Simulation of Lévy processes.

A Lévy process $\{X_t\}_{t\geq 0}$ is a continuous-time stochastic process with independent, stationary increments: it represents the motion of a point whose successive displacements are random and independent, and statistically identical over different time intervals of the same length. A Lvy process may thus be viewed as the continuous-time analog of a random walk. Lévy processes are currently being used to model several finance and risk processes, all due to their versatility and extensive theoretical results. The subclass of Lévy processes we are interested in during this project, is the one with the following decomposition:

$$X_t = \mu t + \sigma B_t + \sum_{i=1}^{N_t} Y_i,$$

where

- μ is called the linear drift of the process
- $\sigma \geq 0$ is called the Gaussian intensity of the process
- $\{B_t\}_{t\geq 0}$ is a standard Brownian motion (that is, a continuous-time stochastic process with independent and stationary increments and with $B_t \sim N(0,t)$ for all $t \geq 0$ and $B_0 = 0$)
- $\{N_t\}$ is a Poisson process of intensity $\lambda \geq 0$ with interarrival times $T_1, T_2, T_3, \ldots$ (recall that $T_i \sim Exp(\lambda)$) and arrival times $S_1, S_2, S_3, \ldots$ (recall that $S_n = \sum_{i=1}^{\infty} T_i$), and
- $Y_1, Y_2, Y_3 \ldots$ is a sequence of independent and identically distributed random variables with distribution F.

One can say that we are interested in simulating a Brownian motion with jumps that occur at random times. During the Stochastic Simulation course we were able to simulate all the previous random components, except for $\{B_t\}$. Numerical simulation of a Brownian motion is difficult, and in most cases, inefficient. The purpose of this project is to use some theoretical results in order to simulate a discrete "skeleton" of $\{X_t\}_{t\geq 0}$ (say $\{(P_i, A_i, M_i)\}_{i\in\mathbb{N}}$, a 3-coordinate discrete stochastic process) which contains useful information of the original continuous-time process. Basically, for each $i \in \mathbb{N}$:

- P_i is equal to $X_{S_i^-}$, that is, equal to the process $\{X_t\}_{t\geq 0}$ **p**rior to its i-th jump,

- A_i is equal to X_{S_i}, that is, equal to the process $\{X_t\}_{t\geq 0}$ **a**fter its i-th jump, and
- M_i is equal to $\sup_{0\leq s\leq S_i} X_s$, that is, the **m**aximum of $\{X_t\}_{t\geq 0}$ up to the time of its i-th jump.

Although this method will be numerically efficient and easy to implement, notice that we will only be able to describe the process $\{X_t\}$ at the time of its jumps (that is, at $S_1, S_2, S_3...$), and **not** between them. This is a drawback of our method.

First, let us state a Theorem which will be the building block of this project.

Theorem 1. *Let $T \sim Exp(\lambda)$,*

$$V = \max_{0\leq t\leq T} \mu t + \sigma B_t, \quad \textit{and} \quad W = \left(\max_{0\leq t\leq T} \mu t + \sigma B_t\right) - (\mu T + \sigma B_T).$$

Then V and W are independent with $V \sim Exp(\phi_1)$, $W \sim Exp(\phi_2)$, where

$$\phi_1 = \frac{\mu}{\sigma^2} + \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2\lambda}{\sigma^2}} \quad \textit{and} \quad \phi_2 = -\frac{\mu}{\sigma^2} + \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2\lambda}{\sigma^2}}.$$

Notice that in particular

$$\mu T + \sigma B_T = V - W.$$

Thus, instead of simulating the path of $\{\mu t + \sigma B_t\}_{t\geq 0}$ (the continuous part of $\{X_t\}_{t\geq 0}$) between jumps, we are able to simulate its maximum and its value prior to the jump by only simulating exponentially distributed random variables. All in all, the method boils down to the next steps. First, set $(P_0, A_0, M_0) = (0,0,0)$. Then the next steps need to be simulated recursively using the following for $i \geq 1$.

1. Simulate V_i and W_i according to the distributions of Theorem 1. Remember that V_i will be the highest point that $\{\mu t + \sigma B_t\}_{t\geq 0}$ reaches before the next interarrival time T_i, and $V_i - W_i$ is the value of that process at time T_i.
2. Simulate Y_i, which corresponds to the size of the $i-th$ jump.
3. Set

$$P_i = A_{i-1} + (V_i - W_i), \tag{1}$$
$$A_i = A_{i-1} + (V_i - W_i) + Y_i, \tag{2}$$
$$M_i = \max\{M_{i-1}, A_{i-1} + V_i, A_{i-1} + (V_i - W_i) + Y_i\}. \tag{3}$$

The tasks for this project are the next ones:

1. Implement the previous algorithm through a simulation with some fixed parameters of your choice up until 1000 jumps in order to get a simulation of $(P_{1000}, A_{1000}, M_{1000})$ (that is the value of $\{X_t\}_{t\geq 0}$ prior to its 1000-th jump, after its 1000-th jump, and the maximum value attained up to the 1000-th jump.). Repeat this 100 times in order to get histograms for $(P_{1000}, A_{1000}, M_{1000})$.

2. Carefully study and explain the reasoning behind the algorithm. described previously, specially equations (1), (2) and (3). HINT: By looking at the simulation step by step, things should get much clearer. Pathwise explanations are required.

3. Experiment with different values of μ, σ, λ (extremely big or extremely small, combining scenarios). Plot results and comment.

4. Experiment with different F's (use exponential, Erlang, hyperexponential or Pareto). Plot results an comment.

5. First passage probabilities are defined as $\mathbb{P}(\sup_{t\geq 0} X_t > a)$ for some fixed $a \geq 0$, that is, the probability that a Lévy process will ever upcross level a. This is specially useful when working with finance and risk models (the probability that a stock or a risk reserve reaches certain level). How can we estimate them? Which coordinate of our discrete skeleton $\{(P_i, A_i, M_i)\}_{i\in\mathbb{N}}$ is useful to estimate first passage probabilities? Estimate first passage probabilities for some of the previous processes for levels $a = 10, a = 1000, a = 100000$.