Force between two uniformly magnetized / polarized spheres

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Simple arguments show that the magnetic force between two uniformly magnetized spheres is identical to the force between two point magnetic dipoles located at the centers of these spheres, and that the electric force between two uniformly polarized spheres is identical to the force between two point electric dipoles.

I. INTRODUCTION

Small dipolar neodymium magnet spheres are used both in and out of the classroom to teach principles of mathematics, physics, chemistry, biology, and engineering [1, 2]. They offer engaging hands-on exposure to principles of magnetism and are particularly useful in studying lattice structures, where they offer greater versatility than standard ball-and-stick models because they can connect at a continuous range of angles. They have spawned a grassroots learning community dedicated to sharing photos and tutorials of beautiful magnetic sculptures, some made from thousands of magnets, including models of molecules, fractals, and Platonic solids [3]. YouTube magnet sphere videos have attracted over a hundred million views [2].

Computer simulations of dynamic interactions between magnet spheres assume point dipole interactions for simplicity [4, 5], or treat extended magnets as aggregates of such point dipoles [6]. Dipolar magnetic interactions have also been studied in assemblies of spherical nanoparticles [7]. Analytical solutions have been found for the force between permanent magnets of various non-spherical shapes, such as cubes and cylinders[1–7].

Of interest are the force and torque between two uniformly magnetized spheres. Calculating the magnetic field produced by a uniformly magnetized sphere is a standard undergraduate exercise. The equivalence of this field, outside the sphere, to the point magnetic dipole field is well known [9]. But direct calculations of the force exerted by this field on another uniformly magnetized sphere are challenging, and have been done only in three special cases.

The first is for two uniformly magnetized spheres with magnetizations that are perpendicular to the line through the spheres [13]. The second is for two uniformly magnetized spheres with parallel magnetizations that make an arbitrary angle with the line through the sphere centers [11]. The third is for two identical uniformly magnetized spheres with one of the magnetizations parallel to the line through the spheres [12].

All three calculations yield a force that is identical to the force between two point magnetic dipoles.

This paper has two purposes: (1) to prove that the force between two uniformly magnetized spheres with arbitrary sizes, positions, magnetizations, and orientations is identical to the force between two point magnetic dipoles, and (2) to discuss implications of this equivalence for physics education.

Proofs of this equivalence, if they exist in modern literature or textbooks, have eluded our searches. Popular online discussions of the force between magnets discuss the force between magnetic dipoles, cylindrical bar magnets, and nearby magnetized surfaces, but not the force between spheres [15, 16].

We present three proofs of this equivalence, each of which draws upon different facets of the undergraduate physics curriculum. The first is a simple symmetry argument based on Newton's third law (Sec. IV). The second is a direct integration of the magnetic force $\mathbf{F} = -\boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B})$ over a sphere (Sec. VI). The third uses potential theory to examine changes in the total magnetic energy of the two-sphere system, and relies on Green's theorem (Sec. V). All three rely on the equivalence of the field of a uniformly magnetized sphere to the field of a point dipole.

We also discuss the non-central nature of the force between two magnet spheres / dipoles, namely, that this force is not generally directed along the line through the sphere centers. Newton's third law is not a general law of nature, and applies for central forces but not necessarily for non-central forces [14]. As shown in Sec. VI, it does apply to the force between magnet spheres / dipoles.

In Sec. X, we identify ranges of angles leading to attractive vs. repulsive forces. We discuss the role of torques between magnetic spheres and demonstrate that a two-magnet system can exchange momentum with the electromagnetic fields: We attach two magnet spheres rigidly to the ends of a rod with the magnetizations parallel to each other but oriented at a 45° angle to the rod. When allowed to rotate freely about its center of mass, we observe that the rod accelerates from rest in response to this torque, borrowing angular momentum from the electromagnetic fields. We propose two undergraduate exercises. Finally, we confirm that pinching a ring of magnets into a triangle requires actively orienting the magnets so that they will form stable triangle corners, something that is known in the magnet sphere learning community.

We treat the magnetization of each magnet as fixed, and neglect the demagnetization of one magnet by the field produced by another magnet. This assumption is appropriate for "hard" magnetic materials with high coercivities that are used to make permanent magnets. In the B vs. H ferromagnetic hysteresis loop, the coercivity H_C is defined as the value of H at which B falls to zero [10]. High coercivity therefore implies high resistance to demagnetization by external magnetic fields. The alloy used to make permanent neodymium magnets, $Nd_2Fe_{14}B$, has high coercivity [26].

Several countries, including the United States, have banned the sale of 5-mm nickel-coated neodymium magnets marketed as a desk toys (under the trade names BuckyBalls, Zen Magnets, Neoballs, etc.) following reports of intestinal injuries from ingestion of these magnets. But 2.5-mm magnets may be still be purchased [27], and magnet spheres of various sizes, including 5-mm diameter spheres, may be purchased from industrial suppliers [28–30].

The equivalence of the sphere-sphere interaction with the dipole-dipole interaction should be helpful for research on arrays of nanoparticles and permanent magnets, in dynamic simulations of magnet sphere interactions, as well as in efforts to use magnet spheres in science education.

The purpose of the paper is to calculate the force $\mathbf{F}_{12}$ of sphere 1 on sphere 2, which is not generally directed along the line between the sphere centers (Fig. 1). In Sec. VIII, we review a standard, brute-force, method of calculating this force. In Sec. **??**, we present a simple argument based on Newton's third law that shows that this force is identical to the force between two dipoles. In Sec. 5, we confirm this result by considering the total magnetic energy of the two-sphere system.

II. POINT DIPOLES

In this section, we review the fields of and forces between two point dipoles, and introduce notation used in subsequent sections. The magnetic field at position $\mathbf{r}$ due to a point magnetic dipole $\mathbf{m}$ at the origin is given by [8]

$$\mathbf{B}(\mathbf{m};\mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{3\mathbf{m}\cdot\mathbf{r}}{r^5}\mathbf{r} - \frac{\mathbf{m}}{r^3}\right), \quad (1)$$

