What Is Calculus?

- Calculus is the study of continuous changes and limits

Calculus
- differential calculus
- integral calculus

Old name : Calculus of infinitesimal

A system of symbolic expression

close to 0

Achilles' heel

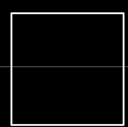
ex 1. Zeno's Paradox : Achilles can never catch up w/ a turtle.

This is obviously false

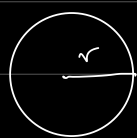
The infinite many number may not sum up to infinite.

ex 2. Gongsun Long

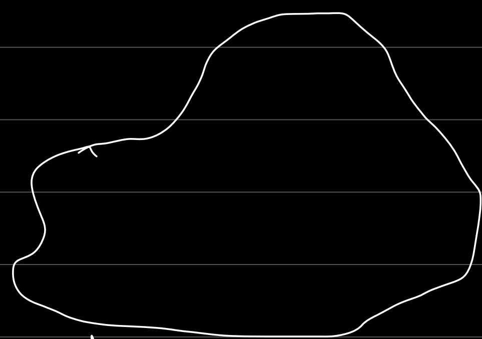
ex 3. Area of shapes



a



Area a^2 πr^2



what is the area?

How to define area?

using Integral.

Functions!

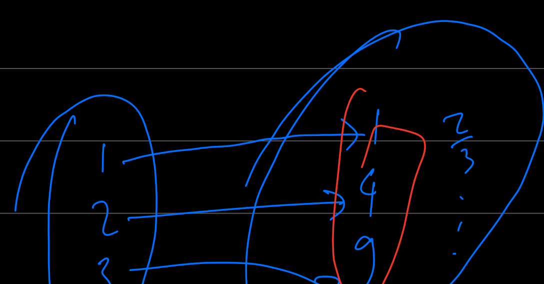
function Domain codomain

Notation: $f : D \to Y$

ex: Let $D = \{1, 2, 3\}$

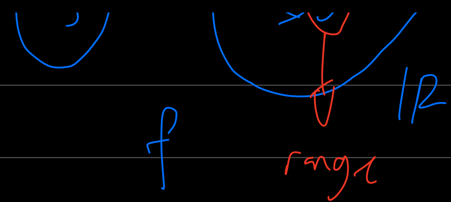
$Y = \mathbb{R}$

$y = f(x) = x^2$



dependent variable

independent variables



Note: Usually, the co-domain is set to be $\mathbb{R}$. It's different from the image or range of the function

$$\text{range}(f) = \{ f(x) : x \in D \} \subseteq Y$$

Rates of Change

1. Average Rates of Change

ex. Drop an object ; The distance dropped, and is denoted by $y = 4.9t^2$ in meter.

Q: Average speed between the 1st and 3rd second?

A: A. S $= \dfrac{\text{Change in distance}}{\text{Change in time}}$

$$= \frac{y(3) - y(1)}{3-1} = \frac{4.9\,(3-1)}{3-1}$$

$$= 19.6 \text{ m/s}$$

$\Delta y =$

$y(3) - y(-1)$: the change in the dependent variable y

$\Delta x =$

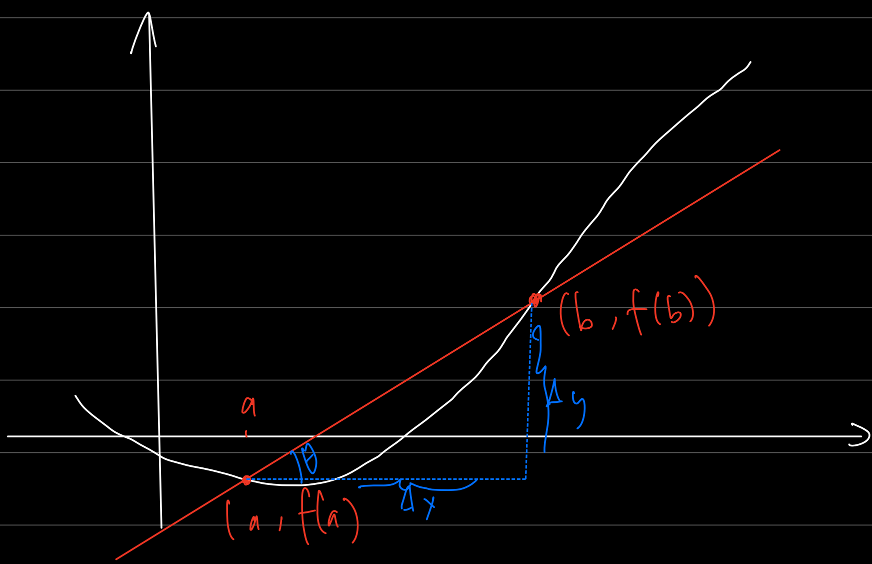
$3 - 1$: the change in the independent ... x

