Notes:

2k Factorial Experiments

Key Learning Points

1. Describe the mathematical relationship between causal factors.
2. Create a mathematical model, and DOE report.
3. Utilize 2k factorial experiments in an improvement project.

What is a 2k Factorial Experiment?

A 2k factorial experiment ensures that all factors are tested in a balanced design to ensure unaliased factor effect and interaction estimates. A factor effect or interaction is "aliased" when the estimate of the effect is influenced by another effect. Each factor is tested on only two levels, a high and a low, therefore 2k factorial experiments can also be called a 2 level factorial experiment.

The goal is to obtain a mathematical relationship which allows you to identify not only critical factors, but the best levels for those factors.

Investigate

2k Factorial Experiments allow you to investigate a large number of factors simultaneously. They are a specialized case of full factorial designs.

Easy to Analyze

2k designs are used most frequently in DOE applications because they are very easy to analyze and lend themselves well to sequential studies.

DOE Vocabulary

- k1 x k2 x k3 ... Factorial: Description of the basic design. The number of ks is the number of factors. The value of each k is the number of levels of interest for that factor. In 2k experiments, you will always have the value of k1=k2=k3=2. A 23 = 2 x 2 x 2, similarly a 22 = 2 x 2, etc… These types of basic designs indicate that all factors are investigated at only two levels.
- Repetition: Running several experimental runs consecutively using the same treatment combination.
- Replication: Replicating the entire experiment. Replication automatically implies that you do NOT run several experimental runs consecutively using the same treatment combination.
- Residual Error: the variability that remains after all the main effects and interactions have been identified.
- Lack-of-Fit: an indication of the fit of a prediction model. If the lack- of- fit p-value is less than your selected a-level, evidence exists that your model does not accurately fit the data.
- Pure Error: The variation in the data (sum of squares) that can only be estimated via true repeat or replicate measurements, either via center points or via repeats/replicate sample size selection.

Types of 2k Factorials

There are two types of 2k Factorials.

One Observation Per Treatment Combination:

- Usually low statistical power
- Use normal plots or Pareto charts instead of F-tests
- Error estimates are possible after dropping unimportant factors

More Than One Observation Per Treatment Combination (known as repeats or replicates):

- Better estimates of error
- Better statistical power
- F-tests, normal plots, and Pareto charts can be used

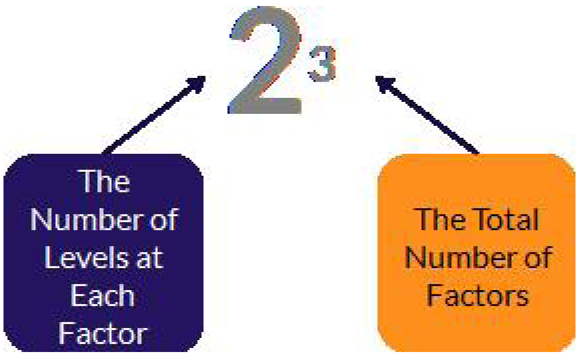
Factorial Design

A factorial design is notated as:
$\text{Level}_{\text{Factor}}$

You can calculate the total number of runs needed for a 2k Factorial experiment by:

Notes:

(# of levels$_{factor1}$) x (# of levels$_{factor2)}$ x (# of levels$_{factor3}$) x (…)



Example: General Factorial Design

4 factors: factors 1 & 4 have 2 levels, factors 2 & 3 have 5 levels each

2 x 5 x 5 x 2 = 100 runs

Example: Factorial Design at Two Levels

3 factors, each with 2 levels

2 x 2 x 2 = 8 runs = 2^3

Standard Order of 2^k Designs

2k factorials refer to k factors, each with 2 levels. A 2^2 factorial is a 2x2 factorial. This design has two factors with two levels and can be executed in only 2x2 or 4 runs. Likewise, a 2^3 factorial has 3 factors, each with two levels. This experiment can be done in 2x2x2 or 8 runs.

The design matrix for 2k factorials are usually shown in standard order. The low level of a factor is designed with a "-" or -1 and the high level is designated with a "+" or 1.

