7.3

16. $y = \theta^3 e^{-2\theta} \cos 5\theta$

$\frac{dy}{d\theta} = 3\theta^2 e^{-2\theta} \cos 5\theta - (\theta^3 2e^{-2\theta} \cos 5\theta - \theta^3 e^{-2\theta} 5 \sin 5\theta)$

24. $y = \int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t \, dt$

$\frac{dy}{dx} = \ln e^{2x} (2e^{2x}) - \ln e^{4\sqrt{x}} \left(\frac{2}{\sqrt{x}} e^{4\sqrt{x}} \right)$

$= 4xe^{2x} - 8e^{4\sqrt{x}}$

27. $e^{2x} = \sin(x + 3y)$

$2e^{2x} = \cos(x+3y)\left(1 + 3\frac{dy}{dx}\right)$

$2e^{2x} = \cos(x+3y) + 3\cos(x+3y)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{2e^{2x} - \cos(x+3y)}{3\cos(x+3y)}$

30. $\int (2e^x - 3e^{-2x})\, dx$

$= \int 2e^x\, dx - \int 3e^{-2x}\, dx$

$= 2\int e^x\, dx - 3\int e^{-2x}\, dx$

$= 2e^x - 3\left(\frac{1}{-2} e^{-2x}\right) + C$

$= 2e^x + \frac{3}{2} e^{-2x} + C$

43. $\int_0^{\frac{\pi}{4}} (1 + e^{\tan\theta}) \sec^2\theta\, d\theta$

Let $u = \tan\theta$

$\frac{du}{d\theta} = \sec^2\theta$

$d\theta = \frac{du}{\sec^2\theta}$

$\theta = 0, \; u = 0$

$\theta = \frac{\pi}{4}, \; u = 1$

$\int_0^1 (1 + e^u) \cancel{\sec^2\theta}\, \frac{du}{\cancel{\sec^2\theta}}$

$= u + e^u \Big|_0^1$

$= u + e^u \Big|_0^1$

$= 1 + e - 1$

$= e$

48. $\int_0^{\sqrt{\ln\pi}} 2x\, e^{x^2}\cos(e^{x^2})\,dx$

Let $u = e^{x^2}$

$\frac{du}{dx} = 2xe^{x^2}$

$\frac{du}{2xe^{x^2}} = dx$

$x = \sqrt{\ln\pi} \to u = e^{\ln\pi}$
$= \pi$

$x = 0 \to u = e^0$
$= 1$

$\int_1^{\pi} \cancel{2xe^{x^2}} \cos u \; \frac{du}{\cancel{2xe^{x^2}}}$

$= \sin u \Big|_1^{\pi}$

$= 0 - \sin 1$

$= -\sin 1$

50. $\int \frac{dx}{1+e^x}$

$= \int \frac{1}{1+e^x}\, dx$

$=$ Let $u = 1+e^x$

$\frac{du}{dx} = e^x$
$= u-1$

$dx = \frac{du}{u-1}$

$\int \frac{1}{u} \frac{du}{u-1}$

$= \frac{1}{u(u-1)}\, du$

$= 1 = \frac{A}{u} + \frac{B}{(u-1)}$

$1 = A(u-1) + B(u)$

$u=1 \to 1 = B$, $B=1$, $u=0 \to 1 = -A$, $A=-1$

$\int -\frac{1}{u} + \frac{1}{(u-1)}\, du$

$$= -\ln u + \ln(u-1) + C$$

$$= -\ln(1+e^x) + \ln(e^x) + C$$

59. $y = x^{\pi}$

$$\frac{dy}{dx} = \pi x^{\pi-1}$$

66. $y = 5^{-\cos 2t}$

$$\frac{dy}{dt} = 2\sin 2t \; 5^{-\cos 2t} \ln 5$$

73. $y = \log_3\left(\left(\frac{x+1}{x-1}\right)^{\ln 3}\right)$

$$= (\ln 3)\log_3\left(\frac{x+1}{x-1}\right)$$

$$= (\cancel{\ln} 3)\cdot\frac{\ln(x+1/x-1)}{\cancel{\ln 3}}$$

$$= \ln\left(\frac{x+1}{x-1}\right)$$

$$\frac{dy}{dx} = \frac{x-1-x-1}{(x-1)^2}\cdot\frac{x-1}{x+1}$$

$$= \frac{-2}{(x-1)^{\cancel{2}}}\cdot\frac{\cancel{x-1}}{x+1}$$

$$= \frac{-2}{x^2-1}$$

