# Welfare and revenue in budget-constrained markets

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joint work with Simon Finster and Paul Goldberg

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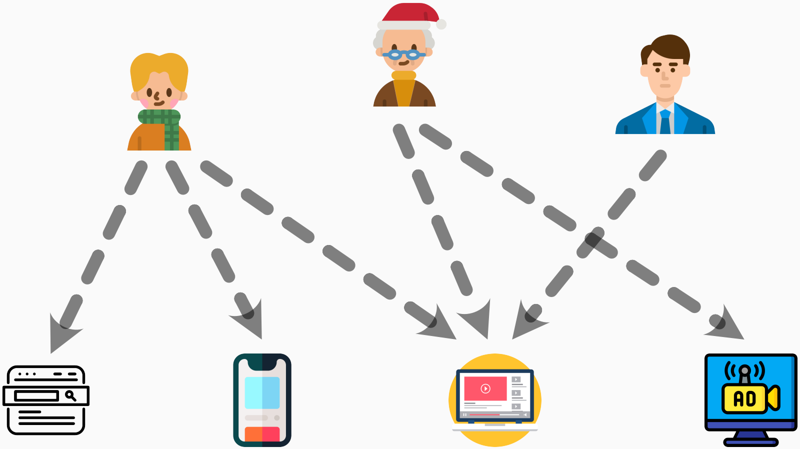
# Buying ad slots



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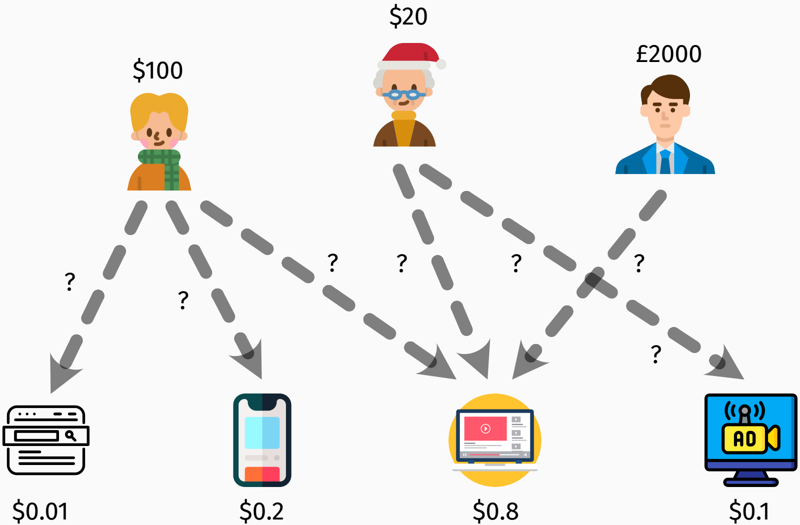
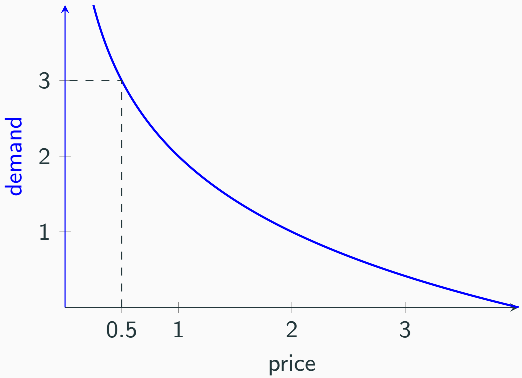


# Example: Revenue and welfare diverge




One buyer and supply 3.

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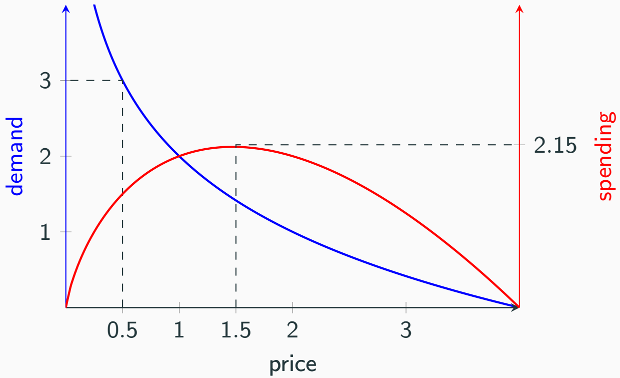


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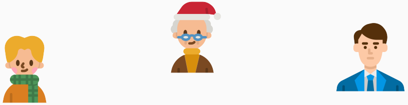
# The market