2^2 Factorial	
Temp	Conc
-1	-1
1	-1
-1	1
1	1

Notes:

2^3 Factorial		
Temp	*Conc*	*Cat*
-1	-1	-1
1	-1	-1
-1	1	-1
1	1	-1
-1	-1	1
1	-1	1
-1	1	1
1	1	1

Notes:

Standard Order (Yates Algorithm)

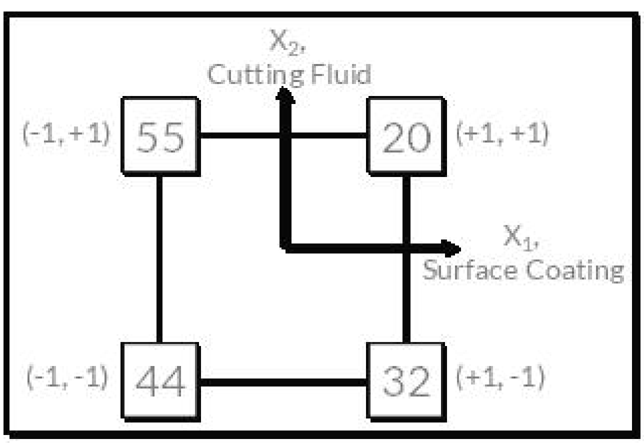
Standard Order (Yates Algorithm) is a factorial design in which the first column of the design matrix consists of successive minus and plus signs, the second column of successive pairs of minus and plus signs, the third column of four minus signs followed by four plus signs, etc.

In general, the Kth column consists of 2^{k-1} minus signs followed by 2^{k-1} plus signs. The order of the run should be random.

Example: 2x2 Full Factorial Experiment

A 2^2 factorial experiment was conducted to determine the effects that tool surface coating (x^1) and the application of cutting fluid (x^2) might have on the surface finish of a turned part. The low level for each of the two variables represented the absence of the technology in question (i.e., no surface coating, no cutting fluid). The high level represented the technology at some commercially recommended level. The response is the part surface finish in microns (R_a value). Note that smaller values are better.

X_1	X_2	R_a
-	-	44
+	-	32
-	+	55
+	+	20



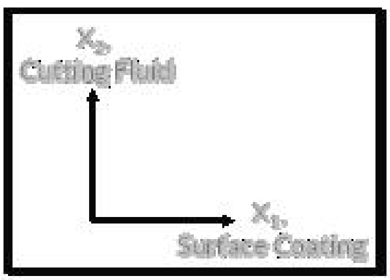
Main Effect

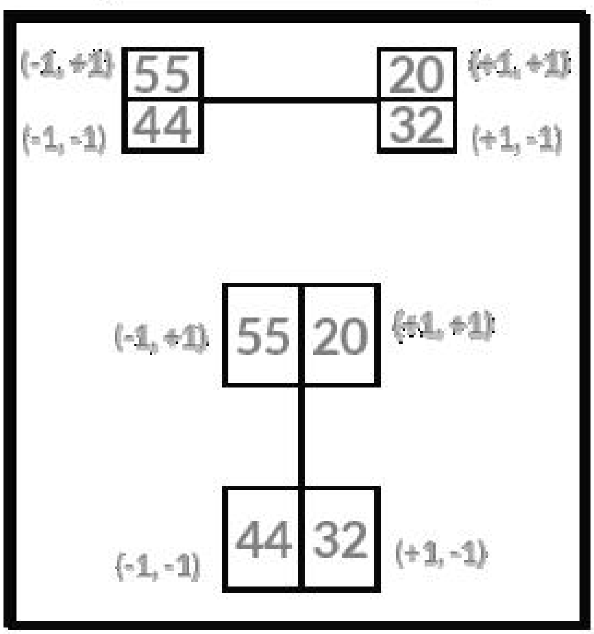
Main effect of surface coating, X_1:

- E_c = [(32+20)/2] - [(44+55)/2] = –23.5
- The use of surface coating on the cutting tool has a negative effect of 23.5 microns.

Main effect of cutting fluid, X_2:

- E_f = [(55+20)/2] - [(44+32)/2] = –0.5
- The use of cutting fluid has a negative effect of 0.5 microns on surface finishing.





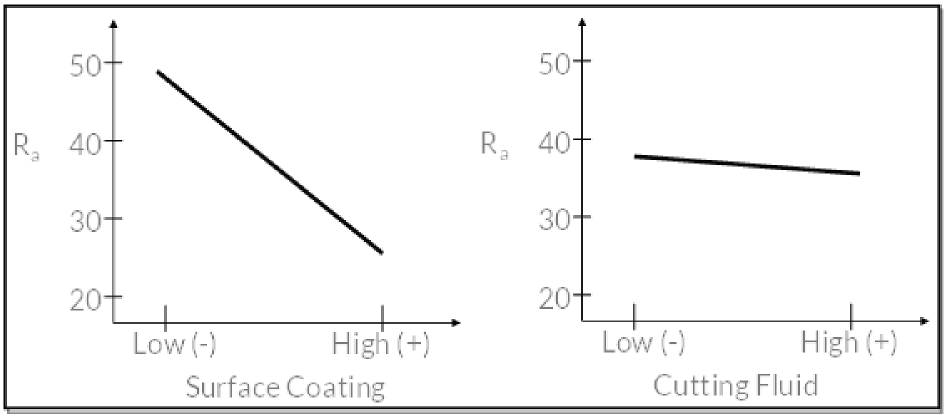
What is a Main Effect?

