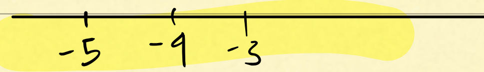
4.2

15. a) ii

$y = x^2 + 8x + 15$

$= (x+5)(x+3)$

$x = -5 \quad x = -3$



$y' = 2x + 8$

$= 2(x+4)$

$x = -4$

b) $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$

$f'(x) = nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \dots + a_1$

We say that $f(x)$ is continuous at the interval $x = [a, b]$, and have a absolute maximum or minimum value.

then $f(a) = f(b) = 0$, where $f(x)$ has an absolute maximum or minimum value at $x = c$ in the interval $[a,b]$, therefore $f'(c) = 0$

$\therefore$ So, there lies a zero of $nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \dots + a_1$ between every two zeros of $x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$

18. If $f(x)$ is a cubic polynomial with 4 or more zeros,

by Rolle's Theorem:

$f'(x)$ = has 3 or more zeros

$f''(x)$ = has 2 or more zeros

$f'''(x)$ = has at least one zero.

This is a contradiction since:

when $f(x)$ is a cubic polynomial, $f'''(x)$ is a non-zero constant.

20. $f(x) = x^3 + \frac{4}{x^2} + 7, (-\infty, 0)$

$f'(x) = 3x^2 + \frac{8}{x^3} = 0$

$x^2\left(3x + \frac{8}{x^5}\right) = 0$

$x^2 = 0$

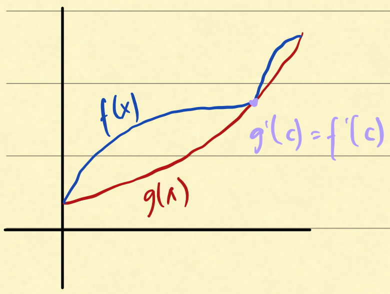
28. $f(0) = 5$, and $f'(x) = 2$ for all x

Must $f(x) = 2x + 5$ for all x?

* Yes, since the derivative of $f(x)$ has to be 2. If the coefficient of x is not 2, then $f'(x) \neq 2$. We can prove that there has to be the constant 5 by using $f(0) = 5$. We know that there should be $2x$. But when we use $f(0)$, the answer should be $f(0) = 5$. So, $f(x) = 2x+5$ is a must for all x.

60. f and g are diff on $[a,b]$, and $f(a) = g(a)$, $f(b) = g(b)$.

$$f'(c) = \frac{f(b) - f(a)}{b-a} \qquad g'(c) = \frac{g(b) - g(a)}{b-a}$$



So, $f'(c) = g'(c)$

61. $f'(x) \leq 1$ for $1 \leq x \leq 4$. Show that $f(4) - f(1) \leq 3$

$$f'(x) = \frac{f(4) - f(1)}{4-1} \leq 1$$

$$= \frac{f(4) - f(1)}{3} \leq 1$$

$$= f(4) - f(1) \leq 3$$

63. Show that $|\cos x - 1| \leq |x|$ for all x values.

Consider $f(t) = \cos t$ on $[0, x]$

Rolle's theorem:

$$f'(t) = \frac{\cos x - \cos 0}{x} = \frac{\cos x - 1}{x}$$

in the interval $[0, x]$ $\exists$ $x = c$

$f'(c) = 0$

$f'(c) = -\sin c$

we know that $f'(c) = f'(t)$ so, $-\sin c = \dfrac{\cos x - 1}{x}$

$$-1 \leq -\sin c \leq 1$$

$$-1 \leq \frac{\cos x - 1}{x} \leq 1$$

$$-x \leq \cos x - 1 \leq x$$

$$|\cos x - 1| \leq |x|$$

65. Yes since they have the same rate of change which means that they have the same slope.
Both functions also have the same starting point which makes them identical, knowing that they have the same slope.

