Rings and Modules

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Contents

1 Modules

1.1 Generalities

Remark. Conventions:

- All rings are unital.
- All modules are left-modules.
- All homomorphisms are between modules under the same ring.
- I will *not* be very careful about saying map and homomorphism. If a map should obviously be a homomorphism, then it is one.

Definition 1.1 (Homomorphism)**.** Let R be a ring and M, M' be R-modules. A map $f : M \to M'$ is a *homomorphism* if

$$f(x+y) = f(x) + f(y) \qquad \text{and} \qquad f(xy) = f(x)f(y).$$

We denote the set of homomorphism from M to M' by $\mathrm{Hom}(M, M')$.

f is an *epimorphism* if it is surjective and a *monomorphism* if it is injective.

f is an *isomorphism* if there exists a homomorphism $f^{-1} : M' \to M$ so that $f \circ f^{-1} = \mathrm{id}_{M'}$ and $f^{-1} \circ f = \mathrm{id}_M$. Equivalently, f is an isomorphism if it is both monic and epic.

If $M = M'$, then f is called an *endomorphism*. Moreover, if f is also an isomorphism, then f is an *automorphism*.

Definition 1.2 (Kernel & Image)**.** Let $f : M \to N$ be a homomorphism. Then, we define the *kernel of f* to be the set

$$\ker(f) \coloneqq \{\, x \in M \mid f(x) = 0 \,\}$$

and the *image of f* to be the set

$$\mathrm{im}(f) \coloneqq \{\, f(x) \mid x \in M \,\}.$$

Observe that $\ker(f)$ is a submodule of M and $\mathrm{im}(f)$ is a submodule of N.

Definition 1.3 (Quotient/Factor Modules)**.** Let M be an R-module and $N \subset M$ a submodule. Then, M/N is the set of all additive cosets $\overline{x} = x + N$, and is an R-module with the operations

$$\overline{x} + \overline{y} \coloneqq \overline{x+y} \qquad \text{and} \qquad \overline{x} \cdot \overline{y} \coloneqq \overline{xy}.$$

We remark that these operations are well-defined, and leave the checking of such to the reader.

Definition 1.4 (Cokernel & Coimage)**.** If $f : M \to N$ is a homomorphism, we define its *cokernel* and *coimage* by

$$\mathrm{coker}(f) \coloneqq N/\,\mathrm{im}(f) \qquad \text{and} \qquad \mathrm{coim}(f) \coloneqq M/\ker(f)$$

In particular, observe that if N is a submodule of M and $j : N \to M$ is the canonical inclusion, then

$$\mathrm{coker}(f) = M/j(N) = M/N.$$

Theorem 1.1 (Universal Property of the Cokernel)**.** Let $f : N \to M$ be a homomorphism and $\pi : M \to \operatorname{coker}(f) = M/\operatorname{im}(f)$ be the canonical projection. Then,

1. $\pi \circ f = 0$.
2. If $\varphi : M \to P$ is another homomorphism satisfying $\varphi \circ f = 0$, then there is a unique homomorphism $\psi : \operatorname{coker}(f) \to P$ so that $\psi \circ \pi = \varphi$, i.e. the following diagram commutes:

$$\begin{tikzcd} N \arrow[r, "f"] & M \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & \operatorname{coker}(f) = M/\operatorname{im}(f) \arrow[d, dashed, "\psi"] \\ & & P \end{tikzcd}$$

3. And moreover, if φ and P satisfy properties (1) and (2) in the same way as π and $\operatorname{coker}(f)$, then ψ is an isomorphism.

Proof. 1. Let $x \in N$. Then,

$$(\pi \circ f)(x) = \pi(f(x)) = f(x) + \operatorname{im}(f) = 0 + \operatorname{im}(f),$$

since $f(x) \in \operatorname{im}(f)$.

2. We define the map

$$\psi : \operatorname{coker}(f) \to P, \qquad \psi(x + \operatorname{im}(f)) = \varphi(x).$$

- ψ *is well-defined.* Let $x + \operatorname{im}(f) = x' + \operatorname{im}(f)$. Then, $x = x' + i$ for some $i \in \operatorname{im}(f)$, so

  $$\psi(x + \operatorname{im}(f)) = \varphi(x) = \varphi(x' + i) = \varphi(x') + \varphi(i) = \varphi(x') = \psi(x' + \operatorname{im}(f)),$$

  since $\varphi \circ f = 0$.

