Rescaling a gravitational simulation

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Abstract

We reproduce some results on the rescaling of the phase-space evolution of the collapse of a 1d self-gravitating system and argue that there are long-lived quasi stationary states during the system's relaxation. The fact that the graph must obey Liouville's theorem, implying that sections initially distinct must remain so at all later times, together with the approximate fit of the rescaled diagrams and the invariance of the general shape to externl mean fields, seem to suggest that rescaling a distribution to give an estimate for its inner evolution at a later time can give a qualitatively accurate resut, while not taking into account the exact radial stretching and turnaround time.

1 N-body simulation

1.1 Description of the model

In this experiment we analyse a one-dimensional self-gravitating system[1]. In this 1D gravitational setup, the (attractive) force between two sheets of mass m_1 and m_2 is

$$F = 2\pi G m_1 m_2$$

and hence the potential is always positive and increasing with distance,

$$\phi(x) = 2\pi G m |x| \tag{1}$$

for a sheet of mass m at the origin.

The approach of using discrete points in phase space, effectively delta functions in the probability $f(x, v)$, presents several problems. First of all, the overcrowding of the particles in real space when the system is near collapse leads the simulation to slow down and very often to stall. Secondly, any slight deviation of any particles near the center leads to chaotic phase mixing at the center of the phase space diagram.

In this experiment we make use of a simple Vlasov-Poisson solver[2]. This system is numerically more stable, and the advance of the distribution is entirely determined by the timestep we choose, so we can more accurately model the behaviour of a continuous mass distribution as opposed to initially sampling points from our chosen distribution.

We begin with the collisionless Boltzmann equation,

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{d\phi(x)}{dx}\frac{\partial f}{\partial v} = 0 \tag{2}$$

sampled on a discrete Nx by Nv grid of points. Hence we model f as having constant values within each grid square of phase space. We then evolve the grid forwards in time according to the flux balance method, with symmetric updates (half-step in x, step in v, half-step in x). For example, the v step is

$$\frac{\partial f}{\partial t} - \frac{d\phi(x)}{dx}\frac{\partial f}{\partial v} = 0$$

$$f(x_i, v_j, t + \Delta t) = f(x_i, v_j, t) - D_v f(x_i, v_j) + D_v f(x_i, v_{j-1})$$

where Df is the flux conservation term, derived from the flow of phase-space fluid which in one dimension is

$$Df(x_i) \approx \frac{1}{\Delta x} \int_{\frac{\Delta x}{2}-v\Delta t}^{\frac{\Delta x}{2}} [g(x_i) + (g(x_{i+1}) - g(x_{i-1}))\frac{h}{2\Delta x}]dh$$

using a first-order expansion for $f(x)$ around $f(x_i)$.

We now use the model to look at the dynamics of a collapsing self-gravitating system in one spatial dimension. In our simulation, we use a 2000 by 2000 grid of points with timesteps of $0.025s$.

1.2 Phase space winding

If we initialise the system to have uniform spatial density, and a finite width in velocity space, the phase space sheet rotates about itself without ever coiling. This illustrates the case of total collapse without mixing, where each sheet's acceleration is entirely determined by its position relative to the other sheets, so we have infinite spatial density at the origin at a time T_c where

$$T_c = \frac{1}{\sqrt{2\pi G\rho_0}}$$

By giving an initial density perturbation towards the center, we initialise a system that undergoes violent relaxation due to the inner particles collapsing earlier, and therefore lose more energy than they gained when the outer particles begin to collapse, causing them to have a smaller oscillation time and winding up the phase space sheet (phase mixing). The repeating nature of the pattern suggests that we can rescale the distribution to at a time t_1 so that it can fit inside the distribution at a later time t_2, thereby giving us an approximate way to evolve the central part of the distribution forward in time.

