

# Algebraic Structure of Quantum Controlled States and Operators

Edwin Agnew

Department of Computer Science  
University of Oxford

Lia Yeh

Department of Computer Science and Technology  
University of Cambridge\*  
ly404@cam.ac.uk

Richie Yeung

Department of Computer Science  
University of Oxford  
richie.yeung@cs.ox.ac.uk

Quantum control is an important logical primitive of quantum computing programs, and an important concept for graphical rewriting in quantum graphical calculi. In this work, we investigate the algebraic structure of *controlled diagrams* in the ZXW-calculus — diagrams extended with an additional qubit wire for triggering an operation on or off. By formalising these properties of quantum control as a higher-order map, we enable powerful new graphical rewrite rules, going beyond prior calculi for which the allowed parameters are restricted to phases.

First, we prove that controlled square matrices form a ring, and thus admit expressive rewrite rules. We also show that controlled states form a ring, which is isomorphic to the ring of multilinear polynomials. Putting these together, we have completeness for polynomials over same-size square matrices, implying that these rules suffice to perform any factorisation of any qubit Hamiltonian.

## 1 Introduction

Controlling or branching to different possible linear maps, relations, or channels is important across quantum information and quantum computation, and has been studied through many different approaches. In quantum algorithms common techniques are block encodings [12, 20] and linear combination of unitaries [6], while a number of formalisations have included routed quantum circuits [28], the many-worlds calculus [5], categorifying signal flow diagrams [3], and classical and quantum control in quantum modal logic [23].

The question we are interested in is how quantum graphical calculi such as the ZX [7], ZW [8], and ZH [2] calculus can be augmented to support properties of quantum control. An early use of controlled state diagrams was for proving constructive and rational angle ZX calculus completeness [16]. More recently, controlled state and controlled matrix diagrams have been applied to addition and differentiation of ZX diagrams [15], differentiating and integrating ZX diagrams for quantum machine learning [29], Hamiltonian exponentiation and simulation [24], and non-linear optical quantum computing [10]. To sum ZX diagrams, these works have used controlled states along with the W generator from the ZW calculus.

Given how useful controlled diagrams are, a natural question to ask is why they work: What their underlying mathematical structures are, and which equational rewrites they satisfy.

First, we show that the set of all controlled  $n$ -partite states defines a commutative ring. We introduce  $\boxplus$  which defines an Abelian group and  $\boxtimes$  which defines a commutative monoid, and show that  $\boxtimes$  distributes over  $\boxplus$ . The fragment of the qubit ZW calculus corresponding to controlled states, which we call *arithmetic ZXW diagrams* hence defines a ring which we prove is isomorphic to multilinear polynomials  $\mathbb{C}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$ , and prove completeness for. Analogously, we show that the set of all controlled square matrices on  $n$  qubits defines a non-commutative ring  $(\tilde{M}^n, \blacktriangle, \blacklozenge)$ .

We add controlled square matrices and rewrite rules for them to the ZXW-calculus, in which we plug controlled states into each control wire of controlled square matrices. We prove their completeness and that this is isomorphic to multivariate polynomials over same-size square matrices. Commutativity of controlled square matrices holds in the special case that the controls target mutually exclusive sectors, allowing copying of arbitrary controlled

---

\*This work was done while LY was at the University of Oxford.

diagrams. As a result, we can factor multivariate polynomials over same-size square matrices. This means we now have the ability to factor any Hamiltonian in the ZXW-calculus [24], even with all its terms black-boxed.

Applying these findings, we present a quantum complexity theory result concerning efficient rewriting of quantum states to any arithmetic ZXW diagram, and prove that the higher-order map Ctrl which sends square matrices to controlled square matrices satisfies some important algebraic properties. In sum, these algebraic properties of quantum control give rise to powerful new graphical reasoning capabilities.

## 2 Preliminaries

This section introduces the ZXW-calculus, and how controlled diagrams are defined in it. The ZXW-calculus is a graphical formalism for qudit computation, unifying the ZX and ZW calculi and synthesising their relative strengths. The ZXW-calculus consists of diagrams built from a small number of generators and equipped with a complete set of rewrite rules, which enables all equalities between linear maps to be proven diagrammatically. Diagrams are to be read top to bottom or, in later sections, left to right.

### 2.1 The ZXW-Calculus

The qubit ZXW-calculus is built from the following generators:

$$\begin{aligned} \llbracket \begin{array}{|c|} \hline | \\ \hline \end{array} \rrbracket &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \llbracket \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \rrbracket &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \llbracket \begin{array}{c} \curvearrowright \end{array} \rrbracket &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} & \llbracket \begin{array}{c} \cup \end{array} \rrbracket &= \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \quad (2.1)$$

$$\begin{aligned} \llbracket \begin{array}{c} n \\ \diagup \quad \diagdown \\ \boxed{c} \\ \diagdown \quad \diagup \\ m \end{array} \rrbracket &= |0^m\rangle\langle 0^n| + c|1^m\rangle\langle 1^n|, c \in \mathbb{C} & \llbracket \begin{array}{c} \blacktriangle \end{array} \rrbracket &= |00\rangle\langle 0| + |01\rangle\langle 1| + |10\rangle\langle 1| \end{aligned} \quad (2.2)$$

$$\llbracket \begin{array}{|c|} \hline \blacksquare \\ \hline \end{array} \rrbracket = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (2.3)$$

For simplicity, we introduce the following additional notation:

$$\begin{aligned} \begin{array}{c} \dots \\ \diagup \quad \diagdown \\ \alpha \\ \diagdown \quad \diagup \\ \dots \end{array} &:= \begin{array}{c} \dots \\ \diagup \quad \diagdown \\ \boxed{e^{i\alpha}} \\ \diagdown \quad \diagup \\ \dots \end{array} & \begin{array}{c} \dots \\ \diagup \quad \diagdown \\ \dots \\ \diagdown \quad \diagup \\ \dots \end{array} &:= \begin{array}{c} \dots \\ \diagup \quad \diagdown \\ 0 \\ \diagdown \quad \diagup \\ \dots \end{array} = \begin{array}{c} \dots \\ \diagup \quad \diagdown \\ 1 \\ \diagdown \quad \diagup \\ \dots \end{array} \end{aligned} \quad (2.4)$$

$$\begin{aligned} \begin{array}{c} \dots \\ \diagup \quad \diagdown \\ \alpha \\ \diagdown \quad \diagup \\ \dots \end{array} &:= \begin{array}{c} \dots \\ \diagup \quad \diagdown \\ \blacksquare \quad \blacksquare \\ \diagdown \quad \diagup \\ \dots \end{array} & \begin{array}{c} \dots \\ \blacktriangle \end{array} &:= \begin{array}{c} \dots \\ \diagup \quad \diagdown \\ \blacktriangle \\ \diagdown \quad \diagup \\ \dots \end{array} \end{aligned} \quad (2.5)$$

Equations in ZXW apply diagrammatic rewrites which prove equalities of the underlying matrices. The complete rule set is given in Figure 1. Several important lemmas are found in Appendix A.

### 2.2 Controlled Diagrams

Following [24], we cover the definitions of controlled states and controlled square matrices, and arithmetic on them. Note that this is a different definition of controlled states to Ref. [15] in which controlling on  $|0\rangle$  is  $|+\rangle^{\otimes n}$  instead of  $|0\rangle^{\otimes n}$ ; this choice appears to make a substantial difference in the algebraic properties, which we discuss in Remark 2.

ZX Rules			
			(S1)
	(S2)		(K1)
	(Ept)		(K2)
	(B2)		(Zer)
	(K0)		(H)
ZW Rules			
	(Pcy)		(ADD)
	(Sym)		(Aso)
	(BZW)		(WW)
ZXW Rules			
	(Bs0)		(TA)
	(Bs1)		(HD)
Controlled ZXW Ring Rules			
	(Mcpy)		(Mcom)

Figure 1: The ZXW Calculus Rules, where  $k \in \{0, 1\}$  and  $a \in \mathbb{C}$ . The ZX rules, ZW rules, and ZXW rules are sufficient for ZXW completeness, while the controlled ZXW ring rules are additionally required for Theorem 5.1.

