APPM 4600 Homework 9

3 November 2024

1. We have the system

$$A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ \text{and } \mathbf{x} = \begin{bmatrix} u \\ v \end{bmatrix}.$$

We wish to minimize the error $||A\mathbf{x} - b||$. We form the Gram matrx

$$G = A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

with inverse

$$(A^T A)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

We then solve the system $A^T A\mathbf{x} = A^T \mathbf{b}$, which yields

$$\mathbf{X} = \begin{bmatrix} u \\ v \end{bmatrix} = (A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix},$$

that is, we have least squares solution

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}.$$

2. We have the system

$$A\mathbf{x} - \mathbf{c} = \mathbf{b}, \tag{1}$$

where

$$A = \begin{bmatrix} 1 & 3 \\ 6 & -1 \\ 4 & 0 \\ 2 & 7 \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

We wish to minimize the quantity

$$E^2 = b_1^2 + 4b_2^2 + 25b_3^2 + 9b_4^2,$$

from which we construct the diagonal weight matrix

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

We multiple our system (1) by this weight on the left and have

$$DA\mathbf{x} - D\mathbf{c} = D\mathbf{b}.$$

The error term that we wish to minimize is now

$$E^2 = ||D\mathbf{b}||_2,$$

which is the same problem as finding the least squares solution to the system

$$DA\mathbf{x} = D\mathbf{c}.$$

As in problem (1), we have the least squares solution

$$\begin{aligned} \mathbf{x} = (A^T D^T DA)^{-1} A^T D^T D\mathbf{c} &= \left(\begin{bmatrix} 1 & 12 & 20 & 6 \\ 3 & -2 & 0 & 21 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 12 & -2 \\ 20 & 0 \\ 6 & 21 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 3 \\ 12 & -2 \\ 20 & 0 \\ 6 & 21 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 15 \\ 12 \end{bmatrix} \\ &= \dots \\ &\approx \begin{bmatrix} 0.6536 \\ 0.3929 \end{bmatrix}. \end{aligned}$$

3. (a) We wish to show that $\{1, x, x^2, \dots, x^n\}$ is linearly independent, that is, we wish to show that the system

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = 0 \tag{2}$$

has only the trivial solution $0 = c_0 = c_1 = c_2 = \cdots = c_n$. Since (2) is identically zero, all of its derivatives must also be zero, that is,

$$\begin{aligned}
0 &= c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1} + c_n x^n \\
0 &= c_1 + 2c_2 x + 3c_3 x^2 + \cdots + (n-1) c_{n-1} x^{n-2} n c_n x^{n-1} \\
0 &= 2c_2 + 6c_3 x + 12 c_4 x^2 + \cdots + (n-1)(n-2) c_{n-1} x^{n-3} + n(n-1) c_n x^{n-2} \\
&\dots \\
0 &= ((n-1)!) c_{n-1} + (n!) c_n x \\
0 &= (n!) c_n.
\end{aligned}$$

This is an upper triangular system which we can solve via back substitution. The last equation trivially gives $c_n = 0$, the next $c_{n-1} = 0$, and so on to give $0 = c_n = c_{n-1} = \cdots = c_2 = c_1 = c_0$.
Thus, $\{1, x, x^2, \dots, x^n\}$ is linearly independent.

(b) We wish to show that $\{1, \cos x, \cos 2x, \dots, \cos nx, \sin x, \sin 2x, \dots, \sin nx\}$ is linearly independent, that is, we wish to show that the system

$$0 = a_0 + b_1 \cos x + b_2 \cos 2x + \cdots + b_n \cos nx + c_1 \sin x + c_2 \sin 2x + \cdots + c_n \sin nx \tag{3}$$

has only the trivial solution $0 = a_0 = b_1 = b_2 = \cdots = b_n = c_1 = c_2 = \cdots = c_n$. Since (3) is identically zero, so must be all of its derivatives,

$$\begin{aligned}
0 &= a_0 + b_1 \cos x + b_2 \cos 2x + \cdots + b_n \cos nx + c_1 \sin x + c_2 \sin 2x + \cdots + c_n \sin nx \\
0 &= -b_1 \sin x - 2b_2 \sin 2x + \cdots - n b_n \sin nx + c_1 \cos x + 2c_2 \cos 2x + \cdots + n c_n \cos nx \qquad (4) \\
0 &= -b_1 \cos x - 4b_2 \cos 2x + \cdots - n^2 b_n \cos nx - c_1 \sin x - 4c_2 \sin 2x + \cdots - n^2 c_n \sin nx \\
0 &= b_1 \sin x + 8b_2 \sin 2x + \cdots + n^3 b_n \sin nx - c_1 \cos x - 8c_2 \cos 2x + \cdots - n^3 c_n \cos nx \qquad (5) \\
&\dots
\end{aligned}$$

We add together (4) and (5) to get

$$0 = 6b_2 \sin 2x + \cdots + (n^3 - n) b_n \sin nx - 6c_2 \cos 2x + \cdots - (n^3 - n) c_n \cos nx,$$

which has eliminated the b_1 and c_1 terms. We can proceed in a similar manner on this equation until we isolate b_n and c_n, that is, we have

$$\begin{aligned}
0 &= b_n \cos nx + c_n \sin nx \\
&= n\left(-b_n \sin nx + c_n \cos nx\right),
\end{aligned}$$

which is a system with only the solution $b_n = c_n = 0$ (we can see this by considering both equations at $x = 0$). In a similar manner, back-substitution yields successively $b_{n-1} = c_{n-1} = 0, \dots, b_1 = c_1 = 0$. Then (3) becomes $a_0 = 0$. Thus, $\{1, \cos x, \cos 2x, \dots, \cos nx, \sin x, \sin 2x, \dots, \sin nx\}$ is linearly independent.

