Diffusion-Limited Aggregation [and Discussion]

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Diffusion-limited aggregation

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Improved algorithms have been developed for both off-lattice and hypercubic lattice diffusion-limited aggregation (DLA) in dimensionalities (d) 3–8 and for two-dimensional off-lattice DLA. In two-dimensional off-lattice DLA a fractal dimensionality (D) of about 1.71 was obtained for clusters containing up to 10^6 particles. This is significantly larger than the value of $(d^2+1)/(d+1)$ ($\frac{5}{3}$ for $d=2$) predicted by mean field theories. For $d>2$ the off-lattice simulations give results that are consistent with the mean field theories. For $d=3$ and $d=4$ the effects of lattice anisotropy can easily be seen for clusters containing 3×10^6 and 10^6 sites respectively and the effective fractal dimensionalities are slightly smaller for the lattice model clusters than for the off-lattice clusters. Results are also presented for two-, three- and four-dimensional lattice model clusters with noise reduction.

INTRODUCTION

Pattern-formation processes have been of considerable scientific interest and practical importance for many decades. Interest in the growth of complex structures under non-equilibrium conditions has been stimulated by several recent developments. The dissemination of the concepts of fractal geometry (Mandelbrot 1982) and related ideas have provided us with ways of describing a very broad range of irregular structures in quantitative terms. It has been shown that even very simple nonlinear systems and models (see, for example, May 1976; Lorenz 1963; Feigenbaum 1978) can lead to complex, often chaotic, behaviour that can frequently be described in terms of fractal geometry. In addition, it has been shown that simple models for growth and aggregation processes frequently lead to disorderly structures that exhibit a spatially chaotic fractal geometry. In particular, the introduction of the diffusion-limited aggregation (DLA) model (Witten & Sander 1981) led to a renewed interest in models for growth, aggregation and morphogenesis. The increased power, availability and ease of use of digital computers played a crucial role in all of these developments.

In the DLA models, particles are added, one at a time, to a growing cluster or aggregate of particles via random-walk trajectories originating from outside of the region occupied by the cluster. This simple process leads to the formation of a random fractal structure with a fractal dimensionality that is substantially smaller than that of the space or lattice in which the aggregate is embedded. Although the name diffusion-limited aggregation is now well accepted, the DLA model does not describe growth from a finite-density field obeying the diffusion equation. Instead, it describes a random growth process in which the growth probabilities at the surface of the growing cluster are controlled by a scalar field ϕ that obeys the Laplace equation $\nabla^2\phi = 0$. The random walkers are used here to simulate the scalar field ϕ that obeys the Laplace equation with the boundary conditions $\phi = 0$ on the cluster and $\phi = 1$ at infinity. The close relation between the DLA process and growth controlled by a laplacian or harmonic field is made explicit in the dielectric breakdown model (Niemeyer *et al.* 1984).

The DLA model is important because it provides a basis for understanding a wide variety of phenomena such as electrodeposition, random dendritic growth, fluid–fluid displacement in Hele–Shaw cells and porous media, dissolution of porous media, dielectric breakdown and possibly biological processes such as the growth of nerve cells and blood vessels (for reviews see Ball 1986*a*; Meakin 1988; Matsushita 1989). Despite the apparent simplicity of the DLA model, it provides an important theoretical challenge that has not yet been fully met. Despite the introduction of several promising new approaches (Turkevich & Scher 1985; Ball *et al.* 1985; Halsey *et al.* 1986; Ball 1986*b*; Procaccia & Zeitak 1988; Bohr *et al.* 1989) we do not yet have a completely satisfactory theory for DLA. In fact, at the present time, there is still substantial uncertainty concerning even the correct qualitative description of DLA clusters generated by computer simulations and these uncertainties are not likely to be reduced in the absence of a better theoretical understanding.

Despite the practical difficulties of determining the asymptotic scaling behaviour of DLA clusters from finite (often quite small-scale) simulations, much of our knowledge and understanding of DLA still comes directly from computer simulations. The main purpose of this paper is to describe some recent advances in this area with an emphasis on simulations using spaces and lattices with euclidean dimensionalities $(d) \geqslant 3$, a subject that has been neglected in the past.

The DLA model

A simple two-dimensional square lattice diffusion-limited aggregation model is illustrated in figure 1. This figure shows a small cluster with a maximum radius of $r_{\max}$ measured from the original seed or growth site. Particles are launched one at a time from a randomly selected position on a 'launching' circle of radius $r_{\max}+5$ lattice units and follow random walk trajectories on the lattice. The particle may follow a trajectory (like t_1 in the figure) that eventually brings it to an unoccupied perimeter site. In this event the perimeter site is filled and the cluster grows. Alternatively, the particle may follow a trajectory (like t_2) that eventually moves it a long distance from the cluster (a distance of $kr_{\max}$, where k is 3 in figure 1). If this happens, the trajectory is terminated and a new random-walk trajectory is started from a randomly selected position on the launching circle. This procedure is repeated many times until a large cluster has been generated. Algorithms similar to this can be used to generate clusters containing a few thousand sites or particles (Witten & Sander 1981).