$$\left[ \begin{array}{c} | \\ \vdots \\ \text{\tiny $\tilde M$} \\ \vdots \\ | \end{array} \right] = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}$$
$$\begin{array}{c} \bullet \\ | \\ \boxed{\tilde{M}} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \bullet \\ | \\ \boxed{\tilde{M}} \end{array} \pi = \begin{array}{c} \boxed{M} \end{array} \quad (2.6)$$
$$\begin{array}{c} \mathbb{C}^2 \\ \mathbb{C}^{2^n} \end{array} \rightarrow \begin{array}{c} \mathbb{C}^2 \\ \mathbb{C}^{2^n} \end{array} \xrightarrow{\text{Ctrl}} \begin{array}{c} \mathbb{C}^2 \\ \mathbb{C}^{2^n} \end{array} \xrightarrow{M} \begin{array}{c} \mathbb{C}^2 \\ \mathbb{C}^{2^n} \end{array} = \begin{array}{c} \mathbb{C}^2 \\ \mathbb{C}^{2^n} \end{array} \xrightarrow{\text{Ctrl}} \begin{array}{c} \mathbb{C}^2 \\ \mathbb{C}^{2^n} \end{array} \xrightarrow{\tilde{M}} \begin{array}{c} \mathbb{C}^2 \\ \mathbb{C}^{2^n} \end{array} \quad (2.7)$$

$$\text{Ctrl} \boxed{X} = \boxed{\tilde{X}} = \text{CNOT} \quad (2.8)$$
$$\left[ \begin{array}{c} | \\ | \\ | \\ | \\ | \end{array} \right] \left[ \begin{array}{c} | \\ \triangle \\ | \\ | \\ | \end{array} \right] = \left[ \begin{array}{cc} 1 & | \\ 0 & | \\ \vdots & \psi \\ 0 & \psi \\ 0 & | \end{array} \right]$$
$$\begin{array}{c} \bullet \\ | \\ \triangle \psi \\ | \\ \dots \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \dots \end{array} \quad \begin{array}{c} \bullet \pi \\ | \\ \triangle \psi \\ | \\ \dots \end{array} = \begin{array}{c} \triangle \psi \\ | \\ \dots \end{array} \quad (2.9)$$

$$(2.10)$$

The addition and multiplication of controlled states are defined similarly to controlled matrix arithmetic, except that a layer of  $\blacktriangledown$ s are appended at the bottom to preserve the number of outputs. The role of  $\blacktriangledown$  is to *copy* controlled diagrams, as we will show in Section 3.

**Proposition 2.** Given controlled states  $\tilde{\psi}$  and  $\tilde{\phi}$ , we define addition  $\tilde{\psi} \boxplus \tilde{\phi}$  and multiplication  $\tilde{\psi} \boxtimes \tilde{\phi}$  operations on them to result in the controlled states:

(2.11)

**Remark 1.** Ref. [24] defined this addition with red spiders at the bottom instead of  $\blacktriangledown$ s; the linear map is the same, being  $\widetilde{\phi + \psi}$ . In choosing to use  $\blacktriangledown$ , we will soon define the arithmetic fragment of the ZXW-calculus.

Removing the  $\blacktriangledown$ 's from the bottom of this multiplication gives the controlled diagram for the tensor product in Hilbert space  $\widetilde{\psi \otimes \phi}$ , as noted in Ref. [15].

Although the interpretation of  $\tilde{\psi} \boxtimes \tilde{\phi}$  is not a nice expression in terms of  $\psi$  and  $\phi$ , what we will show in this work is that this multiplication is not that of the linear maps, but of multiplication in the ring of *multilinear polynomials* which we will identify with these controlled diagrams.

### 3 Ring Axioms for Controlled Diagrams

In this section, we reverse-engineer the underlying algebraic properties of controlled state and controlled square matrix diagrams. This builds up to diagrams for the unique normal form for states used for the first proofs of complete axiomatisation for qubit graphical calculi [13, 14]. All proofs in this section can be found in Appendix B.1. Note that while the ZXW calculus is complete, and Lemmas 3.1, 3.2, 3.4, and 3.5 hold for any concrete realisations of  $\tilde{M}$ , for abstract  $\tilde{M}$ s they can only be verified via the plugging in of basis states.

Let  $\tilde{M}_n$  be the set of controlled square matrices on  $n$  qubits. The goal of this section is to prove that the addition and multiplication operations introduced above induce a ring on  $\tilde{M}_n$ . By Proposition 1, the addition and multiplication of controlled matrices is just the controlled addition and multiplication of the underlying matrices so the fact that the ring properties hold is not particularly surprising. What is more interesting is how easily these properties can be proven with a small subset of the ZXW rules. Likewise, we show that the set of controlled  $n$ -qubit states  $\tilde{S}_n$  also forms a ring. The first lemma enables us to copy controlled matrices.

**Lemma 3.1.** For any square matrix  $M$ ,

(Mcpy)

Now we show that controlled matrix addition and multiplication satisfy the ring axioms. Associativity of  $+$ ,  $\times$  follow immediately from (Aso, S1), respectively. Commutativity of addition follows from the commutativity of matrix addition and Proposition 1.

**Lemma 3.2.** Let  $M_1, M_2$  be  $n \times n$  matrices.

$$\begin{array}{c} \text{---} \blacktriangle \text{---} \\ | \\ \vdots \quad \boxed{\tilde{M}_1} \quad \vdots \quad \boxed{\tilde{M}_2} \quad \vdots \\ \text{---} \end{array} = \begin{array}{c} \text{---} \blacktriangle \text{---} \\ | \\ \vdots \quad \boxed{\tilde{M}_2} \quad \vdots \quad \boxed{\tilde{M}_1} \quad \vdots \\ \text{---} \end{array} \quad (3.1)$$

**Lemma 3.3.** The additive identity is defined as  $\text{red circle} \otimes I_n$ :

$$\begin{array}{c} \text{---} \blacktriangle \text{---} \\ | \\ \vdots \quad \boxed{\tilde{M}_1} \quad \vdots \quad \text{red circle} \\ \text{---} \end{array} \stackrel{(A.1)}{=} \begin{array}{c} \text{---} \\ | \\ \vdots \quad \boxed{\tilde{M}_1} \quad \vdots \\ \text{---} \end{array} \quad (3.2)$$

The multiplicative identity is defined very similarly as  $\text{green circle} \otimes I_n$ . The existence of additive inverses relies on the copying lemma from before.

**Lemma 3.4.** The additive inverse of  $\tilde{M}$  is  $\boxed{-1} \circ \tilde{M}$ .

**Lemma 3.5.** The addition and multiplication operations of controlled matrices distribute:

$$\begin{array}{c} \text{---} \text{green circle} \text{---} \\ | \\ \text{---} \blacktriangle \text{---} \\ | \\ \vdots \quad \boxed{\tilde{M}_1} \quad \vdots \quad \boxed{\tilde{M}_2} \quad \vdots \quad \boxed{\tilde{M}_3} \quad \vdots \\ \text{---} \end{array} = \begin{array}{c} \text{---} \blacktriangle \text{---} \\ | \\ \text{---} \text{green circle} \text{---} \quad \text{---} \text{green circle} \text{---} \\ | \quad | \\ \vdots \quad \boxed{\tilde{M}_1} \quad \vdots \quad \boxed{\tilde{M}_2} \quad \vdots \quad \boxed{\tilde{M}_1} \quad \vdots \quad \boxed{\tilde{M}_3} \quad \vdots \\ \text{---} \end{array} \quad (3.3)$$

Combining the lemmas of this section shows that controlled matrices form a ring. A similar result can be shown for controlled states. Once again, we start with the ability to copy controlled states.

**Lemma 3.6.** For any state  $\psi$ ,

$$\begin{array}{c} \text{---} \blacktriangle \text{---} \\ | \\ \text{---} \text{green circle} \text{---} \\ | \\ \text{---} \blacktriangle \text{---} \\ | \\ \text{---} \psi \text{---} \\ | \\ \vdots \end{array} = \begin{array}{c} \text{---} \blacktriangle \text{---} \\ | \\ \text{---} \text{green circle} \text{---} \quad \text{---} \text{green circle} \text{---} \\ | \quad | \\ \text{---} \blacktriangle \text{---} \quad \text{---} \blacktriangle \text{---} \\ | \quad | \\ \text{---} \psi \text{---} \quad \text{---} \psi \text{---} \\ | \quad | \\ \vdots \quad \vdots \end{array} \quad (3.4)$$

Many of the ring axioms follow directly from basic ZXW rules. For example we can show commutativity of addition as follows:


**Lemma 3.7.** For  $n$ -partite states  $\psi_1, \psi_2$ ,  $\tilde{\psi}_1 \boxplus \tilde{\psi}_2 = \tilde{\psi}_2 \boxplus \tilde{\psi}_1$ .

Associativity of  $\boxplus$  follows similarly, using (Aso). Next we have the additive identity.

**Lemma 3.8.**  $\tilde{\psi} \boxplus \tilde{0} = \tilde{\psi}$ .

The additive inverse is defined similarly to the case of controlled matrices.





**Lemma 3.9.** For a controlled state  $\tilde{\psi}$ , its additive inverse is  $\tilde{\psi} \circ \boxed{-1}$ .

Associativity and commutativity of  $\boxtimes$  follow as before, using (S1) for . Finally, we must prove distributivity.

**Lemma 3.10.**  $\tilde{\psi}_1 \boxtimes (\tilde{\psi}_2 \boxplus \tilde{\psi}_3) = (\tilde{\psi}_1 \boxtimes \tilde{\psi}_2) \boxplus (\tilde{\psi}_1 \boxtimes \tilde{\psi}_3)$ .

**Remark 2.** A different addition and multiplication for controlled states was defined in Ref. [16]. There corresponded to entry-wise addition and multiplication of statevectors, while our  $\boxplus$  and  $\boxtimes$  correspond to addition and multiplication of polynomials in bijective correspondence to controlled states, which we show next.

