Force between uniformly magnetized / polarized spheres

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Simple arguments show that the magnetic force between two uniformly magnetized spheres is identical to the force between two point magnetic dipoles at the same distance and with the same magnetic orientations as these spheres, and that the electric force between two uniformly polarized spheres is identical to the force between two point electric dipoles.

I. INTRODUCTION

Small dipolar neodymium magnet spheres are used both in and out of the classroom to teach principles of mathematics, physics, chemistry, biology, and engineering [1, 2]. They offer engaging hands-on exposure to principles of magnetism and are particularly useful in studying lattice structures, where they offer greater versatility than standard ball-and-stick models because they can connect at a continuous range of angles. They have spawned a grassroots learning community dedicated to sharing photos and tutorials of beautiful magnetic sculptures, some made from thousands of magnets, including models of molecules, fractals, and Platonic solids [3]. YouTube magnet sphere videos have attracted over a hundred million views [2].

Computer simulations of dynamic interactions between magnet spheres assume point dipole interactions for simplicity [4, 5], or treat extended magnets as aggregates of such point dipoles [6]. Dipolar magnetic interactions have also been studied in assemblies of spherical nanoparticles [7]. Analytical solutions have been found for the force between permanent magnets of various nonspherical shapes, such as cubes and cylinders [18–25]. Calculating the interaction energy of two point magnetic dipoles is a standard undergraduate exercise, and the force between two such dipoles has been well studied [27–32].

Of interest are the force and torque between two uniformly magnetized spheres. Calculating the magnetic field produced by a uniformly magnetized sphere is a standard undergraduate exercise. The equivalence of this field, outside the sphere, to the point magnetic dipole field is well known [9]. But direct calculations of the force exerted by this field on another uniformly magnetized sphere are challenging, and have been done only in three special cases.

The first is for two uniformly magnetized spheres with magnetizations that are perpendicular to the line through the spheres [13]. The second is for two uniformly magnetized spheres with parallel magnetizations that make an arbitrary angle with the line through the sphere centers [11]. The third is for two identical uniformly magnetized spheres with one of the magnetizations parallel to the line through the spheres [12].

All three calculations yield a force that is identical to the force between two point magnetic dipoles.

This paper has two purposes: (1) to prove that the force between two uniformly magnetized spheres with arbitrary sizes, positions, magnetizations, and orientations is identical to the force between two point magnetic dipoles, and (2) to discuss implications of this equivalence for physics education.

Proofs of this equivalence, if they exist in modern literature or textbooks, have eluded our searches. Popular online discussions of the force between magnets discuss the force between magnetic dipoles, cylindrical bar magnets, and nearby magnetized surfaces, but not the force between spheres [15, 16]. Some individuals have suspected, or may have known, that the forces are equivalent [8]. But this knowledge is not general, based on our discussions with colleagues, our discussions with individuals who are currently working in the field of ferromagnetic interactions, and on our search of the journal literature and textbooks. A simple result with wide applicability deserves to be widely known. This is the motivation behind this paper.

We present three proofs of this equivalence, each of which draws upon different facets of the undergraduate physics curriculum. The first is a simple symmetry argument based on Newton's third law (Sec. III A). The second is a direct integration of the magnetic force $\mathbf{F} = -\boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B})$ over a sphere (Sec. III B). The third uses potential theory to examine changes in the total magnetic energy of the two-sphere system, and relies on Green's theorem (Sec. III C). All three rely on the equivalence of the field of a uniformly magnetized sphere to the field of a point dipole.

We also discuss the non-central nature of the force between two magnet spheres / dipoles, namely, that this force is not generally directed along the line through the sphere centers. While non-central forces sometimes violate Newton's third law [14, 37, 38], we show that the force between magnet spheres / dipoles obeys this law. We also show that the torques between two dipoles are *not* equal and opposite.

Because the force between two dipoles is non-central and obeys Newton's third law, one might ask whether the torques produced by these forces could produce angular acceleration in an isolated rigid two-dipole system that rotates about an axis passing through its center of mass. Fascinating examples of the exchange between electromagnetic and mechanical momentum are the subject of considerable study [33–36, 38]. But we show that the torques between the dipoles exactly cancel the torques produced by these forces.

The forces and torques between two magnets can seem mysterious. Students may be familiar with the central attractive force between a magnet with its north pole facing the south pole of another magnet, and the repulsive central force for two magnets with facing north poles, but they often cannot explain the origin of these forces, or predict the direction of the non-central forces and torques between magnets oriented in arbitrary directions. In this paper, we present tools intended to help students develop a better understanding of the forces and torques between magnets, and their intimate relationship with the energy of interaction. These tools include instructive figures and a computer simulation that enables students to animate the motion of a uniformly magnetized sphere in response to the forces and torques produced by another uniformly magnetized sphere. We offer this simulation freely to the physics community.

In Sec. **??**, we identify ranges of angles leading to attractive vs. repulsive forces. We discuss the role of torques between magnetic spheres and demonstrate that a two-magnet system can exchange momentum with the electromagnetic fields: We attach two magnet spheres rigidly to the ends of a rod with the magnetizations parallel to each other but oriented at a 45° angle to the rod. When allowed to rotate freely about its center of mass, we observe that the rod accelerates from rest in response to this torque, borrowing angular momentum from the electromagnetic fields. We propose two undergraduate exercises. Finally, we confirm that pinching a ring of magnets into a triangle requires actively orienting the magnets so that they will form stable triangle corners, something that is known in the magnet sphere learning community.

A thorough understanding of the interactions between point dipoles is helpful in understanding sphere-sphere interactions...

We measure the force as a function of distance.

We treat the magnetization of each magnet as fixed,

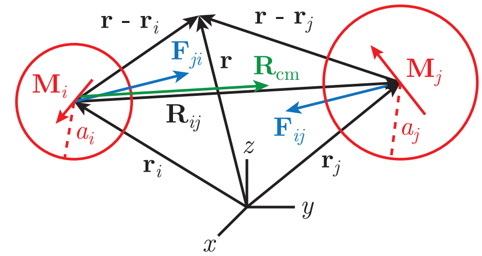

FIG. 1. Diagram showing two uniformly magnetized spheres with positions $\mathbf{r}_1$ and $\mathbf{r}_2$, radii a_1 and a_2, magnetizations $\mathbf{M}_1$ and $\mathbf{M}_2$, and paired forces $\mathbf{F}_1$ and $\mathbf{F}_2$. Shown also are the position vector $\mathbf{r}$ and relative position vectors $\mathbf{r} - \mathbf{r}_1$, $\mathbf{r} - \mathbf{r}_2$, and $\mathbf{R}_{ij} = \mathbf{r}_2 - \mathbf{r}_1$. To consider the forces between two point dipoles, replace spheres $\mathbf{M}_1$ and $\mathbf{M}_2$ by dipoles $\mathbf{m}_1$ and $\mathbf{m}_2$ at the same locations.

and neglect the demagnetization of one magnet by the field produced by another magnet. This assumption is appropriate for "hard" magnetic materials with high coercivities that are used to make permanent magnets. In the B vs. H ferromagnetic hysteresis loop, the coercivity H_C is defined as the value of H at which B falls to zero [10]. High coercivity therefore implies high resistance to demagnetization by external magnetic fields. The alloy used to make permanent neodymium magnets, $Nd_2Fe_{14}B$, has high coercivity [39].

Several countries, including the United States, have banned the sale of 5-mm nickel-coated neodymium magnets marketed as a desk toys (under the trade names BuckyBalls, Zen Magnets, Neoballs, etc.) following reports of intestinal injuries from ingestion of these magnets. But 2.5-mm magnets may be still be purchased [40], and magnet spheres of various sizes, including 5-mm diameter spheres, may be purchased from industrial suppliers [41–43].

The equivalence of the sphere-sphere interaction with the dipole-dipole interaction should be helpful for research on arrays of nanoparticles and permanent magnets, in dynamic simulations of magnet sphere interactions, as well as in efforts to use magnet spheres in science education.

The purpose of the paper is to calculate the force $\mathbf{F}_{12}$ of sphere 1 on sphere 2, which is not generally directed along the line between the sphere centers (Fig. 1). In Sec. **??**, we review a standard, brute-force, method of calculating this force. In Sec. **??**, we present a simple argument based on Newton's third law that shows that this force is identical to the force between two dipoles. In Sec. 5, we confirm this result by considering the total magnetic energy of the two-sphere system.

II. POINT DIPOLES

In this section, we review the fields of and forces between point dipoles, and introduce notation used in subsequent sections. The magnetic field at position $\mathbf{r}$ due to a point magnetic dipole $\mathbf{m}$ at the origin is given by [26]

$$\mathbf{B}(\mathbf{m};\mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{3\mathbf{m}\cdot\mathbf{r}}{r^5}\mathbf{r} - \frac{\mathbf{m}}{r^3}\right), \tag{1}$$

where $r = |\mathbf{r}|$. This field can be obtained from a scalar potential according to

$$\mathbf{B}(\mathbf{m};\mathbf{r}) = -\boldsymbol{\nabla}\varphi(\mathbf{m};\mathbf{r}), \tag{2}$$

where

$$\varphi(\mathbf{m};\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\cdot\mathbf{r}}{|\mathbf{r}|^3}. \tag{3}$$

We now consider the force between two dipoles, $\mathbf{m}_i$ and $\mathbf{m}_j$, at positions $\mathbf{r}_i$ and $\mathbf{r}_j$. Equation (1) enables us to write the field produced by dipole $\mathbf{m}_i$ as

$$\mathbf{B}_i(\mathbf{r}) = \mathbf{B}(\mathbf{m}_i;\mathbf{r}-\mathbf{r}_i), \tag{4}$$

where $\mathbf{r}-\mathbf{r}_i$ is the position vector relative to dipole $\mathbf{m}_i$ (Fig. 1).