2 Results of rescaling

We evolve the distribution forward for 15000 time-steps, which took roughly 150 minutes on a standard CPU. We then split the particles into two sets, the inner set belonging to those particles that seem to cluster around the center in a core, as opposed to the outer ones that form the wound-up halo. The splitting is arbitrary, in this case the middle 25% were chosen as the inner set, which were the particles that never achieve a spatial position larger than 8 on our scale (rounding any phase space density below 0.001 down to zero).

To get a qualitative understanding of the effect of the outer particles on the evolution of innermost ones, we evolve the middle set of particles under its own field, and then under the addition of different external fields. These comparisons are made to illustrate the discrepance we obtain by rescaling the distribution for a later time, as by doing so we have effectively ignored the mass that would have been present at a larger radius prior to the rescaling. Due to the symmetry of the problem, any mass at a distance $|x| > |x_0|$ will have no effect on the motion of mass within the radius x_0. Hence the deviation from a self-consistent evolution of the central particles is due to the overlap in distributions, which in general changes over time.

We note that the two main changes caused by different external mean fields (both static and evolving) are:

- a change in the turnaround time, or roughly speaking the period of oscillation at a given radius
- changes to the deformation of the phase space spiral.

Given that the internal particles evolve independently of the outer mass, disregarding changes in timescale and stretching, the argument is that once we reach the slowly varying distribution the pattern will remain self-similar beyond times at which the simulation breaks down due to numerical errors accumulating (in spite of constraining the energy and probability, the distribution 'diffuses' through phase-space). Windings cannot merge together due to Liouville's theorem, and furthermore energy conservation implies that the mass is confined to the central part of the potential. This is similar to the QSS (quasi stationary state) achieved due to relaxation that is described in [4].

We see that the deformation compounds over time, making nesting a single configuration n times impractical. By matching the rescaling with the decaying oscillation period, we are able to obtain a

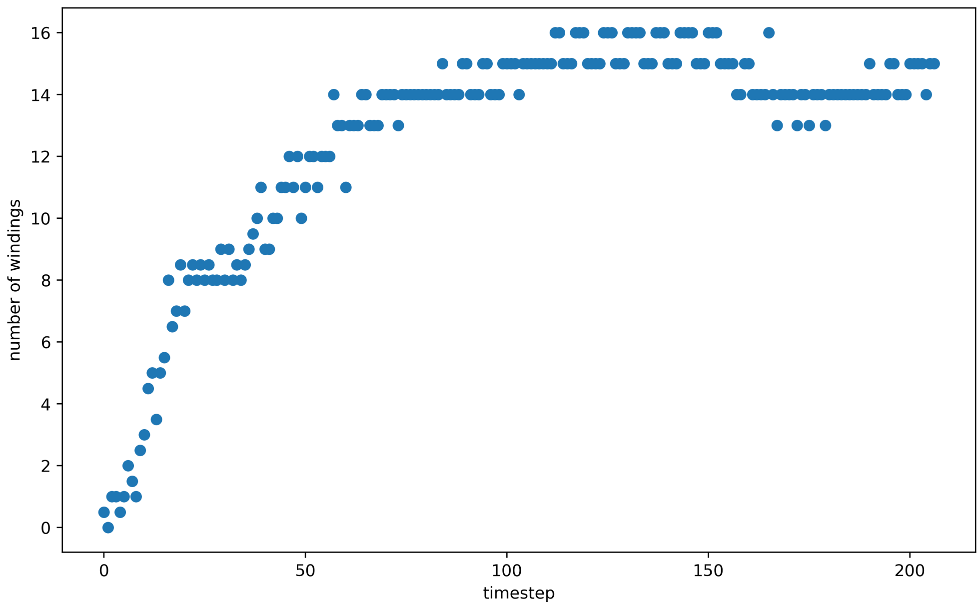

Figure 1: Number of crossings of the spiral along the x-axis, calculated by taking a strip 8 cells wide in v-space and finding the peaks of the resulting density profile. There is some instability towards larger times due to the peaks diffusing in phase space, causing the outermost peaks to gradually vanish, but notwithstanding this the number of windings quickly flattens out. After this stage the pattern is slowly time-varying. (Windings are calculated by $(n-1)//2$ with n the total number of crossings, to exclude the central peak.)