## 4 Isomorphism between the Ring of Controlled States and Multilinear Polynomials

It's been known since 2011 that ,  can be used to add and multiply number states  $|a\rangle$ , respectively [9]. In the previous section we saw that ,  can moreover be used to copy controlled diagrams. In this section, we explain this connection by demonstrating that controlled states are in fact isomorphic to multilinear polynomials. This being a bijection is a well-known folklore result in the study of entangled states, but to the best of our inquiries we are not aware of a proof. More generally, Ref. [31] presented Cartesian Distributive Categories exemplified by polynomial circuits, which are isomorphic to polynomials over arbitrary commutative semirings or rings; their proof is non-constructive, giving explicit proof only for the case of Boolean circuits [30]. Our proof hinges on a normal form inspired by the recent proof of completeness for the ZXW calculus [19], suggesting that much of the expressive power of the ZXW calculus comes from this algebraic structure.

Firstly, we describe how to interpret certain ZXW diagrams as polynomials. Consider the diagrams:

$$\begin{array}{c} \text{triangle with dot} \\ \swarrow \quad \searrow \\ \boxed{-1} \end{array} \qquad \begin{array}{c} \text{triangle with dot} \\ \swarrow \quad \searrow \\ \boxed{2} \quad \boxed{3} \end{array} \tag{4.1}$$

If we treat the bottom wires as an indeterminate  $x$ , we can read these bottom-up as computing  $x - 1$  and  $2x + 3$ , respectively. Moreover, since these diagrams are both controlled states, they can be added together, yield a diagram resembling  $3x + 2$ :

$$\begin{array}{c} \text{triangle with dot} \\ \swarrow \quad \searrow \\ \boxed{-1} \quad \boxed{2} \quad \boxed{3} \\ \swarrow \quad \searrow \\ x \end{array} \stackrel{(A.9)}{=} \begin{array}{c} \text{triangle with dot} \\ \swarrow \quad \searrow \\ \boxed{1} \quad \boxed{2} \quad \boxed{-1} \quad \boxed{3} \\ \swarrow \quad \searrow \\ x \end{array} \stackrel{(A.9)}{=} \begin{array}{c} \text{triangle with dot} \\ \swarrow \quad \searrow \\ \boxed{3} \quad \boxed{2} \\ \swarrow \quad \searrow \\ x \end{array} \tag{4.2}$$

When trying to multiply these diagrams, rather than getting  $(x-1)(2x+3) = 2x^2 + x - 3$ , we instead get  $x-3$ .

(4.3)

The reason for the missing  $2x^2$  term is that Lemma A.15 implies  $x^2 = 0$ . Other than that, controlled state arithmetic appears to faithfully reflect polynomial arithmetic. To help formalise this correspondence, we introduce the following definition.

**Definition 4.1.** A ZXW diagram with a single input on top is **arithmetic** if it contains only  $|$ ,  $\times$  wires,  $\blacktriangle$ ,  $\circ$ ,  $\blacktriangledown$  nodes and  $\boxed{a}$  boxes.

**Remark 3.** This fragment of the ZXW-calculus defines a subcategory; adding  $\circ$ , this is an instance of a Cartesian Distributive Category as defined in Ref. [31].

To interpret an arithmetic ZXW diagram as an arithmetic expression, read  $\blacktriangle$  as  $+$ ,  $\circ$  as  $\times$ ,  $\boxed{a}$  as the number  $a$ ,  $\blacktriangledown$  as fanout and output/bottom wires as variables  $x_1, \dots, x_n$  numbered from left to right. The following lemma establishes that all arithmetic diagrams are controlled states:

**Lemma 4.1.** For any arithmetic diagram  $A$ ,

(4.4)

*Proof.* By definition, other than wires  $A$  contains only  $\blacktriangle$ ,  $\circ$ ,  $\blacktriangledown$ , and  $\boxed{a}$ . All  $\boxed{a}$ 's can be removed with (Ept).

Meanwhile all the spiders copy  $\circ$  due to (Bs0, K0, A.1) respectively. □

Just as it is typical to represent a polynomial in normal form as a sum of products, it is possible to rewrite every arithmetic diagram into a normal form as a single  $\blacktriangle$ , followed by a layer of  $\circ$ , followed by a layer of  $\boxed{a}$ ,  $\blacktriangledown$ .

**Definition 4.2.** An  $n$ -output arithmetic diagram is said to be written in **polynomial normal form** (PNF) if it is of the form:

(4.5)



The  $i$ th coefficient  $a_i$  is connected to the  $k$ th  $\nabla$  iff the  $k$ th bit in the binary expansion of  $i$  is 1.

This normal form is familiar from completeness of all linear maps for qubits [14] and for qudits [19]. The reason we introduce the definition of a PNF is that it is an arithmetic diagram and therefore has a more immediate arithmetic interpretation. The reason for the specific connectivity condition is that it enables a diagram in PNF to directly represent its own matrix.

**Proposition 3.** The diagram in equation (4.5) equals the matrix 
$$\begin{bmatrix} 1 & a_0 \\ 0 & a_1 \\ \dots & \dots \\ 0 & a_{2^n-1} \end{bmatrix}$$

*Proof.* See Appendix B. □

Thus, every controlled state can be represented as at least one arithmetic diagram (namely, its PNF). Moreover, we now show that any other arithmetic diagram can always be rewritten to its PNF.

**Theorem 4.1.** All arithmetic diagrams can be rewritten into PNF through application of ZXW rules. Therefore, those ZXW-calculus rules applied suffice for completeness for the arithmetic fragment of the ZXW-calculus.

*Proof.* We present an algorithm to rewrite any arithmetic diagram to PNF in Appendix B.2. □

## 4.1 Isomorphism

At last we can prove the isomorphism. Recall that  $\tilde{\mathcal{S}}_n$  is the ring of controlled states with  $n$  outputs. Throughout, we shall let  $\mathcal{P}_n$  denote the multilinear ring  $\mathbb{C}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$ .

**Theorem 4.2.** There is an isomorphism  $\mathcal{P}_n \simeq \tilde{\mathcal{S}}_n$

First, we shall define the map  $\phi_n : \mathcal{P}_n \rightarrow \tilde{\mathcal{S}}_n$  before proving it induces an isomorphism.  $\phi_n$  is defined to map an arbitrary polynomial  $p(x_1, \dots, x_n) = a_0 + a_1 x_1 + \dots + a_{2^n-1} x_1 x_2 \dots x_n$  to the PNF in equation (4.5).

Some important special cases are mapping scalars  $a \in \mathbb{C}$  and indeterminates  $x_i$ :

$$\phi_n(a) = \begin{array}{c} \boxed{a} \\ | \\ \text{---} \text{---} \text{---} \end{array} \quad \phi_n(x_1) = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad , \phi_n(x_2) = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad , \dots, \phi_n(x_n) = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad (4.6)$$

The proof that  $\phi_n$  is a homomorphism resembles equations (4.2) and (4.3), but with greater generality. That  $\phi_n$  is a bijection relies on proposition 3. The full proof is found in Appendix B.

## 5 Completeness for Factoring Controlled Operators

Instead of the indeterminates being complex numbers represented by  $\boxed{a}$ 's, we can let them be same-size matrices represented by controlled square matrix diagrams. We then have that:

**Theorem 5.1.** ZXW diagrams where the outputs of an arithmetic ZW diagram are each plugged into controls of same-size controlled matrices, are isomorphic to multivariate polynomials over same-size square matrices with complex number coefficients. The rules for their completeness are the same subset of ZXW rules used for completeness for arithmetic diagrams in the ZXW-calculus in Theorem 4.1, plus the controlled square matrix as a generator along with the four rewrite rules for it in Definition 2.1, Lemma 3.1, and Lemma 3.2. (See Figure 1.)

*Proof.* The proof is by the same algorithm for rewriting to PNF as Theorem 4.1, modifying step (6) to copy controlled square matrices using Lemma 3.1, using Lemma 3.2 to commute controlled square matrices whose controls act on mutually exclusive sectors. □

## 6 Applications

### 6.1 Factorising Hamiltonians

As an application, we leverage both our rewrite rules for arithmetic ZW diagrams, and for controlled diagrams, to *factor* them. For example, for same size square matrices  $I, A, B$  and  $a, b, c \in \mathbb{C}$ :

$$p(\tilde{A}, \tilde{B}) = aI + \widetilde{bA^2} + cBA =$$

$$= aI + \widetilde{(bA + cB)A}$$
(6.1)

Factoring Hamiltonians is important to optimise quantum algorithms for chemistry and physics simulations. However, previous graphical rewrites for factoring Hamiltonians had only been doable for Hamiltonians with concretely-specified matrix terms [24]. This completeness result guarantees that for any Hamiltonian, even if its matrix terms are black-box, these graphical rewrite rules are capable of deriving any of its possible factorisations.