4.3

38. a) $g(x) = x^{\frac{2}{3}}(x+5)$

$$g'(x) = \frac{2}{3}x^{-\frac{1}{3}}(x+5) + x^{\frac{2}{3}}$$

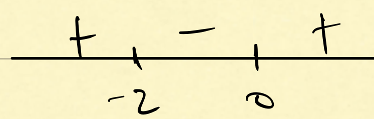
$$= \frac{2}{3\sqrt[3]{x}}(x+5) + x^{\frac{2}{3}}$$

$$= \frac{2x+10}{3\sqrt[3]{x}} + x^{\frac{2}{3}} = 0$$

$$2x + 10 + 3x = 0$$

$$5x + 10 = 0$$

$x = -2 \rightarrow$ Critical point



$$f(-1) = \frac{2(-1)+10}{3\sqrt[3]{-1}} + (-1)^{\frac{2}{3}} = -$$

decreasing at $(-\infty, -2) \cup (0, \infty)$

increasing at $(-2, 0)$

b) $x = -2 \quad x = 0$

$g(-2) = 9.2 \longrightarrow$ local max at $(-2, 9.2)$

$g(0) = 0 \longrightarrow$ local min at $(0,0)$

51. $g(x) = \dfrac{x-2}{x^2-1}, \quad 0 \le x < 1$

a) $g'(x) = \dfrac{1(x^2-1) - (x-2)(2x)}{(x^2-1)^2}$

$= \dfrac{x^2-1-2x^2+4x}{(x^2-1)^2}$

$= \dfrac{-x^2+4x-1}{(x^2-1)^2} = 0$

$-x^2+4x-1=0$

$\dfrac{-4 \pm \sqrt{16-4(-1)(-1)}}{-2} = \dfrac{-4 \pm \sqrt{16-4}}{-2}$

$= \dfrac{-4 \pm 2\sqrt{3}}{-2}$

$x = 2+\sqrt{3}$

$\cancel{x} = 2-\sqrt{3}$

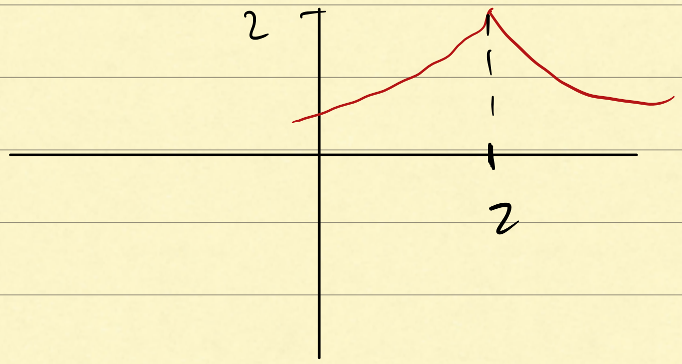
local max $\qquad$ local min

$f(0) = 2 \qquad f(2-\sqrt{3}) = \dfrac{\sqrt{3}}{4\sqrt{3}-6}$

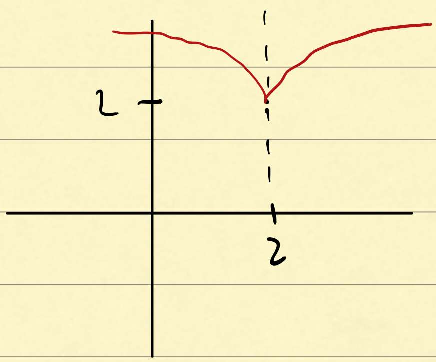
6. there is an absolute minimum at $x = 2-\sqrt{3}$ which is $\dfrac{\sqrt{3}}{4\sqrt{3}-6}$

No absolute max

65. a)

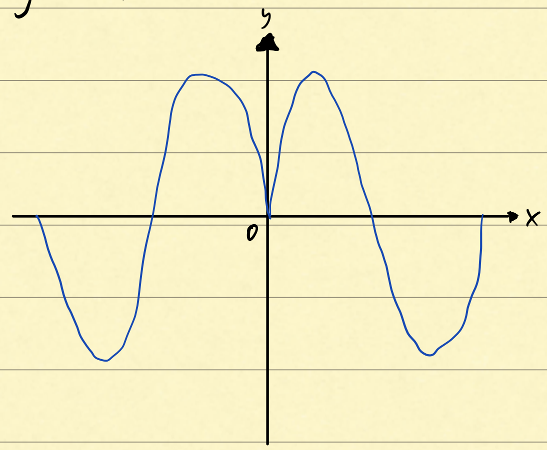


b)



