1 Summary

- This document is a first step towards looking at the effects of population or cohort-specific productivity shocks within an RBC framework. In Section 2, I first set out a very simple two-generation general equilibrium OLG model with exogenous mortality between generations. This is mainly to help think about shocks in a closed-form framework and to develop intuition for how one might model these mortality issues. In Section 3, I then develop a slightly richer (though still too simple to take seriously) model that has three overlapping generations and allows for endogenous labour supply decisions. I then parameterise and simulate that model to study the effect of shocks to the between-generation mortality rates, the productivity of the middle-aged cohort, and the birth rate.

- The model is still overly simple and there are many things that would need to be given more thought before it could be taken to data.[1] With that said, it is quite helpful to (i) develop intuition about the effects of these types of shocks (ii) force the reader to think about the different ways that one might model this and the pros and cons of this approach.

2 Two Period OLG Model with Mortality (Closed-Form)

- There are two generations, young (y) and old (o). Young agents supply a fixed unit of labour and divide their wages between consumption and savings. Old agents spend their savings (and do not supply labour).[2]

- In period t, there are N_t^y young agents and N_t^o old agents. In the next period, an exogenous share $1 - d_{t+1}$ of the young agents survive and become old agents, while the remaining d_{t+1} die. The rate of growth of young agents is exogenous and given by n_{t+1}. Finally, TFP A_t grows at exogenous rate g_{t+1}. Transitions are thus given by:

$$
\begin{aligned}
N_{t+1}^o &= (1 - d_{t+1}) N_t^y \\
N_{t+1}^y &= (1 + n_{t+1}) N_t^y \\
A_{t+1} &= (1 + g_{t+1}) A_t
\end{aligned}
$$

- The government distributes the capital that was owned by those who were young in period $t-1$ but did not survive to period t evenly among the surviving old people.[3] This takes place through lump sum transfers, τ_t, which households take as exogenous. Defining s_{t-1} as savings per young in period $t-1$ and r_t as the return on savings, transfers are given by

$$
\begin{aligned}
\tau_t &= \frac{s_{t-1} N_{t-1}^y d_t (1 + r_t)}{N_t^o} \\
&= \frac{d_t}{1 - d_t} s_{t-1} (1 + r_t)
\end{aligned}
$$

- Households solve the following problem:

$$
\max_{\substack{c_t^y \geq 0, \\ c_{t+1}^o \geq 0}} \ln(c_t^y) + \beta (1 - d_{t+1}) \ln(c_{t+1}^o) \tag{1}
$$

[1] High among these, though certainly not an exhaustive list, are to allow partial depreciation, add uncertainty and productivity shocks, think seriously about the most sensible way of modelling generations (particularly with regard to the birth rate), take parameterisation more seriously, and to think about the other major changes happening at the time of the war.

[2] This initial model is a special case of Ludwig and Vogel (2010) in J Pop Econ. Be careful with this paper. There is a typo in most of the main propositions (confirmed to me by the authors over email), meaning that the analytical solutions here do not exactly correspond to a specialised version of their model.

[3] A more general version would allow agents to access an annuity market to insure against the mortality risk.

s.t.

$$c_t^y + s_t = w_t \tag{2}$$
$$c_{t+1}^o = (1 + r_{t+1})s_t + \tau_{t+1} \tag{3}$$

- Taking the FOC gives the Euler equation

$$\frac{c_{t+1}^o}{c_t^y} = \beta(1 - d_{t+1})(1 + r_{t+1}) \tag{4}$$

- Re-arranging and solving give slight modifications of the standard solutions for the two-period lifecycle model:

$$c_t^y = \frac{w_t + \frac{\tau_{t+1}}{1+r_{t+1}}}{1 + \beta(1 - d_{t+1})}$$
$$s_t = \frac{\beta(1 - d_{t+1})w_t - \frac{\tau_{t+1}}{1+r_{t+1}}}{1 + \beta(1 - d_{t+1})}$$
$$c_{t+1}^o = (\beta(1 - d_{t+1})(1 + r_{t+1}))\frac{w_t + \frac{\tau_{t+1}}{1+r_{t+1}}}{1 + \beta(1 - d_{t+1})}$$

- Firms operate in a competitive market and have access to a Cobb-Douglas production technology given by $Y_t = K_t^\alpha (A_t N_t)^{1-\alpha}$. All capital depreciates at the end of the period, so competitive factor markets imply that factors earn their marginal product:

$$1 + r_t = \alpha k_t^{\alpha-1}$$
$$w_t = (1 - \alpha)A_t k_t^\alpha$$

where $k_t \equiv \frac{K_t}{A_t N_t^y}$.

