Introduction to Bayesian Data Analysis

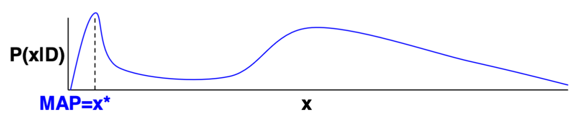
MIE223

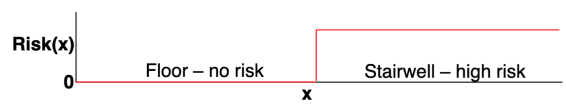
Winter 2025

1 Bayesian Data Analysis

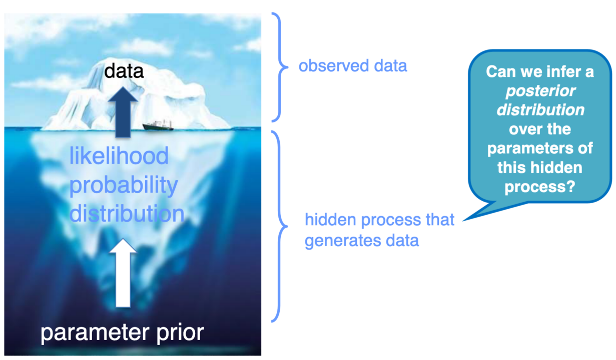
1.1 Bayesian Motivation: Bayesian Decision Theory

- Robot has belief P(x—D) over position
  - D consists of noisy range finder readings
- Associate Risk(x) w position x (e.g., stairs!)
  - MAP Risk = Risk(x*) = 0 (Max Posterior)
  - E(p(x—d))(Risk(x))
  - Full Bayesian Risk = x Risk(x)p(x—D) ¿ 0
- Which risk estimate would you use?





1.2 Bayesian Generative View of Data



1.3 A coin tossing example

- Suppose we know nothing about coins except that each tossing event produces a head with some unknown probability p and a tail with probability 1-p.
  - Our model of a coin has one parameter, p.
- If we observe 1 heads in 1 toss, what is p?
- If we observe 5 heads in 10 tosses, what is p?
- If we observe 53 heads in 100 tosses, what is p?

- How confident are we in each case?

- The frequentist answer (also called maximum likelihood): Pick the value of p that makes the observation of 53 heads and 47 tails most probable.
  - This value is p=0.53

1.4 A coin tossing example: maximum likelihood (frequentist estimation)

probability of a particular sequence containing 53 heads and 47 tails.

$$P(D) = p^{53}(1-p)^{47}$$

$$\frac{dP(D)}{dp} = 53p^{52}(1-p)^{47} - 47p^{53}(1-p)^{46}$$

$$= \left(\frac{53}{p} - \frac{47}{1-p}\right)\left[p^{53}(1-p)^{47}\right]$$

$$= 0 \;\; if \;\; p = .53$$

1.5 Some problems with picking the parameters that are most likely to generate the data

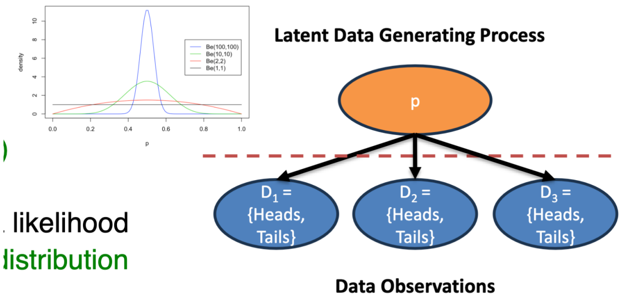
- What if we only tossed the coin once and we got 1 head?
  - Is p=1 a sensible answer?
  - Surely p=0.5 is a much better answer.
- Is it reasonable to give a single answer?
  - If we don't have much data, we are unsure about p.
  - Our computations of probabilities will work much better if we take this uncertainty into account.

