Rings and Modules Ethan Zhang

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1 Modules

1.1 Generalities

Remark. Conventions:

- All rings are unital.
- All modules are left-modules.
- All homomorphisms are between modules under the same ring.
- I will *not* be very careful about saying map and homomorphism. If a map should obviously be a homomorphism, then it is one.

Definition 1.1 (Homomorphism). Let R be a ring and M, M' be R-modules. A map $f: M \to M'$ is a homomorphism if

$$f(x+y) = f(x) + f(y)$$
 and $f(xy) = f(x)f(y)$.

We denote the set of homomorphism from M to M' by Hom(M, M').

f is an epimorphism if it is surjective and a monomorphism if it is injective.

f is an isomorphism if there exists a homomorphism $f^{-1}: M' \to M$ so that $f \circ f^{-1} = \mathrm{id}_{M'}$ and $f^{-1} \circ f = \mathrm{id}_{M}$. Equivalently, f is an isomorphism if it is both monic and epic.

If M = M', then f is called an *endomorphism*. Moreover, if f is also an isomorphism, then f is an automorphism.

Definition 1.2 (Kernel & Image). Let $f: M \to N$ be a homomorphism. Then, we define the kernel of f to be the set

$$\ker(f) := \{ x \in M \mid f(x) = 0 \}$$

and the $image \ of \ f$ to be the set

$$\operatorname{im}(f) := \{ f(x) \mid x \in M \}.$$

Observe that ker(f) is a submodule of M and im(f) is a submodule of N.

Definition 1.3 (Quotient/Factor Modules). Let M be an R-module and $N \subset M$ a submodule. Then, M/N is the set of all additive cosets $\overline{x} = x + N$, and is an R-module with the operations

$$\overline{x} + \overline{y} \coloneqq \overline{x + y}$$
 and $\overline{x} \cdot \overline{y} \coloneqq \overline{xy}$.

We remark that these operations are well-defined, and leave the checking of such to the reader.

Definition 1.4 (Cokernel & Coimage). If $f: M \to N$ is a homomorphism, we define its *cokernel* and *coimage* by

$$\operatorname{coker}(f) := N/\operatorname{im}(f)$$
 and $\operatorname{coim}(f) := M/\ker(f)$

In particular, observe that if N is a submodule of M and $j: N \to M$ is the canonical inclusion, then

$$\operatorname{coker}(f) = M/j(N) = M/N.$$

Theorem 1.1 (Universal Property of the Cokernel). Let $f: N \to M$ be a homomorphism and $\pi: M \to \operatorname{coker}(f) = M/\operatorname{im}(f)$ be the canonical projection. Then,

- 1. $\pi \circ f = 0$.
- 2. If $\varphi: M \to P$ is another homomorphism satisfying $\varphi \circ f = 0$, then there is a unique homomorphism $\psi: \operatorname{coker}(f) \to P$ so that $\psi \circ \pi = \varphi$, i.e. the following diagram commutes:

$$N \xrightarrow{f} M \xrightarrow{\pi} \operatorname{coker}(f) = M/\operatorname{im}(f)$$

$$\downarrow^{\psi}$$

$$P$$

3. And moreover, if φ and P satisfy properties (1) and (2) in the same way as π and $\operatorname{coker}(f)$, then ψ is an isomorphism.

Proof. 1. Let $x \in N$. Then,

$$(\pi \circ f)(x) = \pi(f(x)) = f(x) + im(f) = 0 + im(f),$$

since $f(x) \in \text{im}(f)$.

2. We define the map

$$\psi : \operatorname{coker}(f) \to P, \qquad \psi(x + \operatorname{im}(f)) = \varphi(x).$$

- ψ is well-defined. Let $x + \operatorname{im}(f) = x' + \operatorname{im}(f)$. Then, x = x' + i for some $i \in \operatorname{im}(f)$, so $\psi(x + \operatorname{im}(f)) = \varphi(x) = \varphi(x' + i) = \varphi(x') + \varphi(i) = \varphi(x') = \psi(x' + \operatorname{im}(f))$, since $\varphi \circ f = 0$.
- ψ is a homomorphism. Let $x + \operatorname{im}(f), y + \operatorname{im}(f) \in \operatorname{coker}(f)$. Then,

$$\psi((x + \operatorname{im}(f)) + (y + \operatorname{im}(f))) = \psi((x + y) + \operatorname{im}(f))$$

$$= \varphi(x + y)$$

$$= \varphi(x) + \varphi(y)$$

$$= \psi(x + \operatorname{im}(f)) + \psi(y + \operatorname{im}(f))$$

and

i.e. $\psi' = \psi$.

$$\begin{split} \psi((x+\operatorname{im}(f))(y+\operatorname{im}(f))) &= \psi(xy+\operatorname{im}(f)) \\ &= \varphi(xy) \\ &= \varphi(x)\varphi(y) \\ &= \psi(x+\operatorname{im}(f))\psi(y+\operatorname{im}(f)). \end{split}$$

• ψ is unique. Suppose that ψ' : $\operatorname{coker}(f) \to P$ is another map that makes the diagram commute. Then, for every $x + \operatorname{im}(f) \in \operatorname{coker}(f)$, we have

$$\psi'(x+\operatorname{im}(f)) = (\psi' \circ \pi)(x) = \varphi(x) = (\psi \circ \pi)(x) = \psi(x+\operatorname{im}(f)),$$

3. By hypothesis, $\varphi \circ f = 0$, so we obtain by (2) a unique homomorphism $\psi : \operatorname{coker}(f) \to P$ such that $\varphi = \psi \circ \pi$. Similarly, we assumed that φ and P satisfy property (2) in the same way as π and $\operatorname{coker}(f)$, so we also get a unique homomorphism $\psi' : P \to \operatorname{coker}(f)$ so that $\pi = \psi' \circ \varphi$. Then, $\pi = \psi' \circ \psi \circ \pi$ and $\varphi = \psi \circ \psi' \circ \varphi$. By applying (2) with π and $\operatorname{coker}(f)$, we get $\psi' \circ \psi = \operatorname{id}_{\operatorname{coker}(f)}$, and by applying the hypothesis to φ and P, we get $\psi \circ \psi' = \operatorname{id}_P$. Thus, ψ is indeed an isomorphism, with inverse ψ' .