Def : $y = f(x)$, the average rates of change of y with respect to x from $x = a$ to $x = b$ is given by

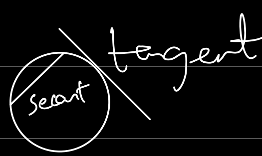
$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

One may also write. $b - a = \Delta x$

$$\frac{\Delta y}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{\Delta x}$$



def: A secant line to a curve is a line connecting two pts of the curve



$\frac{\Delta y}{\Delta x} = \tan\theta$, the slope of the secant line

Instantaneous Rates of Change

Q: What is the speed of the dropping Object at the time $t=1$

Idea: Use the average speed we define from $\underset{a}{t=1}$ to $\underset{b}{t=b}$

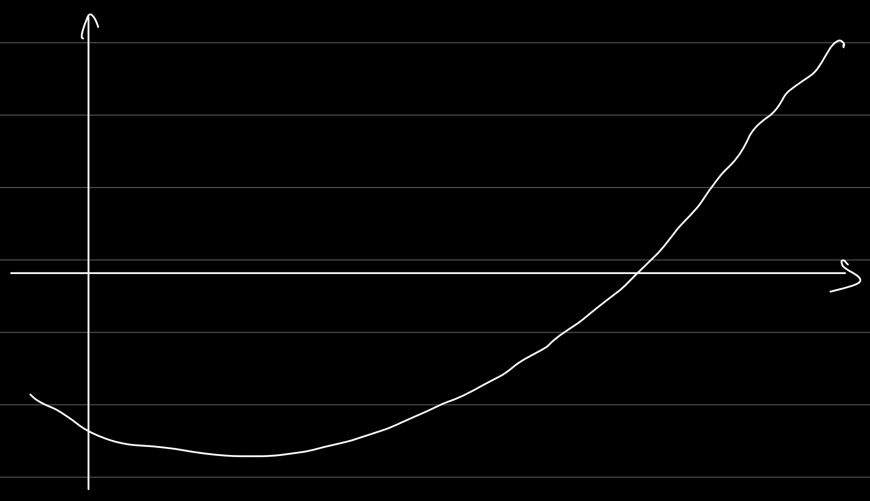
When b is very close to 1

$y = 4.9t^2$ $\quad a = 1$

b	0.9	0.99	0.999	0.001
$\frac{\Delta y}{\Delta x} = \frac{f(b)-f(1)}{b-1}$	$\frac{4.9(0.9^2-1^2)}{0.9-1}$ $\downarrow$ $\underline{9.31}$	9.751	$\vdots$	9.8049

It seems that the average speed tends to 9.8 as b getting closer to 1

The instantaneous rates of change of the function $y=f(t)$ at $t=a$ is "the limit of the average rates of change from t=a to t=b, as b approaches a"