### 6.2 The Control Higher-Order Map

In quantum circuits, quantum control is realized through controlled gates. In this section, we show that we can reason with controlled gates by a straightforward construction on top of our ring of controlled square matrices. We define the higher-order map  $\text{Ctrl}$  which takes a square matrix  $M : V \rightarrow V$  to its controlled square diagram  $V \otimes \mathbb{C}^2 \rightarrow V \otimes \mathbb{C}^2$ . In the functorial box notation of [18], we write:

$$\text{Ctrl}(M) = \tilde{M}$$
(6.2)

We prove in Appendix B.3 that composition of controlled operations in sequence and in parallel is well-behaved.

**Proposition 4.**

$$\text{Ctrl}(M_1) \circ \text{Ctrl}(M_2) = \text{Ctrl}(M_1 \circ M_2)$$
(6.3)

**Proposition 5.**

Diagram (6.4) illustrates the equivalence between two ways of representing a controlled operation. On the left, a large light blue box contains two smaller boxes,  $M_1$  and  $M_2$ , stacked vertically. A single control wire, labeled 'Ctrl', enters from the bottom and connects to both  $M_1$  and  $M_2$ . On the right, the same operation is shown as two separate light blue boxes, one for  $M_1$  and one for  $M_2$ . Each box has its own control wire, both labeled 'Ctrl', which are connected to the same input control line. The two diagrams are separated by an equals sign.

(6.4)

Furthermore, successive applications of Ctrl recovers the standard notion of multiple-control, which computes the AND of the control qubits:

**Proposition 6.**

Diagram (6.5) illustrates the equivalence between a controlled operation with multiple controls and a nested controlled operation. On the left, a box labeled  $\tilde{M}$  has a single control input  $V$  and a single output  $W$ . Above the box, there are two control wires, each labeled  $\mathbb{C}^2$ , which are connected to a small circle with an AND symbol ( $\wedge$ ). On the right, a large light blue box contains a smaller light blue box labeled  $M$ . The outer box has four control wires, each labeled  $\mathbb{C}^2$ , and a single control input  $V$  and output  $W$ . The inner box  $M$  has a single control input labeled 'Ctrl'. The two diagrams are separated by an equals sign.

(6.5)

## 7 Conclusion

To conclude, we proved completeness for all controlled  $n$ -partite states, which we showed form a commutative ring isomorphic to multilinear polynomials. Also, we showed that all controlled  $n$ -qubit square matrices form a non-commutative ring. Furthermore, we have completeness for plugging controlled states into the control wires of controlled diagrams, isomorphic to all multivariate polynomials over same-size square matrices, with application to factoring Hamiltonians. When the controls target mutually exclusive sectors, a rewrite rule can be applied to copy any controlled diagram, and thereby factor any Hamiltonian.

We showed that it is unlikely that arbitrary quantum states or circuits can be efficiently rewritten to any arithmetic ZXW diagram, as this would imply  $\text{RP} = \text{NQP}$ . This opens up connections between quantum circuit complexity and the far better understood algebraic complexity. We have shown that every (controlled) state computes a polynomial; hence, we can interpret a universal fragment of the ZXW-calculus as corresponding to arithmetic circuits. This generalises Ref. [4], which found an algebraic interpretation of a certain fragment of ZW calculus. This line of reinterpreting quantum circuits as computing polynomials rather than unitary matrices offers a new perspective on quantum computation.

In another direction, we can apply completeness for polynomials isomorphic to controlled states to study entanglement. It can be easily shown diagrammatically that the polynomials corresponding to entangled (non-separable) states are exactly those that cannot be factored into irreducibles containing only variables corresponding to Alice's subsystem or only corresponding to Bob's subsystem. Since there are efficient algorithms for polynomial factorisation [11], this gives rise to a novel entanglement classification algorithm for pure states. Further developing this into a more refined algebraic theory of entanglement, building on the work in Ref. [1], could offer further insights.

The natural next step is extend our results to controlled *qudit* diagrams. While the diagrams being controlled are over qudits, we can consider control in the qubit subspace, as done in the ZXW-calculus completeness proof for any qudit dimension [19]. A starting guess would be that qudit controlled states are isomorphic to polynomials  $\mathbb{C}^{d-1}[x_1, \dots, x_n]/(x_1^d, \dots, x_n^d)$  due to the Hopf law between Z and W. Qudit multiple-control would likely have more complex structure than the qubit case here, considering the constructions for all prime-dimensional  $d$ -ary classical reversible gates built in Ref. [22].

Last, we are interested in exploring how to embed these new semantics for quantum controlled states and matrices into a functional programming language like in Ref. [21], or translated to an equational theory for a quantum programming language like in Ref. [25]. We would like to try sector-preserving channels [27] and scoped effects [17] as approaches to better formulate the semantics of multiple-control. We are also curious about reconciling the interpretation of diagrammatic differentiation of our arithmetic polynomial circuits by the approach in Ref. [31], with that of quantum circuits and ZX diagrams in Refs. [26, 29, 15].

## 8 Acknowledgements

We thank Bob Coecke for motivating our exploration of these topics. We are grateful to all the collaboration with Matt Wilson in discovering the functorial properties of control. We thank Razin Shaikh, Hany (Quanlong) Wang, Itai Leigh, Tim Forrer, Aleks Kissinger, and Frank (Peng) Fu for insightful discussions. We also thank the anonymous reviewers who have given feedback on this work for their helpful comments. LY is thankful for funding from the Google PhD Fellowship, and previously from the Basil Reeve Graduate Scholarship at Oriel College with the Clarendon Fund. RY would like to thank Simon Harrison for his generous support via the Wolfson Harrison UKRI Quantum Foundation Scholarship.

## References

- [1] AGNEW, E. Quantum Polynomials in the ZXW Calculus. Master’s thesis, University of Oxford, 2023.
- [2] BACKENS, M., AND KISSINGER, A. ZH: A Complete Graphical Calculus for Quantum Computations Involving Classical Non-linearity. In *Proceedings of the 15th International Conference on Quantum Physics and Logic, Halifax, Canada, 3-7th June 2018* (2019), P. Selinger and G. Chiribella, Eds., vol. 287 of *Electronic Proceedings in Theoretical Computer Science*, Open Publishing Association, pp. 23–42.
- [3] BAEZ, J. C., AND ERBELE, J. Categories in control, 2015.
- [4] CARETTE, T., MOUTOT, E., PEREZ, T., AND VILMART, R. Compositionality of planar perfect matchings, 2023.
- [5] CHARDONNET, K., DE VISME, M., VALIRON, B., AND VILMART, R. The many-worlds calculus, 2023.
- [6] CHILDS, A. M., AND WIEBE, N. Hamiltonian simulation using linear combinations of unitary operations. *Quantum Info. Comput.* 12, 11–12 (nov 2012), 901–924.
- [7] COECKE, B., AND DUNCAN, R. Interacting quantum observables: categorical algebra and diagrammatics. *New Journal of Physics* 13 (2011), 043016.
- [8] COECKE, B., AND KISSINGER, A. The compositional structure of multipartite quantum entanglement. In *International Colloquium on Automata, Languages, and Programming* (2010), Springer, pp. 297–308.
- [9] COECKE, B., KISSINGER, A., MERRY, A., AND ROY, S. The ghz/w-calculus contains rational arithmetic. *arXiv preprint arXiv:1103.2812* (2011).
- [10] DE FELICE, G., SHAIKH, R. A., POÓR, B., YEH, L., WANG, Q., AND COECKE, B. Light-matter interaction in the zxw calculus. *arXiv preprint arXiv:2306.02114* (2023).
- [11] FORBES, M. A., AND SHPILKA, A. Complexity theory column 88: Challenges in polynomial factorization. *ACM SIGACT News* 46, 4 (2015), 32–49.
- [12] GILYÉN, A., SU, Y., LOW, G. H., AND WIEBE, N. Quantum singular value transformation and beyond: exponential improvements for quantum matrix arithmetics. In *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing* (New York, NY, USA, 2019), STOC 2019, Association for Computing Machinery, p. 193–204.

