Centrality, Connected Components, and Communities

MIE223

Winter 2025

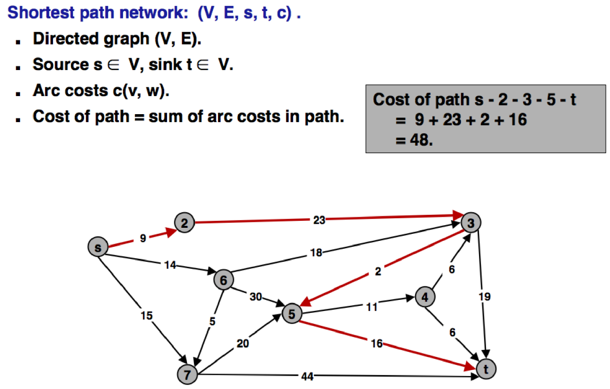
1 Centrality, Connected Components, and Communities

Note 1. centrality can be determined by degree

1.1 what does degree not capture

cut vertices connect two components but does not have a high degree

1.2 shortest path in a network



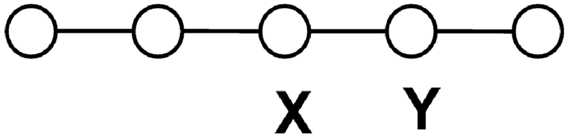
Shortest path network: (V, E, s, t, c) .

- Directed graph (V, E).
- Source s ∈ V, sink t ∈ V.
- Arc costs c(v, w).
- Cost of path = sum of arc costs in path.

Cost of path s - 2 - 3 - 5 - t
= 9 + 23 + 2 + 16
= 48.

1.3 Betweenness: centrality capturing brokerage

Intuition: how many pairs of individuals would have to go through you in order to reach one another in the minimum number of hops?



1.4 Betweenness: definition

$$C_B(i) = \sum_{j<k} g_{jk}(i)/g_{jk}$$

Where g_{jk} = the number of shortest paths connecting *jk*
$g_{jk}(i)$ = the number that vertex *i* is on.

Usually normalized by:

$$C'_B(i) = C_B(i)/\underbrace{[(n-1)(n-2)/2]}_{\text{number of pairs of vertices excluding the vertex itself}}$$

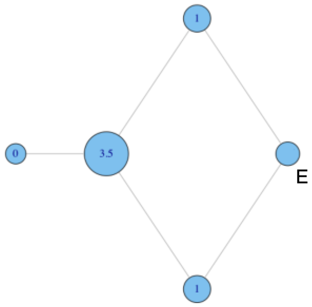
a betweeness centrality for node i is the number of nodes other than i who have to through i to interact with each other

g_{jk}(i) is a subset of g_{jk}

3 ordered pairs with 3 nodes equates to 9 paths

1.5 quiz questions

- What is the betweenness of node E?



a) 0.5

b) 1

c) 1.5

d) 2

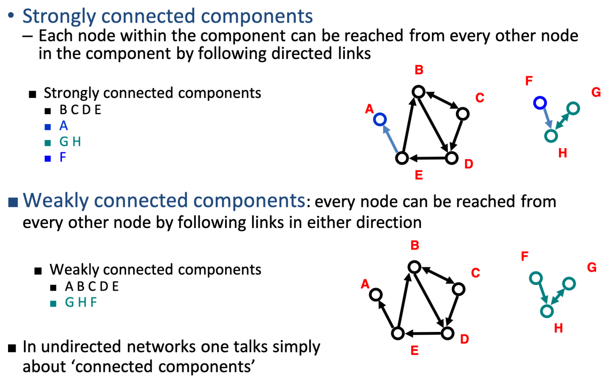
what is betweenness of node E? a) 0.5 to get from 1 to 1 you can go through 3.5 or E

q2: G high betweenness low degree, A opposite

2 Community detection

Can we discover community structure in an automated way?

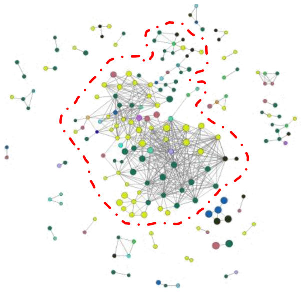
2.1 Connected components



- Strongly connected components
  - Each node within the component can be reached from every other node in the component by following directed links
  - Strongly connected components
    - B C D E
    - A
    - G H
    - F
- Weakly connected components: every node can be reached from every other node by following links in either direction
  - Weakly connected components
    - A B C D E
    - G H F
- In undirected networks one talks simply about 'connected components'

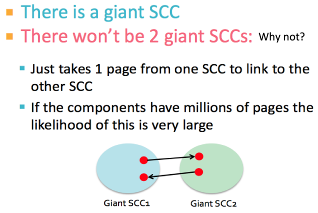
2.2 Giant component

Note 2. if the largest component encompasses a significant fraction of the graph, it is called the giant component