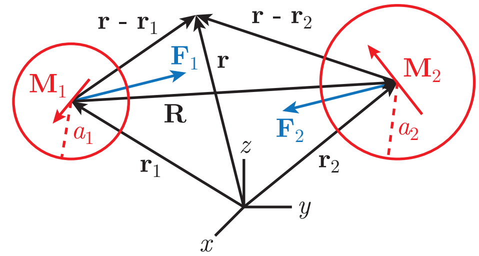

FIG. 1. Diagram showing two uniformly magnetized spheres with positions $\mathbf{r}_1$ and $\mathbf{r}_2$, radii a_1 and a_2, magnetizations $\mathbf{M}_1$ and $\mathbf{M}_2$, and paired forces $\mathbf{F}_1$ and $\mathbf{F}_2$. Shown also are the position vector $\mathbf{r}$ and relative position vectors $\mathbf{r}-\mathbf{r}_1$, $\mathbf{r}-\mathbf{r}_2$, and $\mathbf{R}_{ij} = \mathbf{r}_2 - \mathbf{r}_1$. To consider the forces between two point dipoles, replace spheres $\mathbf{M}_1$ and $\mathbf{M}_2$ by dipoles $\mathbf{m}_1$ and $\mathbf{m}_2$ at the same locations.

where $r = |\mathbf{r}|$. This field can be obtained from scalar potential according to

$$\mathbf{B}(\mathbf{m};\mathbf{r}) = -\boldsymbol{\nabla}\varphi(\mathbf{m};\mathbf{r}), \quad (2)$$

where

$$\varphi(\mathbf{m};\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\cdot\mathbf{r}}{|\mathbf{r}|^3}. \quad (3)$$

We now consider the force between two dipoles, $\mathbf{m}_1$ and $\mathbf{m}_2$, at positions $\mathbf{r}_1$ and $\mathbf{r}_2$. Equation (1) enables us to write the field produced by dipole $\mathbf{m}_i$ as

$$\mathbf{B}_i(\mathbf{r}) = \mathbf{B}(\mathbf{m}_i;\mathbf{r}-\mathbf{r}_i), \quad (4)$$

for $i = 1, 2$, where $\mathbf{r}-\mathbf{r}_i$ is the position vector relative to dipole $\mathbf{m}_i$ (Fig. 1).

The interaction energy between dipole $\mathbf{m}_j$ and the magnetic field produced by dipole $\mathbf{m}_i$ is given by

$$\begin{aligned} U_{ij} &= -\mathbf{m}_j\cdot\mathbf{B}_i(\mathbf{r}_j) && (5)\\ &= \frac{\mu_0}{4\pi}\left[\frac{\mathbf{m}_i\cdot\mathbf{m}_j}{{R_{ij}}^3} - 3\frac{(\mathbf{m}_i\cdot\mathbf{R}_{ij})(\mathbf{m}_j\cdot\mathbf{R}_{ij})}{{R_{ij}}^5}\right], && (6)\end{aligned}$$

where $\mathbf{R}_{ij} = \mathbf{r}_j - \mathbf{r}_i$ is the vector from $\mathbf{m}_i$ to $\mathbf{m}_j$, and $\mathbf{B}_i(\mathbf{r}_j)$ is the field produced by dipole $\mathbf{m}_i$ evaluated at the position of dipole $\mathbf{m}_j$.

The force of dipole $\mathbf{m}_i$ on dipole $\mathbf{m}_j$ follows as

$$\begin{aligned} \mathbf{F}_{ij} &= -\boldsymbol{\nabla}_j U_{ij} && (7)\\ &= \frac{3\mu_0}{4\pi {R_{ij}}^5}\Big[(\mathbf{m}_i\cdot\mathbf{R}_{ij})\,\mathbf{m}_j + (\mathbf{m}_j\cdot\mathbf{R}_{ij})\,\mathbf{m}_i \\ &\quad + (\mathbf{m}_i\cdot\mathbf{m}_j)\,\mathbf{R}_{ij} - 5\frac{(\mathbf{m}_i\cdot\mathbf{R}_{ij})\,(\mathbf{m}_j\cdot\mathbf{R}_{ij})}{{R_{ij}}^2}\mathbf{R}_{ij}\Big]. && (8)\end{aligned}$$

Here, $\boldsymbol{\nabla}_j$ is the gradient with respect to $\mathbf{r}_j$. $\mathbf{F}_j$ is not a central force, namely, it is not generally parallel to the vector $\mathbf{R}_{ij}$ between the dipoles.

The force $\mathbf{F}_{ji} = -\boldsymbol{\nabla}_i U_{ji}$ of $\mathbf{m}_j$ on dipole $\mathbf{m}_i$ follows similarly from $U_{ji} = -\mathbf{m}_i\cdot\mathbf{B}_j(\mathbf{r}_i)$. A little algebra shows that $U_{ji} = U_{ji}$ and $\mathbf{F}_{ji} = -\mathbf{F}_{ij}$, confirming that Newton's third law applies to the force between point magnetic dipoles.

The torque of $\mathbf{m}_i$ on $\mathbf{m}_j$ is given by

$$\boldsymbol{\tau}_{ij} = \mathbf{m}_j \times \mathbf{B}_i(\mathbf{r}_j) \tag{9}$$

$$= \frac{\mu_0}{4\pi}\left(\frac{3\mathbf{m}_i \cdot \mathbf{R}_{ij}}{{R_{ij}}^5}\mathbf{m}_j \times \mathbf{R}_{ij} - \frac{\mathbf{m}_j \times \mathbf{m}_i}{{R_{ij}}^3}\right). \tag{10}$$

This torque is not generally equal and opposite to the torque $\boldsymbol{\tau}_{ji} = \mathbf{m}_i \times \mathbf{B}_j(\mathbf{r}_i)$ on $\mathbf{m}_i$.

III. MAGNETIZED SPHERES

We consider the magnetic force between two uniformly magnetized spheres with arbitrary sizes, positions, magnetizations, and orientations (Fig. 1). Sphere i, denoted by S_i, has position vector $\mathbf{r}_i$, radius R_i, magnetization $\mathbf{M}_i$, and total magnetic dipole moment

$$\mathbf{m}_i = \frac{4}{3}\pi R_i^3 \mathbf{M}_i, \tag{11}$$

where $i = 1, 2$. This sphere produces a magnetic field

$$\mathbf{B}_i(\mathbf{r}) = \mathbf{B}(\mathbf{m}_i; \mathbf{r} - \mathbf{r}_i) \text{ for } |\mathbf{r} - \mathbf{r}_i| > R_i, \tag{12}$$

where $\mathbf{r} - \mathbf{r}_i$ is the position vector relative to the sphere center.