The slope of the tangent line

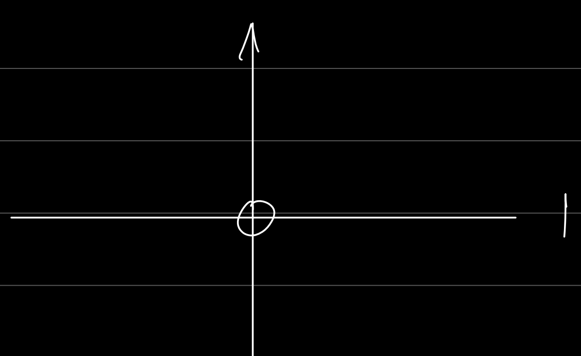
ex: $y = 4.9t^2$ $\quad t=1$

speed at $t=1$ is $\displaystyle\lim_{b\to 1} \frac{4.9b^2 - 4.9}{b-1}$

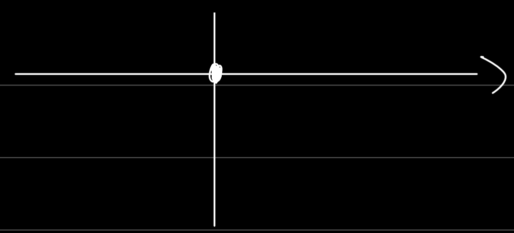
$$= \lim_{b\to 1} \frac{4.9\cancel{(b-1)}(b+1)}{\cancel{b-1}}$$

$$= \lim_{b\to 1} 4.9(b+1) = 9.8$$

The limit is a kind of approach and it has nothing to do w/ the value of the function at that single pt.



$$f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$



$$\lim_{x \to 0} f(x) = 1 \neq 0$$

Summary:

Average rates of change $\rightleftharpoons$ the slope of secant line.

Instantenous rates of change $\rightleftharpoons$ the slope of tangent line

Limits: let $f: D \to \mathbb{R}$ be a fn ($D \subseteq \mathbb{R}$) defined "near" a pt $x = c$ (no need

1

for the pt itself)

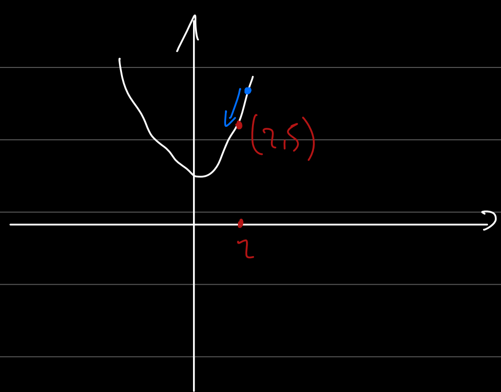
Notation: $\lim_{x \to c} f(x) = L$

"the limit of $f(x)$ as x approaches c is equal to L"

Rk: L does not necessarily equal to $f(c)$

ex: $f(x) = x^2 + 1$

$\lim_{x \to 2} x^2 + 1 = 5$



$x = 2.1$, $f(x) = 2.1^2 + 1$

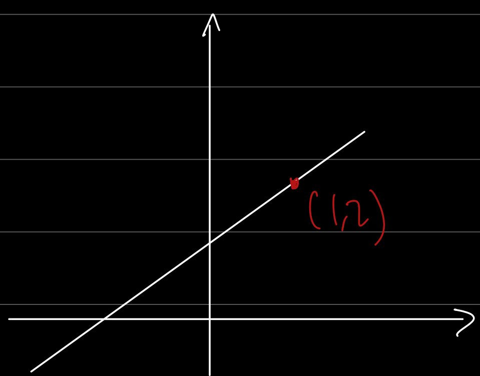
$x = 2.001$, $f(x) = 2.001^2 + 1$

ex: $f(x) = x^2 - 1$ $\quad D = \{x : x \neq 1\}$

$\overline{x-1}$: ... $1\}$)

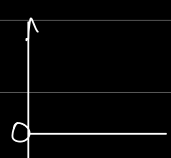
$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1} = 2$$

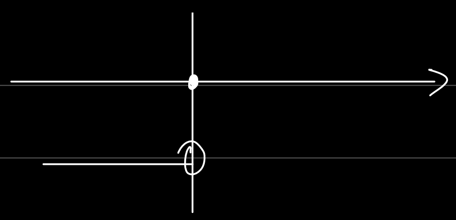
Fn not defined at $x=1$, but the limit still exist. given the fuction is defined on all numbers near $x=1$



non ex: $f(x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$ "Sign function"

$\lim_{x\to 0} f(x)$ DNE (short for does not exist)





approach 0 from the right : 1

approach 0 from the left : −1

Rk: Limit, if exist, is unique.