The efficiency of the DLA algorithm can be dramatically improved by allowing the random walker to take long steps when it is far from any of the sites occupied by the cluster (Ball & Brady 1985; Meakin 1985). If the random walker is enclosed

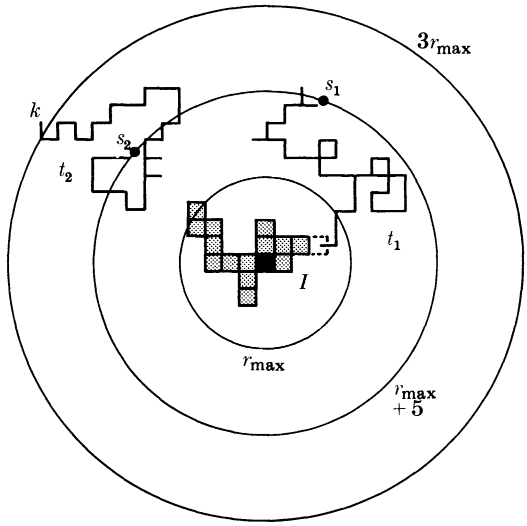

FIGURE 1. A simple model for two-dimensional square-lattice DLA. The filled square represents the original seed or growth site, the shaded squares represent sites that have already been filled by the growth process and the open site with broken border is an unoccupied perimeter site that is entered by the random walker following trajectory t_1.

in a region with no occupied (absorbing) sites or particles and with a simple shape (at the centre of a circle or square for example), we can replace the random walk within that region by a single step to an appropriately selected position at its edge. For example, if the random walker is the centre of an empty circle, we can immediately transfer the particle to a randomly selected position on the circle. To take advantage of the ensuing reduction in the number of steps in the random walk, we must have an efficient way of determining the distance from the random walker to nearest occupied site(s) on the cluster (Ball & Brady 1985; Meakin 1985). For two-dimensional clusters containing about 10^5 particles or sites this procedure reduced the computer time needed to grow a cluster by a factor of about 1000. In addition, the distance kr_{max} at which trajectories are terminated can be made very large (100 r_{max} is typical of recent algorithms) thus eliminating any bias that may arise by terminating the trajectories too close to the cluster. Clusters containing more than about 10^5 particles or sites cannot be represented by filled and empty sites on a square lattice. Instead, the cluster is represented by a hierarchy of 'maps' on different length scales (L_n) that are updated as needed as the cluster grows. The nth level map consists of a lattice whose elements represent regions of size $L_n \times L_n$. If the lattice element is 'empty' or 'off', then a random walker in this region can take a step of length L_n. If the lattice element is 'full' or 'on', then the more detailed map at level $n-1$ is examined to determine if the walker may take a step of length $L_{n-1} = \frac{1}{2}L_n$. The maps on shorter and shorter length scales are examined until the random walker is allowed to make a move or until the lowest level is reached. At the lowest (most detailed) level each element of the map contains information about the exact location of sites or particles that can be used to determine if the walker has contacted the cluster or if a step of the minimum length (one lattice unit for lattice models or a distance on the order of the particle radius for off-lattice models) should be made. To reduce information storage requirements only those portions of the lower-level, more detailed maps that are needed (regions close to the cluster) are constructed and updated. Figures 2 and 3 show two-dimensional square lattice and off-lattice clusters generated by

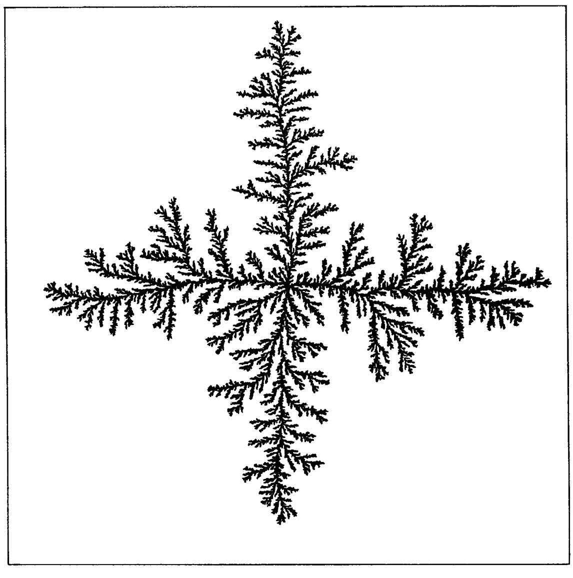

FIGURE 2. A 1.27×10^7 site square DLA cluster grown by using the algorithm of Ball & Brady (1985).

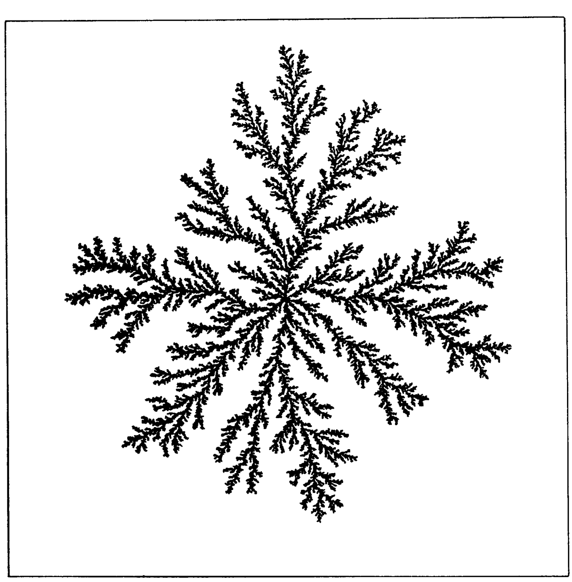

FIGURE 3. A 10^6 particle two-dimensional off-lattice DLA cluster.

using algorithms of this type. Figure 2 shows a 1.27×10^7 site square-lattice DLA cluster grown by using the algorithm of Ball & Brady (1985). In this case the effects of the weak lattice anisotropy on the overall shape of the cluster can easily be seen. Figure 3 shows a 10^6 particle off-lattice DLA cluster (Tolman & Meakin 1989). Similar algorithms have been developed for DLA in hypercubic lattices and spaces with dimensionalities of 3–8 (Tolman & Meakin 1989). The details concerning the structure of the hierarchy of maps and the way information is stored at the lowest level vary with d and the size of the clusters that are required. However, the general structure of the algorithms is very similar to that described above.

Results

Two-dimensional DLA

Computer simulations have been used quite extensively to investigate the structure of both lattice and off-lattice models for DLA. Much of this work has been reviewed quite recently (Meakin 1988) and the structure of square-lattice DLA clusters containing up to 4×10^6 sites has been investigated (Meakin *et al.* 1987). Here some more recent results are presented.