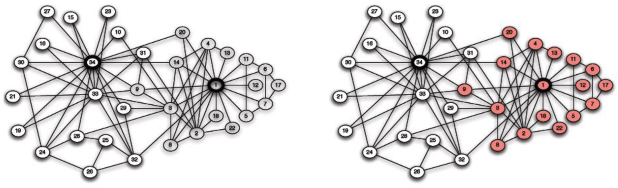
great exam question: generalize equation for betweeness for edges How many paths pass through this edge rather than the node generalize the equation to directed graphs

2.3 No 2 giant SCCs



- There is a giant SCC
- There won't be 2 giant SCCs: Why not?
  - Just takes 1 page from one SCC to link to the other SCC
  - If the components have millions of pages the likelihood of this is very large

2.4 Splitting Zachary Karate Club



(a) *Karate club network*

(b) *After a split into two clubs*

do until (2 disconnected graphs) compute betweennesses first for the given graph remove edge has highest betweenness

2.5 betweenness clustering

Algorithm

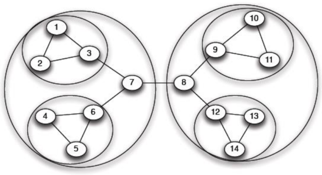
- Compute betweenness for all edges
- while (betweenness of any edge ¿ threshold):
- remove edge with highest betweenness
- recalculate betweenness

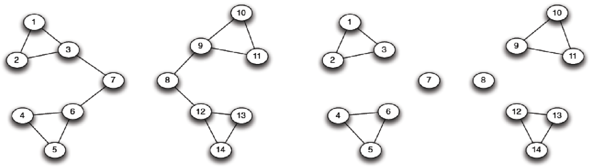
Betweenness needs to be recalculated at each step

- removal of an edge can impact the betweenness of another edge
- very expensive: all pairs shortest path – O(N^3)
- may need to repeat up to N times
- does not scale to more than a few hundred nodes, even with the fastest algorithms

2.6 betweenness clustering:

successively remove edges of highest betweenness (the bridges, or local bridges), breaking up the network into separate components

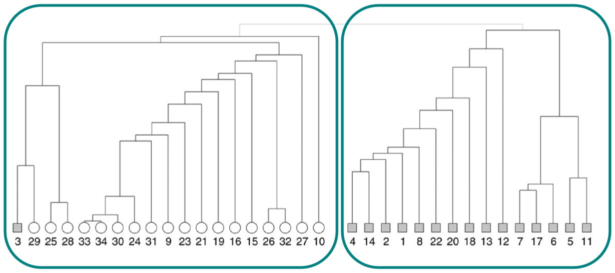




(a) *Step 1*

(b) *Step 2*

2.7 betweenness clustering algorithm & the karate club data set



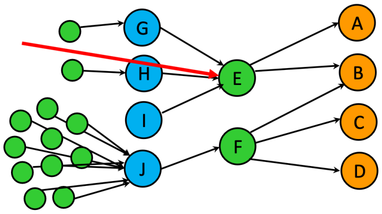
dendrogram: tree diagram that shows the hierarchical relationship between objects

3 Google's PageRank

3.1 PageRank: Intuition

Shouldn't E's vote be worth more than F's?

How many levels should we consider?



Imagine a contest for The Web's Best Page

- Initially, each page has one vote
- Each page votes for all the pages it has a link to
- To ensure fairness, pages voting for more than one page must split their vote equally between them
- Voting proceeds in rounds; in each round, each page has the number of votes it received in the previous round
- In practice, it's a little more complicated - but not much!

3.2 PageRank

Each page i is given a rank x$_i$
Goal: Assign the x$_i$ such that the rank of each page is governed by the ranks of the pages linking to it:

$$x_i = \sum_{j \in B_i} \frac{1}{N_j} x_j$$

Rank of page *i*

Rank of page *j*

How do we compute the rank values?

Every page *j* that links to *i*

Number of links out from page *j*

3.3 Iterative PageRank (simplified)

Initialize all ranks to be equal, e.g.:

$$x_i^{(0)} = \frac{1}{n}$$

Iterate until convergence

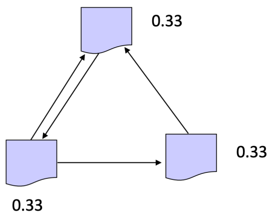
$$x_i^{(k+1)} = \sum_{j \in B_i} \frac{1}{N_j} x_j^{(k)}$$

3.4 Example

Example: Step 0

Initialize all ranks to be equal

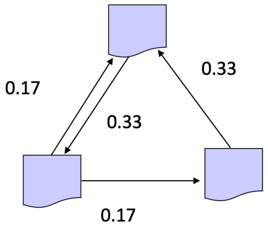
$$x_i^{(0)} = \frac{1}{n}$$



Example: Step 1

Propagate weights across out-edges

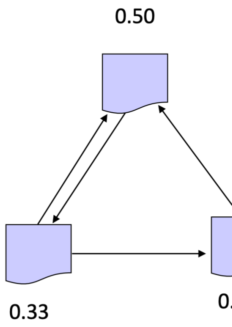
$$x_i^{(k+1)} = \sum_{j \in B_i} \frac{1}{N_j} x_j^{(k)}$$



