11.7

Motion in Space

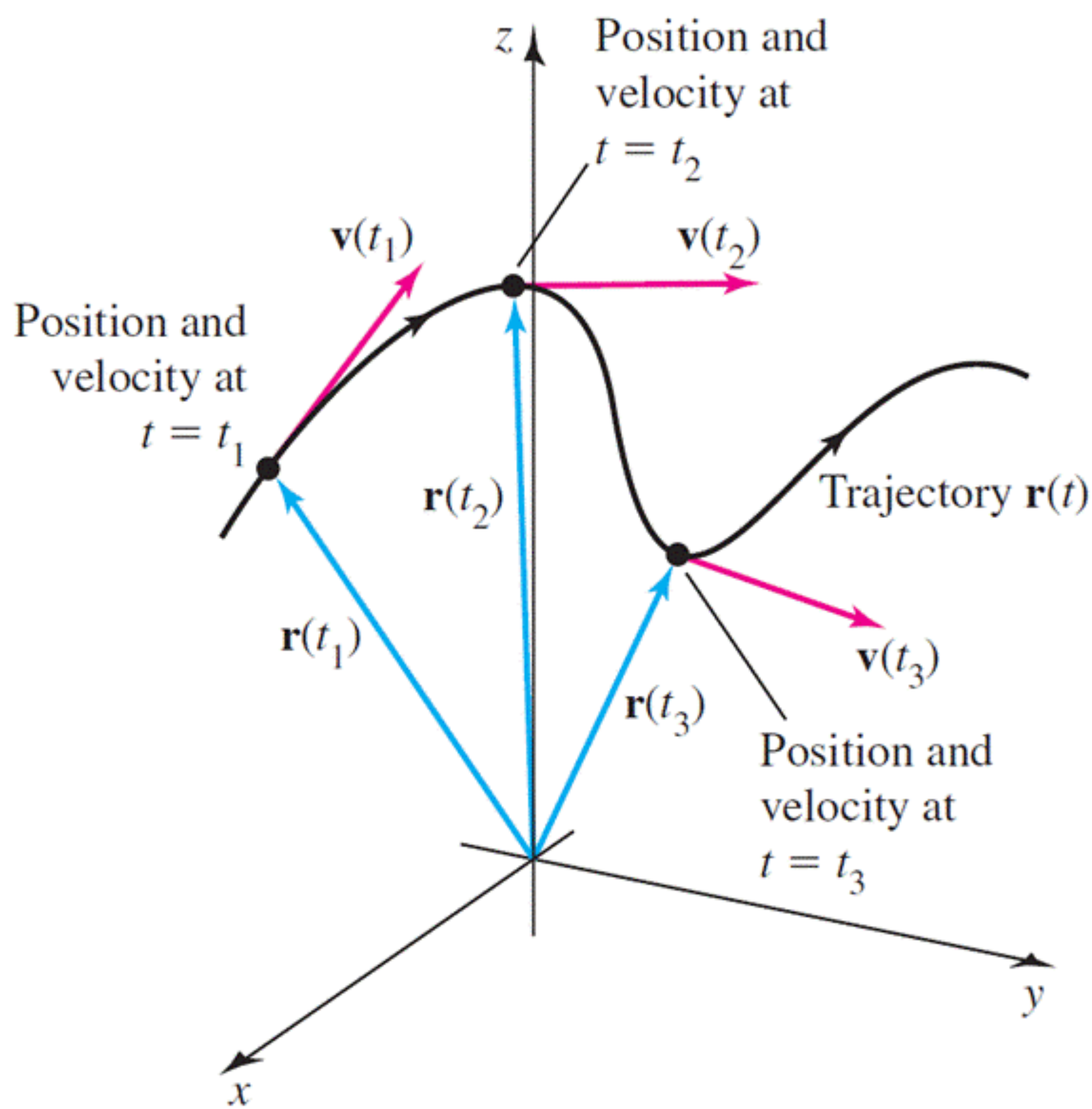

Position and velocity at $t = t_2$
$\mathbf{v}(t_1)$
$\mathbf{v}(t_2)$
Position and velocity at $t = t_1$
$\mathbf{r}(t_2)$
Trajectory $\mathbf{r}(t)$
$\mathbf{r}(t_1)$
$\mathbf{v}(t_3)$
$\mathbf{r}(t_3)$
Position and velocity at $t = t_3$
z
y
x

FIGURE 11.81

DEFINITION **Position, Velocity, Speed, Acceleration**

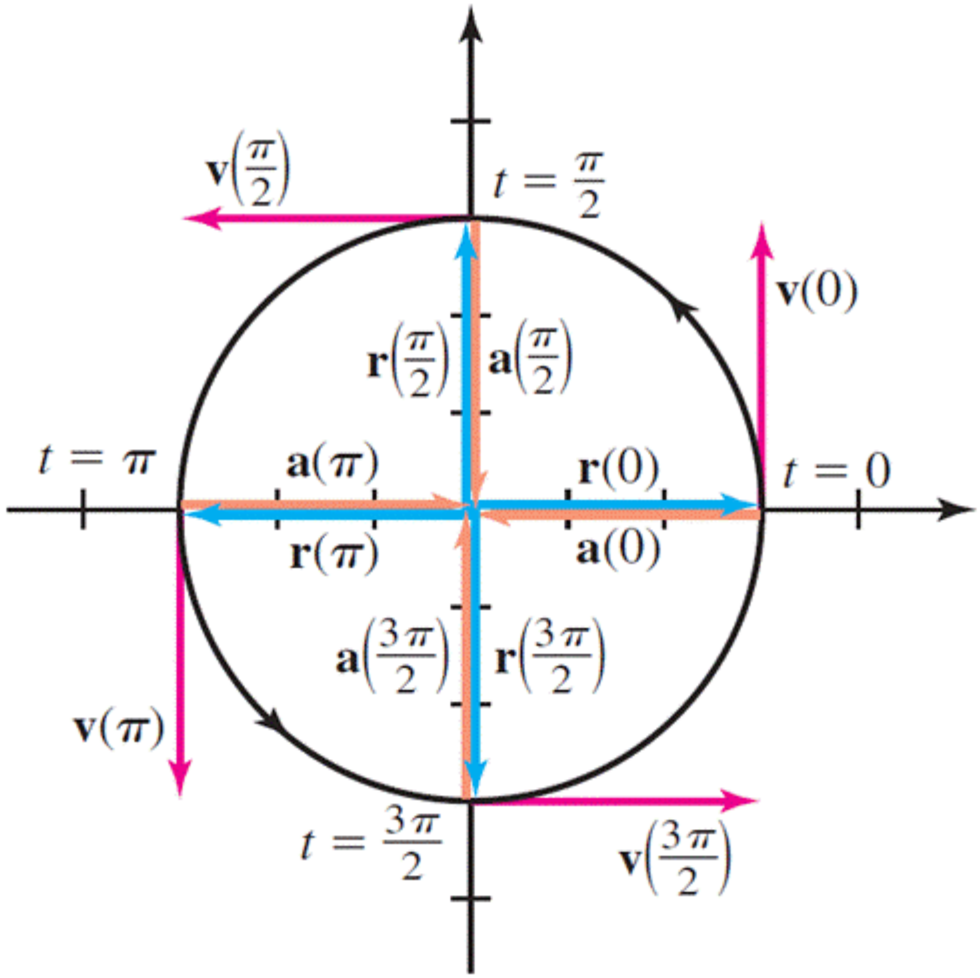
Let the **position** of an object moving in three-dimensional space be given by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $t \geq 0$. The **velocity** of the object is