A main effect is the average change in the output variable produced by a change in the levels of a factor.

Notes:

Main Effect Plot

Which factor has a greater effect?



The Y axis is the output (surface finish), the X axis is the factor.

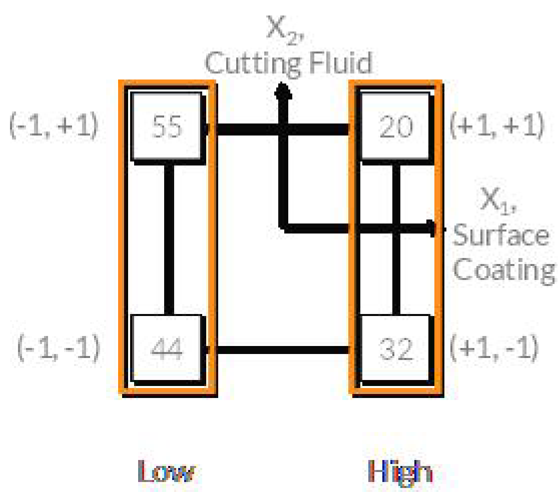
Interaction

- A two-factor interaction occurs when the effect that one factor has on the response depends on the level of a second factor.

  At low level (-) of the X_1 factor, the effect for X_2 is: Cutting Fluid = 55 - 44 = 11

- At high level (+) of the X_1 factor, the effect for X_2 is: Cutting Fluid = 20 - 32 = -12

  Since the effect of X_2 on surface finish depends on the level of X_1, it is said that there is an interaction.



Notes:

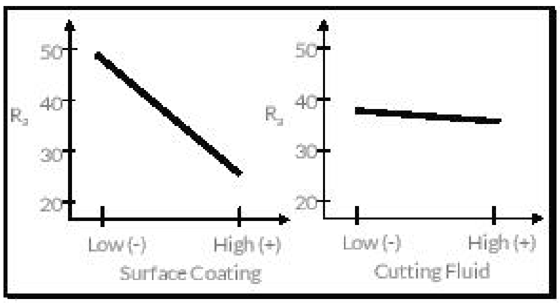
Example: Conclusion

Statistical Conclusion

- Surface coating brings a larger effect on the surface finishing than the cutting fluid.
- The 2-way interaction is significant.

Practical Conclusion

- Introduction of the surface coating technology brings large improvement to surface finishing.
- Cutting fluid used alone shows no significant effect.
- If further surface finishing improvement is desired, use both surface coating and cutting fluid technologies.



E_{CF} = [(44+20)/2] - [(32+55)/2] = -11.5

Step 1: Define the Problem

The Problem:

A process engineer would like to determine the effect of Quench Temperature, Quench Time, and Quench Oil types on the hardness of a steel shaft.

Step 2: State the Hypotheses

- H_o: Effect of Quench Temp = 0
- H_a: Effect of Quench Temp ≠ 0
- H_o: Effect of Quench Time = 0
- H_a: Effect of Quench Time ≠ 0
- H_o: Effect of Quench Oil = 0
- H_a: Effect of Quench Oil ≠ 0

Notes:

Step 3: State the Factors and Levels of Interest

For this experiment the factors are Quench Temp, Quench Time, Quench Oil Brand.

Independent Variables (Xs) (Factors)	Level (-)	Level (+)
Quench Temp (C)	160	180
Quench Time (sec.)	5	15
Quench Oil Brand	A	B

Step 4: Create an Appropriate Experimental Data sheet

Temp	Time	Oil
-	-	-
+	-	-
-	+	-
+	+	-
-	-	+
+	-	+
-	+	+
+	+	+

This is an example of a 2k factorial experiment with only one observation per treatment combination (experimental run).

- Minitab: Stat > DOE > Factorial > Analyze Factorial Design
  - Number of Factors: 3
  - Designs: Full Factorial
  - Factors: Temp, Time, Oil

Step 5: Run the Experiment and Collect the Data

For this example, 1 experimental run was completed and hardness was ascertained.

