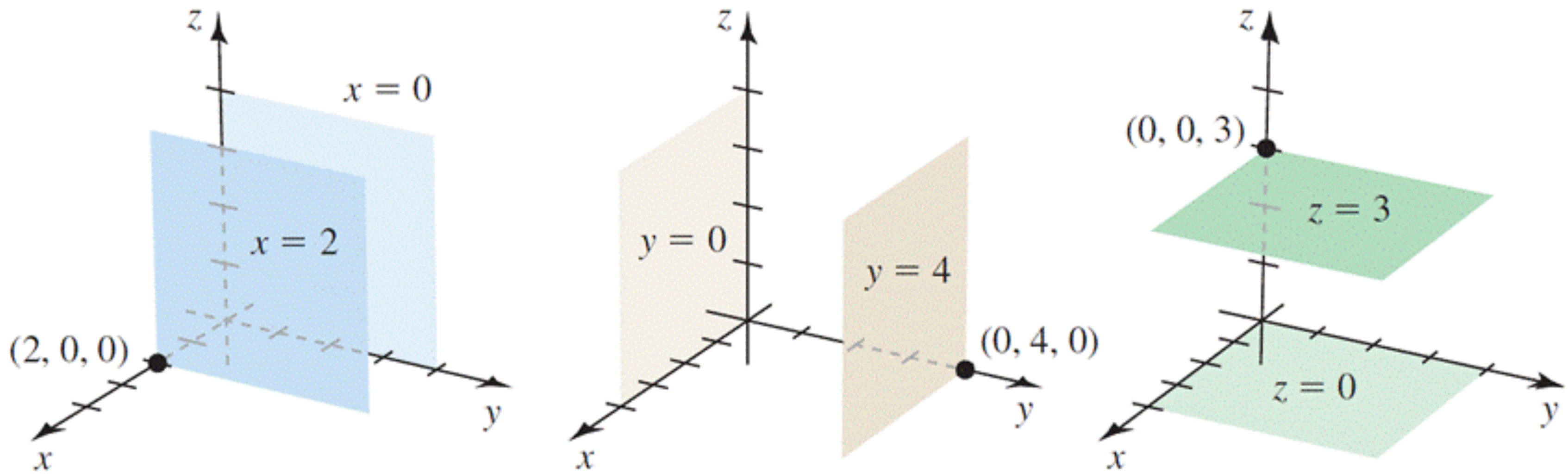
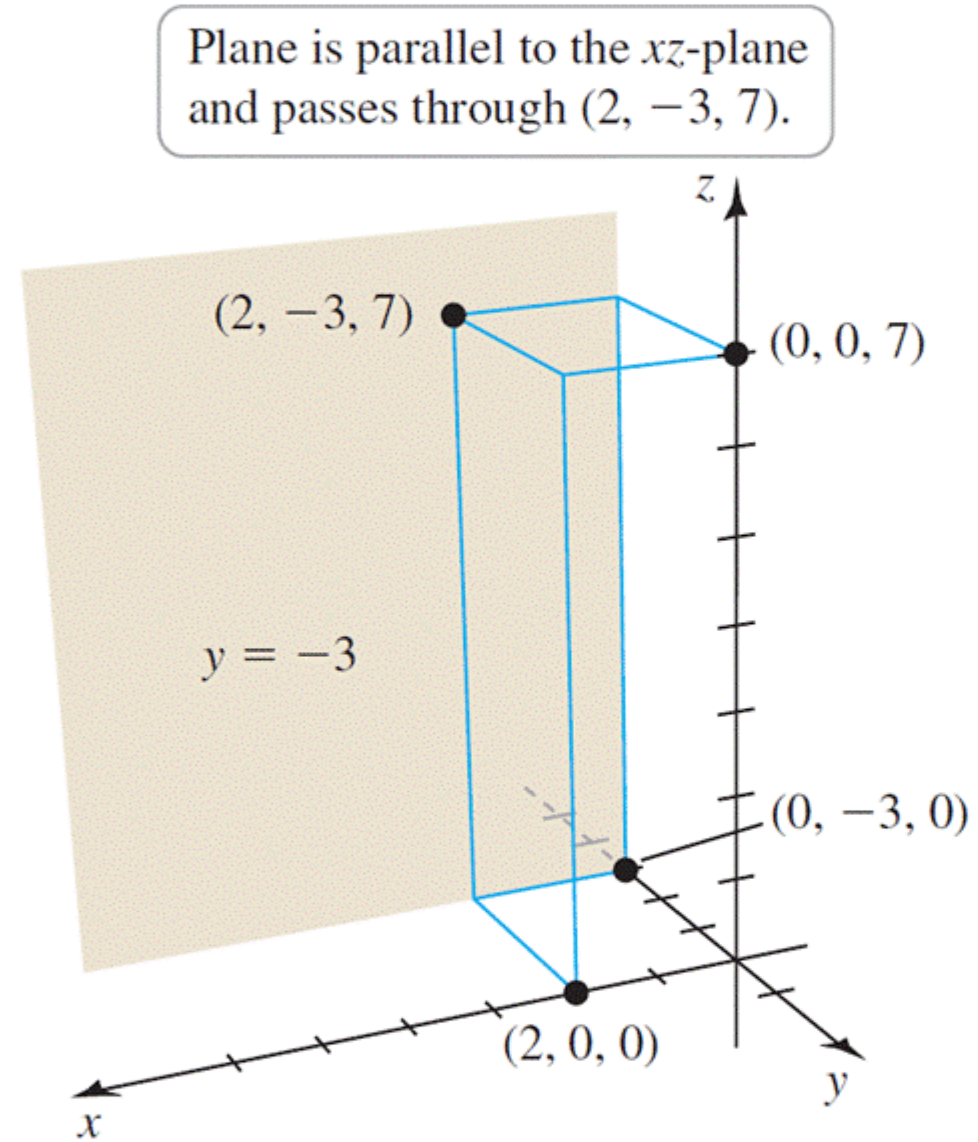
