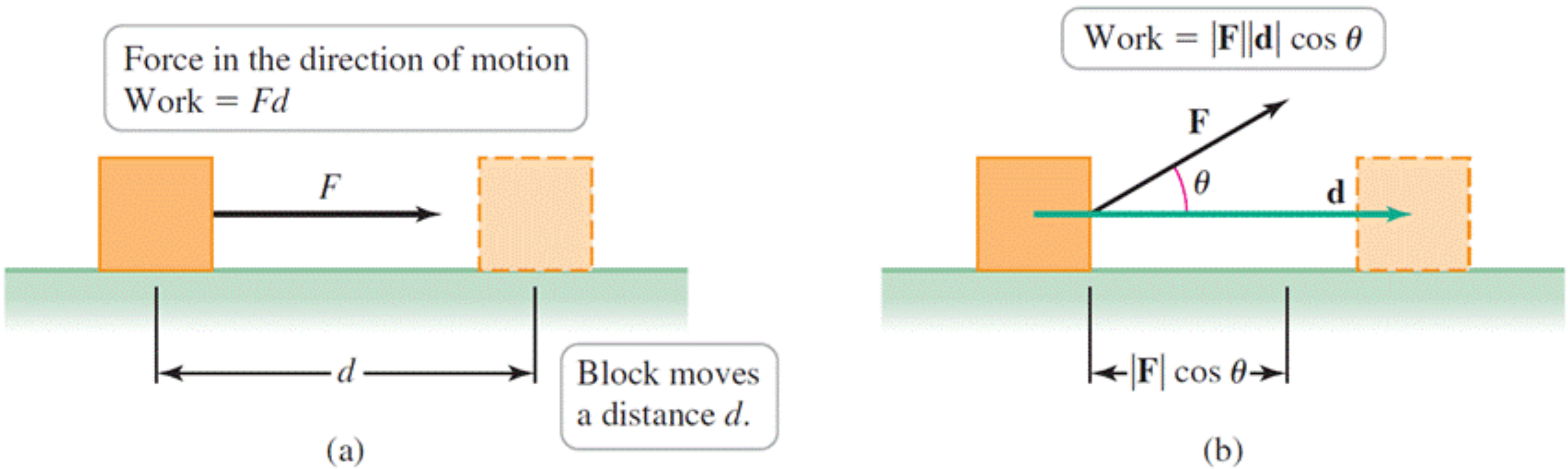
11.3

Dot Products


Force in the direction of motion
Work = *Fd*
F
d
Block moves
a distance *d*.
(a)
Work = |**F**||**d**| cos θ
F
θ
d
|**F**| cos θ
(b)
FIGURE 11.43

DEFINITION **Dot Product**

Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in two or three dimensions, their **dot product** is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ with $0 \leq \theta \leq \pi$ (Figure 11.44). If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$, and θ is undefined.

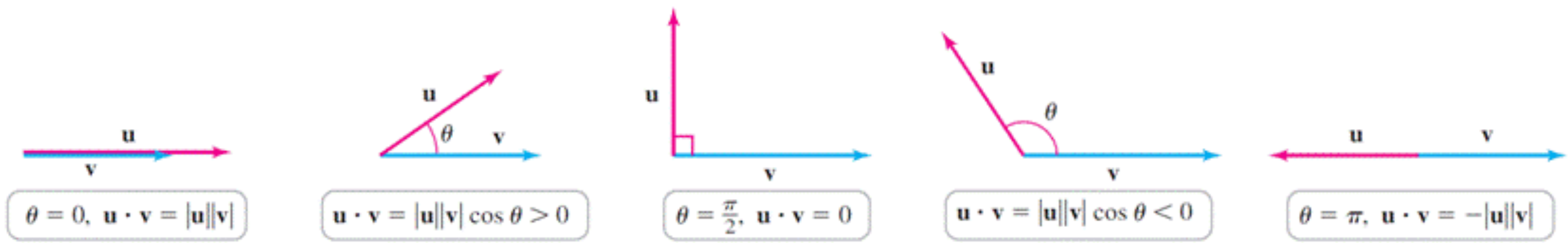

u
v
$\theta = 0,\ \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|$
u
θ
v
$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta > 0$
u
v
$\theta = \frac{\pi}{2},\ \mathbf{u} \cdot \mathbf{v} = 0$
u
θ
v
$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta < 0$
u
v
$\theta = \pi,\ \mathbf{u} \cdot \mathbf{v} = -|\mathbf{u}||\mathbf{v}|$

FIGURE 11.44

DEFINITION Orthogonal Vectors

Two vectors **u** and **v** are **orthogonal** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. The zero vector is orthogonal to all vectors. In two or three dimesions, two nonzero orthogonal vectors are perpendicular to each other.

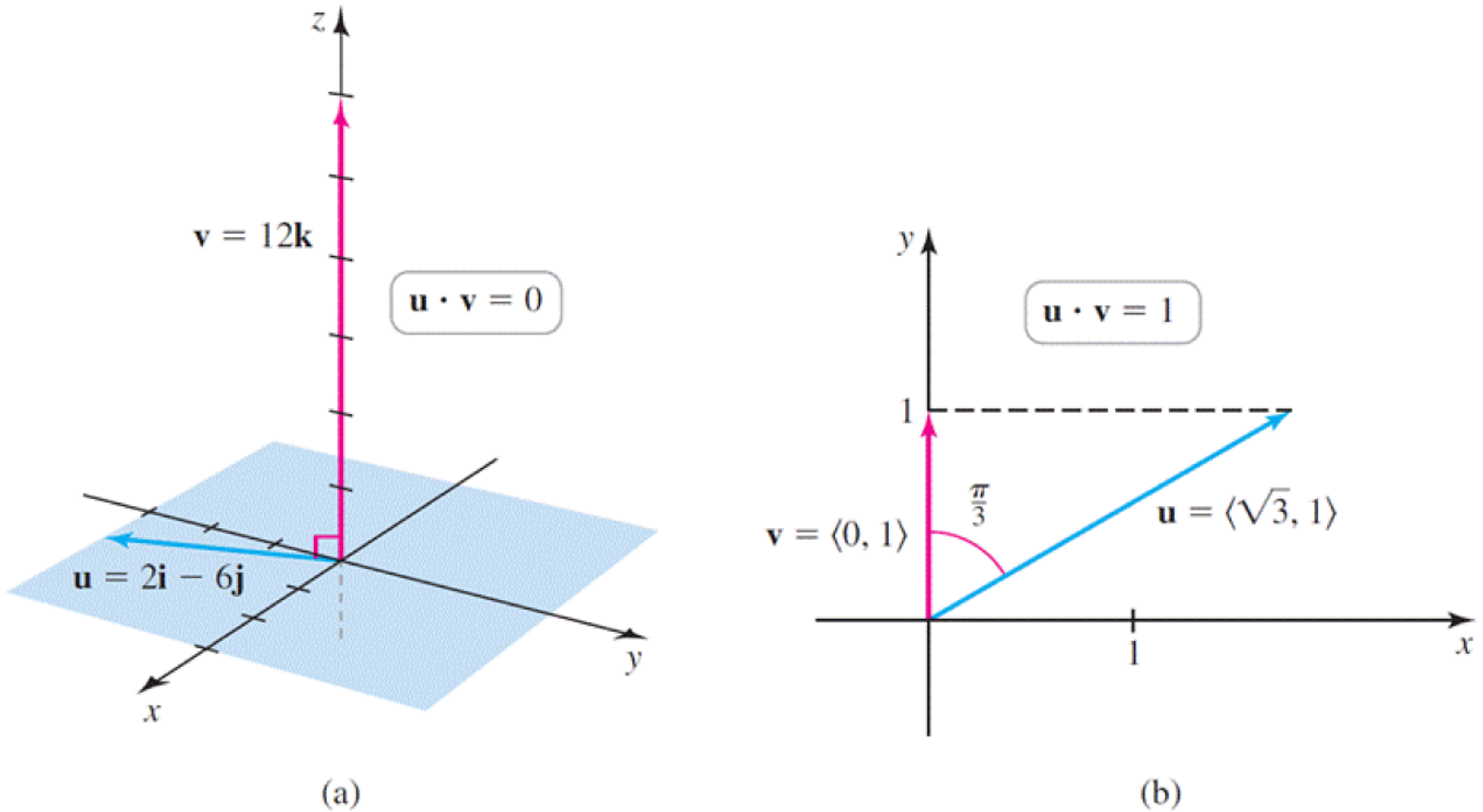


FIGURE 11.45

Related Exercises 9–12

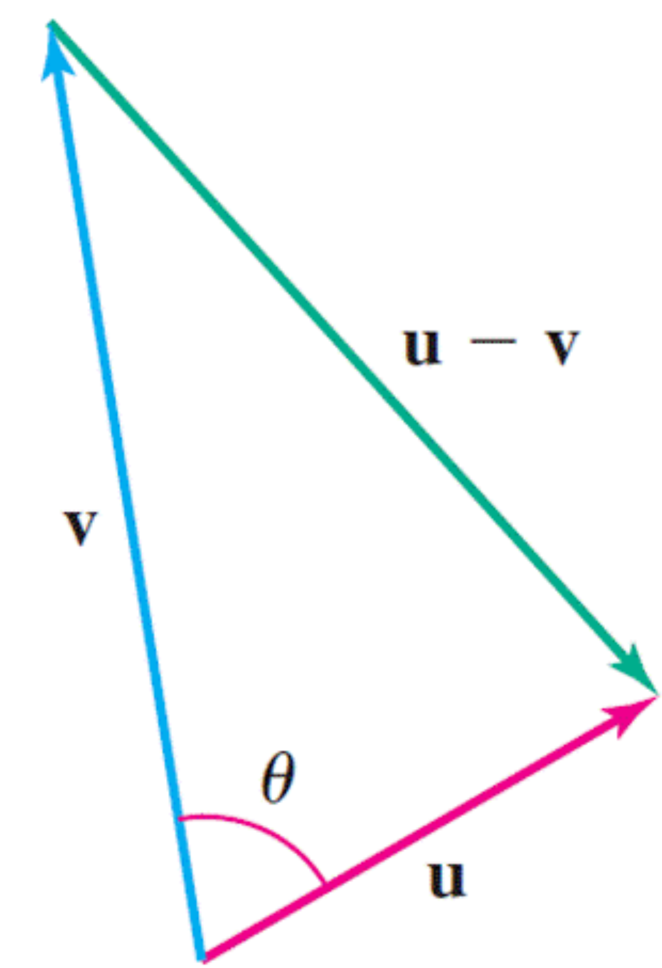


FIGURE 11.46

THEOREM 11.1 Dot Product

Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$,

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

THEOREM 11.2 Properties of the Dot Product

Suppose **u**, **v**, and **w** are vectors and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ Commutative property
2. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$ Associative property
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ Distributive property