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.$$

The **speed** of the object is the scalar function

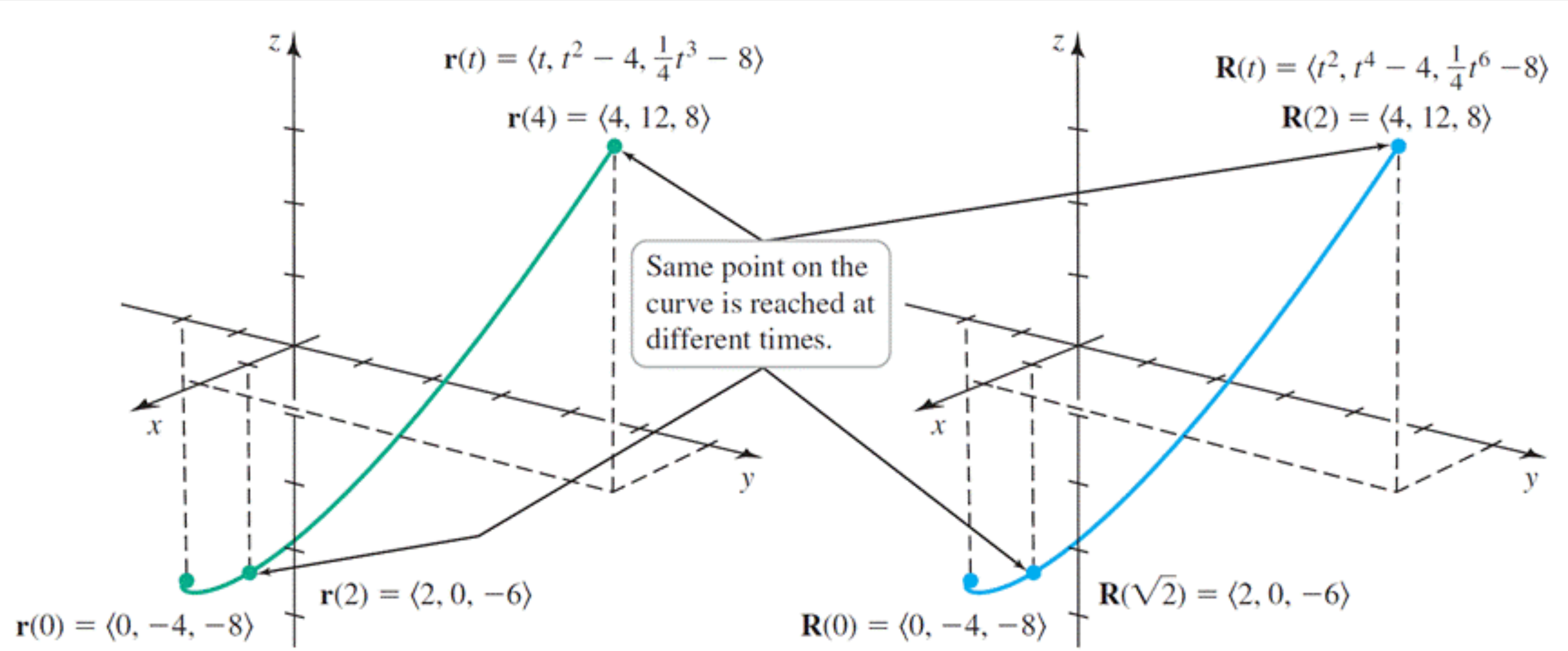
$$|\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}.$$

The **acceleration** of the object is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.



Circular motion: At all times $\mathbf{a}(t) = -\mathbf{r}(t)$ and $\mathbf{v}(t)$ is orthogonal to $\mathbf{r}(t)$ and $\mathbf{a}(t)$.

FIGURE 11.83



$\mathbf{r}(t) = \langle t, t^2 - 4, \frac{1}{4}t^3 - 8 \rangle$
$\mathbf{r}(4) = \langle 4, 12, 8 \rangle$
$\mathbf{R}(t) = \langle t^2, t^4 - 4, \frac{1}{4}t^6 - 8 \rangle$
$\mathbf{R}(2) = \langle 4, 12, 8 \rangle$
Same point on the curve is reached at different times.
$\mathbf{r}(2) = \langle 2, 0, -6 \rangle$
$\mathbf{r}(0) = \langle 0, -4, -8 \rangle$
$\mathbf{R}(\sqrt{2}) = \langle 2, 0, -6 \rangle$
$\mathbf{R}(0) = \langle 0, -4, -8 \rangle$
z
x
y
z
x
y

