

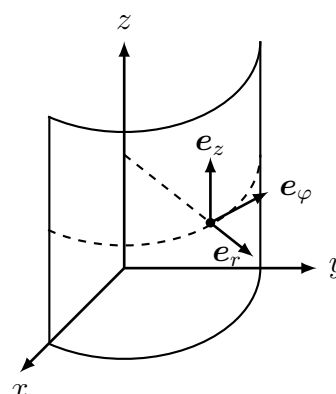
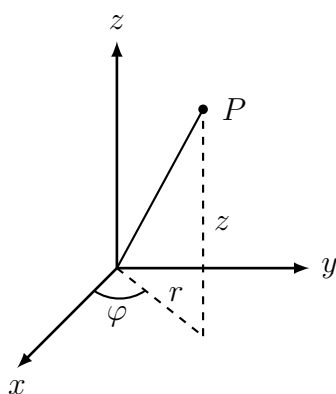
常用数学公式

区艺锋

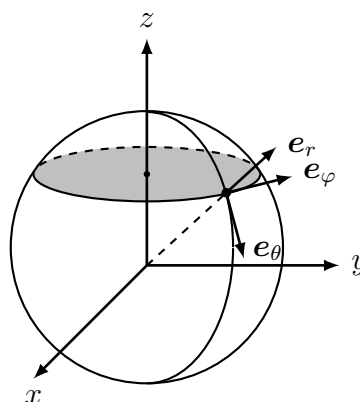
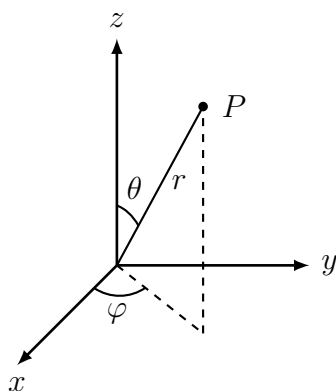
1 曲线坐标系

1.1 变量定义与基矢方向

柱坐标系



球坐标系



1.2 线元与体元

对于一般的曲线坐标系来说，线元和体元的表达式为

$$d\mathbf{l} = h_1 du_1 \mathbf{e}_1 + h_2 du_2 \mathbf{e}_2 + h_3 du_3 \mathbf{e}_3$$

$$dV = h_1 h_2 h_3 du_1 du_2 du_3$$

在直角坐标系中

$$u_1 = x, u_2 = y, u_3 = z$$

$$h_1 = 1, h_2 = 1, h_3 = 1$$

在柱坐标系中

$$u_1 = r, u_2 = \varphi, u_3 = z$$

$$h_1 = 1, h_2 = r, h_3 = 1$$

在球坐标系中

$$u_1 = r, u_2 = \theta, u_3 = \varphi$$

$$h_1 = 1, h_2 = r, h_3 = r \sin \theta$$

2 高斯公式和斯托克斯公式

高斯公式

$$\oint_{\partial \Omega} \mathbf{f} \cdot d\mathbf{S} = \iiint_{\Omega} (\nabla \cdot \mathbf{f}) dV$$

斯托克斯公式

$$\oint_{\partial \Sigma} \mathbf{f} \cdot d\mathbf{l} = \iint_{\Sigma} (\nabla \times \mathbf{f}) \cdot d\mathbf{S}$$

3 梯度、散度、旋度与拉普拉斯算符

3.1 通式

梯度

$$\nabla \psi = \frac{1}{h_1} \frac{\partial \psi}{\partial u_1} \mathbf{e}_1 + \frac{1}{h_2} \frac{\partial \psi}{\partial u_2} \mathbf{e}_2 + \frac{1}{h_3} \frac{\partial \psi}{\partial u_3} \mathbf{e}_3$$

散度

$$\nabla \cdot \mathbf{f} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 f_1) + \frac{\partial}{\partial u_2} (h_3 h_1 f_2) + \frac{\partial}{\partial u_3} (h_1 h_2 f_3) \right]$$

旋度

$$\nabla \times \mathbf{f} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 f_1 & h_2 f_2 & h_3 f_3 \end{vmatrix}$$

拉普拉斯算符

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \psi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial u_3} \right) \right]$$

3.2 柱坐标系

$$\begin{aligned}\nabla\psi &= \frac{\partial\psi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\varphi}\mathbf{e}_\varphi + \frac{\partial\psi}{\partial z}\mathbf{e}_z \\ \nabla\cdot\mathbf{f} &= \frac{1}{r}\frac{\partial}{\partial r}(rf_r) + \frac{1}{r}\frac{\partial f_\varphi}{\partial\varphi} + \frac{\partial f_z}{\partial z} \\ \nabla\times\mathbf{f} &= \left(\frac{1}{r}\frac{\partial f_z}{\partial\varphi} - \frac{\partial f_\varphi}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial f_r}{\partial z} - \frac{\partial f_z}{\partial r}\right)\mathbf{e}_\varphi + \left[\frac{1}{r}\frac{\partial}{\partial r}(rf_\varphi) - \frac{1}{r}\frac{\partial f_r}{\partial\varphi}\right]\mathbf{e}_z \\ \nabla^2\psi &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2}\end{aligned}$$