A quite large number (221) of 10^6 particle off-lattice DLA clusters were grown and the radius of gyration (R_g) of each of the clusters was determined for each 5% increment in the cluster size (s). The dependence of $\ln(R_g)$ on $\ln(s)$ is quite linear for clusters containing more than a few hundred particles and it is apparent that the dependence of R_g on s can be described very well by the power law

$$R_g \sim s^{\beta}. \tag{1}$$

The corresponding effective fractal dimensionality D_β ($D_\beta = 1/\beta$) was obtained by least-squares fitting the coordinates $(\ln(R_g), \ln(s))$ by a straight line for clusters in the size range $s_1 \leqslant s \leqslant s_2$ ($s_2 = 1.05^9\, s_1$). Figure 4 shows the dependence of the exponent β on s $((s_1 s_2)^{\frac{1}{2}})$ obtained in this way.

It is apparent that β is essentially independent of s. By least-squares fitting the dependence of R_g on s for clusters in the size range $10^5 \leqslant s \leqslant 10^6$ a value of 0.5837 ± 0.0014 was obtained for β corresponding to a fractal dimensionality of

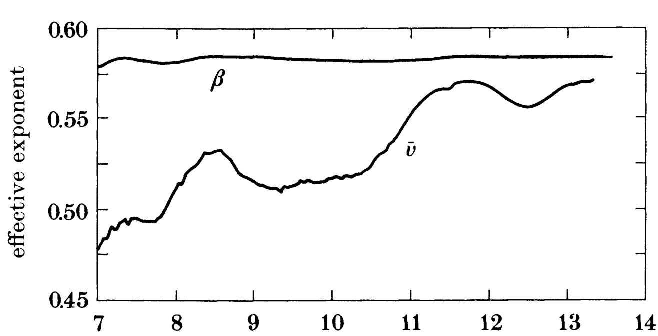

FIGURE 4. Dependence of the exponents β and $\bar{\upsilon}$, which describe the growth of the radius of gyration and width of the active zone on the cluster size s. The results for β and $\bar{\upsilon}$ were obtained from 221 and 93 10^6 particle off-lattice two-dimensional DLA clusters respectively.

1.713 ± 0.004. This result is in excellent agreement with that obtained earlier ($\beta \simeq 0.584$; Meakin & Sander 1985, and unpublished results). It appears that this is by far the most accurately known exponent associated with any DLA model and there is no evidence that the asymptotic value for the fractal dimensionality D_β is different from that obtained from quite small clusters for two-dimensional off-lattice DLA.

For some of the 10^6 particle clusters we also measured the width of the active zone, ξ (variance in the deposition radius; Plischke & Racz 1984) as a function of s. The dependence of ξ on s can be represented by the power law

$$\xi \sim s^{\bar{v}}. \tag{2}$$

Using quite small square-lattice DLA clusters, Plischke & Racz (1984) found an effective value of about 0.484 for $\bar{v}$. However, the effective value for $\bar{v}$ increases with increasing cluster size and reaches a value of about 0.54 for off-lattice clusters in the size range 25000–50000 particles. This suggests that in the asymptotic ($s \rightarrow \infty$) limit $\bar{v}$ might be equal to β (Meakin & Sander 1985). Figure 4 also shows the results obtained for $\bar{v}$ from 93 10^6 particle off-lattice clusters. It is apparent that $\bar{v}$ continues to increase with increasing cluster size and reaches a value quite close to that found for β for clusters containing 10^6 particles.

We have also investigated the cluster-size distribution (N_s) for the incremental growth in DLA clusters. The clusters are grown to a size of s_1 sites or particles and then an additional s_2 particle is added. The quantity N_s is then the number of clusters of size s in the incremental growth consisting of the last s_2 particles added. We might expect (Racz & Vicsek 1983; Matsushita & Meakin 1988) that the dependence of N_s on s should be a power law

$$N_s \sim s^{-\tau} \tag{3}$$

with τ given by

$$\tau = 1 + D_{\mathrm{i}}/D, \tag{4}$$

where D_{i} is the dimensionality of the old growth/new growth interface (Meakin & Witten 1983). Figure 5 shows the results obtained from 104 10^5 particle off-lattice DLA clusters for $s_1 + s_2 = 10^5$ and three different values for s_1 (25000, 50000 and 75000). It is apparent from figure 5 that the dependence of N_s on s is not really algebraic. This may be a consequence of the fact that the old growth/new growth interface is not fully saturated and there are a lot of small clusters that will grow as s_2 is further increased. However, the results shown in figure 5 do suggest that τ might have a value of about 1.68. It has been argued that the fractal dimensionality of the old growth/new growth interface should be 1.0 and in this event we would expect to obtain a value of $1+\beta$ or about 1.58 for τ. A value of 1.68 indicates that D_{i} should have a value of about 1.16, which is in good agreement with simulation results (a value of 1.13 was found for D_{i} by Meakin & Witten (1983) using square-lattice DLA clusters). Results very similar to those shown in figure 5 were obtained from 435 10^5 site square-lattice DLA clusters.

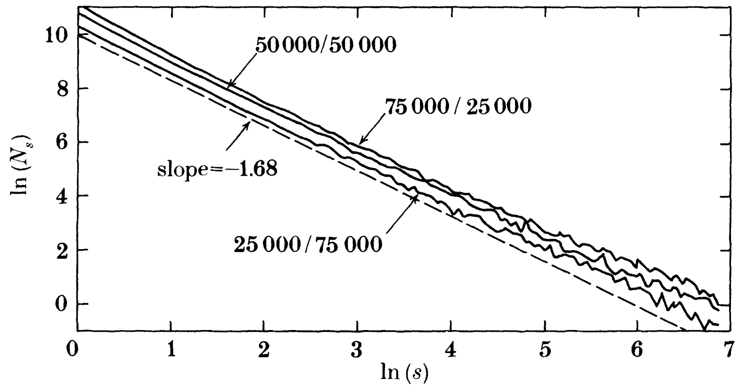

FIGURE 5. The cluster-size distributions N_s for the incremental growth obtained from 104 10^5 particle off-lattice DLA clusters. The three curves show the results obtained for $s_1 = 25000$, 50000 and 75000 and $s_2 = 75000$, 50000 and 25000 respectively, where s_1 is the number of particles in the old growth and s_2 is the number of particles in the new or incremental growth.