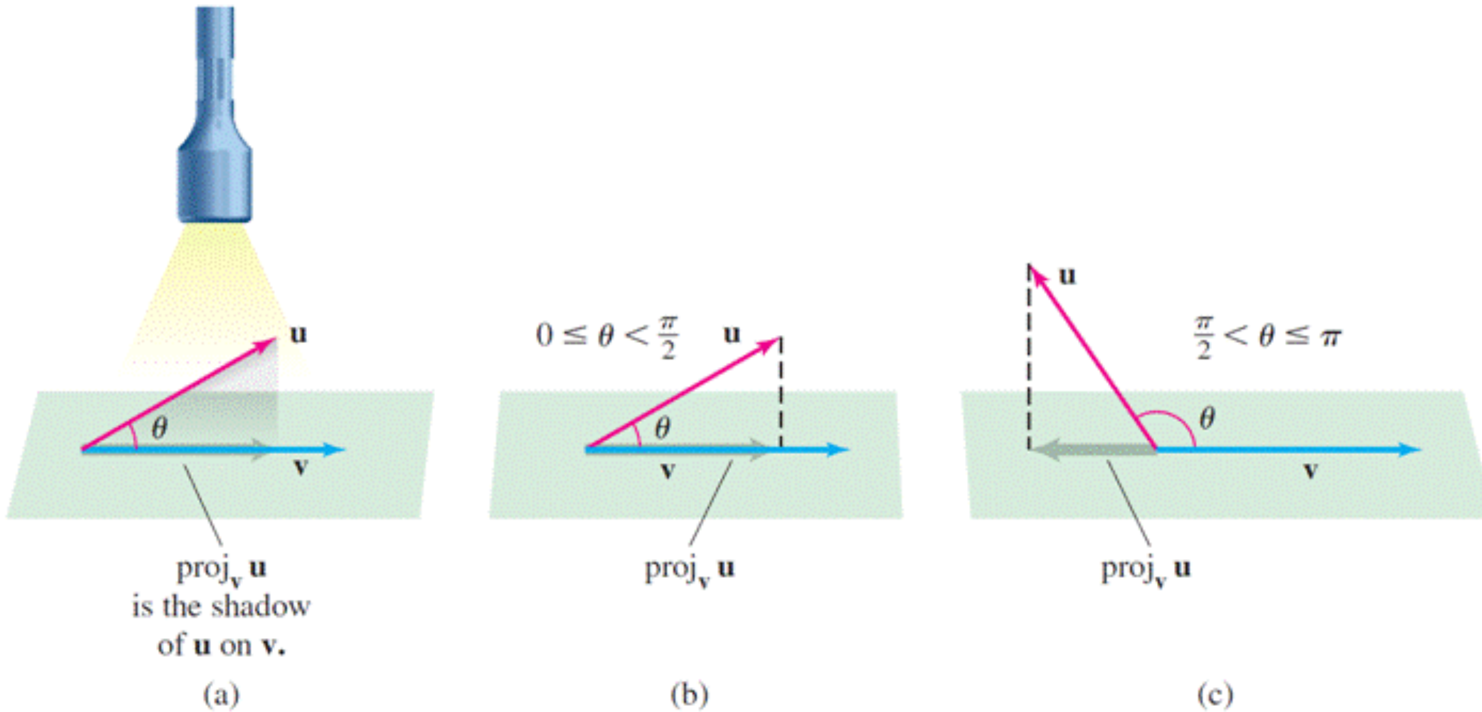
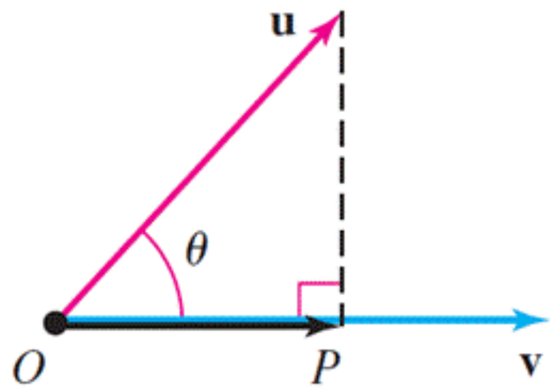

u
θ
v
proj$_{\mathbf{v}}$ **u**
is the shadow
of **u** on **v**.
(a)
$0 \le \theta < \frac{\pi}{2}$
u
θ
v
proj$_{\mathbf{v}}$ **u**
(b)
u
$\frac{\pi}{2} < \theta \le \pi$
θ
v
proj$_{\mathbf{v}}$ **u**
(c)

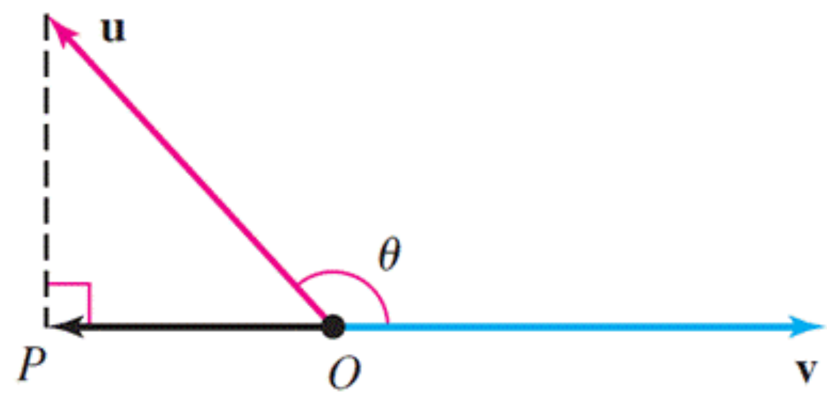
FIGURE 11.47



$0 \le \theta < \frac{\pi}{2}$

$\mathrm{scal}_{\mathbf{v}}\, \mathbf{u} = |\mathbf{u}| \cos\theta > 0$

(a)



$\frac{\pi}{2} < \theta \le \pi$

$\mathrm{scal}_{\mathbf{v}}\, \mathbf{u} = |\mathbf{u}| \cos\theta < 0$

(b)

FIGURE 11.48

DEFINITION (Orthogonal) Projection of u onto v

The **orthogonal projection of u onto v**, denoted $\operatorname{proj}_{\mathbf{v}}\mathbf{u}$, where $\mathbf{v} \neq \mathbf{0}$, is

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta \left(\frac{\mathbf{v}}{|\mathbf{v}|}\right).$$

The orthogonal projection may also be computed with the formulas

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \operatorname{scal}_{\mathbf{v}}\mathbf{u}\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v},$$

where the **scalar component of u in the direction of v** is

$$\operatorname{scal}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}.$$

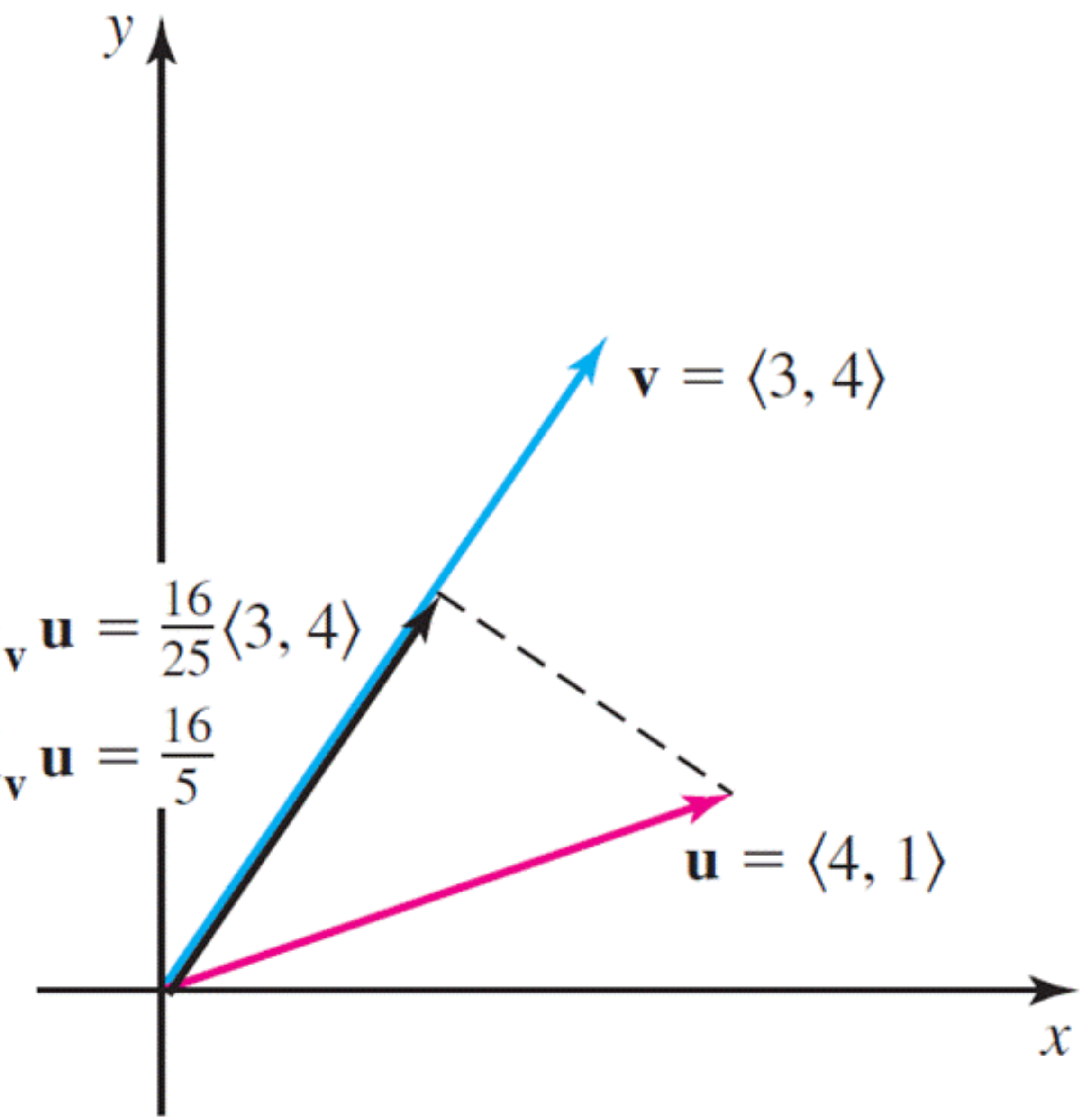
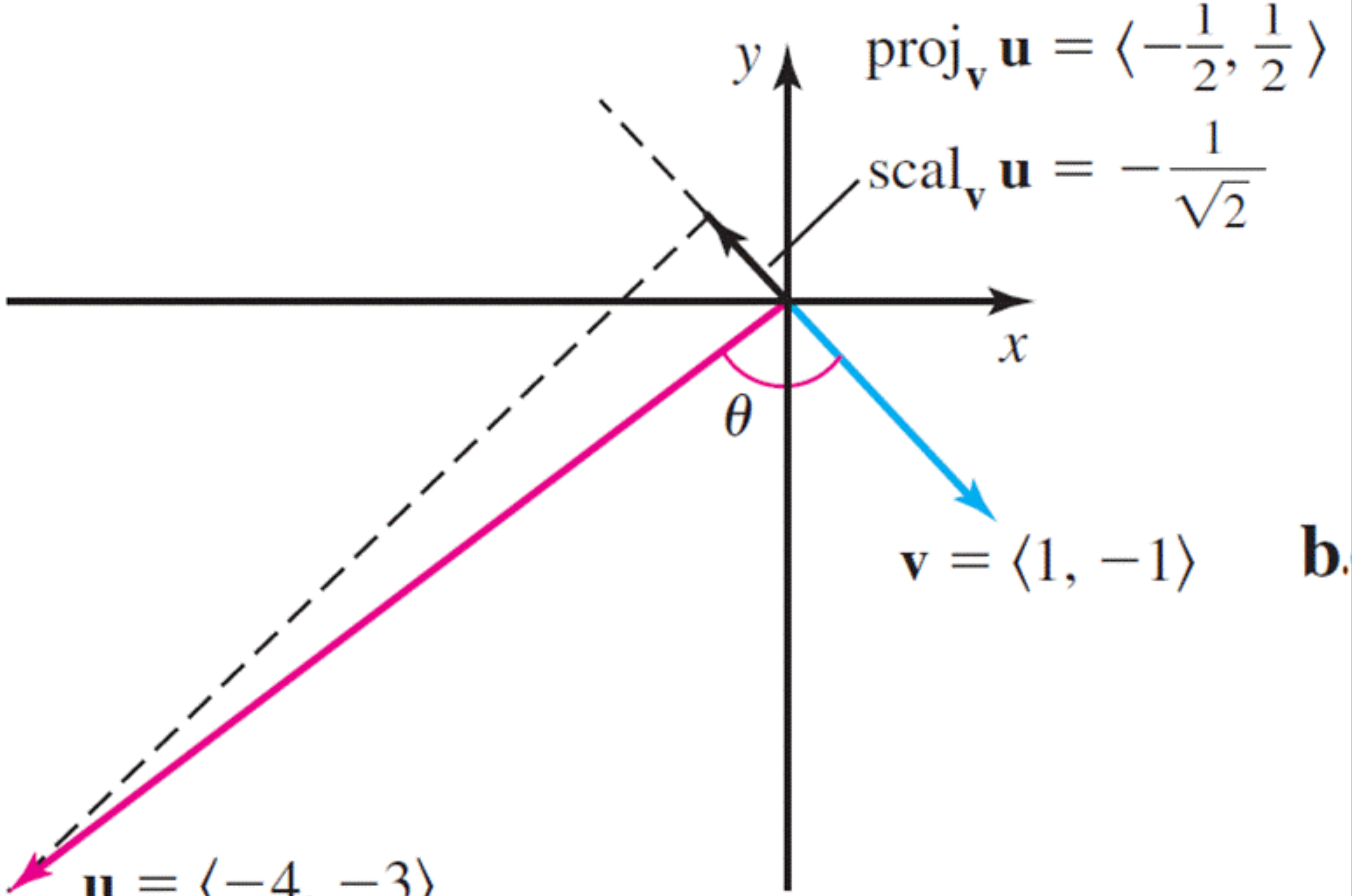

