DP Formulation Overview

Problem Statement

Typically in prediction market binary options, for any given event/outcome with probability `c`, you can purchase like so:

- Buy 1 YES contract with price `c * (1 + buy spread)` that pays 1 if outcome occurs, 0 otherwise
- Buy 1 NO contract with price `(1 - c) * (1 + buy spread)` that pays 0 if outcome occurs, 1 otherwise

For vast majority of events, a resolution date is set such that the YES and NO contracts resolve at that time.
Note that in some you can't hold YES and NO simultaneously (buy NO = sell existing YES and vice versa).

Once contracts are owned, they can also be sold:

- Sell 1 YES contract with price `c * (1 - sell spread)`
- Sell 1 NO contract with price `(1 - c) * (1 - sell spread)`

Variable Definitions

Time Horizon:

- $t \in [0, T]$ be the trading time steps (T = last trading time step, resolve immediately after)

Cash wealth:

- $W_t \in [0, \infty)$ be the wealth level at time t

Contracts:

- $x_t \in [0, \infty)$ be the number of contracts held at time t
  - $x_t < 0$ = holding $|x_t|$ NO contracts
  - $x_t > 0$ = holding $|x_t|$ YES contracts
- $c_t \in [0, 1]$ be the contract price at time t

Portfolio Value:

- $P_{t,start}$ be the portfolio value at the start of time t
  - $P_{t,start} = \begin{cases} W_{t-1} + x_{t-1}c_t & x_{t-1} \geq 0 \\ W_{t-1} - x_{t-1}(1-c_t) & x_{t-1} < 0 \end{cases}$
- $P_{t,end}$ be the portfolio value at the end of time t ($P_{t,start}$ minus spread costs)
  - $P_{t,end} = \begin{cases} W_t + x_t c_t & x_t \geq 0 \\ W_t - x_t(1-c_t) & x_t < 0 \end{cases}$

Portfolio Allocation at start of time t (State Variable):

- $\theta_t \in [-1, 1] = \begin{cases} \frac{x_{t-1}c_t}{P_{t,start}} & x_{t-1} \geq 0 \\ \frac{x_{t-1}(1-c_t)}{P_{t,start}} & x_t < 0 \end{cases}$

Portfolio Allocation at end of time t (Action Variable):

- Defined in terms of $P_{t,end}$:
  - $b'_t \in [-1, 1] = \begin{cases} \frac{x_t c_t}{P_{t,end}} & x_t \geq 0 \\ \frac{x_t(1-c_t)}{P_{t,end}} & x_t < 0 \end{cases}$
- Defined in terms of $P_{t,start}$:
  - $b_t = b'_t * \frac{P_{t,end}}{P_{t,start}} = \begin{cases} \frac{x_t c_t}{P_{t,start}} & x_t \geq 0 \\ \frac{x_t(1-c_t)}{P_{t,start}} & x_t < 0 \end{cases}$

Mark to Market (Price transition from t-1 to t)

- YES Contract:
  $R_{\text{YES},t} = \frac{c_t}{c_{t-1}}$
- NO Contract: $R_{\text{NO},t} = \frac{1-c_t}{1-c_{t-1}}$

Market Spreads

- YES Sell Spread: $\gamma_{\text{YES},s}$
- YES Buy Spread: $\gamma_{\text{YES},b}$
- NO Sell Spread: $\gamma_{\text{NO},s}$
- NO Buy Spread: $\gamma_{\text{NO},b}$

Goal

Since DP generates a Policy and Value table, in order to leverage the properties of a bounded DP with θ_t and b'_t, we need to express everything in terms of θ_t,b'_t, c_t.

Let our value function we seek to maximize be $V_T = \mathbb{E}[log(W_T)]$. In other words we seek to maximize the terminal expected log wealth at resolution time, since this maximizes long-term growth while penalizing excessively risky bets leading to drawdowns.

Initial Condition

- $t = 0, c_0 \in [0, 1], W_0$
- $x_0 = 0, P_{t,start} = W_0, \theta_0 = 0, b'_0 = 0, b_0 = 0$

Terminal Condition:

Let p be the subjective (trader's) probability of outcome occuring; equivalently trader's value of YES contract.