Three-dimensional DLA

Relatively few studies of the structure of three-dimensional DLA clusters have been carried out. An effective fractal dimensionality (D_β) of about 2.49 has been obtained from approximately 100 50000 site cubic lattice DLA clusters (P. Meakin & L. M. Sander, unpublished work). Figure 6 shows a projection of a 3×10^6 site cubic-lattice DLA cluster work onto a plane and a cross section through the origin of the cluster along a parallel plane. The effects of the weak cubic-lattice anisotropy (that are not readily apparent in clusters containing 10^4 sites) are quite evident in

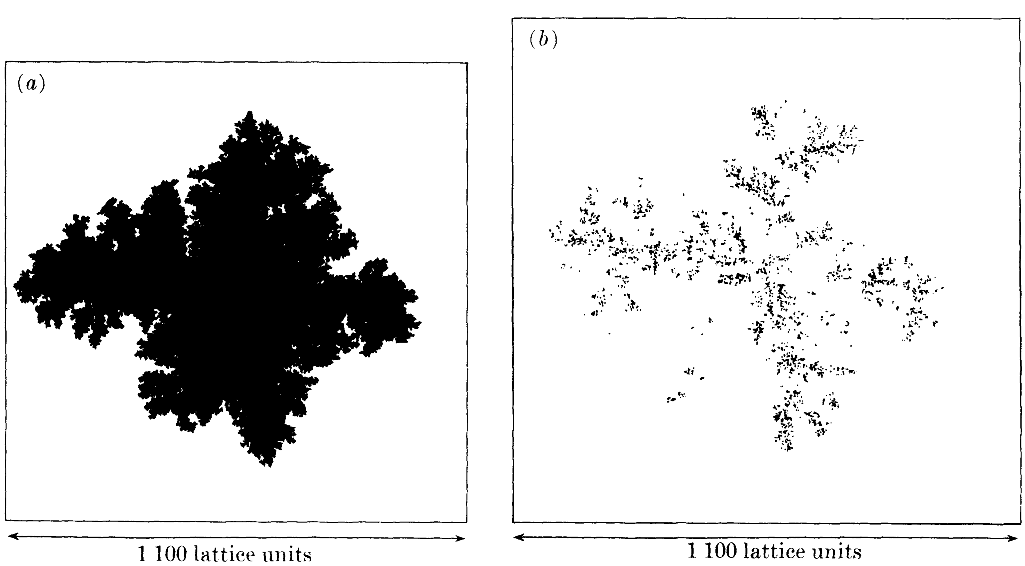

FIGURE 6. A projection (a) and a cross section through the origin (b) for a 3×10^6 site cubic-lattice DLA cluster.

this figure. The overall dimension (linear size) of this cluster is the same as that for square-lattice DLA clusters containing about 5×10^4 sites. It is at about this size that square-lattice DLA clusters have a diamond-like shape which eventually evolves into a cross-like shape (figure 1). Because a quite large amount of computer time (about 10 h of CPU time on an IBM 3090 computer) is required to grow 3×10^6 site clusters, more quantitative results were obtained from 98 1250000 site cubic-lattice DLA clusters, 138 300000 site clusters, 482 100000 site clusters and 169 100000 particle three-dimensional off-lattice DLA clusters. Figure 7 shows the cluster-size dependence of the effective fractal dimensionality D_β obtained from the 100000 particle off-lattice clusters and the 1250000 site cubic-lattice clusters. For the off-lattice clusters the fractal dimensionality is very close to the value of 2.50 ($(d^2+1)/(d+1)$) obtained from mean field theories (Muthukumar 1983; Tokuyama & Kawasaki 1984; Matsushita *et al.* 1986). For the cubic-lattice model D_β has an effective value slightly smaller than 2.50 (about 2.48 for 10^6 site clusters). This is presumably a consequence of lattice anisotropy and similar effects have been seen in square-lattice DLA (Meakin 1986; Meakin *et al.* 1987) but it appears from figure 7 that enormous clusters would be required to see an effective fractal dimensionality significantly smaller than 2.48.

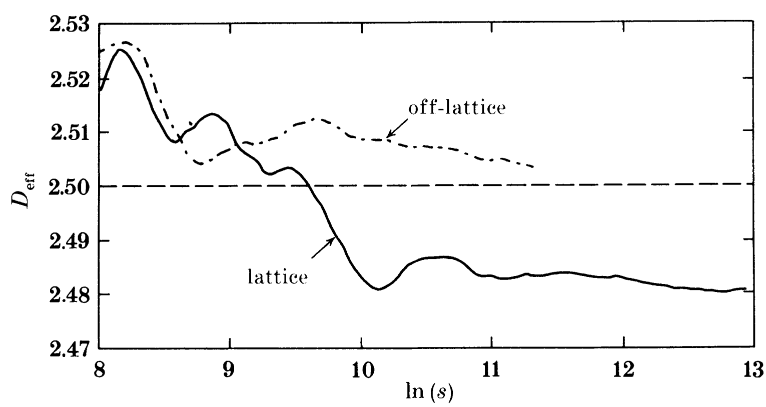

FIGURE 7. Dependence of the effective fractal dimensionality (D_β) on the cluster size s obtained from off-lattice and cubic-lattice DLA clusters. The broken line indicates the mean field theory value of 2.50.