- Household savings are invested in the capital market so market clearing implies $K_{t+1} = s_t N_t^y$.

- The transition equation for capital is thus given by

$$\begin{aligned} k_{t+1}(k_t) &= \frac{s_t N_t^y}{A_{t+1} N_{t+1}^o} \\ &= \frac{s_t}{A_{t+1}(1 + n_{t+1})} \\ &= \frac{\beta(1 - d_{t+1})w_t - \frac{\tau_{t+1}}{1+r_{t+1}}}{A_{t+1}(1 + n_{t+1})(1 + \beta(1 - d_{t+1}))} \\ &= \frac{\beta(1 - d_{t+1})^2(1 - \alpha)}{(1 + g_{t+1})(1 + n_{t+1})(1 + \beta(1 - d_{t+1})^2)} k_t^\alpha \end{aligned} \tag{5}$$

and final consumption and savings values are given by

$$c_t^y(k_t) = \frac{(1 - \alpha)A_t}{1 + \beta(1 - d_{t+1})^2} k_t^\alpha$$
$$s_t(k_t) = \frac{\beta(1 - d_{t+1})^2(1 - \alpha)A_t}{1 + \beta(1 - d_{t+1})^2} k_t^\alpha$$
$$c_{t+1}^o(k_t) = \frac{\beta^\alpha \alpha (1 - d_{t+1})^{2\alpha-1} A_t (1 - \alpha)^\alpha (1 + n_{t+1})^{1-\alpha} (1 + g_{t+1})^{1-\alpha}}{(1 + \beta(1 - d_{t+1})^2)^\alpha} k_t^{\alpha^2}$$

Note that the expression for savings is always positive, so the household will always be a net saver (otherwise they would have no income when old, which is not a solution).

- Note that c_t^y and s_t do not depend on n_{t+1} or g_{t+1} (these operate in the same way). This is because the income and substitution effects cancel each other out exactly (since we are using log utility and full depreciation).[4] So a known exogenous shock to the labour supply next period does not increase or decrease savings.
- However, c_t^y is increasing in the death rate d_{t+1} and s_t is decreasing in the death rate, which captures the fact that the household is effectively discounting the future at a higher rate and thus prefers to consume now. The effect on c_{t+1}^o is ambiguous. On the one hand, the household saved less last period. On the other hand, they receive more from the government if they do survive. If $\alpha \leq 0.5$ then c_{t+1}^o unambiguously increases if d_{t+1} rises; if $\alpha > 0.5$ then the effect is ambiguous - it is more likely to be positive for low β and (locally) high d_{t+1}, and negative otherwise. Intuitively, if α is low then savings are high and thus a small change in the death rate can have a large change in what the household gets paid.
- This model does not allow us to say much about unanticipated shocks to the death rate. This is because the old only provide capital, and we have assumed that the capital stock is not affected since it just gets redistributed. The only effect that this shock would have would be to make the surviving old people richer as they would get more capital. If there was instead an unanticipated shock to the birth rate, then the immediate effect would be that the old people would be worse off (as their capital would have a lower return due to the limited labour supply) while the young people would be better off (as there is now more capital).

2.1 Extension: Labour Supply of Old Agents

- One extension is to allow the old generation to work. In particular, suppose that they also supply labour inelastically, and output is now given by $Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$ where $L_t \equiv N_t^y + \lambda_t N_t^o$. We might think of impairment (both physical and mental) among survivors as a negative shock to λ_t.
- The budget constraints now become

$$\begin{aligned} c_t^y + s_t &= w_t \\ c_{t+1}^o &= (1 + r_{t+1})s_t + \tau_{t+1} + \lambda_{t+1} w_{t+1} \end{aligned}$$

- The presence of this additional term makes the algebra somewhat lengthier, but it remains tractable. The initial expression for savings become

$$s_t = \frac{\beta(1 - d_{t+1})w_t - \frac{\tau_{t+1} + \lambda_{t+1} w_{t+1}}{1 + r_{t+1}}}{1 + \beta(1 - d_{t+1})}$$

