# Solution - Robust optimization

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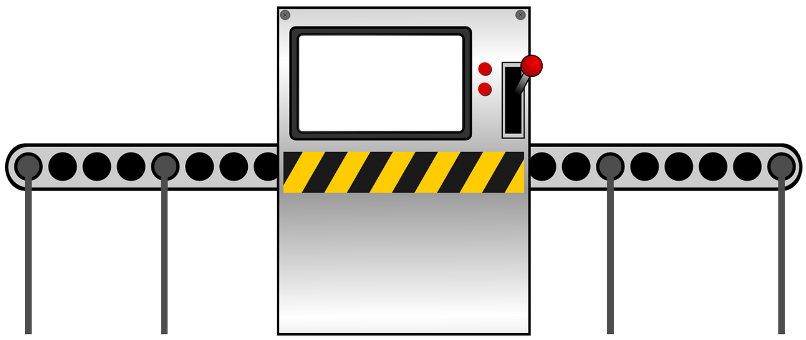
## Production planning



You work for a production company and support them with optimizing their capacity and production schedule for a new factory.

The company has $p \in P$ different products that are produced on different machine types $m \in M$. Not each product can be produced on each machine, i.e., parameter $a_{p,m} = 1$, if product $p$ can be produced on machine type $m$ and $a_{p,m} = 0$ otherwise. As you are opening a new factory, you also have to decide how many machines of type $m$ you want to buy. The price is $c_m^M$ for one machine of type $m \in M$. Each machine of type $m \in M$ provides $T_m$ hours of production.

The production costs are $c_p^P$ for each $p \in P$. The targeted production quantities $d_p$ for each product $p \in \mathcal{P}$ for the next year are given. Because we consider the entire year, we approximate the production quantities as continuous values.

The production time of product $p \in P$ on machine type $m \in M$ is uncertain. You know that the expected production time is $\bar{t}_{p,m}$ and the deviation (positive and negative) can be up to $t_{p,m}$. From experience from other factories, we can conclude that for each machine type $m \in M$ not more than 30% of the products that can be produced on machine type $m$ will have a deviation from the expected production time.

### Task

Formulate a robust optimization model that decides the number of machines and production quantities for each product and machine to have minimal cost and cover the demand in all cases of production time deviation. Use a budget of uncertainty.

Hint: Start with writing down the deterministic model with $\tilde{t}_{p,m}$ as placeholder for the uncertain parameter, before introducing the budget of uncertainty.

Solve your robust model with GAMS using the file `robust_model.gms` which already has some data input. Compare the solution to the following cases:

- no deviation from the expected production time is included in the model ($\Gamma_m = 0$)
- full deviation from the expected production time is included in the model (all products on that machine type can deviate at the same time)

## Solution

Variables
$y_m$ Number of machines of type $m$
$x_{p,m}$ Production amount of product $p$ on machine type $m$

Model without budget of uncertainty:

$$\min \sum_{m\in M} \left[ c^M_m y_m + \sum_{p\in P} c^P_p x_{p,m} \right] \tag{1a}$$
$$s.t. \sum_{m\in M} x_{p,m} \geq d_p \qquad \forall p \in P \tag{1b}$$
$$\sum_{p\in P} \tilde{t}_{p,m} x_{p,m} \leq T_m y_m \qquad \forall m \in M, \tilde{t}_{p,m} \in [\bar{t}_{p,m} - t_{p,m}, \bar{t}_{p,m} + t_{p,m}] \tag{1c}$$
$$x_{p,m} \leq BigM_{m,p} a_{p,m} y_m \qquad \forall m \in M, p \in P \tag{1d}$$
$$y_m \geq 0 \text{ and integer} \qquad \forall m \in M \tag{1e}$$
$$x_{p,m} \geq 0 \qquad \forall m \in M, p \in P \tag{1f}$$

where $\tilde{t}_{p,m}$ represents the uncertain production time and $BigM_{m,p}$ a large enough constant (here: $\frac{T_m}{\bar{t}_{p,m} - t_{p,m}}$).
The model (1a)-(1f) does not model that maximum 30% of the products per machine type take their worst case values.

In the remainder, we use a budget of uncertainty $\Gamma_m = 0.3 \sum_{p\in P} a_{p,m}$ per machine type.