The interaction energy between dipole $\mathbf{m}_j$ and the magnetic field produced by dipole $\mathbf{m}_i$ is given by

$$\begin{aligned} U_{ij} &= -\mathbf{m}_j\cdot\mathbf{B}_i(\mathbf{r}_j) && (5)\\ &= \frac{\mu_0}{4\pi}\left[\frac{\mathbf{m}_i\cdot\mathbf{m}_j}{{R_{ij}}^3} - 3\frac{(\mathbf{m}_i\cdot\mathbf{R}_{ij})(\mathbf{m}_j\cdot\mathbf{R}_{ij})}{{R_{ij}}^5}\right], && (6)\end{aligned}$$

where $\mathbf{R}_{ij} = \mathbf{r}_j - \mathbf{r}_i$ is the vector from $\mathbf{m}_i$ to $\mathbf{m}_j$, and $\mathbf{B}_i(\mathbf{r}_j)$ is the field produced by dipole $\mathbf{m}_i$ evaluated at the position of dipole $\mathbf{m}_j$.

The force of dipole $\mathbf{m}_i$ on dipole $\mathbf{m}_j$ follows as

$$\begin{aligned} \mathbf{F}_{ij} &= -\boldsymbol{\nabla}_j U_{ij} && (7)\\ &= \frac{3\mu_0}{4\pi {R_{ij}}^5}\Bigg[(\mathbf{m}_i\cdot\mathbf{R}_{ij})\,\mathbf{m}_j + (\mathbf{m}_j\cdot\mathbf{R}_{ij})\,\mathbf{m}_i \\ &\quad + (\mathbf{m}_i\cdot\mathbf{m}_j)\,\mathbf{R}_{ij} - 5\frac{(\mathbf{m}_i\cdot\mathbf{R}_{ij})\,(\mathbf{m}_j\cdot\mathbf{R}_{ij})}{{R_{ij}}^2}\mathbf{R}_{ij}\Bigg]. && (8)\end{aligned}$$

Here, $\boldsymbol{\nabla}_j$ is the gradient with respect to $\mathbf{r}_j$. $\mathbf{F}_{ij}$ is not a central force, namely, it is not generally parallel to the vector $\mathbf{R}_{ij}$ between the dipoles.

The force $\mathbf{F}_{ji} = -\boldsymbol{\nabla}_i U_{ji}$ of $\mathbf{m}_j$ on dipole $\mathbf{m}_i$ follows similarly from $U_{ji} = -\mathbf{m}_i\cdot\mathbf{B}_j(\mathbf{r}_i)$. A little algebra shows that $U_{ji} = U_{ji}$ and

$$\mathbf{F}_{ji} = -\mathbf{F}_{ij}, \tag{9}$$

confirming that Newton's third law applies to the force between point magnetic dipoles.

The torque of $\mathbf{m}_i$ on $\mathbf{m}_j$ is given by

$$\begin{aligned} \boldsymbol{\tau}_{ij} &= \mathbf{m}_j\times\mathbf{B}_i(\mathbf{r}_j) && (10)\\ &= \frac{\mu_0}{4\pi}\left(\frac{3\mathbf{m}_i\cdot\mathbf{R}_{ij}}{{R_{ij}}^5}\mathbf{m}_j\times\mathbf{R}_{ij} + \frac{\mathbf{m}_i\times\mathbf{m}_j}{{R_{ij}}^3}\right). && (11)\end{aligned}$$

This torque is not generally equal and opposite to the torque of $\mathbf{m}_j$ on $\mathbf{m}_i$, given by

$$\begin{aligned} \boldsymbol{\tau}_{ji} &= \mathbf{m}_i\times\mathbf{B}_j(\mathbf{r}_i) && (12)\\ &= \frac{\mu_0}{4\pi}\left(\frac{3\mathbf{m}_j\cdot\mathbf{R}_{ij}}{{R_{ij}}^5}\mathbf{m}_i\times\mathbf{R}_{ij} - \frac{\mathbf{m}_i\times\mathbf{m}_j}{{R_{ij}}^3}\right). && (13)\end{aligned}$$

The force $\mathbf{F}_{ij}$ between two dipoles is non-central and obeys Newton's third law. One might therefore wonder whether the associated torques produce angular acceleration in an isolated rigid two-dipole system that rotates freely about an axis passing through its center of mass, located relative to $\mathbf{m}_i$ by $\mathbf{R}_{\rm cm}$ (Fig. 1). We imagine joining the two dipoles rigidly to each other in a way that preserves their relative magnetic orientations, and imagine allowing this system to spin freely about its center of mass. The net torque on the two-dipole system about an axis passing through this center of mass is given by

$$\boldsymbol{\tau}_{net} = (\mathbf{R}_{ij}-\mathbf{R}_{\rm cm})\times\mathbf{F}_{ij} + (-\mathbf{R}_{\rm cm})\times\mathbf{F}_{ji} + \boldsymbol{\tau}_{ij} + \boldsymbol{\tau}_{ji}, \tag{14}$$

where the first term gives the torque produced by $\mathbf{F}_{ij}$, the second term gives the torque produced by $\mathbf{F}_{ji}$, and the third and fourth terms are the torques that the dipoles produce on each other. Appealing to Eqs. (8), (9), (11), and (13) shows immediately that $\boldsymbol{\tau}_{\rm net} = 0$. Thus, the torques $\boldsymbol{\tau}_{ij}$ and $\boldsymbol{\tau}_{ji}$, despite not generally being equal and opposite, conspire to cancel the torques produced by the non-central forces $\mathbf{F}_{ij}$ and $\mathbf{F}_{ji}$, which are equal and opposite.

Energy considerations assist in the understanding of the forces and torques on dipoles. The force $\mathbf{F}_{ij} = -\boldsymbol{\nabla}_j U_{ij}$ acts in the direction of maximum *decrease* of the energy $U_{ij} = -\mathbf{m}_j\cdot\mathbf{B}_i(\mathbf{r}_j)$, that is, $\mathbf{F}_{ij}$ acts in the direction of maximum *increase* in $\mathbf{m}_j\cdot\mathbf{B}_i$. In other words, $\mathbf{F}_{ij}$ is in the direction of the virtual displacement of $\mathbf{m}_j$ that gives the largest increase in $\mathbf{m}_j\cdot\mathbf{B}_i = m_j B_i\cos\theta$, where θ is the angle between $\mathbf{m}_j$ and $\mathbf{B}_i$. If $\theta < \pi/2$ and $\mathbf{m}_j\cdot\mathbf{B}_i > 0$, its increase may be accomplished by moving $\mathbf{m}_j$ in a direction of *increasing* field strength B_i. If $\theta > \pi/2$ and $\mathbf{m}_j\cdot\mathbf{B}_i < 0$, its increase may be accomplished by moving $\mathbf{m}_j$ in a direction of *decreasing* field strength B_i. Regardless of the value of θ and the sign of $\mathbf{m}_j\cdot\mathbf{B}_i$, its increase may be accomplished by moving $\mathbf{m}_j$ into a region where $\mathbf{m}_j$ and $\mathbf{B}_i$ are closer to parallel (smaller θ).

The energy determines whether $\mathbf{F}_{ij}$ is attractive or repulsive. The radial component of $\mathbf{F}_{ij}$,

$$F_{ij}^R = \hat{\mathbf{R}}_{ij}\cdot\mathbf{F}_{ij}, \tag{15}$$

implies an attractive force if $F_{ij}^R < 0$ and a repulsive force $F_{ij}^R > 0$, where $\hat{\mathbf{R}}_{ij} = \mathbf{R}_{ij}/R_{ij}$. Equations (7), (8), and (5) enable us to rewrite F_{ij}^R in the simple form,

$$F_{ij}^R = \frac{3U_{ij}}{R_{ij}} = -\frac{3\mathbf{m}_j\cdot\mathbf{B}_i(\mathbf{r}_j)}{R_{ij}}. \tag{16}$$

Thus, $\mathbf{F}_{ij}$ is attractive when $U_{ij} < 0$ and $\mathbf{m}_j\cdot\mathbf{B}_i > 0$, and repulsive when $U_{ij} > 0$ and $\mathbf{m}_j\cdot\mathbf{B}_i < 0$. Accordingly, only the angle θ is needed to determine whether $\mathbf{F}_{ij}$ is repulsive or attractive. If $\theta < \pi/2$, the force is attractive, drawing $\mathbf{m}_j$ closer to $\mathbf{m}_i$ where the fields are stronger, and increasing $\mathbf{m}_j \cdot \mathbf{B}_i > 0$ by *increasing* B_i. If $\theta > \pi/2$, the force is repulsive, pushing $\mathbf{m}_j$ away from $\mathbf{m}_i$, into regions with weaker fields, and increasing $\mathbf{m}_j \cdot \mathbf{B}_i < 0$ by *decreasing* B_i. These results agree with the arguments in the previous paragraph.

For many particles,...

It's useful to define

III. MAGNETIZED SPHERES

We consider the magnetic force between two uniformly magnetized spheres with arbitrary sizes, positions, magnetizations, and orientations (Fig. 1). Sphere i, denoted by S_i, has position vector $\mathbf{r}_i$, radius R_i, magnetization $\mathbf{M}_i$, and total magnetic dipole moment

$$\mathbf{m}_i = \frac{4}{3}\pi R_i^3 \mathbf{M}_i, \tag{17}$$

where $i = 1,\ 2$. This sphere produces a magnetic field

$$\mathbf{B}_i(\mathbf{r}) = \mathbf{B}(\mathbf{m}_i; \mathbf{r} - \mathbf{r}_i) \text{ for } |\mathbf{r} - \mathbf{r}_i| > R_i, \tag{18}$$

where $\mathbf{r} - \mathbf{r}_i$ is the position vector relative to the sphere center.

We consider two uniformly magnetized spheres with positions $\mathbf{r}_i$, radii R_i, and magnetizations $\mathbf{M}_i$, for $i = 1,\ 2$. Outside of sphere i (for $|\mathbf{r} - \mathbf{r}_i| > R_i$), its magnetic field is given Eq. (1) according to

$$\mathbf{B}_i(\mathbf{r}) = \mathbf{B}(\mathbf{m}_i; \mathbf{r} - \mathbf{r}_i), \tag{19}$$

and its scalar potential by Eq. (3) according to

$$\varphi_i(\mathbf{r}) = \varphi(\mathbf{m}_i; \mathbf{r} - \mathbf{r}_i). \tag{20}$$

A. Newton's Third Law

A five-step argument involving Newton's third law shows that the force between two spheres with magnetizations $\mathbf{M}_1$ and $\mathbf{M}_2$ is identical to the force between two point dipoles with magnetic moments $\mathbf{m}_1$ and $\mathbf{m}_2$ at the same locations as the spheres, and obeying Eq. (17), as follows:

1. Let $\mathbf{F}_2$ represent the force of dipole $\mathbf{m}_1$ on dipole $\mathbf{m}_2$. This force is produced by the field $\mathbf{B}_1$ from dipole $\mathbf{m}_1$ (Fig. 2a).

