Lecture 7, Tuesday, Sept/27/2022

Outline:

- Proof of the Chain Rule
- Linearization
- Differentials
- Related Rates
- Extreme values of fns.

Proof of Chain Rule:

$y = f(x)$, $z = g(y)$, Then $\frac{dz}{dx} = g'(f(x)) \cdot f'(x)$

Pf: Define: $\varphi(y) = \begin{cases} \frac{g(y) - g(y_0)}{y - y_0} & y \neq y_0 \\ g'(y_0) & y = y_0 \end{cases}$

By def, $\varphi(y)$ cts at $y = y_0$

$$\therefore \frac{g(\overset{y}{f(x)}) - g(\overset{y_0}{f(x_0)})}{x - x_0} = \boxed{\varphi(f(x))}\,\boxed{\frac{f(x) - f(x_0)}{x - x_0}} \quad (*)$$

As $x \to x_0$, $f(x) \to f(x_0)$, thus

$$\varphi(f(x)) \to \varphi(f(x_0)) = \varphi(y_0) = g'(f(x_0))$$

$$\therefore \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} = g'(f(x_0)) \cdot f'(x_0)$$

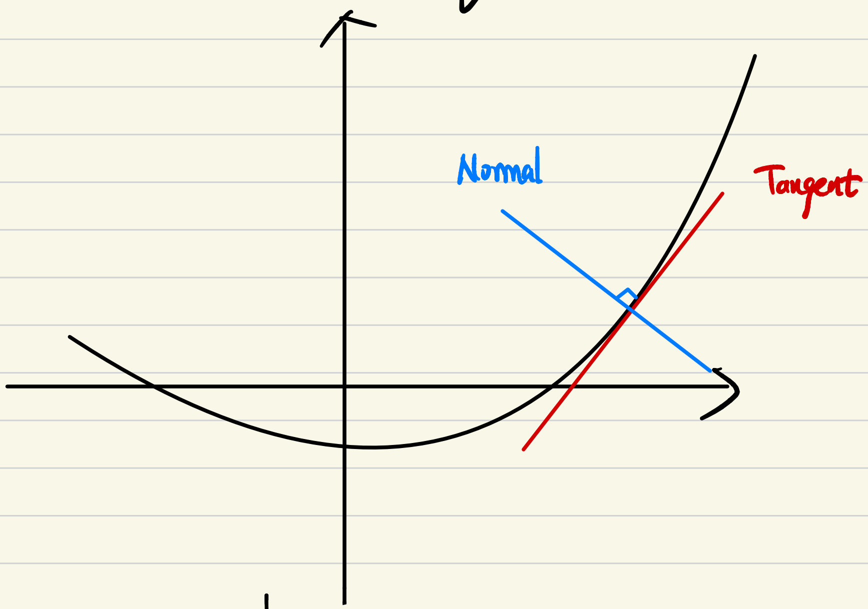
Key: Avoid the case $f(x) = f(x_0)$ (or $y = y_0$)

If $f(x) = f(x_0)$, $(*)$ has 0 on both side

Otherwise, use def of φ

Linearization

① Normal line and tangent line



normal line is perpendicular to the tangent line.

In numerical, the slope multiplication is (−1). Or vertical / horizontal

Equations for tangent / normal line

$$y = \underbrace{k}_{\text{slope } (f'(x) \text{ or } -\frac{1}{f'(x)})}x + \underbrace{b}$$ determined by passing $(x, f(x))$

eg: $y=x^2$ at $(1,1)$.

pf: Tangent line: $y'(1)=2$

$y=2x+b$. As passing $(1,1)$, we have $b=-1$

$y=2x-1$ $\quad y-1=2(x-1)$

Normal line

$y=-\frac{1}{2}x+b$, $b=\frac{3}{2}$, $y=-\frac{1}{2}x+\frac{3}{2}$

$y-1=-\frac{1}{2}(x-1)$

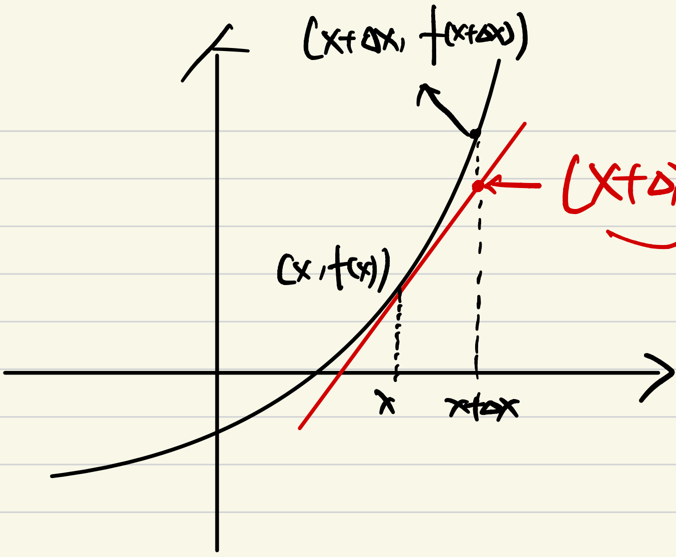
More specific: passing $(c, f(c))$, $f'(c)\neq 0,\infty$