match with the innermost particles, up to the resolution of the model. The fact that the presence of external (uniform or otherwise) fields seems to only impact the winding speed and deformation of the spiral (see Figure 4), suggests that the inner particles (larger than a minimal infinitesimal radius) will qualitatively follow the general shape of the rescaling for a large number of forward steps.

3 Conclusions

Rescaling at earlier times is qualitatively correct (3). This however cannot be used as a predictor for the future state of the distribution since the winding speed of the inner layers of the halo increases at these early times of the relaxation (1), and furthermore the amplitudes of oscillation change by significantly different ratios over these times, as seen in 2.

Plotting the distribution at later times, and following the pattern of an increasing rescaling factor (tending, we suppose, towards 1), 4 shows distributions at different times plotted on top of each other with a scale factor of 1. While not exact, a periodic pattern seems to emerge, in which successive peaks branch off from the central spike, representing the halos slowly spiralling outwards. We have an almost repeating pattern separated by approximately 380 timesteps.

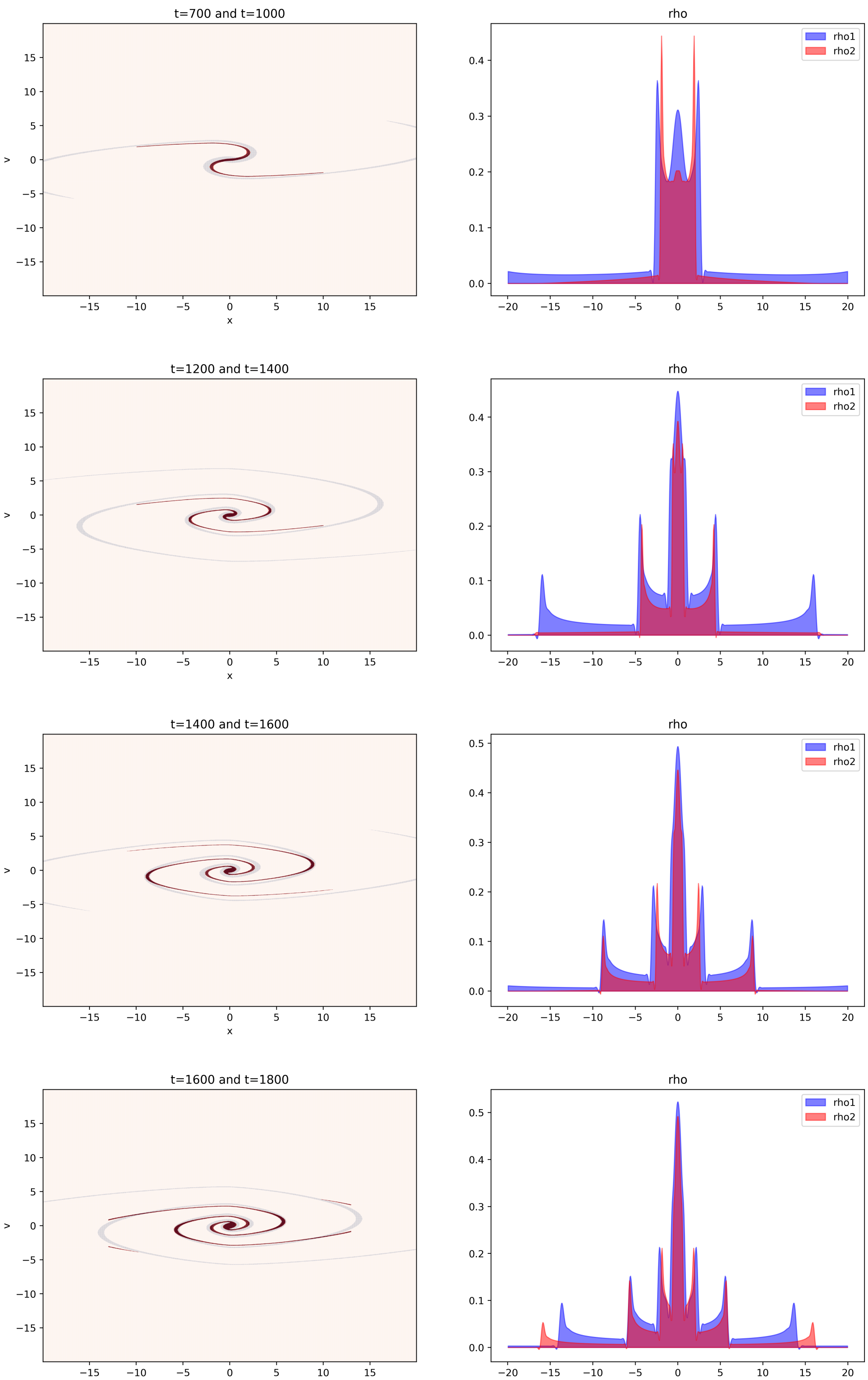

Figure 2: Rescaling at a single interval. Each dark plot has been rescaled by a given factor, and is kept centered with the light plot and is not rotated. 'Rho2' in each case is the density of the rescaled inner plot, which is also rescaled in the x-direction but keeps the same amplitude as it originally had. The scaling factor is chosen such that the central peaks in the real-space density graph line up at the first possible instance, i.e. at the peaks closest to the central spike. The smaller time is that of the dark plot. We can see the matching pattern between successive rescaled plots, and also the decreasing time difference between these in agreement with the decreasing time of oscillation in Figure 1; there are no times in between each pair in which the shape of the central spiral matches, eg. the innermost winding advances by 2π between times. The top two plots have a scaling of 0.5, the third of 0.55 and the last 0.65. This seems to suggest that over time, the similarity of the spiral happens on a larger scale, as more layers of winding appear in the halo. We also note how for the later plots, matching the inner spikes leads to the outer ones falling out of place due to a different degree of radial stretching of the particles. There is nevertheless a remarkable fit between the inner layers of particles.

