Math 320-3: Midterm 1 Practice
Northwestern University, Spring 2015

1. Give an example, with justification, of each of the following.

(a) A limit $\lim_{(x,y,z,w)\to(0,0,0,0)} f(x,y,z,w)$ which does not exist.
(b) A function $f : \mathbb{R}^2 \to \mathbb{R}$ which is not differentiable at $\mathbf{0}$ but whose partial derivatives exist.
(c) A function $f : \mathbb{R}^2 \to \mathbb{R}$ such that f_{xy} and f_{yx} exist but are not continuous at $(0,0)$.
(d) A differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is not continuously differentiable.
(e) A differentiable function $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that $Df(\mathbf{x})$ is the same matrix for all $\mathbf{x} \in \mathbb{R}^3$.

2. For a function $f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{a} \in \mathbb{R}^n$, prove that $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ exists if and only if $\lim_{\mathbf{x}\to\mathbf{a}} f_i(\mathbf{x})$ exists for each $i = 1, \ldots, m$.

3. Wade, 11.2.2. Suppose that $f, g : \mathbb{R} \to \mathbb{R}^m$ are differentiable at a and there is a $\delta > 0$ such that $g(x) \neq \mathbf{0}$ for all $0 < |x - a| < \delta$. If $f(a) = g(a) = \mathbf{0}$ and $Dg(a) \neq \mathbf{0}$, prove that

$$\lim_{x\to a} \frac{\|f(x)\|}{\|g(x)\|} = \frac{\|Df(a)\|}{\|Dg(a)\|}.$$

4. Determine whether or not the function

$$f(x,y,z) = \begin{cases} xyz + x + y + z + y^2z + 1 & (x,y,z) \neq (0,0,0) \\ 1 & (x,y,z) = (0,0,0) \end{cases}$$

is differentiable at $(0,0,0)$.

5. Wade, 11.2.7. Prove that

$$f(x,y) = \begin{cases} \frac{x^3 - xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is continuous on $\mathbb{R}^2$ and has first-order partial derivatives everywhere on $\mathbb{R}^2$, but f is not differentiable at $(0,0)$.

6. Wade, 11.4.3. Suppose that $k \in \mathbb{N}$ and that $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of order k; that is, that $f(\rho\mathbf{x}) = \rho^k f(\mathbf{x})$ for all $x \in \mathbb{R}^n$ and all $\rho \in \mathbb{R}$. If f is differentiable on $\mathbb{R}^n$, prove that

$$x_1 \frac{\partial f}{\partial x_1}(\mathbf{x}) + \cdots + x_n \frac{\partial f}{\partial x_n}(\mathbf{x}) = kf(\mathbf{x})$$

for all $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

7. Suppose that $g : \mathbb{R}^k \to \mathbb{R}^n$ and $F : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ are differentiable on their domains and that for any $\mathbf{x} \in \mathbb{R}^k$, $F(\mathbf{x}, g(\mathbf{x})) = \mathbf{0}$. If $DF_{\mathbf{y}}(\mathbf{x}, g(\mathbf{x}))$ is invertible, show that

$$Dg(\mathbf{x}) = -[DF_{\mathbf{y}}(\mathbf{x}, g(\mathbf{x}))]^{-1} DF_{\mathbf{x}}(\mathbf{x}, g(\mathbf{x}))$$

for any $\mathbf{x} \in \mathbb{R}^k$. To clarify the notation, we are denoting the variables making up the domain $\mathbb{R}^k \times \mathbb{R}^n$ of F by $(\mathbf{x}, \mathbf{y})$ where $\mathbf{x} \in \mathbb{R}^k$ and $\mathbf{y} \in \mathbb{R}^n$, so that $DF_{\mathbf{x}}$ denotes the partial Jacobian matrix obtained by differentiating with respect to the $\mathbf{x}$ variables alone and $DF_{\mathbf{y}}$ the partial Jacobian matrix obtained by differentiating with respect to the $\mathbf{y}$ variables alone. So, this problem is meant to derive the formula we gave in class for the Jacobian matrix of the implicit function defined by the Implicit Function Theorem.

Hint: Apply the chain rule to the composition $\mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ of the functions

$$\mathbf{x} \mapsto (\mathbf{x}, g(\mathbf{x})) \text{ and } (\mathbf{x}, \mathbf{y}) \mapsto F(\mathbf{x}, \mathbf{y}).$$

8. Wade, 11.5.3. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are differentiable on $\mathbb{R}^n$ and that there exist $r > 0$ and $\mathbf{a} \in \mathbb{R}^n$ such that $Dg(\mathbf{x})$ is the identity matrix for all $\mathbf{x} \in B_r(\mathbf{a})$. Prove that there is a function $h : B_r(\mathbf{a})\backslash\{\mathbf{a}\} \to B_r(\mathbf{x})$ such that

$$\frac{|f(g(\mathbf{x})) - f(g(\mathbf{a}))|}{\|\mathbf{x} - \mathbf{a}\|} \leq \|Df((g \circ h)(\mathbf{x}))\|$$

for all $\mathbf{x} \in B_r(\mathbf{a})\backslash\{\mathbf{a}\}$.

9. Show that for h, k, ℓ close enough to 0, the expression

$$2e + e(h + k + \ell)$$

approximates $e^{(1+h)(1+k)} + e(1 + \ell)$ to 2 decimal places. You may use without proof the fact that if the absolute values of the entries of a 3×3 matrix are all less than or equal to $M > 0$, then the norm of that matrix is less than or equal to $3M\sqrt{3}$.

10. Find a point (x_0, y_0, z_0, w_0) in $\mathbb{R}^4$ near which there exist continuously differentiable functions $x(z, w)$ and $y(z, w)$ such that the quadruple $(x(z, w), y(z, w), z, w)$, for (z, w) in some open set W containing (z_0, w_0), satisfies

$$y \sin \pi x - zxw = 1 \text{ and } ye^{z+w} - x + y = 1.$$

Then, for the function $g : W \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ defined by $g(z, w) = (x(z, w), y(z, w))$, compute the Jacobian matrix $Dg(z_0, w_0)$.