- Solving yields the capital transition equation, expressing in terms of capital per labour unit, as

$$\begin{aligned} k_{t+1}(k_t) &= \frac{\beta(1 - d_{t+1})^2 \alpha(1 - \alpha)}{(1 + g_{t+1})\left[\alpha(1 + n_{t+1})(1 + \beta(1 - d_{t+1})^2) + \lambda_{t+1}(1 - d_{t+1})(1 + \alpha\beta(1 - d_{t+1})^2)\right]} k_t^\alpha \\ k_{t+1}(k_t) &= \frac{\alpha(1 - \alpha)}{(1 + g_{t+1})\left[\alpha(1 + n_{t+1})\phi_{t+1} + \lambda_{t+1}(1 - d_{t+1})(\phi_{t+1} - (1 - \alpha))\right]} k_t^\alpha \end{aligned}$$

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[4] For $u(c) = \frac{c^{1-\nu} - 1}{1 - \nu}$ we get $c_t^y = \frac{w_t + \frac{\tau_{t+1}}{1 + r_{t+1}}}{1 + \beta(1 - d_{t+1})(1 + r_{t+1})^{\frac{1-\nu}{\nu}}}$. Since the $(1 + r_{t+1})$ in the numerator will cancel in equilibrium from τ_{t+1}, we now see that this is responsive to the return on savings. For $\nu < 1$, the substitution effect will dominate and an increase in n_{t+1}, which increases r_{t+1}, will lead to an increase in savings and a decrease in consumption while young.

where $\phi_{t+1} \equiv 1 + \frac{1}{\beta(1-d_{t+1})^2}$.

- We can see that this nests the simple version as the special case where $\lambda_t = 0$ for all t. After a lot of algebra, we can also get the final savings as

$$s_t = \frac{\alpha(1-\alpha)A_t(\frac{1+n_{t+1}}{1-d_{t+1}} + \lambda_{t+1})}{\alpha\frac{1+n_{t+1}}{1-d_{t+1}}\phi_{t+1} + \lambda_{t+1}(\phi_{t+1} - (1-\alpha))} k_t^\alpha$$

$$s_t = \frac{\beta(1-d_{t+1})^2(\frac{1+n_{t+1}}{1-d_{t+1}} + \lambda_{t+1})}{\beta(1-d_{t+1})^2(\frac{1+n_{t+1}}{1-d_{t+1}} + \lambda_{t+1}) + \frac{1+n_{t+1}}{1-d_{t+1}} + \frac{\lambda_{t+1}}{\alpha}}(1-\alpha)A_t k_t^\alpha$$

- I leave aside more detailed analysis of this model for now, since analytical methods will become messy and simulation methods are used in the next section for a richer model. If we pursue this further, it may be worth studying this case in more detail (and possibly trying to extend to three generations).

3 Three Generations with Endogenous Labour

- This section makes two main changes and then simulates the model, as well as a few shocks, using Dynare. The first change is that I allow for three generations. I think it is probably possible to get a closed-form solution for this akin to the one in the previous sections, though it would involve a considerable amount of algebra. The second change is that I make labour supply endogenous. Endogenous labour supply is almost impossible to get in closed form (the only – very special – case that I am aware of is only tractable because it ends up with labour supply as always constant).

- Agents live for three periods: young, middle, and old. Labour supply is chosen by both young and middle, where λ_t determines the relative productivity of middle compared to young. There are now two death rates: d_t^m and d_t^o which capture the transitions from young to middle and middle to old, respectively. Government re-allocation of capital behaves in the same way as before, where the re-allocation is within the generation, so we have τ_t^m and τ_t^o. Capital is still rented each period from savings and fully depreciates at the end of the period.