FIGURE 11.84

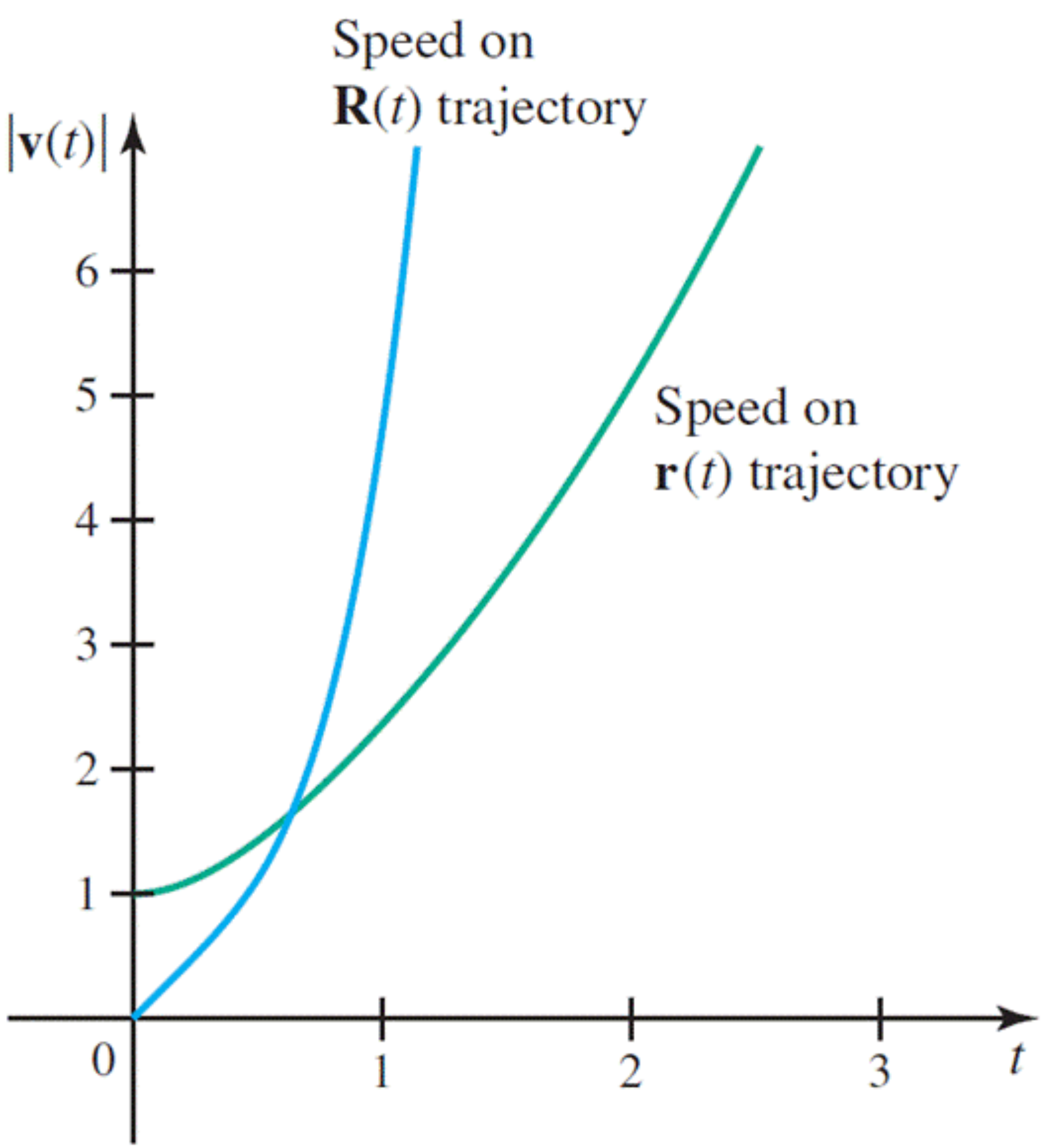

Speed on
R(t) trajectory
|v(t)|
6
5
4
3
2
1
0
1
2
3
t
Speed on
r(t) trajectory