2. Sphere $\mathbf{M}_1$ produces the same field $\mathbf{B}_1$, and therefore exerts the same force $\mathbf{F}_2$ on dipole $\mathbf{m}_2$ (Fig. 2b).

3. Newton's third law gives the force $\mathbf{F}_1 = -\mathbf{F}_2$ of dipole $\mathbf{m}_2$ on sphere $\mathbf{M}_1$ (Fig. 2c). This force is produced by the field $\mathbf{B}_2$ from dipole $\mathbf{m}_2$ (Fig. 2c).

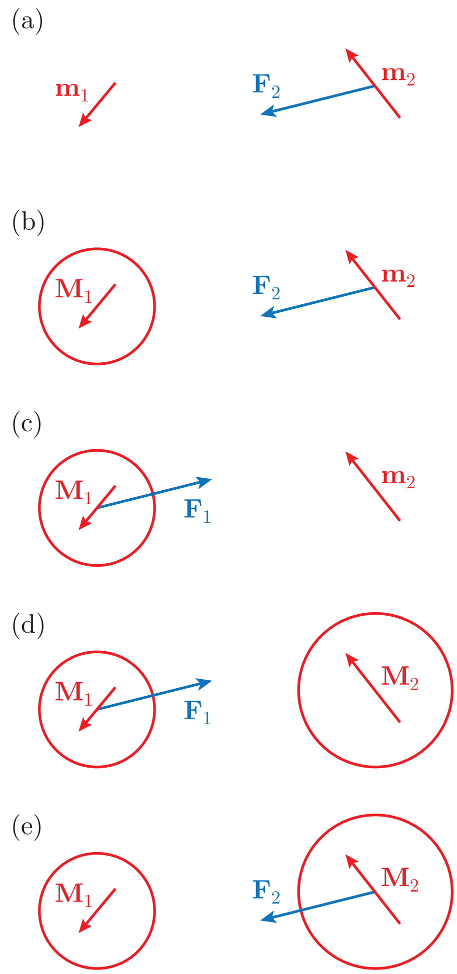

FIG. 2. Diagram illustrating the five steps of the Newton's third law argument showing that the force between two uniformly magnetized spheres is identical to the force between two point dipoles.

4. Sphere $\mathbf{M}_2$ produces the same field $\mathbf{B}_2$, and therefore exerts the same force $\mathbf{F}_1$ on sphere $\mathbf{M}_1$ (Fig. 2d).

5. Again applying Newton's third law shows that the force $\mathbf{F}_2 = -\mathbf{F}_1$ of sphere $\mathbf{M}_1$ on sphere $\mathbf{M}_2$ is identical to the force of dipole $\mathbf{m}_1$ on dipole $\mathbf{m}_2$ (Figs. 2e, 2a).

B. Direct integration

Using (the field is the same for a point dipole and a uniformly magnetized sphere centered at $\mathbf{r}_i$)

$$\int_{S_i} d\mathbf{r}' \mathbf{B}(\mathbf{M}_i; \mathbf{r} - \mathbf{r}') = \mathbf{B}(\mathbf{m}_i; \mathbf{r} - \mathbf{r}_i),$$

the force can be directly integrated over S_2

$$\begin{aligned}
\mathbf{F}_2 &= -\nabla_2 \int_{S_2} \mathbf{B}(\mathbf{m}_1; \mathbf{r}-\mathbf{r}_1)\cdot \mathbf{M}_2 d\mathbf{r}. \\
&= -\nabla_2 \int_{S_2} \frac{\mu_0}{4\pi}\left[\frac{3\mathbf{m}_1\cdot(\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|^5}(\mathbf{r}-\mathbf{r}_1) - \frac{\mathbf{m}_1}{|\mathbf{r}-\mathbf{r}_1|^3}\right]\cdot \mathbf{M}_2 d\mathbf{r} \\
&= \nabla_2 \int_{S_2} \frac{\mu_0}{4\pi}\left[\frac{3\mathbf{M}_2\cdot(\mathbf{r}_1-\mathbf{r})}{|\mathbf{r}_1-\mathbf{r}|^5}(\mathbf{r}_1-\mathbf{r}) - \frac{\mathbf{M}_2}{|\mathbf{r}_1-\mathbf{r}|^3}\right] d\mathbf{r}\cdot \mathbf{m}_1 \\
&= \nabla_2 \left[\int_{S_2} \mathbf{B}(\mathbf{M}_2; \mathbf{r}_1-\mathbf{r}) d\mathbf{r}\right]\cdot \mathbf{m}_1 \\
&= \nabla_2 \mathbf{B}(\mathbf{m}_2; \mathbf{r}_1-\mathbf{r}_2)\cdot \mathbf{m}_1 [\text{you can see this is equal to } -\mathbf{F}_{12} = \nabla_1 \mathbf{B}(\mathbf{m}_1; \mathbf{r}_2-\mathbf{r}_1)\cdot \mathbf{m}_2] \\
&= -\nabla_2 \mathbf{B}(\mathbf{m}_1; \mathbf{r}_1-\mathbf{r}_2)\cdot \mathbf{m}_2 (\text{equalt to the force on the point dipole 2})
\end{aligned}$$

To calculate the force between two magnetized spheres, we consider the energy of interaction between the magnetic field $\mathbf{B}_1$ produced by S_1 and a volume element d^3x of S_2, with dipole moment $d\mathbf{m}_2 = \mathbf{M}_2 d^3x$. This energy is given by $dU_{12} = -d\mathbf{m}_2\cdot\mathbf{B}_1 = -\mathbf{M}_2\cdot\mathbf{B}_1 d^3x$. The corresponding force on this element is $d\mathbf{F}_{12} = -\boldsymbol{\nabla}(dU_{12}) = \boldsymbol{\nabla}(\mathbf{M}_2\cdot\mathbf{B}_1)\, d^3x$. Therefore, the total force on S_2 is a volume integral over this sphere,

$$\mathbf{F}_{12} = \int_{S_2} \boldsymbol{\nabla}(\mathbf{M}_2\cdot\mathbf{B}_1)\, d^3x, \tag{21}$$

and the associated total magnetic interaction energy is

$$U_{12} = -\int_{S_2} \mathbf{M}_2\cdot\mathbf{B}_1 d^3x. \tag{22}$$

This technique has been used to calculate the force between two uniformly magnetized spheres in special cases. Cite examples here...

C. Potential theory

The force on the sphere i can be obtained from $\mathbf{F}_i = -\nabla_i U$ where ∇_i denotes the gradient operator with respect to $\mathbf{r}_i$. Here the magnetic field energy is given by

$$\begin{aligned}
U(\mathbf{r}_1,\mathbf{r}_2) &= \frac{1}{2\mu_0}\int (\mathbf{B}_1+\mathbf{B}_2)^2 d\mathbf{r} \\
&= \frac{1}{2\mu_0}\int (B_1^2 + 2\mathbf{B}_1\cdot\mathbf{B}_2 + B_2^2) d\mathbf{r} \\
&= U_1 + U_{12} + U_2
\end{aligned}$$

Obviously, the integrals $U_i = (2\mu_0)^{-1}\int B_i^2 d\mathbf{r}$ $(i = 1,2)$ do not depend on $\mathbf{r}_1$ and $\mathbf{r}_2$ and therefore the force comes from

$$\begin{aligned}
U_{12} &= \frac{1}{\mu_0}\int \mathbf{B}_1\cdot\mathbf{B}_2 d\mathbf{r} \\
&= \frac{1}{\mu_0}\Big(\int_{S_1} + \int_{S_2} + \int_{\text{outside}}\Big)\nabla\varphi_1\cdot\nabla\varphi_2 d\mathbf{r}
\end{aligned}$$

For the integral over the sphere 1, we use $\nabla^2\varphi_2 = 0$ and the Gauss' theorem

$$\int_{S_1} \nabla\varphi_1\cdot\nabla\varphi_2 d\mathbf{r} = \int_{S_1} \nabla\cdot(\varphi_1\nabla\varphi_2) d\mathbf{r} = \int_{\partial S_1} \varphi_1\nabla\varphi_2\cdot\hat{\mathbf{n}}_1 dA$$

where ∂S_1 denotes the boundary of S_1 and $\hat{\mathbf{n}}_1$ is the unit vector on ∂S_1. Note that $\varphi_1\nabla\varphi_2$ is continuous across ∂S_1. Similarly,

$$\int_{S_2} \nabla\varphi_1\cdot\nabla\varphi_2 d\mathbf{r} = \int_{S_1} \nabla\cdot(\varphi_2\nabla\varphi_1) d\mathbf{r} = \int_{\partial S_2} \varphi_2\nabla\varphi_1\cdot\hat{\mathbf{n}}_2 dA.$$

Since $\varphi_1 = \varphi_1^{\text{point}}$ (the superscript 'point' denotes 'due to the point dipole) and $\nabla\varphi_2 = \nabla\varphi_2^{\text{point}}$ on ∂S_1 and $\varphi_2 = \varphi_2^{\text{point}}$ and $\nabla\varphi_1 = \nabla\varphi_1^{\text{point}}$ on ∂S_2, the energy U_{12} is the same for the point dipoles and for the spheres ($U_{12} = U_{12}^{\text{point}}$) and so is the force ($\mathbf{F}_i = \mathbf{F}_i^{\text{point}}$). (Note that $U_i \neq U_i^{\text{point}}$ because $\nabla^2\varphi_i^{\text{point}} \neq 0$ at $\mathbf{r} = \mathbf{r}_i$ for point dipoles. Nevertheless the force is the same.)

D. Stress tensor

The global force on an object can be calculated by integrating the Maxwell stress tensor over an arbitrary surface surrounding the object. The stress tensor depends only on the magnetic field.