tangent: $\dfrac{y-f(c)}{x-c}=f'(c)$

normal: $\dfrac{y-f(c)}{x-c}=-\dfrac{1}{f'(c)}$

Tangent: $y=f(c)+f'(c)(x-c)$

Write as $\underset{\sim}{L_c}$ at the pt c.



Only need the information at X!!

Q: We know that $\sqrt[3]{27}=3$.

What is $\sqrt[3]{27.1}$? 3.003699

Try use this tangent line to estimate!

$$y=f(x)=\sqrt[3]{x}=x^{\frac{1}{3}}$$

$$y'=\frac{1}{3}x^{-\frac{2}{3}}, \quad \text{at } x=27.$$

$$y'(27)=\frac{1}{3}\frac{1}{(\sqrt[3]{27})^2}=\frac{1}{27}$$

$$\sqrt[3]{27.1}=f(27+0.1)$$

Let $x=27$, $\Delta x=0.1$

$$\approx f(27)+f'(27)\Delta x$$

$$=3+\frac{0.1}{27}\approx 27.003703$$

Linear Approximation

Def: Assume $f(x)$ is differentiable at $x=a$,

$L_a(x) = f(a) + f'(a)(x-a)$ is the

Linearization of f at $x=a$.

This gives an approximation near $x=a$

$f(x) \approx L_a(x)$ called the standard linear approximation of f near a, and a is the center of the approximation.

eg: Approximate $(1+y)^k$ for small y. Centered at 1.

Pf: Let $f(x)=x^k$, $f'(x)=kx^{k-1}$.

At $x=1$, $f(1)=1$, $f'(1)=k$

$L_1(x) = 1+k(x-1)$

$\therefore f(1+y) \approx L_1(1+y) = 1+k(1+y-1) = \underbrace{1+ky}_{\text{approximation}}$

eg: $\sqrt{1.01} = (1+0.01)^{\frac{1}{2}} \approx 1.005$

$(1.00498756\cdots)$

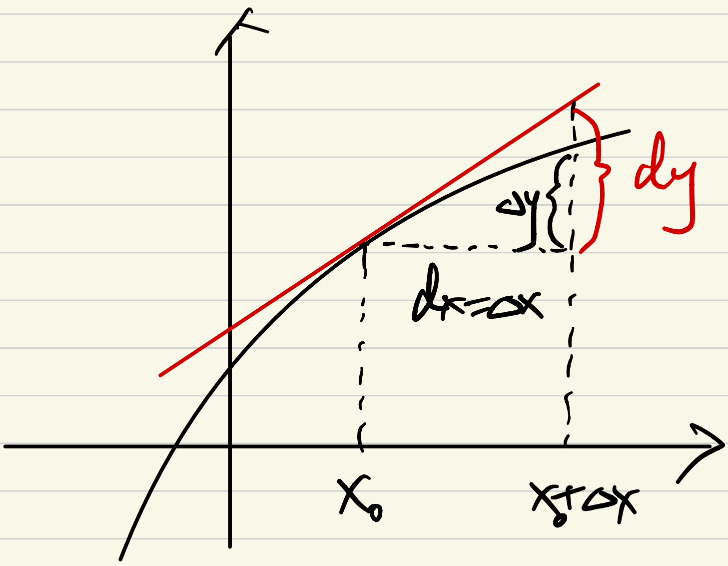
$\sqrt[3]{1+5x^4} = (1+5x^4)^{\frac{1}{3}} \approx 1+\frac{5}{3}x^4$ (x is small)

Differentials:

<u>def</u>: Let $y=f(x)$ differentiable. The differential dx is an independent variable and the differential dy is given by

or df $\quad dy = f'(x)\,dx$ depends on x and dx

Note: dy is not Δy.



Both Δy and dy depend on both x_0 and Δx (dx)

$dy = f'(x_0)\cdot(\Delta x)$

$= f'(x_0)\cdot dx$

eg. $y = f(x) = x^5 + 37x$. $x=1$, $dx = 0.2$

$$dy = f'(x)\,dx = (5x^4+37)\Big|_{x=1}(0.2)$$
$$= 8.4$$

Differentials can be used to approximate changes

eg: For a circle w/ radius 10m. Change to 10.1m will change the area by ____.

