Cookbook for Robust Optimization

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March 31th, 2019

1 Introduction

The standard mixed integer linear programming (MILP) is:

$$\begin{aligned}
\min \quad & \mathbf{c}^T\mathbf{x} + \mathbf{f}^T\mathbf{y} && (1)\\
\text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \geq \mathbf{b} && (2)\\
& \mathbf{y} \in \mathbf{Y} && (3)\\
& \mathbf{x} \geq 0 && (4)
\end{aligned}$$

where the $\mathbf{y}$ is vector of integer variables.

The problem with uncertainty in coefficients of continuous decision variables left-hand-side of parts of the constraints, becomes:

$$\begin{aligned}
\min \quad & \mathbf{c}^T\mathbf{x} + \mathbf{f}^T\mathbf{y} && (5)\\
\text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \geq \mathbf{b} && (6)\\
& \left(\overline{\mathbf{A}} \pm \hat{\mathbf{A}}\right)\mathbf{x} + \overline{\mathbf{B}}\mathbf{y} \leq \overline{\mathbf{b}} && (7)\\
& \mathbf{y} \in \mathbf{Y} && (8)\\
& \mathbf{x} \geq 0 && (9)
\end{aligned}$$

where the $\mathbf{y}$ is vector of integer variables.

After linearization, the problem becomes:

$$\begin{aligned}
\min \quad & \mathbf{c}^T\mathbf{x} + \mathbf{f}^T\mathbf{y} && (10)\\
\text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \geq \mathbf{b} && (11)\\
& \begin{cases} \overline{\mathbf{A}}_i^T\mathbf{x} + \Gamma_i\lambda_i + \boldsymbol{\mu}_i^T\mathbf{1} + \overline{\mathbf{B}}_i\mathbf{y} \leq \overline{b}_i \\ \lambda_i + \boldsymbol{\mu}_i \geq \hat{\mathbf{A}}_i \circ \mathbf{z} \quad \text{(total } J \text{ rows)} \\ \lambda_i \geq 0 \\ \boldsymbol{\mu}_i \geq \mathbf{0} \quad \text{(total } J \text{ rows)} \end{cases} \qquad \forall i \in I(\text{row}) && (12)\\
& -\mathbf{z} \leq \mathbf{x} \leq \mathbf{z} \quad \text{(total } J \text{ rows)} && (13)\\
& \mathbf{z} \geq \mathbf{0} && (14)\\
& \mathbf{y} \in \mathbf{Y} && (15)\\
& \mathbf{x} \geq 0 && (16)
\end{aligned}$$

2 Examples

2.1 Production Planning

You work for a production company and support them with optimizing their capacity and production schedule for a new factory.

The company has $p \in P$ different products that are produced on different machine types $m \in M$. Not each product can be produced on each machine, i.e., parameter $a^m(p) = 1$, if product p can be produced on machine type m and $a^m(p) = 0$ otherwise. As you are opening a new factory, you also have to decide how many machines of type m you want to buy. The price is cm^m for one machine of type $m \in M$. Each machine of type m ∈ M provides Tm hours of production.

The production costs are cp^m for each $p \in P$. The targeted production quantities $d(p)$ for each product $p \in P$ for the next year are given. Because we consider the entire year, we approximate the production quantities as continuous values.

The production time of product $p \in P$ on machine type $m \in M$ is uncertain. You know that the expected production time is $\overline{t^m(p)}$ and the deviation (positive and negative) can be up to $t^m(p)$. From experience from other factories, we can conclude that for each machine type $m \in M$ not more than 30% of the products that can be produced on machine type m will have a deviation from the expected production time.

Formulate a robust optimization model that decides the number of machines and production quantities for each product and machine to have minimal cost and cover the demand in all cases of production time deviation. Use a budget of uncertainty.

Set	Definition	Size
P	different products	10
M	different machines	4

Table 1. Categories of Sets and Their Attributes

Decision Variable	Definition	Stage	Value Range
y^m	different products	First	$\{0,1,2,...,10\}$
$\mathrm{x}^m(p)$	different machines	First	$\mathbb{R}^+$

Table 2. Categories of Decision Variables and Their Attributes

The problem becomes:

$$
\begin{aligned}
\min \quad & \sum_M y^m cm^m + \sum_P \left[cp(p) \sum_M x^m(p) \right] && && (17)\\
\text{s.t.} \quad & \sum_M x^m(p) \geq d(p) && \forall p \quad \text{(Cover all demand, \$)} && (18)\\
& x^m(p) \leq a^m(p) y^m \frac{1}{\varepsilon} && \forall p, m \quad \text{(machine produces some products)} && (19)\\
& \sum_P \left[\overline{t^m(p)} \pm t^m(p) \right] x^m(p) \leq T^m y^m && \forall m \quad \text{(Limited machine production time, hour)} && (20)\\
& x^m(p) \in \mathbb{R}^+ && && (21)\\
& y^m \in \{0,1,2,...,10\} && && (22)
\end{aligned}
$$