- [13] HADZIHASANOVIC, A. The algebra of entanglement and the geometry of composition, 2017.
- [14] HADZIHASANOVIC, A., NG, K. F., AND WANG, Q. Two complete axiomatisations of pure-state qubit quantum computing. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science* (New York, NY, USA, 2018), LICS '18, Association for Computing Machinery, p. 502–511.
- [15] JEANDEL, E., PERDRIX, S., AND VESHCHZEROVA, M. Addition and differentiation of zx-diagrams, 2024.
- [16] JEANDEL, E., PERDRIX, S., AND VILMART, R. A generic normal form for zx-diagrams and application to the rational angle completeness, 2018.
- [17] LINDLEY, S., MATACHE, C., MOSS, S., STATON, S., WU, N., AND YANG, Z. Scoped effects as parameterized algebraic theories, 2024.
- [18] MELLIÈS, P.-A. Functorial boxes in string diagrams. In *International Workshop on Computer Science Logic* (2006), Springer, pp. 1–30.
- [19] POÓR, B., WANG, Q., SHAIKH, R. A., YEH, L., YEUNG, R., AND COECKE, B. Completeness for arbitrary finite dimensions of zxw-calculus, a unifying calculus. *arXiv preprint arXiv:2302.12135* (2023).
- [20] RALL, P. Quantum algorithms for estimating physical quantities using block encodings. *Phys. Rev. A* 102 (Aug 2020), 022408.
- [21] RENNELA, M., AND STATON, S. Classical Control, Quantum Circuits and Linear Logic in Enriched Category Theory. *Logical Methods in Computer Science Volume 16, Issue 1* (Mar. 2020).
- [22] ROY, P., VAN DE WETERING, J., AND YEH, L. The qudit zh-calculus: Generalised toffoli+hadamard and universality. *Electronic Proceedings in Theoretical Computer Science* 384 (Aug. 2023), 142–170.
- [23] SATI, H., AND SCHREIBER, U. The quantum monadology, 2023.
- [24] SHAIKH, R. A., WANG, Q., AND YEUNG, R. How to sum and exponentiate hamiltonians in zxw calculus. *arXiv preprint arXiv:2212.04462* (2022).
- [25] STATON, S. Algebraic effects, linearity, and quantum programming languages. *SIGPLAN Not.* 50, 1 (jan 2015), 395–406.
- [26] TOUMI, A., YEUNG, R., AND DE FELICE, G. Diagrammatic differentiation for quantum machine learning. *Electronic Proceedings in Theoretical Computer Science* 343 (Sept. 2021), 132–144.
- [27] VANRIETVELDE, A., AND CHIRIBELLA, G. Universal control of quantum processes using sector-preserving channels. *Quantum Information and Computation* 21, 15 & 16 (Nov. 2021), 1320–1352.
- [28] VANRIETVELDE, A., KRISTJÁNSSON, H., AND BARRETT, J. Routed quantum circuits. *Quantum* 5 (2021), 503.
- [29] WANG, Q., YEUNG, R., AND KOCH, M. Differentiating and integrating zx diagrams with applications to quantum machine learning, 2022.
- [30] WILSON, P., AND ZANASI, F. Reverse derivative ascent: A categorical approach to learning boolean circuits. *Electronic Proceedings in Theoretical Computer Science* 333 (Feb. 2021), 247–260.
- [31] WILSON, P., AND ZANASI, F. An axiomatic approach to differentiation of polynomial circuits. *Journal of Logical and Algebraic Methods in Programming* 135 (2023), 100892.

## Appendix A Basic Lemmas

The following two lemmas follow immediately from the bra-ket definition of  $\blacktriangle$ :

**Lemma A.1.**

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \blacktriangle \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} | \\ | \\ | \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \blacktriangle \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad (\text{A.1})$$

**Lemma A.2.**

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \blacktriangle \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad (\text{A.2})$$

**Lemma A.3.**

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad (\text{A.3})$$

*Proof.*

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad (\text{A.4})$$

□

**Lemma A.4.**

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad (\text{A.5})$$

*Proof.*

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad (\text{A.6})$$

□

**Lemma A.5.**

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad (\text{A.7})$$

*Proof.*

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad (\text{A.8})$$

□

**Lemma A.6.**

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad (\text{A.9})$$

*Proof.*

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \end{array} \stackrel{(BZW)}{=} \begin{array}{c} \text{Diagram 4} \end{array} \stackrel{(ADD)}{=} \begin{array}{c} \text{Diagram 5} \end{array} \quad (A.10)$$

□

**Lemma A.7.**

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} \quad (A.11)$$

*Proof.*

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} \stackrel{(A.9)}{=} \begin{array}{c} \text{Diagram 3} \end{array} \stackrel{(A.7)}{=} \begin{array}{c} \text{Diagram 4} \end{array} \quad (A.12)$$

□

**Lemma A.8.**

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} \quad (A.13)$$

*Proof.*

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} \stackrel{(BZW)}{=} \begin{array}{c} \text{Diagram 3} \end{array} \stackrel{(Aso)}{=} \begin{array}{c} \text{Diagram 4} \end{array} \quad (A.14)$$

$$\begin{array}{c} \text{Diagram 5} \end{array} \stackrel{(A.9)}{=} \begin{array}{c} \text{Diagram 6} \end{array} = \dots = \begin{array}{c} \text{Diagram 7} \end{array} \stackrel{(A.9)}{=} \begin{array}{c} \text{Diagram 8} \end{array}$$

□

**Lemma A.9.**

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} \quad (A.15)$$

*Proof.*

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \end{array} \stackrel{(TA)}{=} \begin{array}{c} \text{Diagram 4} \end{array} \quad (A.16)$$

$$\begin{array}{c} \text{Diagram 5} \end{array} \stackrel{(BZW)}{=} \begin{array}{c} \text{Diagram 6} \end{array} \stackrel{(K0)}{=} \begin{array}{c} \text{Diagram 7} \end{array} \stackrel{(Bs0)}{=} \begin{array}{c} \text{Diagram 8} \end{array} \stackrel{(Ept)}{=} \begin{array}{c} \text{Diagram 9} \end{array}$$

□

**Lemma A.10.**

$$\begin{array}{c} \text{Diagram 10} \end{array} = \begin{array}{c} \text{Diagram 11} \end{array} \quad (A.17)$$

*Proof.*

$$\begin{array}{c} \text{Diagram 12} \end{array} \stackrel{(BZW)}{=} \begin{array}{c} \text{Diagram 13} \end{array} \stackrel{(WW)}{=} \begin{array}{c} \text{Diagram 14} \end{array} \quad (A.18)$$

$$\begin{array}{c} \text{Diagram 15} \end{array} \stackrel{(BZW)}{=} \begin{array}{c} \text{Diagram 16} \end{array} \stackrel{(BZW)}{=} \begin{array}{c} \text{Diagram 17} \end{array} = \begin{array}{c} \text{Diagram 18} \end{array}$$

□

**Lemma A.11.**

$$\begin{array}{c} \text{Diagram 19} \end{array} = \begin{array}{c} \text{Diagram 20} \end{array} \quad (A.19)$$

*Proof.*

$$\begin{array}{c} \text{Diagram 21} \end{array} \stackrel{(Pcy)}{=} \begin{array}{c} \text{Diagram 22} \end{array} = \begin{array}{c} \text{Diagram 23} \end{array} \quad (A.20)$$

$$\begin{array}{c} \text{Diagram 24} \end{array} \stackrel{(TA)}{=} \begin{array}{c} \text{Diagram 25} \end{array} \stackrel{(A.5)}{=} \begin{array}{c} \text{Diagram 26} \end{array} = \begin{array}{c} \text{Diagram 27} \end{array}$$

□



**Lemma A.12.**

(A.21)

*Proof.*

(A.22)

□

## Appendix B Main Proofs

### B.1 Proofs for Section 3

#### Proof of Lemma 3.1

*Proof.* First of all, using (BZW) we can rewrite the LHS to

(B.1)

Then clearly

(B.2)

Meanwhile,

(B.3)

Thus the two sides are equal over the Z basis and so are equal as diagrams.

□

#### Proof of Lemma 3.2

*Proof.* We prove by plugging red and commutativity of matrix addition. By definition of controlled matrices, plugging  $\begin{array}{c} \bullet \\ | \end{array}$  gives  $I_n$  on both sides. Meanwhile, plugging  $\begin{array}{c} \bullet \\ | \end{array}$  gives:

$$\begin{aligned}
 & \begin{array}{c} \bullet \\ \curvearrowright \\ \begin{array}{|c|} \hline \tilde{M}_1 \\ \hline \vdots \\ \hline \end{array} \quad \begin{array}{|c|} \hline \tilde{M}_2 \\ \hline \vdots \\ \hline \end{array} \end{array} = \begin{array}{|c|} \hline M_1 \\ \hline \vdots \\ \hline \end{array} + \begin{array}{|c|} \hline M_2 \\ \hline \vdots \\ \hline \end{array} \\
 & = \begin{array}{|c|} \hline M_2 \\ \hline \vdots \\ \hline \end{array} + \begin{array}{|c|} \hline M_1 \\ \hline \vdots \\ \hline \end{array} = \begin{array}{c} \bullet \\ \curvearrowright \\ \begin{array}{|c|} \hline \tilde{M}_2 \\ \hline \vdots \\ \hline \end{array} \quad \begin{array}{|c|} \hline \tilde{M}_1 \\ \hline \vdots \\ \hline \end{array} \end{array}
 \end{aligned} \tag{B.4}$$