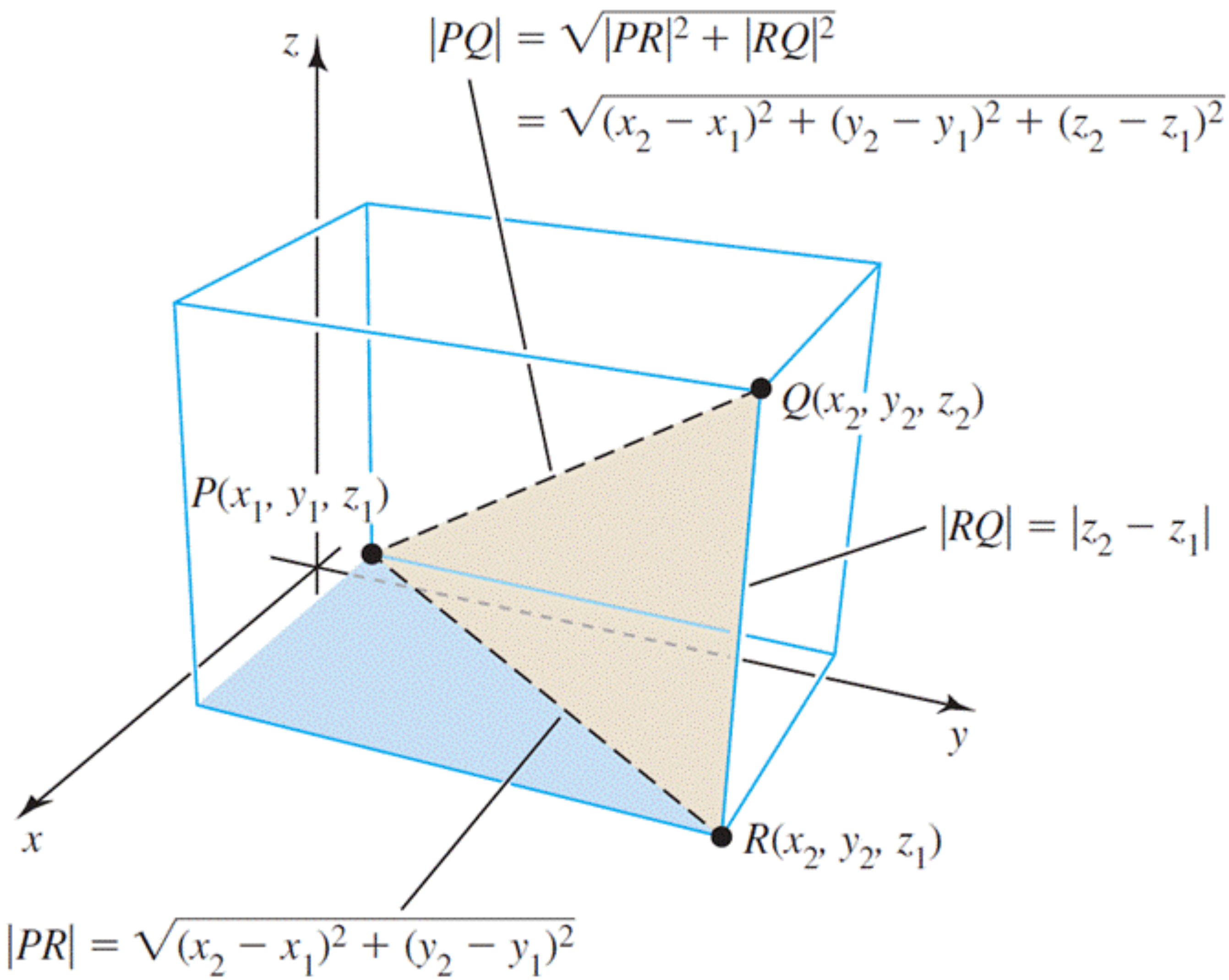