FIGURE 11.85

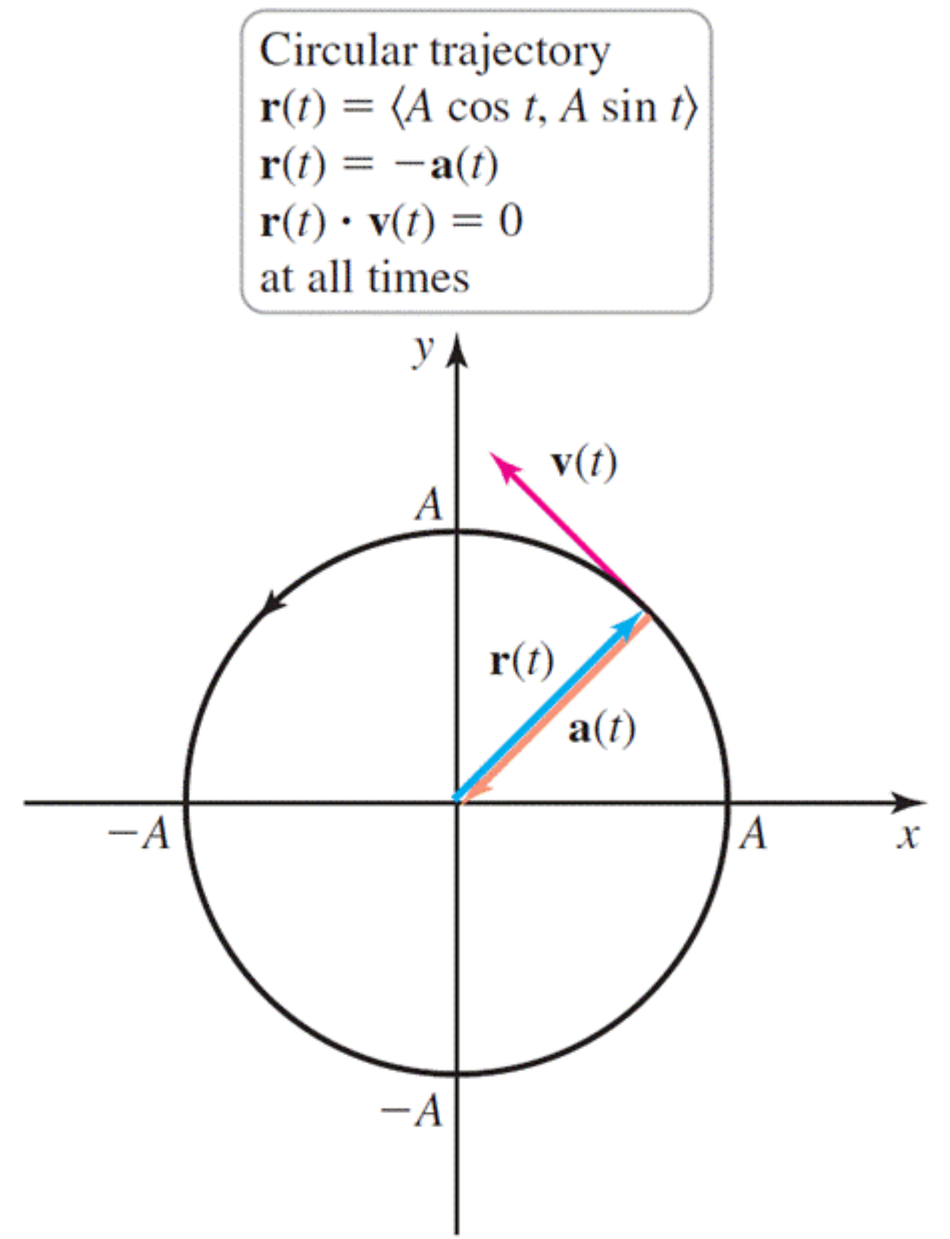
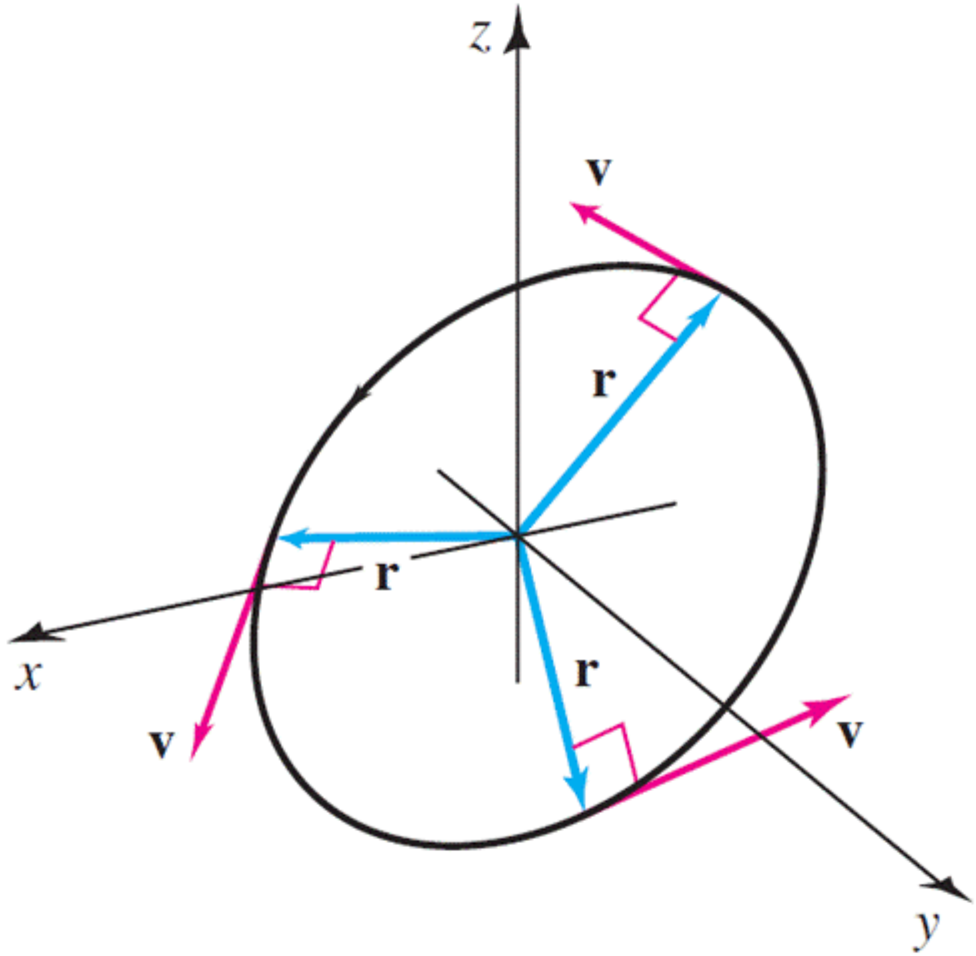
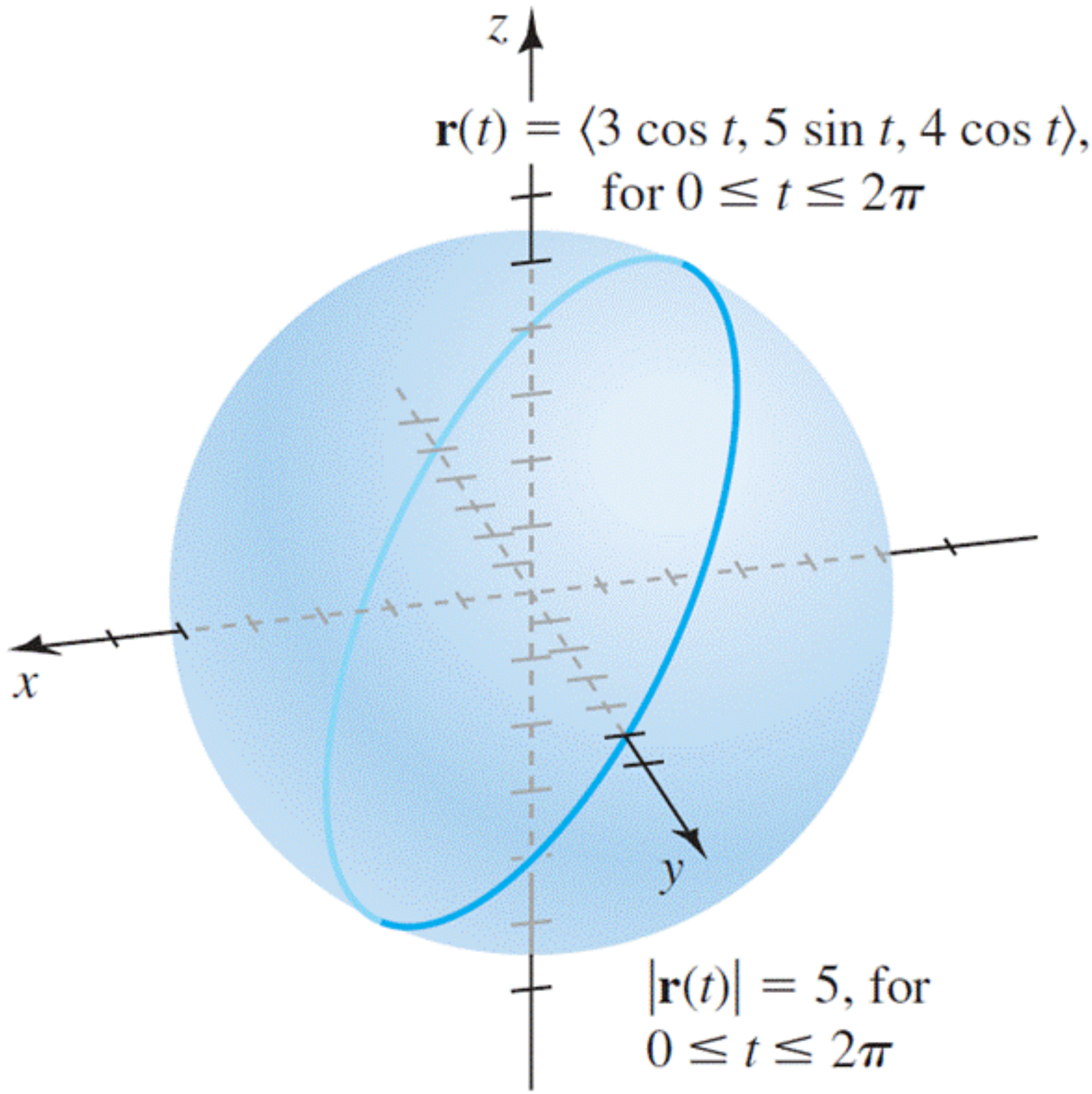