- To get a BGP, we need preferences that conform with the restrictions in KPR 1988 in JME. For simplicity, I use the special case where $\sigma \to 1$, which is just log utility for consumption and CRRA utility of leisure with parameter $\mu \neq 1$. At some point, it will probably be best to use the Hansen-Rogerson labour setup as we are likely talking about high aggregate employment fluctations with low individual labour supply elasticity. I continue to employ no uncertainty for simplicity. Consumers thus solve the following problem:

$$\begin{aligned}
\max_{\substack{c_t^y, c_{t+1}^m, c_{t+2}^o \\ h_t^y, h_{t+1}^m}} \quad & \ln(c_t^y) + \eta\frac{(1-h_t^y)^{\mu-1} - 1}{\mu - 1} \\
& + \beta(1-d_{t+1}^m)\left[\ln(c_{t+1}^m) + \eta\frac{(1-h_{t+1}^m)^{\mu-1} - 1}{\mu-1}\right] \\
& + \beta^2(1-d_{t+1}^m)(1-d_{t+2}^o)\ln(c_{t+2}^o)
\end{aligned}$$

s.t.

$$c_t^y + s_t^y = w_t h_t^y \tag{6}$$
$$c_{t+1}^m + s_{t+1}^m = w_{t+1}\lambda_{t+1} h_{t+1}^m + s_t^y(1 + r_{t+1}) + \tau_{t+1}^m \tag{7}$$
$$c_{t+2}^o = s_{t+1}^m(1 + r_{t+2}) + \tau_{t+2}^o \tag{8}$$

- Substituting the constraints into the objective function and taking FOCs gives 12 equations in 12 unknowns that define the solution. These are written below, and for convenience I have normalised the variables to be the stationary equivalents (mostly involving dividing by A_t).

$$\begin{aligned}
\frac{c_{t+1}^m}{c_t^y} &= \frac{\beta(1 - d_{t+1}^m)(1 + r_{t+1})}{1 + g_{t+1}} \\
\frac{c_{t+1}^o}{c_t^m} &= \frac{\beta(1 - d_{t+1}^o)(1 + r_{t+1})}{1 + g_{t+1}} \\
h_t^y &= 1 - \left(\eta \frac{c_t^y}{w_t}\right)^{\frac{1}{\mu}} \\
h_t^m &= 1 - \left(\eta \frac{c_t^m}{w_t \lambda_t}\right)^{\frac{1}{\mu}} \\
c_t^y &= w_t h_t^y - s_t^y \\
c_t^m &= w_t \lambda_t h_t^m + \frac{s_{t-1}^y(1 + r_t)}{1 + g_t} + \tau_t^m - s_t^m \\
c_t^o &= \frac{s_{t-1}^m(1 + r_t)}{1 + g_t} + \tau_t^o \\
\tau_t^m &= \frac{s_{t-1}^y(1 + r_t)\frac{d_t^m}{1 - d_t^m}}{1 + g_t} \\
\tau_t^o &= \frac{s_{t-1}^m(1 + r_t)\frac{d_t^o}{1 - d_t^o}}{1 + g_t} \\
w_t &= (1 - \alpha)k_t^\alpha \\
r_t &= \alpha k_t^{\alpha - 1} - 1 \\
k_t &= \frac{s_{t-1}^y(1 + n_{t-1}) + s_{t-1}^m(1 - d_{t-1})}{(1 + g_t)(h_t^y(1 + n_t)(1 + n_{t-1}) + \lambda_t h_t^m(1 - d_t)(1 + n_{t-1}))}
\end{aligned}$$

$$\{c_t^y, c_t^m, c_t^o, s_t^y, s_t^m, h_t^y, h_t^m, k_t, w_t, r_t, \tau_t^m, \tau_t^o\}$$

- To simulate shocks, we need to parameterise the model. I set $\alpha = 0.33$ because this is standard. The Frisch labour elasticity is given by $\frac{1}{\mu} \cdot \frac{1-h_t}{h_t}$. Since labour supply in equilibrium is typically around one-third, to get a standard Frisch elasticity of around 0.5 we need to set $\mu = 4$. Based on the simulations, I set $\eta = 0.8$, which yields labour supplies of around one-third. I interpret one period as being 20 years (completely arbitrary) and set the base levels of d^m and d^o to be 0.1 each, which is also somewhat arbitrary. I set $\lambda = 1$ for lack of a reason to do otherwise. I set $n = (1 + 0.01)^{20} - 1$ to reflect 1% growth per year and $g = 0$ for now for simplicity. I then set $\beta = 0.9$ since this yields steady state real interest rates of around 2% per year.