FIGURE 11.30



Plane is parallel to the xz-plane and passes through (2, −3, 7).
z
(2, −3, 7)
(0, 0, 7)
y = −3
(0, −3, 0)
(2, 0, 0)
y
x

FIGURE 11.31



z
$|PQ| = \sqrt{|PR|^2 + |RQ|^2}$
$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
$Q(x_2, y_2, z_2)$
$P(x_1, y_1, z_1)$
$|RQ| = |z_2 - z_1|$
y
x
$R(x_2, y_2, z_1)$
$|PR| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

FIGURE 11.32

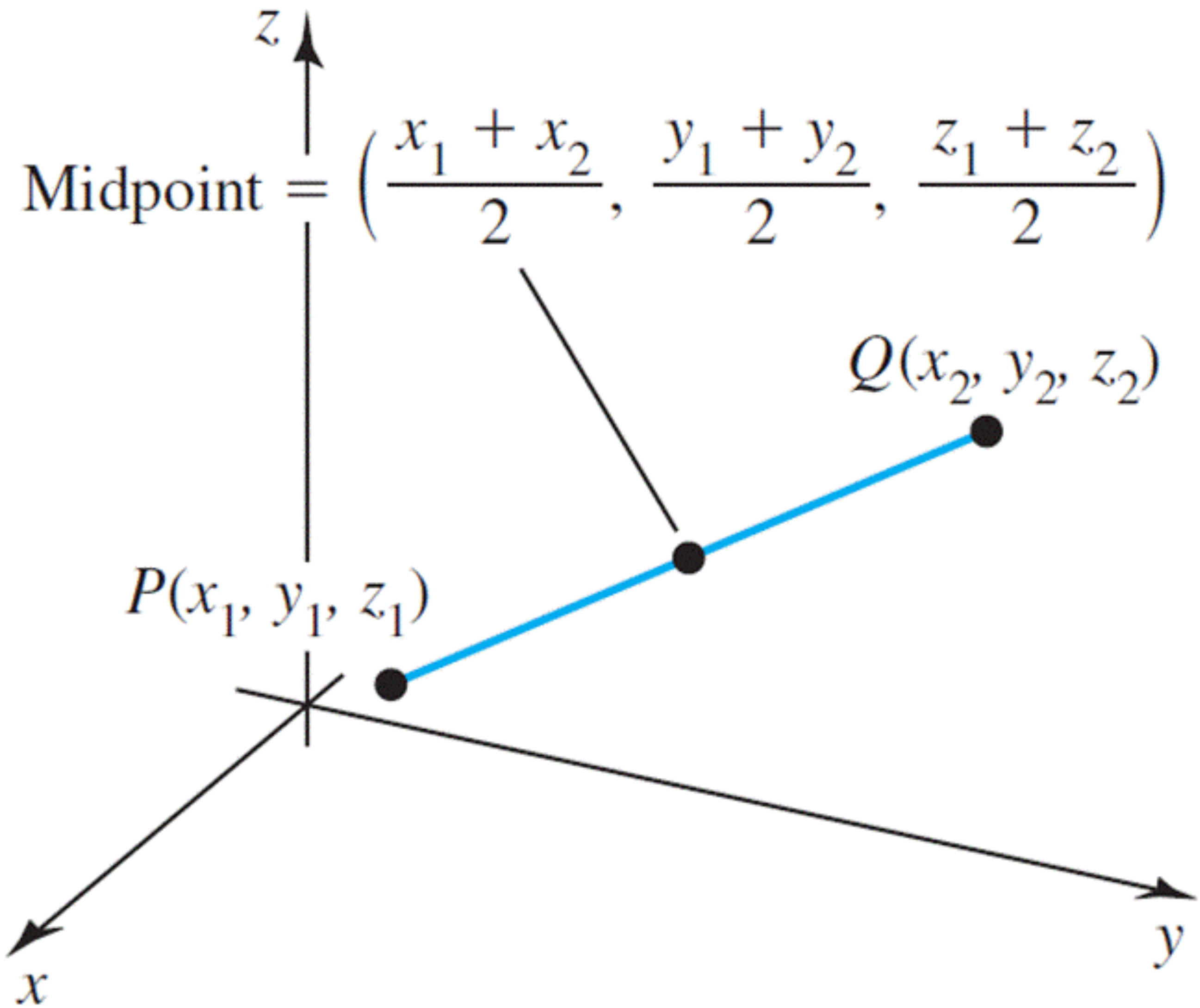
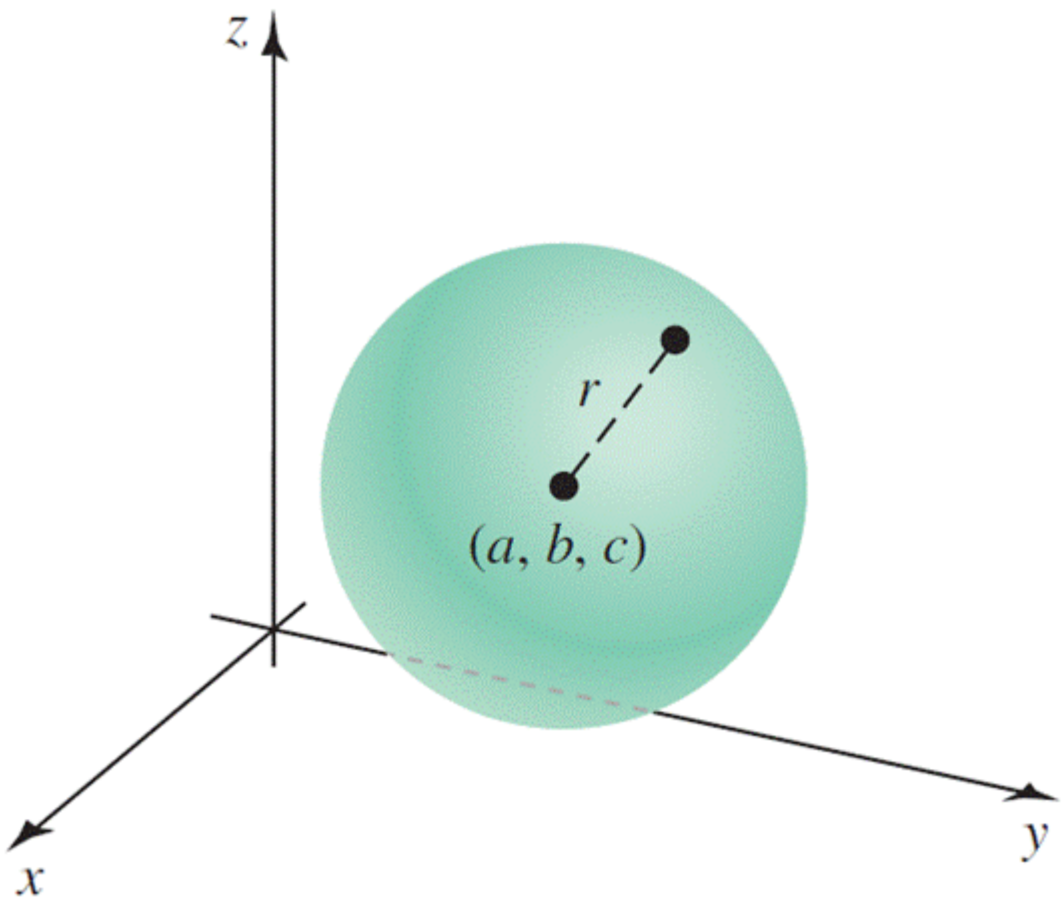

