Notes for "Understanding quantum mechanics 2"

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Introduction

In the video, we formulate classical mechanics on phase space. There are several different but equivalent ways to do this. In particular, one can choose how to represent the time-dependence of physical quantities. One may either[1]

1. describe time evolution by a time-dependent phase space point $(x(t), p(t))$, or
2. interpret the phase space point as specifying the initial condition (x_0, p_0) at a fixed reference time t_0 and account for time evolution by making the observables on phase space explicitly time-dependent.

As we will prove below, the two options are related by a (time-dependent) canonical transformation and are therefore equivalent for the formulation of classical mechanics.

The choice of option (1) or (2) is fully analogous to the choice between the Schrödinger picture (corresponding to 1) and the Heisenberg picture (corresponding to 2) in the formulation of quantum mechanics. In the video, I use option (2), mainly because I plan to emphasize the Heisenberg picture in later episodes.

As option (1) might be more familiar to people and in order to resolve any confusion, we will now look at both formalisms in turn and describe how to convert one description into the other. The mathematical notation in the following sections is more verbose than is customary in the physics literature. I erred on this side in order to make the definitions and arguments of all expressions as unambiguous as possible.

Formalism 1: Time-dependent phase space point $(x(t), p(t))$

The phase space in our example will be $\mathcal{P}_1 = \mathbb{R}^2$ with coordinates x and p. By abuse of notation we use the same letters for the two functions parameterizing the *phase space trajectory*, i.e. the time-dependent phase space point $(x(t), p(t))$:

$$\begin{aligned} x &: \mathbb{R} \to \mathbb{R}, \quad t \mapsto x(t) \\ p &: \mathbb{R} \to \mathbb{R}, \quad t \mapsto p(t) \end{aligned}$$

A generic observable A will be modeled as a real function on phase space[2]:

$$A : \mathcal{P}_1 \to \mathbb{R}, \quad (x, p) \mapsto A(x, p)$$

The particular observables X and P, representing unknown position and momentum, simply project to the first and second coordinate of the phase space point, respectively:

$$\begin{aligned} X &: \mathcal{P}_1 \to \mathbb{R}, \quad (x, p) \mapsto x \\ P &: \mathcal{P}_1 \to \mathbb{R}, \quad (x, p) \mapsto p \end{aligned}$$

We also define the Hamiltonian as an observable H:

$$H : \mathcal{P}_1 \to \mathbb{R}, \quad (x, p) \mapsto H(x, p).$$

[1] As we do in the video, we will consider only the simplest example of a single degree of freedom with position x and momentum p. The formalism can easily be generalized to multiple degrees of freedom.

[2] One may allow observables to also depend explicitly on time, even in this formalism. This is typically not needed for describing closed systems and would add only confusion to the present exposition. If needed, it can be done without changing anything essential about the formalism. One only needs to add partial time-derivatives $\frac{\partial A}{\partial t}$ in the appropriate places.

For example, a linear harmonic oscillator would have

$$H(x,p) = \alpha x^2 + \beta p^2$$

for some fixed α, $\beta \in \mathbb{R}_{>0}$. We can also write this in terms of observables as

$$H = \alpha X^2 + \beta P^2.$$

Given two observables A, B, we define their *Poisson bracket* $\{A, B\}$ as:

$$\{A, B\} : \mathcal{P}_1 \to \mathbb{R}, \quad (x,p) \mapsto \frac{\partial A}{\partial x}(x,p)\frac{\partial B}{\partial p}(x,p) - \frac{\partial A}{\partial p}(x,p)\frac{\partial B}{\partial x}(x,p),$$

or more compactly as

$$\{A, B\} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x}.$$

Plugging in the definitions of X and P, we find that these two observables satisfy the *canonical relation*

$$\{X, P\} = 1.$$

How does an observable A change with time? Strictly speaking, in this formalism it does not change with time at all, since it is just a fixed function on phase space. What we are interested in physically, however, is how the predicted value of an observable changes with time if we know the phase space trajectory $(x(t), p(t))$. This change with time is

$$\frac{d}{dt}A(x(t), p(t)) = \frac{\partial A}{\partial x}\Big|_{\substack{x=x(t)\\p=p(t)}}\frac{dx}{dt} + \frac{\partial A}{\partial p}\Big|_{\substack{x=x(t)\\p=p(t)}}\frac{dp}{dt}.$$

Note that this is not an observable (i.e. it is not a real function on phase space) in the sense of our formalism but it is just a function of time (for a given phase space trajectory, i.e. for given *functions* $x, p : \mathbb{R} \to \mathbb{R}$).