4.4

7. $y = \sin|x|, -2\pi \leq x \leq 2\pi$



$\sin|x| = \sin x$ if $x \geq 0$

$\sin|x| = -\sin x$ if $x < 0$

* The graph is rising on $(-\frac{3\pi}{2}, -\frac{\pi}{2})$, $(0, \frac{\pi}{2})$ and $(\frac{3\pi}{2}, 2\pi)$
* The graph is falling on $(-2\pi, -\frac{3\pi}{2})$, $(-\frac{\pi}{2}, 0)$ and $(\frac{\pi}{2}, \frac{3\pi}{2})$

* points of inflection is $(-\pi, 0)$ and $(\pi, 0)$
* Local minimum is -1 at $x = \pm\frac{3\pi}{2}$, 0 at $x = 0$
* Local max is 1 at $x = \pm\frac{\pi}{2}$, 0 at $x = \pm 2\pi$
* Concave up on $(-2\pi, -\pi)$ and $(\pi, 2\pi)$
* Concave down on $(-\pi, 0)$ and $(0, \pi)$

19. $y = 1 - 9x - 6x^2 - x^3$

$y' = -3x^2 - 12x - 9$

$y'' = -6x - 12$

$y'' = 0$

$-6x - 12 = 0$

$-6x = 12$

$x = -2 \longrightarrow$ Inflection point

$y' = -3x^2 - 12x - 9$

$0 = 3x^2 + 12x + 9$

$= (3x+3)(x+3)$

$x = -1 \qquad x = -3$

$y(-1) = 1 - 9(-1) - 6(-1)^2 - (-1)^3$

$= 1 + 9 - 6 + 1$

$= 5 \longrightarrow (-1, 5)$ local max

$y(-3) = 1 - 9(-3) - 6(-3)^2 - (-3)^3$

$= 1 + 27 - 54 + 27$

$= 1 \longrightarrow (-3, 1)$ local min

35. $y = x^{\frac{2}{3}}\left(\frac{5}{2}-x\right)$

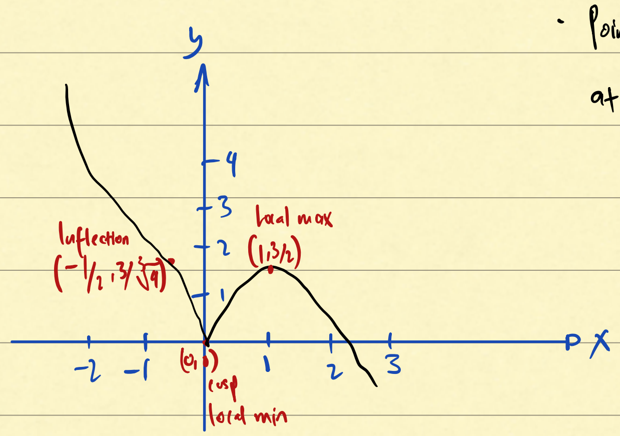
$= \frac{5}{2}x^{\frac{2}{3}} - x^{\frac{5}{3}}$

$y' = \frac{5}{3}x^{-\frac{1}{3}} - \frac{5}{3}x^{\frac{2}{3}}$

$y'' = -\frac{5}{9}x^{-\frac{4}{3}} - \frac{10}{9}x^{-\frac{1}{3}}$

$= \frac{-5}{9}x^{-\frac{4}{3}}(1+2x)$

- The graph is rising in the coordinate $(0,1)$, falling on $(-\infty,0)$ and $(1,\infty)$.
- The local minimum is 0 at $x=0$
- The local maximum is $\frac{3}{2}$ at $x=1$
- Concave up on $(-\infty,-\frac{1}{2})$
- Concave down on $(-\frac{1}{2},0)$ and $(0,\infty)$
- Point inflection at $x=-\frac{1}{2}$ and a cusp at $x=0$



88. $y = \dfrac{x^3+x-2}{x-x^2}$

$= \dfrac{\cancel{(x-1)}(x^2+x+2)}{x\cancel{(x-1)}(-1)}$

$= \dfrac{x^2+x+2}{-x}$

$y' = \dfrac{(2x+1)(-x)-(x^2+x+2)(-1)}{-x^2}$

$= \dfrac{-2x^2-x+x^2+x+2}{-x^2}$

$= \dfrac{-x^2+2}{-x^2}$

$y' = 0$

$-x^2 + 2 = 0$

$-x^2 = -2$

$x = \pm\sqrt{2}$

$x = \sqrt{2} \longrightarrow f(\sqrt{2}) = \dfrac{(\sqrt{2})^2 + \sqrt{2} + 2}{-\sqrt{2}}$