- One seller and multiple buyers $J$
- Multiple *divisible* goods $N := \{1, \ldots, n\}$ available with supply of 1 each
- **Bundles** are vectors $\boldsymbol{x} \in [0, 1]^n$
- **Market outcome**
  - anonymous (non-negative) **prices** $\boldsymbol{p} \in \mathbb{R}^n$
  - **allocation** $(\boldsymbol{x}^j)_{j \in J}$ with $\sum_{j \in J} \boldsymbol{x}^j \leq \mathbf{1}$

## The market — Buyer preferences

- Linear **valuation** $v(\boldsymbol{x}) = \boldsymbol{r} \cdot \boldsymbol{x}$
- **Quasi-linear utility** $u(\boldsymbol{p}; \boldsymbol{x}) = v(\boldsymbol{x}) - \boldsymbol{p} \cdot \boldsymbol{x}$
- At prices $\boldsymbol{p} \in \mathbb{R}^n$, buyer demands utility-maximising bundle not exceeding budget $\beta$.

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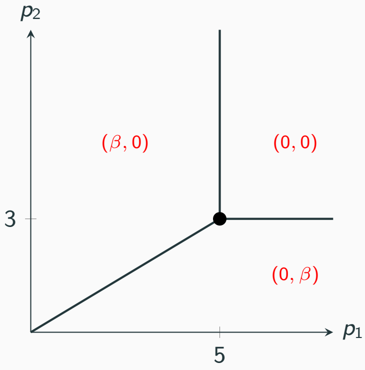
**Geometric perspective**

- Demand divides price-space into $n + 1$ regions corresponding to goods $1, \ldots, n$ and *nothing*.
- Within each region, buyer spends entire budget on this good




Preferences of single buyer with $\boldsymbol{r} = (5, 3)$ and budget $\beta = 1$

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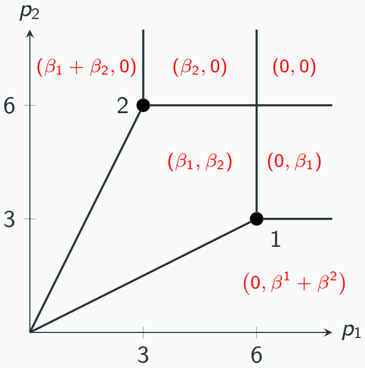
## Aggregate demand

- Aggregate demand/spending of multiple buyers is Minkowski sum of individual demands




Preferences of two buyers

# Objectives

A market outcome $\boldsymbol{p}$ and $(\boldsymbol{x}^j)_{j \in J}$ is **envy-free** if $\boldsymbol{x}^j$ maximises buyer $j$'s utility at prices $\boldsymbol{p}$.

1. **Revenue maximisation.** Envy-free market outcome that maximises revenue $\sum_{j \in J} \boldsymbol{p} \cdot \boldsymbol{x}^j$.
2. **Welfare maximisation.** Envy-free market outcome with allocation maximising welfare $\sum_{j \in J} \boldsymbol{r}^i \cdot \boldsymbol{x}^j$.
3. **Competitive Equilibrium.** Envy-free market outcome satisfying market-clearing: $\sum_{j \in J} \boldsymbol{x}^j = \mathbf{1}$.

(No strategic bidding!)

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**First and second welfare theorems**

1. The allocation of a competitive equilibrium maximises welfare (is efficient).
2. Any market-clearing prices together with a welfare-maximising (efficient) allocation form a competitive equilibrium.

# Our contributions

Recall: maximising revenue benefits seller, maximising welfare also takes into account buyers.

**Main theorem**

In our market setting, market-clearing prices $\boldsymbol{p}^*$ exist, are unique, and maximise revenue.

**Corollary**

*Market-clearing prices $\boldsymbol{p}^*$ are buyer-optimal among all revenue-maximising outcomes.*

**Corollary**

*We can find $\boldsymbol{p}^*$ 'efficiently' (using our algorithm, or existing algorithms).*

We focus on finding $\boldsymbol{p}^*$, but computing corresponding allocation is easy. details

# Related work

**Arctic auction**

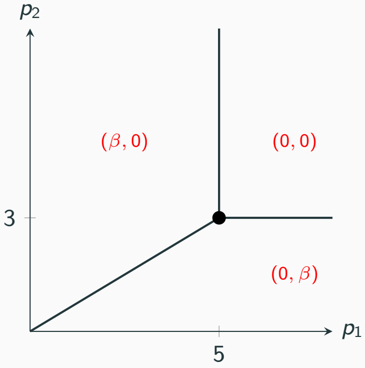
- Arctic auction belongs to family of Product-Mix Auctions by Klemperer [2008, 2010, 2018]
- Primary objective: envy-free revenue maximisation
- Applications include sovereign debt restructuring (in collaboration with IMF)