Pf: $A(r) = \pi r^2$. $A'(r) = 2\pi r$

$$dA = A'(r)\,dr = 2\pi \times 10 \times 0.1\,(m^2) = 2\pi\,(m^2)$$

$$\Delta A = \pi\,(10.1^2 - 10^2) \approx 2.01\pi\ (m^2)$$

Derivative as a quotient?

We have $dy = f'(x)\,dx$, or $\frac{dy}{dx} = f'(x)$

Not exactly a quotient!

eg: $y = f(x)$, $z = g(t)$.

No meanings!

$$f'(x)g'(t) = \frac{dy}{dx}\cdot\frac{dz}{dt} \neq \boxed{\frac{dy}{dt}\cdot\frac{dz}{dx}}$$

Error of Standard Linear Approximation:

When Δx is small, the error

$$\text{Error} = f(x_0+\Delta x) - L(x_0+\Delta x)$$

$$= f(x_0+\Delta x) - f(x_0) - f'(x_0)\Delta x$$

$$= \Delta x \left[\underbrace{\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}}_{\text{secant } \frac{\Delta y}{\Delta x}} - \underbrace{f'(x_0)}_{\frac{dy}{dx} \text{ tangent}} \right]$$

Call it $\varepsilon = \begin{cases} \frac{\Delta y}{\Delta x} - \frac{dy}{dx} & \Delta x \neq 0 \\ 0 & \Delta x = 0 \end{cases}$

Note: $\lim_{\Delta x \to 0} \varepsilon = \lim_{\Delta x \to 0} \left[\frac{\Delta y}{\Delta x} - f'(x_0)\right] = f'(x_0) - f'(x_0)$

$$= 0$$

Again, what is ε?

$$\varepsilon = \frac{f(x+\Delta x) - L_x(x+\Delta x)}{\Delta x}$$

Error does not only goes to 0, but faster than Δx.

$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$: limit of the quotient

$\frac{dy}{dx}$: quotient of limit

Also we have $\Delta y = f'(x_0)\Delta x + \varepsilon \Delta x$

$\updownarrow$

$dy = f'(x_0)\, dx$

Proof of Chain Rule: (Another)

$y = f(x)$, $z = g(y)$. Show that $\frac{dz}{dx}(x_0) = g'(f(x_0))\, f'(x_0)$

Pf: By def, we have

$$\Delta y = f'(x_0)\Delta x + \varepsilon_1 \Delta x \quad \left(\lim_{\Delta x \to 0} \varepsilon_1 = 0 \right)$$

$$\Delta z = g'(f(x_0))\Delta y + \varepsilon_2 \Delta y \quad \left(\lim_{\Delta y \to 0} \varepsilon_2 = 0 \right)$$

$$= \left(g'(f(x_0)) + \varepsilon_2\right)\left(f'(x_0) + \varepsilon_1\right) \Delta x$$

$$= g'(f(x_0)) f'(x_0) \Delta x + \left[\varepsilon_1 \varepsilon_2 + \varepsilon_2 f'(x_0) + \varepsilon_1 g'(f(x_0))\right] \Delta x.$$

On the other hand:

$$\Delta z = (g \circ f)'(x_0)\Delta x + \varepsilon_3 \Delta x$$

Compare these two, only need to show

$$\lim_{\Delta x \to 0} \left(\varepsilon_1 \varepsilon_2 + \varepsilon_2 f'(x_0) + \varepsilon_1 g'(f(x_0))\right) = 0$$

or $\lim_{\Delta x \to 0} \varepsilon_2 = 0$ ✓

Related Rates.

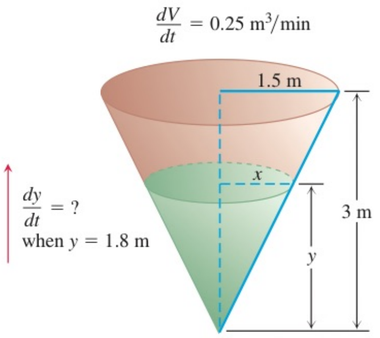


FIGURE 3.32 The geometry of the conical tank and the rate at which water fills the tank determine how fast the water level rises (Example 1).

EXAMPLE 1 Water runs into a conical tank at the rate of $0.25\ \text{m}^3/\text{min}$. The tank stands point down and has a height of 3 m and a base radius of 1.5 m. How fast is the water level rising when the water is 1.8 m deep?