3.3 球坐标系

$$\begin{aligned}\nabla\psi &= \frac{\partial\psi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\mathbf{e}_\varphi \\ \nabla\cdot\mathbf{f} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2f_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta f_\theta) + \frac{1}{r\sin\theta}\frac{\partial f_\varphi}{\partial\varphi} \\ \nabla\times\mathbf{f} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta f_\varphi) - \frac{\partial f_\theta}{\partial\varphi}\right]\mathbf{e}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial f_r}{\partial\varphi} - \frac{\partial}{\partial r}(rf_\varphi)\right]\mathbf{e}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rf_\theta) - \frac{\partial f_r}{\partial\theta}\right]\mathbf{e}_\varphi \\ \nabla^2\psi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}\end{aligned}$$

4 矢量分析

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = (\mathbf{c}\cdot\mathbf{b})\mathbf{a} - (\mathbf{c}\cdot\mathbf{a})\mathbf{b}$$

$$\nabla(\varphi\psi) = \varphi\nabla\psi + \psi\nabla\varphi$$

$$\nabla\cdot(\varphi\mathbf{f}) = (\nabla\varphi)\cdot\mathbf{f} + \varphi\nabla\cdot\mathbf{f}$$

$$\nabla\times(\varphi\mathbf{f}) = (\nabla\varphi)\times\mathbf{f} + \varphi\nabla\times\mathbf{f}$$

$$\nabla(\mathbf{f}\cdot\mathbf{g}) = \mathbf{f}\times(\nabla\times\mathbf{g}) + (\mathbf{f}\cdot\nabla)\mathbf{g} + \mathbf{g}\times(\nabla\times\mathbf{f}) + (\mathbf{g}\cdot\nabla)\mathbf{f}$$

$$\nabla\cdot(\mathbf{f}\times\mathbf{g}) = (\nabla\times\mathbf{f})\cdot\mathbf{g} - \mathbf{f}\cdot(\nabla\times\mathbf{g})$$

$$\nabla\times(\mathbf{f}\times\mathbf{g}) = (\mathbf{g}\cdot\nabla)\mathbf{f} + (\nabla\cdot\mathbf{g})\mathbf{f} - (\mathbf{f}\cdot\nabla)\mathbf{g} - (\nabla\cdot\mathbf{f})\mathbf{g}$$

$$\nabla \cdot \nabla \varphi = \nabla^2 \varphi$$

$$\nabla \times \nabla \varphi = 0$$

$$\nabla \cdot (\nabla \times \mathbf{f}) = 0$$

$$\nabla \times (\nabla \times \mathbf{f}) = \nabla(\nabla \cdot \mathbf{f}) - \nabla^2 \mathbf{f}$$

5 傅里叶级数

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}$$

其中

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx$$

$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx$$

6 积分

$$\int_{-l}^l \cos^2 \frac{n\pi x}{l} dx = l$$

$$\int_{-l}^l \sin^2 \frac{n\pi x}{l} dx = l$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\int_0^{\infty} e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \quad (a > 0)$$

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$\int_0^{\pi} \sin nx dx = \frac{1 - (-1)^n}{n}$$

$$\int_0^{\pi} x \sin nx dx = -\frac{(-1)^n \pi}{n}$$

$$\int_0^{\pi} x^2 \sin nx dx = \frac{2[(-1)^n - 1]}{n^3} - \frac{(-1)^n \pi^2}{n}$$

$$\int_0^{\pi} \cos nx dx = 0$$

$$\int_0^{\pi} x \cos nx dx = \frac{(-1)^n - 1}{n^2}$$

$$\int_0^{\pi} x^2 \cos nx dx = \frac{(-1)^n 2\pi}{n^2}$$

7 复级数

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad (|z| < \infty)$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \quad (|z| < \infty)$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1)$$

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \quad (|z| < \infty)$$

8 三角函数

8.1 积化和差

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin (\alpha + \beta) + \sin (\alpha - \beta)]$$

$$\cos \alpha \sin \beta = \frac{1}{2} [\sin (\alpha + \beta) - \sin (\alpha - \beta)]$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos (\alpha + \beta) + \cos (\alpha - \beta)]$$

$$\sin \alpha \sin \beta = -\frac{1}{2} [\cos (\alpha + \beta) - \cos (\alpha - \beta)]$$

8.2 和差化积

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\sin \alpha - \sin \beta = 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

8.3 和差角公式

$$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

9 其他

在球坐标系中

$$\nabla \cdot \left(\frac{1}{r^2} \mathbf{e}_r \right) = 0$$

$$\nabla \frac{1}{r} = -\frac{1}{r^2} \mathbf{e}_r$$

$$\nabla^2 \frac{1}{r} = -4\pi \delta(\mathbf{r})$$