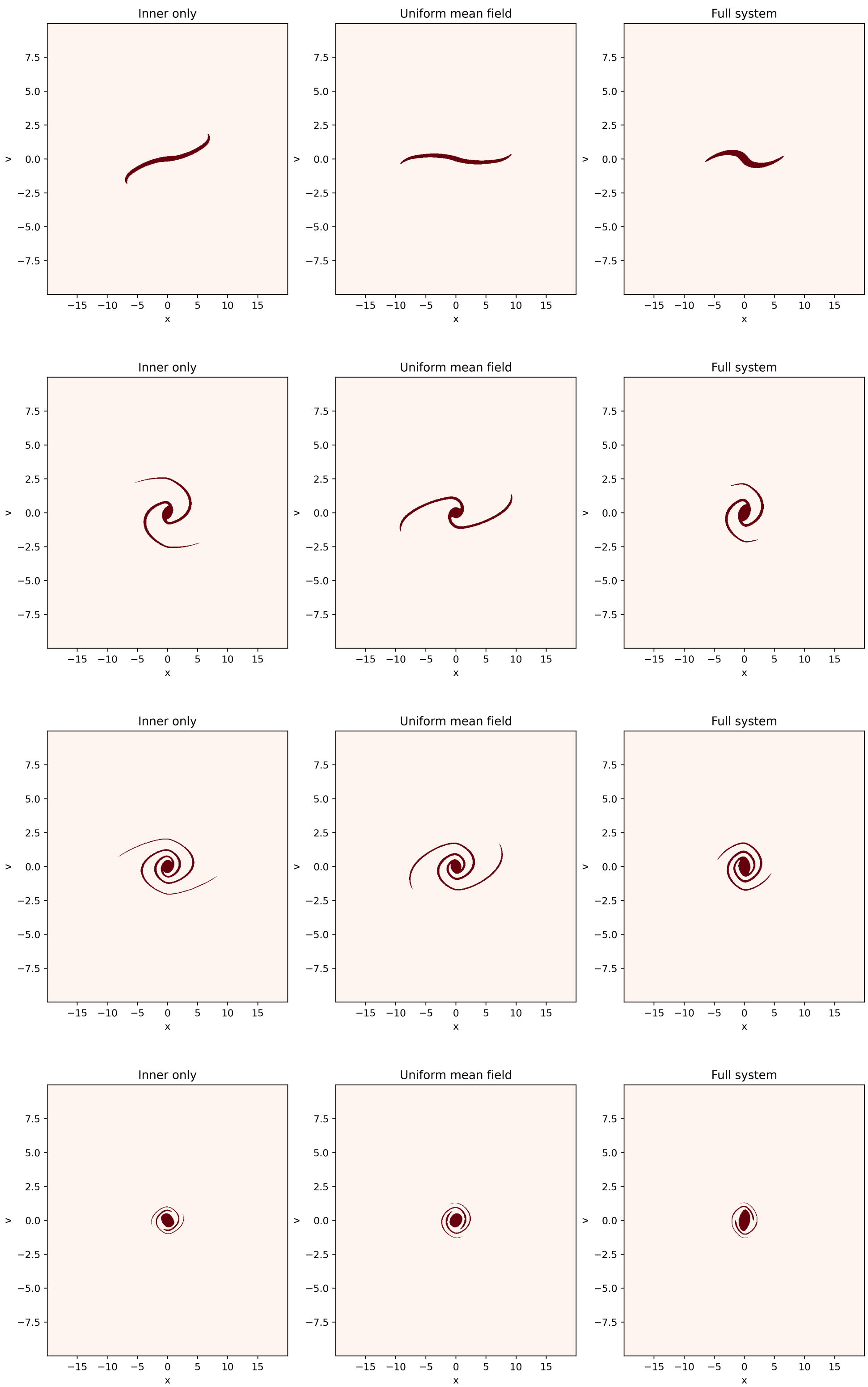

Figure 3: Comparison of the evolution of the inner 25% of particles under different fields. Each horizontal row is at the same timestep. The left column represents the particles evolving entirely under their own field, while the middle column has an additional unvarying density of value 0.02 that is constant in space and stretches across the whole spatial dimension. The right column shows the middle particles of the full evolution in phase space, i.e. without the outer 75% of phase fluid being plotted. This represents the external field that is also evolving over time. Unsurprisingly, the right column shows the fastest winding since the external mass is greatest, leading to a faster collapse towards the center and thus a shorter oscillation time - note that this is not apparent from these graphs but becomes evident when tracing the evolution over small time intervals. More importantly, the general shape and even evolution stage seems very similar between plots, indicating that the presence of particles of a larger oscillation amplitude does not change the qualitative shape of the distribution or the time and length scales over which it evolves. With these initial conditions, any plots beyond the bottom one simply give a more strongly separated core-halo pattern, since the inner spiral ceases to be resolved.