y
$\mathbf{v} = \langle 3, 4 \rangle$
$\text{proj}_{\mathbf{v}}\, \mathbf{u} = \frac{16}{25}\langle 3, 4 \rangle$
$\text{scal}_{\mathbf{v}}\, \mathbf{u} = \frac{16}{5}$
$\mathbf{u} = \langle 4, 1 \rangle$
x

FIGURE 11.49



y
$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \langle -\frac{1}{2}, \frac{1}{2} \rangle$
$\text{scal}_{\mathbf{v}}\,\mathbf{u} = -\frac{1}{\sqrt{2}}$
x
θ
$\mathbf{v} = \langle 1, -1 \rangle$
b.
$\mathbf{u} = \langle -4, -3 \rangle$

FIGURE 11.50

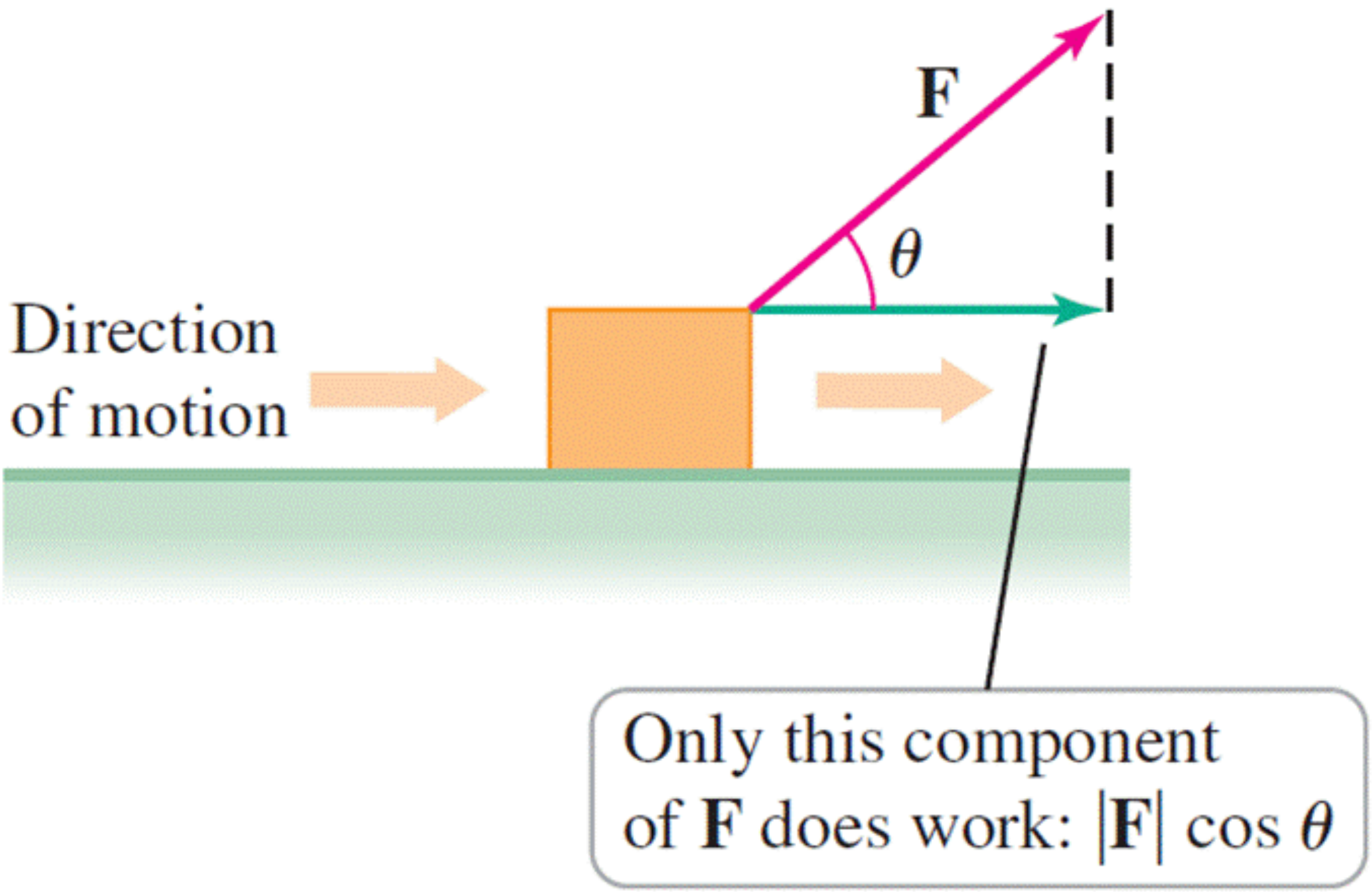

F
θ
Direction
of motion
Only this component
of F does work: |F| cos θ

FIGURE 11.51

DEFINITION **Work**

Let a constant force **F** be applied to an object, producing a displacement **d**. If the angle between **F** and **d** is θ, then the **work** done by the force is