In the second method we use the fact that in a given volume the momentum in the fields must be conserved without any external fields. Any external fields that do exist can be measured at the boundaries of our volume. Therefore integrating the stress of the total magnetic field over our volume will yield the force on the volume [45, 46]. Given that two uniformly magnetized spheres produce the same total magnetic field as two magnetic dipoles (again assuming $m = 4\pi a^3 M/3$) outside of the spheres and we may draw our arbitrary volumes outside the spheres, the forces on each magnetic sphere must be the same as the forces on the dipoles *regardless of either sphere's orientation*.

The FIELD outside a uniformly magnetized sphere is the same as the field of a point dipole at the center. In this case the total field (except inside either sphere) is the same as that of two dipoles. So the stress tensor (except inside either sphere) is the same as it would be for two dipoles. In particular, the stress tensor on (say) the plane bisecting the line of centers is identical for two spheres as for two dipoles. But the force (on either one) is the integral of the stress tensor over this plane (the contribution from the hemispherical surface at "infinity" vanishes). Therefore, the force must be the same as between two dipoles, regardless of their orientation. Am I missing something?

We are grateful to David Griffiths for this proof. Kirk McDonald said it was obvious. Invite David to co-author the paper?

IV. PEDAGOGICAL TOOLS

In all cases except for the two unstable repulsive cases (Examples 3 and 6), the forces and torques on a dipole or spherical magnet bring it eventually to the stable minimum-energy state discussed in Example 1. If $\mathbf{m}_2$ is allowed to freely move and rotate in space, the two unstable repulsive cases will not be possible to realize physically because any slight perturbation in the position or orientation of $\mathbf{m}_2$ will lead eventually to its attraction to the minimum-energy state.

nterpreis straightforward in some cases and subtle in others. Fig. 3(a), position A, shows

In other cases, applying these principles in understanding Fig. 3 is straightforward is subtle in other cases.

It is a standard undergraduate exercise to calculate the magnetic field at position $\mathbf{r}$ produced by a sphere of radius R and uniform magnetization $\mathbf{M}$, and located at the coordinate origin [9]. Outside of the sphere (for $r > R$), this field is identical to the dipole field given by Eq. (1), where

$$\mathbf{m} = \int_V \mathbf{M} dV = \frac{4\pi}{3} R^3 \mathbf{M} \tag{23}$$

is the total dipole moment of the sphere, and the integral is over the sphere volume. Inside the sphere, the field is uniform and is given by $\mathbf{B} = 2\mu_0 \mathbf{M}/3$.

V. DYNAMICS

A. Dynamical Equations

We consider the dynamical interactions between N identical spheres ($i = 1, 2, 3, \cdots, N$), each with diameter D, mass $\tilde{m}$, and magnetic moment of magnitude $m = |\mathbf{m}|$. The magnetic force on mass j is given by

$$\mathbf{F}_j = \sum_{i=1,\ i\neq j}^{N} \mathbf{F}_{ij}, \tag{24}$$

where $\mathbf{F}_{ij}$ is given by Eq. (8) and where $j = 1, 2, \cdots, N$. Newton's second law gives

$$\mathbf{F}_j + \mathbf{F}_j^{\text{other}} = \tilde{m} \mathbf{a}_j, \tag{25}$$

where $\mathbf{a}_j = d\mathbf{v}_j/dt$ is the acceleration of mass j, $\mathbf{v}_j = d\mathbf{r}_j/dt$ is its velocity, and $\mathbf{F}_j^{\text{other}}$ are other forces of interest.

The magnetic torque on mass j is given by

$$\boldsymbol{\tau}_j = \sum_{i=1, i\neq j}^{N} \boldsymbol{\tau}_{ij}, \tag{26}$$

where $\boldsymbol{\tau}_{ij}$ is given by Eq. (11). The rotational form of Newton's second law demands that

$$\boldsymbol{\tau}_j + \boldsymbol{\tau}_j^{\text{other}} = I \boldsymbol{\alpha}_j, \tag{27}$$

where $\boldsymbol{\alpha}_j = d\boldsymbol{\omega}_j/dt$ is the angular acceleration of mass j, $\boldsymbol{\omega}_j$ is its angular velocity, $\boldsymbol{\tau}_j^{\text{other}}$ are other torques of interest, and

$$I = \frac{2}{5} \tilde{m} \left(\frac{D}{2}\right)^2 \tag{28}$$

is its moment of inertia.

We define dimensionless variables to render the calculations pertinent to spheres of any size and magnetization, and to simplify computations. We scale the length by D, the force by $F_0 = 3\mu_0 m^2/2\pi D^4$, and the time by $T_0 = \sqrt{\tilde{m} D/F_0}$. Here, F_0 is the force between two magnets in the minimum-energy state with the north pole of one touching the south pole of the other, and T_0 is the time scale for two magnets to collide starting from rest at separations of the order of $2D$. We define (primed) dimensionless coordinates according to

$$\mathbf{F} = F_0 \mathbf{F}' \tag{29}$$

$$\mathbf{B} = (F_0 D/m) \mathbf{B}' \tag{30}$$

$$\boldsymbol{\tau} = F_0 D \boldsymbol{\tau}' \tag{31}$$

$$\mathbf{m} = m \hat{\mathbf{m}} \tag{32}$$

$$t = T_0 t' \tag{33}$$

$$\mathbf{r} = D \mathbf{r}' \tag{34}$$

$$\mathbf{v} = d\mathbf{r}/dt = (D/T_0) \mathbf{v}' \tag{35}$$

$$\mathbf{a} = d\mathbf{v}/dt = (D/T_0^2) \mathbf{a}' \tag{36}$$

$$\boldsymbol{\omega} = (1/T_0) \boldsymbol{\omega}' \tag{37}$$

$$\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt = (1/T_0^2) \boldsymbol{\alpha}' \tag{38}$$

In dimensionless units, the magnet diameter and the center-to-center distance between touching magnets are both 1.

In dimensionless units, we have

$$\mathbf{B}'(\hat{\mathbf{m}}; \mathbf{r}') = \frac{1}{6} \left(\frac{3\hat{\mathbf{m}} \cdot \mathbf{r}'}{r'^5} \mathbf{r}' - \frac{\hat{\mathbf{m}}}{r'^3} \right), \tag{39}$$

$$\mathbf{B}_i'(\mathbf{r}') = \mathbf{B}(\hat{\mathbf{m}}_i; \mathbf{r}' - \mathbf{r}_i'), \tag{40}$$

$$\mathbf{a}_j' = \mathbf{F}_j^{\text{other}'} + \sum_{i=1,\ i\neq j}^{N} \mathbf{F}_{ij}' \tag{41}$$

$$\begin{aligned} \mathbf{F}_{ij}' = & \frac{1}{2{R_{ij}}'^5} \Big[\left(\hat{\mathbf{m}}_i \cdot \mathbf{R}_{ij}'\right) \hat{\mathbf{m}}_j + \left(\hat{\mathbf{m}}_j \cdot \mathbf{R}_{ij}'\right) \hat{\mathbf{m}}_i \\ & + \left(\hat{\mathbf{m}}_i \cdot \hat{\mathbf{m}}_j\right) \mathbf{R}_{ij}' - 5 \frac{\left(\hat{\mathbf{m}}_i \cdot \mathbf{R}_{ij}'\right)\left(\hat{\mathbf{m}}_j \cdot \mathbf{R}_{ij}'\right)}{{R_{ij}}'^2} \mathbf{R}_{ij}' \Big]. \end{aligned} \tag{42}$$

$$\boldsymbol{\alpha}_j' = 10 \left(\boldsymbol{\tau}_j^{\text{other}'} + \sum_{i=1,\ i\neq j}^{N} \boldsymbol{\tau}_{ij}' \right) \tag{43}$$

$$\boldsymbol{\tau}_{ij}' = \frac{\hat{\mathbf{m}}_i \cdot \mathbf{R}_{ij}'}{2{R_{ij}}'^5} \hat{\mathbf{m}}_j \times \mathbf{R}_{ij}' + \frac{\hat{\mathbf{m}}_i \times \hat{\mathbf{m}}_j}{6{R_{ij}}'^3} \tag{44}$$

B. 2D Geometry

It is instructive to investigate the magnetic interactions between two spheres whose positions and magnetic orientations are confined to the x-y plane. Dipole $\mathbf{m}_1$ is held at fixed position $\mathbf{r}'_1 = 0$ and fixed orientation $\hat{\mathbf{m}}_1 = \hat{\mathbf{x}}$, and produces a static magnetic field $\mathbf{B}_1$. Dipole $\mathbf{m}_2$ moves and rotates in response to this field, with dimensionless position

$$\mathbf{R}'_{12} = \mathbf{r}'_2 - \mathbf{r}'_1 = R\cos\Phi\hat{\mathbf{x}} + R\sin\Phi\hat{\mathbf{y}}, \tag{45}$$

magnetic orientation

$$\hat{\mathbf{m}}_2(t) = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}, \tag{46}$$

angular velocity $\boldsymbol{\omega} = \hat{\mathbf{z}}d\phi/dt$, and angular acceleration $\boldsymbol{\alpha} = \hat{\mathbf{z}}d\phi/dt$. Here, R and Φ give the time-dependent dimensionless position of $\mathbf{m}_2$ in cylindrical coordinates and ϕ gives its time-dependent magnetic orientation. Under these conditions, $\mathbf{F}_{12}$ is confined to the x-y plane, and $\boldsymbol{\tau}_{12}$ is confined to the z direction.

C. Magnetic Forces and Torques

Figure 3 shows the dipole magnetic field $\mathbf{B}_1$ produced by $\mathbf{m}_1$, together with the magnetic forces and torques on $\mathbf{m}_2$ for various positions and orientations of $\mathbf{m}_2$ [44]. These results illustrate principles discussed in Sec. II:

- The force and torque on $\mathbf{m}_2$ act in directions that increase $\mathbf{m}_2 \cdot \mathbf{B}_1$ most rapidly.
- The force on $\mathbf{m}_2$ is attractive if $\mathbf{m}_2 \cdot \mathbf{B}_1 > 0$ and repulsive when $\mathbf{m}_2 \cdot \mathbf{B}_1 < 0$.