We consider two uniformly magnetized spheres with positions $\mathbf{r}_i$, radii R_i, and magnetizations $\mathbf{M}_i$, for $i = 1, 2$. Outside of sphere i (for $|\mathbf{r} - \mathbf{r}_i| > R_i$), its magnetic field is given Eq. (1) according to

$$\mathbf{B}_i(\mathbf{r}) = \mathbf{B}(\mathbf{m}_i; \mathbf{r} - \mathbf{r}_i), \tag{13}$$

and its scalar potential by Eq. (3) according to

$$\varphi_i(\mathbf{r}) = \varphi(\mathbf{m}_i; \mathbf{r} - \mathbf{r}_i). \tag{14}$$

IV. NEWTON'S THIRD LAW

A five-step argument involving Newton's third law shows that the force between two spheres with magnetizations $\mathbf{M}_1$ and $\mathbf{M}_2$ is identical to the force between two point dipoles with magnetic moments $\mathbf{m}_1$ and $\mathbf{m}_2$ at the same locations as the spheres, and obeying Eq. (11), as follows:

1. Let $\mathbf{F}_2$ represent the force of dipole $\mathbf{m}_1$ on dipole $\mathbf{m}_2$. This force is produced by the field $\mathbf{B}_1$ from dipole $\mathbf{m}_1$ (Fig. 2a).

2. Sphere $\mathbf{M}_1$ produces the same field $\mathbf{B}_1$, and therefore exerts the same force $\mathbf{F}_2$ on dipole $\mathbf{m}_2$ (Fig. 2b).

3. Newton's third law gives the force $\mathbf{F}_1 = -\mathbf{F}_2$ of dipole $\mathbf{m}_2$ on sphere $\mathbf{M}_1$ (Fig. 2c). This force is produced by the field $\mathbf{B}_2$ from dipole $\mathbf{m}_2$ (Fig. 2c).

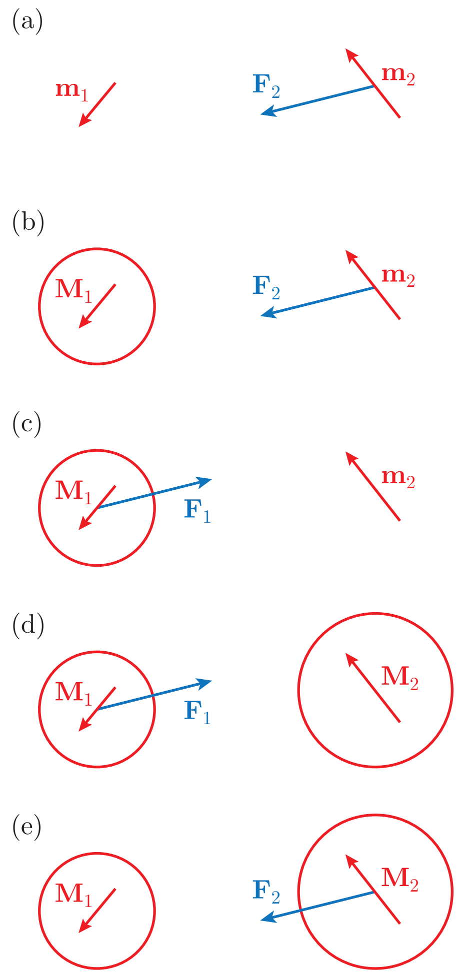

FIG. 2. Diagram illustrating the five steps of the Newton's third law argument showing that the force between two uniformly magnetized spheres is identical to the force between two point dipoles.

4. Sphere $\mathbf{M}_2$ produces the same field $\mathbf{B}_2$, and therefore exerts the same force $\mathbf{F}_1$ on sphere $\mathbf{M}_1$ (Fig. 2d).

5. Again applying Newton's third law shows that the force $\mathbf{F}_2 = -\mathbf{F}_1$ of sphere $\mathbf{M}_1$ on sphere $\mathbf{M}_2$ is identical to the force of dipole $\mathbf{m}_1$ on dipole $\mathbf{m}_2$ (Figs. 2e, 2a).

V. POTENTIAL THEORY

The force on the sphere i can be obtained from $\mathbf{F}_i = -\nabla_i U$ where ∇_i denotes the gradient operator with respect to $\mathbf{r}_i$. Here the magnetic field energy is given by

$$\begin{aligned} U(\mathbf{r}_1, \mathbf{r}_2) &= \frac{1}{2\mu_0}\int (\mathbf{B}_1 + \mathbf{B}_2)^2 d\mathbf{r} \\ &= \frac{1}{2\mu_0}\int (B_1^2 + 2\mathbf{B}_1 \cdot \mathbf{B}_2 + B_2^2) d\mathbf{r} \\ &= U_1 + U_{12} + U_2 \end{aligned}$$

Obviously, the integrals $U_i = (2\mu_0)^{-1}\int B_i^2 d\mathbf{r}$ $(i = 1, 2)$ do not depend on $\mathbf{r}_1$ and $\mathbf{r}_2$ and therefore the force comes from

$$\begin{aligned} U_{12} &= \frac{1}{\mu_0}\int \mathbf{B}_1 \cdot \mathbf{B}_2 d\mathbf{r} \\ &= \frac{1}{\mu_0}\left(\int_{S_1} + \int_{S_2} + \int_{\text{outside}}\right)\nabla\varphi_1 \cdot \nabla\varphi_2 d\mathbf{r} \end{aligned}$$

For the integral over the sphere 1, we use $\nabla^2\varphi_2 = 0$ and the Gauss' theorem

$$\int_{S_1} \nabla\varphi_1\cdot\nabla\varphi_2 d\mathbf{r} = \int_{S_1} \nabla\cdot(\varphi_1\nabla\varphi_2)d\mathbf{r} = \int_{\partial S_1} \varphi_1\nabla\varphi_2\cdot\hat{\mathbf{n}}_1 dA$$

where ∂S_1 denotes the boundary of S_1 and $\hat{\mathbf{n}}_1$ is the unit vector on ∂S_1. Note that $\varphi_1\nabla\varphi_2$ is continuous across ∂S_1. Similarly,

$$\int_{S_2} \nabla\varphi_1\cdot\nabla\varphi_2 d\mathbf{r} = \int_{S_1} \nabla\cdot(\varphi_2\nabla\varphi_1)d\mathbf{r} = \int_{\partial S_2} \varphi_2\nabla\varphi_1\cdot\hat{\mathbf{n}}_2 dA.$$

Since $\varphi_1 = \varphi_1^{\text{point}}$ (the superscript 'point' denotes 'due to the point dipole) and $\nabla\varphi_2 = \nabla\varphi_2^{\text{point}}$ on ∂S_1 and $\varphi_2 = \varphi_2^{\text{point}}$ and $\nabla\varphi_1 = \nabla\varphi_1^{\text{point}}$ on ∂S_2, the energy U_{12} is the same for the point dipoles and for the spheres ($U_{12} = U_{12}^{\text{point}}$) and so is the force ($\mathbf{F}_i = \mathbf{F}_i^{\text{point}}$). (Note that $U_i \neq U_i^{\text{point}}$ because $\nabla^2\varphi_i^{\text{point}} \neq 0$ at $\mathbf{r} = \mathbf{r}_i$ for point dipoles. Nevertheless the force is the same.)