$$W = |\mathbf{F}||\mathbf{d}| \cos \theta = \mathbf{F} \cdot \mathbf{d}.$$

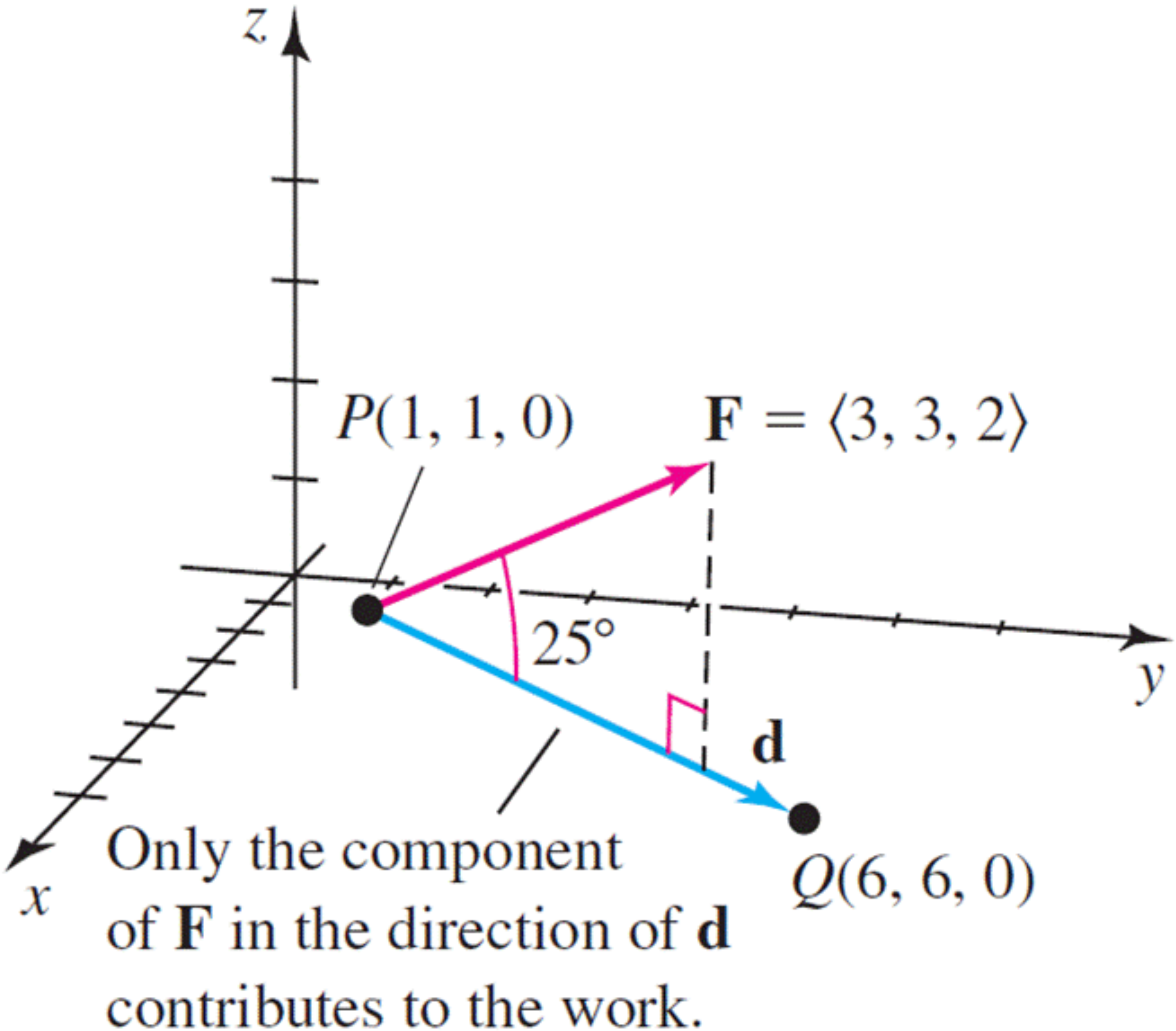

z
$P(1, 1, 0)$
$\mathbf{F} = \langle 3, 3, 2 \rangle$
25°
y
d
x
Only the component
of **F** in the direction of **d**
contributes to the work.
$Q(6, 6, 0)$

FIGURE 11.52

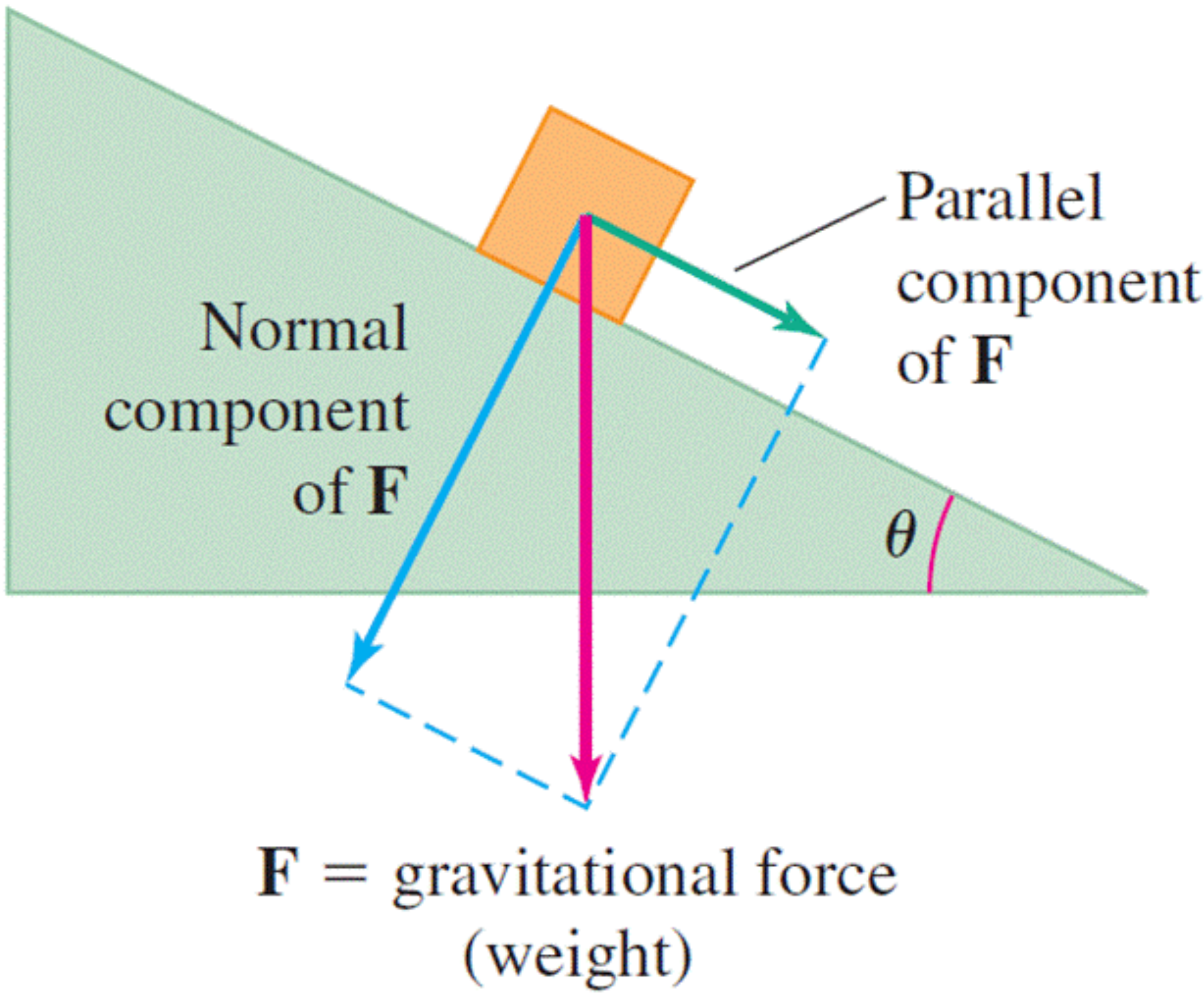
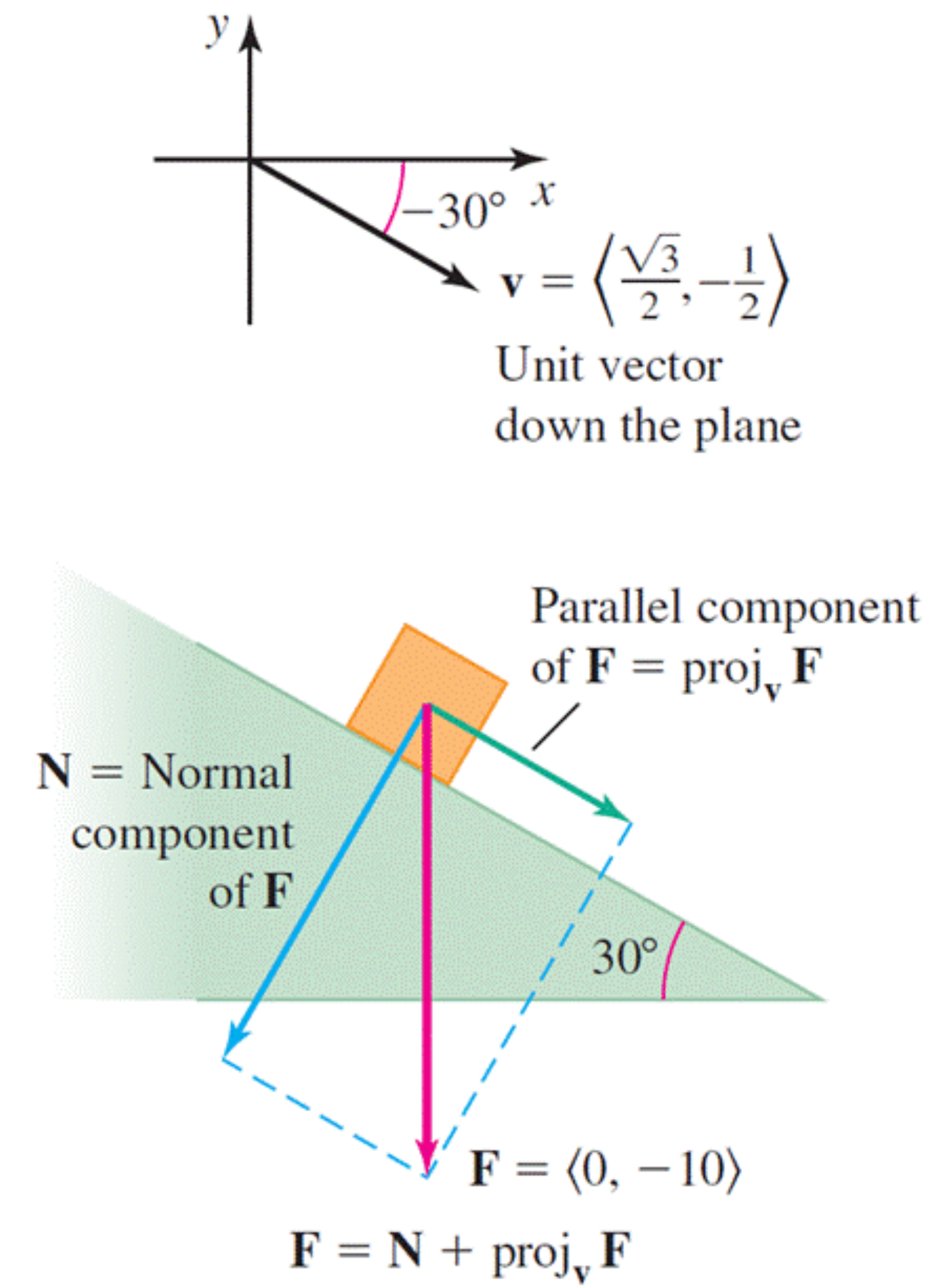

Parallel
component
of **F**
Normal
component
of **F**
θ
F = gravitational force
(weight)

FIGURE 11.53



y
x
−30°
$\mathbf{v} = \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$
Unit vector down the plane
Parallel component of $\mathbf{F} = \text{proj}_{\mathbf{v}} \mathbf{F}$
$\mathbf{N}$ = Normal component of $\mathbf{F}$
30°
$\mathbf{F} = \langle 0, -10 \rangle$
$\mathbf{F} = \mathbf{N} + \text{proj}_{\mathbf{v}} \mathbf{F}$