The forces and torques shown in Fig. 3 fall into six main categories:

1. In configurations A and E in panel (a), $\mathbf{m}_1$ and $\mathbf{m}_2$ are co-linear and parallel to each other, and $\mathbf{m}_2$ is parallel to $\mathbf{B}_1$, giving the product $\mathbf{m}_2 \cdot \mathbf{B}_1 = m_2 B_1 > 0$. To increase this product most rapidly, $\mathbf{m}_1$ attracts $\mathbf{m}_2$ into its vicinity, where the field is stronger. Since $\mathbf{m}_2$ is already aligned with $\mathbf{B}_1$, there is no torque. This is the familiar attractive example of two co-linear magnets with the north pole of one magnet facing the south pole of the other. When co-linear, parallel spherical magnets are released from rest, they eventually come into contact with each other with the north pole of one touching the south pole of the other. This is the minimum-energy, stable state of the two-magnet system.

2. In configurations B, D, F, and H in panel (a), the force on $\mathbf{m}_2$ is attractive ($\mathbf{m}_2 \cdot \mathbf{B}_1 > 0$) and non-central. This force acts to increase $\mathbf{m}_2 \cdot \mathbf{B}_1$ by moving the dipole into a polar field region where the field is stronger and is parallel to $\mathbf{m}_2$. The torque acts to rotate $\mathbf{m}_2$ into alignment with the local field, which also increases $\mathbf{m}_2 \cdot \mathbf{B}_1$. Configurations A, B, E, and F in panel (b), configurations B and F in panel (c), and configurations B, C, F, and G in panel (d) also have $\mathbf{m}_2 \cdot \mathbf{B}_1 > 0$, and behave similarly.

3. Configurations C and G in panel (a) are repulsive and unstable. Here, $\mathbf{m}_1$ and $\mathbf{m}_2$ are parallel but not co-linear. Instead, $\mathbf{m}_1$ and $\mathbf{m}_2$ are perpendicular to the line through their centers, and $\mathbf{B}_1$ is antiparallel to them, giving $\mathbf{m}_2 \cdot \mathbf{B}_1 = -m_2 B_1$. To increase this product, the force on $\mathbf{m}_2$ acts to push it into regions where the field is *weaker*. This is the familiar repulsive example of two parallel magnets lined up next to each other with their adjacent north and south poles repelling each other. There is no torque on $\mathbf{m}_2$ because it is antiparallel to $\mathbf{B}_1$. But this configuration is unstable; any perturbation in the position or orientation of $\mathbf{m}_2$ leads to a torque that aligns it with $\mathbf{B}_1$, whence the force becomes attractive.

4. In configurations C, D, G, and H in panel (b), the force is non-central and repulsive ($\mathbf{m}_2 \cdot \mathbf{B}_1 < 0$). To increase this product, the force on $\mathbf{m}_2$ acts to move it toward *weaker* fields, and toward regions where $\mathbf{m}_2$ is better aligned with the field, and the torque acts to rotate $\mathbf{m}_2$ into alignment with the local field. The net effect is to move $\mathbf{m}_2$ into a configuration in which $\mathbf{m}_2 \cdot \mathbf{B}_1 > 0$ and the force becomes attractive. Configurations D and H of panel (c), configurations A, D, E, and H in panel (d), and configurations B, D, F, and H in panel (e) also have $\mathbf{m}_2 \cdot \mathbf{B}_1 < 0$, and behave similarly.

5. In configurations A, C, E, and G in panel (c), $\mathbf{m}_2 \perp \mathbf{B}_1$, $\mathbf{m}_2 \cdot \mathbf{B}_1 = 0$, and the force on $\mathbf{m}_2$ has no radial component – it is neither attractive nor repulsive. Its azimuthal component acts to move $\mathbf{m}_2$ toward a region where $\mathbf{m}_2$ is better aligned with the field, thereby increasing $\mathbf{m}_2 \cdot \mathbf{B}_1$. In these configurations, a force toward or away from $\mathbf{m}_1$ would not change $\mathbf{m}_2 \cdot \mathbf{B}_1$ because $\mathbf{m}_2$ would remain perpendicular to $\mathbf{B}_1$. The torque, as always, acts to rotate $\mathbf{m}_2$ into alignment with the local field, tending to a force with an attractive radial component.

6. In configurations A and E of panel (e), $\mathbf{m}_1$ and $\mathbf{m}_2$ are co-linear and antiparallel to each other, and $\mathbf{m}_2$ is antiparallel to $\mathbf{B}_1$, giving the product $\mathbf{m}_2 \cdot \mathbf{B}_1 = -m_2 B_1$ as in Example 3. To increase this product most rapidly, $\mathbf{m}_1$ acts to repel $\mathbf{m}_2$ into regions where the field is *weaker*. This is the familiar repulsive example of two co-linear magnets with north poles, or south poles, facing each other. There is no torque on $\mathbf{m}_2$ because it is antiparallel to $\mathbf{B}_1$. As in Example 3, this configuration is unstable because any perturbation in the position or

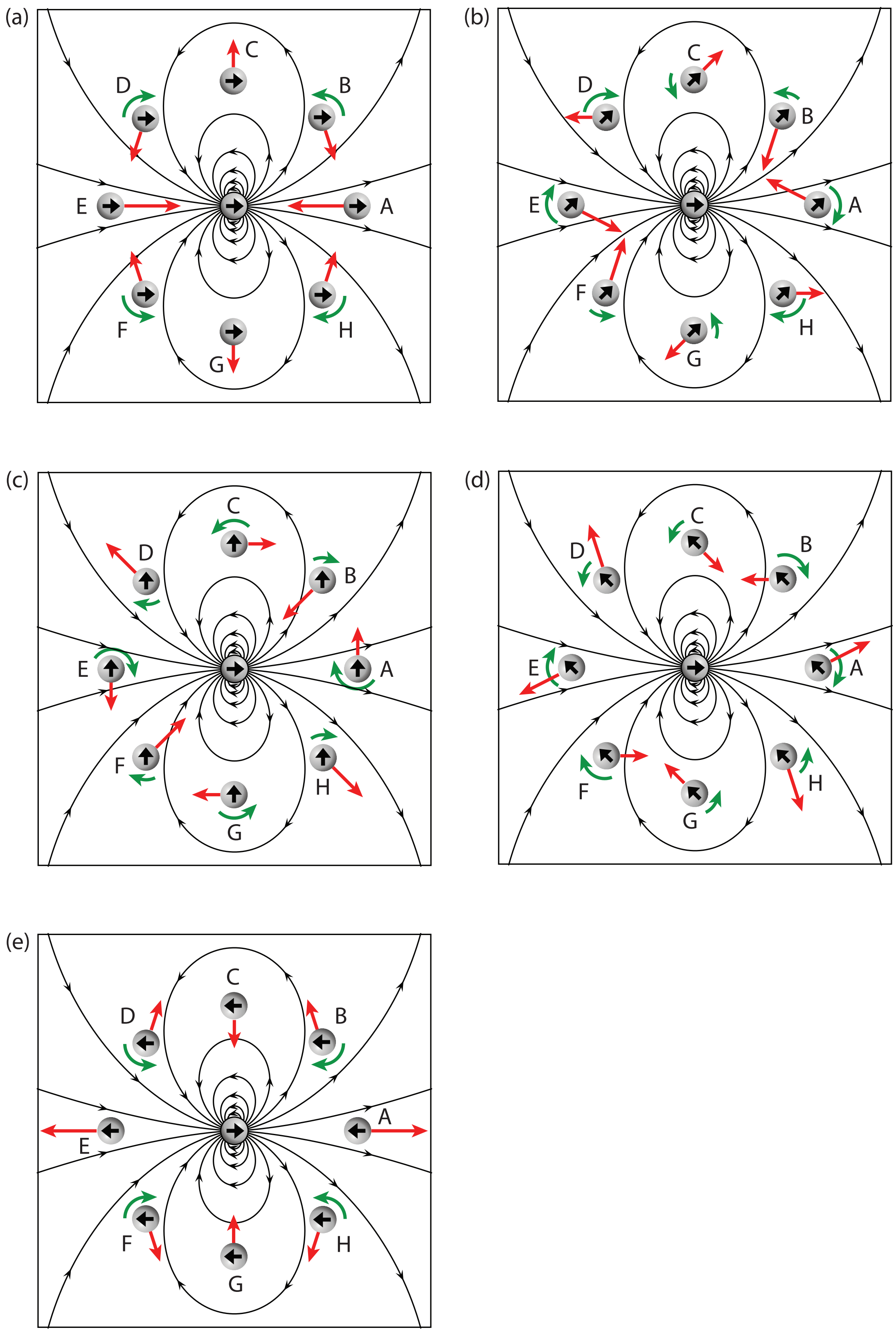

FIG. 3. Magnetic field lines produced by dipole $\mathbf{m}_1$, at the center of each panel, and the resulting magnetic forces and torques on dipole $\mathbf{m}_2$ at various positions and orientations, as given by Eqs. (39), (42), and (44). Shown are drawings for $\mathbf{m}_2$ and $\mathbf{m}_1$ differing by angles 0 (a), $\pi/4$ (b), $\pi/2$, (c) $3\pi/4$ (d), and π (e). For each panel, there are eight positions of $\mathbf{m}_2$, labeled A-H, spaced evenly around a circle of diameter $R = d$. Force vectors are shown with their lengths proportional to the force magnitude. Torques are indicated by clockwise and counterclockwise circular arcs, with the arc length increasing with torque magnitude, and with no arc if the torque is zero. The figure applies both to point dipoles and to uniformly magnetized spheres with total magnetic moments given by Eq. (17).

orientation of $\mathbf{m}_2$ leads to a torque that tends to align it with $\mathbf{B}_1$, whence the force becomes attractive.