Solution Figure 3.32 shows a partially filled conical tank. The variables in the problem are

V = volume (m^3) of the water in the tank at time t (min)

x = radius (m) of the surface of the water at time t

y = depth (m) of the water in the tank at time t.

We assume that V, x, and y are differentiable functions of t. The constants are the dimensions of the tank. We are asked for dy/dt when

$$y = 1.8\ \text{m} \quad \text{and} \quad \frac{dV}{dt} = 0.25\ \text{m}^3/\text{min}.$$

The water forms a cone with volume

$$V = \frac{1}{3}\pi x^2 y.$$

This equation involves x as well as V and y. Because no information is given about x and dx/dt at the time in question, we need to eliminate x. The similar triangles in Figure 3.32 give us a way to express x in terms of y:

$$\frac{x}{y} = \frac{1.5}{3} \quad \text{or} \quad x = \frac{y}{2}.$$

Therefore, we find

$$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3$$

to give the derivative

$$\frac{dV}{dt} = \frac{\pi}{12}\cdot 3y^2\frac{dy}{dt} = \frac{\pi}{4}y^2\frac{dy}{dt}.$$

At $y = 1.8$ m.

$$0.25 = \frac{\pi}{4} \times 1.8^2 \cdot \frac{dy}{dt}$$

or $\frac{dy}{dt} = \frac{25}{81\pi} \approx 0.098$ (m/min)

Extreme Value of Functions.

Def: Let $f: D \to \mathbb{R}$. Then f has an absolute maximum if $\exists\, c \in D$ s.t.

$$f(x) \leq f(c) \quad \forall x \in D$$

absolute minimum, if $\exists\, c \in D$ s.t.

$$f(x) \geq f(c) \quad \forall x \in D$$

Rk: — Another name: global maximum / minimum.

- Extremum meas max/min.
- Plural form: maxima, minima, extrema
- Depends on the domain!

eg: (1) $f(x) = x^2$ on $\mathbb{R}^2$ no

(2) $f(x) = x^2$ on $[0, 2]$ max ✓ min ✗

(3) $f(x) = x^2$ on $[0, 1]$ max ✓ min ✓

(4) $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ on $[-1, 1]$ max ✗ min ✗

Thm: (Extreme Value Thm) f is cts on closed interval, then f has both global maximum and minimum,

i.e.: $\exists x_1, x_2 \in [a,b]$, $f(x_1)=m$, $f(x_2)=M$

s.t. $m \leq f(x) \leq M \quad \forall x \in [a,b]$

Cor: Image of a closed interval under cts fn is a closed interval.

<u>Proof</u>: Something in Analysis.

Local maximum / minimum:

Say $f(x)$ has a local maximum at
local minimum

$x=c$, if $\exists \alpha > 0$ s.t.

$f(c) \geq f(x)$, $\forall x \in (c-\alpha, c) \cup (c, c+\alpha)$

$f(c) \leq f(x)$

RK: ① must be an interior pt.

② Global extremum must be local extremum

How to locate local extremum?

Def: (Critical Point): $f: D \to \mathbb{R}$, an interior pt $x=c$ is said to be a critical pt of f, if:

(1) $f'(c)=0$ or (2) $f'(c)$ DNE

eg: $f(x)=\begin{cases} |x| & x<1 \\ 1 & x\geq 1 \end{cases}$

Thm: Local extremum happens on critical pts.

Local extremum pt is critical pt.

RK: Critical pts may not be local extremum

eg: $f(x)=|x|$. on $[-1,1]$

$f(x)=x^3$ on $[-1,1]$

Pf: Suppose $x=c$ is a local maximum.

If $f'(c)$ DNE, ✓

Otherwise, we have

$$\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}$$

$\because$ $f(c)$ local maximum, $f(x)-f(c)\leq 0$

$$\therefore \lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}\geq 0, \quad \lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}\leq 0$$

To have them equal, they must $=0$

Thus $f'(c)=0$

Strategy for finding global extremum.

1. Evaluate f at all critical pts and end pts

2. Take the largest and smallest.

eg1. $f: [-2, 4] \to \mathbb{R}$ extremum

$f(x) = 2x^3 - 3x^2 - 12x + 15$

pf: All potential pts to check:

① end pts: $f(-2) = -16 - 12 + 24 + 15 = 11$

$f(4) = 128 - 48 - 48 + 15 = 47$ max

② critical pts: $2, -1$

$f'(x) = 6x^2 - 6x - 12 = 0$

$= 6(x-2)(x+1) = 0$

$f(2) = 16 - 12 - 24 + 15 = -5$ min

$f(-1) = -2 - 3 + 12 + 15 = 22$