**Fisher markets**

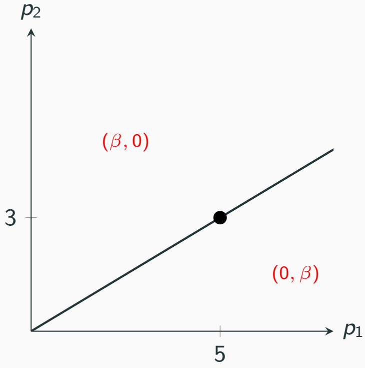
- Eisenberg-Gale [1959] convex program
- Variety of combinatorial algorithms [Devanur et al., 2008; Orlin, 2010; Adsul et al., 2012]
- Quasi-Fisher markets (and other preferences) studied, e.g., by [Chen et al., 2007; Murray, 2020; Gao and Kroer, 2020]
- Objectives: competitive and Nash equilibria
- Efficient algorithms for CE in quasi-Fisher markets [Chen et al., 2007] and [Gao and Kroer, 2020]

## Related work — Fisher markets

Original Fisher market identical to our setting, apart from . . .




Our setting




Fisher market

. . . buyers in Fisher markets don't value money.

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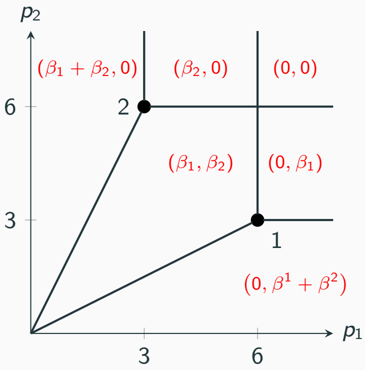


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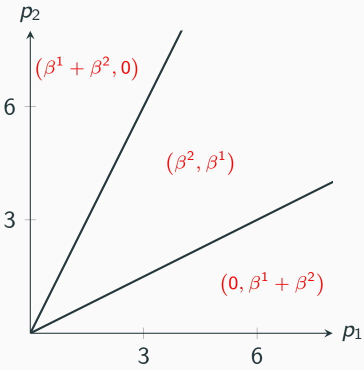


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# Related work — Arctic auction

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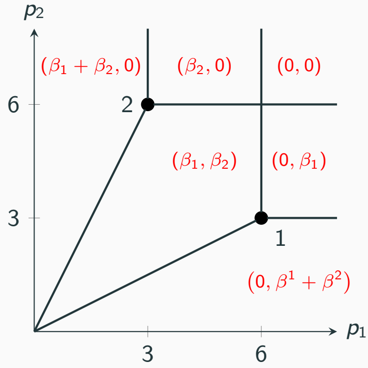
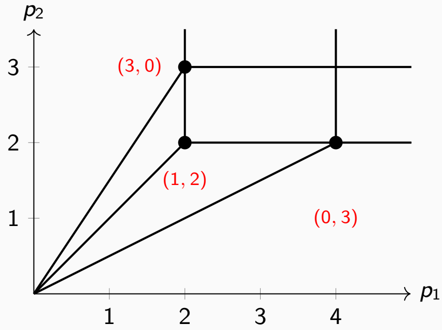
**Arctic Product-Mix Auction** designed by Paul Klemperer for Government of Iceland to exchange blocked accounts for other financial assets (e.g. cash or bonds of different quality)

- **Bidding language** allows each buyer to submit preferences corresponding to the aggregate of our demand type
- Primary objective: *envy-free revenue maximisation*
- General version of Arctic Auction allows seller to specify *supply costs* (we don't)
- "DotEcon-Klemperer algorithm" finds revenue-maximising prices [Fichtl, 2022]



# Feasible prices



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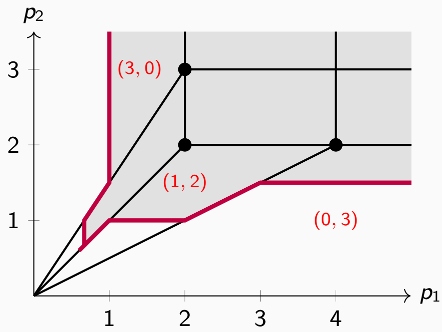