D. Time Dependence

Studying the time-dependent motion of $\mathbf{m}_2$ under the influence of $\mathbf{m}_1$ introduces another level of complexity, and understanding, to the problem owing to the translational and rotational inertia of $\mathbf{m}_2$. A force in a particular direction does not mean that $\mathbf{m}_2$ moves in that direction; it means that the change in its linear momentum is in that direction. Similarly, a torque in a particular direction does not mean that $\mathbf{m}_2$ turns in that direction; it means that the change in its angular momentum is in that direction. A torque that acts to align $\mathbf{m}_2$ with $\mathbf{B}_1$ can eventually lead to angular momentum that acts to align these vectors. But the very angular momentum that carries $\mathbf{m}_2$ into alignment with $\mathbf{B}_1$ continues to rotate the sphere in the same direction until $\mathbf{m}_2$ is out of alignment with $\mathbf{B}_1$ again, when a torque in the opposite direction begins to slow the rotation in an attempt to bring the sphere back into alignment. In the absence of dissipation, such oscillations can continue indefinitely, as shown below. In a similar way, forces that strive to bring the sphere into regions where $\mathbf{m}_2$ is better aligned with $\mathbf{B}_1$ give it linear momentum that can carry the sphere beyond these regions.

To investigate one possible mechanism for damping these oscillations, we include three types of kinetic friction. We confine magnets to the top surface of a horizontal table, with the downward gravitational force $\tilde{m}g$ on a magnet equaling the upward normal force on it, and with a kinetic friction force $\mathbf{f}_t = -\mu_t \tilde{m} g \hat{\mathbf{v}}$ directed opposite to the direction of motion of the sphere, $\hat{\mathbf{v}}$. Here, μ_t is the coefficient of kinetic friction between a magnet and the table and g is the local acceleration of gravity.

The second type of kinetic friction is a frictional torque $\boldsymbol{\tau}_t = -\tilde{\mu}_t \tilde{m} g D \hat{\boldsymbol{\omega}}$ on a magnet that is spinning on the table, where $\tilde{\mu}_t$ is the associated coefficient of friction. This torque is directed opposite to the direction $\hat{\boldsymbol{\omega}}$ of angular velocity of the sphere.

The third type of kinetic friction is friction between two magnet spheres that are in contact with each other, and are sliding against each other. Such sliding can occur because $\mathbf{m}_2$ is rotating but not translating, because $\mathbf{m}_2$ is translating (tangential to $\mathbf{m}_1$) but not rotating, or because $\mathbf{m}_2$ is both translating and rotating. This produces a force $\mathbf{f}_m = -\mu_m |\mathbf{F}_{12} \cdot \hat{\mathbf{R}}| \hat{\mathbf{v}}_t$ directed opposite to the tangential velocity vector $\mathbf{v}_t = \mathbf{v} + \boldsymbol{\omega}_2 \times (\mathbf{r}_c - \mathbf{r}_2)$ of $\mathbf{m}_2$ at the point of contact, where $\mathbf{f}_m$ is applied. Here, $\mathbf{r}_c$ is the position vector of the point of contact, whence $|\mathbf{r}_c - \mathbf{r}_2| = D/2$. Also, $\mathbf{F}_{12} \cdot \hat{\mathbf{R}} \leq 0$ is the normal component of the magnetic force and $\hat{\mathbf{v}}_t = \mathbf{v}_t / v_t$. Since $\mathbf{f}_m$ is applied along the tangent to the surface of $\mathbf{m}_2$, it also exerts a torque $\boldsymbol{\tau}_m = (\mathbf{r}_c - \mathbf{r}_2) \times \mathbf{f}_m$ on $\mathbf{m}_2$.

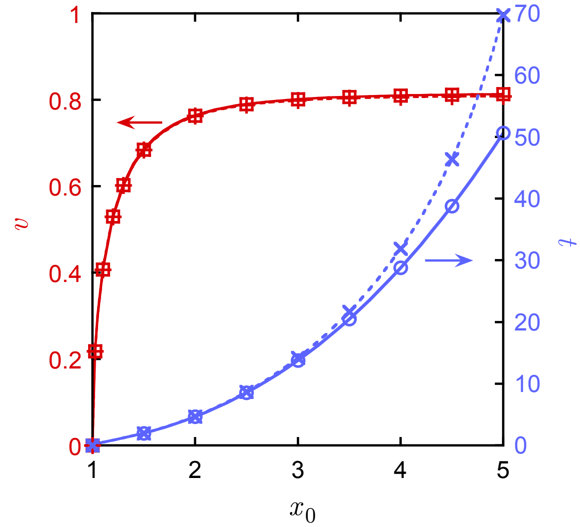

FIG. 4. Dimensionless collision speed v and elapsed time t for a uniformly magnetized sphere with dipole moment $\mathbf{m}_2 = m_2 \hat{\mathbf{x}}$ that is released from rest at initial position $x_0 \hat{\mathbf{x}}$, and attracted by a second uniformly magnetized sphere with dipole moment $\mathbf{m}_1 = m_1 \hat{\mathbf{x}}$ that is held fixed at the origin (Fig. 3a, configuration A). For v, the solid trace is given by Eq. (55) for $\gamma = 0$ (no friction), the dashed trace is given by Eq. (55) for $\gamma = 0.001$ (friction pertinent to Zen Magnets), the open squares are given by fourth-order Runge-Kutta integration (animation program MAGNETS) for $\gamma = 0$, and the plus symbols are given by MAGNETS for $\gamma = 0.001$. For t, the solid trace is given by integrating Eq. (54)] for $\gamma = 0$, the dashed trace is given by integrating Eq. (54)] for $\gamma = 0.001$, the open circles are given by MAGNETS for $\gamma = 0$, and the plus symbols are given by MAGNETS for $\gamma = 0.001$.

Thus, we insert

$$\mathbf{F}^{\text{other}} = \mathbf{f}_t + \mathbf{f}_m \tag{47}$$

and

$$\boldsymbol{\tau}^{\text{other}} = \boldsymbol{\tau}_t + \boldsymbol{\tau}_m \tag{48}$$

into Eqs. (25) and (27), noting that $\mathbf{f}_m$ and $\boldsymbol{\tau}_m$ apply only when $\mathbf{m}_2$ is in contact with $\mathbf{m}_1$.

Three simple examples help to develop understanding of the motion:

Example 1: Translation. The first example is to release sphere 2 from rest at initial position $\mathbf{R}'_{12}(0) = x_0 \hat{\mathbf{x}}$ and orientation $\hat{\mathbf{m}}_2 = \hat{\mathbf{x}}$ (Fig. 3a, configuration A). The objective is to determine the elapsed time and speed at which it collides with sphere 1, which is held fixed at the origin with $\hat{\mathbf{m}}_1 = \hat{\mathbf{x}}$. In this case, $\mathbf{m}_2$ experiences no torque and moves in the $-x$ direction with time-dependent position $\mathbf{R}'_{12}(t) = x(t) \hat{\mathbf{x}}$ satisfying Eqs. (41)-(42)

$$\frac{dv_x}{dt} = \gamma - \frac{1}{x^4}, \tag{49}$$

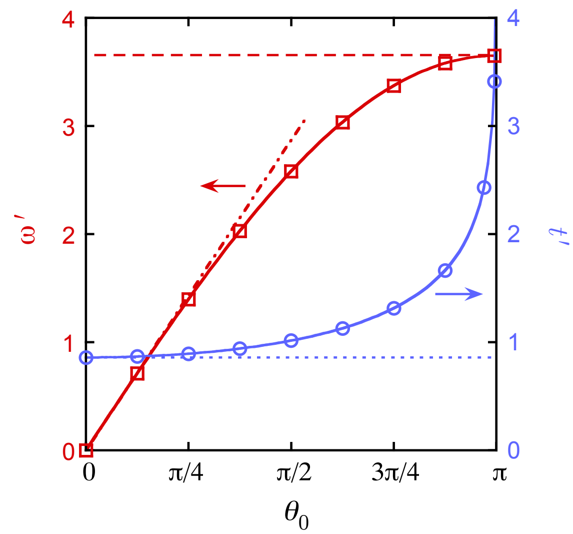

FIG. 5. Dimensionless collision speed $v' = vT/D$ and elapsed time $t' = t/T$ for a sphere of diameter D and magnetic moment $\mathbf{m}_2 = m_2\hat{\mathbf{x}}$ released from rest at initial position $\mathbf{R}_{ij} = x_0\hat{\mathbf{x}}$, and attracted by a second sphere of diameter D and magnetic moment $\mathbf{m}_1 = m_1\hat{\mathbf{x}}$ that is held fixed at the origin (Fig. 3, panel (a), configuration A). Need to explain x_0'.

where $v_x = dx/dt$. Here,

$$\gamma = \frac{f_t}{F_0} = \frac{2\pi\mu_t\tilde{m}gD^4}{3\mu_0 m^2} \tag{50}$$

is the magnitude of the dimensionless (constant) kinetic friction force of the table on $\mathbf{m}_2$. Equation (49) is valid for $v_x < 0$ in order for the friction force to oppose the motion. For $v_x > 0$, the sign of the friction term must be reversed. Writing

$$\frac{dv_x}{dt} = \frac{dv_x}{dx}\frac{dx}{dt} = v_x\frac{dv_x}{dx} \tag{51}$$

enables us to rewrite Eq. (49) as

$$\int v_x dv_x = \int \left(\gamma - \frac{1}{x^4}\right) dx. \tag{52}$$

Integrating and applying the initial conditions $x(0) = x_0$ and $v_x(0) = 0$ gives

$$v_x = -\left[\frac{2}{3}\left(\frac{1}{x^3} - \frac{1}{x_0^3}\right) + 2\gamma\,(x - x_0)\right]^{1/2} \tag{53}$$

Inserting $v_x = dx/dt$ and integrating allows us to find the time at collision,

$$t = \int_1^{x_0} dx \left[\frac{2}{3}\left(\frac{1}{x^3} - \frac{1}{x_0^3}\right) + 2\gamma\,(x - x_0)\right]^{-1/2}. \tag{54}$$

The integral is from $x = 1$, where the collision occurs, to $x_0 \geq 1$, the initial position of $\mathbf{m}_2$. Equation (53) yields the corresponding speed $v = -v_x$ at the time of collision,

$$v = \left[\frac{2}{3}\left(1 - \frac{1}{x_0^3}\right) + 2\gamma\,(1 - x_0)\right]^{1/2}. \tag{55}$$

The value of γ can be estimated for Zen Magnets, for which $D = 5$ mm, $m = 0.05$ A·m^2, and $\tilde{m} = 0.5$ g [2]. A precise value of the friction kinetic friction coefficient μ_t is not needed in order to assess the role of friction. Kinetic friction coefficients generally lie between 0 and 1, and we simply take $\mu_t = 0.5$. Inserting these values, together with $g = 9.8$ m/s^2 and $\mu_0 = 4\pi \times 10^{-7}$ N/A^2, into Eq. (50) gives $\gamma = 0.001$.