VI. DIRECT INTEGRATION

Using (the field is the same for a point dipole and a uniformly magnetized sphere centered at $\mathbf{r}_i$)

$$\int_{S_i} d\mathbf{r}'\mathbf{B}(\mathbf{M}_i; \mathbf{r} - \mathbf{r}') = \mathbf{B}(\mathbf{m}_i; \mathbf{r} - \mathbf{r}_i),$$

the force can be directly integrated over S_2

$$\begin{aligned} \mathbf{F}_2 &= -\nabla_2 \int_{S_2} \mathbf{B}(\mathbf{m}_1; \mathbf{r} - \mathbf{r}_1) \cdot \mathbf{M}_2 d\mathbf{r}. \\ &= -\nabla_2 \int_{S_2} \frac{\mu_0}{4\pi}\left[\frac{3\mathbf{m}_1\cdot(\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|^5}(\mathbf{r}-\mathbf{r}_1) - \frac{\mathbf{m}_1}{|\mathbf{r}-\mathbf{r}_1|^3}\right]\cdot\mathbf{M}_2 d\mathbf{r} \\ &= \nabla_2 \int_{S_2} \frac{\mu_0}{4\pi}\left[\frac{3\mathbf{M}_2\cdot(\mathbf{r}_1-\mathbf{r})}{|\mathbf{r}_1-\mathbf{r}|^5}(\mathbf{r}_1-\mathbf{r}) - \frac{\mathbf{M}_2}{|\mathbf{r}_1-\mathbf{r}|^3}\right] d\mathbf{r}\cdot\mathbf{m}_1 \\ &= \nabla_2\left[\int_{S_2} \mathbf{B}(\mathbf{M}_2; \mathbf{r}_1 - \mathbf{r})d\mathbf{r}\right]\cdot\mathbf{m}_1 \\ &= \nabla_2\mathbf{B}(\mathbf{m}_2; \mathbf{r}_1 - \mathbf{r}_2)\cdot\mathbf{m}_1[\text{you can see this is equal to } -\mathbf{F}_1 = \nabla_1\mathbf{B}(\mathbf{m}_1; \mathbf{r}_2 - \mathbf{r}_1)\cdot\mathbf{m}_2] \\ &= -\nabla_2\mathbf{B}(\mathbf{m}_1; \mathbf{r}_1 - \mathbf{r}_2)\cdot\mathbf{m}_2(\text{equalt to the force on the point dipole 2}) \end{aligned}$$

VII. FROM ELECTRIC DIPOLES

Consider two spheres with positive and negative charges and shifted by $\mathbf{d}$ (for a point dipole let $d \to 0$ with $2qd$ kept constant). Since the forces between spheres are the same as between point particles, the forces between spherical dipoles are the same as forces between point dipoles. We can immediately see that the forces between spherically symmetric dipole spheres are the same as point dipoles.

VIII. FORCE DENSITY

To calculate the force between two magnetized spheres, we consider the energy of interaction between the magnetic field $\mathbf{B}_1$ produced by S_1 and a volume element d^3x of S_2, with dipole moment $d\mathbf{m}_2 = \mathbf{M}_2 d^3x$. This energy is given by $dU_{12} = -d\mathbf{m}_2 \cdot \mathbf{B}_1 = -\mathbf{M}_2 \cdot \mathbf{B}_1 d^3x$. The corresponding force on this element is $d\mathbf{F}_{12} = -\boldsymbol{\nabla}(dU_{12}) = \boldsymbol{\nabla}(\mathbf{M}_2 \cdot \mathbf{B}_1)\, d^3x$. Therefore, the total force on S_2 is a volume integral over this sphere,

$$\mathbf{F}_{12} = \int_{S_2} \boldsymbol{\nabla}(\mathbf{M}_2 \cdot \mathbf{B}_1)\, d^3x, \tag{15}$$

and the associated total magnetic interaction energy is

$$U_{12} = -\int_{S_2} \mathbf{M}_2 \cdot \mathbf{B}_1 d^3x. \tag{16}$$

This technique has been used to calculate the force between two uniformly magnetized spheres in special cases. Cite examples here...

IX. PROBLEMS

An instructive undergraduate exercise is to determine the magnetic field produced by a chain of magnets. A chain of magnets produces a stronger magnetic field than a single magnet. But how much stronger? The answer comes through superposition, a key principle of physics, which states that the net magnetic field is the vector sum of fields from all sources. In contrast with iron and steel, neodymium magnets have high coercivity, meaning that they have a high resistance to demagnetization by external magnetic fields [26]. We can therefore assume that nearby magnets do not affect the magnetization of such magnets, and we can simply add up the magnetic fields produced by all of the magnets in the chain in order to determine the net magnetic field. The following exercise is suitable for advanced physics laboratory students:

Problem 1: (a) Use a gauss meter to measure the north and south polar magnetic fields of four magnet spheres. (b) Deduce their magnetic moments from Eq. (**??**), assuming that they are uniformly magnetized. (c) Apply the principle of superposition to compare predictions and measurements of the axial magnetic field at the end of chains of two, three, and four magnets.

The following exercise is suitable for advanced undergraduate physics students, and is pertinent to angle strain in organic molecules (Sec. **??**).

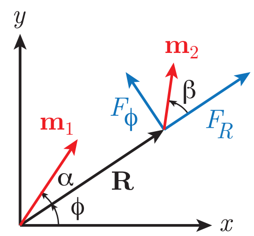

FIG. 3. Diagram illustrating the angles used to study coplanar dipole moments, with dipole $\mathbf{m}_1$ at the origin and dipole $\mathbf{m}_2$ at position $\mathbf{R}_{ij}$, at azimuthal angle ϕ. Angles α and β give the orientations of $\mathbf{m}_1$ and $\mathbf{m}_2$ relative to $\mathbf{R}_{ij}$. Also shown are the radial and azimuthal components of $\mathbf{F}_2 = F_R\hat{\mathrm{R}} + F_\phi\hat{\phi}$, the force on $\mathbf{m}_2$.

Problem 2: Calculate the ground-state energies of symmetric rings of 3, 4, 5, 6, 7, and 8 uniformly magnetized spheres. Treat the spheres as magnetic dipoles, each with magnetic moment $\mathbf{m}$. Include only the energies of nearest neighbor interactions. Show that the energy per magnet decreases with increasing ring size.