1.6 The Bayesian Perspective

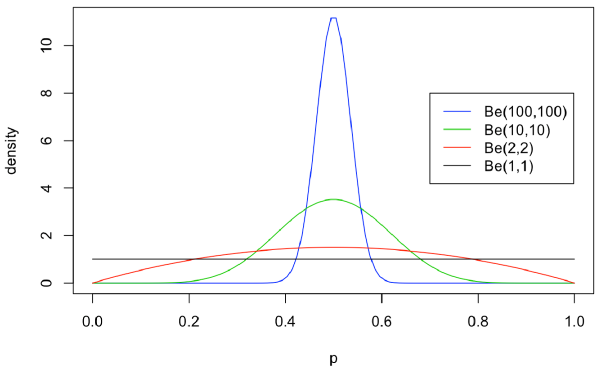
- Rather than taking the most likely estimate of p (i.e. maximum likelihood)...
- Let's build a Bayesian posterior distribution over our belief in p
- The more data we see, the more "peaked" our belief in the correct value of p

1.7 Generative Graphical Model for Biased Coin

- We have one latent parameter p
  - With a prior distribution (Beta)
- Generates data via a likelihood
  - With a Bernoulli distribution
- Infer posterior distribution
  - Also Beta since this is a conjugate prior-posterior pair



How likely is HTH given p? p(1-p)p.



1.8 Using a distribution over parameter values

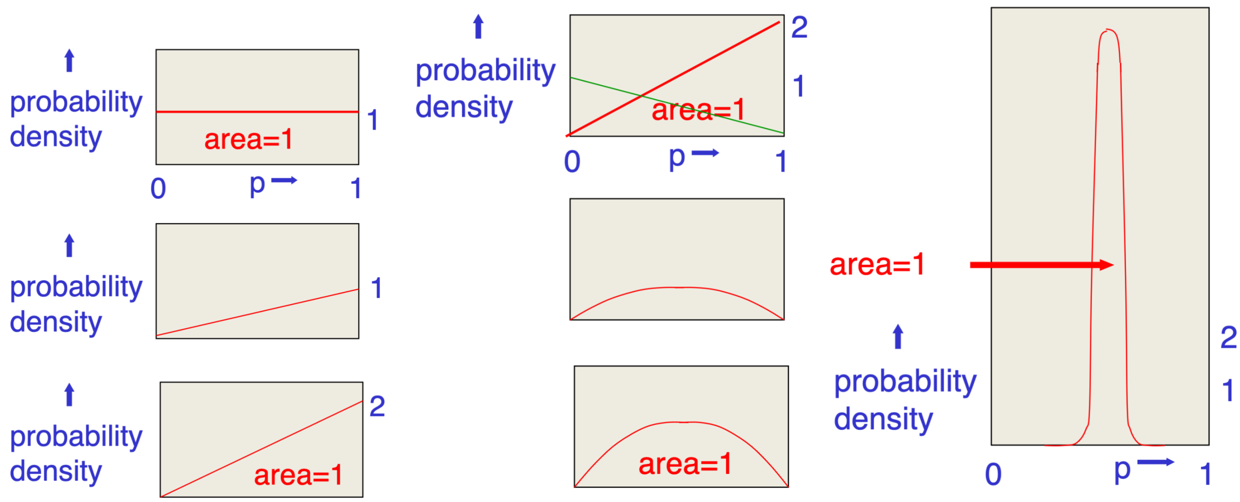
- Start with a prior distribution over p. In this case we used a uniform distribution.
- Multiply the prior probability of each parameter value by the probability of observing a head given that value (likelihood).
- Then scale up all of the probability densities so that their integral comes to 1. This gives the posterior distribution.

1.9 Lets do it again: Suppose we get a tail

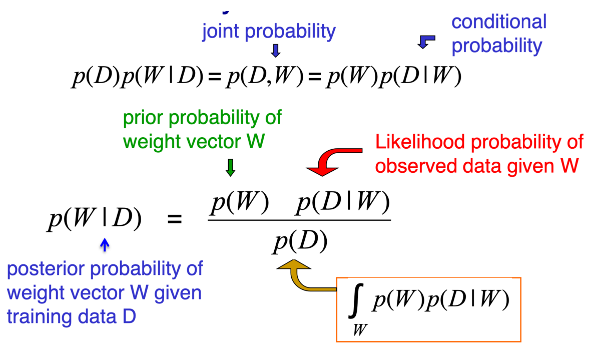
- Start with a prior distribution over p.
- Multiply the prior probability of each parameter value by the probability of observing a tail given that value.
- Then renormalize to get the posterior distribution. Look how sensible it is!