FIGURE 11.54

11.4

Cross Products

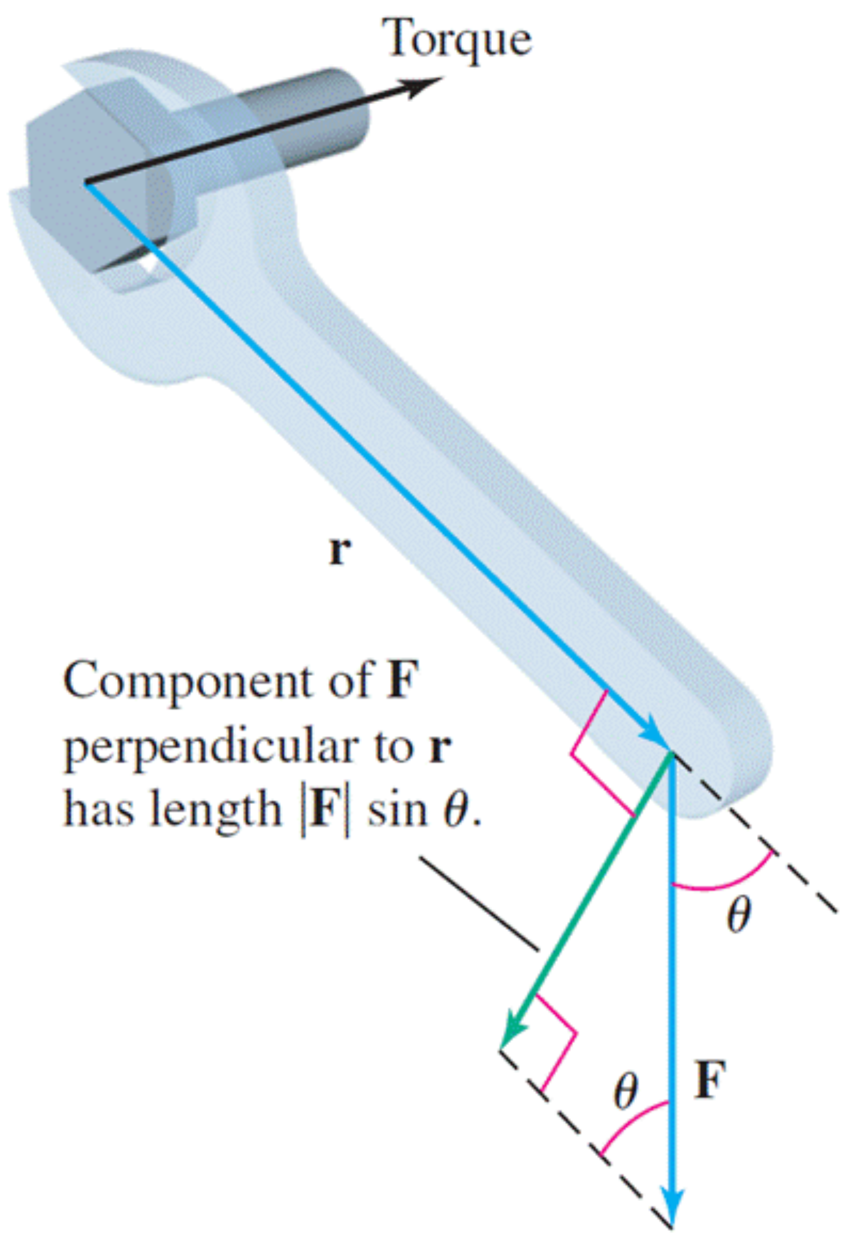

Torque
r
Component of **F** perpendicular to **r** has length $|\mathbf{F}| \sin \theta$.
θ
θ
F

FIGURE 11.55

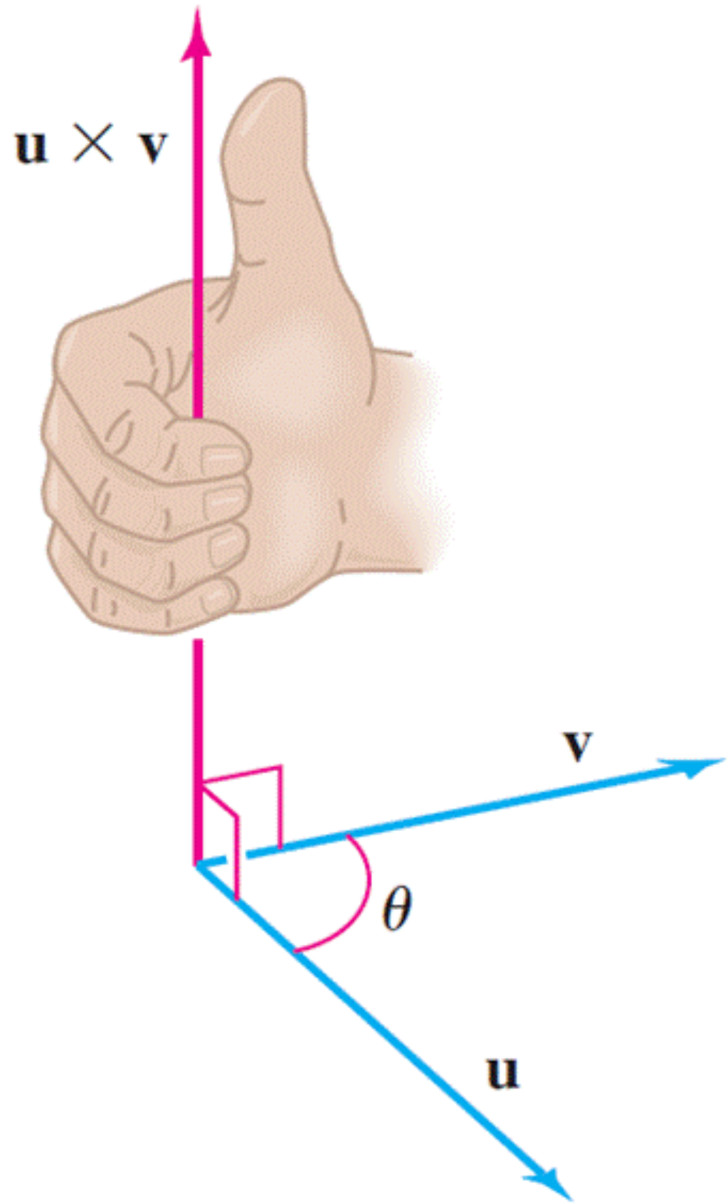


FIGURE 11.56

DEFINITION Cross Product

Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^3$, the **cross product** $\mathbf{u} \times \mathbf{v}$ is a vector with magnitude

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta,$$

where $0 \leq \theta \leq \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$. The direction of $\mathbf{u} \times \mathbf{v}$ is given by the **right-hand rule**: When you put the vectors tail to tail and let the fingers of your right hand curl from $\mathbf{u}$ to $\mathbf{v}$, the direction of $\mathbf{u} \times \mathbf{v}$ is the direction of your thumb, orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ (Figure 11.56). When $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, the direction of $\mathbf{u} \times \mathbf{v}$ is undefined.

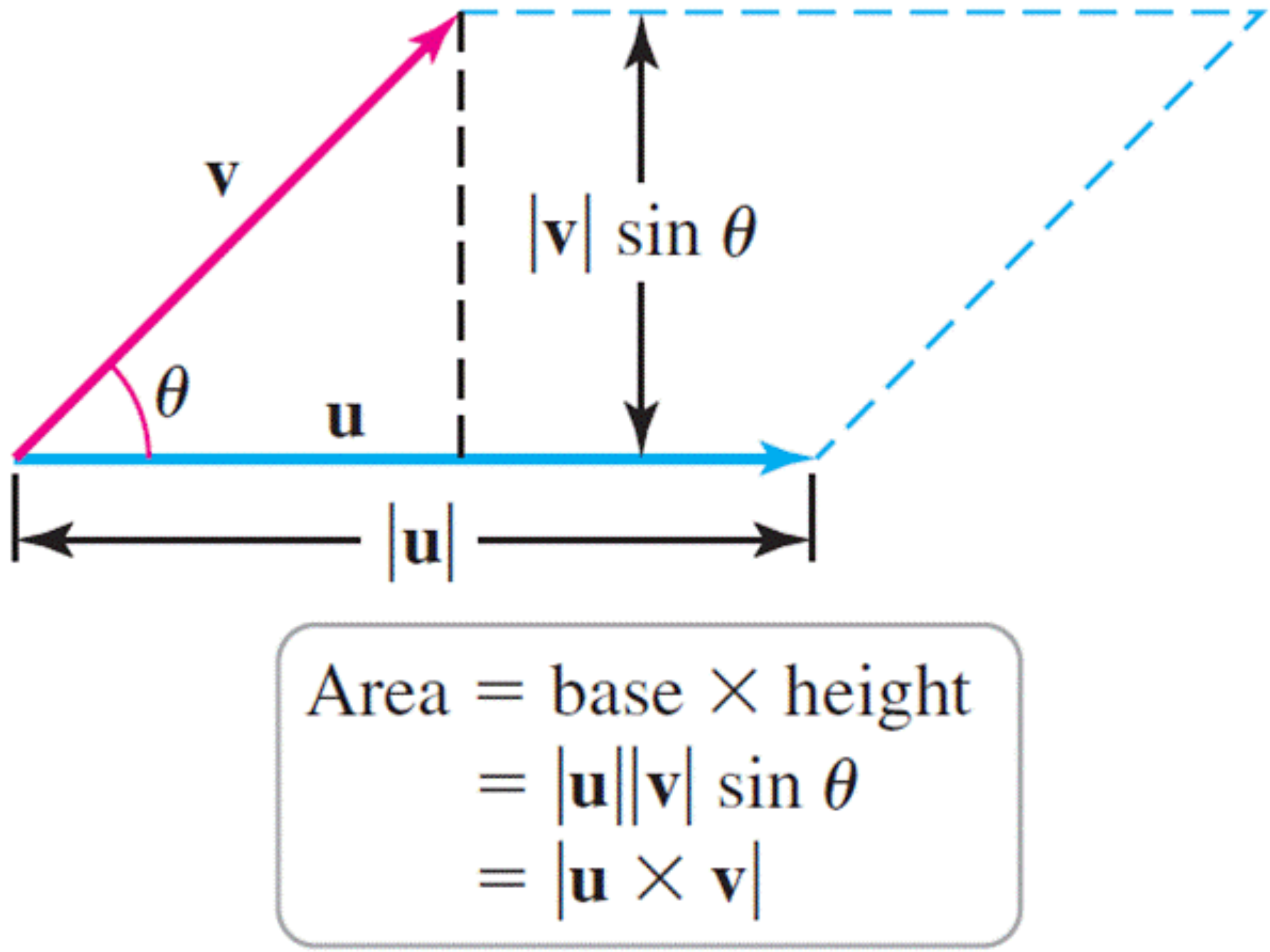


FIGURE 11.57

THEOREM 11.3 Geometry of the Cross Product

Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbf{R}^3$.