X. DISCUSSION

The case of coplanar $\mathbf{R}_{ij}$, $\mathbf{m}_1$, and $\mathbf{m}_2$ is instructive and applicable both to point dipoles and to uniformly magnetized spheres with total magnetic moments given by Eq. (11), as long as the spheres do not overlap. In the x-y plane, we place $\mathbf{m}_1$ at the origin ($\mathbf{r}_1 = 0$), place $\mathbf{m}_2$ at position $\mathbf{R}_{ij} = \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{r}_2$, let ϕ measure the angle of $\mathbf{R}_{ij}$ counterclockwise from the $+x$ axis, let α measure the angle of $\mathbf{m}_1$ counterclockwise from $\mathbf{R}_{ij}$, and let β measure the angle of $\mathbf{m}_2$ counterclockwise from $\mathbf{R}_{ij}$ (Fig. 3). The corresponding force on $\mathbf{m}_2$ is in the same plane,

$$\mathbf{F}_2(R, \alpha, \beta) = F_R\hat{\mathrm{R}} + F_\phi\hat{\phi}, \tag{17}$$

with radial and azimuthal components given by

$$F_R = -F_0 \left(\frac{d}{R}\right)^4 \frac{\cos\alpha\cos\beta + \cos(\alpha+\beta)}{2} \tag{18}$$

and

$$F_\phi = F_0 \left(\frac{d}{R}\right)^4 \frac{\sin(\alpha+\beta)}{2}. \tag{19}$$

Here,

$$F_0 = \frac{3\mu_0 m_1 m_2}{2\pi d^4} \tag{20}$$

is the maximum force between dipoles separated by a distance d. $\mathbf{F}_2$ is attractive if $F_R < 0$, repulsive if $F_R > 0$, and central if $\mathbf{F}_\phi = 0$. Equation (10) gives the z-directed torque on $\mathbf{m}_2$,

$$\boldsymbol{\tau}_2(R, \alpha, \beta) = \tau_0 \left(\frac{d}{R}\right)^3 \frac{\sin(\alpha-\beta) - 3\sin(\alpha+\beta)}{4}\,\hat{\mathrm{z}}, \tag{21}$$

where

$$\tau_0 = \frac{\mu_0 m_1 m_2}{2\pi d^3} \tag{22}$$

is the maximum torque between dipoles separated by a distance d.

Figure 4 illustrates these coplanar results. It shows the dipole magnetic fields $\mathbf{B}_1$ produced by $\mathbf{m}_1$, and shows the forces and torques on $\mathbf{m}_2$ for various positions and orientations of $\mathbf{m}_2$ [31].

Correct interpretation of Fig. 4 requires an understanding of the connection between the force, the torque, and the energy. The force $\mathbf{F}_2 = -\boldsymbol{\nabla}_2 U_{12}$ acts in the direction of maximum *decrease* of the energy $U_{12} = -\mathbf{m}_2 \cdot \mathbf{B}_1(\mathbf{r}_2)$, that is, $\mathbf{F}_2$ acts in the direction of maximum *increase* in $\mathbf{m}_2 \cdot \mathbf{B}_1$ [Sec. II]. Thus, $\mathbf{F}_2$ is in the direction of the virtual displacement of $\mathbf{m}_2$ that increases $\mathbf{m}_2 \cdot \mathbf{B}_1$ most quickly. The torque $\boldsymbol{\tau}_2$ requires no virtual displacements; $\boldsymbol{\tau}_2 = \mathbf{m}_2 \times \mathbf{B}_1$ acts simply to rotate the dipole in place to bring it parallel to the local field $\mathbf{B}_1$, thereby maximizing $\mathbf{m}_2 \cdot \mathbf{B}_1$ at the current location of the dipole. In summary, the force and torque on a dipole will act in directions that *increase* $\mathbf{m}_2 \cdot \mathbf{B}_1$ most rapidly, thereby *decreasing* the energy.

We illustrate these concepts by considering the configurations in Fig. 4:

1. In configurations A and E in panel (a), $\mathbf{m}_1$ and $\mathbf{m}_2$ are co-linear and parallel to each other, and $\mathbf{m}_2$ is parallel to $\mathbf{B}_1$, giving the product $\mathbf{m}_2 \cdot \mathbf{B}_1 = m_2 B_1$. To increase this product most rapidly, $\mathbf{m}_1$ attracts $\mathbf{m}_2$ into its vicinity, where the field is stronger. Since $\mathbf{m}_2$ is already aligned with $\mathbf{B}_1$, there is no torque. This is the familiar attractive example of two co-linear magnets with the north pole of one magnet facing the south pole of the other. When co-linear, parallel spherical magnets are released from rest, they eventually come into contact with each other with the north pole of one touching the south pole of the other. This is the minimum-energy, stable state of the two-magnet system.

2. In configurations B, D, F, and H in panel (a), the angle θ between $\mathbf{m}_2$ and $\mathbf{B}_1$ is acute ($\theta < \pi/2$), and $\mathbf{m}_2 \cdot \mathbf{B}_1 = m_1 B_1 \cos\theta > 0$. The force on $\mathbf{m}_2$ is attractive and non-central ($F_R < 0$ and $F_\phi \neq 0$), and increases $\mathbf{m}_2 \cdot \mathbf{B}_1$ by moving the dipole into a polar field region where the field is stronger and is parallel to $\mathbf{m}_2$. The torque rotates $\mathbf{m}_2$ into alignment with the local field, which also increases $\mathbf{m}_2 \cdot \mathbf{B}_1$. If a dipole in configuration B, D, F, or H is released