We have also measured the width of the active zone (ξ) for the three-dimensional DLA clusters. For the off-lattice model $\bar{\nu}$ increases from a value of about 0.31 for clusters containing a few thousand particles (a result in good agreement with that obtained by Racz & Plischke (1985) from cubic-lattice DLA clusters of the same size) to a value of about 0.34 for clusters containing about 10^5 particles. For cubic-lattice DLA the exponent $\bar{\nu}$ continues to increase with increasing s above 10^5 sites, but in this case the contribution of the growing cluster anisotropy apparent in figure 6 may be important.

Figure 8 shows the cluster size distribution in the incremental growth obtained from 138 300000 site cubic-lattice clusters for old growth/new growth sizes of 75000/225000, 150000/150000 and 225000/75000 sites. Similar results were

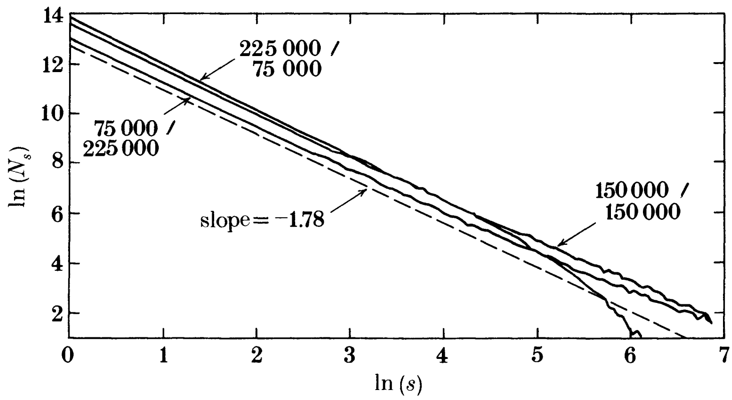

FIGURE 8. Cluster-size distribution in the incremental growth obtained from 300000 site cubic-lattice DLA clusters. The sizes of the old growth and the new growth or incremental growth (s_1/s_2) are shown by each of the curves.

obtained from larger (6×10^5 sites) and smaller (10^5 sites) clusters and from off-lattice simulations. The cluster-size distributions in the incremental growth cannot be described accurately by equation (3) (with a cut-off at large sizes). However, it is apparent from figure 8 that the size distribution can be described approximately by equation (3) with an exponent (τ) of about 1.8. This would correspond to an interface dimensionality of about 2.0. This result is in accord with a direct measurement of D_i (Meakin & Witten 1983) with relatively small cubic-lattice clusters.

DLA *in dimensionalities greater than three*

Clusters containing about 10^5 particles or sites have been grown for $d = 4$–6 by using both off-lattice and hypercubic-lattice models. Figure 9 shows a projection and a cross section for a 10^6 site four-dimensional hypercubic-lattice model cluster. The effects of the lattice anisotropy can be clearly seen in figure 9*a*, *b*. For all of these models the dependence of the cluster radii of gyration on cluster size has been measured to obtain the effective fractal dimensionality D_β. For $d = 2$–5 the effective value for D_β is smaller than the 'mean field' value of $(d^2+1)/(d+1)$ for small clusters. As the cluster size increases, D_β increases and exceeds the mean field value. As the cluster continues to grow, D_β decreases and eventually reaches a value very close to the mean field value. For lattice models a substantially smaller effective fractal dimensionality can be seen for large two-dimensional clusters and a slightly smaller value for $d = 3$ and 4 is reached at the largest attainable cluster sizes. Figure 10 shows the dependence of D_β on s obtained from 82 five-dimensional off-lattice clusters and 241 five-dimensional hypercubic-lattice model clusters each containing 10^5 particles or sites. In this case the effective fractal dimensionality for the largest cluster sizes is quite close to the mean field value of 4.333 for both the lattice and off-lattice models. It appears that for $d \geqslant 5$ clusters containing 10^5 or fewer sites are in the fluctuation-dominated régime and that the hypercubic-lattice anisotropy has little effect on the structure of clusters of this size.

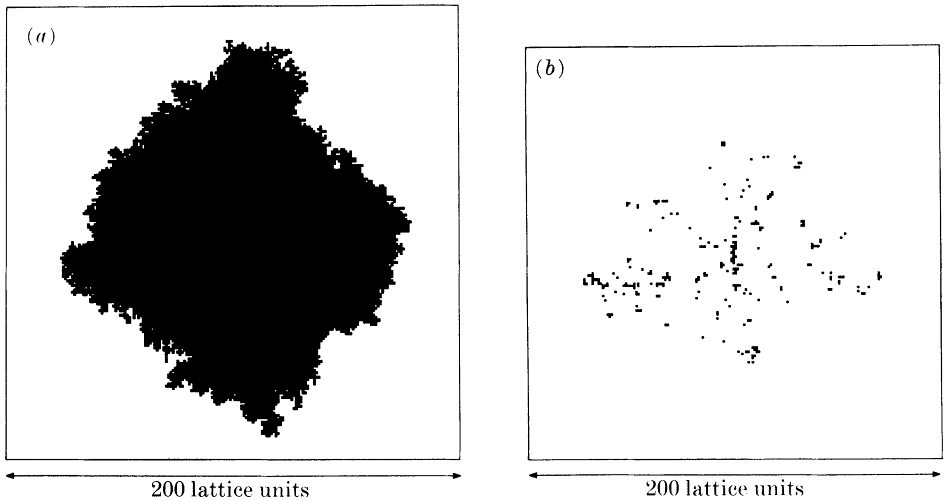

FIGURE 9. A 10^6 site four-dimensional hypercubic-lattice DLA cluster. Figure 9*a* shows a projection of the cluster onto a plane and (*b*) shows a cross section through the cluster that intersects the original seed or growth site.