- We first consider a shock where n_t becomes -0.5 for one period (unexpectly). Most of the shocks used here are probably too large, but this is helpful to build intuition. The shocks are done such that the economy is in steady state in the period before the shock, and the agents correctly anticipate that the exogenous variables return to normal in all subsequent periods after the shock. Figure 1 shows the effects of this shock. The capital-labour ratio

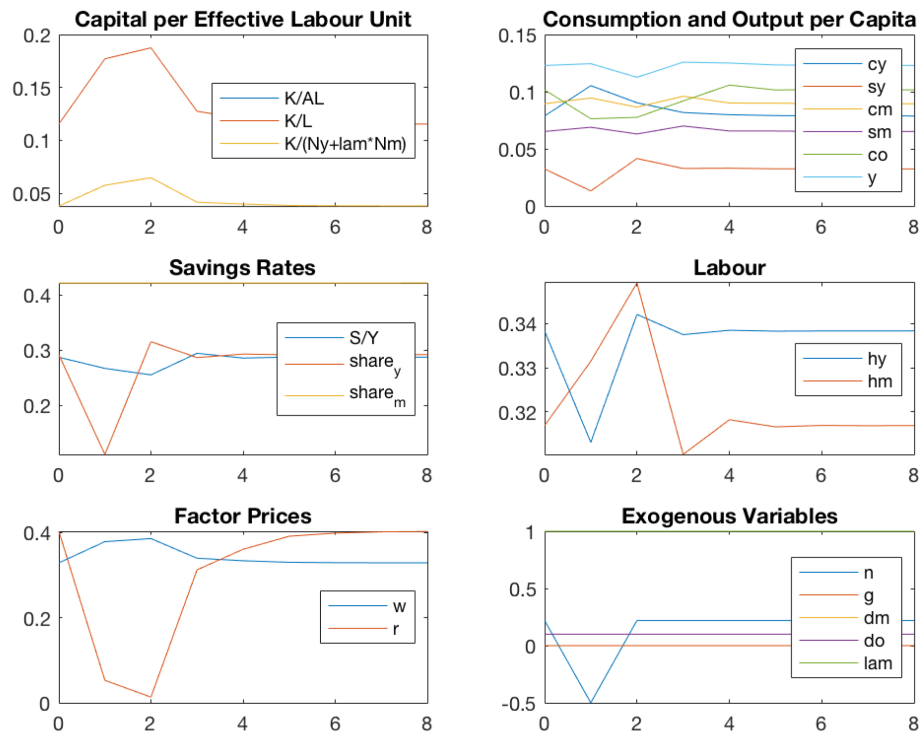

Figure 1: One-period shock to n.

increases for 2 periods since there remains lots of capital from the current middle-aged population who bring it with them when they become old. This leads the interest rate to fall dramatically and the wage rate to rise. The aggregate savings rate falls a little bit, driven by a very large decline in the savings rate of the young. The young consume a lot more and decrease labour, which presumably reflects the income effect dominating the substitution effect; the middle respond by increasing labour supply. These changes are small, but the directions presumably reflect the fact that the young prefer to supply labour in the next period since the wage will still be high (as there is still a lot of capital from the current middle), while the middle want to take advantage of the high wage immediately as they can't next period. Most of the variables are close to their steady state values by period 3, which reflects the fact that the populations are now balanced again. There would presumably be greater persistence if we allowed capital to not fully depreciate. Figure 2 shows the same thing but has $g = (1 + 0.005)^{20} - 1$.

- Figure 3 shows the effects of d_t^m being shocked to 0.6 for one period, reflecting a large amount of deaths in the middle-aged population. Figure 4 shows the effect of λ_t being shocked to 0.5 (from 1) for one period, reflecting the effects of a large decrease in labour productivity of middle-aged workers. These two shocks have very similar effects, which reflects the fact that both are reducing the effective middle-aged labour supply (though the second one has a meaningful negative impact on GDP/capita). Combining the two would thus amplify the effects. Finally, Figure 5 combines all three of these shocks (to n, d^m, λ) at once.