- ψ *is a homomorphism.* Let $x + \operatorname{im}(f), y + \operatorname{im}(f) \in \operatorname{coker}(f)$. Then,

  $$\begin{aligned} \psi((x + \operatorname{im}(f)) + (y + \operatorname{im}(f))) &= \psi((x + y) + \operatorname{im}(f)) \\ &= \varphi(x + y) \\ &= \varphi(x) + \varphi(y) \\ &= \psi(x + \operatorname{im}(f)) + \psi(y + \operatorname{im}(f)) \end{aligned}$$

  and

  $$\begin{aligned} \psi((x + \operatorname{im}(f))(y + \operatorname{im}(f))) &= \psi(xy + \operatorname{im}(f)) \\ &= \varphi(xy) \\ &= \varphi(x)\varphi(y) \\ &= \psi(x + \operatorname{im}(f))\psi(y + \operatorname{im}(f)). \end{aligned}$$

- ψ *is unique.* Suppose that $\psi' : \operatorname{coker}(f) \to P$ is another map that makes the diagram commute. Then, for every $x + \operatorname{im}(f) \in \operatorname{coker}(f)$, we have

  $$\psi'(x + \operatorname{im}(f)) = (\psi' \circ \pi)(x) = \varphi(x) = (\psi \circ \pi)(x) = \psi(x + \operatorname{im}(f)),$$

  i.e. $\psi' = \psi$.

3. By hypothesis, $\varphi \circ f = 0$, so we obtain by (2) a unique homomorphism $\psi : \operatorname{coker}(f) \to P$ such that $\varphi = \psi \circ \pi$. Similarly, we assumed that φ and P satisfy property (2) in the same way as π and $\operatorname{coker}(f)$, so we also get a unique homomorphism $\psi' : P \to \operatorname{coker}(f)$ so that $\pi = \psi' \circ \varphi$. Then, $\pi = \psi' \circ \psi \circ \pi$ and $\varphi = \psi \circ \psi' \circ \varphi$. By applying (2) with π and $\operatorname{coker}(f)$, we get $\psi' \circ \psi = \mathrm{id}_{\operatorname{coker}(f)}$, and by applying the hypothesis to φ and P, we get $\psi \circ \psi' = \mathrm{id}_P$. Thus, ψ is indeed an isomorphism, with inverse ψ'. $\square$

In the particular case where N is a submodule of M, the Universal Property of the Cokernel gives rise to the Universal Property of the Quotient:

Theorem 1.2 (Universal Property of the Quotient)**.** Let N be a submodule of M and $\pi : N \to M/N$ the canonical projection. Then, given any homomorphism $f : M \to L$ with $f \circ j = 0$, we obtain a unique homomorphism $\overline{f} : M/N \to L$ so that $f = \overline{f} \circ \pi$.

Proof. Let $j : N \to M$ be the canonical inclusion and observe that $M/N = \operatorname{coker}(j)$. Thus, by the Universal Property, there is a unique map $\overline{f} : M/N \to L$ so that

$$\begin{tikzcd} N \arrow[r, "j"] & M \arrow[r, "\pi"] \arrow[dr, "f"'] & M/N \arrow[d, dashed, "\overline{f}"] \\ & & L \end{tikzcd}$$

commutes. ◻

Thus, we obtain the classic isomorphism theorems:

Theorem 1.3 (1st Isomorphism Theorem)**.** Let $f : N \to M$ be a homomorphism of R-modules. Then, there is a unique isomorphism $\psi : N/\ker(f) = \operatorname{coim}(f) \to \operatorname{im}(f)$.

Proof. Let $f' : N \to \operatorname{im}(f)$ be given by $f'(x) \coloneqq f(x)$.