92. $\int \frac{x 2^{x^2}}{1 + 2^{x^2}} dx$

Let $u = 2^{x^2}$

$\ln(u) = x^2 \ln(2)$

$\frac{1}{u} \frac{du}{dx} = 2x \ln(2)$

$\frac{du}{dx} = 2x \ln(2)\, u$

$= 2x \ln(2)\, 2^{x^2}$

$dx = \frac{du}{2x \ln(2) u}$

$\int \frac{\cancel{x}\cancel{u}}{1+u} \frac{du}{2\cancel{x}\ln(2)\cancel{u}}$

$\frac{1}{2\ln(2)} \int \frac{1}{1+u} du$

$\frac{1}{2\ln(2)} \cdot \ln(1+u) + C$

$= \frac{1}{2\ln(2)} \cdot \ln(1 + 2^{x^2}) + C$

100. $\int_1^e \frac{2 \ln 10 \log_{10} x}{x} dx$

Let $u = \log x$

$\log_e \log_e x$

$\log_e \quad \frac{1}{x} dx = du$

$= \frac{dx}{x} = \log_e 10\, du$

$= 2 \cos e^{10} \log_e 10 \int u\, du$

$= \cancel{2} (\ln 10)^2 \frac{u^2}{\cancel{2}}$

$= (\ln 10)^2 (\log x)^2 \Big|_1^e$

$= (\ln 10)^2 (\log e)^2 \rightarrow \frac{(\ln 10)^2}{(\ln 10)} = 1$

114. $y = t^{\sqrt{t}}$

$\ln(y) = \ln(t^{\sqrt{t}})$

$\ln(y) = \sqrt{t} \ln(t)$

$\frac{1}{y} \frac{dy}{dt} = \frac{1}{2\sqrt{t}} \ln(t) + \frac{\sqrt{t}}{t}$

$\frac{dy}{dt} = \frac{1}{2\sqrt{t}} \ln(t) + \frac{\sqrt{t}}{t} \cdot y \rightarrow \frac{dy}{dt} = \left(\frac{1}{2\sqrt{t}} \ln(t) + \frac{\sqrt{t}}{t}\right) t^{\sqrt{t}}$

116. $y = x^{\sin x}$

$\ln(y) = \sin x \ln(x)$

$\frac{1}{y}\frac{dy}{dx} = \cos x \ln(x) + \frac{\sin x}{x}$

$\frac{dy}{dx} = \left(\cos x \ln(x) + \frac{\sin x}{x}\right) y$

$\frac{dy}{dx} = \left(\cos x \ln(x) + \frac{\sin x}{x}\right) x^{\sin x}$

136. a) $f(x) = e^x$

Concave up when $f''(x) > 0$

$f'(x) = e^x$

$f''(x) = e^x > 0$

b) $0 < a < b$ then

$$e^{(\ln a+\ln b)/2}\cdot(\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x\,dx < \frac{e^{\ln a}+e^{\ln b}}{2}\cdot(\ln b - \ln a).$$

BC tangent

$= \frac{\ln a + \ln b}{2}$

$y = e\left(x - \frac{\ln a + \ln b}{2}\right) + e^{(\ln a+\ln b)/2}$

Area AEFB

$\frac{e^{\ln a} + e^{\ln b}}{2}(\ln b - \ln a)$

Area ABCD

$\frac{1}{2}(AB+CD)\,AD$

$\frac{1}{2}\left(\frac{\ln a - \ln b}{2} + \frac{\ln b - \ln a}{2} + 2\right) e^{(\ln a + \ln b)/2} \cdot (\ln b - \ln a)$

$= e^{(\ln a + \ln b)/2} \cdot (\ln a + \ln b)$

So, since the area ABCD < Area below curve from $x = \ln a$ to $x = \ln b$

< Area AEFD,

$e^{(\ln a + \ln b)/2} (\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x \, dx < \frac{e^{\ln a} + e^{\ln b}}{2} \cdot (\ln b - \ln a)$

c) $e^{(\ln a + \ln b)/2} (\ln b - \ln a) < e^x \Big|_{\ln a}^{\ln b} < \frac{e^{\ln a} + e^{\ln b}}{2} (\ln b - \ln a$

$= e^{\frac{1}{2}\ln(ab)} < \left(e^{\ln b} - e^{\ln a}\right) / (\ln b - \ln a) < \frac{a+b}{2}$