Figure 4 shows results for v and t as a function of x_0, for no table friction ($\gamma = 0$) and for table friction pertinent to Zen Magnets ($\gamma = 0.001$). For $x_0 = 1$, $\mathbf{m}_2$ is initially in contact with $\mathbf{m}_1$, and $v = t = 0$. As x_0 increases from 1, v increases rapidly at first and slowly thereafter, reflecting the rapid decay of the $1/r^4$ magnetic force with increasing distance between spheres - starting $\mathbf{m}_2$ four or more sphere diameters from $\mathbf{m}_1$ (at initial position $x_0 = 4$ and larger) makes little difference on the collision speed because the magnetic force is weak at large distances, and this force contributes little to final speed of $\mathbf{m}_2$. On the other hand, t increases rapidly at these large distances because $\mathbf{m}_2$, starting from rest, spends a lot of time traveling slowly before reaching the regions near $\mathbf{m}_2$ where the force is strong. Values for t were obtained from Eq. (54) by numerical integration with $dx = 10^{-6}$. The role of friction is negligible for small x_0 because the magnetic force, which is proportional to $1/x^4$, overwhelms the table friction force, which is constant and small. The role of friction on v remains negligible for $x_0 \approx 4$ because at these distances, it simply serves to oppose the magnetic force, which is already weak and plays little role in the collision speed. Friction plays a role on the elapsed time for $x_0 \approx 4$, because it reduces the already small speed of $\mathbf{m}_2$ at these distances, where most of the time is spent.

Example 2: Rotation. While Example 1 considers translation without rotation, Example 2 considers rotation without translation. We place sphere 2 at a fixed location $\mathbf{R}'_{12}(t) = x_0\hat{\mathbf{x}}$ and consider only the magnetic torque and the table friction torque on it, ignoring the magnetic force and the accompanying translation in order to isolate the rotational behavior. We release sphere 2 from rest at an angle ϕ_0, measured counterclockwise from the $+x$ axis, and allow sphere 2 to rotate freely about the z axis according to $\hat{\mathbf{m}}_2(t) = \cos\phi(t)\hat{\mathbf{x}} + \sin\phi(t)\hat{\mathbf{y}}$, with sphere 1 again held fixed at the origin with $\hat{\mathbf{m}}_1 = \hat{\mathbf{x}}$. According to Eqs. (43) and (44), the time-dependent orientation angle $\phi(t)$ obeys

$$\frac{d\omega_z}{dt} = 10\tilde{\gamma} - \frac{10}{3x_0^3}\sin\phi, \tag{56}$$

$\omega_z = d\phi/dt$ is the z-component of the angular velocity. Here,

$$\tilde{\gamma} = \frac{\tau_t}{F_0 D} = \frac{2\pi\tilde{\mu}_t\tilde{m}gD^4}{3\mu_0 m^2} \tag{57}$$

is the magnitude of the dimensionless (constant) kinetic friction torque of the table on sphere 2. This constant is identical to γ, given by Eq. (50), except to replace μ_t by $\tilde{\mu}_t$. To evaluate the role of rotational friction for Zen Magnets, we therefore take $\tilde{\gamma} = \gamma = 0.001$. Equation (56) is valid for $\omega_z < 0$; the sign of the friction term must be reversed for $\omega_z > 0$ in order for the friction torque to oppose the motion. We insert

$$\frac{d\omega_z}{dt} = \frac{d\omega_z}{d\phi}\frac{d\phi}{dt} = \omega_z \frac{d\omega_z}{dx} \tag{58}$$

into Eq. (56) and rewrite it to yield

$$\int \omega_z d\omega_z = 10 \int \left(\tilde{\gamma} - \frac{\sin\phi}{3x_0^3} \right) d\phi. \tag{59}$$

Integrating and applying the initial conditions $\omega_z(0) = 0$ and $\phi(0) = \phi_0$ yields the time-dependent angular velocity,

$$\omega_z = -\left[20\tilde{\gamma}\,(\phi - \phi_0) + \frac{20}{3x_0^3}\,(\cos\phi - \cos\phi_0) \right]^{1/2} \tag{60}$$

Inserting $\omega_z = d\phi/dt$ and integrating gives

$$\int dt = \int_0^{\phi_0} d\phi \left[20\tilde{\gamma}\,(\phi - \phi_0) + \frac{20}{3x_0^3}\,(\cos\phi - \cos\phi_0) \right]^{-1/2}. \tag{61}$$

The integral is from $\phi = 0$, where $\mathbf{m}_2$ crosses the x axis, to ϕ_0, the initial angle. Equation (60) yields the corresponding speed $\omega = -\omega_z$ at the axis crossing,

$$\omega = \left[-20\tilde{\gamma}\phi_0 + \frac{20}{3x_0^3}\,(1 - \cos\phi_0) \right]^{1/2} \tag{62}$$

Example 3: Rotation with Magnet-Magnet Friction. Example 3 again considers rotation without rotation, but this time with $\mathbf{m}_2$ in contact with $\mathbf{m}_1$, and experiencing the associated magnet-to-magnet frictional torque.

There are 3 kinds of kinetic friction in the problem, as we discussed:

1. ball-table friction force: A translational friction force directed opposite the translational velocity. This is the kinetic friction force on a ball sliding on a table, applied at the center of the ball, and does not affect the rotation of the ball.

2. ball-table friction torque: A rotational friction torque directed opposite the angular velocity. This is the kinetic friction torque on a ball rotating on a table about an axis perpendicular to the table, and does not affect the translation of the ball.

3. ball-ball friction force and torque: If m2 is in contact with m1, and if m2 is sliding against m1, m2 will experience a kinetic friction force f at the point of contact, directed opposite to the direction of the slide. For example, if m2 is at (1,0) and is not rotating, but is sliding along m1 in the +y direction, then m2 will experience a force f applied at the point of contact, directed in the y direction. This force will slow the translation, and will exert a counterclockwise (positive) torque equal to fD/2 that will cause the ball to begin to rotate. A second example: If m2 is at (1,0) and is not translating but is rotating clockwise (negative angular velocity), then it will again experience a force f applied at the point of contact, directed in the y direction. This force will accelerate the ball in the y direction, and will exert a counterclockwise (positive) torque equal to fD/2 that will slow the rotation. To determine whether this force plays a role, we need only to figure out whether m2 is sliding along m1 at its point of contact, and the force will be in the direction opposite to the slide.

So, the ball-ball friction force and torque is the toughest to implement. We can try the problem without it, and implement it later if we think we need to.

Boyd

and continues to rotate the sphere out of alignment with $\mathbf{B}_1$

Heres my proposal: Produce 6 short 2-d videos, each with just two magnet spheres, that relate to Figure 4 of the paper (see attached). In all 6 videos, one of the magnet spheres is held fixed at the origin during the entirety of the simulation, with its axis held fixed in the +x direction, like the central magnet in each panel of Fig. 4 (called m1). Each of the 6 simulations will start with the second magnet (called m2) in Fig. 4 at a different position and/or orientation, and will track its progress until it is forced out of the frame or reaches a stable minimum-energy equilibrium state in contact with m1, as discussed in paragraph 1 on page 5 of the paper. Ignore static friction, kinetic friction, gravity, the phase of the moon, etc. Once m2 makes contact with m1, torques on m2 are still pertinent, as is the component of the force of m1 on m2 that is tangential to the interface. (The component of the magnetic force that is perpendicular to this interface will be balanced by a normal force from m1.) Keeping m1 fixed in space and keeping its orientation fixed spares us from worrying about conservation of momentum during the collision - nothing that m2 does to m1 has the slightest effect on its position and orientation. These videos will therefore show what happens if you glue a Zen Magnet to the table and release another Zen Magnet from various positions, and in various orientations, nearby. Each of the 6 simulations starts with m1 and m2 separated by the same center-tocenter distance r. We could try r = 3d (three magnet diameters) and see how well it works, and move to r = 2d or r = 4d if needed. Each of the 6 simulations will illustrate and confirm, in spectacular manner, the principles taught in paragraphs 16 on pages 5 and 7 of the paper, and illustrated in Fig. 4, with simulation 1 corresponding to paragraph 1, simulation 2 to paragraph 2, etc., as follows:

1. Configuration A in panel (a). 2. Configuration B in panel (a). 3. Configuration C in panel (a). Also, a second simulation with m2 in the same location as C in panel (a), but rotated counterclockwise by 10 degrees (just enough to be visible). 4. Configuration C in panel

(b). 5. Configuration A in panel (c). 6. Configuration A in panel (e). Also, a second simulation with m2 in the same location as A in panel (e), but rotated counter-clockwise by 10 degrees (just enough to be visible).

An instructive undergraduate exercise is to determine the magnetic field produced by a chain of magnets. A chain of magnets produces a stronger magnetic field than a single magnet. But how much stronger? The answer comes through superposition, a key principle of physics, which states that the net magnetic field is the vector sum of fields from all sources. In contrast with iron and steel, neodymium magnets have high coercivity, meaning that they have a high resistance to demagnetization by external magnetic fields [39]. We can therefore assume that nearby magnets do not affect the magnetization of such magnets, and we can simply add up the magnetic fields produced by all of the magnets in the chain in order to determine the net magnetic field. The following exercise is suitable for advanced physics laboratory students:

Problem 1: (a) Use a gauss meter to measure the north and south polar magnetic fields of four magnet spheres. (b) Deduce their magnetic moments from Eq. (**??**), assuming that they are uniformly magnetized. (c) Apply the principle of superposition to compare predictions and measurements of the axial magnetic field at the end of chains of two, three, and four magnets.