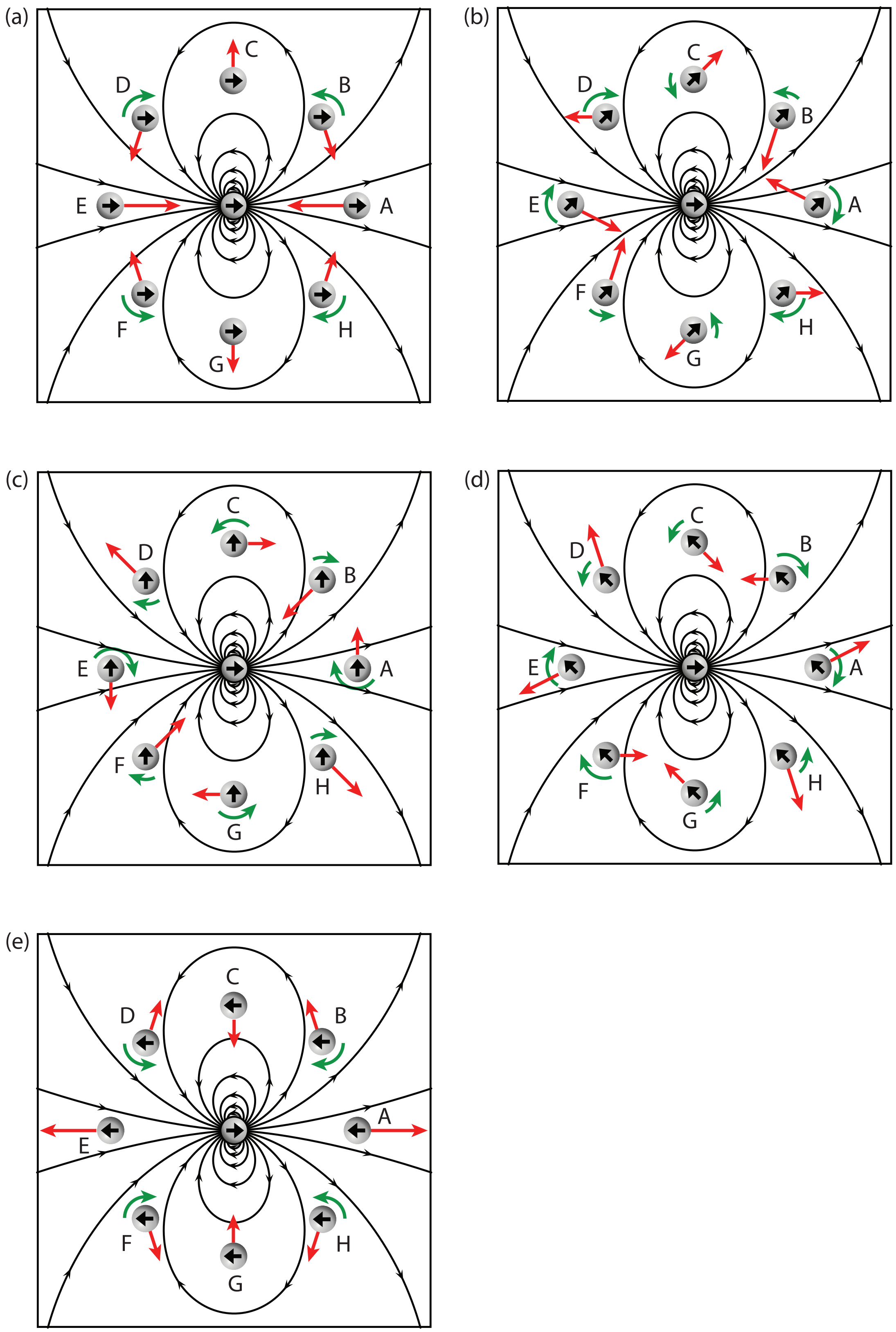

FIG. 4. Magnetic field lines produced by dipole $\mathbf{m}_1$, at the center of each panel, and the resulting forces and torques on dipole $\mathbf{m}_2$ at various positions and orientations, as given by Eqs. (1), (17), and (21). Shown are drawings for $\mathbf{m}_2$ and $\mathbf{m}_1$ differing by angles 0 (a), $\pi/4$ (b), $\pi/2$, (c) $3\pi/4$ (d), and π (e). For each panel, there are eight positions of $\mathbf{m}_2$, labeled A-H, spaced evenly around a circle of diameter $R = d$. Force vectors are shown with their lengths proportional to the force magnitude. Torques are indicated by clockwise and counterclockwise circular arcs, with the arc length increasing with torque magnitude, and with no arc if the torque is zero. The figure applies both to point dipoles and to uniformly magnetized spheres with total magnetic moments given by Eq. (11).

from rest, the magnetic force and torque bring it eventually to the minimum-energy state discussed in Example 1. Configurations A, B, E, and F in panel (b), configurations B and F in panel (c), and configurations B, C, F, and G in panel (d) also have $\mathbf{m}_2 \cdot \mathbf{B}_1 > 0$, and behave similarly.

3. Configurations C and G in panel (a) are repulsive and unstable. Here, $\mathbf{m}_1$ and $\mathbf{m}_2$ are parallel but not co-linear. Instead, $\mathbf{m}_1$ and $\mathbf{m}_2$ are perpendicular to the line through their centers, and $\mathbf{B}_1$ is antiparallel to them, giving $\mathbf{m}_2 \cdot \mathbf{B}_1 = -m_2 B_1$. To increase this product, $\mathbf{m}_1$ repels $\mathbf{m}_2$ into regions where the field is *weaker*. This is the familiar repulsive example of two parallel magnets lined up next to each other with their adjacent north and south poles repelling each other. There is no torque on $\mathbf{m}_2$ because it is antiparallel to $\mathbf{B}_1$. But this configuration is unstable; any perturbation in the position or orientation of $\mathbf{m}_2$ leads to a torque that aligns it with $\mathbf{B}_1$, whence the force becomes attractive.

4. In configurations C, D, G, and H in panel (b), the angle θ between $\mathbf{m}_2$ and $\mathbf{B}_1$ is obtuse ($\theta > \pi/2$), yielding $\mathbf{m}_2 \cdot \mathbf{B}_1 < 0$. To increase this product, a non-central, repulsive force on $\mathbf{m}_2$ moves it toward *weaker* fields, and toward regions where $\mathbf{m}_2$ is better aligned with the field, and the torque rotates $\mathbf{m}_2$ into alignment with the local field. The net effect is to move $\mathbf{m}_2$ into a configuration in which $\mathbf{m}_2 \cdot \mathbf{B}_1 > 0$ and the force becomes attractive, whence $\mathbf{m}_2$ eventually finds its way into the minimum-energy state described in Example 1. Configurations D and H of panel (c), configurations A, D, E, and H in panel (d), and configurations B, D, F, and H in panel (e) also have $\mathbf{m}_2 \cdot \mathbf{B}_1 < 0$, and behave similarly.

5. In configurations A, C, E, and G in panel (c), $\mathbf{m}_2 \perp \mathbf{B}_1$, $\mathbf{m}_2 \cdot \mathbf{B}_1 = 0$, and the force on $\mathbf{m}_2$ has no radial component – it is neither attractive nor repulsive. Its azimuthal component moves $\mathbf{m}_2$ toward a region where $\mathbf{m}_2$ is better aligned with the field, thereby increasing $\mathbf{m}_2 \cdot \mathbf{B}_1$. In these configurations, a force toward or away from $\mathbf{m}_1$ would not change $\mathbf{m}_2 \cdot \mathbf{B}_1$ because $\mathbf{m}_2$ would remain perpendicular to $\mathbf{B}_1$. The torque, as always, rotates $\mathbf{m}_2$ into alignment with the local field, which leads eventually to a force with an attractive radial component.