1.10 Lets do it another 98 times

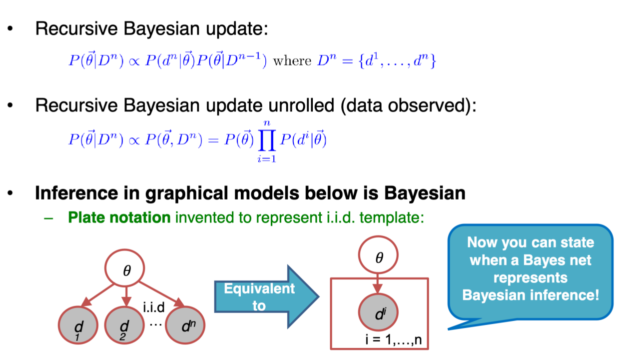
- After 53 heads and 47 tails we get a very sensible posterior distribution that has its peak at 0.53 (assuming a uniform prior)



1.11 Bayes Theorem



1.12 Aside: Bayesian Graphical Model Notation



bernoulli processes are iid.

1.13 Sanner on the Chalkboard

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \tag{1}$$

It's proportional to P(B | A)P(A) because P(B) is a constant.

posterior is proportional to likelihood times prior.
proportional to:

$$\prod_{i=1}^{n} P(x_i|\theta)P(\theta) \tag{2}$$

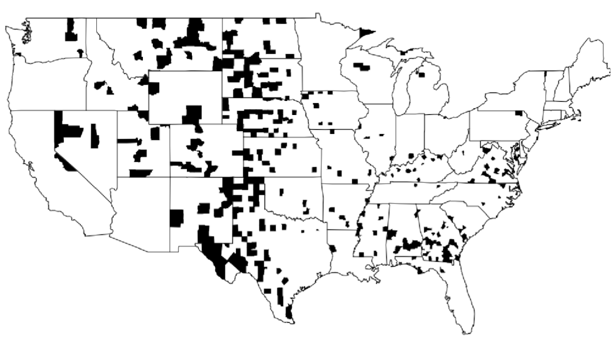
$$P(p) = \begin{cases} 1 & \text{if } 0 \leq p \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$P(D_1 = \text{Head}|p) = p$
With enough data, the dirac delta function will peak at the true value of p. It also happens to be the maximum likelihood estimate.

1.14 Why Bayesian Data Analysis? Sparse Data

- Choropleth at right shows countries with lowest 10% death rates for kidney cancer
- Should we live in the Midwest to avoid kidney cancer?
- What went wrong? Small data due to small populations. A lot of people just don't get kidney cancer. We come up with a prior distribution. Then we can update our local beliefs based on the data.

Lowest kidney cancer death rates



Source: Bayesian Data Analysis (Gelman et al, 2013)

1.15 Why Bayesian Data Analysis? Small Data

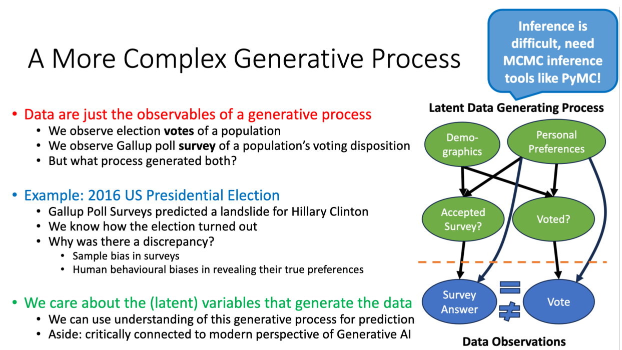
- Why Bayesian? Small Data. (Gelman, 2005)
  - Sample sizes are never large.
  - If N is too small to get a sufficiently-precise estimate, you need to get more data (or make more assumptions).
  - But once N is "large enough", you can start subdividing the data to learn more (for example, in a public opinion poll, once you have a good estimate for the entire country, you can estimate among men and women, northerners and southerners, different age groups, etc.).
  - N is never enough because if it were "enough" you'd already be on to the next problem for which you need more data.

1.16 The Bayesian framework

The Bayesian framework assumes that we always have a prior distribution for everything.

- The prior may be very vague.
- When we see some data, we combine our prior distribution with a likelihood term to get a posterior distribution.
- The likelihood term takes into account how probable the observed data is given the parameters of the model.
  - It favors parameter settings that make the data likely.
  - It fights the prior
  - With enough data the likelihood terms always wins.

1.17 A More Complex Generative Process



Survey answer was inconsistent with the voting outcome. People who wanted to vote for Trump in 2016 did not admit it. You have the observed data, so you can find the posterior distribution. This may reveal the percentage of Trump voters who would answer the survey.