Circular trajectory
$\mathbf{r}(t) = \langle A \cos t, A \sin t \rangle$
$\mathbf{r}(t) = -\mathbf{a}(t)$
$\mathbf{r}(t) \cdot \mathbf{v}(t) = 0$
at all times
y
$\mathbf{v}(t)$
A
$\mathbf{r}(t)$
$\mathbf{a}(t)$
$-A$
A
x
$-A$

FIGURE 11.86



On a trajectory on which $|\mathbf{r}(t)|$ is constant, $\mathbf{v}$ is orthogonal to $\mathbf{r}$ at all points.

FIGURE 11.87



z
$\mathbf{r}(t) = \langle 3\cos t, 5\sin t, 4\cos t\rangle$,
for $0 \le t \le 2\pi$
x
y
$|\mathbf{r}(t)| = 5$, for
$0 \le t \le 2\pi$

FIGURE 11.88

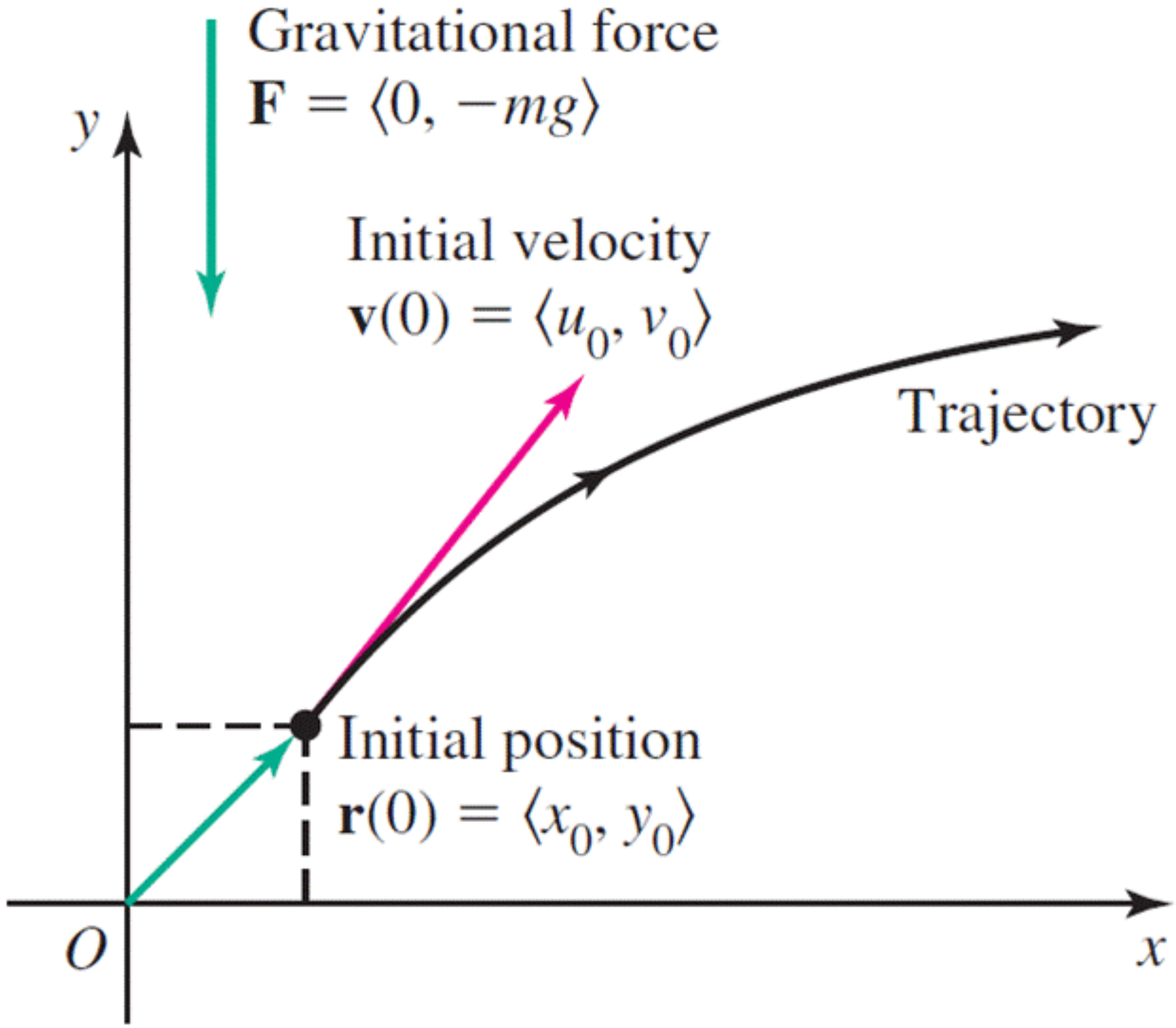

Gravitational force
$\mathbf{F} = \langle 0, -mg \rangle$
y
Initial velocity
$\mathbf{v}(0) = \langle u_0, v_0 \rangle$
Trajectory
Initial position
$\mathbf{r}(0) = \langle x_0, y_0 \rangle$
O
x