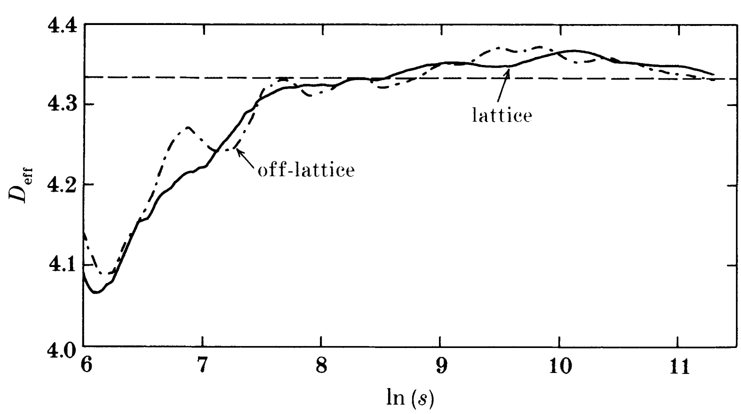

FIGURE 10. The cluster-size dependence of the effective fractal dimensionality D_β obtained from 100000 sites or particles five-dimensional hypercubic-lattice and off-lattice DLA clusters. The mean field theory value of 4.333 is indicated by a broken horizontal line.

For $d \geqslant 6$ we appear to see only the first part of the dependence of D_β on s. D_β increases to a value that exceeds the mean field value of $(d^2+1)/(d+1)$ but does not decrease. Figure 11 shows an example of this behaviour obtained from 28 70000 site seven-dimensional hypercubic-lattice DLA clusters. It is not surprising that the asymptotic behaviour is not seen in this case because a cluster of 70000 sites has a radius of gyration of only about 6.5 lattice units and a maximum radius of about twice that. For $d \geqslant 6$ the dependence of D_β on s is the same (within the statistical uncertainties of our simulations) for both lattice and off-lattice models.

The cluster-size distribution has been measured for the incremental growth in both the four- and five-dimensional models. Values of about 1.70 and 1.65

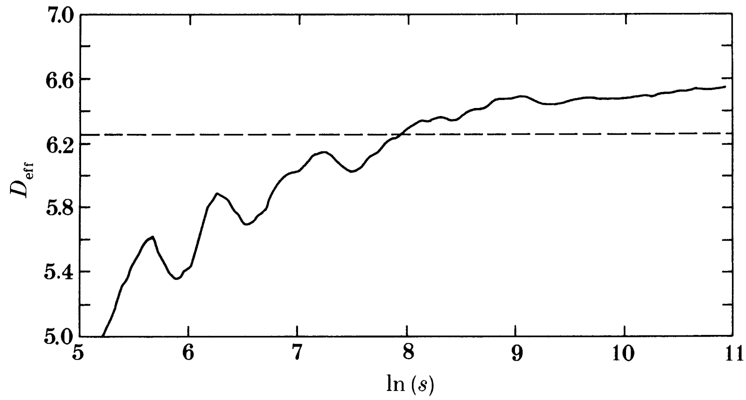

FIGURE 11. Cluster-size dependence of the effective fractal dimensionality obtained from seven-dimensional hypercubic-lattice DLA clusters. The horizontal line indicates the fractal dimensionality of 6.25 predicted by the mean field theory.

respectively were obtained for the exponent τ, which describes the cluster-size distribution for the incremental growth. With equation (4) these results indicate an interface dimensionality (D_{i}) of about 2.4 for $d = 4$ and about 2.8 for $d = 5$. Because of the small spatial extent (overall size) of these clusters, the uncertainties associated with these values are quite large.

Noise-reduced DLA

In the noise-reduced DLA model (Tang 1985; Kertesz & Vicsek 1985) random walkers are used to contact the surface of a growing cluster by using the DLA

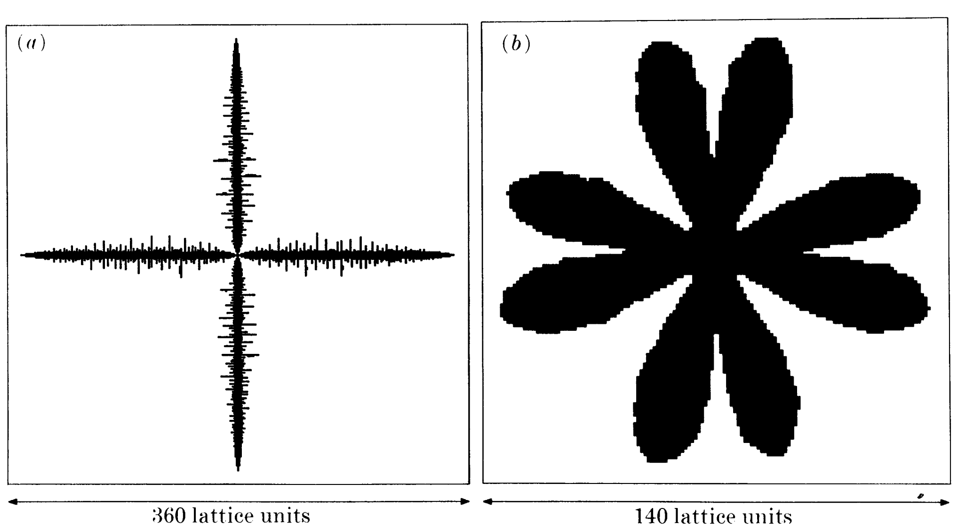

FIGURE 12. Clusters grown with random walkers and a noise-reduction parameter (m) of 10000 with DLA boundary conditions (a) and dielectric breakdown model boundary conditions (b). Both clusters eventually evolve into complex patterns, but for (a) this occurs via side branching and for (b) via tip splitting.

