MAT 1001 Midterm Exam, 10:00am -Noon, March 27, 2021

Your Name and Student ID:

Your Lecture Class(e.g, L1) and **your tutorial class** (e.g, T01):

Instruction: (i) This is a closed-book and closed-notes exam; no calculators, no dictionaries and no cell phones; (ii) Show your work unless otherwise instructed–a correct answer without showing your work when required shall be given no credits; (iii) Write down ALL your work and your answers(including the answers for short questions) in the Answer Book.

1. (27 points) **Short Questions** (for these questions, NO need to show your work, just write down your answers in the Answer Book; NO partial credits for each question)

   (i). Which of the following statements is false?

   (a) Let f be a continuous function defined for all real numbers. If $a_1 = a$ and $a_{n+1} = f(a_n)$ (for all $n \geq 1$) define a convergent sequence $\{a_n\}$, then f has a fixed point (i.e. $f(x_0) = x_0$ for some x_0).

   (b) If the sequence $\{a_1, a_3, a_5, \ldots\}$ converges to L_1 and $\{a_2, a_4, a_6, \ldots\}$ converges to L_2 with $L_1 \neq L_2$, then $\{a_n\}_{n\geq 1}$ cannot converge.

   (c) If $\sum |a_n|$ diverges, then $\sum a_n$ cannot converge.

   (ii). Which of the following statements is false?

   (a) Given an alternating series, if it does not satisfy the conditions in the alternating series test, then it must diverge.

   (b) Rearranging the terms (which means changing the order of the terms) in $\displaystyle\sum_{n=1}^{\infty}(-1)^n(1/n^{\pi})$ will never change the value of the series.

   (c) If $\sum a_n$ is convergent but $\sum b_n$ is divergent, then $\sum(a_n + b_n)$ must be divergent.

   (iii). Which of the following is the interval of convergence for the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{6(5+\sin x)^n}$?

   (a) $\{x|\ x$ is not an integer multiple of $2\pi\}$.

   (b) $\{x|\ x$ is not an integer multiple of $\pi\}$.

   (c) $-\infty < x < \infty$.

   (d) diverges for all x.

   (iv). Which of the following is the Maclaurin series for $\cos 2x$?

(a) $\sum_{n=0}^{\infty} \frac{(-1)^{2n} 2^{2n} x^{2n}}{n!}$.

(b) $\sum_{n=0}^{\infty} \frac{(-1)^{2n} 2^{2n} x^{2n}}{(2n)!}$.

(c) $\sum_{n=0}^{\infty} \frac{(-1)^{n} 2^{2n} x^{2n}}{n!}$.

(d) $\sum_{n=0}^{\infty} \frac{(-1)^{n} 2^{2n} x^{2n}}{(2n)!}$.

(v). Which of the following is the Taylor series generated by $f(x) = \frac{1}{x^2}$ at $x = 8$?

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n n (x-8)^n}{8^{n+2}}$.

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(x-8)^n}{8^{n+1}}$.

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(x-8)^n}{8^{n+2}}$.

(d) $\sum_{n=0}^{\infty} \frac{(-1)^n n (x-8)^n}{8^{n+1}}$.

(vi). $\frac{2}{3} - \frac{2^3}{3^3 \cdot 3!} + \frac{2^5}{3^5 \cdot 5!} - \frac{2^7}{3^7 \cdot 7!} + \ldots =$

(a) $\sin \frac{3}{2}$.

(b) $\cos \frac{2}{3}$.

(c) $\ln \frac{2}{3}$.

(d) $\sin \frac{2}{3}$.

(vii). Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be nonzero vectors in space. Which of the following statements is false?

(a) $\vec{u} \times \vec{v} = \vec{0}$ is equivalent to $\vec{u} = k\vec{v}$ for some scalar k.

(b) The vector $\vec{u} \times (\vec{v} \times \vec{w})$ is parallel to the plane spanned by $\vec{v}$ and $\vec{w}$.

(c) $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|$ if $\vec{u}$ and $\vec{v}$ are parallel.

(viii). Which of the following statements is false?

(a) If we want to expand a function $f(x)$ as a power series about $x = 0$ in the interval $(-1, 1)$, then necessarily f has to be infinitely differentiable on the interval (i.e., derivative of all orders of f exist on the interval).

