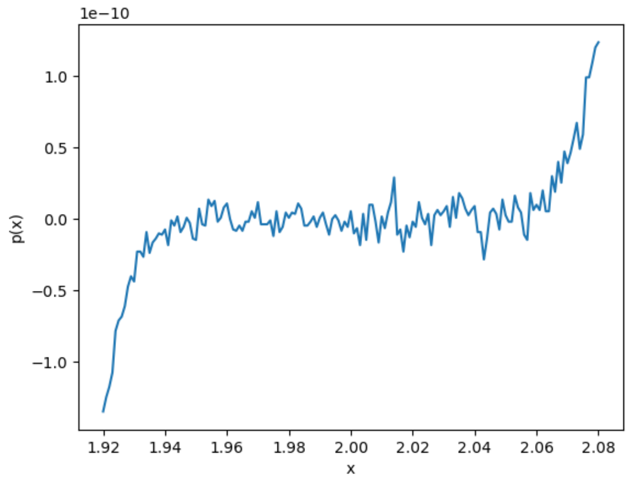
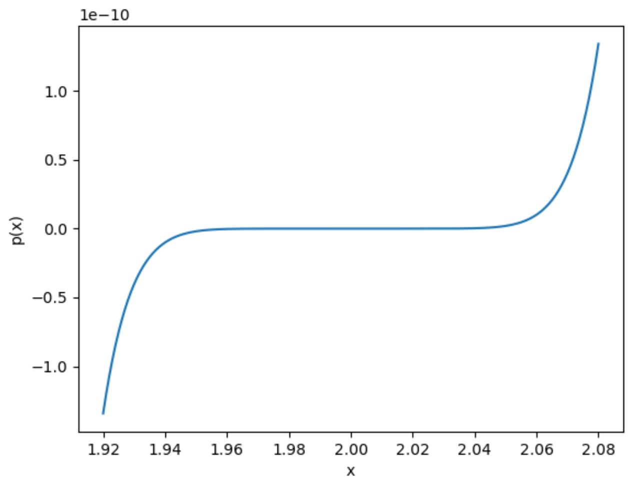
APPM 4600 Homework 1

6 September 2024

1. (a) A plot of $p(x) = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512$ evaluated via its expanded form is shown below.



(b) A plot of $p(x) = (x-2)^9$ evaluated in that form is shown below.



(c) The second plot is much closer to being correct. When evaluated via the expanded form, the high order terms (which largely cancel near $x \approx 2$) become very large and cause a loss of precision in the floating point representation of the value of the function.

2. (a) We can evaluate as

$$\begin{aligned}\sqrt{x+1} - 1 &= \frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{\sqrt{x+1}+1} \\ &= \frac{x+1-1}{\sqrt{x+1}+1} \\ &= \frac{x}{\sqrt{x+1}+1}.\end{aligned}$$

(b) We can evaluate via the sum to product identity as

$$\sin(x) - \sin(y) = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right).$$

(c) We can evaluate as

$$\begin{aligned}\frac{\cos(x)-1}{\sin(x)} &= \frac{(\cos(x)-1)(\cos(x)+1)}{\sin(x)(\cos(x)+1)}\\ &= \frac{\cos^2(x)-1}{\sin(x)(\cos(x)+1)}\\ &= \frac{\sin(x)}{cos(x)+1}.\end{aligned}$$

3. (a) We have the Taylor expansion

$$\begin{aligned}P_2(x) &= (1+x+x^3)\cos(x)\\ &= (1+x+x^3)\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\dots\right)\\ &\approx 1+x-\frac{1}{2}x^2.\end{aligned}$$

We evaluate at $x = 0.5$ and find $P_2(0.5) = 1.375$. From Taylor's inequality, we have that the error is bound as

$$|f(0.5) - P_2(0.25)| \leq \frac{|M|}{3!}(0.5)^3,$$

where $|f^{(3)}(x)| \leq M$ on $x \in [0, 0.5]$. We have

$$\begin{aligned}f'(x) &= (1+3x^2)\cos(x) - (1+x+x^3)\sin(x)\\ f''(x) &= 6x\cos(x) - 2(1+3x^2)sin(x) + (1+x+x^3)\cos(x)\\ f'''(x) &= 6\cos(x) - 6x\sin(x) - 12x\sin(x) - (1+3x^2)\cos(x) - (1+x+x^3)\sin(x),\end{aligned}$$

which we can bound as $|f'''(x)| < 3$ on $x \in [0, 0.5]$. Thus, our error bound is

$$|f(0.5) - P_2(0.25)| \leq \frac{3}{3!}(0.5)^3 = \left(\frac{1}{2}\right)^4 = 0.0625.$$

The actual error is

$$|f(0.5) - P_2(0.5)| = |1.426071 - 1.375| = 0.05107,$$

which is less than our error bound.

(b) From Taylor's theorem, we have the error bound

$$|f(x) - P_2(x)| \leq \frac{|M|}{3!}(x)^3,$$

where $|f^{(3)}(x)| \leq M$ on $x \in [0, x]$. If we have $|x| < 0.5$, we can use $M = 3$ from before.