StdOrder	RunOrder	CenterPt	Blocks	Temp	Time	Oil	Hardness
1	1	1	1	160	5	A	60
2	2	1	1	180	5	A	72
3	3	1	1	160	15	A	54
4	4	1	1	180	15	A	68
5	5	1	1	160	5	B	52
6	6	1	1	180	5	B	83
7	7	1	1	160	15	B	45
8	8	1	1	180	15	B	80

Notes:

Step 6a: Construct Experimental Design Spreadsheet

Notes:

Construct the ANOVA Table and use the appropriate graphical tool to evaluate the data. Calculate the effects of the experiment by hand. First, look at Temperature.

Simply add the yields associated with (-1), and the yields associated with (1) and calculate the average (Sum/4). The "1s" and "-1s" in the temperature column are called the "contrast" for the main effect of temperature.

Temp	Time	Oil	Hardness
(-) 160	(-) 5	(-) A	60
(+) 180	(-) 5	(-) A	72
(-) 160	(+) 15	(-) A	54
(+) 180	(+) 15	(-) A	68
(-) 160	(-) 5	(+) B	52
(+) 180	(-) 5	(+) B	83
(-) 160	(+) 15	(+) B	45
(+) 180	(+) 15	(+) B	80

Calculations

This step shows the independent effects of Temperature, Time, and Oil.

	Temp	Time	Oil	Hardness
	(-) 160	(-) 5	(-) A	60
	(+) 180	(-) 5	(-) A	72
	(-) 160	(+) 15	(-) A	54
	(+) 180	(+) 15	(-) A	68
	(-) 160	(-) 5	(+) B	52
	(+) 180	(-) 5	(+) B	83
	(-) 160	(+) 15	(+) B	45
	(+) 180	(+) 15	(+) B	80
Total (-)	(60+54+52+45)= (-)211	(60+72+52+83) = (-)267	(60+72+54+68) = (-)254	
Total (+)	(72+68+83+80) = 303	(54+68+45+80) =247	(52+83+45+80) = 260	
Sum	-211 + 303 = 92	-267 + 247 = -20	-254 + 260 = 6	
Average Effect	23	-5	1.5	

Main Effect Calculations

Temp Main Effect:	[(72+68+83+80) - (60+54+52+45)]/4 = (303-211)/4 = 23
Time Main Effect:	[(54+68+45+80) - (60+72+52+83)]/4 = (247-267)/4 = -5

Oil Main Effect:	[(52+83+45+80) - (60+72+54+68)]/4 = (260-254)/4 = 1.5

Notes:

The Factor's Effect on Hardness

Now you have calculated the independent main effect each factor (Temperature, Time, and Oil Type) on the hardness of the steel.

- Temperature's Average Effect: 23
- Time's Average Effect: -5
- Oil's Average Effect: 1.5

Minitab will supply individual p-values for each factor, similar to conducting three 2-sample t-tests.

Temp	Time	Oil	Hardness
(-) 160	(-) 5	(-) A	60
(+) 180	(-) 5	(-) A	72
(-) 160	(+) 15	(-) A	54
(+) 180	(+) 15	(-) A	68
(-) 160	(-) 5	(+) B	52
(+) 180	(-) 5	(+) B	83
(-) 160	(+) 15	(+) B	45
(+) 180	(+) 15	(+) B	80
23	-5	1.5	

Calculating Interaction Contrasts

The benefit of factorial experiments is that they provide the ability to assess "interactions" between factors." Is there a particular combination of input settings that improve Hardness over and above the singular (main) effects?"

The interaction contrast is derived by multiplying the columns to be represented.

	Temp	Time	Oil	Temp* Time	Temp* Oil	Time * Oil	Temp* Time* Oil	Hardness
	(-) 160	(-) 5	(-) A	(+)	(+)	(+)	(-)	60
	(+) 180	(-) 5	(-) A	(-)	(-)	(+)	(+)	72
	(-) 160	(+) 15	(-) A	(-)	(+)	(-)	(+)	54
	(+) 180	(+) 15	(-) A	(+)	(-)	(-)	(+)	68
	(-) 160	(-) 5	(+) B	(+)	(+)	(-)	(-)	52
	(+) 180	(-) 5	(+) B	(+)	(-)	(-)	(+)	83
	(-) 160	(+) 15	(+) B	(-)	(+)	(+)	(-)	45
	(+) 180	(+) 15	(+) B	(+)	(-)	(+)	(+)	80

	Temp	Time	Oil	Temp* Time	Temp* Oil	Time * Oil	Temp* Time* Oil	Hardness
Total (-)	(60+54+52+45)= (-)211	(60+72+52+83) = (-)267	(60+72+54+68) = (-)254	(72+54+83+45) = (-) 254	(72+68+52+45) = (-)237	(54+68+52+83) = (-)257	(60+68+83+45) = (-)256	
Total (+)	(72+68+83+80) = 303	(54+68+45+80) =247	(52+83+45+80) = 260	(60+68+52+80) = 260	(60+54+83+80) = 277	(60+72+45+80) = 257	(72+54+52+80) = 258	
Sum	-211 + 303 = 92	-267 + 247 = -20	-254 + 260 = 6	-254 + 260 = 6	-237 + 277 = 40	-257 + 257 = 0	-256 + 258 = 2	
Average Effect	23	-5	1.5	1.5	10	0	0.5	

Notes:

Notes on Calculating Significance

As a rule, you cannot estimate standard error and obtain p-values for estimates if your design is ”saturated”. In a saturated design, the number of factors the experimenter is trying to estimate is one less than the number of runs.