Case 1 (Own YES): With p probability $|X_T|$ YES contracts resolve to 1, but in any case keep W_T cash wealth

$$V_T = plog(W_T + |X_T|) + (1 - p)log(W_T)$$

$$V_T = plog(W_T + \frac{|\theta_T|}{1 - |\theta_T|} * W_T/C_T) + (1 - p)log(W_T)$$

$$V_T = log(W_T) + plog(1 + \frac{|\theta_T|}{c_T(1 - |\theta_T|)})$$

Case 2 (Own NO): With $1 - p$ probability $|X_T|$ NO contracts resolve to 1, but in any case keep W_T cash wealth

$$V_T = plog(W_T) + (1 - p)log(W_T + |X_T|)$$

$$V_T = plog(W_T) + (1-p)log(W_T + \frac{|\theta_T|}{1-|\theta_T|} * W_T/(1-c_T))$$

$$V_T = log(W_T) + (1-p)log(1 + \frac{|\theta_T|}{(1-c_T)(1-|\theta_T|)})$$

θ_{t+1} Update (Portfolio Mark to Market):

Case 1 (Own YES)

$$\theta_{t+1} = \frac{x_t c_{t+1}}{W_t + x_t(c_{t+1})}$$

$$\theta_{t+1} = \frac{x_t c_{t+1}/P_{t,end}}{W_t/P_{t,end} + x_t c_{t+1}/P_{t,end}}$$

$$\theta_{t+1} = \frac{b'_t * R_{\text{YES},t}}{(1-b'_t) + b'_t * R_{\text{YES},t}}$$

Case 2 (Own NO)

$$\theta_{t+1} = \frac{x_t(1-c_{t+1})}{W_t - x_t(1-c_{t+1})}$$

$$\theta_{t+1} = \frac{x_t(1-c_{t+1})/P_{t,end}}{W_t/P_{t,end} + x_t(1-c_{t+1})/P_{t,end}}$$

$$\theta_{t+1} = \frac{b'_t * R_{\text{NO},t}}{(1+b'_t) - b'_t * R_{\text{NO},t}}$$

More Generally:

$$\theta_{t+1} = \frac{b'_t * R_{\text{NO},t}}{(1-|b'_t|) + |b'_t| * R_{\text{NO},t}}$$

Portfolio Value Update

$P_{t,end} * (1 - |b_t'|) = P_{t,start} * (1 - |\theta_t|) * (\frac{W_t}{W_{t-1}})$

$\frac{P_{t,end}}{P_{t,start}} = \frac{(1-|\theta_t|)*\frac{W_t}{W_{t-1}}}{1-|b_t'|}$

Wealth Update (with Spread):

Let $\mathrm{B} = \frac{b_t'}{1-|b_t'|}, \quad \Theta = \frac{\theta_t}{1-|\theta_t|}, \quad y = \frac{W_t}{W_{t-1}}.$

And note that

$b_t = (\frac{b_t'}{1-|b_t'|})(1 - |\theta_t|)(\frac{W_t}{W_{t-1}}).$

Case 1:

$x_t \geq x_{t-1},\ x_{t-1} < 0,\ x_t < 0$

We sell $x_t - x_{t-1}$ NO contracts

$$W_t = W_{t-1} + (x_t - x_{t-1})(1 - c_t)(1 - \gamma_{NO,s})$$

$$W_t = W_{t-1} + \frac{W_{t-1}}{1 - |\theta_t|} * (b_t - \theta_t) * (1 - \gamma_{NO,s})$$

$$W_t = W_{t-1} * (1 + \frac{b_t - \theta_t}{1 - |\theta_t|} * (1 - \gamma_{NO,s}))$$

$$y = 1 + (\mathrm{B}y - \Theta) * (1 - \gamma_{NO,s})$$

$$y - y\mathrm{B}(1 - \gamma_{NO,s}) = 1 - \Theta(1 - \gamma_{NO,s})$$

$$y = \frac{1 - \Theta(1 - \gamma_{NO,s})}{1 - \mathrm{B}(1 - \gamma_{NO,s})}$$

Case 2:

$x_t \geq x_{t-1},\ x_{t-1} < 0,\ x_t \geq 0$

We sell x_{t-1} NO contracts and buy x_t YES contracts

$$W_t = W_{t-1} - x_{t-1}(1 - c_t)(1 - \gamma_{NO,s}) - x_t c_t (1 + \gamma_{\mathrm{YES},b})$$

$$W_t = W_{t-1} - \frac{W_{t-1}}{1 - |\theta_t|} * \theta_t * (1 - \gamma_{NO,s}) - \frac{W_{t-1}}{1 - |\theta_t|} * b_t * (1 + \gamma_{\mathrm{YES},b})$$

$$W_t = W_{t-1}(1 - \frac{\theta_t(1 - \gamma_{NO,s}) + b_t(1 + \gamma_{\mathrm{YES,b}})}{1 - |\theta_t|})$$

$$y = 1 - \Theta(1 - \gamma_{NO,s}) - \mathrm{B}y(1 + \gamma_{\mathrm{YES},b})$$

$$y(1 + B(1 - \gamma_{\mathrm{YES},b})) = 1 - \Theta(1 - \gamma_{NO,s})$$

$$y = \frac{1 - \Theta(1 - \gamma_{NO,s})}{1 + \mathrm{B}(1 + \gamma_{\mathrm{YES},b})}$$

Case 3:

$x_t \geq x_{t-1},\ x_{t-1} \geq 0,\ x_t \geq 0$

We buy $x_t - x_{t-1}$ YES contracts

$$W_t = W_{t-1} - (x_t - x_{t-1})(c_t)(1 + \gamma_{\mathrm{YES},b})$$

$$W_t = W_{t-1} - \frac{W_{t-1}}{1 - |\theta_t|} * (b_t - \theta_t) * (1 + \gamma_{\mathrm{YES},b})$$

$$W_t = W_{t-1} * (1 - \frac{b_t - \theta_t}{1 - |\theta_t|} * (1 + \gamma_{\mathrm{YES},b}))$$

$$y = 1 - (\mathrm{B}y - \Theta) * (1 + \gamma_{\mathrm{YES},b})$$

$$y = \frac{1 + \Theta(1 + \gamma_{\mathrm{YES},b})}{1 + \mathrm{B}(1 + \gamma_{\mathrm{YES},b})}$$

Case 4:

$x_t < x_{t-1},\ x_{t-1} \geq 0,\ x_t \geq 0$

We sell $-(x_t - x_{t-1})$ YES contracts

$$W_t = W_{t-1} - (x_t - x_{t-1})(c_t)(1 - \gamma_{\mathrm{YES},s})$$

$$W_t = W_{t-1} - \frac{W_{t-1}}{1 - |\theta_t|} * (b_t - \theta_t) * (1 - \gamma_{\mathrm{YES},s})$$

$$W_t = W_{t-1} * (1 - \frac{b_t - \theta_t}{1 - |\theta_t|} * (1 - \gamma_{\mathrm{YES},s}))$$

By symmetry to Case 3:

$$y = \frac{1 + \Theta(1 - \gamma_{\mathrm{YES},s})}{1 + \mathrm{B}(1 - \gamma_{\mathrm{YES},s})}$$

Case 5:

$x_t < x_{t-1},\ x_{t-1} \geq 0,\ x_t < 0$

We sell x_{t-1} YES contracts and buy x_t NO contracts

$$W_t = W_{t-1} + x_{t-1}(c_t)(1 - \gamma_{\mathrm{YES},s}) + x_t(1 - c_t)(1 + \gamma_{NO,b})$$

$$W_t = W_{t-1} + \frac{W_{t-1}}{1-|\theta_t|} * \theta_t * (1-\gamma_{\text{YES},s}) + \frac{W_{t-1}}{1-|\theta_t|} * b_t * (1+\gamma_{NO,b})$$

$$W_t = W_{t-1} * (1 + \frac{\theta_t(1-\gamma_{\text{YES},s}) + b_t(1+\gamma_{NO,b})}{1-|\theta_t|})$$

$$y = 1 + \Theta(1-\gamma_{\text{YES,s}}) + \mathrm{B}y(1+\gamma_{NO,b})$$

$$y = \frac{1+\Theta(1-\gamma_{\text{YES},s})}{1-\mathrm{B}(1+\gamma_{NO,b})}$$

Case 6:

$x_t < x_{t-1},\ x_{t-1} < 0,\ x_t < 0$

We buy $-(x_t - x_{t-1})$ NO contracts

$$W_t = W_{t-1} + (x_t - x_{t-1})(1-c_t)(1+\gamma_{NO,b})$$

$$W_t = W_{t-1} + \frac{W_{t-1}}{1-|\theta_t|} * (b_t - \theta_t) * (1+\gamma_{NO,b})$$

$$W_t = W_{t-1} * (1 + \frac{b_t - \theta_t}{1-|\theta_t|} * (1+\gamma_{NO,b}))$$

By symmetry to Case 1:

$$y = \frac{1-\Theta(1+\gamma_{NO,b})}{1-\mathrm{B}(1+\gamma_{NO,b})}$$

Note: Θ and B can be thought of as starting and ending coverage ratio (contract value / cash wealth value).

More generally we have:

$$y = \frac{1+|\Theta|(1+\gamma_{\Theta})}{1+|\mathrm{B}|(1+\gamma_{\mathrm{B}})}$$

or

$$y = \frac{\frac{1}{1+\gamma_\Theta} + |\Theta|}{\frac{1}{1+\gamma_\mathrm{B}} + |\mathrm{B}|}$$

where γ is positive if buying and negative if selling. So long as $|\Theta|, |\mathrm{B}| \in [0, \infty)$ and $\gamma \in (-1, 1)$, y is always positive (W_t will never flip negative).

Recursion Step

At each $t \in [0..T]$,

$$V_t = log(W_{t-1}) + v_t(\theta_t, c_t)$$

Where $v_t(\theta_t, c_t)$ is the growth function of log wealth between the start and end of time t, $log(W_{t-1})$ is the cash wealth at start of time t.

$$V_{t+1} = log(W_t) + v_{t+1}(\theta_{t+1}, c_{t+1})$$

Recursion Step (Bellman Equation):

$$V_{t+1} = log(W_{t-1}) + \max_{b'_t \in [-1,1]} \mathbb{E}_{c_t | c_{t-1}}[log(y) + v_{t+1}(\theta_{t+1}, c_{t+1})]$$

$$v_t(\theta_t, c_t) = \max_{b'_t \in [-1,1]} \mathbb{E}_{c_t | c_{t-1}}[log(y) + v_{t+1}(\theta_{t+1}, c_{t+1})]$$

Runtime Analysis

Space Complexity

Hence our space complexity for policy (and value) grid is

$$O(T * |\theta| * |c|)$$

where $|\theta|$ and $|c|$ are bounded between -1 and 1, and can be represented as a constant number of grid points; so really space complexity is

$$O(T * N^2)$$

if both values are represented with the same granularity for some constant N.

Time Complexity

And our time complexity for generating such a policy is

$$O(T * |\theta| * |b'| * |c|^2)$$

Since for each element in the DP grid we try out $|b'|$ different allocations and taking expected value over $|c|$ different prices. $|b'|$ is also bounded by -1 and 1, and if it also is represented with the same granularity as the other bounded values we have runtime complexity of

$$O(T * N^4)$$

which makes the problem tractable. Note that this is with a full price transition matrix; if we were to swap out with a binomial/trinomial price transition model runtime would be less.

Optimized Implementation

We observe that the expectation term $\mathbb{E}_{c_{t+1}|c_t}[v_{t+1}(\theta_{t+1}, c_{t+1})]$ depends on b'_t and c_t but **not on** θ_t. This enables factorization:

Precompute expectations: For all (b'_t, c_t) pairs, compute:

$$F(b'_t, c_t) = \mathbb{E}_{c_{t+1}|c_t}[v_{t+1}(\theta_{t+1}(b'_t, c_t, c_{t+1}), c_{t+1})]$$

Complexity: $O(N_b \times N_c \times N_c) = O(N^3)$

Optimize allocations: For all (θ_t, c_t) pairs:

$$v_t(\theta_t, c_t) = \max_{b'}[\log(y_t(\theta_t, b'_t)) + F(b'_t, c_t)]$$

Complexity: $O(N_\theta \times N_b) = O(N^2)$

Total per time step: $O(N^3)$, giving overall complexity of:

$$O(T \times N^3)$$