(c) We have

$$\begin{aligned}\int_0^1 f(x)\,\mathrm{d}x &\approx \int_0^1 P_2(x)\,\mathrm{d}x\\ &= \int_0^1 \left(1+x-\frac{1}{2}x^2\right)\mathrm{d}x\\ &= \left(x+\frac{1}{2}x^2-\frac{1}{6}x^3\right)_0^1\\ &= \frac{4}{3}.\end{aligned}$$

(d) We known that $|f^{(3)}(x)| \leq 16$ on $x \in [0,1]$, so the error is bounded by

$$\begin{aligned} E &\leq \int_0^1 \frac{16}{3!} x^3 \, \mathrm{d}x \\ &= \frac{8}{3} \left(\frac{1}{4} x^4 \right)_0^1 \\ &= \frac{2}{3}. \end{aligned}$$

4. (a) The "bad" root will be the one where the numerator of the quadratic formula is near zero. Since b is negative, this is the root

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

We compute the square root to three decimal places,

$$\sqrt{b^2 - 4ac} = \sqrt{56^2 - 4} \approx 55.964,$$

and find root

$$r_2 = \frac{1}{4} (56 - 55.964) = 0.018.$$

The actual root is $r_2 \approx 0.01786284073$, which is 0.77% error.
The other root is computed as

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{56 + \sqrt{56^2 - 4}}{2} \approx \frac{56 + 55.964}{2} \approx 55.982,$$

which is a 0.00025% error from the real root.

(b) We have that

$$\begin{aligned} ax^2 + bx + c &= (x - r_1)(x - r_2) \\ &= x^2 - (r_1 + r_2)x + r_1 r_2, \end{aligned}$$

which implies that $b = -r_1 - r_2$ and $c = r_1 r_2$. These give the two relations

$$\begin{aligned} r_2 &= -b - r_1 \\ r_2 &= \frac{c}{r_1}. \end{aligned}$$

The second gives a better estimate for the root,

$$r_2 = \frac{c}{r_1} \approx \frac{1}{55.982} = 0.01786288450,$$

which is a 0.00025% error from the real root.

5. (a) The absolute error is bounded as

$$|\Delta y| \leq |\Delta x_1| + |\Delta x_2|,$$

and the relative error as

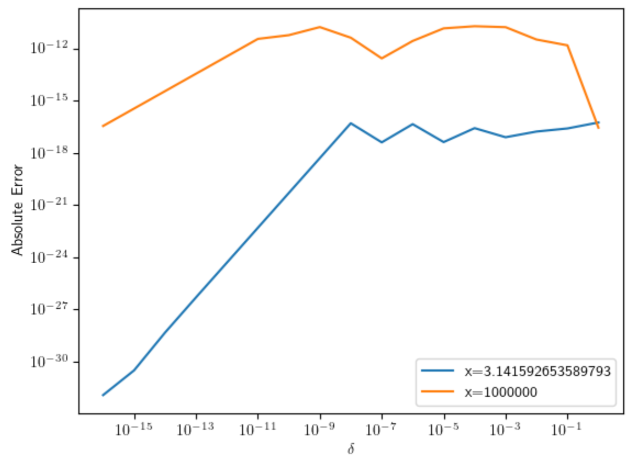
$$\frac{|\Delta y|}{|y|} \leq \frac{|\Delta x_1| + |\Delta x_2|}{|x_1 - x_2|}.$$

The relative error is large when x_1 and x_2 are close in value, and the Δx terms become comparable in magnitude to $x_1 - x_2$.

(b) We can evaluate as

$$\cos(x+\delta) - \cos(x) = -2\sin\left(\frac{2x+\delta}{2}\right)\sin\left(\frac{\delta}{2}\right). \tag{1}$$

A plot of the difference in the evaluation of the two expressions is shown below for $x = \pi$ and $x = 10^6$.



Notice that for $x = \pi$ and δ small, the difference is linear with δ—the non-corrected expression evaluates to zero, and the difference is simply the value of the corrected expression. As δ increases the subtraction becomes less problematic and the absolute error between the two expressions stops growing. For $x = 10^6$, the two terms that are subtracted are large and the error is significant even for small δ.

(c) Let $f(x) = \cos(x)$. We have the derivatives

$$f'(x) = -\sin(x)$$
$$f''(x) = -\cos(x),$$

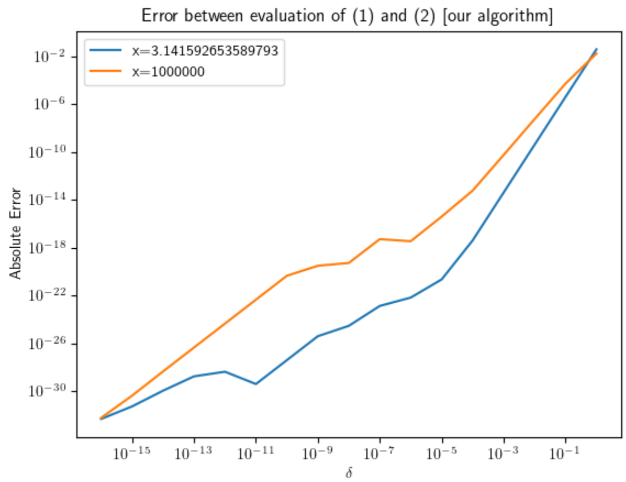
so the Taylor expansion yields

$$\cos(x+\delta) - \cos(x) \approx -\delta\sin(x) - \frac{\delta^2}{2}\cos(x).$$

We modify the second order term to be evaluated at $x + \frac{\delta}{2}$, which splits the range $[x, x+\delta]$ over which we concerned about evaluation, yielding

$$\cos(x+\delta) - \cos(x) \approx -\delta\sin(x) - \frac{\delta^2}{2}\cos(x + \frac{\delta}{2}) \tag{2}$$

The absolute difference between evaluating via (1) and (2) is plotted below.



Notice that our algorithm (2) has low absolute different for small δ for both $x = \pi$ and $x = 10^6$. It performs poorly with large δ, where the higher order terms we dropped from the Taylor expansion become more significant.