The following exercise is suitable for advanced undergraduate physics students, and is pertinent to angle strain in organic molecules (Sec. **??**).

Problem 2: Calculate the ground-state energies of symmetric rings of 3, 4, 5, 6, 7, and 8 uniformly magnetized spheres. Treat the spheres as magnetic dipoles, each with magnetic moment $\mathbf{m}$. Include only the energies of nearest neighbor interactions. Show that the energy per magnet decreases with increasing ring size.

VI. UNIFORMLY POLARIZED SPHERES

Consider two spheres with positive and negative charges and shifted by $\mathbf{d}$ (for a point dipole let $d \to 0$ with $2qd$ kept constant). Since the forces between spheres are the same as between point particles, the forces between spherical dipoles are the same as forces between point dipoles. We can immediately see that the forces between spherically symmetric dipole spheres are the same as point dipoles.

VII. CONCLUSIONS

May be able to extend these arguments to spheres with non-uniform, spherically symmetric magnetizations $\mathbf{M}(r)$.

The preceding discussion elucidates the forces and torques between magnets in a variety of configurations.

VIII. ACKNOWLEDGMENTS

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[1] David A. Richter, Expert Report, "Teaching Geometry with Magnet Sphere Kits," in the matter of Zen Magnets, LLC, CPSC Docket No. 12-2, Item 124, Exhibit 3, 10/20/2014, http://www.cpsc.gov/en/Recalls/Recall-Lawsuits/Adjudicative-Proceedings/ (accessed Feb. 9, 2016).

[2] Boyd F. Edwards, "Expert Report: Educational Value of Neodymium Magnet Spheres in the matter of Zen Magnets, LLC, CPSC Docket No. 12-2," Item 124, Exhibit 4, 10/20/2014, http://www.cpsc.gov/en/Recalls/Recall-Lawsuits/Adjudicative-Proceedings/ (accessed Feb. 9, 2016).

[3] The Zen Gallery, curated by Shihan Qu, http://zenmagnets.com/gallery/ (accessed Feb. 9, 2016).

[4] "Simulating toy magnet balls? (eg: Zen Magnets)," http://www.physicsforums.com/showthread.php?t=596513 (accessed Feb. 9, 2016).

[5] G. Helgesen, T. T. Skjeltorp, P. M. Mors, R. Botet, and R. Jullien, "Aggregation of Magnetic Microspheres: Experiments and Simulations," Phys. Rev. Lett. **61** (15), 1736-1739 (1988).

[6] Bernhard Thomaszewski, Andreas Gumann, Simon Pabst, Wolfgang Straber, "Magnets in Motion," ACM Transactions on Graphics **27** (5), Article 5 (2008).

[7] M. Varón, M. Beleggia, T. Kasama, R. J. Harrison, R. E. Dunin-Borkowski, V. F. Puntes, & C. Frandsen, "Dipolar Magnetism in Ordered and Disordered Low-Dimensional Nanoparticle Assemblies," Scientific Reports **3** (1234), 1-5 (2013).

[8] P. C. Clemmow, *An Introduction to Electromagnetic Theory*, (Cambridge University Press, London, 1973), p. 284. Clemmow suggests that he knows of the sphere-dipole equivalence, and knows how to explain it, by including the following exercise in his book; "A permanent magnet in the form of a uniformly magnetized sphere is placed with its centre at O in a field due to other permanent magnets. Explain why the force on the sphere is the force that would be exerted (in the absence of the sphere) on a certain dipole at O. Two permanent magnets, each consisting of a uniformly magnetized material between concentric spheres of radii a and b ($> a$), are placed with

their directions of magnetization along the line of centres, which are distance d ($> 2b$) apart. Find the force between the magnets if each has magnetization $\mathbf{M}$."

[9] Roald K. Wangsness, *Electromagnetic Fields*, 2nd edition (Wiley, New York, NY, 1986), p. 326.

[10] Roald K. Wangsness, *Electromagnetic Fields*, 2nd edition (Wiley, New York, NY, 1986), p. 340.

[11] D. Vokoun and M. Beleggia, "Forces between arrays of permanent magnets of basic geometric shapes," Journal of Magnetism and Magnetic Materials **350**, 174-178 (2014). This publication quotes the result of an unpublished calculation of the magnetic interaction between two identical spheres with parallel magnetizations. Details of this calculation were sent privately to us by D. Vokoun on February 8, 2016. We independently completed a calculation for the special case of two identical uniformly magnetized spheres with parallel magnetizations that are parallel to the line between the sphere centers.

[12] W. Booth and B. Edwards, unpublished.

[13] M. Beleggia and M. De Graef, "General magnetostatic shape-shape interactions," Journal of Magnetism and Magnetic Materials **285**, L1-L10 (2005).

[14] Jerry B. Marion and Stephen T. Thornton, "Classical Dynamics of Particles and Systems," 4th edition (Saunders College Publishing, Fort Worth, Texas, 1995), p. 50. This reference mentions that the force between moving charged particles violates Newton's third law.
Such particles can exchange energy with the fields. For example, consider particle 1, positively charged and moving in the $+x$ direction, and particle 2, positively charged and moving in the $+y$ direction, at the instant that particle 1 is at the coordinate origin and particle 2 is on the $+y$ axis. At that instant, the electric forces are central and repulsive, but particle 1 exerts a non-central magnetic force on particle 2 in the $+x$ direction, while particle 2 exerts no magnetic force on particle 1. See http://physics.stackexchange.com/questions/138095/newtons-third-law-exceptions (accessed February 10, 2016).

[15] Wikipedia, *Force between Magnets*, https://en.wikipedia.org/wiki/Force_between_magnets (accessed February 10, 2016).

[16] Attraction/repulsion force between two equal magnets?, http://www.eng-tips.com/viewthread.cfm?qid=341932 (accessed February 10, 2016).

[17] Freeman J. Dyson, "Feynman's proof of the Maxwell equations, Am. J. Phys. **58** (3), 209211 (1990).

[18] G. Akoun and J. Yonnet, IEEE Trans. Magn. **20**, 1962 (1984).

[19] T. H. Boyer, Am. J. Phys. **56**, 688 (1988).

[20] A. Kruusing, J. Magn. Magn. Mater. **234**, 545 (2001).

[21] D. Vokoun, M. Beleggia, L. Heller, and P. Sittner, J. Magn. Magn. Mater. **321**, 3758 (2009).

[22] J. S. Agashe and D. P. Arnold, J. Phys. D. Appl. Phys. **42**, 099801 (2009).

[23] S. Sanz and L. Garcia-Tabares, IEEE Trans. Appl. Supercond. **20**, 846 (2010).

[24] J. Camacho and V. Sosa, Rev. Mex. Fsica E **59**, 8 (2013).

[25] M. Beleggia, S. Tandon, Y. Zhi, M. De Graef, "On the magnetostatic interactions between nanoparticles of arbitrary shape," Journal of Magnetism and Magnetic Materials, **278**, 270-284 (2004).

[26] David Griffiths, *Introduction to Electrodynamics*, 3rd edition (Pearson Education, Delhi, India, 1999), p. 246.

[27] David Griffiths, *Introduction to Electrodynamics*, 3rd edition (Pearson Education, Delhi, India, 1999), p. 282.

[28] Kar W. Young, Peter B. Landecker, and Daniel D. Villani, "An analytic solution for the force between two magnetic dipoles," Magnetic and Electrical Separation, **9**, 39-52 (1998)

[29] Timothy H. Boyer, "The force on a magnetic dipole," Am. J. Phys. **56**, 688 (1988).

[30] K. R. Brownstein, "Force exerted on a magnetic dipole," Am. J. Phys. **61**, 940 (1993).

[31] Jack B. Greene and Frank G. Karioris, "Force on a Magnetic Dipole," Am. J. Phys. **39**, 172 (1971).

[32] Lev Vaidman, "Torque and force on a magnetic dipole," Am. J. Phys. **58**, 978 (1990).

[33] David J. Griffiths and V. Hnizdo, "Mansuripur's paradox," Am. J. Phys. **81**, 570 (2013).

[34] David J. Griffiths and V. Hnizdo, "The torque on a dipole in uniform motion," Am. J. Phys. **82**, 251 (2014).

[35] V. Hnizdo, "Hidden mechanical momentum and the field momentum in stationary electromagnetic and gravitational systems," Am. J. Phys. **65**, 515 (1997).

[36] Timothy H. Boyer, "Classical interaction of a magnet and a point charge: The Shockley-James paradox," Phys. Rev. E **91**, 013201 (2015).

[37] Richard P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* Vol. II, (Addison-Wesley, Boston, Massachusetts, 1964), p. 26-5. Here, Feynman discusses the example of the non-central force between two positively charged particles, one moving in the $+x$ direction and the other moving in the $+y$ direction. At the instant that particle 1 is at the coordinate origin and particle 2 is on the $+y$ axis, the electric forces are central and repulsive, but particle 1 exerts a non-central magnetic force on particle 2 in the $+x$ direction, while particle 2 exerts no magnetic force on particle 1. Thus, Newton's third law does not apply, raising a paradox about the origin of the extra mechanical momentum.

[38] Adam Caprez and Herman Batelaan, "Feynmans Relativistic Electrodynamics Paradox and the Aharonov-Bohm Effect," Found. Phys. **39**, 295-306 (2009). Here, the authors show that the extra mechanical momentum of Feynman's paradox is hidden in the electromagnetic fields.

[39] Coercivity, http://en.wikipedia.org/wiki/Coercivity.

[40] Compliance Magnets, http://micromagnets.com

[41] K&J Magnetics, Inc., http://www.kjmagnetics.com/

[42] Alibaba, http://www.alibaba.com.

[43] Applied Magnets, http://www.magnet4less.com

[44] The vector magnetic fields shown in Fig. 3 are courtesy Wikipedia and Wikimedia Commons, https://en.wikipedia.org/wiki/File:VFPt_dipole_point.svg.

[45] David Griffiths, *Introduction to Electrodynamics*, 3rd edition (Pearson Education, Delhi, India, 1999), pp. 351-354.

[46] Jack Vanderlinde, *Classical Electromagnetic Theory*, 1st edition (John Wiley & Sons, New York, New York, 1993), pp. 81-84.