$= e^{\ln \sqrt{ab}} < \frac{b-a}{(\ln b - \ln a)} < \frac{a+b}{2}$ $(e^{\ln a} = a \text{ \& } e^{\ln b} = b)$

$= \sqrt{ab} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}$

193. a) $y = \ln x$

$y' = \frac{1}{x}$

$x = e$

$y = \ln e$

$y' = \frac{1}{e}$

$\frac{y-1}{x-e} = \frac{1}{e}$

$y - 1 = \frac{x}{e} - 1$

$y = \frac{x}{e}$

b) $\ln x$ intersects $\frac{x}{e}$ at $x = e$

- Slope of $\frac{x}{e}$ will always be $\frac{1}{e}$
- Slope of $\ln x = \frac{1}{x}$ for $x > e$ is $\frac{1}{x} < \frac{1}{e}$
- Slope of tangent > slope of $\ln x$ for $x > e$

They don't intersect for $x > e$ and $\frac{x}{e}$ lies above $\ln x$.

slope $\frac{x}{e}$ < slope $\ln x$

So, $\ln x < \frac{x}{e}$

c) $\ln x < \frac{x}{e} \to e \ln x < x \to \ln x^e < e$

d) $\ln x^e < x \to e^{(\ln x^e)} < e^x \to x^{e \ln e} < e^x$

$= x^e < e^x$

7.5

5. $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

$= \lim_{x \to 0} \frac{\sin x}{2x}$

$= \lim_{x \to 0} \frac{\cos x}{2}$

$= \frac{1}{2}$

15. $\lim_{x \to 0} \frac{8x^2}{\cos x - 1}$

$= \lim_{x \to 0} \frac{16x}{-\sin x}$

$= \lim_{x \to 0} \frac{16}{-\cos x}$

$= -16$

20. $\lim_{x \to 1} \frac{x-1}{\ln x - \sin \pi x}$

$= \lim_{x \to 1} \frac{1}{\frac{1}{x} - \pi \cos \pi x}$

$= \frac{1}{1 + \pi}$

26. $\lim_{x \to (\pi/2)^-} \left(\frac{\pi}{2} - x\right) \tan x$

$= \lim_{x \to (\pi/2)^-} \left(\frac{\pi}{2} - x\right) \frac{\sin x}{\cos x}$

$= \lim_{x \to (\pi/2)^-} (\pi/2 - x) \sin x / \cos x$

$= \lim_{x \to (\pi/2)^-} \frac{-(\sin x) - (\pi/2 - x) \cos x}{-\sin x}$

$= \lim_{x \to (\pi/2)^-} 1 - (\pi/2 - x) \cot x$

$= 1 - (\pi/2 - \pi/2) \cot \frac{\pi}{2} = 1$

27. $\lim_{\theta \to 0} \frac{3^{\sin\theta} - 1}{\theta}$

$= \lim_{\theta \to 0} \cos\theta \, 3^{\sin\theta} \ln 3$

$= \ln 3$

30. $\lim_{x \to 0} \frac{3^x - 1}{2^x - 1}$

$= \lim_{x \to 0} \frac{3^x \ln 3}{2^x \ln 2}$

$= \frac{\ln 3}{\ln 2}$

32. $\lim_{x \to \infty} \frac{\log_2 x}{\log_3(x+3)}$

$= \lim_{x \to \infty} \frac{\frac{1}{x \ln(2)}}{\frac{1}{(x+3)\ln(3)}}$

$= \lim_{x \to \infty} \frac{(x+3)\ln(3)}{x \ln(2)}$

$= \lim_{x \to \infty} \frac{\ln 3}{\ln 2} \to \log_2(3)$

38. $\lim_{x \to 0^+} (\ln x - \ln \sin x)$

$\lim_{x \to 0^+} \ln x - \lim_{x \to 0^+} \ln \sin x$

$= \infty - \infty = 0$

40. $\lim_{x \to 0^+} \left(\frac{3x+1}{x} - \frac{1}{\sin x} \right)$

$= \lim_{x \to 0^+} \left(\frac{3x \sin x + \sin x - x}{x \sin x} \right)$

$= \lim_{x \to 0^+} \frac{3 \sin x + 3x \cos x + \cos x - 1}{\sin x + x \cos x}$

$= \lim_{x \to 0^+} \frac{3 \cos x + 3 \cos x + 3x \sin x - \sin x}{\cos x + \cos x - x \sin x} = \frac{6}{2} = 3$