6. In configurations A and E of panel (e), $\mathbf{m}_1$ and $\mathbf{m}_2$ are co-linear and antiparallel to each other, and $\mathbf{m}_2$ is antiparallel to $\mathbf{B}_1$, giving the product $\mathbf{m}_2 \cdot \mathbf{B}_1 = -m_2 B_1$ as in Example 3. To increase this product most rapidly, $\mathbf{m}_1$ repels $\mathbf{m}_2$ into regions where the field is *weaker*. This is the familiar repulsive example of two co-linear magnets with north poles, or south poles, facing each other. There is no torque on $\mathbf{m}_2$ because it is antiparallel to $\mathbf{B}_1$. As in Example 3, this configuration is unstable because any perturbation in the position or orientation of $\mathbf{m}_2$ leads to a torque that aligns it with $\mathbf{B}_1$, whence the force becomes attractive.

In all cases except for the two unstable repulsive cases (Examples 3 and 6), the forces and torques on a dipole or spherical magnet bring it eventually to the stable minimum-energy state discussed in Example 1. If $\mathbf{m}_2$ is allowed to freely move and rotate in space, the two unstable repulsive cases will not be possible to realize physically because any slight perturbation in the position or orientation of $\mathbf{m}_2$ will lead eventually to its attraction to the minimum-energy state.

nterpreis straightforward in some cases and subtle in others. Fig. 4(a), position A, shows

In other cases, applying these principles in understanding Fig. 4 is straightforward is subtle in other cases.

For $\mathbf{R}_{ij} \parallel \mathbf{m}_1 \perp \mathbf{m}_2$, $\mathbf{F}_2(d, 0, \pi/2) = (F_0/2)\hat{\phi}$ is non-central and has no radial component, and $\boldsymbol{\tau}(d, 0, \pi/2) = -\tau_0 \hat{z}$ tends to align the dipole with the field. The case of both dipoles making an angle $\alpha = \beta = \pi/4$ with $\mathbf{R}_{ij}$ gives a non-central force $\mathbf{F}_2(d, \pi/4, \pi/4) = F_0(-\hat{\mathrm{R}}/4 + \hat{\phi}/2)$ whose azimuthal component exceeds its radial component, and a torque $\boldsymbol{\tau} = -(3\tau_0/4)\hat{z}$. The case of both dipoles making an angle $\alpha = \beta = \pi/2$ with $\mathbf{R}_{ij}$ gives a non-central force $\mathbf{F}_2(d, \pi/4, \pi/4) = F_0(-\hat{\mathrm{R}}/4 + \hat{\phi}/2)$ whose azimuthal component exceeds its radial component, and a torque $\boldsymbol{\tau} = -(3\tau_0/4)\hat{z}$.

It is a standard undergraduate exercise to calculate the magnetic field at position $\mathbf{r}$ produced by a sphere of radius R and uniform magnetization $\mathbf{M}$, and located at the coordinate origin [9]. Outside of the sphere (for $r > R$), this field is identical to the dipole field given by Eq. (1), where

$$\mathbf{m} = \int_V \mathbf{M} dV = \frac{4\pi}{3} R^3 \mathbf{M} \tag{23}$$

is the total dipole moment of the sphere, and the integral is over the sphere volume. Inside the sphere, the field is uniform and is given by $\mathbf{B} = 2\mu_0 \mathbf{M}/3$.

XI. SIMULATION

We simulate the interactions between N identical spheres ($i = 1, 2, 3, \cdots, N$), each with diameter D, mass M, and magnetic moment m. The magnetic force on mass j is given by

$$\mathbf{F}_j = \sum_{i=1,\ i \neq j}^{N} \mathbf{F}_{ij}, \tag{24}$$

where $\mathbf{F}_{ij}$ is given by Eq. (8). If this is the only force acting on mass j, then Newton's second law demands that

$$\mathbf{F}_j = M \mathbf{a}_j, \tag{25}$$

where $\mathbf{a}_j$ is the acceleration of mass j.

The magnetic torque on mass j is given by

$$\boldsymbol{\tau}_j = \sum_{i=1, i\neq j}^{N} \boldsymbol{\tau}_{ij}, \tag{26}$$

where $\boldsymbol{\tau}_{ij}$ is given by Eq. (10). If this is the only torque acting on mass j, then the rotational form of Newton's second law demands that

$$\boldsymbol{\tau}_j = I\boldsymbol{\alpha}_j, \tag{27}$$

where $\boldsymbol{\alpha}_j$ is the angular acceleration of mass j, and

$$I = \frac{2}{5}M\left(\frac{D}{2}\right)^2 \tag{28}$$

is its moment of inertia.

We scale the length by D the time by $T = \sqrt{MD/F_0}$, which gives the time scale for two magnets to come together from distances of the order of D, and define (primed) dimensionless coordinates according to

$$\begin{aligned}
\mathbf{F} &= F_0\mathbf{F}' && (29)\\
\boldsymbol{\tau} &= F_0 D\boldsymbol{\tau}' && (30)\\
\mathbf{m} &= m\mathbf{m}' && (31)\\
t &= Tt' && (32)\\
\mathbf{r} &= D\mathbf{r}' && (33)\\
\mathbf{v} &= d\mathbf{r}/dt = (D/T)\mathbf{v}' && (34)\\
\mathbf{a} &= d\mathbf{v}/dt = (D/T^2)\mathbf{a}' && (35)\\
\omega &= d\phi/dt = (1/T)\omega' && (36)\\
\alpha &= d\omega/dt = (1/T^2)\alpha' && (37)\\
& && (38)
\end{aligned}$$

In these units, the magnet diameter is 1, and the closest approach of one magnet to another (center to center) is also 1.

In dimensionless units, we have

$$\mathbf{a}'_j = \sum_{i=1,\ i\neq j}^{N} \mathbf{F}'_{ij}, \tag{39}$$

$$\begin{aligned}
\mathbf{F}'_{ij} =& \frac{1}{2{R_{ij}}'^5}\Big[\left(\mathbf{m}'_i\cdot\mathbf{R}'_{ij}\right)\mathbf{m}'_j + \left(\mathbf{m}'_j\cdot\mathbf{R}'_{ij}\right)\mathbf{m}'_i \\
&+ \left(\mathbf{m}'_i\cdot\mathbf{m}'_j\right)\mathbf{R}'_{ij} - 5\frac{\left(\mathbf{m}'_i\cdot\mathbf{R}'_{ij}\right)\left(\mathbf{m}'_j\cdot\mathbf{R}'_{ij}\right)}{{R_{ij}}'^2}\mathbf{R}'_{ij}\Big].
\end{aligned} \tag{40}$$

$$\boldsymbol{\alpha}'_j = 10\sum_{i=1,\ i\neq j}^{N} \boldsymbol{\tau}'_{ij} \tag{41}$$

$$\boldsymbol{\tau}'_{ij} = \frac{\mathbf{m}'_i\cdot\mathbf{R}'_{ij}}{2{R_{ij}}'^5}\mathbf{m}'_j\times\mathbf{R}'_{ij} - \frac{\mathbf{m}'_j\times\mathbf{m}'_i}{6{R_{ij}}'^3} \tag{42}$$

For 2d motion in the x-y plane, the last two equations have components only in the z direction. In fact, these equations are not generally valid in 3d.