Let $j : \ker(f) \to N$ be the canonical inclusion and $\pi : N \to N/\ker(f)$. Note that $f' \circ j = f \circ j = 0$, by definition. Hence, by the Universal Property of the Quotient, we obtain a unique map $\overline{f} : N/\ker(f) \to \operatorname{im}(f)$ so that

$$\begin{tikzcd} \ker(f) \arrow[r, "j"] & N \arrow[r, "\pi"] \arrow[dr, "f'"'] & M/N \arrow[d, dashed, "\overline{f}"] \\ & & \operatorname{im}(f) \end{tikzcd}$$

commutes.

Since f' is surjective, so is $\overline{f}$. Moreover, the kernel of $\overline{f}$ is

$$\begin{aligned} \ker(\overline{f}) = \{\, x + \ker(f) \mid \overline{f}(x + \ker(f)) = (\overline{f} \circ \pi)(x) = f(x) = 0 \,\} \\ = \{\, x + \ker(f) \mid x \in \ker(f) \,\} = \{\, 0 + \ker(f) \,\}, \end{aligned}$$

so $\overline{f}$ is also injective. Thus, $\overline{f}$ is indeed an isomorphism. ◻

Theorem 1.4 (2nd (Diamond) Isomorphism Theorem)**.** Let P and N be submodules of M. Then,

$$\frac{N}{N \cap P} \cong \frac{N + P}{P}.$$

Proof. Let $i : N \cap P \to N$ and $j : N \to N + P$ be the canonical inclusions, and let $\varphi : N + P \to (N + P)/P$ and $\pi : N \to N/(N \cap P)$ be the canonical projections. Observe that $\operatorname{im}(i) = N \cap P \subset P = \ker(\varphi \circ j)$, so $(\varphi \circ j) \circ i = 0$. Thus, by the Universal Property of the Quotient, there is a unique map $\psi : N/(N \cap P) \to (N + P)/P$ so that the following diagram commutes:

$$\begin{tikzcd}
 & N \cap P \arrow[d, "i"] & \\
 & N \arrow[dl, "\pi"'] \arrow[dr, "j"] & \\
N/(N \cap P) \arrow[dr, dashed, "\psi"'] & & N + P \arrow[dl, "\varphi"] \\
 & (N + P)/P &
\end{tikzcd}$$

We first note that if $(n + p) + P \in (N + P)/P$ (where $n \in N$ and $p \in P$), then

$$(n + p) + P = (n + P) + (p + P) = (n + P) + (0 + P) = n + P = (\varphi \circ j)(p),$$

i.e. $\varphi \circ j$ is surjective. Thus, so is ψ, since $\psi \circ \pi = \varphi \circ j$.

Moreover, we observe that if $n \in N$ and

$$0 = \psi(n + N \cap P) = (\psi \circ \pi)(n) = (j \circ \varphi)(n) = n + P,$$

then $n \in P$, too, i.e. $n \in N \cap P$ and thus, $\ker(\psi) = 0$. Therefore, ψ is injective and indeed an isomorphism. $\square$

Theorem 1.5 (3rd Isomorphism Theorem)**.** Let $P \subset N \subset M$ be submodules of M. Then,

$$\frac{M}{N} \cong \frac{M/P}{N/P}.$$

Proof. Let $j : N \to M$ be the canonical inclusion, and let $\pi : M \to M/N$, $\varphi : M \to M/P$, and $\psi : (M/P) \to (M/P)/(N/P)$ be the canonical projections. Then, observe that $\operatorname{im}(\varphi \circ j) = N/P = \ker(\psi)$, i.e. $\psi \circ \varphi \circ j = 0$. So, by the Universal Property of the Quotient, we obtain a unique map $\lambda : M/N \to (M/P)/(N/P)$ such that

$$\begin{tikzcd}
 & N \arrow[d, "j"] & \\
 & M \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\
M/N \arrow[dr, dashed, "\lambda"'] & & M/P \arrow[dl, "\psi"] \\
 & (M/P)/(N/P) &
\end{tikzcd}$$

commutes.

Obviously, $\psi \circ \varphi$ is surjective, so λ must be as well. Moreover, if

$$0 = \lambda(m + N) = (\lambda \circ \pi)(m) = (\psi \circ \varphi)(m),$$

then $m \in N$, so $\ker(\lambda) = 0$ and λ is injective. Thus, λ is indeed an isomorphism. $\square$