In this case, there are three main effects, three two-factor interactions and one three-factor interaction. So without replicating the experiment, there are only eight total test runs, and the design is saturated. The best option is to replicate your experiment. If that is not a good option for the experimenter, there is another choice.

Minitab will calculate Lenth’s PSE analysis, which calculates a pseudo standard error and a corresponding significance level. This will be displayed on a normal probability plot or Pareto chart, with significance identified by Minitab. In subsequent analyses, run a reduced model that only uses the most significant factors. Then the traditional p-values will be available.

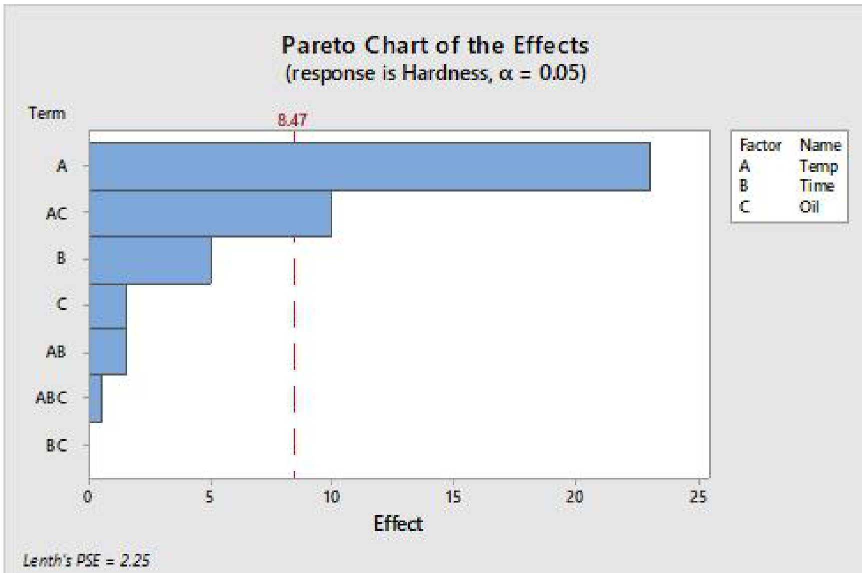
It is appropriate to use a hierarchical model. That means that if you are including atwo-factor interaction, you must also include each of the two corresponding main effects, even if they are not significant.

Step 6b: Construct the ANOVA Table for the Full Model

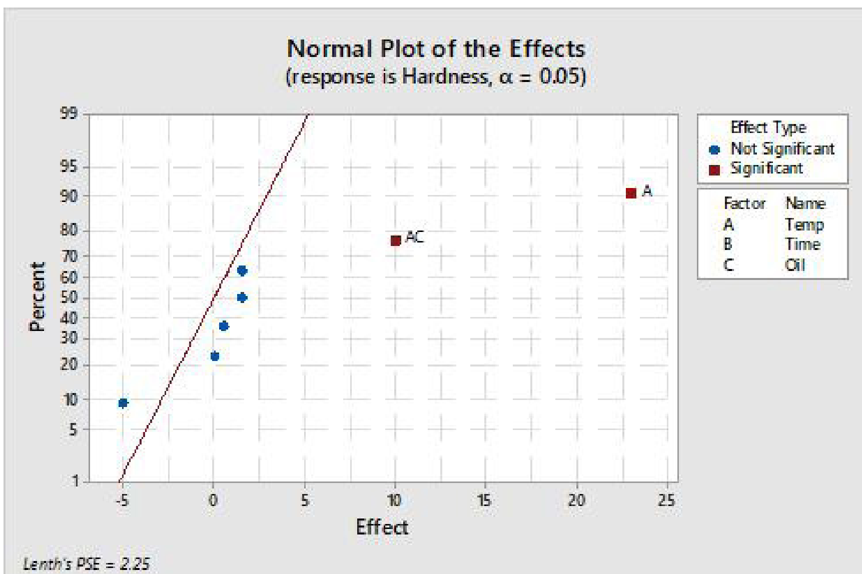
- **Minitab: Stat > DOE > Factorial > Analyze Factorial Design**
    - Response: Hardness
    - Graphs > Effects Plot
    - Check off Normal and Pareto