FIGURE 11.89

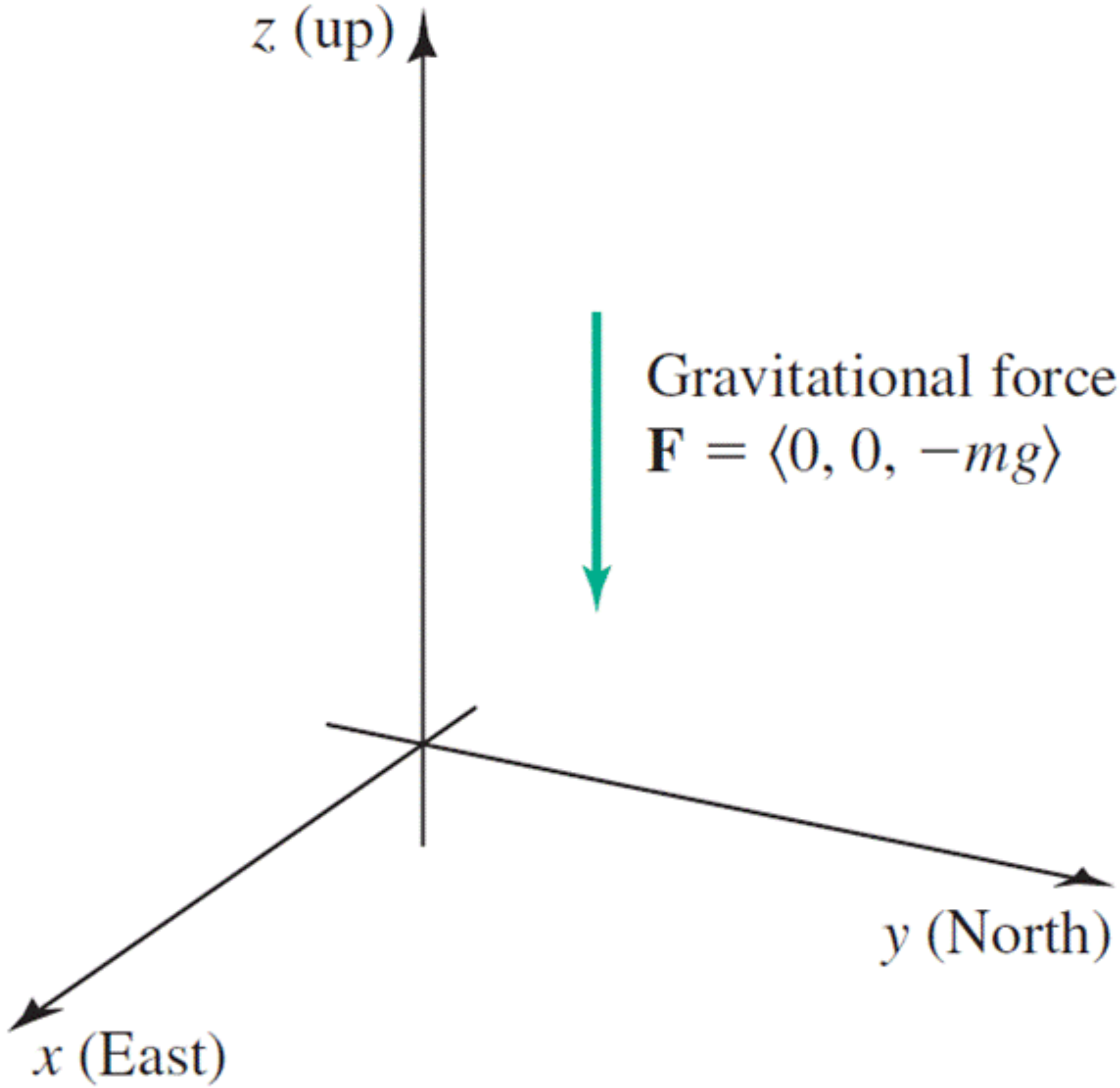


FIGURE 11.93

11.8

Length of Curves

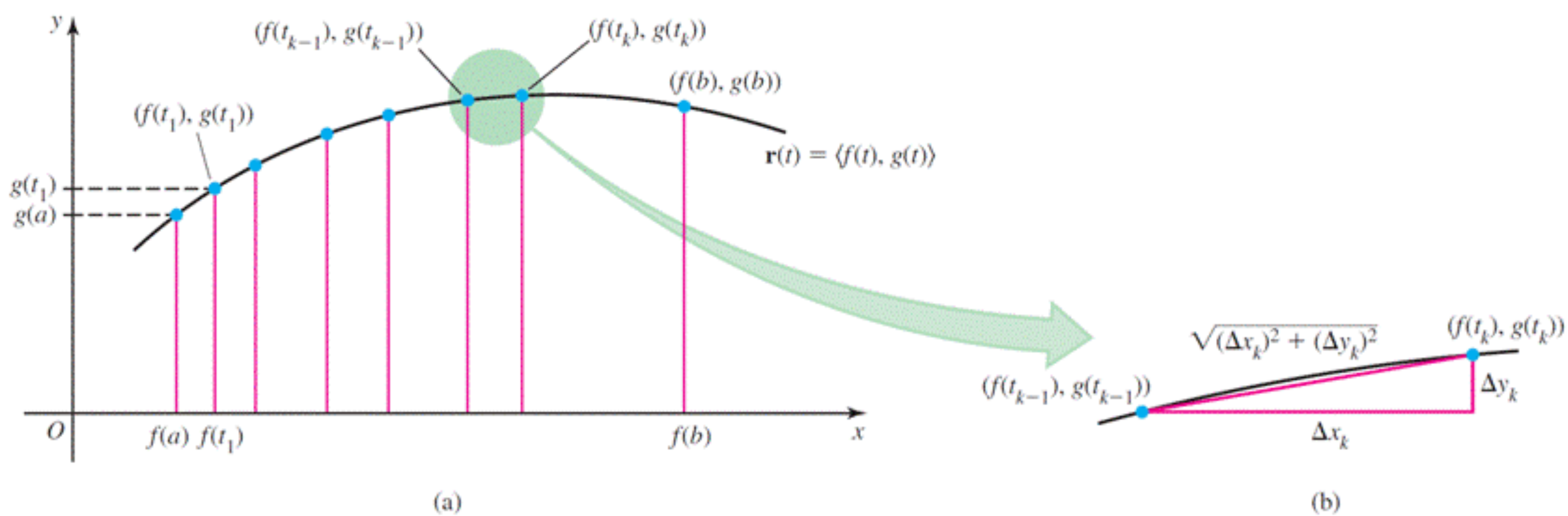
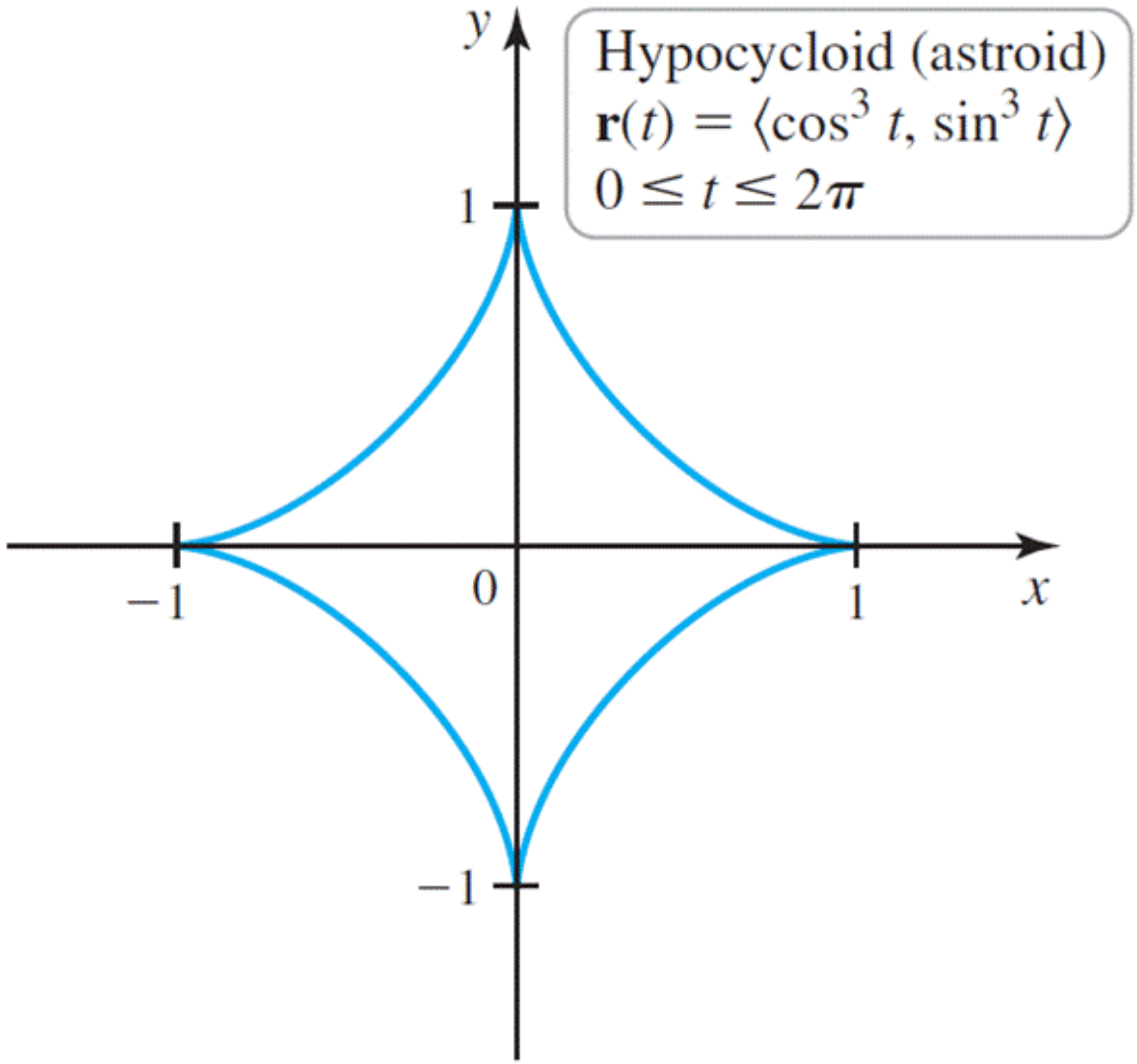

y
$(f(t_{k-1}), g(t_{k-1}))$
$(f(t_k), g(t_k))$
$(f(b), g(b))$
$(f(t_1), g(t_1))$
$\mathbf{r}(t) = \langle f(t), g(t) \rangle$
$g(t_1)$
$g(a)$
O
$f(a)$ $f(t_1)$
$f(b)$
x
(a)
$\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$
$(f(t_k), g(t_k))$
$(f(t_{k-1}), g(t_{k-1}))$
Δy_k
Δx_k
(b)