This somewhat unsatisfactory situation clears up when we apply the Hamiltonian equations of motion,

$$\frac{dx}{dt}(t) = \frac{\partial H}{\partial p}\Big|_{\substack{x=x(t)\\p=p(t)}}, \tag{1}$$

$$\frac{dp}{dt}(t) = -\frac{\partial H}{\partial x}\Big|_{\substack{x=x(t)\\p=p(t)}}. \tag{2}$$

Plugging these equations into $\frac{d}{dt}A(x(t), p(t))$, we get

$$\begin{aligned}\frac{d}{dt}A(x(t), p(t)) &= \frac{\partial A}{\partial x}\Big|_{\substack{x=x(t)\\p=p(t)}}\frac{\partial H}{\partial p}\Big|_{\substack{x=x(t)\\p=p(t)}} - \frac{\partial A}{\partial p}\Big|_{\substack{x=x(t)\\p=p(t)}}\frac{\partial H}{\partial x}\Big|_{\substack{x=x(t)\\p=p(t)}}\\ &= \{A, H\}\Big|_{\substack{x=x(t)\\p=p(t)}}.\end{aligned}$$

We see that the change of A with time at time t really depends only on the values of x and p *at time* t and not on the whole phase space trajectory. For each observable A we can therefore find an observable $\dot{A}$, namely

$$\dot{A} = \{A, H\},$$

such that for any phase space trajectory $(x(t), p(t))$ solving the equations of motion we have for all t:

$$\frac{d}{dt}A(x(t), p(t)) = \dot{A}\Big|_{\substack{x=x(t)\\p=p(t)}}.$$

Using these notations, we can write the equations of motion as

$$\dot{X} = \{X, H\},$$
$$\dot{P} = \{P, H\}.$$

Formalism 2: Time-independent phase space point (x_0, p_0)

We now want to interpret phase space differently, namely we define it as $\mathcal{P}_2 = \mathbb{R}^2$ with coordinates x_0 and p_0 and we consider a phase space point (x_0, p_0) to represent the initial condition of the physical system at a fixed reference time t_0. There is no notion of a phase space trajectory in this version of the formalism.

A generic observable A will be modeled as a real function of phase space *and time*:

$$A : \mathcal{P}_2 \times \mathbb{R} \to \mathbb{R}, \quad ((x_0, p_0), t) \mapsto A((x_0, p_0), t)$$

In order to connect this version of the formalism to the formalism 1 of the previous section, we demand for any observable A that corresponds to an observable A_1 in formalism 1 and for any phase space trajectory $(x(t), p(t))$ in formalism 1 that satisfies the equations of motion such that $(x(t_0), p(t_0)) = (x_0, p_0)$ that

$$A((x_0, p_0), t) = A_1(x(t), p(t)) \tag{3}$$

for all times t. In particular, this implies

$$\begin{aligned} X((x_0, p_0), t) &= X_1(x(t), p(t)) = x(t), && (4)\\ P((x_0, p_0), t) &= P_1(x(t), p(t)) = p(t), && (5)\\ X((x_0, p_0), t_0) &= x_0, && (6)\\ P((x_0, p_0), t_0) &= p_0, && (7) \end{aligned}$$

and for corresponding observables A, A_1:

$$A((x_0, p_0), t) = A_1(x(t), p(t)) = A_1(X((x_0, p_0), t), P((x_0, p_0), t)). \tag{8}$$

Given two observables A, B in formalism 2, we define the *new*[3] Poisson bracket $\{A, B\}_0$ as:

$$\{A, B\}_0 : \mathcal{P}_2 \times \mathbb{R} \to \mathbb{R}, \quad ((x_0, p_0), t) \mapsto \frac{\partial A}{\partial x_0}((x_0, p_0), t)\frac{\partial B}{\partial p_0}((x_0, p_0), t) - \frac{\partial A}{\partial p_0}((x_0, p_0), t)\frac{\partial B}{\partial x_0}((x_0, p_0), t),$$

or more compactly as

$$\{A, B\}_0 = \frac{\partial A}{\partial x_0}\frac{\partial B}{\partial p_0} - \frac{\partial A}{\partial p_0}\frac{\partial B}{\partial x_0}.$$

Given two observables A, B in formalism 2 that correspond to observables A_1, B_1 in formalism 1, respectively, we find (by plugging in (8), applying the chain rule, and rewriting surviving terms as Poisson brackets) the following "chain rule" connecting the two Poisson bracket definitions:

$$\{A, B\}_0 = \{A_1, B_1\}|_{\substack{x = X((x_0, p_0), t)\\ p = P((x_0, p_0), t)}} \{X, P\}_0.$$

Note that the first Poisson bracket on the right-hand side is the one defined for formalism 1 with respect to the variables x and p!