Results

Pareto Chart of the Effects



Normal Plot of the Effects



Notes:

Analysis of Variance Table

Factorial Regression: Hardness versus Temp, Time, Oil

Analysis of Variance

Source	DF	Adj SS	Adj MS	F-Value	P-Value
Model	7	1317.50	188.21	*	*
Linear	3	1112.50	370.83	*	*
Temp	1	1058.00	1058.00	*	*
Time	1	50.00	50.00	*	*
Oil	1	4.50	4.50	*	*
2-Way Interactions	3	204.50	68.17	*	*
Temp*Time	1	4.50	4.50	*	*
Temp*Oil	1	200.00	200.00	*	*
Time*Oil	1	0.00	0.00	*	*
3-Way Interactions	1	0.50	0.50	*	*
Temp*Time*Oil	1	0.50	0.50	*	*
Error	0	*	*		
Total	7	1317.50			

Model Summary

S	R-sq	R-sq(adj)	R-sq(pred)
*	100.00%	*	*

Coded Coefficients

Term	Effect	Coef	SE Coef	T-Value	P-Value	VIF
Constant		64.25	*	*	*	
Temp	23.00	11.50	*	*	*	1.00
Time	-5.000	-2.500	*	*	*	1.00
Oil	1.5000	0.7500	*	*	*	1.00
Temp*Time	1.5000	0.7500	*	*	*	1.00
Temp*Oil	10.000	5.000	*	*	*	1.00
Time*Oil	-0.000000	-0.000000	*	*	*	1.00
Temp*Time*Oil	0.5000	0.2500	*	*	*	1.00

Regression Equation in Uncoded Units

Hardness = -100.8 + 1.000 Temp - 3.050 Time - 75.75 Oil + 0.01500 Temp*Time + 0.4500 Temp*Oil - 0.8500 Time*Oil + 0.005000 Temp*Time*Oil

Alias Structure

Factor	Name
A	Temp
B	Time
C	Oil

Aliases
I
A
B
C
AB
AC
BC
ABC

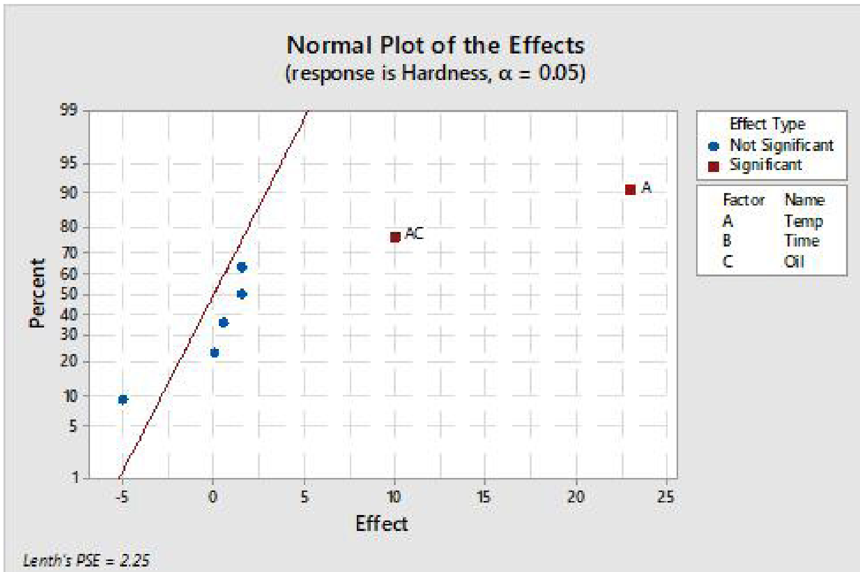
Effects Plot for Hardness

Effects Pareto for Hardness

Notes:

Significant Factors

The significant factors in this experiment are temperature, and the interaction of temperature and oil.



Step 7: Rerun a Reduced Model

One should remove factors in the order as follows and check in between: (1) any interactions (2-way, 3-way, etc.) that are not significant, (2) any main effects that remain insignificant. This is especially true when there are no replicates, and, hence, no measure of error with full design.

Significant factors are the main effects of temperature and time, and the interaction of temperature and oil. The main effect "oil" cannot be removed because of the interaction of temperature and oil.

Step 8: Investigate the Residuals Plots to Ensure Model Fit

Residuals are the difference between the actual Y value and the Y value predicted by the prediction equation.