algorithm. After a random walker has contacted an unoccupied perimeter site, a record of the contact is kept (the contact score associated with that site is incremented by one) and a new random walker is started from a randomly selected position on the launching circle. Only after a perimeter site has been contacted m times in this fashion does it become occupied. After a growth event all of the new perimeter sites are given a score of zero, but the old perimeter sites retain their contact scores, which continue to accumulate. In this model $m = 1$ corresponds to ordinary DLA. Clusters grown in this fashion with small values of m strongly resemble much larger clusters grown with $m = 1$ (figure 1) (Kertesz *et al.* 1986). It is widely believed, but has never been rigorously shown, that the structure of small clusters grown with noise reduction faithfully represents the structure of much

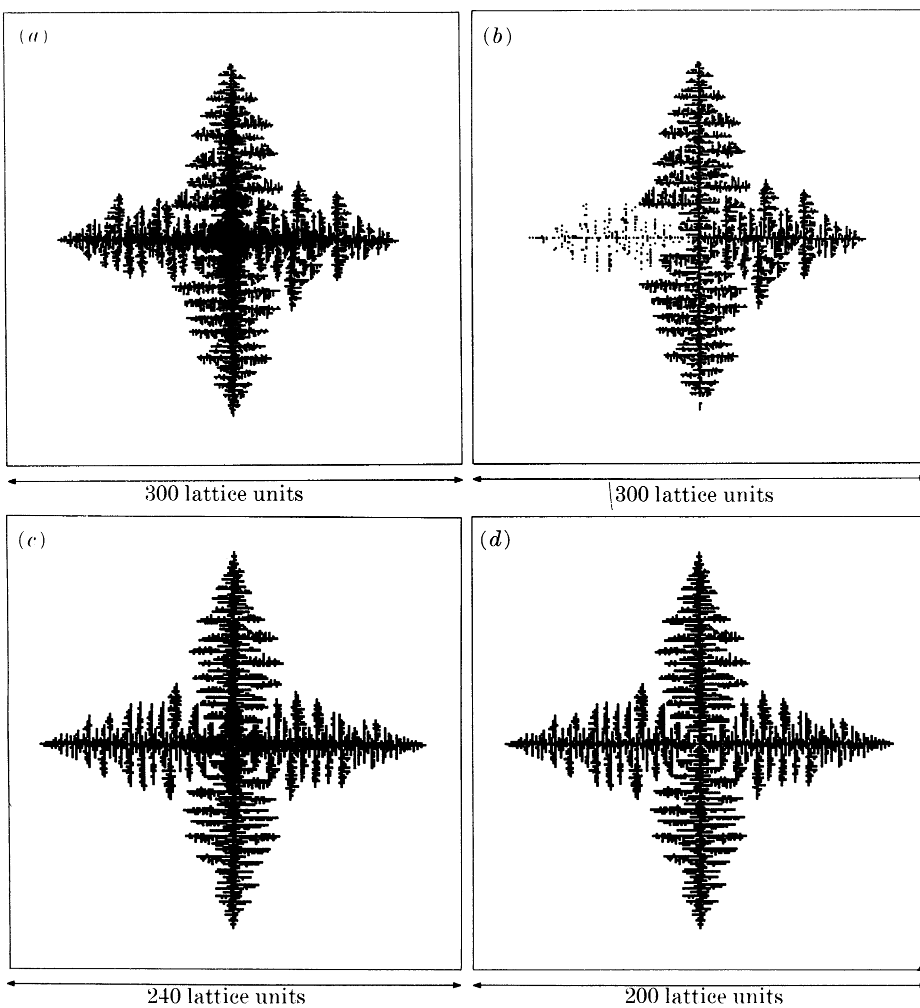

FIGURE 13. Noise-reduced cubic-lattice DLA clusters. (a, b) Projection and a cross section for a 43253 site cluster grown with a noise reduction parameter of 30. In (b) the cross section is through the cluster origin and one of the cluster arms (on the left-hand side) deviates from this plane. (c, d) A projection and cross section for a 28350 site cluster grown with a noise-reduction parameter (m) of 100.

larger clusters grown without noise reduction and that the asymptotic behaviour of lattice models for DLA can be investigated by using noise reduction.

In the dielectric breakdown model the growth probability P_i associated with the ith unoccupied perimeter site is given by

$$P_i \sim n\phi_i, \tag{5}$$

where ϕ_i is the potential (obtained by solving the discretized Laplace equation) at the ith perimeter site and n is the number of occupied nearest neighbours. This model can be implemented with random walkers by allowing the walker to step onto the cluster and occupy the previously visited unoccupied perimeter site (L. Pietronero, personal communication 1985). Ordinarily the differences between the DLA and dielectric breakdown model clusters that result from the different local boundary conditions are quite small. However, for noise-reduced DLA and related models (Nittmann & Stanley 1985) these local boundary conditions become more important (R. C. Ball, personal communication 1987; L. Pietronero, personal communication 1987). This is illustrated in figure 12, which shows clusters grown by using both models (with random walkers) with a noise reduction parameter (m) of 10000. Noise-reduced DLA clusters first grow as compact crosses that begin to side branch when the cluster size s reaches a value proportional to $(\log(m))^3$ (Eckmann *et al.* 1989). The noise-reduced DLA cluster shown in figure 12a has just passed this stage. The dielectric breakdown model cluster on the other hand (figure 12b) grows four arms that undergo tip splitting rather than side branching.

We have also done noise-reduced DLA simulations with three-dimensional cubic and four-dimensional hypercubic lattices. Figure 13 shows a projection and cross section for a 43253 site cubic-lattice cluster grown with a noise-reduction parameter (m) of 30 and for a 28350 site cluster grown with a noise-reduction parameter of 100. The structures generated by these models resemble quite closely those generated by the disorderly growth of cubic crystals under non-equilibrium conditions (Garcia-Ruiz 1986).