Example

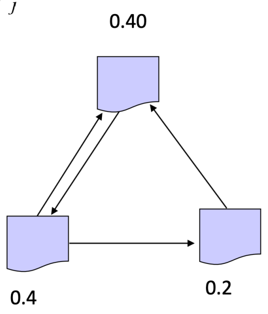
Compute weights based on in-edges

x



Example: Convergence

$$x_i^{(k+1)} = \sum_{j \in B_i} \frac{1}{N_j} x_j^{(k)}$$



3.5 Naïve PageRank Algorithm Restated

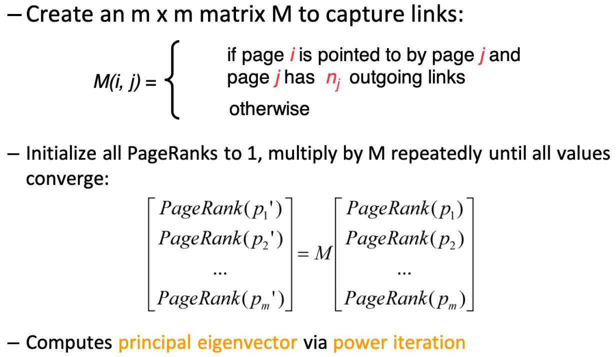
Let

- N(p) = number outgoing links from page p
- B_p = set of pages that back-link to page p

$$PageRank(p) = \sum_{b \in B_p} \frac{1}{N(b)} PageRank(b)$$

Each page b distributes its importance to all of the pages it points to (so we scale by 1/N(b))
Page p’s importance is increased by the importance of its back set

3.6 In Linear Algebra formulation



– Create an m x m matrix M to capture links:

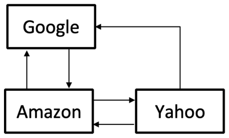
$$M(i, j) = \begin{cases} & \text{if page } i \text{ is pointed to by page } j \text{ and page } j \text{ has } n_j \text{ outgoing links} \\ & \text{otherwise} \end{cases}$$

– Initialize all PageRanks to 1, multiply by M repeatedly until all values converge:

$$\begin{bmatrix} PageRank(p_1') \\ PageRank(p_2') \\ \dots \\ PageRank(p_m') \end{bmatrix} = M \begin{bmatrix} PageRank(p_1) \\ PageRank(p_2) \\ \dots \\ PageRank(p_m) \end{bmatrix}$$

– Computes principal eigenvector via power iteration

3.7 A Brief Example



$$\begin{bmatrix} g' \\ y' \\ a' \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 \\ 1 & 0.5 & 0 \end{bmatrix} * \begin{bmatrix} g \\ y \\ a \end{bmatrix}$$

Running for multiple iterations:

$$\begin{bmatrix} g \\ y \\ a \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0.5 \\ 1.5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0.75 \\ 1.25 \end{bmatrix}, \dots \begin{bmatrix} 1 \\ 0.67 \\ 1.33 \end{bmatrix}$$

Total rank sums to number of pages

3.8 Random Surfer Model

PageRank has an intuitive basis in random walks on graphs
Imagine a random surfer, who starts on a random page and, in each step,

- with probability d, clicks on a random link on the page
- with probability 1-d, jumps to a random page (bored?)

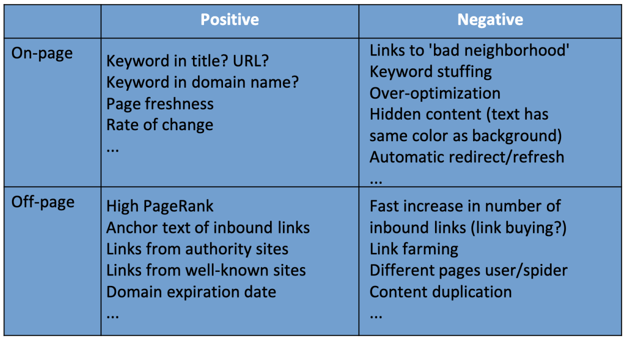
The PageRank of a page can be interpreted as the fraction of steps the surfer spends on the corresponding page
Transition matrix can be interpreted as a Markov Chain

3.9 Recap: PageRank

Estimates absolute 'quality' or 'importance' of a given page based on inbound links

- Can be computed via fixpoint iteration
- Can be interpreted as the fraction of time a 'random surfer' would spend on the page
- Several refinements (not covered here)

An important factor for Google ranking of web pages, but not the only one
Overall ranking is based on many factors (Google: ¿200)



	Positive	Negative
On-page	Keyword in title? URL? Keyword in domain name? Page freshness Rate of change ...	Links to 'bad neighborhood' Keyword stuffing Over-optimization Hidden content (text has same color as background) Automatic redirect/refresh ...
Off-page	High PageRank Anchor text of inbound links Links from authority sites Links from well-known sites Domain expiration date ...	Fast increase in number of inbound links (link buying?) Link farming Different pages user/spider Content duplication ...

Aside: What could be the other 200 factors?

- Note: This is entirely speculative!