$= -1 - 2\sqrt{2}$

$= -3.83 \longrightarrow (\sqrt{2}, -3.83)$ local max

$x = -\sqrt{2} \longrightarrow f(-\sqrt{2}) = \dfrac{(-\sqrt{2})^2 - \sqrt{2} + 2}{\sqrt{2}}$

$= -1 + 2\sqrt{2}$

$= 1.83 \longrightarrow (-\sqrt{2}, 1.83)$ Local min

$-$ $\quad$ $+$ $\quad$ $-$

$-\sqrt{2}$ $\quad$ $\sqrt{2}$

increasing at $(-\sqrt{2}, \sqrt{2})$

decreasing at $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

$y' = \dfrac{x^2 + 2}{x^2}$

$y'' = \dfrac{(-2x)(x^2) - (-x^2+2)(2x)}{(x^2)^2} = \dfrac{-2x^3 + 2x^3 - 4x}{x^4}$

$$= \frac{-9x}{x^4}$$

$y'' = 0$

$$\frac{-9x}{x^4} = 0$$

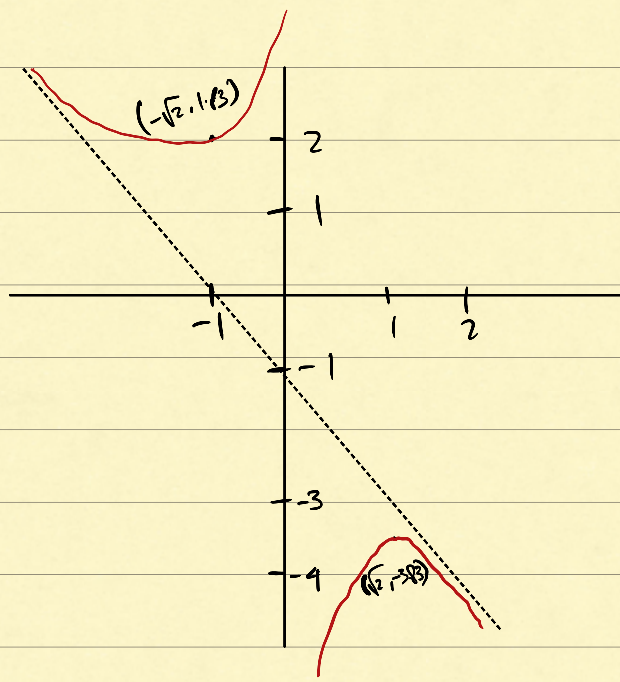
$x = 0$

There is no inflection points since x^{15} discontinues at $x=0$

$$y = \frac{x^2+x+2}{-x}$$

$$= -x - 1 - \frac{2}{x}$$

for $x = \infty$, $y = -x-1$



98. a) * The object is moving away from the origin at $1.5 < t < 4$; $10 < t < 12$; $13.5 < t < 16$

* The object is moving towards the origin at $0 < t < 1.5$; $4 < t < 10$; $12 < t < 13.5$

b) Velocity $= 0$, when the tangent line is horizontal, $t = 4$; $t = 12$; $t = 16$

c) Acceleration $= 0$, when there exist a point of inflection, $t = 2$; $t = 6$; $t = 9$; $t = 11$; $t = 13.5$

d) Acceleration positive, when concaved up at $0<t<1.5$; $6<t<8$; $10<t<13.5$

Acceleration negative, when concaved down at $2<t<6$; $8<t<10$; $13.5<t<16$

100. Increasing at $2<t<5$, $9<t<12$

Decreasing at $0<t<2$, $5<t<9$

108. $y = ax^3 + bx^2 + cx + d$, $a \neq 0$?