Residuals should:

- Be randomly and normally distributed about a mean of zero
- Not correlate with the predicted Y
- Not exhibit trends over time (if data is chronological)

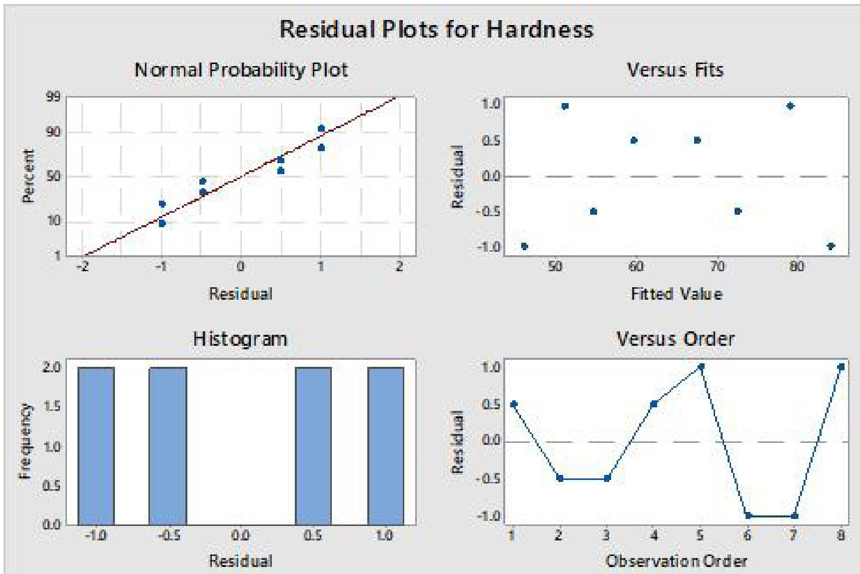
Problems with the residuals would indicate the model is inadequate.

Residuals Plots

- **Minitab: Stat > DOE > Factorial > Analyze Factorial Design**

Notes:

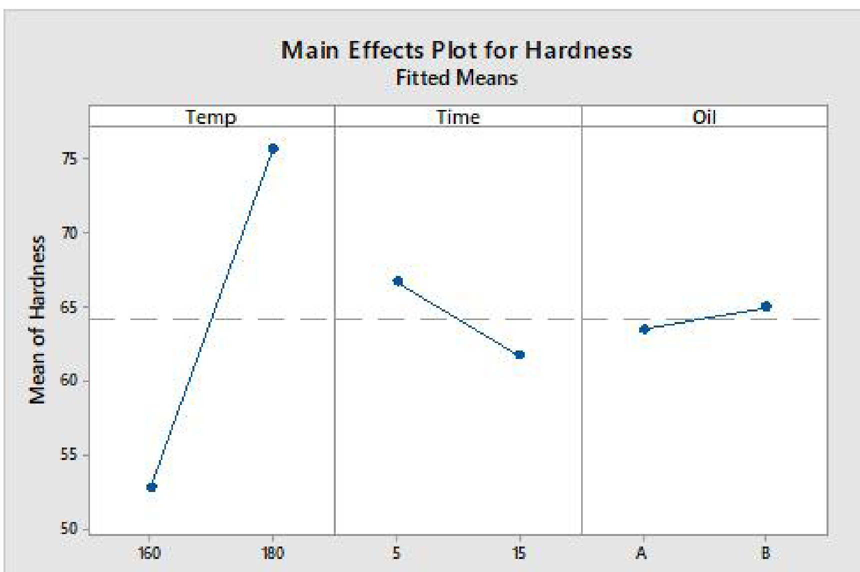
- Response: Hardness
- Graphs > Residual Plots>Four in one



Step 9: Investigate Significant Main Effects and Interactions

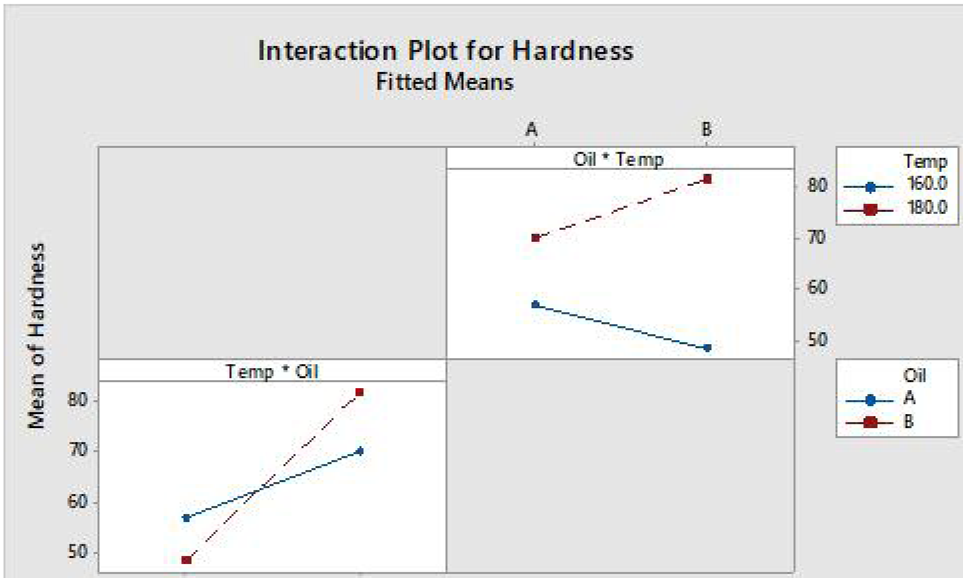
Using the ANOVA table and appropriate graphical tool, investigate significant main effects and interactions.