(b) If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for $x \in (-1, 1)$, then $f(x) = \sum_{n=0}^{2021} c_n x^n + O(x^{2022})$ as $x \to 0$.

(c) If $f(0) = 0 = f^{(n)}(0)$ for all positive integers n, then f has to be identically equal to 0, at least in a small open interval containing 0.

(ix). Write down the parametric equations of the line tangent to the curve $\vec{r}(t) = \cos(e^t)\vec{i} + (3 - t^2)\vec{j} + t\vec{k}$ at $t = 0$:

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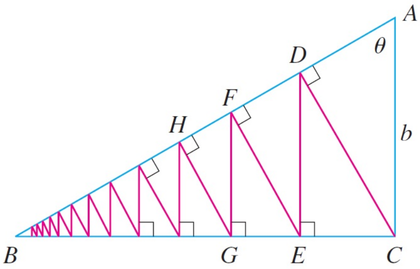
2. (5 points) In the triangle ACB below, the angle at corner C is a right angle, and line segments CD, EF, GH, etc., are parallel, while AC, DE, FG, etc., are parallel. The drawing $CDEFGHI$... continues indefinitely. Is the length

$$|CD| + |DE| + |EF| + |FG| + |GH| + \ldots$$

finite? If so, find it.



3. (24 points) For each of the following series, determine whether it is convergent absolutely, convergent conditionally, or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2 n}$.

   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$.

   (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 5 \cdot 8 \cdot \ldots \cdot (3n-1)}$.

   (d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$, where $0 < p < 1$.

4. (8 points) Determine the interval of convergence for the series $\sum_{n=0}^{\infty} \frac{2^n x^n}{4^n+1}$.

5. (8 points) Find

$$\lim_{x\to 0} \frac{\tan x - \sin x}{(1+x^3)^\pi - 1}.$$

6. (10 points) Let $f(x)$ be continuously differentiable on the finite interval $[a, b]$ (i.e., f' exists and is continuous on $[a, b]$), and suppose f'' exists on the open interval (a, b). Prove the following special case of Taylor's Theorem: there exists $c \in (a, b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(c)}{2!}(b-a)^2.$$

7. (24 points) Consider the cycloid parametrized by

$$x = t - \sin t, \quad y = 1 - \cos t, \qquad t \in [0, 2\pi].$$

   (a) Find its arclength.

(b) Find the area of the surface generated by revolving the cycloid about the x-axis.

(c) Find the curvature of the cycloid at $t = \pi$.

(d) Let the y-axis **point downward** (and the x-axis be horizontal, pointing to the right). Consider a particle sliding frictionlessly on the cycloid under the influence of gravity, from the origin $O = (0, 0)$ to the bottom $B = (\pi, 2)$ of the cycloid, with 0 initial speed. Find the time T it takes for the particle to reach B from O. Length is measured in meters, and time in seconds.

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8. (15 points) Consider the polar curve $r = \sin(2\theta)$, $\theta \in [0, \pi]$.

   (a) Sketch the curve on the xy-plane.

   (b) Compute the slope of the curve at $\theta = \pi/4$.

   (c) Find the area of the region bounded by the polar curve for $0 \le \theta \le \pi/2$.

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9. (16 points) Let $P = (2, 4, 5)$, $Q = (1, 5, 7)$ and $R = (-1, 6, 8)$.

   (a) Find the area of the triangle ΔPQR;

   (b) Find an equation of the plane containing the triangle mentioned above;

   (c) Find parametric equations of the line which is perpendicular to the plane mentioned above, and passes through the origin $(0, 0, 0)$.

   (d) Find the distance from the origin to the plane mentioned above.

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10. (16 points) Let $\vec{r}(t)$, $-\infty < t < \infty$, be the position vector function of a particle moving, with positive speed, along a smooth curve C in space. Suppose $\vec{r}(t)$ is twice differentiable (which means the second order derivative exists) for all t.

    (a) Prove that the speed of the particle is constant if and only if the velocity vector $\vec{v}(t)$ is always orthogonal to the acceleration vector $\vec{a}(t)$.

    (b) Prove that if the curvature of curve C is 0 everywhere, then the curve has to be a straight line.

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