$y' = a.3x^2 + b.2x + c$

$y'' = a.6x + 2b$

$y'' = 0$

$\frac{6ax + 2b}{2} = \frac{0}{2}$

$3ax + b = 0$

$x = \frac{-b}{3a} \rightarrow$ Point of Inflection

$-$ | $+$

$-\frac{b}{3a}$

$\therefore a \neq 0$ since the equation will not exist

111. $y = ax^3 + bx^2 + cx$

$y' = a.3x^2 + b.2x + c$

$y'' = a.6x + 2b$

$y'' = 0$ at $x = 1$

$6a(1) + 2b = 0$

$\frac{6a + 2b}{2} = \frac{0}{2}$

$3a + b = 0$

$b = -3a$

$y' = 0$ at $x = -1, 3$

$3a(-1)^2 + 2b(-1) + c = 0$

$3a - 2b + c = 0$

$3a - 2(-3a) + c = 0$

$3a + 6a + c = 0$

$9a + c = 0$

$c = -9a$

$y = ax^3 + bx^2 + cx$

$11 = a(1)^3 + b(1)^2 + c(1)$

$11 = a + b + c$

$11 = a - 3a - 9a$

$-11a = 11$

$a = -1$

$b = -3(-1)$

$= 3$

$c = -9(-1)$

$= 9$

4.5

12. $V_{cone} = \frac{1}{3}\pi r^2 h$

$r = x$ $\quad h = 3 + y$

$x = \sqrt{3^2 - y^2}$

$x = \sqrt{9 - y^2}$

$$V = \frac{1}{3}\pi \left(\sqrt{9-y^2}\right)^2 (3+y)$$
$$= \frac{1}{3}\pi (9-y^2)(3+y)$$
$$= \frac{1}{3}\pi (27 + 9y - 3y^2 - y^3)$$

$$V' = \cancel{\tfrac{1}{3}}\pi\left(\cancel{9}^{\,3} - \cancel{6}^{\,2}y - \cancel{3y^2}\right)$$
$$0 = \pi(3 - 2y - y^2)$$
$$y^2 + 2y - 3 = 0$$
$$(y+3)(y-1) = 0$$
$$y = -3 \quad y = 1$$

$\longrightarrow$ largest volume $= \frac{1}{3}\pi\left(27 + 9(1) - 3(1)^2 - (1)^3\right)$

$$= \frac{32\pi}{3}$$

15. * Designing a $1000\ cm^3$ right circular cylindrical.

* The top and bottom of radius r will be cut from squares that measure $2r$ units on a side.

* Total amount of aluminium used by the can:

$$A = 8r^2 + 2\pi r h$$

rather than:

$$A = 2\pi r^2 + 2\pi r h$$

Solution:

$V = \pi r^2 h$

$1000 = \pi r^2 h$

$h = \dfrac{1000}{\pi r^2}$

$A = 8r^2 + 2\pi r h = 8r^2 + \dfrac{2000}{r}$

$A' = 16r - \dfrac{2000}{r^2} = 0$

$-16 = \dfrac{-2000}{r^2}$

$\cancel{r = \dfrac{-2000}{-16r^2}}$

$r = \dfrac{125}{r^2}$

$r = 5 \rightarrow h = \dfrac{1000}{\pi r^2}$

$= \dfrac{1000}{25\pi} = \dfrac{40}{\pi} \rightarrow \dfrac{40}{\pi} : \dfrac{8}{\pi}$

20. a) $V = x^2 y$

$y = 276 - 4x$

$V = x^2(276 - 4x)$

$= 276x^2 - 9x^3$

$V' = 552x - 27x^2 = 0$

$3x(184-3x)=0$

$x=0$

$3x=184$

$x=61.33$

$f(61.33)=276-4(61.33)$

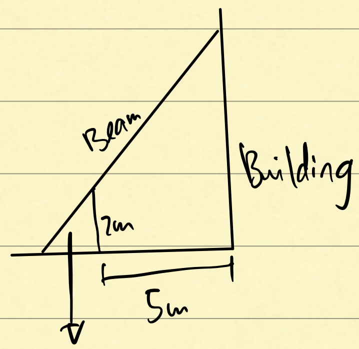
$=276-245.32$

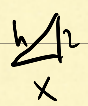
$=30.68$

Dimension is 30.68cm

length of square end is 61.33cm

39.