In 2d, the dipole moments

$$\mathbf{m}'_i = \cos\phi_i\hat{\mathrm{x}} + \sin\phi_i\hat{\mathrm{y}} \tag{43}$$

are unit vectors that involve the magnetic orientation angle ϕ_i for mass i. These orientation angles obey $d^2\phi_i/dt'^2 = \alpha'_i$.

XII. CONCLUSIONS

May be able to extend these arguments to spheres with non-uniform, spherically symmetric magnetizations $\mathbf{M}(r)$.

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[1] David A. Richter, Expert Report, "Teaching Geometry with Magnet Sphere Kits," in the matter of Zen Magnets, LLC, CPSC Docket No. 12-2, Item 124, Exhibit 3, 10/20/2014, http://www.cpsc.gov/en/Recalls/Recall-Lawsuits/Adjudicative-Proceedings/ (accessed Feb. 9, 2016).

[2] Boyd F. Edwards, "Expert Report: Educational Value of Neodymium Magnet Spheres in the matter of Zen Magnets, LLC, CPSC Docket No. 12-2," Item 124, Exhibit 4, 10/20/2014, http://www.cpsc.gov/en/Recalls/Recall-Lawsuits/Adjudicative-Proceedings/ (accessed Feb. 9, 2016).

[3] The Zen Gallery, curated by Shihan Qu, http://zenmagnets.com/gallery/ (accessed Feb. 9, 2016).

[4] "Simulating toy magnet balls? (eg: Zen Magnets)," http://www.physicsforums.com/showthread.php?t=596513 (accessed Feb. 9, 2016).

[5] G. Helgesen, T. T. Skjeltorp, P. M. Mors, R. Botet, and R. Jullien, "Aggregation of Magnetic Microspheres: Experiments and Simulations," Phys. Rev. Lett. **61** (15), 1736-1739 (1988).

[6] Bernhard Thomaszewski, Andreas Gumann, Simon Pabst, Wolfgang Straber, "Magnets in Motion," ACM Transactions on Graphics **27** (5), Article 5 (2008).

[7] M. Varón, M. Beleggia, T. Kasama, R. J. Harrison, R. E. Dunin-Borkowski, V. F. Puntes, & C. Frandsen, "Dipolar Magnetism in Ordered and Disordered Low-Dimensional Nanoparticle Assemblies," Scientific Reports **3** (1234), 1-5 (2013).

[8] P. C. Clemmow, *An Introduction to Electromagnetic Theory*, (Cambridge University Press, London, 1973), p. 284.

[9] Roald K. Wangsness, *Electromagnetic Fields*, 2nd edition (Wiley, New York, NY, 1986), p. 326.

[10] Roald K. Wangsness, *Electromagnetic Fields*, 2nd edition (Wiley, New York, NY, 1986), p. 340.

[11] D. Vokoun and M. Beleggia, "Forces between arrays of permanent magnets of basic geometric shapes," Journal of Magnetism and Magnetic Materials **350**, 174-178 (2014). This publication quotes the result of an unpublished calculation of the magnetic interaction between two identical spheres with parallel magnetizations. Details of this calculation were sent privately to us by D. Vokoun on February 8, 2016. We independently completed a calculation for the special case of two identical uniformly magnetized spheres with parallel magnetizations that are parallel to the line between the sphere centers.

[12] W. Booth and B. Edwards, unpublished.

[13] M. Beleggia and M. De Graef, "General magnetostatic shape-shape interactions," Journal of Magnetism and Magnetic Materials **285**, L1-L10 (2005).

[14] Jerry B. Marion and Stephen T. Thornton, "Classical Dynamics of Particles and Systems," 4th edition (Saunders College Publishing, Fort Worth, Texas, 1995), p. 50. This reference mentions that the force between moving charged particles violates Newton's third law. Such particles can exchange energy with the fields. For example, consider particle 1, positively charged and moving in the $+x$ direction, and particle 2, positively charged and moving in the $+y$ direction, at the instant that particle 1 is at the coordinate origin and particle 2 is on the $+y$ axis. At that instant, the electric forces are central and repulsive, but particle 1 exerts a non-central magnetic force on particle 2 in the $+x$ direction, while particle 2 exerts no magnetic force on particle 1. See http://physics.stackexchange.com/questions/138095/newtons-third-law-exceptions (accessed February 10, 2016).

[15] Wikipedia, *Force between Magnets*, https://en.wikipedia.org/wiki/Force_between_magnets (accessed February 10, 2016).

[16] Attraction/repulsion force between two equal magnets?, http://www.eng-tips.com/viewthread.cfm?qid=341932 (accessed February 10, 2016).

[17] Freeman J. Dyson, "Feynman's proof of the Maxwell equations, Am. J. Phys. **58** (3), 209211 (1990).

[1] G. Akoun and J. Yonnet, IEEE Trans. Magn. **20**, 1962 (1984).

[2] T. H. Boyer, Am. J. Phys. **56**, 688 (1988).

[3] A. Kruusing, J. Magn. Magn. Mater. **234**, 545 (2001).

[4] D. Vokoun, M. Beleggia, L. Heller, and P. Sittner, J. Magn. Magn. Mater. **321**, 3758 (2009).

[5] J. S. Agashe and D. P. Arnold, J. Phys. D. Appl. Phys. **42**, 099801 (2009).

[6] S. Sanz and L. Garcia-Tabares, IEEE Trans. Appl. Supercond. **20**, 846 (2010).

[7] J. Camacho and V. Sosa, Rev. Mex. Fsica E **59**, 8 (2013).

[8] David Griffiths, *Introduction to Electrodynamics*, 3rd edition (Pearson Education, Delhi, India, 1999), p. 246.

[26] Coercivity, http://en.wikipedia.org/wiki/Coercivity.

[27] Compliance Magnets, http://micromagnets.com

[28] K&J Magnetics, Inc., http://www.kjmagnetics.com/

[29] Alibaba, http://www.alibaba.com.

[30] Applied Magnets, http://www.magnet4less.com

[31] The vector magnetic fields shown in Fig. 4 are courtesy Wikipedia and Wikimedia Commons, https://en.wikipedia.org/wiki/File:VFPt_dipole_point.svg.