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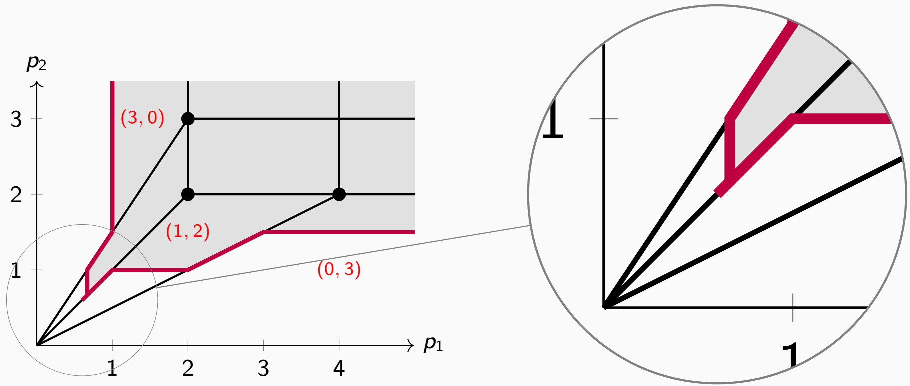

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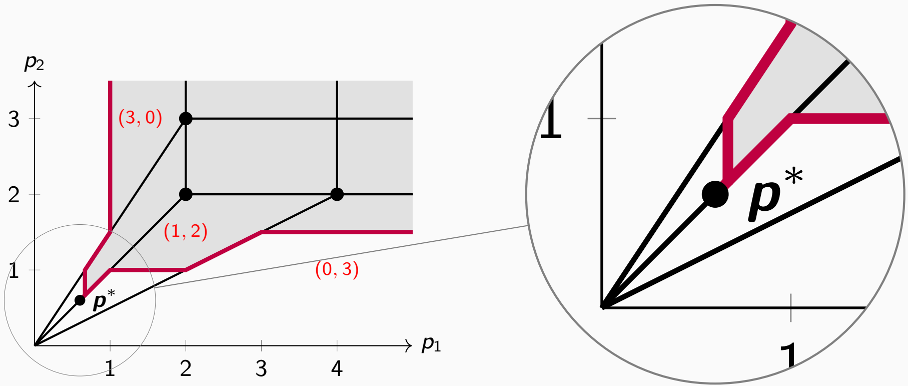

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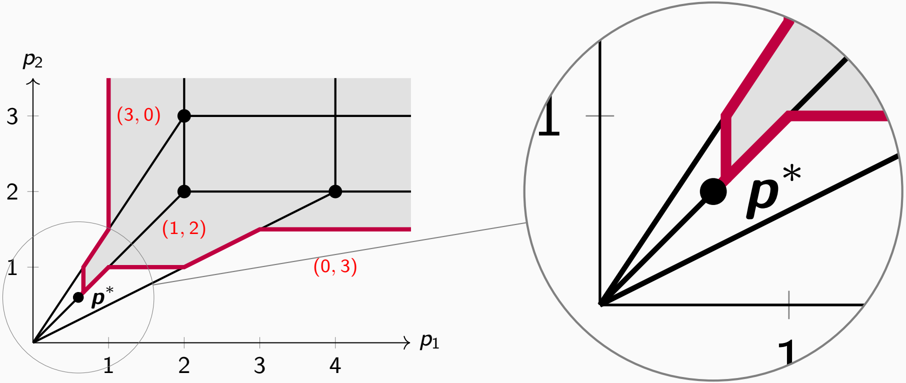


**Proposition**

The feasible region has an elementwise-minimal point $\boldsymbol{p}^*$.

Proof idea: For any feasible $\boldsymbol{p}$ and $\boldsymbol{q}$, their elementwise-minimum $\boldsymbol{p} \wedge \boldsymbol{q}$ is feasible.

# Minimal prices are revenue-maximising

**Lemma**

*For any feasible points $\boldsymbol{p} \leq \boldsymbol{q}$, we can achieve weakly higher revenue at $\boldsymbol{p}$ than at $\boldsymbol{q}$.*

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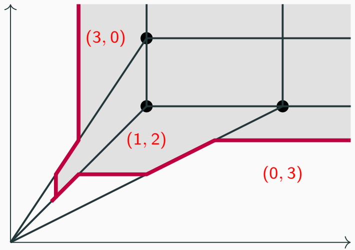
**Intuition**

- Bids spend weakly more at $\boldsymbol{p}$ than at $\boldsymbol{q}$.
- Total spending goes up (weakly).
- Issue: some bids might spend only part of their budget at $\boldsymbol{q}$ and $\boldsymbol{p}$. How much?
- Want to argue that demand change is 'consistent', so these bids spent weakly more.