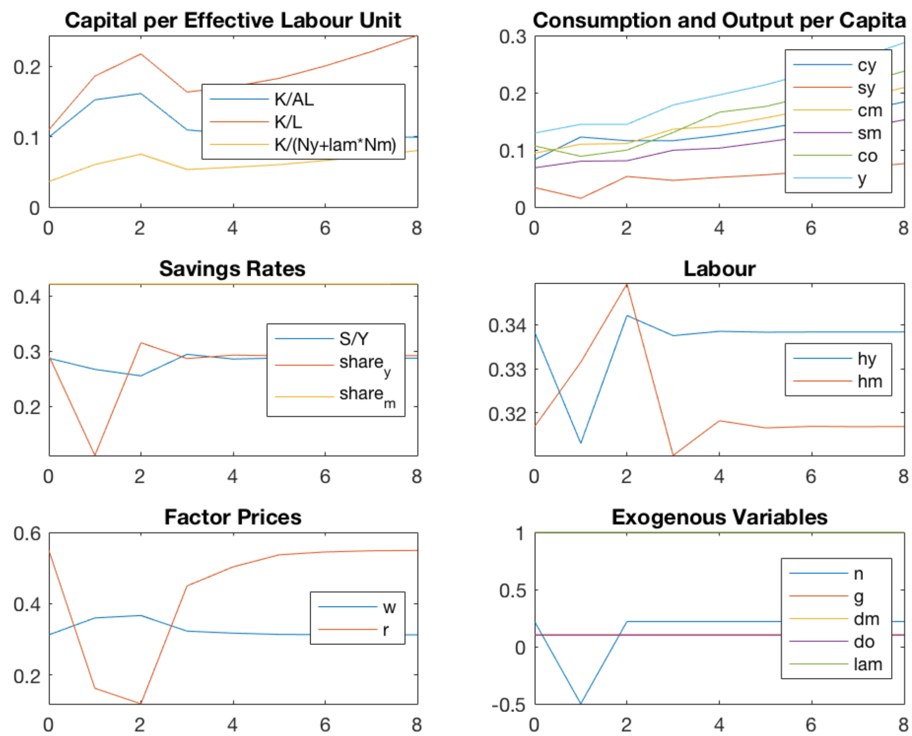

Figure 2: One-period shock to n, with positive TFP growth.

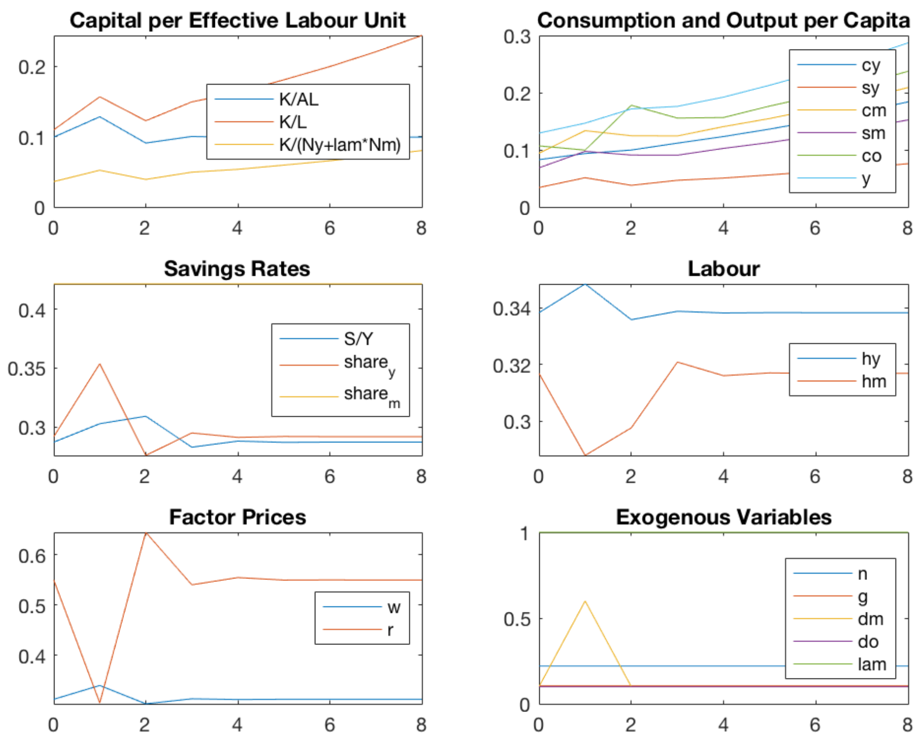

Figure 3: One-period shock to d^m, with positive TFP growth.