z
Midpoint = $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$
$Q(x_2, y_2, z_2)$
$P(x_1, y_1, z_1)$
x
y

FIGURE 11.33



$$\text{Sphere: } (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

$$\text{Ball: } (x-a)^2 + (y-b)^2 + (z-c)^2 \leq r^2$$

FIGURE 11.34

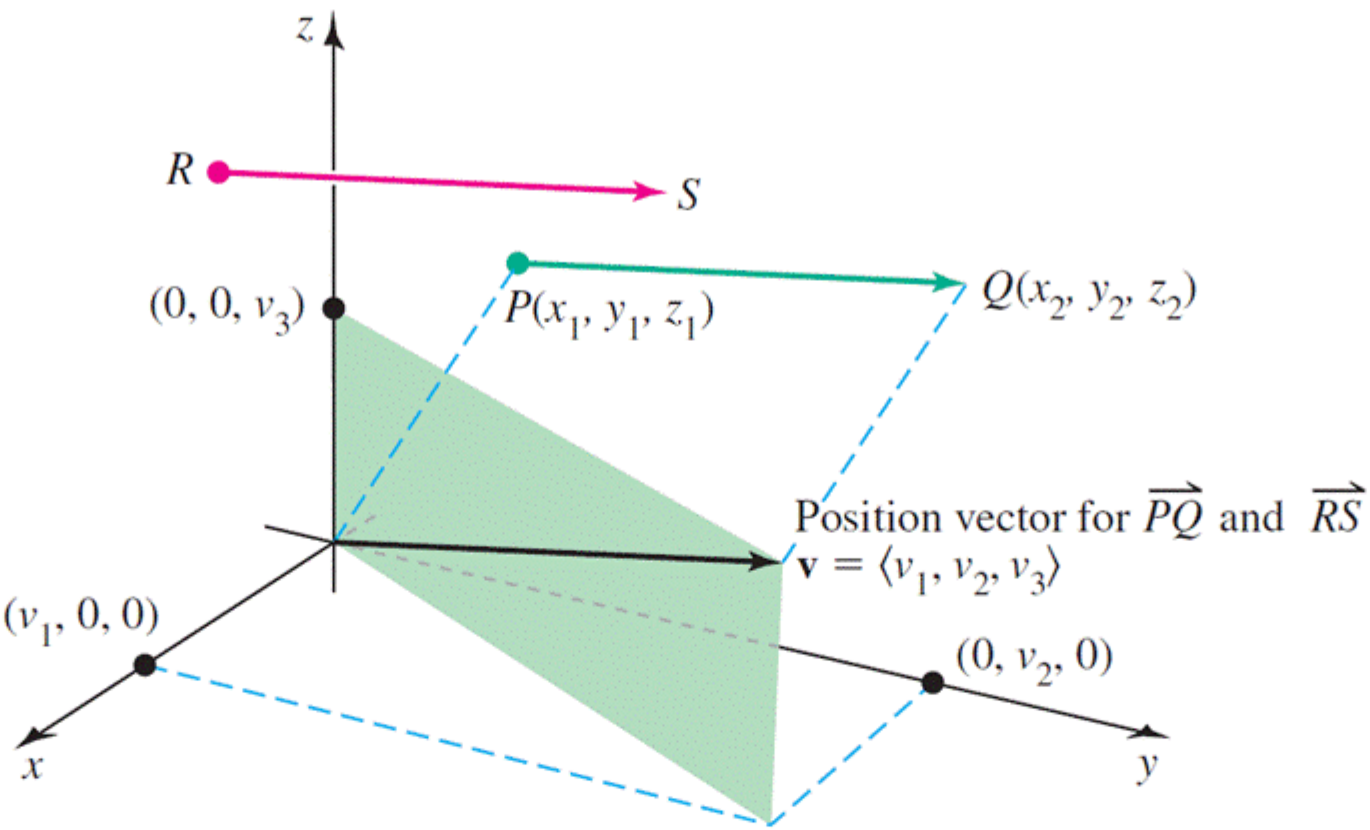
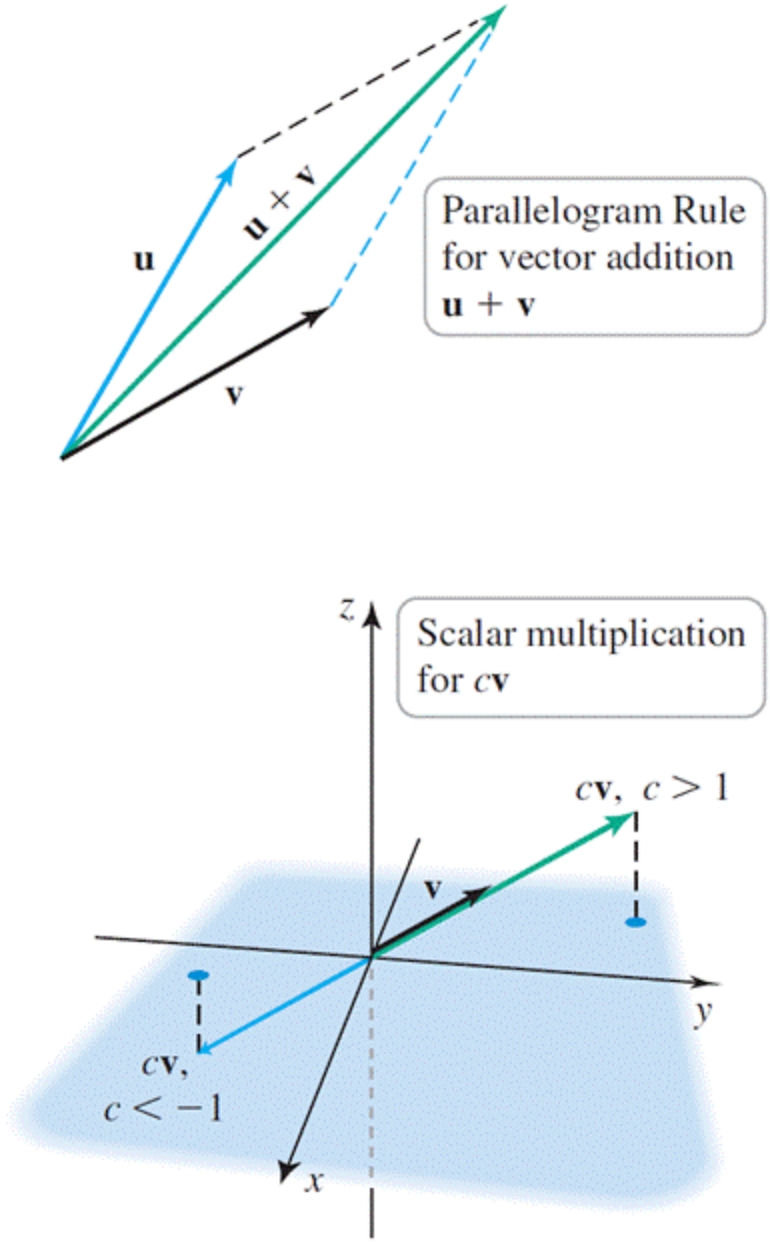
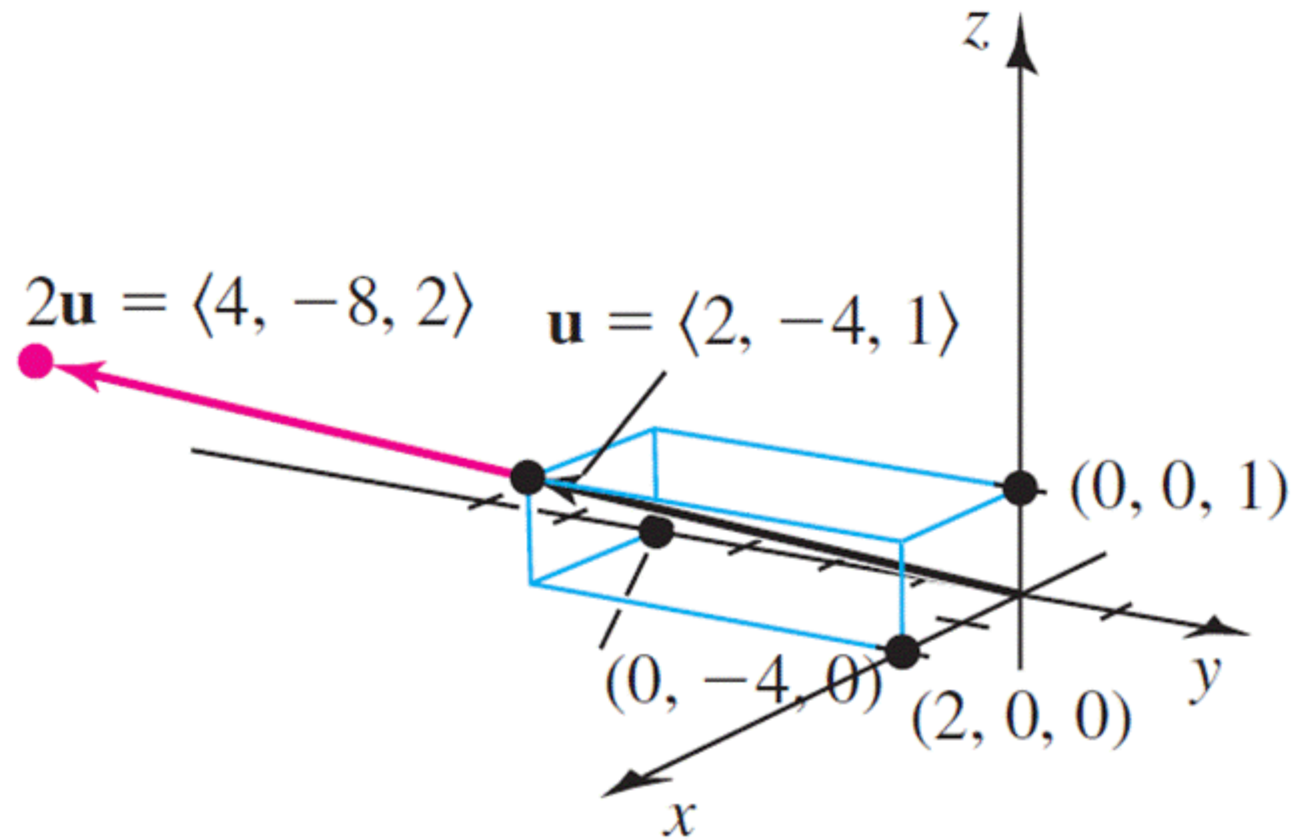