FIGURE 11.95

DEFINITION **Arc Length for Vector Functions**

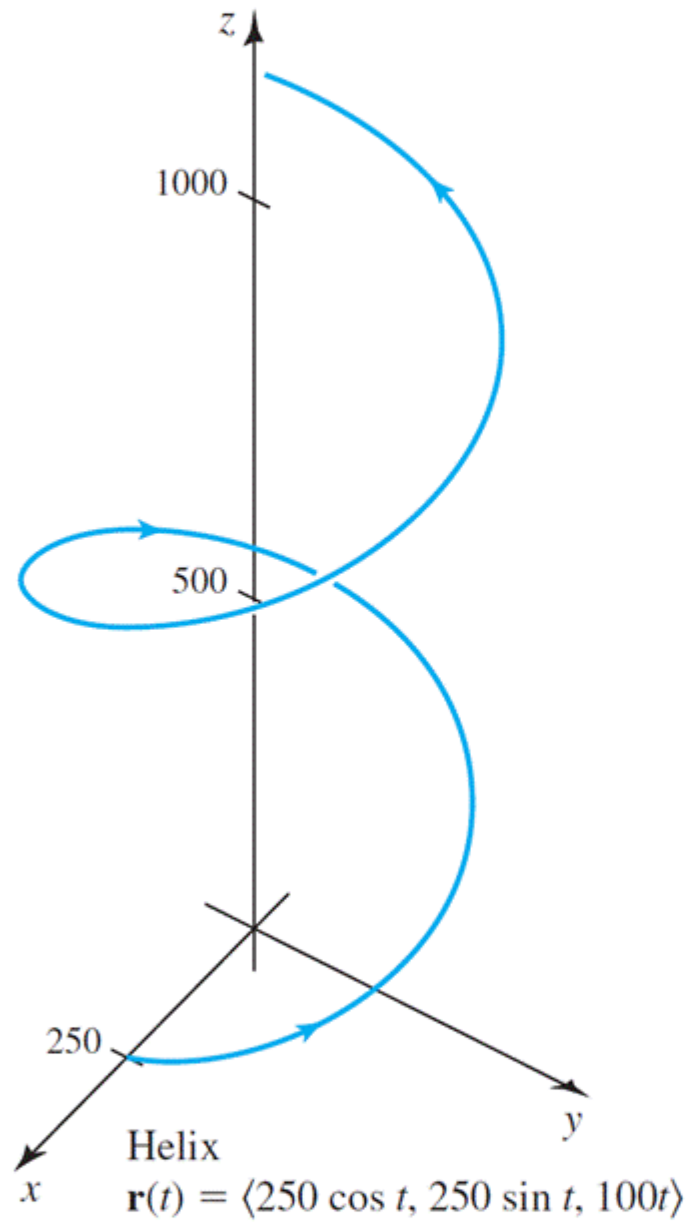
Consider the parameterized curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f', g', and h' are continuous, and the curve is traversed once for $a \le t \le b$. The **arc length** of the curve between $(f(a), g(a), h(a))$ and $(f(b), g(b), h(b))$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt = \int_a^b |\mathbf{r}'(t)|\, dt.$$



Hypocycloid (astroid)
$\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t \rangle$
$0 \le t \le 2\pi$
y
x
1
0
−1
1
−1

FIGURE 11.96


z
1000
500
250
x
y
Helix
$\mathbf{r}(t) = \langle 250 \cos t, 250 \sin t, 100t \rangle$
FIGURE 11.97

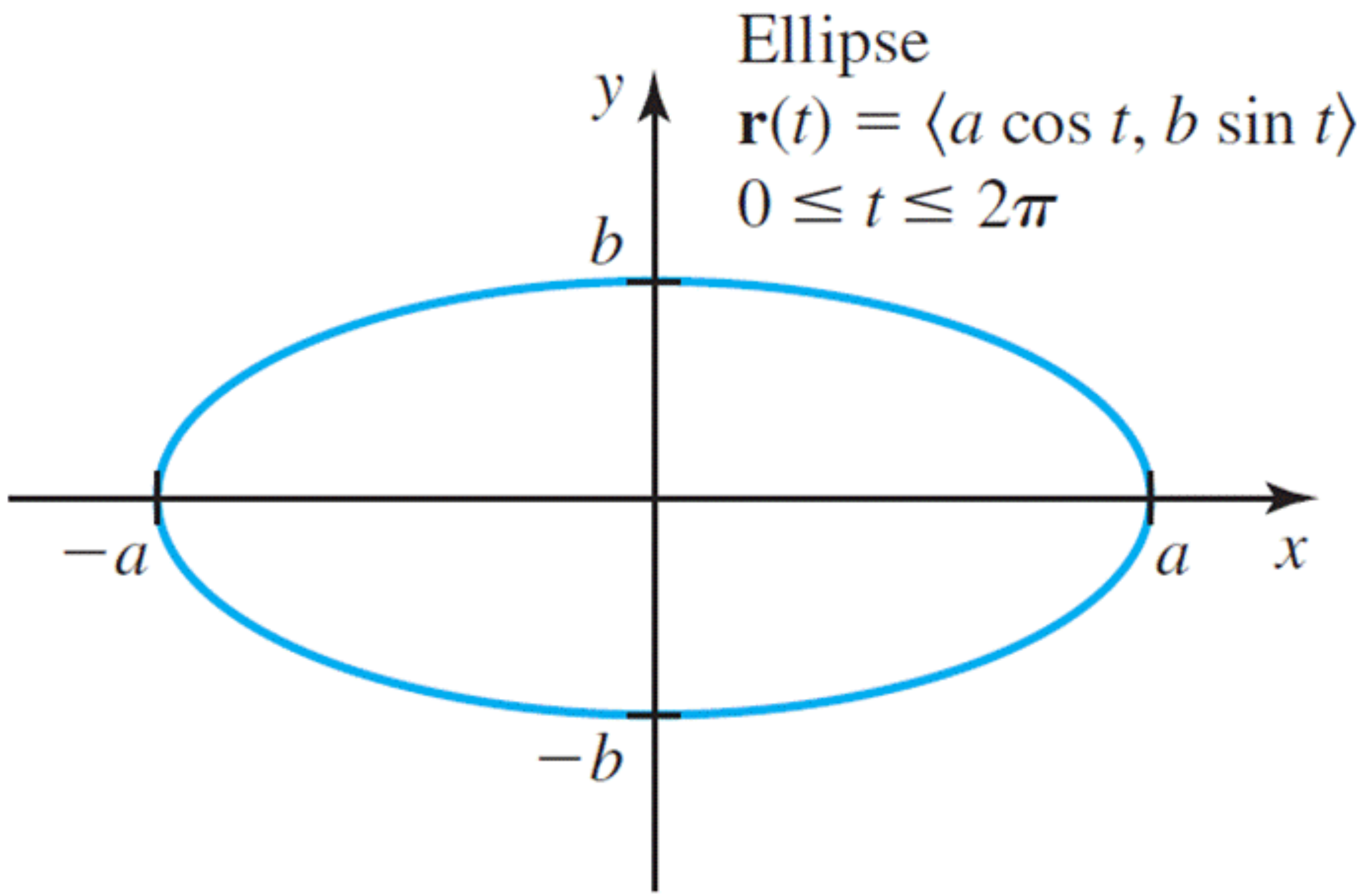

Ellipse
$\mathbf{r}(t) = \langle a \cos t, b \sin t \rangle$
$0 \leq t \leq 2\pi$
y
b
$-a$
a
x
$-b$