□

### Proof of Lemma 3.4

*Proof.*

$$\begin{aligned}
 & \begin{array}{c} \begin{array}{|c|} \hline 1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline -1 \\ \hline \end{array} \\ \curvearrowright \\ \begin{array}{|c|} \hline \tilde{M} \\ \hline \vdots \\ \hline \end{array} \quad \begin{array}{|c|} \hline \tilde{M} \\ \hline \vdots \\ \hline \end{array} \end{array} \stackrel{(Mcpy)}{=} \begin{array}{c} \begin{array}{|c|} \hline 1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline -1 \\ \hline \end{array} \\ \curvearrowright \\ \begin{array}{|c|} \hline \tilde{M} \\ \hline \vdots \\ \hline \end{array} \end{array} \stackrel{(A.11)}{=} \begin{array}{c} \bullet \\ | \\ \begin{array}{|c|} \hline \tilde{M} \\ \hline \vdots \\ \hline \end{array} \end{array} = \begin{array}{c} \bullet \\ | \\ \text{---} \end{array}
 \end{aligned} \tag{B.5}$$

□

### Proof of Lemma 3.5

*Proof.*

$$\begin{aligned}
 & \begin{array}{c} \bullet \\ \curvearrowright \\ \begin{array}{|c|} \hline \tilde{M}_1 \\ \hline \vdots \\ \hline \end{array} \quad \begin{array}{|c|} \hline \tilde{M}_2 \\ \hline \vdots \\ \hline \end{array} \quad \begin{array}{|c|} \hline \tilde{M}_3 \\ \hline \vdots \\ \hline \end{array} \end{array} \stackrel{(BZW)}{=} \begin{array}{c} \begin{array}{|c|} \hline \tilde{M}_1 \\ \hline \vdots \\ \hline \end{array} \quad \begin{array}{|c|} \hline \tilde{M}_2 \\ \hline \vdots \\ \hline \end{array} \quad \begin{array}{|c|} \hline \tilde{M}_3 \\ \hline \vdots \\ \hline \end{array} \end{array}
 \end{aligned} \tag{B.6}$$

$$\stackrel{(Mcpy)}{=} \begin{array}{c} \bullet \\ \curvearrowright \\ \begin{array}{|c|} \hline \tilde{M}_1 \\ \hline \vdots \\ \hline \end{array} \quad \begin{array}{|c|} \hline \tilde{M}_1 \\ \hline \vdots \\ \hline \end{array} \quad \begin{array}{|c|} \hline \tilde{M}_2 \\ \hline \vdots \\ \hline \end{array} \quad \begin{array}{|c|} \hline \tilde{M}_3 \\ \hline \vdots \\ \hline \end{array} \end{array}$$

$$\stackrel{(3.1)}{=} \begin{array}{c} \bullet \\ \curvearrowright \\ \begin{array}{|c|} \hline \tilde{M}_1 \\ \hline \vdots \\ \hline \end{array} \quad \begin{array}{|c|} \hline \tilde{M}_2 \\ \hline \vdots \\ \hline \end{array} \quad \begin{array}{|c|} \hline \tilde{M}_1 \\ \hline \vdots \\ \hline \end{array} \quad \begin{array}{|c|} \hline \tilde{M}_3 \\ \hline \vdots \\ \hline \end{array} \end{array}$$

□

**Proof of Lemma 3.6**

*Proof.* As before, plugging  $|0\rangle$  gives

(B.7)

Meanwhile, plugging  $|1\rangle$  gives

(B.8)

Completing the proof

□

**Proof of Lemma 3.7**

*Proof.*

□

**Proof of Lemma 3.8**

*Proof.* It is clear that is the controlled state  $\tilde{0}$ .

Then we have:

(B.9)

□

**Proof of Lemma 3.9**

*Proof.*  $\tilde{\psi} \circ \boxed{-1}$  is still a controlled state since  $\boxed{-1}$  does nothing to  $\bullet$ . Then  $\tilde{\psi} \circ \boxed{-1}$  inverts  $\tilde{\psi}$  since:

$$\begin{array}{c}
 \begin{array}{c} \text{Diagram 1: } \tilde{\psi} \circ \boxed{-1} \text{ applied to } \tilde{\psi} \\ \text{Diagram 2: } \text{Simplification using (3.4) and (A.11)} \\ \text{Diagram 3: } \text{Further simplification} \\ \text{Diagram 4: } \text{Final result } \tilde{0} \end{array} \\
 \text{(B.10)}
 \end{array}$$

□

**Proof of Lemma 3.9**

*Proof.*

$$\begin{array}{c}
 \tilde{\psi}_1 \boxtimes (\tilde{\psi}_2 \boxplus \tilde{\psi}_3) = \begin{array}{c} \text{Diagram 1} \end{array} \stackrel{(BZW)}{=} \begin{array}{c} \text{Diagram 2} \end{array} \\
 \text{(B.11)}
 \end{array}$$

$$\begin{array}{c}
 = \begin{array}{c} \text{Diagram 3} \end{array} \stackrel{(3.4)}{=} \begin{array}{c} \text{Diagram 4} \end{array} \\
 \text{(B.12)}
 \end{array}$$

$$\begin{array}{c}
 = \begin{array}{c} \text{Diagram 5} \end{array} = \begin{array}{c} \text{Diagram 6} \end{array} \\
 \text{(B.13)}
 \end{array}$$



Where  $D_{even}, D_{odd}$  are PNF diagrams. Since they are over  $n$  variables, we can apply the inductive hypothesis and obtain:

$$D_{even} = \begin{bmatrix} 1 & a_0 \\ 0 & a_2 \\ \dots & \dots \\ 0 & a_{2^{n+1}-2} \end{bmatrix}, D_{odd} = \begin{bmatrix} 1 & a_1 \\ 0 & a_3 \\ \dots & \dots \\ 0 & a_{2^{n+1}-1} \end{bmatrix} \quad (*)$$

Next, plugging red we observe:

Diagram (B.19) illustrates the decomposition of a PNF diagram  $D$  into  $D_{even}$  and  $D_{odd}$  components. The left side shows  $D$  with a red dot on its output line. This is equal to the sum of  $D_{even}$  and  $D_{odd}$  with a red dot on the output of  $D_{odd}$ . The right side shows the same sum, but with a red dot on the output of  $D_{even}$ . The equation is labeled (B.19).

Meanwhile,

Diagram (B.20) illustrates the decomposition of a PNF diagram  $D$  into  $D_{even}$  and  $D_{odd}$  components with a  $\pi$  symbol. The left side shows  $D$  with a  $\pi$  symbol on its output line. This is equal to the sum of  $D_{even}$  and  $D_{odd}$  with a  $\pi$  symbol on the output of  $D_{odd}$ . The right side shows the same sum, but with a  $\pi$  symbol on the output of  $D_{even}$ . The equation is labeled (B.20).

Summing these together,

Diagram (B.21) shows the sum of the previous two equations. The left side is  $D$  with a  $\pi$  symbol on its output line. This is equal to the sum of  $D$  with a red dot on its output line and  $D$  with a  $\pi$  symbol on its output line. The right side shows the sum of  $D_{even}$  with a red dot on its output line and  $D_{odd}$  with a  $\pi$  symbol on its output line. The equation is labeled (B.21).

$$= (D_{\text{even}} \otimes |0\rangle) + (D_{\text{odd}} |1\rangle \langle 1| \otimes |1\rangle) \quad (\text{B.22})$$

$$\stackrel{(*)}{=} \begin{bmatrix} 1 & a_0 \\ 0 & a_2 \\ \dots & \dots \\ 0 & a_{2^{n+1}-2} \end{bmatrix} \otimes |0\rangle + \begin{bmatrix} 0 & a_1 \\ 0 & a_3 \\ \dots & \dots \\ 0 & a_{2^{n+1}-1} \end{bmatrix} \otimes |1\rangle \quad (\text{B.23})$$

$$= \begin{bmatrix} 1 & a_0 \\ 0 & 0 \\ 0 & a_2 \\ 0 & 0 \\ \dots & \dots \\ 0 & a_{2^{n+1}-2} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & a_1 \\ 0 & 0 \\ 0 & a_3 \\ \dots & \dots \\ 0 & 0 \\ 0 & a_{2^{n+1}-1} \end{bmatrix} = \begin{bmatrix} 1 & a_0 \\ 0 & a_1 \\ 0 & a_2 \\ 0 & a_3 \\ \dots & \dots \\ 0 & a_{2^{n+1}-2} \\ 0 & a_{2^{n+1}-1} \end{bmatrix} \quad (\text{B.24})$$

Completing the inductive step. □

## B.2 Proofs for Section 4

### Proof of Theorem 4.1

*Proof.* Let  $A$  be an arithmetic diagram. If  $A = \boxed{a}$ , we are done.

Otherwise,  $A$  has at least one output. First, we shall rewrite  $A$  into three layers, consisting of: (1) a single  $W$  at the top, (2) a layer of  $\circlearrowleft$  and (3) a layer of  $\boxed{a}$ 's and  $\blacktriangledown$ 's. Then we shall collect terms and order the boxes to produce a PNF.