z
R
S
(0, 0, v_3)
$P(x_1, y_1, z_1)$
$Q(x_2, y_2, z_2)$
Position vector for $\overrightarrow{PQ}$ and $\overrightarrow{RS}$
$\mathbf{v} = \langle v_1, v_2, v_3 \rangle$
(v_1, 0, 0)
(0, v_2, 0)
x
y

FIGURE 11.35



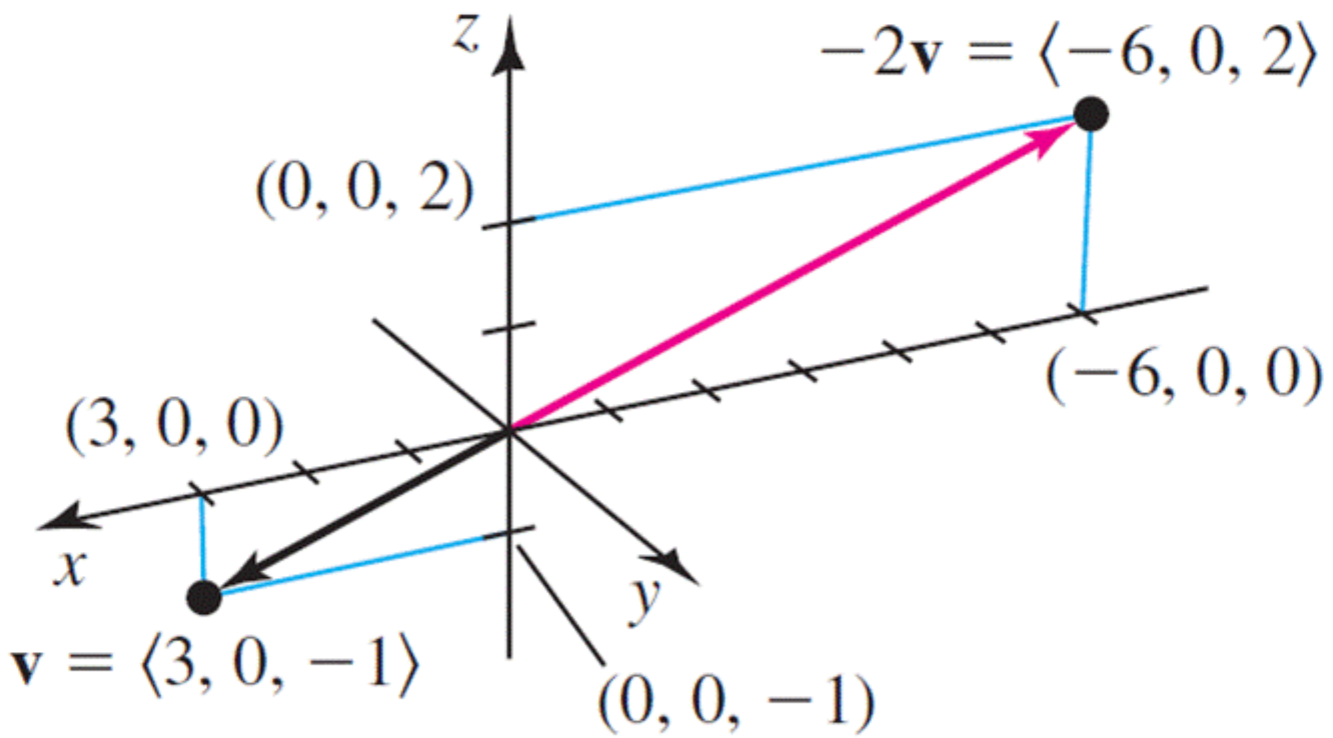
Parallelogram Rule
for vector addition
u + v
u
v
u + v
Scalar multiplication
for cv
z
cv, c > 1
v
y
cv,
c < −1
x

FIGURE 11.36



$2\mathbf{u} = \langle 4, -8, 2 \rangle$
$\mathbf{u} = \langle 2, -4, 1 \rangle$
z
$(0, 0, 1)$
$(0, -4, 0)$
$(2, 0, 0)$
y
x

FIGURE 11.37


z
$-2\mathbf{v} = \langle -6, 0, 2 \rangle$
(0, 0, 2)
(3, 0, 0)
(−6, 0, 0)
x
y
$\mathbf{v} = \langle 3, 0, -1 \rangle$
(0, 0, −1)
FIGURE 11.38