1. The vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
2. If $\mathbf{u}$ and $\mathbf{v}$ are two sides of a parallelogram (Figure 11.57), then the area of the parallelogram is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta.$$

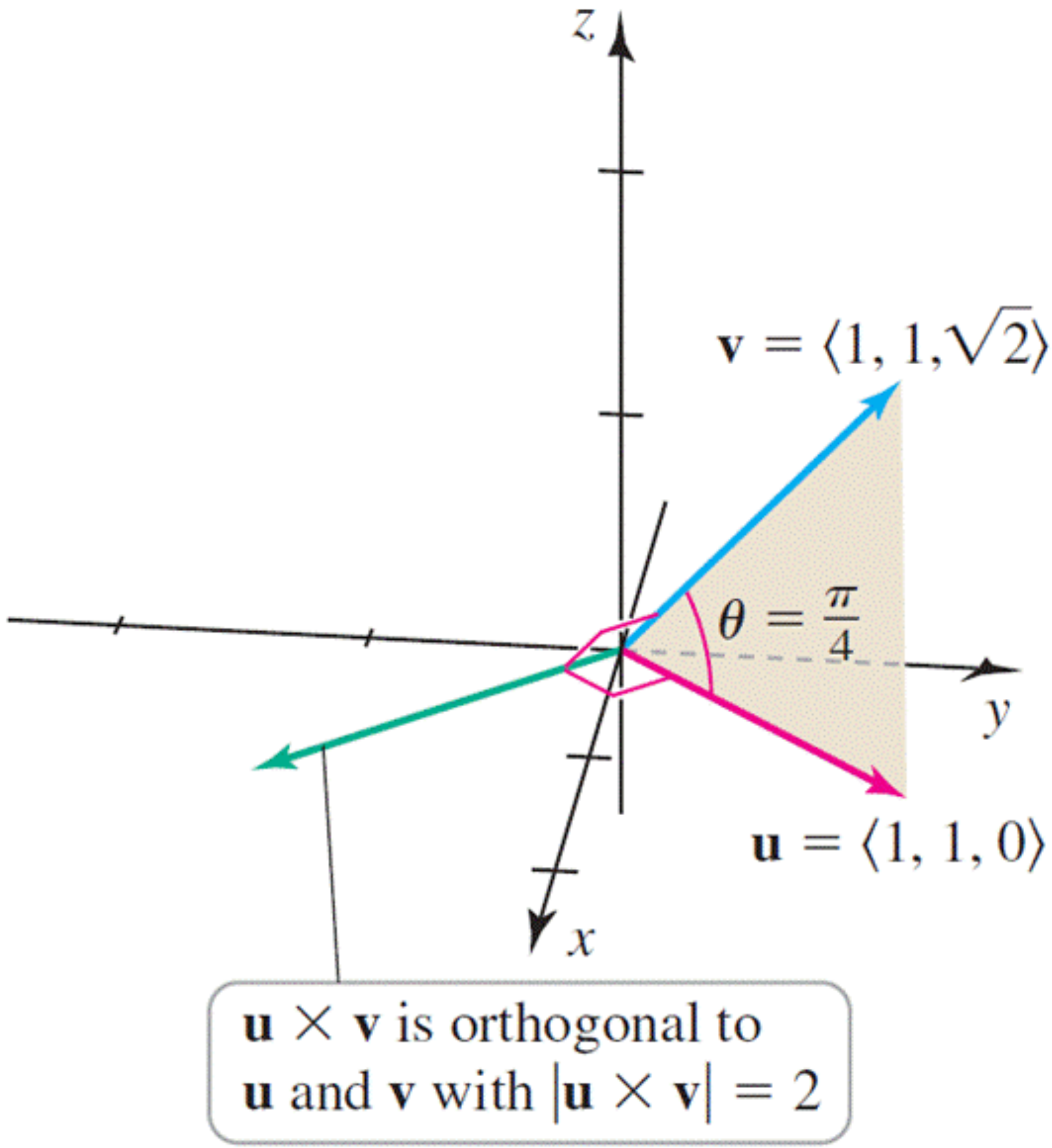
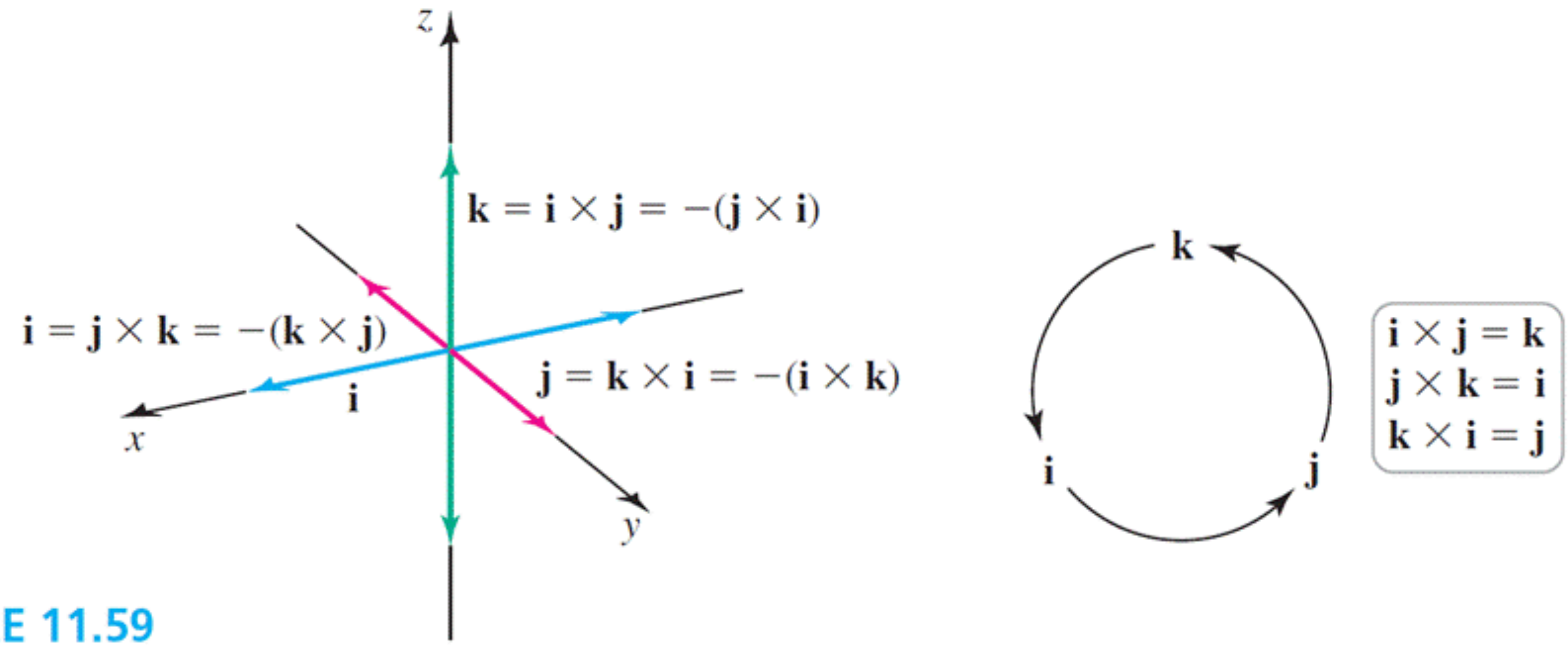

z
$\mathbf{v} = \langle 1, 1, \sqrt{2} \rangle$
$\theta = \frac{\pi}{4}$
y
$\mathbf{u} = \langle 1, 1, 0 \rangle$
x
$\mathbf{u} \times \mathbf{v}$ is orthogonal to
$\mathbf{u}$ and $\mathbf{v}$ with $|\mathbf{u} \times \mathbf{v}| = 2$

FIGURE 11.58

THEOREM 11.4 Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in $\mathbf{R}^3$, and let a and b be scalars.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ — Anticommutative property
2. $(a\mathbf{u}) \times (b\mathbf{v}) = ab(\mathbf{u} \times \mathbf{v})$ — Associative property
3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ — Distributive property
4. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$ — Distributive property



z
$\mathbf{k} = \mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i})$
$\mathbf{i} = \mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j})$
$\mathbf{j} = \mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k})$
$\mathbf{i}$
x
y
$\mathbf{k}$
$\mathbf{i}$
$\mathbf{j}$
$\mathbf{i} \times \mathbf{j} = \mathbf{k}$
$\mathbf{j} \times \mathbf{k} = \mathbf{i}$
$\mathbf{k} \times \mathbf{i} = \mathbf{j}$

FIGURE 11.59

THEOREM 11.5 **Cross Products of Coordinate Unit Vectors**

$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j} \qquad \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