Main Effects Plot

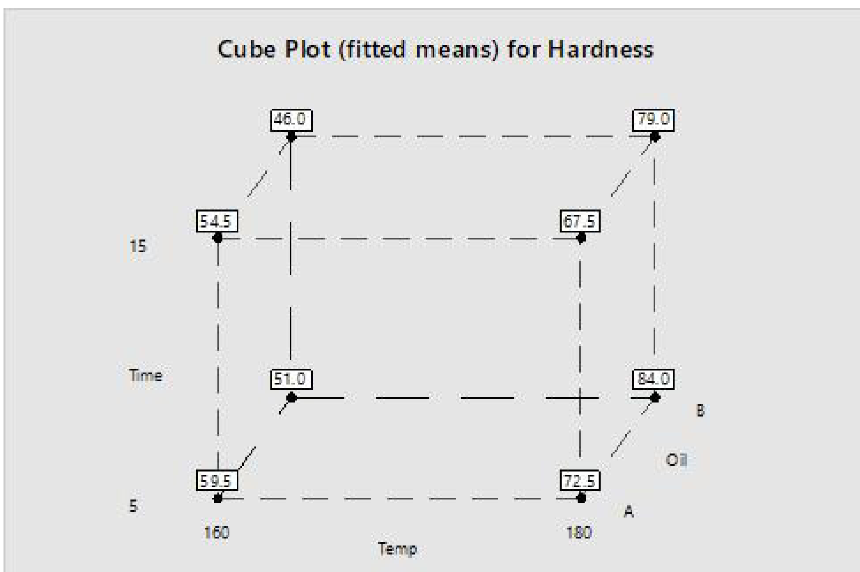


Notes:

Interaction Plot



Cube Plot



Step 10: State the Mathematical Model

Calculate the variation for the main effects and interactions left in the model. State the mathematical model obtained.

Determine the variation that is accounted for by the main effects and interactions left in the model. State the mathematical model obtained. This is quantified by the R^2 value presented in Minitab.

Mathematical Model

Use the coefficients from the analysis to derive the reduced mathematical model (Add the coefficients of the factors.)

Notes:

Notes:

Coded Coefficients

Term	Effect	Coef	SE Coef	T-Value	P-Value	VIF
Constant		64.250	0.456	140.76	0.000	
Temp	23.000	11.500	0.456	25.20	0.000	1.00
Time	-5.000	-2.500	0.456	-5.48	0.012	1.00
Oil	1.500	0.750	0.456	1.64	0.199	1.00
Temp*Oil	10.000	5.000	0.456	10.95	0.002	1.00

Hardness = -126.25 + 1.15 Temp - 0.5 Time - .8425 Oil + .5 Temp*Oil

Step 11: Translate the Statistical Conclusion into Process Terms

Recommendations: To increase Hardness, it is recommended that the process be run with both Quench Temperature & Quench Oil set to their high values. Quench Time should be set to its low value.

- Quench Temp = 180o C
- Oil = Type B
- Quench Time = 5 seconds

Step 12: Replicate Optimum Conditions

Plan the next experiment and/or institutionalize the change.

When Should 2^k Factorial Experiments Be Used?

2^k factorial DOE is used to determine the main effects and interactions of usually no more than five factors. It is the perfect tool for examining the interplay of multiple factors, as the factors are guaranteed to be uncorrelated with each other.

Recommendations

- Use the default generators in Minitab to ensure the design is balanced and orthogonal.
- Run the experimental runs in random order whenever possible.
- Replicate the experiment for more power. Repeated measures from the same run do not give the same estimating power as replicates.
- Perform a verifying experiment to prove your best conditions.

- A true optimum may lie outside the design.

Notes: