Cookbook for Robust Optimization

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1 Introduction

Robust optimization is a field of optimization theory that deals with optimization problems in which a certain measure of robustness is sought against uncertainty that can be represented as deterministic variability in the value of the parameters of the problem itself and/or its solution.[1]

To illustrate the importance of robustness in practical applications, we quote from the case study by Ben-Tal and Nemirovski (2000)[2] on linear optimization problems from the Net Lib library:

> In real-world applications of Linear Programming, one can- not ignore the possibility that a small uncertainty in the data can make the usual optimal solution completely meaningless from a practical viewpoint.

The budget of uncertainty indicates we stipulate that nature will be restricted in its behavior, in that only a subset of the coefficients will change in order to adversely affect the solution.[3]

2 Problem with Box Uncertainty and Budget of Uncertainty

The standard mixed integer linear programming (MILP) is:

$$
\begin{aligned}
\min \quad & \mathbf{c}^T \mathbf{x} + \mathbf{f}^T \mathbf{y} && (1)\\
\text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \geq \mathbf{b} && (2)\\
& \mathbf{y} \in \mathbf{Y} && (3)\\
& \mathbf{x} \geq 0 && (4)
\end{aligned}
$$

where the $\mathbf{y}$ is vector of integer variables.

If continuous variables in some constraints have uncertain coefficients. The problem becomes robust optimization of mixed integer problems. For now, we only deal with box uncertainty. The expression becomes:

$$
\begin{aligned}
\min \quad & \mathbf{c}^T \mathbf{x} + \mathbf{f}^T \mathbf{y} && (5)\\
\text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \geq \mathbf{b} && (6)\\
& \left(\overline{\mathbf{A}} \pm \hat{\mathbf{A}}\right) \mathbf{x} + \overline{\mathbf{B}}\mathbf{y} \leq \overline{\mathbf{b}} && (7)\\
& \mathbf{y} \in \mathbf{Y} && (8)\\
& \mathbf{x} \geq 0 && (9)
\end{aligned}
$$

where the $\overline{\mathbf{A}}$ is vector of mean values for the coefficients of continuous variables, and $\hat{\mathbf{A}}$ are vector of uncertainties. Note that the sign of the constraint is "less equal".

After linearization according to theorem 1 in paper[3], the problem becomes:

$$
\begin{aligned}
\min \quad & \mathbf{c}^T \mathbf{x} + \mathbf{f}^T \mathbf{y} && (10)\\
\text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \geq \mathbf{b} && (11)\\
& \begin{cases} \overline{\mathbf{A}}_i^T \mathbf{x} + \Gamma_i \lambda_i + \boldsymbol{\mu}_i^T \mathbf{1} + \overline{\mathbf{B}}_i \mathbf{y} \leq \overline{b}_i \\ \lambda_i + \boldsymbol{\mu}_i \geq \hat{\mathbf{A}}_i \circ \mathbf{z} \quad \text{(total } J \text{ rows)} \\ \lambda_i \geq 0 \\ \boldsymbol{\mu}_i \geq \mathbf{0} \quad \text{(total } J \text{ rows)} \end{cases} \qquad \forall i \in I(\text{row}) && (12)\\
& -\mathbf{z} \leq \mathbf{x} \leq \mathbf{z} \quad \text{(total } J \text{ rows)} && (13)\\
& \mathbf{z} \geq \mathbf{0} && (14)\\
& \mathbf{y} \in \mathbf{Y} && (15)\\
& \mathbf{x} \geq 0 && (16)
\end{aligned}
$$

where J is the column number of the matrix $\overline{\mathbf{A}}$ and I is the row number. Γ_i is the budget of uncertainty for i-the constraint.

3 Examples

3.1 Production Planning with Uncertainty in Production Efficiency

You work for a production company and support them with optimizing their capacity and production schedule for a new factory.

The company has $p \in P$ different products that are produced on different machine types $m \in M$. Not each product can be produced on each machine, i.e., parameter $a^m(p) = 1$, if product p can be produced on machine type m and $a^m(p) = 0$ otherwise. As you are opening a new factory, you also have to decide how many machines of type m you want to buy. The price is cm^m for one machine of type $m \in M$. Each machine of type m ∈ M provides Tm hours of production.

