Lecture 2. Thursday, Sep/8/22

Outline:

- Definition of limit.
- Continuity
- Limit properties / laws
- Composition of cts fns
- Basic Computation

Limits:

<u>Def</u>: !! Let $f: D \to \mathbb{R}$ be a function defined on an open interval containing c, except possible at c. ($(c-\alpha, c) \cup (c, c+\alpha)$ for some $\alpha > 0$)

Let $L \in \mathbb{R}$ (not $\pm\infty$), then

$$\lim_{x \to c} f(x) = L$$

if $\forall \varepsilon > 0$, there exists a $\delta > 0$, s.t. $\forall x \in D$ w/ $\underbrace{0 <}_{\text{exclude } c \text{ itself}} |x-c| < \delta$, we have $|f(x) - L| < \varepsilon$

RK: ① δ changes w/ ε. usually, When ε small, δ is also small.

② To show limit exist, try using $|x-c|$ to control $|f(x) - L|$
explain control

Informal notation: $f(x) \to L$ as $x \to c$

eg1 Show $\lim_{x \to c} x = c$.

Idea: Use $|x-c|$ to control $|f(x)-L|$ $\nearrow |x-c|$

Pf: $\forall \varepsilon > 0$, pick $\delta = \varepsilon$, s.t.

$\forall x$ w/ $0 < |x-c| < \delta$, we have

$$|x-c| < \varepsilon.$$

$$\therefore \lim_{x \to c} x = c$$

eg2 Show $\lim_{x \to c} x^2 = c^2$

Q: use $|x-c|^{<\delta}$ to control $|x-c||x+c|$.

That is to say, given an $\varepsilon > 0$, how to choose δ?

$$- \quad x^2 - c^2 = \underbrace{(x-c)}_{\checkmark}\underbrace{(x+c)}_{?}$$

→ Find relation between $|x-c|$ and $|x+c|$.

Note: $|x+c| = |\underbrace{x-c}_{\text{what we have}}+2c| \le |x-c| + 2|c|$

Thus: $|x^2-c^2| \le (|x-c|+2|c|)\,|x-c|$

$$< (\delta + 2|c|)\,\delta < \varepsilon$$

Recall that ε small. δ is also small,

we can assume $\delta < 1$.

so $(\delta+2|c|)\,\delta < (1+2|c|)\,\delta < \varepsilon$.

let $\delta = \min\left\{1, \dfrac{\varepsilon}{1+2|c|}\right\}$.

Pf: $\forall \varepsilon > 0$, let $\delta = \min\left\{1, \dfrac{\varepsilon}{1+2|c|}\right\}$

Then if $|x-c| < \delta$, we have

$$|x^2-c^2| \le (1+2|c|)\frac{\varepsilon}{1+2|c|} < \varepsilon$$

$$\therefore \lim_{x \to c} x^2 = c^2$$

eg3 Show that if $\lim_{x\to c} f(x) = L$, then L is unique

Pf: prove by contradiction. Suppose

$\exists\ L_1 \neq L_2$, s.t. $\lim_{x\to c} f(x) = L_1 / L_2$.

$\therefore\ \forall\ \varepsilon>0,\ \exists\ \delta$ s.t. $\forall x,\ 0<|x-c|<\delta$.

$$|f(x)-L_1| < \varepsilon, \quad |f(x)-L_2| < \varepsilon.$$

$$\Rightarrow\ |L_1-L_2| \leqslant |f(x)-L_1| + |L_2-f(x)|$$

$$< 2\varepsilon$$

Let $\varepsilon < \frac{1}{2}|L_1-L_2|$ (arbitrarily chosen)

$\Rightarrow \Leftarrow$

Hard, we will not focus on this ε-δ proofs.

Q: How to compute limit more effeciently?

Continuity:

Recall in general $\lim_{x\to c} f(x) \neq f(c)$

But we can see for a large group of functions, this prop holds.

Def: Let $f: D \to \mathbb{R}$ be a fn on an open nbhd of c (including c, $(c-a, c+a)$)

We say f is cts at $x=c$, if

$$\lim_{x\to c} f(x) = f(c)$$

A fn is cts if it is cts at all pts

Idea: No jumps.

RK: The following fns are all cts.

- Constant
- Polynormial
- Exponential
- logarithm (for $x > 0$)
- $\sin(x)$, $\cos(x)$

eg: $\lim_{x \to c} x^2 = c^2$

Pf: As $f(x) = x^2$ cts, we have

$$\lim_{x \to c} f(x) = f(c) = c^2$$

eg $\lim_{x \to 1} \sin(x) = \sin(1)$

Limit property

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c} [f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x \to c} f(x) = L > 0$.)

eg: $\lim_{x \to 2} \frac{\sqrt{\ln x}}{x^2-3}$

Show every details, could omit when experienced.

$$= \frac{\lim_{x \to 2} \sqrt{\ln x}}{\lim_{x \to 2} (x^2-3)} \quad (\text{law } 5)$$

$$= \frac{\sqrt{\lim_{x \to 2} \ln x}}{\lim_{x \to 2} x^2 - \lim_{x \to 2} 3} \quad \begin{matrix}(\text{law } 7) \\ (\text{law } 2)\end{matrix} \quad = \frac{\sqrt{\ln 2}}{2^2-3} \quad (\text{cts})$$

$$= \sqrt{\ln 2}$$