APPM 4600 Homework 2

14 September 2024

1. (a) Consider the function

$$f(x) = (1+x)^n - 1 - nx.$$

Observe that

$$\lim_{x\to 0}\left|\frac{f(x)}{x}\right| = \lim_{x\to 0}\left|\frac{(1+x)^n - 1 - nx}{x}\right| = \lim_{x\to 0}\left|n(1+x)^{n-1} - n\right| = 0.$$

Thus, $f(x) = o(x)$ as $x \to 0$, which implies that

$$(1+x)^n = 1 + nx + o(x).$$

(b) Observe that

$$\lim_{x\to 0}\left|\frac{x\sin\sqrt{x}}{x^{\frac{3}{2}}}\right| = \lim_{x\to 0}\left|\frac{\sin\sqrt{x} + \frac{1}{2}\sqrt{x}\cos\sqrt{x}}{\frac{3}{2}\sqrt{x}}\right| = \lim_{x\to 0}\left|\frac{\frac{3}{4\sqrt{x}}\cos\sqrt{x} - \frac{1}{4}\sin\sqrt{x}}{\frac{3}{4\sqrt{x}}}\right| = \lim_{x\to 0}\left|\cos\sqrt{x} - \frac{\sqrt{x}}{3}\sin\sqrt{x}\right|$$
$$= 1 \neq 0.$$

Thus, as $x \to \infty$,

$$x\sin\sqrt{x} = O\left(x^{\frac{3}{2}}\right).$$

(c) Observe that

$$\lim_{t\to\infty}\left|\frac{e^{-t}}{\frac{1}{t^2}}\right| = \lim_{t\to\infty}\left|\frac{t^2}{e^t}\right| = \lim_{t\to\infty}\left|\frac{2t}{e^t}\right| = \lim_{t\to\infty}\left|\frac{2}{e^t}\right| = 0.$$

Thus, as $t \to \infty$,

$$e^{-t} = o\left(\frac{1}{t^2}\right).$$

(d) Observe that

$$\lim_{\epsilon\to 0}\left|\frac{\int_0^\epsilon e^{-x^2}\,\mathrm{d}x}{\epsilon}\right| = \lim_{\epsilon\to 0}\left|\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\epsilon e^{-x^2}\,\mathrm{d}x\right| = \lim_{\epsilon\to 0}\left|e^{-\epsilon^2}\right| = 1 \neq 0.$$

Thus, as $\epsilon \to 0$,

$$\int_0^\epsilon e^{-x^2}\,\mathrm{d}x = O(\epsilon).$$

2. (a) We have the exact problem $A\vec{x} = \vec{b}$ with solution $\vec{x} = A^{-1}\vec{b}$. The perturbed problem is $A\vec{x^*} = (\vec{b} + \vec{\Delta b})$ with solution

$$\vec{x^*} = A^{-1}(\vec{b} + \vec{\Delta b}).$$

The change in solution is

$$\begin{aligned}\vec{\Delta x} = \vec{x^*} - \vec{x} = A^{-1}\left(\vec{b} + \vec{\Delta b} - \vec{b}\right) &= A^{-1}\vec{\Delta b} \\ &= \begin{bmatrix} 1 - 10^{10} & 10^{10} \\ 1 + 10^{10} & -10^{10}\end{bmatrix}\begin{bmatrix}\Delta b_1 \\ \Delta b_2\end{bmatrix} \\ &= \begin{bmatrix}\Delta b_1 + 10^{10}\left(\Delta b_2 - \Delta b_1\right) \\ \Delta b_1 + 10^{10}\left(\Delta b_1 - \Delta b_2\right)\end{bmatrix}. \end{aligned} \tag{1}$$

(b) The condition number of A is bounded as

$$\kappa(A) \leq \frac{\sigma_1}{\sigma_n},$$

where σ_1 and σ_2 are the singular values of A. We have that A has singular value decomposition

$$A = \begin{bmatrix} 10^{-10} & 1 \\ -1 & 10^{-10} \end{bmatrix} \begin{bmatrix} 5 \times_1 0^9 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 10^{-10} \end{bmatrix},$$

so we have $\sigma_1 = 5 \times 10^9$ and $\sigma_2 = \frac{1}{2}$ and the condition number is bound as

$$\kappa(A) \leq \frac{\sigma_1}{\sigma_2} = \frac{5 \times_1 0^9}{\frac{1}{2}} = 10^1 0,$$

which suggests that the problem may be poorly conditioned.

(c) The relative error in the solution is $||\vec{\Delta x}||/||\vec{x}||$. The condition number provides an upper bound for the relative error in the solution relative to the perturbation.

From the absolute error (1), we known that the error is dominated by the $\Delta b_1 - \Delta b_2$ term. The error will be larger if the perturbations in each dimension are not matched. The relative error for various perturbations is given below.

Perturbation	Error
$\Delta b_1 = 10^{-5}, \Delta b_2 = 10^{-5}$	1.002×10^{-5}
$\Delta b_1 = 2 \times 10^{-5}, \Delta b_2 = 0.7 \times 10^{-5}$	1.3×10^5

In general, we cannot expect the perturbation terms to be equal, and we observe a very poorly conditioned system.

3. (a)

A. Codes

The code used to compute the results in the table in 2(c) is listed below.

```
#! /usr/bin/env python3
import numpy as np

# pertubations
b1 = 2e-5
b2 = 0.7e-5

# b1 = 1e-5
# b2 = 1e-5

# setup system to solve
Ainv = np.array([[1-10**10, 10**10], [1+10**10, -10**10]])
b = np.array([[1], [1]])
bpert = np.array([[b1], [b2]])
# compute exact solution
x = np.matmul(Ainv, b)
# compute perturbed solution
xpert = np.matmul(Ainv, b+bpert)
# absolute error
xdif = x - xpert
# relative error
rel_error = np.linalg.norm(xdif) / np.linalg.norm(x)

print(rel_error)
```