FIGURE 11.98

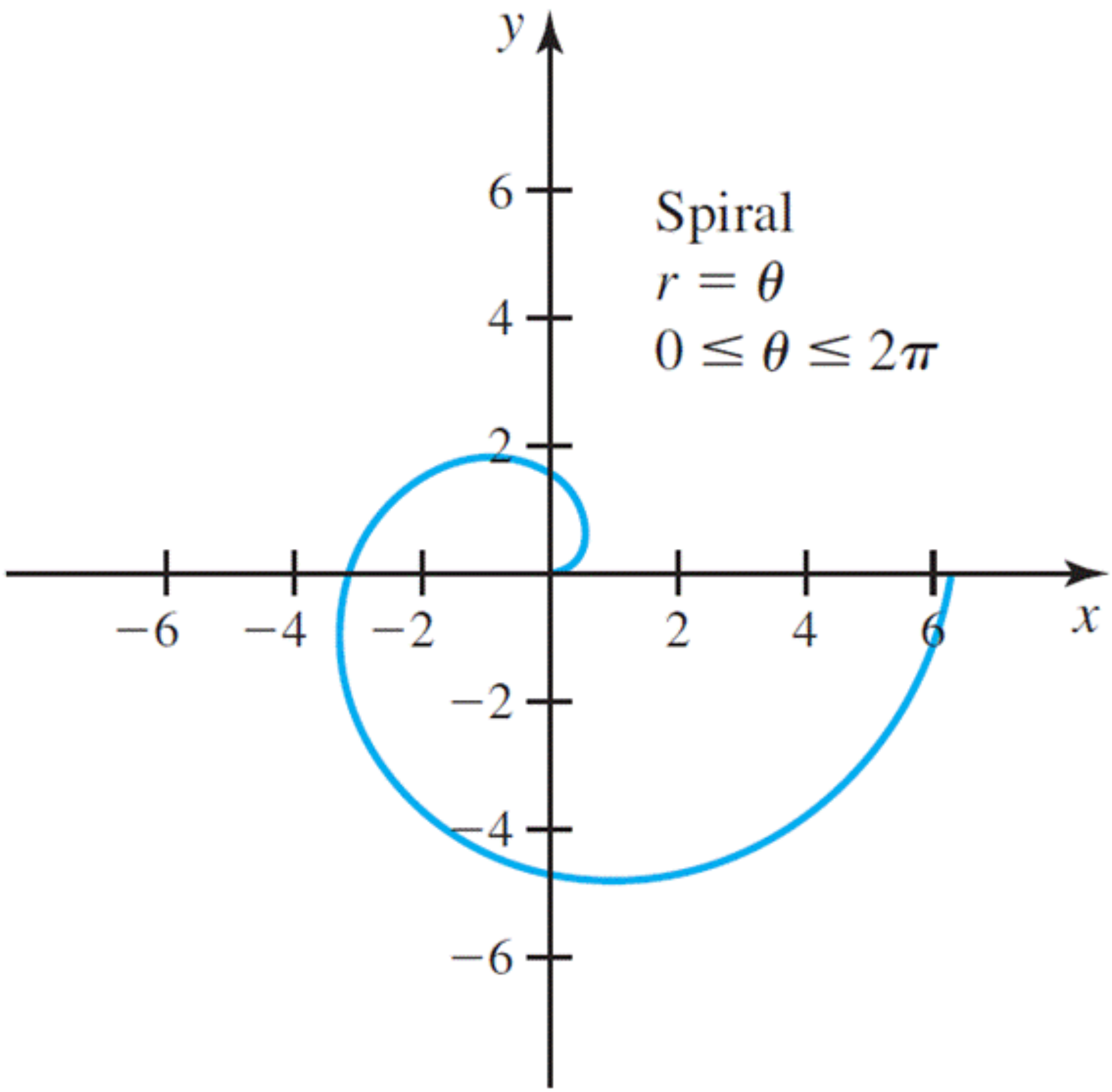


FIGURE 11.99

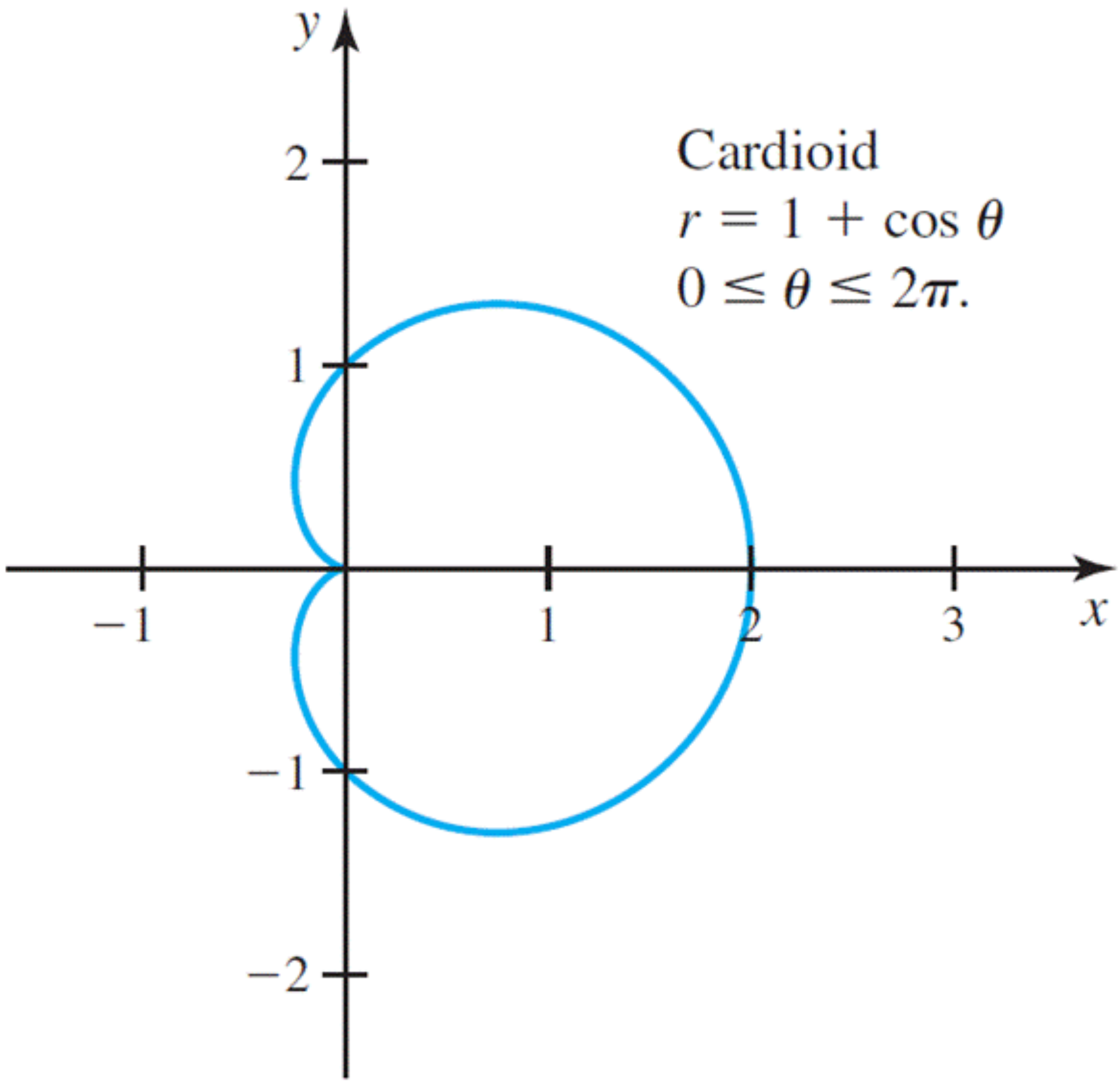


FIGURE 11.100