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**Proof sketch.** Define $S = \{i \in N \mid p_i < q_i\}$. Let $(\boldsymbol{x}^j)_{j \in J}$ and $(\boldsymbol{y}^j)_{j \in J}$ be revenue-maximising allocations at $\boldsymbol{p}$ and $\boldsymbol{q}$.

We construct allocation $(\boldsymbol{z}^j)_{j \in J}$ at $\boldsymbol{p}$ with weakly higher revenue than $(\boldsymbol{y}^j)_{j \in J}$ at $\boldsymbol{q}$.

**Case 1:** For buyers $j$ that demand only goods from $S$ at $\boldsymbol{p}$, set $\boldsymbol{z}^j = \boldsymbol{x}^j$.

**Case 2:** Other buyers demand all goods also demanded at $\boldsymbol{q}$ and no goods in $S$. Set $\boldsymbol{z}^j = \boldsymbol{y}^j$.

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**Corollary**

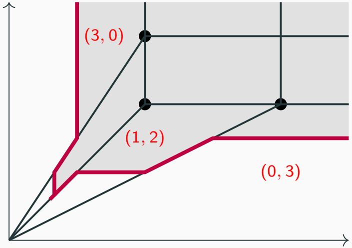
*Revenue is maximised at $\boldsymbol{p}^*$.*

**NB:** In general, there can be many revenue-maximising prices

# Minimal prices clear the market

**Lemma**

*The prices $\mathbf{p}^*$ uniquely clear the market.*



## Minimal prices clear the market

**Lemma**

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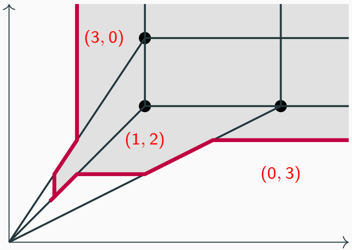
### One proof

- Convex programming duality tells us that market-clearing prices exist.
- Suppose $\boldsymbol{q} \gneq \boldsymbol{p}^*$ is market-clearing.
- Let $\boldsymbol{x} \in [0,1]^n$ be aggregate demand at $\boldsymbol{p}^*$ maximising revenue.
- Revenues are $\sum_{i \in N} q_i$ at $\boldsymbol{q}$ and $\sum_{i \in N} x_i p_i^*$ at $\boldsymbol{p}^*$.
- We know that revenue is maximised at $\boldsymbol{p}^*$, so

$$\sum_{i \in N} q_i \leq \sum_{i \in N} x_i p_i^* \leq \sum_{i \in N} p_i^* < \sum_{i \in N} q_i.$$



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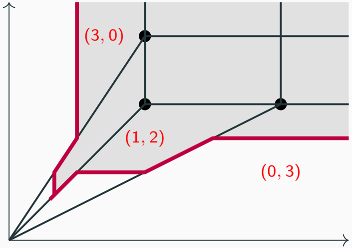
**Price-scaling subroutine**

- Suppose feasible $\boldsymbol{p}$ are not market-clearing
  1. Subroutine finds new allocation at $\boldsymbol{p}$ that exhausts supply of an additional good,
  2. or uniformly scales down prices of some goods while maintaining feasibility and increasing aggregate demand
- This implies that $\boldsymbol{p}^*$ are market-clearing
- Modification of scaling routine by [Adsul et al. 2012] for Fisher markets
  - We scale prices down instead of up
  - Additional pre-processing for bids that may be indifferent between spending and not spending



# Computing the minimal prices

- Repeatedly apply the price-scaling procedure leads to find $\boldsymbol{p}^*$.
- Make progress by exhausting supply of an additional good or reducing price of at least one good.
- Algorithm terminates once supply of every good is exhausted.

**Proposition**

This algorithm finds $\boldsymbol{p}^*$ in *at most* exponential time (in the number of goods and buyers).

Bound is likely to be loose, at least in 'typical' instances. Algorithm performs well in practice.

**Fact**

Can also find $\boldsymbol{p}^*$ in polynomial time with algorithms of [Chen et al. 2007] or [Gao and Kroer, 2020].

# Conclusion

- We've identified an important market setting in which competitive equilibrium maximises revenue (subject to envy-freeness)
- This makes **both** the seller and the buyers / social planner happy, and unifies two research directions.
- Potential applications in advertising, finance, debt restructuring, digital goods, etc.
- **Takeaway:** geometric perspective can be fruitful
- Further work: better running time guarantee for algorithm, or improved (geometric) algorithms

Thank you!

# Appendix

# Computing CE allocation from prices