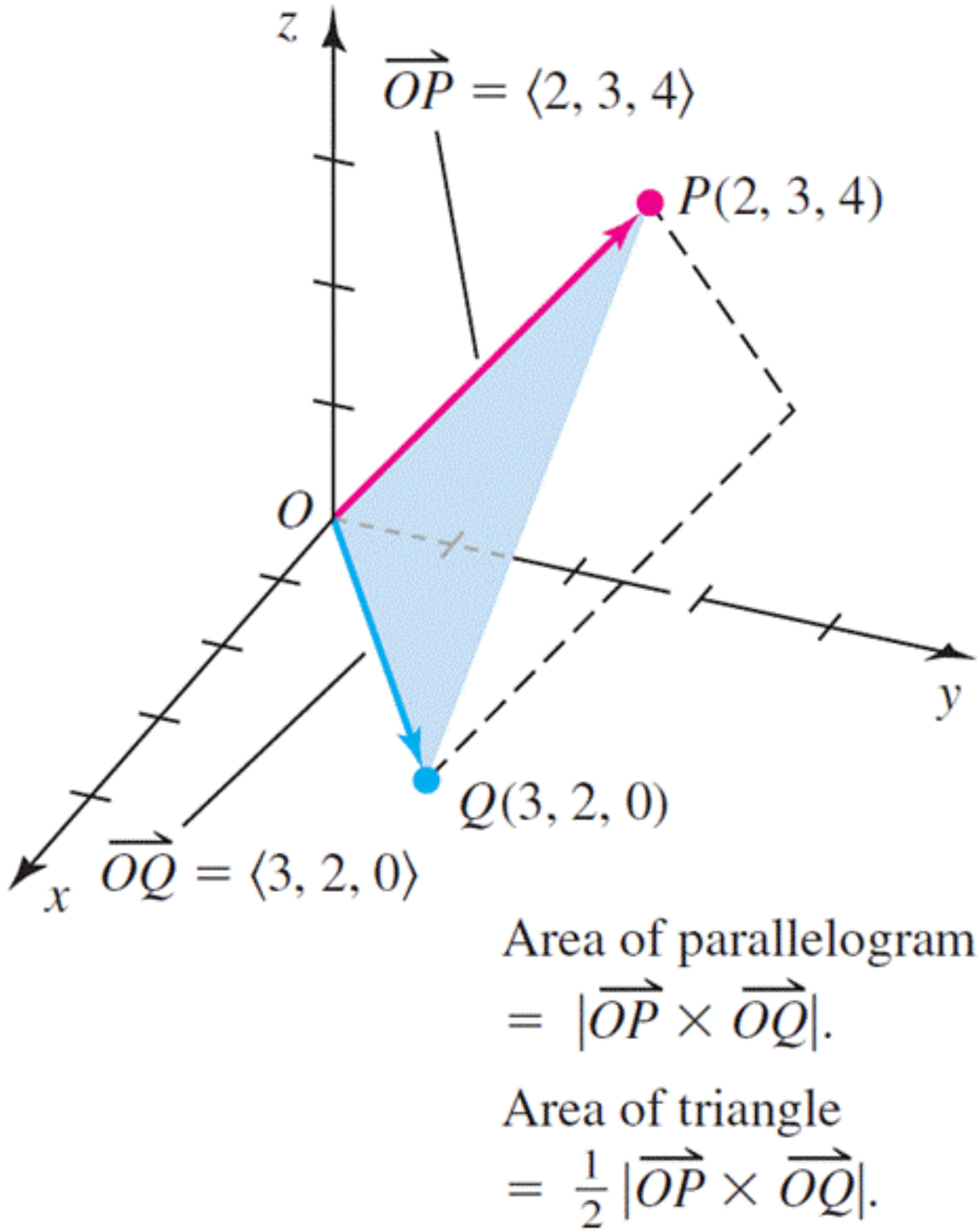
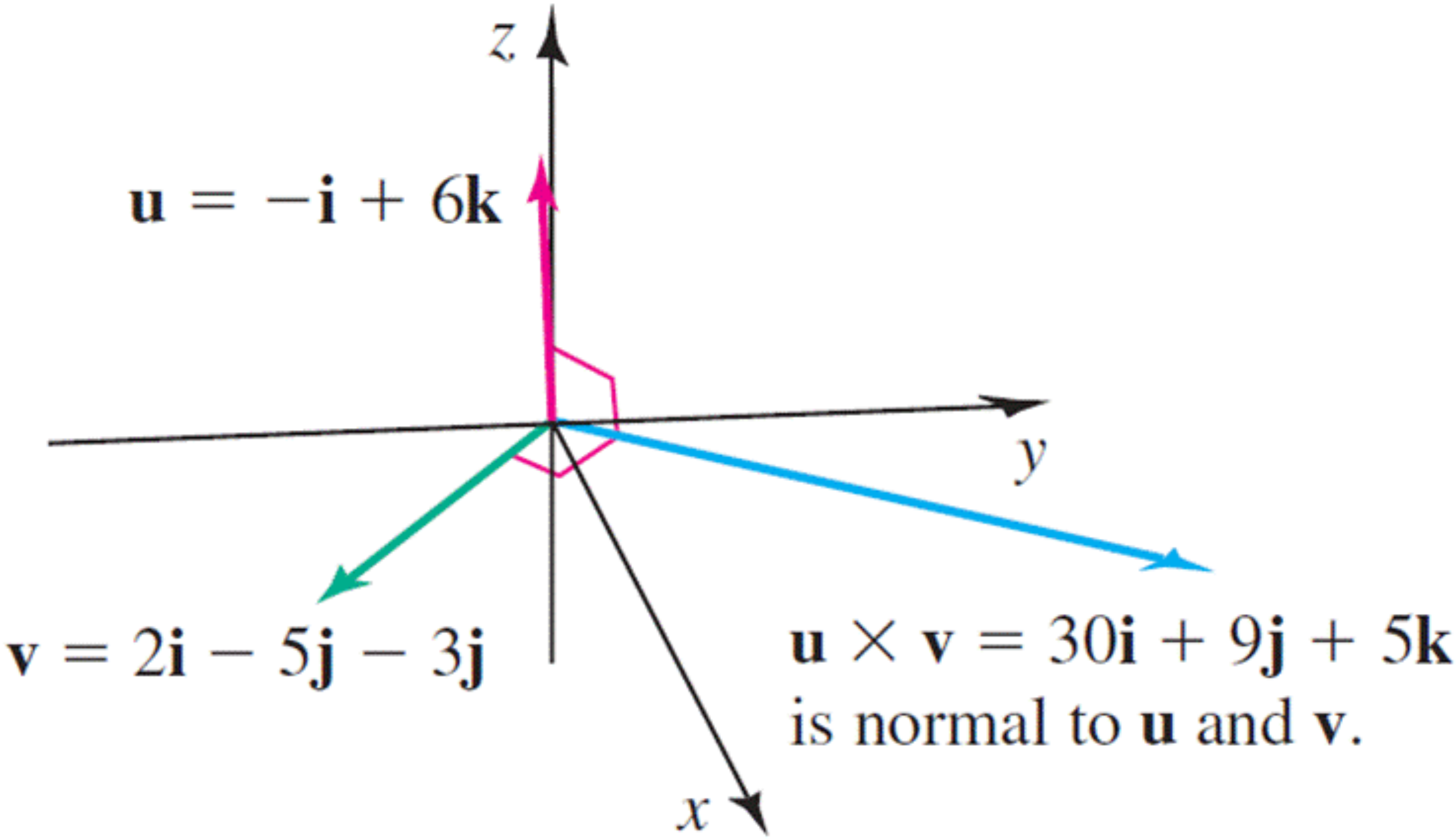

z
$\overrightarrow{OP} = \langle 2, 3, 4 \rangle$
$P(2, 3, 4)$
O
y
$Q(3, 2, 0)$
x
$\overrightarrow{OQ} = \langle 3, 2, 0 \rangle$
Area of parallelogram
$= |\overrightarrow{OP} \times \overrightarrow{OQ}|.$
Area of triangle
$= \frac{1}{2}|\overrightarrow{OP} \times \overrightarrow{OQ}|.$

FIGURE 11.60

THEOREM 11.6 Evaluating the Cross Product

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then,

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{k}.$$



z
$\mathbf{u} = -\mathbf{i} + 6\mathbf{k}$
y
$\mathbf{v} = 2\mathbf{i} - 5\mathbf{j} - 3\mathbf{j}$
$\mathbf{u} \times \mathbf{v} = 30\mathbf{i} + 9\mathbf{j} + 5\mathbf{k}$
is normal to **u** and **v**.
x

FIGURE 11.61

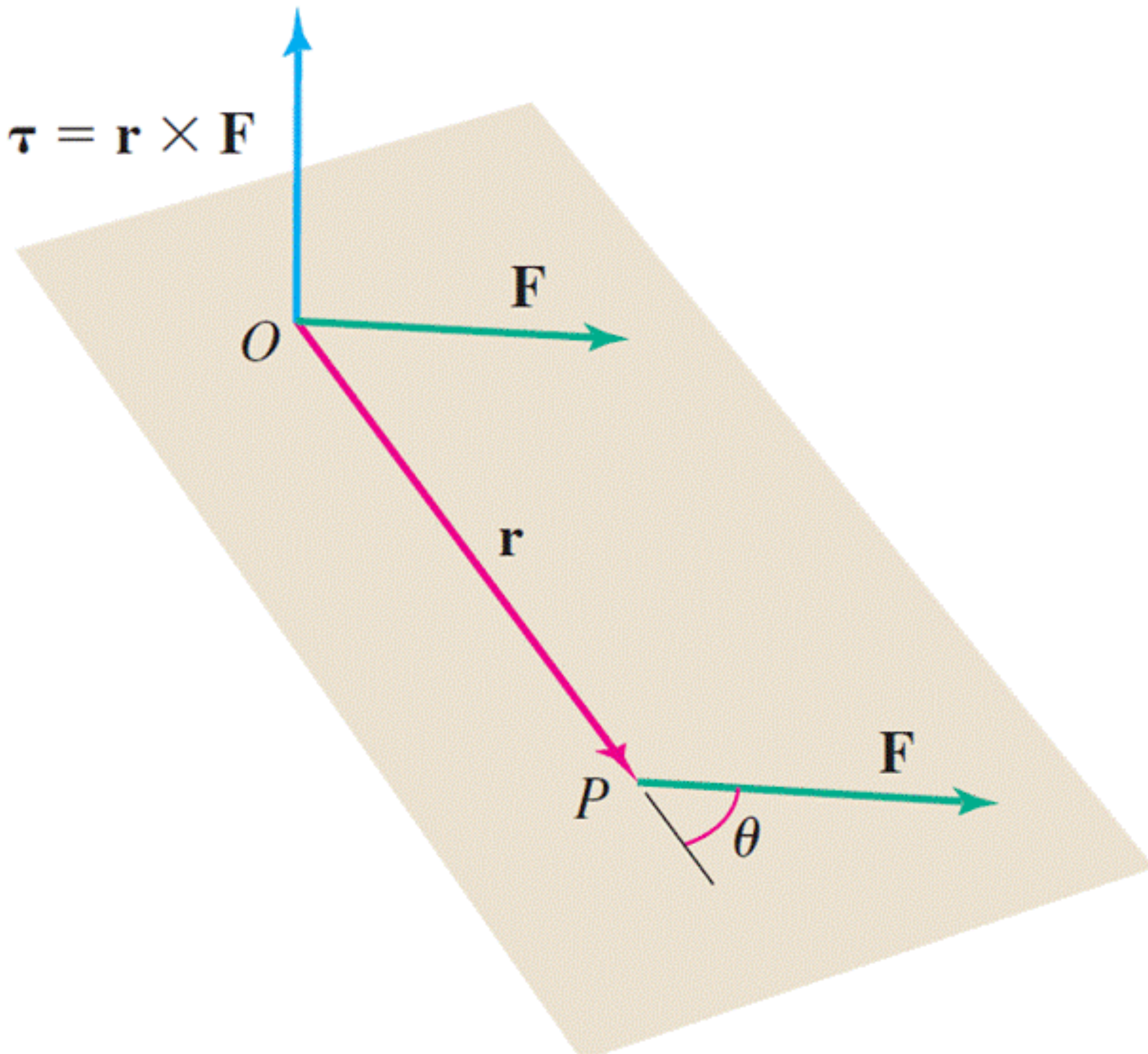
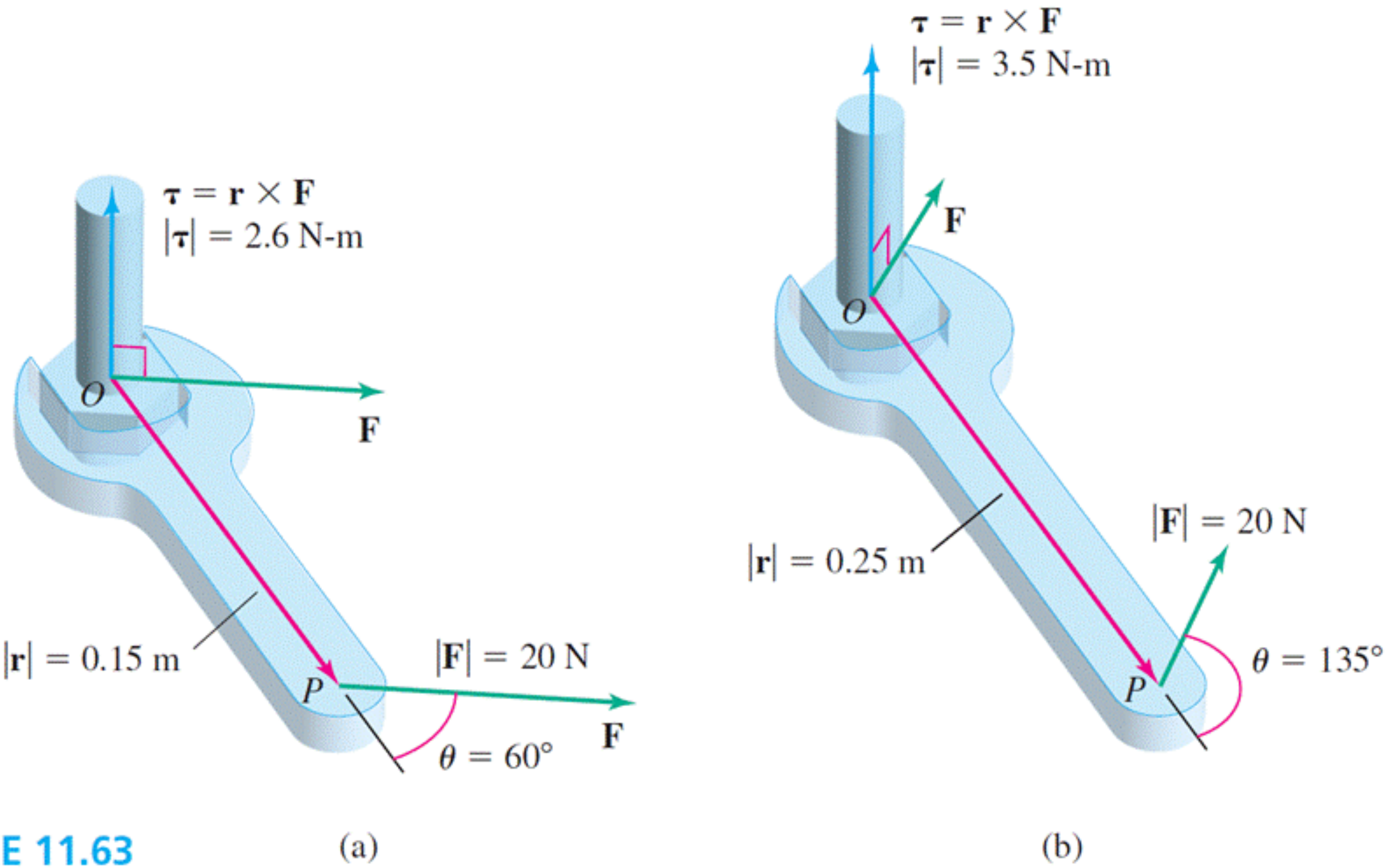