In the particular case where N is a submodule of M, the Universal Property of the Cokernel gives rise to the Universal Property of the Quotient:

Theorem 1.2 (Universal Property of the Quotient). Let N be a submodule of M and $\pi: N \to M/N$ the canonical projection. Then, given any homomorphism $f: M \to L$ with $f \circ j = 0$, we obtain a unique homomorphism $\overline{f}: M/N \to L$ so that $f = \overline{f} \circ \pi$.

Proof. Let $j: N \to M$ be the canonical inclusion and observe that $M/N = \operatorname{coker}(j)$. Thus, by the Universal Property, there is a unique map $\overline{f}: M/N \to L$ so that

$$\begin{array}{ccc}
N & \xrightarrow{j} & M & \xrightarrow{\pi} & M/N \\
\downarrow & & \downarrow & \downarrow \\
f & & \downarrow & \downarrow \\
I & \downarrow &$$

commutes.

Thus, we obtain the classic isomorphism theorems:

Theorem 1.3 (1st Isomorphism Theorem). Let $f: N \to M$ be a homomorphism of R-modules. Then, there is a unique isomorphism $\psi: N/\ker(f) = \mathrm{coim}(f) \to \mathrm{im}(f)$.

Proof. Let $f': N \to \operatorname{im}(f)$ be given by f'(x) := f(x).

Let $j : \ker(f) \to N$ be the canonical inclusion and $\pi : N \to N/\ker(f)$. Note that $f' \circ j = f \circ j = 0$, by definition. Hence, by the Universal Property of the Quotient, we obtain a unique map $\overline{f} : N/\ker(f) \to \operatorname{im}(f)$ so that

$$\ker(f) \xrightarrow{j} N \xrightarrow{\pi} M/N$$

$$f' \downarrow f$$

$$\operatorname{im}(f)$$

commutes.

Since f' is surjective, so is \overline{f} . Moreover, the kernel of \overline{f} is

$$\ker(\overline{f}) = \{ x + \ker(f) \mid \overline{f}(x + \ker(f)) = (\overline{f} \circ \pi)(x) = f(x) = 0 \}$$

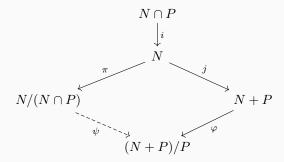
$$= \{ x + \ker(f) \mid x \in \ker(f) \} = \{ 0 + \ker(f) \},$$

so \overline{f} is also injective. Thus, \overline{f} is indeed an isomorphism.

Theorem 1.4 (2nd (Diamond) Isomorphism Theorem). Let P and N be submodules of M. Then,

$$\frac{N}{N \cap P} \cong \frac{N+P}{P}.$$

Proof. Let $i: N \cap P \to N$ and $j: N \to N + P$ be the canonical inclusions, and let $\varphi: N + P \to (N+P)/P$ and $\pi: N \to N/(N \cap P)$ be the canonical projections. Observe that $\operatorname{im}(i) = N \cap P \subset P = \ker(\varphi \circ j)$, so $(\varphi \circ j) \circ i = 0$. Thus, by the Universal Property of the Quotient, there is a unique map $\psi: N/(N \cap P) \to (N+P)/P$ so that the following diagram commutes:



We first note that if $(n+p)+P\in (N+P)/P$ (where $n\in N$ and $p\in P$), then

$$(n+p) + P = (n+P) + (p+P) = (n+P) + (0+P) = n + P = (\varphi \circ j)(p),$$

i.e. $\varphi \circ j$ is surjective. Thus, so is ψ , since $\psi \circ \pi = \varphi \circ j$.

Moreover, we observe that if $n \in N$ and

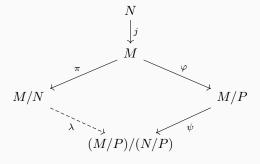
$$0 = \psi(n + N \cap P) = (\psi \circ \pi)(n) = (j \circ \varphi)(n) = n + P,$$

then $n \in P$, too, i.e. $n \in N \cap P$ and thus, $\ker(\psi) = 0$. Therefore, ψ is injective and indeed an isomorphism.

Theorem 1.5 (3rd Isomorphism Theorem). Let $P \subset N \subset M$ be submodules of M. Then,

$$\frac{M}{N} \cong \frac{M/P}{N/P}.$$

Proof. Let $j:N\to M$ be the canonical inclusion, and let $\pi:M\to M/N$, $\varphi:M\to M/P$, and $\psi:(M/P)\to (M/P)/(N/P)$ be the canonical projections. Then, observe that $\operatorname{im}(\varphi\circ j)=N/P=\ker(\psi)$, i.e. $\psi\circ\varphi\circ j=0$. So, by the Universal Property of the Quotient, we obtain a unique map $\lambda:M/N\to (M/P)/(N/P)$ such that



commutes.

Obviously, $\psi\circ\varphi$ is surjective, so λ must be as well. Moreover, if

$$0 = \lambda(m+N) = (\lambda \circ \pi)(m) = (\psi \circ \varphi)(m),$$

then $m \in N$, so $\ker(\lambda) = 0$ and λ is injective. Thus, λ is indeed an isomorphism.