The production costs are cp^m for each $p \in P$. The targeted production quantities $d(p)$ for each product $p \in P$ for the next year are given. Because we consider the entire year, we approximate the production quantities as continuous values.

The production time of product $p \in P$ on machine type $m \in M$ is uncertain. You know that the expected production time is $\overline{t^m(p)}$ and the deviation (positive and negative) can be up to $t^m(p)$. From experience from other factories, we can conclude that for each machine type $m \in M$ not more than 30% of the products that can be produced on machine type m will have a deviation from the expected production time.

Formulate a robust optimization model that decides the number of machines and production quantities for each product and machine to have minimal cost and cover the demand in all cases of production time deviation. Use a budget of uncertainty.

Set	Definition	Size
P	different products	10
M	different machines	4

Table 1. Categories of Sets and Their Attributes

Decision Variable	Definition	Stage	Value Range
y^m	different products	First	$\{0,1,2,...,10\}$
$x^m(p)$	different machines	First	$\mathbb{R}^+$

Table 2. Categories of Decision Variables and Their Attributes

The problem becomes:

$$\min \quad \sum_M y^m cm^m + \sum_P \left[cp(p) \sum_M x^m(p) \right] \tag{17}$$

$$\text{s.t.} \quad \sum_M x^m(p) \geq d(p) \qquad \forall p \quad \text{(Cover all demand, \$)} \tag{18}$$

$$x^m(p) \leq a^m(p) y^m \frac{1}{\varepsilon} \qquad \forall p,m \quad \text{(machine produces some products)} \tag{19}$$

$$\sum_P \left[\overline{t^m(p)} \pm t^m(p) \right] x^m(p) \leq T^m y^m \qquad \forall m \quad \text{(Limited machine production time, hour)} \tag{20}$$

$$x^m(p) \in \mathbb{R}^+ \tag{21}$$

$$y^m \in \{0,1,2,...,10\} \tag{22}$$

It can be transformed into standard robust optimization form by:

$$\mathbf{x} = \left[x^1(1), x^2(1), x^3(1), x^4(1), x^1(2), x^2(2), x^3(2), x^4(2), ... \right]^T \tag{23}$$

$$\mathbf{y} = \left[y^1, y^2, y^3, y^4 \right]^T \tag{24}$$

$$J/K = |P| = 10 \tag{25}$$

$$K = |M| = 4 \tag{26}$$

$$\mathbf{c} = [cp(1), cp(1), cp(1), cp(1), cp(2), cp(2), cp(2), cp(2), ..., cp(10)]^T \qquad (J = 40 \text{ rows}) \tag{27}$$

$$\mathbf{f} = [cm^1, cm^2, cm^3, cm^4]^T \qquad (K = 4 \text{ rows}) \tag{28}$$

$$\overline{\mathbf{A}} = \begin{bmatrix} \overline{t^1(1)} & 0 & 0 & 0 & \dots & \overline{t^1(10)} & 0 & 0 & 0 \\ 0 & \overline{t^2(1)} & 0 & 0 & \dots & 0 & \overline{t^2(10)} & 0 & 0 \\ 0 & 0 & \overline{t^3(1)} & 0 & \dots & 0 & 0 & \overline{t^3(10)} & 0 \\ 0 & 0 & 0 & \overline{t^4(1)} & \dots & 0 & 0 & 0 & \overline{t^4(10)} \end{bmatrix} \quad (K = 4 \text{ rows}, J = 40 \text{ columns}) \tag{29}$$

$$\hat{\mathbf{A}} = \begin{bmatrix} t^1(1) & 0 & 0 & 0 & \dots & t^1(10) & 0 & 0 & 0 \\ 0 & t^2(1) & 0 & 0 & \dots & 0 & t^2(10) & 0 & 0 \\ 0 & 0 & t^3(1) & 0 & \dots & 0 & 0 & t^3(10) & 0 \\ 0 & 0 & 0 & t^4(1) & \dots & 0 & 0 & 0 & t^4(10) \end{bmatrix} \quad (K = 4 \text{ rows}, J = 40 \text{ columns}) \tag{30}$$