FIGURE 11.62



$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$
$|\boldsymbol{\tau}| = 2.6$ N-m
O
$\mathbf{F}$
$|\mathbf{r}| = 0.15$ m
P
$|\mathbf{F}| = 20$ N
$\theta = 60°$
$\mathbf{F}$
(a)
$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$
$|\boldsymbol{\tau}| = 3.5$ N-m
$\mathbf{F}$
O
$|\mathbf{F}| = 20$ N
$|\mathbf{r}| = 0.25$ m
$\theta = 135°$
P
(b)

FIGURE 11.63

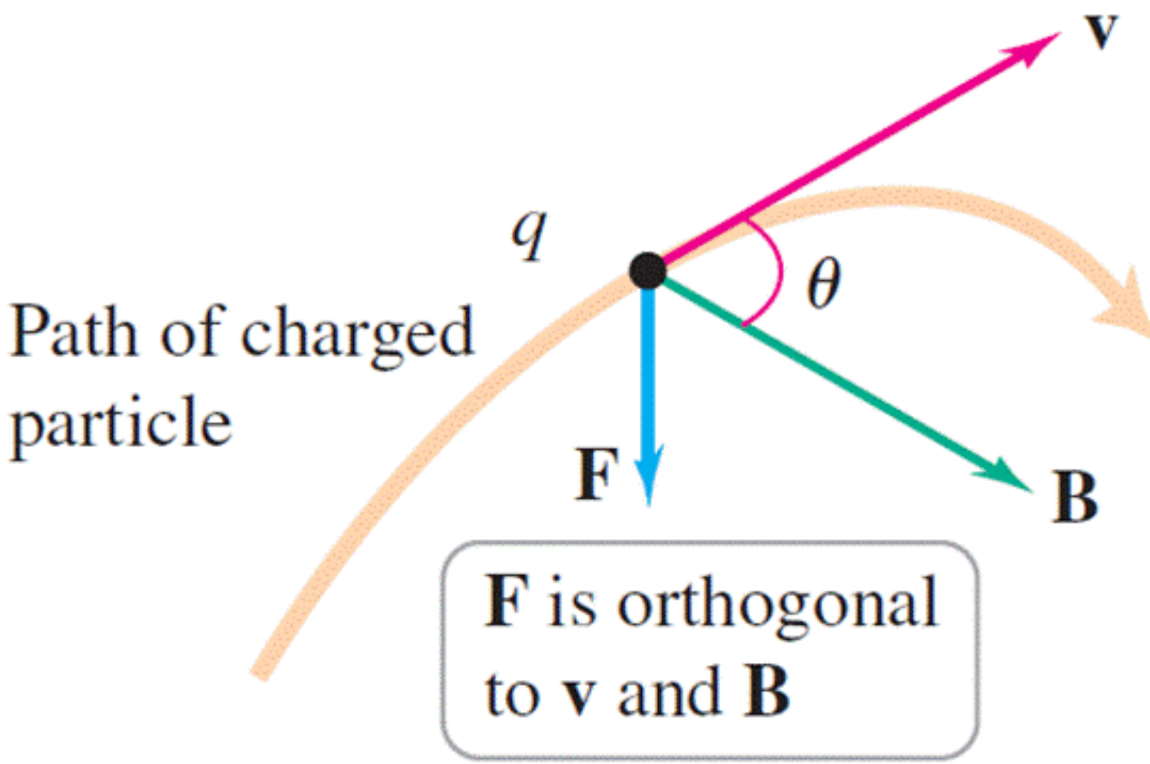


FIGURE 11.64

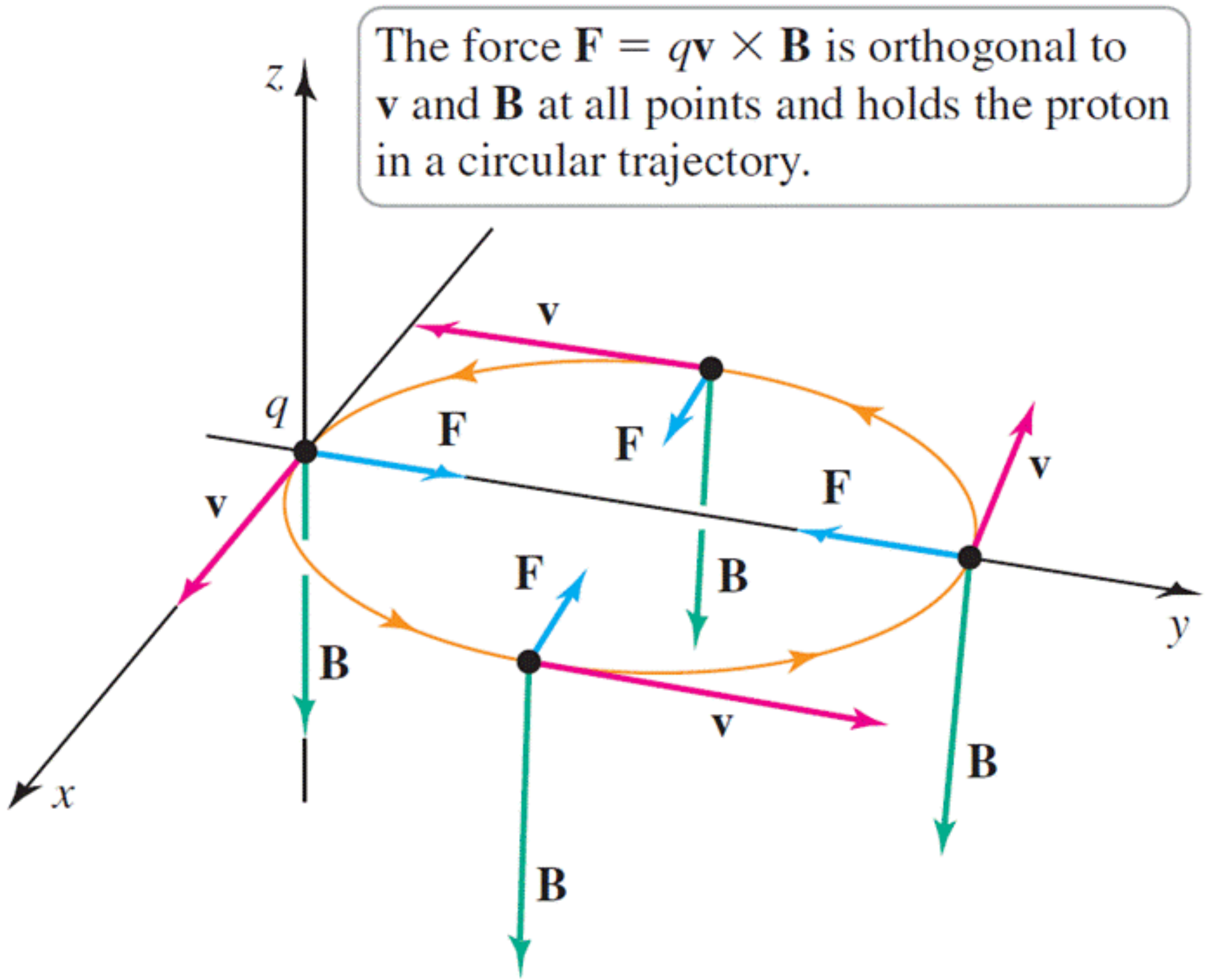

The force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ is orthogonal to $\mathbf{v}$ and $\mathbf{B}$ at all points and holds the proton in a circular trajectory.
z
q
v
F
F
F
F
B
B
B
B
y
x

FIGURE 11.65