Figure 14 shows the dependence of the cluster radius of gyration on the cluster size plotted as $\ln(R_g/s^{\beta^*})$ against $\ln(s)$ where β^* is the mean field radius of gyration

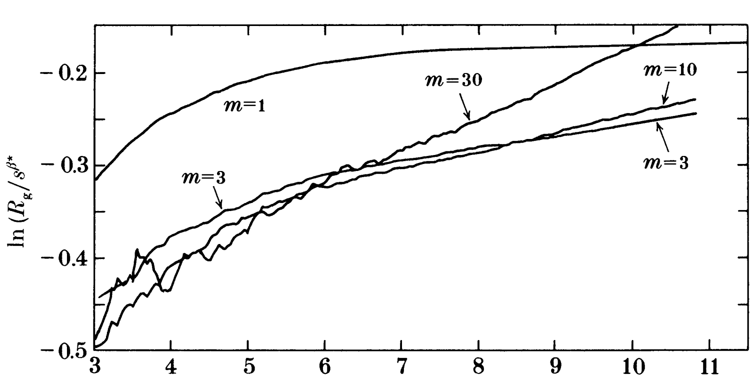

FIGURE 14. Dependence of $\ln(R_g/s^{\beta^*})$ on $\ln(s)$ obtained from four-dimensional DLA clusters with noise-reduction parameters of $m = 1$ (ordinary DLA), 3, 10 and 30. Here β^* is the mean field radius of gyration exponent ($\beta^* = (d+1)/(d^2+1) = \frac{5}{17}$).

exponent ($\frac{5}{17}$). The dependence of $\ln(R_g/\beta^*)$ on $\ln(s)$ seems quite linear (in contrast to the behaviour found for $d = 2$ where strong oscillations are seen). For the largest value of m ($m = 30$) the effective fractal dimensionality obtained from the dependence of R_g on s is about 3.01.

Conclusions

Since the DLA model was introduced about seven years ago by Witten & Sander (1981), the size of clusters that we are able to grow and the speed with which they can grow has increased substantially. This has allowed us to reduce uncertainties caused by both statistical and finite size effects. Part of this improvement has come about as a result of much larger amounts of computer time on much faster computers. However, the major contribution has come from improved algorithms. It seems unlikely that the speed, information storage capability and availability of computers will increase sufficiently during the next 5–10 years to allow us to approach significantly closer to the asymptotic limit and thus resolve many of the outstanding questions concerning both off-lattice and lattice models for DLA. Based on the results reported here, it seems that truly enormous clusters will be required to see the asymptotic effects of lattice anisotropy. It is also probable that the algorithms used to grow DLA clusters will improve at a slower rate during the next five years than they did during the past five years. Clearly a less 'brute force' approach will be needed if further progress is to be made. The use of strategies like noise reduction to see the asymptotic behaviour of DLA could be extremely valuable. However, a better theoretical understanding is needed to use this approach with confidence.

The use of models closely related to DLA to explore non-equilibrium growth processes is still a rapidly growing area. Models of this type can be of considerable value in developing a better understanding of specific physical or chemical processes. Unfortunately, some of the work in this area seems to be poorly motivated and to have little to do with physical reality.

A complete theoretical understanding of DLA still eludes us. A satisfactory theory for a two-dimensional noise-reduced DLA (Eckmann *et al.* 1989) has been developed that leads to quite detailed predictions concerning the cluster structures that have been verified by using computer simulation. Although this is an encouraging development, it does not provide us with an understanding of DLA in the fluctuation dominated or off-lattice régime.

Because of the importance of DLA as a paradigm for non-equilibrium growth and the theoretical challenge posed by this simple-to-define model, it is likely to be the subject of considerable theoretical attention during the next few years. The simulation results presented above for the fractal dimensionality D_β are consistent with the predictions of several mean theories ($D = (d^2+1)/(d+1)$) in the off-lattice or fluctuation dominated régime for $d > 2$. However, for $d = 2$ there is a clear discrepancy between the mean field predictions and the simulation results. If the mean theoretical result is correct for $D \geqslant 3$, a more rigorous justification is needed.

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Discussion

A. Blumen (*University of Bayreuth, F.R.G.*). Is there a transparent argument to understand that a weak anisotropy on a microscopic scale, such as an underlying lattice, leads to such prominent effects (diamond-shaped DLA) on the macroscopic scale?

P. Meakin. Early, small-scale simulations of DLA on a square lattice provided no obvious indications of anisotropy. When larger-scale simulations (Ball & Brady 1985) indicated that lattice anisotropy had an important effect on the overall cluster geometry, this came as a surprise. A recent theoretical analysis of noise-reduced DLA with anisotropy (Eckmann *et al.* 1989) provides an understanding of the shape of DLA clusters with large amounts of noise reduction. If it is accepted that noise reduced DLA reflects the asymptotic (large-size) structure of ordinary square lattice DLA, then the cross-like structure of large DLA clusters (figure 2) seems reasonable. In the noise-reduced DLA model, each site in the cluster can be considered to represent a region of a DLA cluster containing many sites (i.e. the growth process has been 'coarse-grained'). As the noise-reduction parameter (m) is increased, each lattice in the noise-reduced cluster represents a larger and larger region in the ordinary DLA cluster. However, there is no firm theoretical foundation for the correspondence between noise-reduced DLA and square lattice DLA and we do not have a quantitative relation between the noise-reduction parameter (m) and the scale of coarse graining. At present, there is no theoretical understanding (or quantitative simulation results) of how the effect of the weak lattice anisotropy grows as the cluster size increases. However, simulations addressing this question are in progress.

The growth of the anisotropy in lattice models for DLA can probably be understood in terms of the Mullins–Sekerka instability (Mullins & Sekerka 1963). Ball (1986*a*) has discussed how the Mullins–Sekerka instability is modified for the growth of symmetrically arranged arms. This analysis indicates that four arms should be stable and is consistent with the asymptotic shape of square lattice DLA.

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