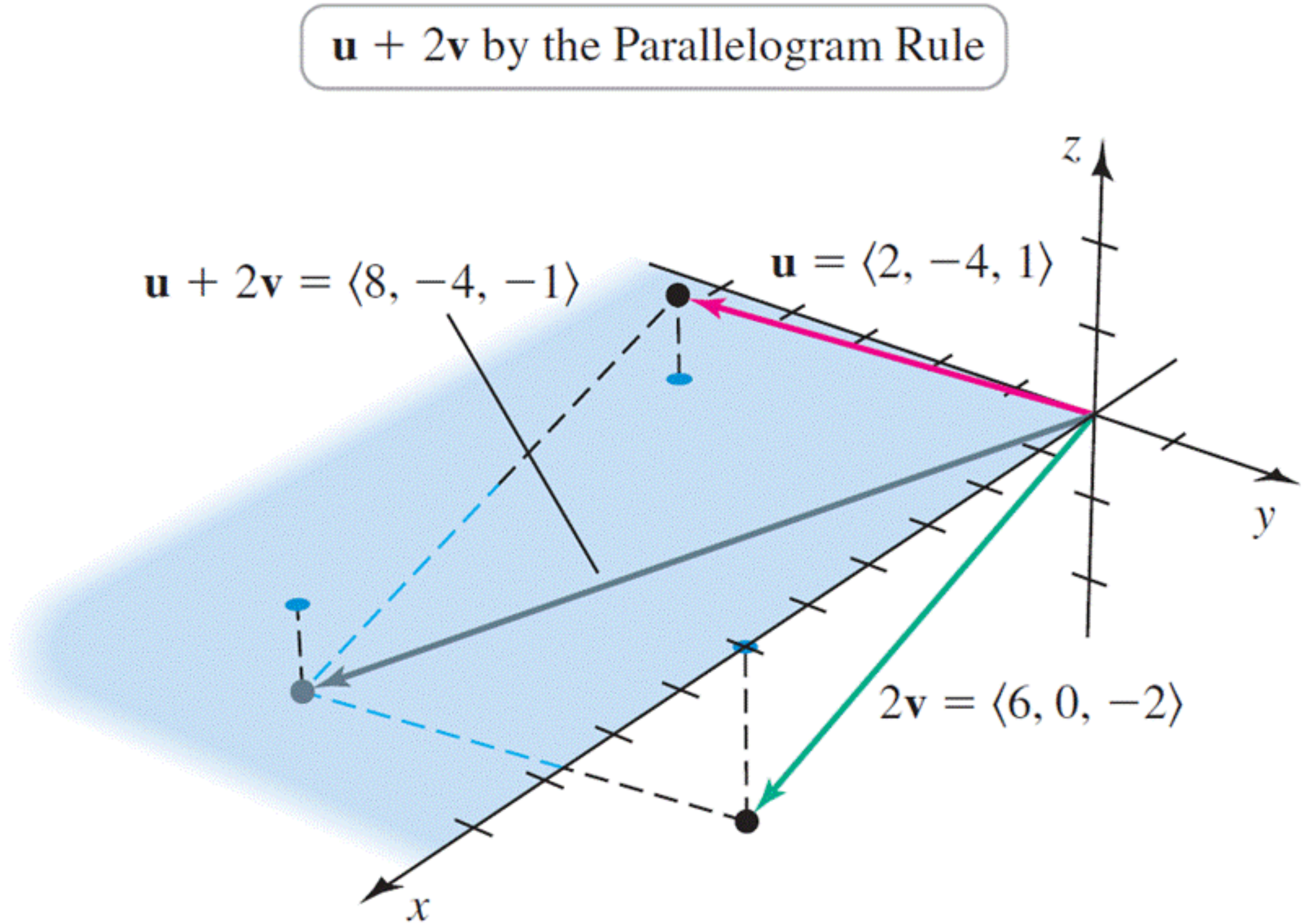


FIGURE 11.39

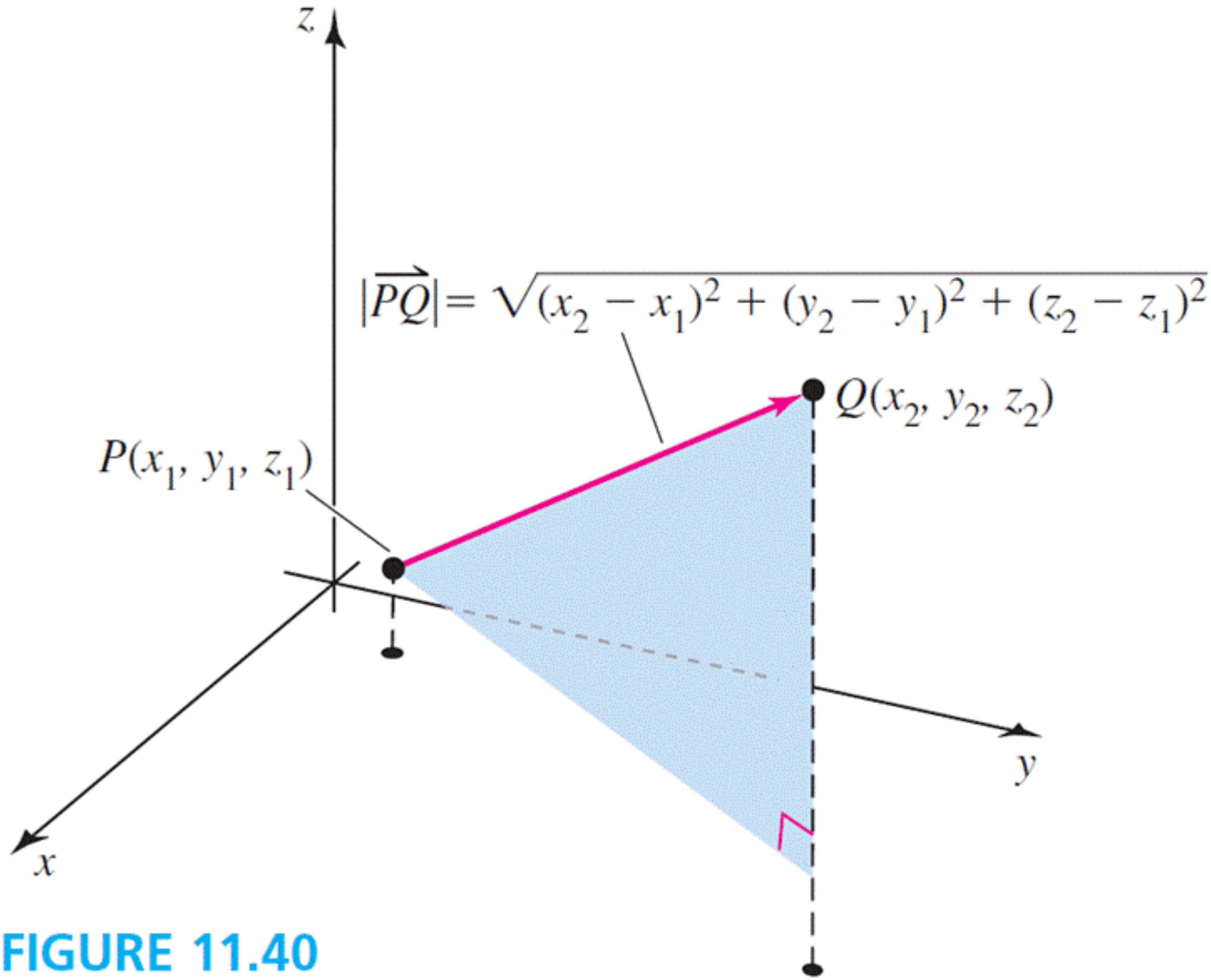
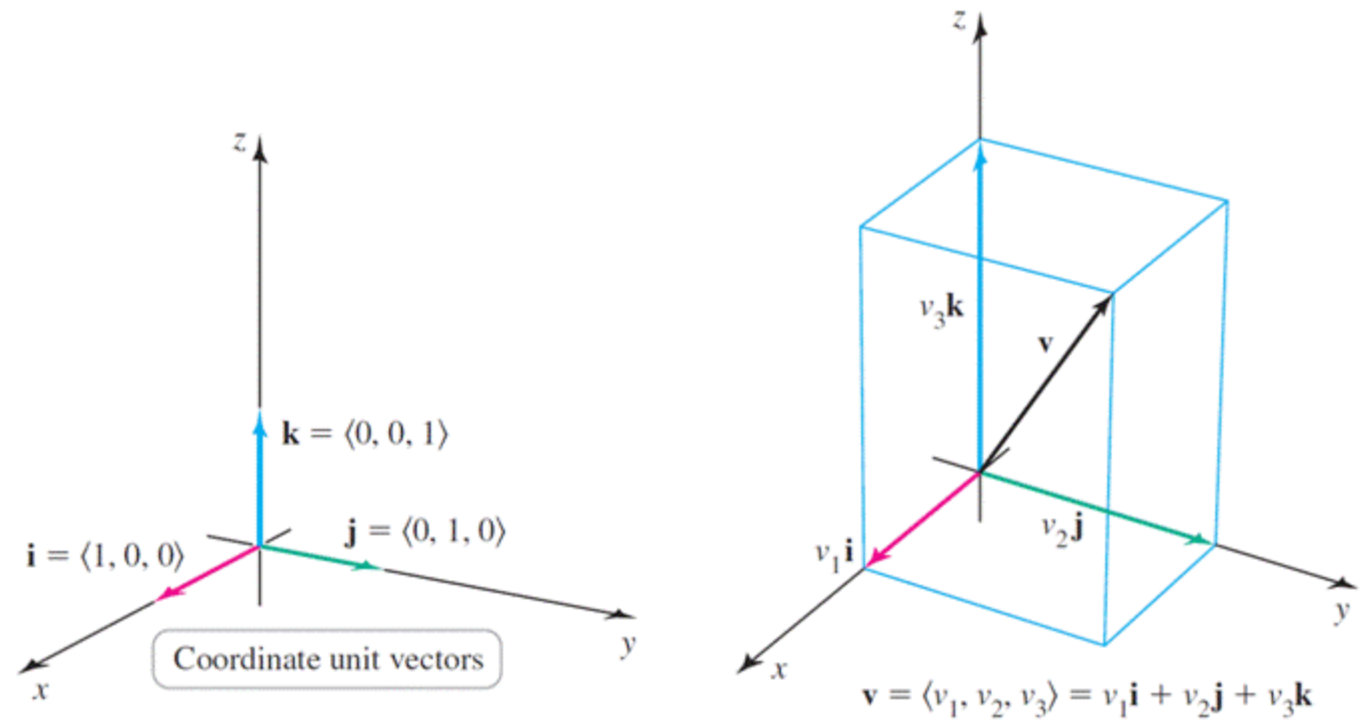

z
$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
$Q(x_2, y_2, z_2)$
$P(x_1, y_1, z_1)$
y
x

FIGURE 11.40

DEFINTION **Magnitude of a Vector**

The **magnitude** (or **length**) of the vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ is the distance from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$:

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$



z
k = ⟨0, 0, 1⟩
i = ⟨1, 0, 0⟩
j = ⟨0, 1, 0⟩
Coordinate unit vectors
x
y
z
$v_3\mathbf{k}$
v
$v_1\mathbf{i}$
$v_2\mathbf{j}$
x
y
$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

FIGURE 11.41