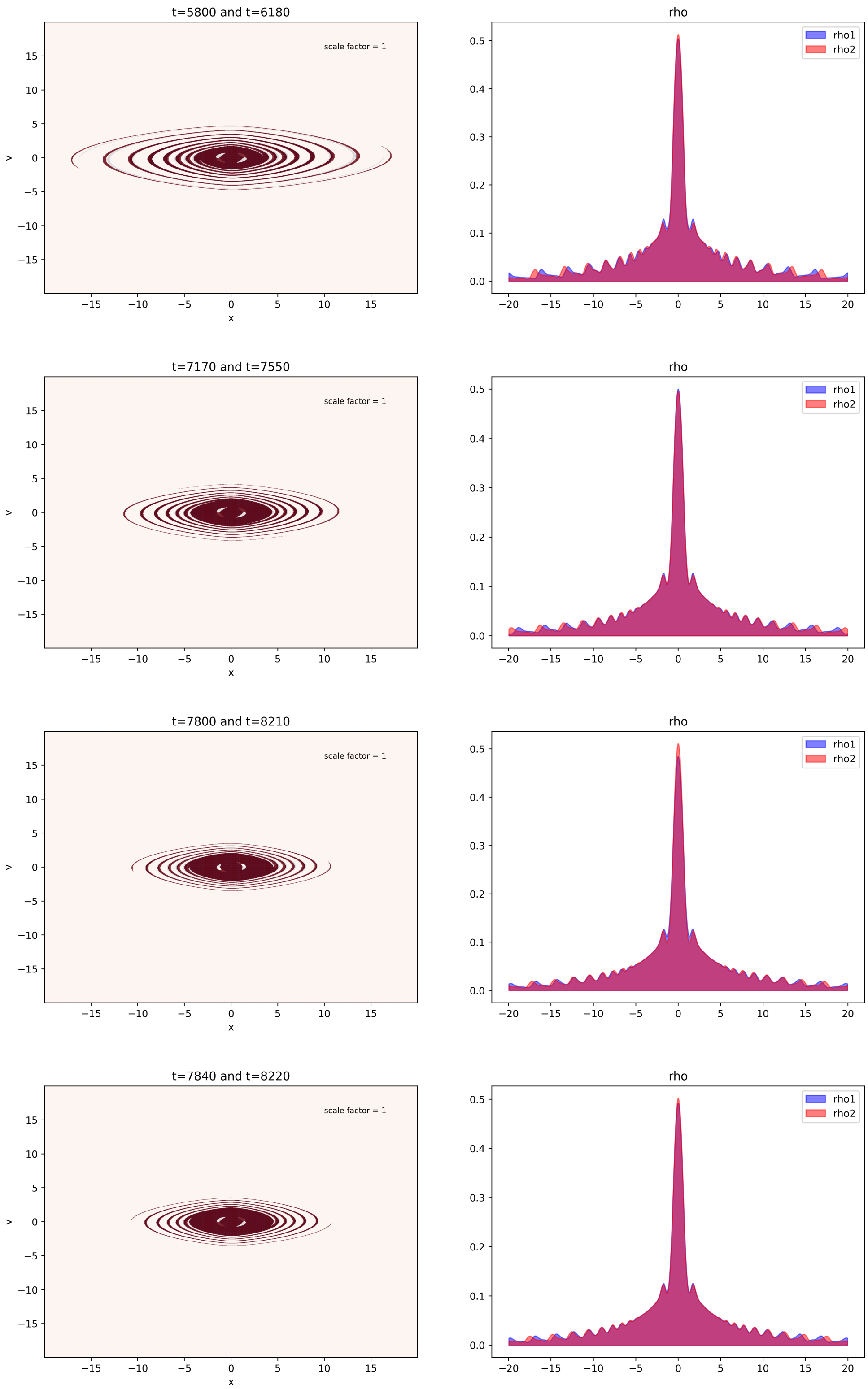

Figure 4: Rescaling in the second stage of relaxation. The radii of oscillation change very slowly, hence (up to the resolution of our model) the scaling factor is close to unity. The oscillation time is also changing slowly, since plotting different distributions at fixed intervals results in an almost repeating pattern. We note that the state is not entirely stationary, as over many timesteps the peaks become shallower and closer together. Over 2500 steps however, the total number of phase space coils only increases by one ($t = 5800$ to $t = 8300$). At later times, the inner patterns match very closely (see the lower graphs).

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