If the top of  $A$  is not already  $\blacktriangle$ , it must be  $\circlearrowleft$ . It cannot be  $\boxed{a}$  since the remaining arithmetic diagram would then have no inputs which is impossible. It cannot be  $\blacktriangledown$  since there is only one input and arithmetic diagrams cannot contain  $\cap$ . Thus we can rewrite:

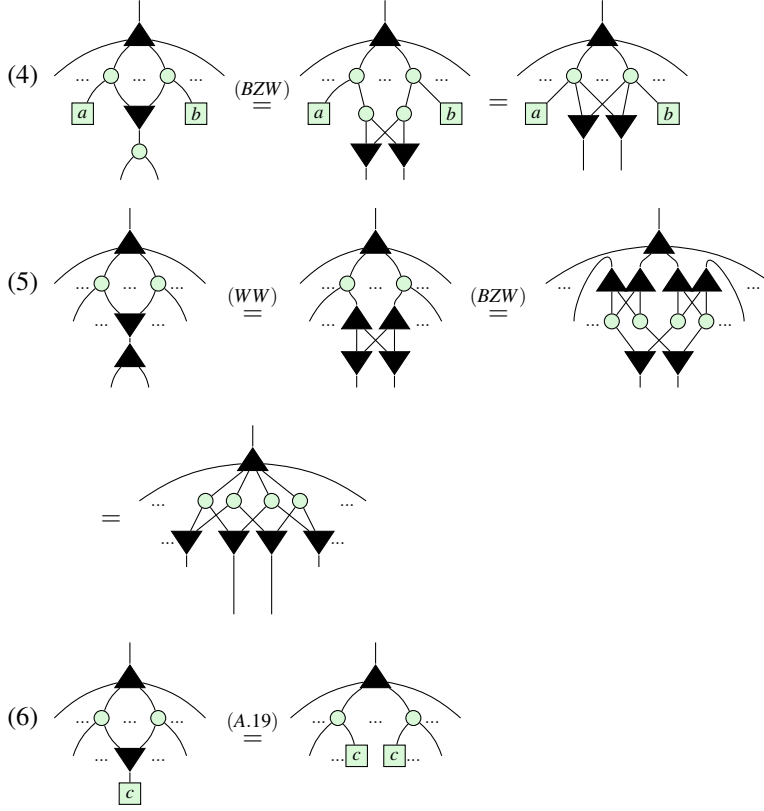
$$(1) \quad \circlearrowleft \stackrel{(A.1)}{=} \boxed{0} \blacktriangle \circlearrowleft$$

(1) guarantees there is a  $W$  at the top. We shall now repeatedly apply rewrites underneath the  $W$  until there are exactly three layers. Assume that fusion is applied as much as possible between each stage and (A.15) is applied

and simplified with (K0) to remove  $\circlearrowleft$  whenever possible. Then for as long as there are at least 4 layers, we can apply one of the following rewrites:

$$(2) \quad \begin{array}{c} \blacktriangle \\ \cap \\ \blacktriangledown \end{array} \stackrel{(A.9)}{=} \begin{array}{c} \blacktriangle \\ \boxed{2} \end{array} = \begin{array}{c} \blacktriangle \\ \circlearrowleft \\ \boxed{2} \end{array}$$

$$(3) \quad \begin{array}{c} \blacktriangle \\ \cap \\ \circlearrowleft \\ \blacktriangle \end{array} \stackrel{(BZW)}{=} \begin{array}{c} \blacktriangle \\ \cap \\ \blacktriangle \cap \blacktriangle \\ \cap \\ \circlearrowleft \cap \circlearrowleft \end{array} = \begin{array}{c} \blacktriangle \\ \cap \\ \circlearrowleft \cap \circlearrowleft \\ \cap \\ \blacktriangledown \cap \blacktriangledown \end{array}$$



Clearly, we can only stop applying these rules once  $A$  is a sum of products of copies. Steps (2) and (3) ensure the top of  $A$  has such a structure and steps (4) - (6) ensure that there is nothing beneath the  $\blacktriangledown$ 's. To see that this will always terminate, observe that (2) and (3) preserve the depth of  $A$  while (4), (5), (6) all decrease it. (2) and (3) can only be applied a finite number of times before another simplification must be used. So repeatedly applying these rewrites must eventually shrink the depth down to 3, as desired. Finally, to put  $A$  in PNF we must:

(7) Collect terms: whenever there are two boxes connected to exactly the same set of  $\blacktriangledown$ 's, use (A.13) to fuse them together.

(8) Pad: use (A.7) to insert  $\boxed{0}$  for any connectivities that do not exist in  $A$ .

(9) Reorder: use (Sym) to reorder coefficients into the canonical order.

Step (7) ensures that every  $\blacktriangledown$  has unique connectivity. Step (8) ensures there are exactly  $2^n$  coefficients so that step (9) can order them in the appropriate way.

Thus  $A$  has been written in PNF, completing the proof.  $\square$

### Proof of Theorem 4.2

*Proof.* First, we show  $\phi_n$  is a homomorphism, i.e.

$$\forall p, q \in \mathcal{P}_n, \phi_n(p + q) = \phi_n(p) \boxplus \phi_n(q), \quad \phi_n(p \times q) = \phi_n(p) \boxtimes \phi_n(q) \quad (\text{B.25})$$

The strategy for the proof will be an induction on  $n$ .



**Base case:** We have not defined controlled states for  $n = 0$ , so the base case begins with  $n = 1$ . Let  $p, q \in \mathcal{P}_1$ . Write as  $p(x_1) = a_0 + a_1x_1, q(x_1) = b_0 + b_1x_1$ , where  $a_0, a_1, b_0, b_1 \in \mathbb{C}$ . Then since  $p + q = a_0 + b_0 + (a_1 + b_1)x_1$ ,

$$\begin{aligned} \phi_1(p) \boxplus \phi_1(q) &= \text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3} = \text{Diagram 4} \quad (\text{B.26}) \\ &\stackrel{(A.9)}{=} \text{Diagram 5} = \text{Diagram 6} = \phi_1(p+q) \end{aligned}$$

Meanwhile, since  $p \times q = a_0b_0 + (a_0b_1 + a_1b_0)x_1$ ,

$$\begin{aligned} \phi_1(p) \boxtimes \phi_1(q) &= \text{Diagram 1} = \text{Diagram 2} \stackrel{(A.17)}{=} \text{Diagram 3} \\ &\stackrel{(A.19)}{=} \text{Diagram 4} \stackrel{(Pcy)}{=} \text{Diagram 5} \stackrel{(A.15)}{=} \text{Diagram 6} \\ &\stackrel{(A.21)}{=} \text{Diagram 7} \stackrel{(A.9)}{=} \text{Diagram 8} = \text{Diagram 9} = \phi_1(p \times q) \quad (\text{B.27}) \end{aligned}$$

Completing the base case.

### Inductive step:

Let  $\text{Hom}(n)$  assert that  $\phi_n$  is a homomorphism. Then for the inductive step we wish to prove that  $\forall n, \text{Hom}(n) \implies \text{Hom}(n+1)$ .

The proof relies on the recursive definition of  $R[x_1, x_2] = R[x_1][x_2]$ , for any ring  $R$ , to rewrite an arbitrary polynomial  $p(x_1, \dots, x_{n+1}) = a_0 + a_1x_{n+1} + \dots + a_{2^{n+1}-1}x_1x_2\dots x_{n+1} \in \mathcal{P}_{n+1}$  as  $p(x_{n+1}) = p_0 + p_1x_{n+1}$ , where  $p_0, p_1 \in \mathcal{P}_n$ . This allows the  $p_i$  to be treated similarly to the scalars in the base case. To emphasise this, they will be drawn in green boxes. To help distinguish when an operation is covered by the inductive hypothesis, the wires for variables

$x_1, \dots, x_n$  will be drawn in blue, while the  $x_{n+1}$  wires will be drawn in black. Thus the inductive hypothesis states that:

$$\phi_n(p_i) \boxplus \phi_n(q_i) = \begin{array}{c} \text{Diagram: Two triangles with inputs } x_1, \dots, x_n \text{ and outputs } p_i, q_i. \end{array} = \begin{array}{c} \text{Diagram: A single triangle with input } x_1, \dots, x_n \text{ and output } p_i + q_i. \end{array} = \phi_n(p_i + q_i) \quad (\text{IH1})$$

$$\phi_n(p_i) \boxtimes \phi_n(q_i) = \begin{array}{c} \text{Diagram: Two triangles with inputs } x_1, \dots, x_n \text{ and outputs } p_i, q_i. \end{array} = \begin{array}{c} \text{Diagram: A single triangle with input } x_1, \dots, x_n \text{ and output } p_i \times q_i. \end{array} = \phi_n(p_i \times q_i) \quad (\text{IH2})$$