4. We wish to show that the following formula generates orthogonal polynomials:

$$\phi_k(x) = (x - b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x), \tag{6}$$

where

$$b_k = \frac{< x\phi_{k-1}, \phi_{k-1} >}{< \phi_{k-1}, \phi_{k-1} >}, \text{ and } c_k = \frac{< x\phi_{k-1}, \phi_{k-2} >}{< \phi_{k-2}, \phi_{k-2} >}.$$

Proof. We prove this by induction. We assume that the base cases ϕ_1 and ϕ_2 are orthogonal.

Assume, by way of induction, that polynomials $\{\phi_1, \phi_2, \ldots, \phi_{k-1}\}$ are orthogonal, that is, $< \phi_n, \phi_m >= 0$ for $n \neq m, n < k, m < k$. We have that $\phi_k(x)$ is of degree k, $\phi_{k-1}(x)$ is of degree $k-1$, and $\phi_{k-2}(x)$ is of degree $k-2$. Therefore, we can write ϕ_k as

$$\phi_k(x) = (x - b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x) - \sum_{i=0}^{k-3} a_i\phi_i(x), \tag{7}$$

for some suitable choice of b_k, c_k, a_i. We take the inner product of (7) with ϕ_j for $j \leq k-3$ and get

$$\begin{aligned}
0 =< \phi_j, \phi_k > &= (x - b_k) < \phi_j, \phi_{k-1} > -c_k < \phi_j, \phi_{k-2} > - \sum_{i=0}^{k-3} a_i < \phi_j, \phi_i > \\
&=< x\phi_j, \phi_{k-1} > -b_k < \phi_j, \phi_{k-1} > -c_k < \phi_j, \phi_{k-2} > - \sum_{i=0}^{k-3} a_i < \phi_j, \phi_i > \\
&=< x\phi_j, \phi_{k-1} > -a_j < \phi_j, \phi_j > \\
&= -a_j < \phi_j, \phi_j >,
\end{aligned} \tag{8}$$

where the third line is due to the fact that $< \phi_n, \phi_m >= 0$ for $n, m \leq k-1$ and $n \neq m$, and the fourth line can be seen by recognizing the previous fact and that $x\phi_j$ is of degree no more than $k-2$. Thus, because $< \phi_j, \phi_j >\neq 0$, we have from (8) that $a_j = 0$ for $j \leq k-3$. Therefore, (7) becomes

$$\phi_k(x) = (x - b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x). \tag{9}$$

We want the inner product of (9) with ϕ_{k-2} and ϕ_{k-1} to be zero. We consider first ϕ_{k-1},

$$\begin{aligned}
0 =< \phi_{k-1}, \phi_k > &= (x - b_k) < \phi_{k-1}, \phi_{k-1} > -c_k < \phi_{k-1}, \phi_{k-2} > \\
&=< x\phi_{k-1}, \phi_{k-1} > -b_k < \phi_{k-1}, \phi_{k-1} >,
\end{aligned}$$

which we can satisfy with the choice

$$b_k = \frac{< x\phi_{k-1}, \phi_{k-1} >}{< \phi_{k-1}, \phi_{k-1} >}.$$

We now turn to the inner product of (9) with ϕ_{k-2},

$$\begin{aligned}
0 =< \phi_{k-2}, \phi_k > &= (x - b_k) < \phi_{k-2}, \phi_{k-1} > -c_k < \phi_{k-2}, \phi_{k-2} > \\
&=< x\phi_{k-2}, \phi_{k-1} > -b_k < \phi_{k-2}, \phi_{k-1} > -c_k < \phi_{k-2}, \phi_{k-2} > \\
&=< x\phi_{k-1}, \phi_{k-2} > -c_k < \phi_{k-2}, \phi_{k-2} >,
\end{aligned}$$

which is satisfied by the choice

$$c_k = \frac{< x\phi_{k-1}, \phi_{k-2} >}{< \phi_{k-2}, \phi_{k-2} >}.$$

Thus, the three term recursion generates orthogonal polynomials.

□

5. We have the formula

$$T_n(x) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right), \tag{10}$$

where $x = \frac{1}{2}(z + \frac{1}{z})$.

The first two polynomials given by (10) are

$$\begin{aligned} T_0(x) &= \frac{1}{2}\left(z^0 + \frac{1}{z^0}\right) = \frac{1}{2}(1+1) = 1, \\ T_1(x) &= \frac{1}{2}\left(z + \frac{1}{z}\right) = x, \end{aligned}$$

which are the first two Chebychev polynomials, as expected.

We will show that (10) satisfies the recurrence from class,

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

We have

$$\begin{aligned} 2xT_{n-1}(x) - T_{n-2}(x) &= 2x\left(\frac{1}{2}\left(z^{n-1} + \frac{1}{z^{n-1}}\right)\right) - \frac{1}{2}\left(z^{n-2} + \frac{1}{z^{n-2}}\right) \\ &= \left(z + \frac{1}{z}\right)\left(\frac{1}{2}\left(z^{n-1} + \frac{1}{z^{n-1}}\right)\right) - \frac{1}{2}\left(z^{n-2} + \frac{1}{z^{n-2}}\right) \\ &= \frac{1}{2}\left(z^n + \frac{1}{z^{n-2}} + z^{n-2} + \frac{1}{z^n} - z^{n-2} - \frac{1}{z^{n-2}}\right) \\ &= \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) \\ &= T_n(x). \end{aligned}$$

Thus, (10) gives the Chebychev polynomials.