It can be transformed into standard robust optimization form by:

$$
\begin{aligned}
\mathbf{x} &= \left[x^1(1), x^2(1), x^3(1), x^4(1), x^1(2), x^2(2), x^3(2), x^4(2), ...\right]^T && && (23)\\
\mathbf{y} &= \left[y^1, y^2, y^3, y^4\right]^T && && (24)\\
J/K &= |P| = 10 && && (25)\\
K &= |M| = 4 && && (26)\\
\mathbf{c} &= [cp(1), cp(1), cp(1), cp(1), cp(2), cp(2), cp(2), cp(2), ..., cp(10)]^T && (J = 40 \text{ rows}) && (27)\\
\mathbf{f} &= [cm^1, cm^2, cm^3, cm^4]^T && (K = 4 \text{ rows}) && (28)
\end{aligned}
$$

$$
\overline{\mathbf{A}} = \begin{bmatrix}
\overline{t^1(1)} & 0 & 0 & 0 & \dots & \overline{t^1(10)} & 0 & 0 & 0 \\
0 & \overline{t^2(1)} & 0 & 0 & \dots & 0 & \overline{t^2(10)} & 0 & 0 \\
0 & 0 & \overline{t^3(1)} & 0 & \dots & 0 & 0 & \overline{t^3(10)} & 0 \\
0 & 0 & 0 & \overline{t^4(1)} & \dots & 0 & 0 & 0 & \overline{t^4(10)}
\end{bmatrix} \quad (K = 4 \text{ rows}, J = 40 \text{ columns}) \tag{29}
$$

$$
\hat{\mathbf{A}} = \begin{bmatrix}
t^1(1) & 0 & 0 & 0 & \dots & t^1(10) & 0 & 0 & 0 \\
0 & t^2(1) & 0 & 0 & \dots & 0 & t^2(10) & 0 & 0 \\
0 & 0 & t^3(1) & 0 & \dots & 0 & 0 & t^3(10) & 0 \\
0 & 0 & 0 & t^4(1) & \dots & 0 & 0 & 0 & t^4(10)
\end{bmatrix} \quad (K = 4 \text{ rows}, J = 40 \text{ columns}) \tag{30}
$$

$$
\overline{\mathbf{B}} = \begin{bmatrix}
-T^1 & 0 & 0 & 0 \\
0 & -T^2 & 0 & 0 \\
0 & 0 & -T^3 & 0 \\
0 & 0 & 0 & -T^4
\end{bmatrix} \quad (K = 4 \text{ rows}, K = 4 \text{ columns}) \tag{31}
$$

$$
\mathbf{A} = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & -1
\end{bmatrix} \qquad (J/K+J=50 \text{ rows}, J=40 \text{ columns}) \tag{32}
$$

$$
\mathbf{B} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
a^1(1)\frac{1}{\varepsilon} & 0 & 0 & 0 \\
0 & a^2(1)\frac{1}{\varepsilon} & 0 & 0 \\
0 & 0 & a^3(1)\frac{1}{\varepsilon} & 0 \\
0 & 0 & 0 & a^4(1)\frac{1}{\varepsilon} \\
a^1(2)\frac{1}{\varepsilon} & 0 & 0 & 0 \\
0 & a^2(2)\frac{1}{\varepsilon} & 0 & 0 \\
0 & 0 & a^3(2)\frac{1}{\varepsilon} & 0 \\
0 & 0 & 0 & a^4(2)\frac{1}{\varepsilon} \\
\vdots & \vdots & \vdots & \vdots
\end{bmatrix} \qquad (J/K+J=50 \text{ rows}, K=4 \text{ columns}) \tag{33}
$$

$$
\mathbf{b} = \Bigg[\underbrace{d(1), d(2), ..., d(10)}_{\text{total } J/K=10}, \underbrace{0, 0, ..., 0}_{\text{total } J=40}\Bigg]^T \tag{34}
$$

To solve it in a traditional robust optimization manner, the problem becomes:

$$
\begin{aligned}
\min \quad & \sum_M y^m cm^m + \sum_P \left[cp(p) \sum_M x^m(p)\right] && && (35) \\
\text{s.t.} \quad & \sum_M x^m(p) \geq d(p) && \forall p \quad \text{(Cover all demand, \$)} && (36) \\
& x^m(p) \leq a^m(p) y^m \frac{1}{\varepsilon} && \forall p, m \quad \text{(machine produces some products)} && (37) \\
& \begin{cases}
\sum_P \overline{t^m(p)} x^m(p) + \Gamma^m \lambda^m + \sum_P \mu^m(p) \leq T^m y^m \\
\lambda^m + \mu^m(p) \geq t^m(p) z^m \quad \forall p \\
-z^m \leq x^m(p) \leq z^m \quad \forall p \\
\lambda^m \geq 0 \\
\mu^m(p) \geq 0 \quad \forall p \\
z^m \geq 0
\end{cases} && \forall m \quad \text{(Limited machine production time, hour)} && (38) \\
& x^m(p) \in \mathbb{R}^+ && && (39) \\
& y^m \in \{0, 1, 2, ..., 10\} && && (40)
\end{aligned}
$$

For more simplification, because $x^m(p)$ is always positive:

$$\begin{aligned}
\min \quad & \sum_M y^m cm^m + \sum_P \left[cp(p) \sum_M x^m(p) \right] && && (41)\\
\text{s.t.} \quad & \sum_M x^m(p) \geq d(p) && \forall p \quad \text{(Cover all demand, \$)} && (42)\\
& x^m(p) \leq a^m(p) y^m \frac{1}{\varepsilon} && \forall p, m \quad \text{(machine produces some products)} && (43)\\
& \begin{cases} \sum_P \overline{t^m(p)} x^m(p) + \Gamma^m \lambda^m + \sum_P \mu^m(p) \leq T^m y^m \\ \lambda^m + \mu^m(p) \geq t^m(p) x^m(p) \quad \forall p \\ \lambda^m \geq 0 \\ \mu^m(p) \geq 0 \quad \forall p \end{cases} && \forall m \quad \text{(Limited machine production time, hour)} && (44)\\
& x^m(p) \in \mathbb{R}^+ && && (45)\\
& y^m \in \{0, 1, 2, ..., 10\} && && (46)
\end{aligned}$$