45. $\underset{t\to\infty}{\ell} \dfrac{e^t + t^2}{e^t - t}$

$= \underset{t\to\infty}{\ell} \dfrac{e^t + 2t}{e^t - 1}$

$= \underset{t\to\infty}{\ell} \dfrac{e^t + 2}{e^t} = \underset{t\to\infty}{\ell} \dfrac{e^t}{e^t} = 1$

54. $\underset{x\to e^+}{\ell} ((\ln x)^{1/(x-e)}$

$= (\ln e)^{1/e - e}$

$= 1^\infty = 1$

62. $\underset{x\to\infty}{\ell} \left(\dfrac{x^2+1}{x+2}\right)^{1/x}$

$= \underset{x\to\infty}{\ell} e^{\ln\left(\frac{x^2+1}{x+2}\right)^{1/x}}$

$= e^{\underset{x\to\infty}{\ell} \frac{1}{x}\ln\left(\frac{x^2+1}{x+2}\right)}$

$= \underset{x\to\infty}{\ell} \ln\left(\dfrac{x^2+1}{x+2}\right)$

$= \underset{x\to\infty}{\ell} \dfrac{2x(x+2) - (x^2+1)}{(x+2)^2}$

$= \underset{x\to\infty}{\ell} \dfrac{x^2+4x-1}{x^2+4x+4} = 1$

$= e^1 = e$

66. $\lim_{x \to 0^+} \sin x \cdot \ln x$

$= \lim \dfrac{\cos x}{x} = 1$

68. $\lim_{x \to 0^+} \left(\dfrac{\sqrt{x}}{\sqrt{\sin x}}\right)$

$= \lim_{x \to 0^+} \sqrt{\dfrac{x}{\sin x}}$

$= \sqrt{\lim_{x \to 0^+} \dfrac{x}{\sin x}}$

$= \sqrt{\lim_{x \to 0^+} \dfrac{1}{\cos x}}$

$= \sqrt{\dfrac{1}{1}} = 1$

72. $\lim_{x \to -\infty} \dfrac{2^x + 4^x}{5^x - 2^x}$

$= \lim_{x \to -\infty} \left(\dfrac{2^x + 4^x}{5^x - 2^x}\right) \dfrac{1}{5^x}$

$= \lim_{x \to -\infty} \dfrac{\frac{2^x + 4^x}{5^x}}{1 - \frac{2^x}{5^x}}$

$= \dfrac{\lim_{x \to -\infty} \frac{2^x + 4^x}{5^x}}{\lim_{x \to -\infty} 1 - \frac{2^x}{5^x}} = \dfrac{0}{1} = 0$

80. $\lim_{x \to 0} \left(\dfrac{\tan 2x}{x^3} + \dfrac{a}{x^2} + \dfrac{\sin bx}{x}\right) = 0$

$= \lim_{x \to 0} \left(\dfrac{\tan \cancel{2x}}{\cancel{2x}} \cdot \dfrac{2x}{x^3} + \dfrac{a}{x^2} + \dfrac{\sin \cancel{bx}}{\cancel{bx}} \cdot b\right)$

$= \lim_{x \to 0} \left(\dfrac{2x}{x^3} + \dfrac{a}{x^2} + b\right) = 0$

$\because \lim_{x \to 0} \dfrac{\tan \theta}{\theta} = 0$

$\lim_{x \to 0} \dfrac{\sin \theta}{\theta} = 0$

$\Rightarrow \lim_{x \to 0} \left(\dfrac{2 + a}{x^2} + b\right) = 0$

if $x = 0$, ∞

But the equation will be zero

if $2 + a = 0$

$b = 0$

$a = -2$

$b = 0$

85. $\lim_{k \to \infty} \left(1 + \frac{r}{k}\right)^k = e^r$

$\lim_{x \to \infty} a_k = \lim_{k \to \infty} \left(1 + \frac{r}{k}\right)^k$

$= \lim_{k \to \infty} \left[\left(1 + \frac{r}{k}\right)^{\frac{k}{r}}\right]^r$

$= e^r$

88. $f(x) = e^{-1/x^2}$

$f'(x) = e^{-1/x^2} \cdot \frac{2}{x^3}$

$e^{-1/x^2} = f(x) = 0$

$f'(0) = f(0) \cdot \frac{2}{x^3}$

$= 0$