Let  $p(x_{n+1}) = p_0 + p_1 x_{n+1}$  and  $q(x_{n+1}) = q_0 + q_1 x_{n+1}$ , where  $p_0, p_1, q_0, q_1 \in \mathcal{P}_n$ . According to our hypothesis,

$$\phi_{n+1}(p) = \phi_{n+1}(p_0) \boxplus (\phi_{n+1}(p_1) \boxtimes \phi_{n+1}(x_{n+1})) \quad (\text{B.28})$$

and likewise for  $\phi_{n+1}(q)$ . We first verify correctness of the constructed controlled diagram

$$\begin{array}{c} \text{Diagram: A triangle with input } x_1, \dots, x_n, x_{n+1} \text{ and output } p. \end{array} = \begin{array}{c} \text{Diagram: A triangle with inputs } x_1, \dots, x_n, x_{n+1} \text{ and outputs } p_0, p_1. \end{array} \quad (\text{B.29})$$

because it results in  $|0\dots 0\rangle$  when the control is  $|0\rangle$ , and in the state corresponding to  $p$  when the control is  $|1\rangle$ .

$$\begin{array}{c} \text{Diagram: A triangle with input } x_1, \dots, x_n, x_{n+1} \text{ and output } p. \end{array} = \begin{array}{c} \text{Diagram: A triangle with inputs } x_1, \dots, x_n, x_{n+1} \text{ and outputs } p_0, p_1. \end{array} = \begin{array}{c} \text{Diagram: A triangle with inputs } x_1, \dots, x_n, x_{n+1} \text{ and outputs } p_0, p_1. \end{array} = \begin{array}{c} \text{Diagram: A triangle with inputs } x_1, \dots, x_n, x_{n+1} \text{ and outputs } p_0, p_1. \end{array} \quad (\text{B.30})$$

$$\begin{array}{c} \text{Diagram: A triangle with input } x_1, \dots, x_n, x_{n+1} \text{ and output } p. \end{array} = \begin{array}{c} \text{Diagram: A triangle with inputs } x_1, \dots, x_n, x_{n+1} \text{ and outputs } p_0, p_1. \end{array} = \begin{array}{c} \text{Diagram: A triangle with inputs } x_1, \dots, x_n, x_{n+1} \text{ and outputs } p_0, p_1. \end{array} + \begin{array}{c} \text{Diagram: A triangle with inputs } x_1, \dots, x_n, x_{n+1} \text{ and outputs } p_0, p_1. \end{array} \quad (\text{B.31})$$

$$= \begin{array}{c} \text{Diagram: A triangle with inputs } x_1, \dots, x_n, x_{n+1} \text{ and outputs } p_0, p_1. \end{array} + \begin{array}{c} \text{Diagram: A triangle with inputs } x_1, \dots, x_n, x_{n+1} \text{ and outputs } p_0, p_1. \end{array} = \begin{array}{c} \text{Diagram: A triangle with inputs } x_1, \dots, x_n, x_{n+1} \text{ and outputs } p_0, p_1. \end{array} + \begin{array}{c} \text{Diagram: A triangle with inputs } x_1, \dots, x_n, x_{n+1} \text{ and outputs } p_0, p_1. \end{array}$$

Applying this in the inductive step for constructing a controlled diagram of  $p+q$  from those of  $p$  and  $q$ :

$$\begin{aligned}
 \phi_{n+1}(p) \boxplus \phi_{n+1}(q) &= \text{Diagram 1} = \text{Diagram 2} \\
 &= \text{Diagram 3} = \text{Diagram 4} \\
 &= \text{Diagram 5} \stackrel{(BZW)}{=} \text{Diagram 6} \\
 &\stackrel{(\text{??})}{=} \text{Diagram 7} = \phi_{n+1}((p_0+q_0) + (p_1+q_1) x_{n+1}) = \phi_{n+1}(p+q)
 \end{aligned}
 \tag{B.32}$$

The diagrams are as follows:

- Diagram 1:** A controlled diagram for  $\phi_{n+1}(p) \boxplus \phi_{n+1}(q)$ . It has a top control line and two bottom data lines labeled  $x_1, \dots, x_n, x_{n+1}$ . The diagram consists of two triangular blocks, one labeled  $\bar{p}$  and the other  $\bar{q}$ , connected by a crossing.
- Diagram 2:** A more complex diagram with two yellow shaded regions. Each region contains a triangular block labeled  $\bar{p}_0, \dots, \bar{p}_1$  and  $\bar{q}_0, \dots, \bar{q}_1$ . The bottom data lines are  $x_1, \dots, x_n, x_{n+1}$ .
- Diagram 3:** A diagram with two green circles on the top control line. The bottom data lines are  $x_1, \dots, x_n, x_{n+1}$ .
- Diagram 4:** A diagram with two green circles on the top control line. The bottom data lines are  $x_1, \dots, x_n, x_{n+1}$ .
- Diagram 5:** A diagram with two green circles on the top control line. The bottom data lines are  $x_1, \dots, x_n, x_{n+1}$ .
- Diagram 6:** A diagram with two green circles on the top control line. The bottom data lines are  $x_1, \dots, x_n, x_{n+1}$ .
- Diagram 7:** A diagram with two green circles on the top control line. The bottom data lines are  $x_1, \dots, x_n, x_{n+1}$ .

Similarly, for multiplication:

This completes the inductive step, proving that  $\forall n > 1$ ,  $\phi_n$  is a homomorphism.

Finally, to see  $\phi_n$  is an isomorphism, we use Theorem 4.1 to write an arbitrary controlled state in PNF:

$$\begin{bmatrix} 1 & a_0 \\ 0 & a_1 \\ \dots & \dots \\ 0 & a_{2^n-1} \end{bmatrix} = \text{Diagram} \quad (\text{B.33})$$

Then all we have to do is interpret it as the image of a polynomial:

$$\text{Diagram} = \text{Diagram} \quad (\text{B.34})$$

$$= \phi_n(a_0) + \phi_n(a_1 x_n) + \dots + \phi_n(a_{2^n-1} x_1 x_2 \dots x_n) \quad (\text{B.35})$$

$$= \phi_n(a_0 + a_1 x_n + \dots + a_{2^n-1} x_1 x_2 \dots x_n) \quad (\text{B.36})$$

□

### B.3 Proofs for Section 6.2

#### Proof of Proposition 4

*Proof.* Ctrl is sound with regards to sequential composition:

$$\begin{aligned} & \text{Diagram} = \text{Diagram} \quad (\text{B.37}) \\ & = \text{Diagram} \stackrel{(*)}{=} \text{Diagram} = \text{Diagram} \end{aligned}$$

Where  $(*)$  follows from Proposition 1. □

#### Proof of Proposition 5

*Proof.* Consider the morphism:  $\phi_{V,W} : \text{Ctrl}(V) \otimes \text{Ctrl}(W) \rightarrow \text{Ctrl}(V \otimes W)$ :

$$\phi_{V,W} = \text{Diagram} \quad (\text{B.38})$$

By conjugating by  $\phi$ , Ctrl is sound under parallel composition of any  $M_1 : V \rightarrow V$ ,  $M_2 : W \rightarrow W$ :

(B.39)

Parallel composition is associative by associativity of the tensor product and of  $\phi$ :

(B.40)

□

### Proof of Proposition 6

*Proof.* Plugging basis states:

(B.41)

Diagrammatic equation (B.42) showing the reduction of a controlled operator with an AND gate to a standard controlled operator. The equation consists of three parts connected by equals signs. The first part shows a box labeled  $\tilde{M}$  with two control lines, each having a green circle and an AND gate symbol ( $\wedge$ ). The second part shows the same box  $\tilde{M}$  with the AND gates simplified to single green circles, labeled with the rule  $(K0)$ . The third part shows a box labeled  $\tilde{M}$  with two control lines, each having a single green circle, labeled with the rule  $(B.44)$ . The final result is a box labeled  $M$  with two control lines, each having a single green circle, labeled with the rule  $(A.1)$ .

□

The AND gate is a primitive in the ZH-calculus [2]. We can also represent the binary AND gate in the ZXW-calculus, by deMorgan's law of the diagram for binary OR given in Proposition 3.

**Lemma B.1.**

Diagrammatic equation (B.43) defining the AND gate in the ZXW-calculus. The left side shows a single wire with an AND gate symbol ( $\wedge$ ). The right side shows a wire with a green circle and two arrows pointing to it from two red circles, labeled with the rule  $(B.43)$ .

*Proof.* We can verify this computes the AND gate by computing on basis states.

Diagrammatic equation (B.44) showing the verification of the AND gate on basis states. The equation consists of three parts connected by equals signs. The first part shows a wire with a green circle and two arrows pointing to it from two red circles, labeled with the rule  $(Bs0)$ . The second part shows a wire with a green circle and two arrows pointing to it from two red circles, labeled with the rule  $(K0)$ . The third part shows a wire with a green circle and two arrows pointing to it from two red circles, labeled with the rule  $(A.1)$ .

Thus  $AND(1, x) = x$ . Since the diagram is clearly commutative, it remains to check  $AND(0, x) = 0$ .

Diagrammatic equation (B.45) showing the verification of the AND gate on basis states. The equation consists of two parts connected by equals signs. The first part shows a wire with a green circle and two arrows pointing to it from two red circles, labeled with the rule  $(A.2)$ . The second part shows a wire with a green circle and two arrows pointing to it from two red circles, labeled with the rule  $(A.1)$ .

□