$h=\sqrt{2^2+x^2}$

$=\sqrt{4+x^2}$

ratio: $\frac{h}{x} : \frac{B}{x+5}$

$B=\frac{h(x+5)}{x}$

$=\frac{\sqrt{4+x^2}(x+5)}{x}$

$=(\sqrt{4+x^2})(1+5x^{-1})$

$B'=\frac{1}{\cancel{2}}(4+x^2)^{-\frac{1}{2}}(\cancel{2}x)(1+5x^{-1})+(-5x^{-2})(\sqrt{4+x^2})$

$B'=0$

$$\frac{x(1+5x^{-1})}{-\sqrt{9+x^2}} - \frac{\sqrt{9+x^2}}{5x^2} = 0$$

$$(x+5)x^2 = 20 + 5x^2$$

$$x^3 + 5x^2 = 20 + 5x^2$$

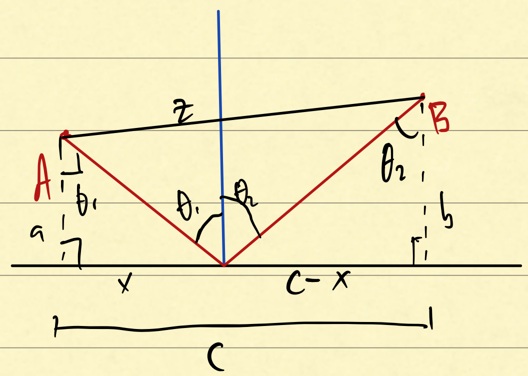
$$x^3 = 20$$

$$x = 20^{\frac{1}{3}}$$

$$\left(1+\frac{5}{20^{\frac{1}{3}}}\right)\left(\sqrt{9+(20^{\frac{1}{3}})^2}\right)$$

$= 9.58\text{cm} \rightarrow$ Shortest length

48.



Suppose $\theta_1 + \theta_2 = 90°$

$$z = \sqrt{a^2+x^2} + \sqrt{b^2+(c-x)^2}$$

$$z' = \frac{\cancel{2}x}{\cancel{2}\sqrt{a^2+x^2}} + \frac{\cancel{2}(c-x)(-1)}{\cancel{2}\sqrt{b^2+(c-x)^2}}$$

$z' = 0$

$$\frac{x}{\sqrt{a^2+x^2}} + \frac{(c-x)(-1)}{\sqrt{b^2+(c-x)^2}} = 0$$

$$\frac{x}{\sqrt{a^2+x^2}} = \frac{c-x}{\sqrt{b^2+(c-x)^2}}$$

$$\sin\theta_1 = \sin\theta_2$$

So, $\theta_1 = \theta_2$

53. a) $A(q) = \frac{km}{q} + cm + \frac{hq}{2}$

$$A'(q) = -\frac{km}{q^2} + \frac{h}{2} = 0$$

$$= \frac{h}{2} = \frac{km}{q^2}$$

$$q^2 = \frac{2km}{h}$$

$$q = \sqrt{\frac{2km}{h}}$$

b) $A(q) = \frac{(k+bq)m}{q} + cm + \frac{hq}{2}$

$$= \frac{km + b\cancel{q}m}{\cancel{q}} + cm + \frac{hq}{2}$$

$$= \frac{km}{q} + bm + cm + \frac{hq}{2}$$

$$= kmq^{-1} + bm + cm + \frac{h}{2} \cdot q$$

$$A'(q) = \frac{-km}{q^2} + \frac{h}{2} = 0$$

$$\frac{h}{2} = \frac{km}{q^2}$$

$$q^2 = \frac{2km}{h}$$

$$q = \sqrt{\frac{2km}{h}}$$

54. $\bar{C}(x) = \frac{C(x)}{x}$ → Average cost of producing x items

If average cost can be minimized, then $\frac{d}{dx}\left(\frac{C(x)}{x}\right) = 0$

$$= \frac{C'(x)\cdot x - C(x)\cdot 1}{x^2} = 0$$

$$= C'(x)\cdot x - C(x) = 0$$

$$C'(x) = \frac{C(x)}{x}$$ → Marginal cost

∴ There is no guarantee in getting minimum cost from the production level, but indicates where the minimum is located

60. a) $V = c(r_0 - r)r^2$

$= cr_0r^2 - cr^3$

$V' = 2cr_0r - 3cr^2$

$V'' = 2c(r_0 - 3r)$

$V'' = 0$

$2c(r_0 - 3r) = 0$

$r = 0$

$r = \frac{2r_0}{3}$

$V' > 0$ if $r < \frac{2r_0}{3}$

$V < 0$ if $r > \frac{2r_0}{3}$

$r = \frac{2r_0}{3}$ is greatest