Reformulate constraint (1c) with budget of uncertainty:

$$\sum_{p\in P} \bar{t}_{p,m} x_{p,m} + \max_{\left\{S_m \cup \{s_m\} | S_m \subseteq P, |S_m| \leq \Gamma_m, s_m \in P \setminus S_m\right\}} \left\{ \sum_{p\in S_m} (t_{p,m}|x_{p,m}|) + (\Gamma_m - \lfloor \Gamma_m \rfloor) t_{s_m,m} |x_{s_m,m}|) \right\} \leq T_m y_m \qquad \forall m \in M \tag{2}$$

Formulate subproblem in (2) as LP for each $m \in M$, which $S_m$ and $s_m$ being the subsets of products that lead to the worst case:

$$\max_{\left\{S_m \cup \{s_m\} | S_m \subseteq P, |S_m| \leq \Gamma_m, s_m \in P \setminus S_m\right\}} \left\{ \sum_{p\in S_m} (t_{p,m}|x_{p,m}|) + (\Gamma_m - \lfloor \Gamma_m \rfloor) t_{s_m,m} |x_{s_m,m}|) \right\}$$
$$\Leftrightarrow$$

$$\max \sum_{p\in P_m} t_{p,m} |x_{p,m}| z_{p,m} \tag{3a}$$
$$s.t. \sum_{p\in P} z_{p,m} \leq \Gamma_m \qquad : \lambda_m \tag{3b}$$
$$0 \leq z_{p,m} \leq 1 \qquad \forall p \in P \qquad : \mu_{p,m} \tag{3c}$$

Dual of (3a)-(3c):

$$\min \Gamma_m \lambda_m + \sum_{p\in P} \mu_{p,m} \tag{4a}$$
$$s.t. \lambda_m + \mu_{p,m} \geq t_{p,m} |x_{p,m}| \qquad \forall p \in P \tag{4b}$$
$$\lambda_m \geq 0 \tag{4c}$$
$$\mu_{p,m} \geq 0 \qquad \forall p \in P \tag{4d}$$

Replace subproblem in (2) with (4a)-(4d) to get the overall robust model.

$$\min \sum_{m \in M} \left[ c^M_m y_m + \sum_{p \in P} c^P_p x_{p,m} \right] \tag{5a}$$

$$s.t. \sum_{m \in M} x_{p,m} \geq d_p \qquad \forall p \in P \tag{5b}$$

$$\sum_{p \in P} \bar{t}_{p,m} x_{p,m} + \Gamma_m \lambda_m + \sum_{p \in P} \mu_{p,m} \leq T_m y_m \qquad \forall m \in M \tag{5c}$$

$$\lambda_m + \mu_{p,m} \geq t_{p,m} x_{p,m} \qquad \forall m \in M, p \in P \tag{5d}$$

$$x_{p,m} \leq BigM_{m,p} a_{p,m} y_m \qquad \forall m \in M, p \in P \tag{5e}$$

$$y_m \geq 0 \text{ and integer} \qquad \forall m \in M \tag{5f}$$

$$x_{p,m} \geq 0 \qquad \forall m \in M, p \in P \tag{5g}$$

$$\lambda_m \geq 0 \qquad \forall m \in M \tag{5h}$$

$$\mu_{p,m} \geq 0 \qquad \forall m \in M, p \in P \tag{5i}$$

Note that we can remove the absolute value in constraint (5d), because $x_{p,m}$ is always positive.

The solution values for the three cases are given below. The production amounts are the same for all three solutions, because we have to cover the demand in each case, which does not change. However, the number of machines per type are different, because the production times are uncertain and can deviate more or less in the different cases. In case with no deviations (factor 0.0), the production time is always the expected production time and we do not have increased production times in the worst case. Therefore, we need less machines. In case with no restrictions on the deviations (factor 1.0) more machines are needed to cover the demand also in the worst case (with long production times). The solution for the restricted budget of uncertainty (factor 0.3) lies in between and leads to less cost and machines compared the very conservative case.

| | | $\Gamma_m = 0.3 \sum_{p \in P} a_{p,m}$ | $\Gamma_m = 0.0 \sum_{p \in P} a_{p,m}$ | $\Gamma_m = 1.0 \sum_{p \in P} a_{p,m}$ |
|---|---|---|---|---|
| Obj | | 29458532 | 29478532 | 30848532 |
| Machines | m1 | 22 | 15 | 7 |
| | m2 | 0 | 0 | 17 |
| | m3 | 12 | 15 | 0 |
| | m4 | 3 | 3 | 17 |
| Production | p1 | 5844 | 5844 | 5844 |
| | p2 | 9313 | 9313 | 9313 |
| | p3 | 5725 | 5725 | 5725 |
| | p4 | 8511 | 8511 | 8511 |
| | p5 | 9465 | 9465 | 9465 |
| | p6 | 27866 | 27866 | 27866 |
| | p7 | 28396 | 28396 | 28396 |
| | p8 | 27394 | 27394 | 27394 |
| | p9 | 27590 | 27590 | 27590 |
| | p10 | 27612 | 27612 | 27612 |