$$\overline{\mathbf{B}} = \begin{bmatrix} -T^1 & 0 & 0 & 0 \\ 0 & -T^2 & 0 & 0 \\ 0 & 0 & -T^3 & 0 \\ 0 & 0 & 0 & -T^4 \end{bmatrix} \quad (K = 4 \text{ rows}, K = 4 \text{ columns}) \tag{31}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & -1 \end{bmatrix} \quad (J/K + J = 50 \text{ rows}, J = 40 \text{ columns}) \tag{32}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a^1(1)\frac{1}{\varepsilon} & 0 & 0 & 0 \\ 0 & a^2(1)\frac{1}{\varepsilon} & 0 & 0 \\ 0 & 0 & a^3(1)\frac{1}{\varepsilon} & 0 \\ 0 & 0 & 0 & a^4(1)\frac{1}{\varepsilon} \\ a^1(2)\frac{1}{\varepsilon} & 0 & 0 & 0 \\ 0 & a^2(2)\frac{1}{\varepsilon} & 0 & 0 \\ 0 & 0 & a^3(2)\frac{1}{\varepsilon} & 0 \\ 0 & 0 & 0 & a^4(2)\frac{1}{\varepsilon} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad (J/K + J = 50 \text{ rows}, K = 4 \text{ columns}) \tag{33}$$

$$\mathbf{b} = \left[\underbrace{d(1), d(2), ..., d(10)}_{\text{total } J/K = 10}, \underbrace{0, 0, ..., 0}_{\text{total } J = 40} \right]^T \tag{34}$$

To solve it in a traditional robust optimization manner, the problem becomes:

$$
\begin{aligned}
\min \quad & \sum_M y^m cm^m + \sum_P \left[cp(p) \sum_M x^m(p) \right] && && (35)\\
\text{s.t.} \quad & \sum_M x^m(p) \geq d(p) && \forall p \quad \text{(Cover all demand, \$)} && (36)\\
& x^m(p) \leq a^m(p) y^m \frac{1}{\varepsilon} && \forall p, m \quad \text{(machine produces some products)} && (37)\\
& \begin{cases} \sum_P \overline{t^m(p)} x^m(p) + \Gamma^m \lambda^m + \sum_P \mu^m(p) \leq T^m y^m \\ \lambda^m + \mu^m(p) \geq t^m(p) z^m \quad \forall p \\ -z^m \leq x^m(p) \leq z^m \quad \forall p \\ \lambda^m \geq 0 \\ \mu^m(p) \geq 0 \quad \forall p \\ z^m \geq 0 \end{cases} && \forall m \quad \text{(Limited machine production time, hour)} && (38)\\
& x^m(p) \in \mathbb{R}^+ && && (39)\\
& y^m \in \{0, 1, 2, ..., 10\} && && (40)
\end{aligned}
$$

For more simplification, because $x^m(p)$ is always positive:

$$
\begin{aligned}
\min \quad & \sum_M y^m cm^m + \sum_P \left[cp(p) \sum_M x^m(p) \right] && && (41)\\
\text{s.t.} \quad & \sum_M x^m(p) \geq d(p) && \forall p \quad \text{(Cover all demand, \$)} && (42)\\
& x^m(p) \leq a^m(p) y^m \frac{1}{\varepsilon} && \forall p, m \quad \text{(machine produces some products)} && (43)\\
& \begin{cases} \sum_P \overline{t^m(p)} x^m(p) + \Gamma^m \lambda^m + \sum_P \mu^m(p) \leq T^m y^m \\ \lambda^m + \mu^m(p) \geq t^m(p) x^m(p) \quad \forall p \\ \lambda^m \geq 0 \\ \mu^m(p) \geq 0 \quad \forall p \end{cases} && \forall m \quad \text{(Limited machine production time, hour)} && (44)\\
& x^m(p) \in \mathbb{R}^+ && && (45)\\
& y^m \in \{0, 1, 2, ..., 10\} && && (46)
\end{aligned}
$$

References

1. Wikipedia contributors. Robust optimization — Wikipedia, the free encyclopedia (2019). [Online; accessed 1-April-2019].
2. Ben-Tal, A. & Nemirovski, A. Robust solutions of linear programming problems contaminated with uncertain data. *Math. programming* **88**, 411–424 (2000).
3. Bertsimas, D. & Sim, M. The price of robustness. *Oper. research* **52**, 35–53 (2004).