Dispersion in Electromagnetic Waveguide

A Final Project for APPM 4350: Fourier Series and Boundary Value Problems

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1 Introduction

2 Waveguide

3 Dispersion

In the previous section, we derived the relationship for the phase velocity in the waveguide,

$$v_p = \frac{\omega}{\beta} = \frac{c}{\sqrt{1 - \frac{\omega_c^2}{\omega^2}}}, \tag{1}$$

which implies

$$\omega^2(\beta) = c^2\beta^2 + \omega_c^2, \tag{2}$$

where ω_c is the cutoff frequency of the waveguide, ω the frequency of operation, and β the phase coefficient.

Notice that v_p is frequency dependent. This means that the velocity with which a signal propagates down the guide depends on its frequency. Consider a signal containing components at multiple frequencies. The shape of that signal is determined by both the amplitude and phase of those components. If the velocity of these components is not the same, then their relative phases will change as they propagate down the guide, resulting in a received signal which is distorted from the original. A signal which is concentrated will tend to spread out, or disperse, with time. We call this effect dispersion, and we call (2) the dispersion relation for the waveguide.

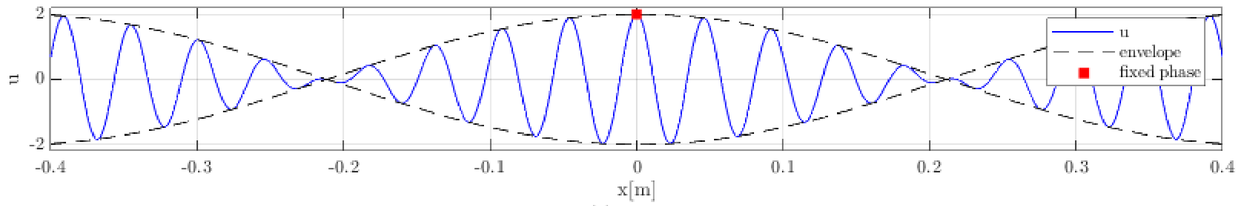
Not all electromagnetic media are dispersive. The free space, for instance, has a linear dispersion relation of the form

$$\omega^2 = c^2\beta^2,$$

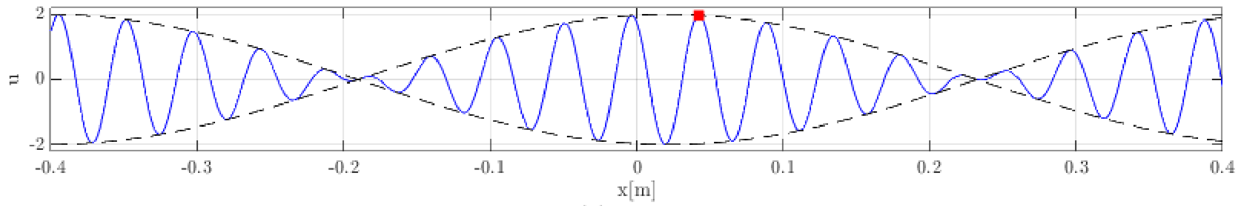
which results in a frequency independent phase velocity. Dispersion emerges either from frequency dependence of properties of the system (such as frequency dependent dielectric constant), called material dispersion, or from non-uniform geometries, called geometric dispersion. Dispersion in waveguides is of the geometric type.

3.1 Group Velocity

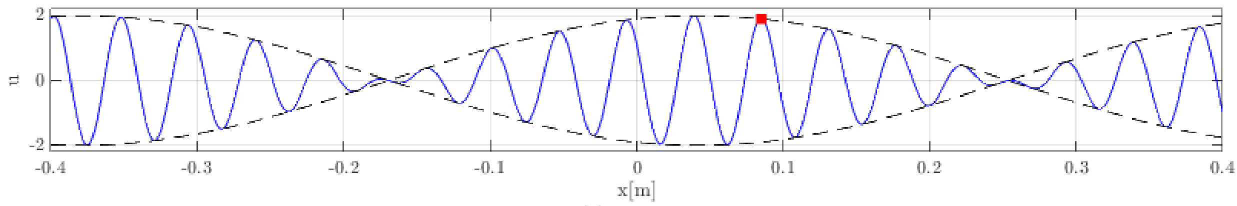
Equation (1) implies that the phase velocity of a frequency above cutoff is greater than the speed of light. Upon first examination, this may seem to contradict physical intuition: we don't expect an electromagnetic wave to propagate faster than light. It turns out that though the phase velocity may be greater than light, the speed of propagation of energy, and therefore also of information, is less than that of light. The speed of energy propagation is described by the group velocity.



(a) t = 0ns



(b) t = 0.1ns



(c) t = 0.2ns

Figure 1: Dispersion of

Consider a signal consisting of two harmonic components separated by some small frequency $\Delta\omega$. The signal consists of an oscillation modulated by a slowly varying envelopse, shown in Figure 1(a). Due to dispersion, the two components must have different phase coefficients, separated by some small $\Delta\beta$. As it propagates, the signal is

$$u(x,t) = \cos(\beta x - \omega t) + \cos((\beta + \Delta\beta)x - (\omega + \Delta\omega)t), \tag{3}$$

where we have from the dispersion relation (2) that

$$\begin{aligned} \omega^2 &= \omega^2(\beta) = c^2\beta^2 + \omega_c^2, \\ (\omega + \Delta\omega)^2 &= \omega^2(\beta + \Delta\beta) = c^2(\beta + \Delta\beta)^2 + \omega_c^2. \end{aligned}$$

Applying the cosine addition rule to (3), we have

$$\begin{aligned} u(x,t) &= 2\cos\left(\frac{\beta + \beta + \Delta\beta}{2}x - \frac{\omega + \omega + \Delta\omega}{2}t\right)\cos\left(\frac{\beta + \Delta\beta - \beta}{2}x - \frac{\omega + \Delta\omega - \omega}{2}t\right) \\ &= 2\cos\left(\left[\beta + \frac{\Delta\beta}{2}\right]x - \left[\omega + \frac{\Delta\omega}{2}\right]t\right)\cos\left(\frac{\Delta\beta}{2}x - \frac{\Delta\omega}{2}t\right) \end{aligned}$$

This solution is plotted in Figure 1. The first cosine term describes the propagation of the carrier of the signal, with velocity

$$v_1 = \frac{2\omega + \Delta\omega}{2\beta + \Delta\beta} \approx \frac{\omega}{\beta} = v_p.$$

The second cosine describes the propagation of the envelope of the signal, with velocity

$$v_2 = \frac{\Delta\omega}{\Delta\beta}.$$

As we take $\beta \to 0$, we call this the group velocity, defined as

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}\beta}.$$

For the waveguide, the group velocity is therefore

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}\beta} = \frac{\mathrm{d}}{\mathrm{d}\beta}\left(\sqrt{c^2\beta^2 + \omega_c^2}\right) = \frac{\sqrt{c^2\beta^2 + \omega_c^2}}{c^2\beta}. \tag{4}$$

We substitute (2) into (4) and have the group velocity in terms of frequency,

$$v_g = c\sqrt{1 - \frac{\omega_c^2}{\omega^2}}.$$

For a frequency above cutoff, group velocity is always slower than the speed of light, as expected from physical principles.

4 Dispersion in Waveguide

5 Measurement