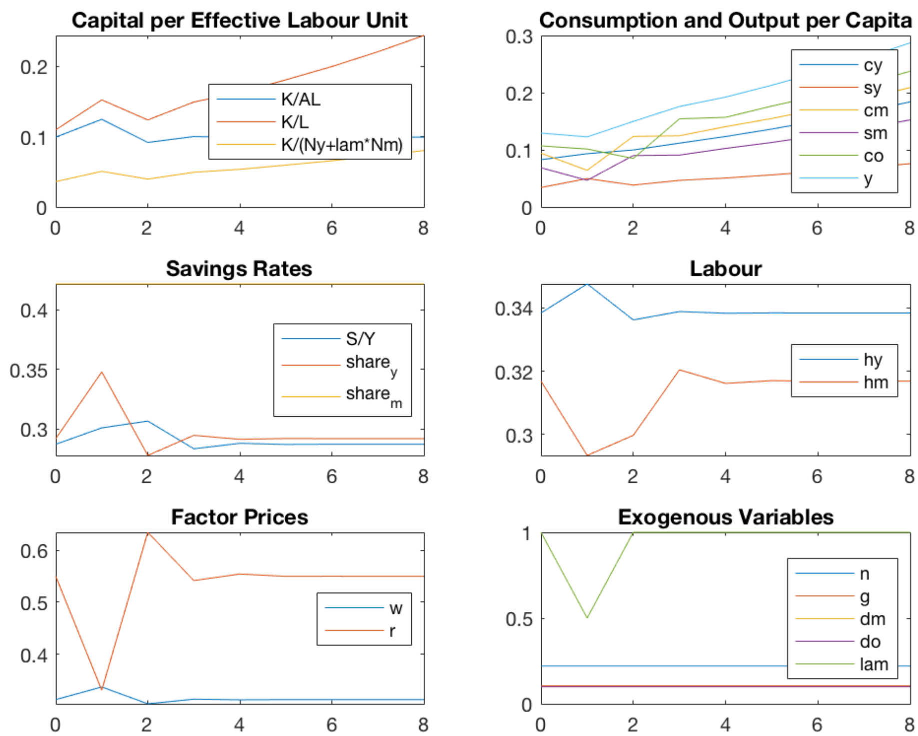

Figure 4: One-period shock to λ_t, with positive TFP growth.

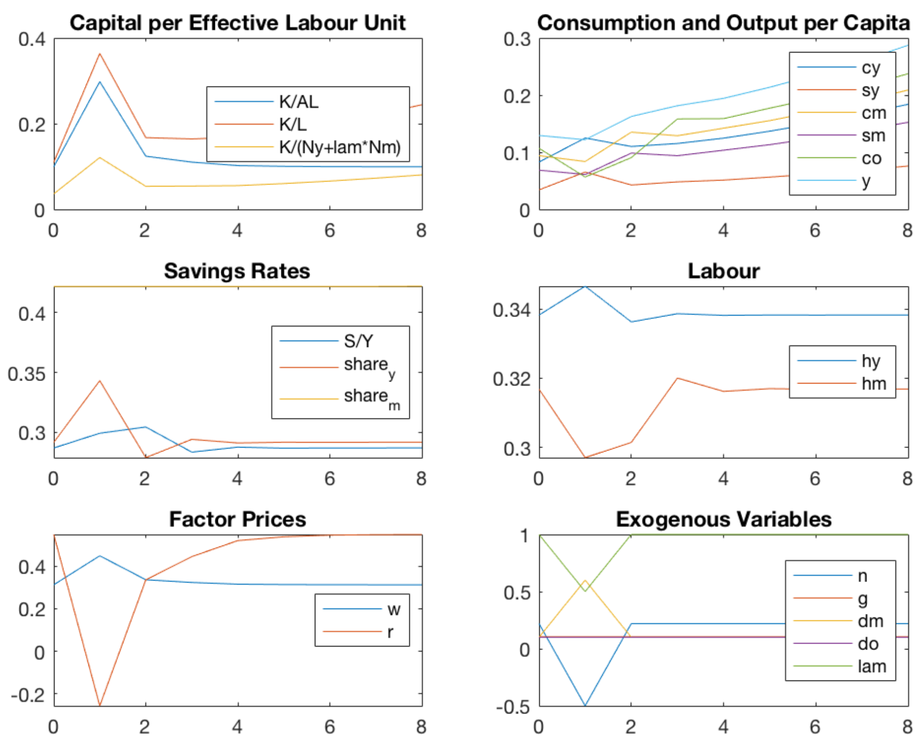

Figure 5: One-period shock to n_t, d_t^m, λ_t, with positive TFP growth.