In particular, we have:

$$\{X, H\}_0 = \{X_1, H_1\}|_{\substack{x = X((x_0, p_0), t)\\ p = P((x_0, p_0), t)}} \{X, P\}_0 = \frac{\partial X}{\partial t}\{X, P\}_0, \tag{9}$$

$$\{P, H\}_0 = \{P_1, H_1\}|_{\substack{x = X((x_0, p_0), t)\\ p = P((x_0, p_0), t)}} \{X, P\}_0 = \frac{\partial P}{\partial t}\{X, P\}_0, \tag{10}$$

where the last step used the equations of motion (following from the equations of motion in formalism 1 by (4) and (5)):

$$\frac{\partial X}{\partial t} = \{X_1, H_1\}|_{\substack{x = X((x_0, p_0), t)\\ p = P((x_0, p_0), t)}} = \left(\frac{\partial H_1}{\partial p}\right)_{\substack{x = X((x_0, p_0), t)\\ p = P((x_0, p_0), t)}}, \tag{11}$$

$$\frac{\partial P}{\partial t} = \{P_1, H_1\}|_{\substack{x = X((x_0, p_0), t)\\ p = P((x_0, p_0), t)}} = -\left(\frac{\partial H_1}{\partial x}\right)_{\substack{x = X((x_0, p_0), t)\\ p = P((x_0, p_0), t)}}. \tag{12}$$

[3] Since the video does not compare the two formalisms, this Poisson bracket on $\mathcal{P}_2$ is just written as $\{A, B\}$ in the video, using the variable names x and p.

From the initial conditions (6) and (7) we see immediately that $\{X,P\}_0 = 1$ for $t = t_0$. This is unsurprising since the two formalisms coincide at time $t = t_0$ and $\{X_1, P_1\} = 1$. We will now proceed to prove that $\{X,P\}_0 = 1$ for all times t, i.e. that the change from formalism 1 to formalism 2 corresponds to a (time-dependent) canonical transformation.

Let $(\bar{x}_0, \bar{p}_0) \in \mathcal{P}_2$ be arbitrary but fixed. Define a function f of time by

$$f(t) = \{X,P\}_0|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}}.$$

Obviously, $f(t_0) = 1$. We will now show[4] that $ff' = 0$, which implies $f' = 0$ and therefore $f = 1$. For all t we have:

$$\begin{aligned}
f(t)f'(t) &= f(t)\left(\{\frac{\partial X}{\partial t}, P\}_0|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}} + \{X, \frac{\partial P}{\partial t}\}_0|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}}\right) && (13)\\
&= \{f(t)\frac{\partial X}{\partial t}, P\}_0|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}} + \{X, f(t)\frac{\partial P}{\partial t}\}_0|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}} && (14)\\
&= \{\{X,P\}_0\frac{\partial X}{\partial t}, P\}_0|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}} + \{X, \{X,P\}_0\frac{\partial P}{\partial t}\}_0|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}} && (15)\\
&\quad - \{\{X,P\}_0, P\}_0|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}}\frac{\partial X}{\partial t}|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}} - \{X, \{X,P\}_0\}_0|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}}\frac{\partial P}{\partial t}|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}} && (16)\\
&= \{\{X,H\}_0, P\}_0|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}} + \{X, \{P,H\}_0\}_0|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}} && (17)\\
&\quad - \{\{X,P\}_0, P\}_0|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}}\frac{\partial H_1}{\partial p}|_{\substack{x=X(\bar{x}_0,\bar{p}_0,t)\\ p=P(\bar{x}_0,\bar{p}_0,t)}} + \{X, \{X,P\}_0\}_0|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}}\frac{\partial H_1}{\partial x}|_{\substack{x=X(\bar{x}_0,\bar{p}_0,t)\\ p=P(\bar{x}_0,\bar{p}_0,t)}} && (18)\\
&= \{\{X,H\}_0, P\}_0|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}} - \{X, \{H,P\}_0\}_0|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}} && (19)\\
&\quad - \{\{X,P\}_0, X\left(\frac{\partial H_1}{\partial x}\right)_{\substack{x=X(\bar{x}_0,\bar{p}_0,t)\\ p=P(\bar{x}_0,\bar{p}_0,t)}} + P\left(\frac{\partial H_1}{\partial p}\right)_{\substack{x=X(\bar{x}_0,\bar{p}_0,t)\\ p=P(\bar{x}_0,\bar{p}_0,t)}}\}_0|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}} && (20)\\
&= \{\{X,P\}_0, H\}_0|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}} && (21)\\
&\quad - \{\{X,P\}_0, X\left(\frac{\partial H_1}{\partial x}\right)_{\substack{x=X(\bar{x}_0,\bar{p}_0,t)\\ p=P(\bar{x}_0,\bar{p}_0,t)}} + P\left(\frac{\partial H_1}{\partial p}\right)_{\substack{x=X(\bar{x}_0,\bar{p}_0,t)\\ p=P(\bar{x}_0,\bar{p}_0,t)}}\}_0|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}} && (22)\\
&= \{\{X,P\}_0, H - X\left(\frac{\partial H_1}{\partial x}\right)_{\substack{x=X(\bar{x}_0,\bar{p}_0,t)\\ p=P(\bar{x}_0,\bar{p}_0,t)}} - P\left(\frac{\partial H_1}{\partial p}\right)_{\substack{x=X(\bar{x}_0,\bar{p}_0,t)\\ p=P(\bar{x}_0,\bar{p}_0,t)}}\}_0|_{\substack{x_0=\bar{x}_0\\ p_0=\bar{p}_0}} && (23)\\
&= 0. && (24)
\end{aligned}$$

where we used the following:

- To get (13), we apply the Leibniz rule for derivation of Poisson brackets to the definition of f.
- To get (14), we use the bilinearity of the Poisson bracket to pull $f(t)$ (which does not depend on (x_0, p_0)) into the brackets.
- To get (15) and (16), we use the Leibniz rule in reverse on the partial derivatives with respect to x_0 and p_0 occurring in the Poisson bracket. Line (16) hereby collects the terms we need to "correct" for changing $f(t)$ (which does not depend on (x_0, p_0)) inside the Poisson bracket to $\{X,P\}_0$ (which does).
- To get (17), we plug in (9) and (10) and for line (18), we plug in (11) and (12).
- To get (19), we use the skew-symmetry of the Poisson bracket ($\{A,B\}_0 = -\{B,A\}_0$). For line (20), we pull the partial derivatives of H_1 (which do not depend on (x_0, p_0)) into the Poisson brackets and combine the brackets using their bilinearity.
- To get (21), we use the Jacobi identity of the Poisson bracket, $\{\{A,B\}_0, C\}_0 + \{\{B,C\}_0, A\}_0 + \{\{C,A\}_0, B\}_0 = 0$.

[4]We use ff' rather than f' alone because this makes it easier to plug in the "chain rule" relations (9) and (10). If anybody knows how to shorten this prove while still keeping it elementary, I'd love to hear about it.

- To get (23), we combine Poisson brackets using their bilinearity.
- We finally get (24) by observing that the partial derivatives with respect to x_0 and p_0 vanish for the right-hand argument of the outermost Poisson bracket (by direct calculation using the chain rule and the definition of H according to (3)). (Note that this expression has the form of a Legendre transform of H to H_1, i.e. it switches the independent variables from x_0, p_0 to x, p.)

We therefore have $\{X, P\}_0 = 1$ for all times t, i.e. the time evolution that connects observables in the two formalisms is a time-dependent canonical transformation and it therefore preserves Poisson brackets for corresponding pairs of observables:

$$\{A, B\}_0 = \{A_1, B_1\}|_{\substack{x=X((x_0,p_0),t) \\ p=P((x_0,p_0),t)}}.$$

The equations of motion take the same compact form in formalism 2 as they did in formalism 1:

$$\dot{X} = \{X, H\}_0,$$
$$\dot{P} = \{P, H\}_0.$$

with the total time derivatives being simply the partial time derivatives in this case, i.e. for any observable A:

$$\dot{A} = \frac{\partial A}{\partial t}.$$