EE 102: Signal Processing and Linear Systems

Homework #10: Filter Design

Name: ______________ **Submission Date:** ______________

Problem 1 An underdamped series RLC circuit is excited by an input at time t_0. The circuit outputs a voltage that is a damped oscillation at its natural frequency. This output signal is modeled as

$$x(t) = A\,e^{-\alpha\,(t-t_0)}\cos\big(\omega_0\,(t-t_0)\big)\,u(t-t_0), \qquad \alpha > 0,\ \omega_0 > 0,$$

where $u(\cdot)$ is the unit step.

(a) (3 points) Write two time-domain properties of $x(t)$ that you can observe directly from the signal equation or by graphing it. You may graph the signal using any software (Python, MATLAB, Desmos) or by hand.

(b) (2 points) Write a signal $\bar{x}(t)$ that when shifted by t_0 gives $x(t)$.

(c) (10 points) Compute the Fourier transform $X(\omega)$ of $x(t)$.

Hint: Use Euler's formula to express the cosine as a sum of complex exponentials — don't solve integration by parts!

(d) (5 points) Compute the Fourier transform $\bar{X}(\omega)$ of $\bar{x}(t)$. Using the time-shift property, prove that $X(\omega) = e^{-j\omega t_0}\bar{X}(\omega)$ to verify your result.

(e) (5 points) Define an ideal low-pass filter as follows:

$$H_{\mathrm{LP}}(\omega) = \begin{cases} 1, & |\omega| \le \Omega_c, \\ 0, & |\omega| > \Omega_c. \end{cases}$$

Sketch $H_{\mathrm{LP}}(\omega)$ for positive and negative ω. Label $\pm\Omega_c$ on your graph.

(f) (5 points) By using the inverse Fourier transform, find the impulse response of the low-pass filter.

Hint: use even symmetry of $H_{\mathrm{LP}}(\omega)$. Solved example 4.5 in Oppenheim & Willsky 2nd Edition has the answer. It is OK to just use the result and approach from the book.

(g) (10 points) We know that if we 'apply' the filter system to the input. That is, if there is a system with impulse response $h_{\mathrm{LP}}(t)$, and we apply an input $x(t)$ to this system to obtain an output $y(t)$. Using the convolution theorem, we know that in the frequency domain, we have that $Y(\omega) = X(\omega)\, H_{\mathrm{LP}}(\omega)$. Compute $Y(\omega)$ and discuss the properties of $Y(\omega)$.

(h) (10 points) Now, let's do the same in time-domain. Write the convolution integral

$$y(t) = (x * h_{\mathrm{LP}})(t) = \int_{-\infty}^{\infty} x(\tau)\, h_{\mathrm{LP}}(t-\tau)\, d\tau.$$

Solve the convolution integral to compute $y(t)$ in time-domain. Compute the inverse Fourier transform of $Y(\omega)$ to obtain $y(t)$ and verify your result.

Important: You will not be able to solve the convolution integral in closed-form. Instead, express your answer as an integral expression that can be evaluated numerically. Things can be quite difficult in time-domain!

(i) (15 points) Using Parseval's theorem, we can define a new type of energy in the frequency domain that is band-limited to $|\omega| \leq \Omega_c$:

$$E_{\mathrm{LP}} = \frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} |X(\omega)|^2\, d\omega,$$

this is the energy contained in the signal $x(t)$ that lies within the passband of the low-pass filter.

Compute E_{LP} in terms of $\alpha, \omega_0, \Omega_c$.

Hint: use $\left|\alpha + j(\omega-\omega_0)\right|^{-2} = \frac{1}{\alpha^2+(\omega-\omega_0)^2}$ and $\int \frac{d\omega}{a^2+(\omega-b)^2} = \frac{1}{a}\arctan\frac{\omega-b}{a}$. You may find it useful to ignore cross terms if $\alpha \ll \omega_0$. How do you justify this approximation physically?

(j) (10 points) Explore other possible filters in frequency domain: low-pass, band-pass, and high-pass. Comment on which one might better isolate the ring-down mode in the RLC circuit. Refer to sketches, $h(t)$, and band-energy calculations. You don't need to solve it out completely but make convincing arguments using Fourier analysis properties.

Problem 2 Validate Problem 1 using computer programs. It is **highly recommended that you start early with the homework so that you can get help during TA's Friday office hour**.

With a sampling rate F_s Hz, we have a time interval of $\Delta t = 1/F_s$ and we can write the continuous-time signal above by substituting $t = n\Delta t$ as a discrete-time signal:

$$x[n] = A\, e^{-\alpha(n-n_0)\Delta t} \cos\big(\omega_0(n-n_0)\Delta t\big)\, u[n-n_0].$$

(a) (5 points) Choose parameters $(A, \alpha, \omega_0, t_0)$ and F_s. Implement the equation above in a function. Then, plot $x[n]$ versus $n\Delta t$. Label your graph. Make sure to choose F_s to be greater than $2 \cdot \frac{\omega_0}{2\pi}$. You will learn why this is important next week.

For a simple start, set $F_s = 2000$ Hz and ensure that the condition above is satisfied.

(b) (10 points) Re-use the previous homework assignment code for DFT implementation to compute the DFT with length N for the signal above (it is also OK to use FFT libraries).

Helpful starting point:

Set $N = 2048$ and $L = 8192$. If your generated $x[n]$ has length $< N$, zero-pad it to length N. If it is longer, keep the first N samples. Then zero-pad from N to L before taking the DFT. Remember that with DFT, the frequency is sampled at discrete frequencies. Use the following frequency grid:

$$\omega_k = \frac{2\pi k F_s}{L}, \quad k = 0, 1, \ldots, L-1.$$

The index corresponding to ω_0 will be

$$k_0 = \text{round}\left(\frac{\omega_0\, L}{2\pi F_s}\right).$$

Plot $|X[k]|$ versus discrete frequency. Indicate indices corresponding to $\pm\omega_0$. You may need to center the spectrum for better visualization (using `fftshift`).

(c) (10 points) Implement the ideal low-pass filter $H[k]$. Compute $y[n]$ using inverse DFT. Plot the filtered signal $y[n]$.

The discrete-time DFT mask is

$$H[k] = \begin{cases} 1, & |\omega_k| \le \Omega_c, \\ 0, & \text{otherwise.} \end{cases}$$

You may start by setting $\Omega_c = 0.8\,\omega_0$.

Compute $Y[k] = X[k] \cdot H[k]$. Compute $y[n]$ by the L-point inverse DFT. Plot $\mathrm{Re}\{y[n]\}$ versus $n\Delta t$.

Optional exploration (not graded):

(d) Implement at least one other filter type (high-pass or band-pass, or something else of your choice). Provide the DFT mask, impulse response, and plot the filtered signal. Comment on how well the filter isolates the ring-down mode in the RLC circuit.

(e) Estimate band-limited input energy by summing $|X[k]|^2$ over passband indices to verify your answer in Problem 1. Find out how well your filter isolates the ringing mode for your chosen parameters. Experiment with different parameters.