EE 102 Week 7, Lecture 1 (Fall 2025)

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1 Goals

By the end of this lecture, you should be able to:

- Understand that Fourier series is the optimal approximator of periodic signals. That is, it minimizes the error energy between the actual signal and the approximated signal.
- Apply Parseval's theorem and Fourier series property to analyze a real-world engineering system.

2 Review

Recall the Fourier Series (FS) synthesis equation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where we sum infinite number of terms to get the signal back *exactly*. But what if we only use a finite number of terms? We can define a truncated sum:

$$x_{\mathrm{FS},N}(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

where N is a positive integer. This is an approximation of the original signal $x(t)$. We can define an error signal

$$e_N(t) = x(t) - x_{\mathrm{FS},N}(t)$$

and the error energy over a period as

$$E_N = \int_{t_0}^{t_0+T} |e_N(t)|^2 dt$$

where T is the period of the signal.

The key idea is that FS gives us an approximation of the signal that minimizes this error energy E_N for a given N. This means that among all possible ways to approximate $x(t)$ using $2N+1$ terms, the FS approximation $x_{\text{FS},N}(t)$ is the best in terms of minimizing the error energy. Further, in the limit of $N \to \infty$, the error energy E_N approaches zero, and the approximation becomes exact.

3 Intution for sums of sinusoids

We have noted many times that the FS gives us a way to write any periodic signal as a sum of sinusoids. But how would this be possible for signals with discontinuities? This is unintuitive because sinusoids are smooth, oscillating functions. So, how can they combine together to form a signal that is dicontinuous, like an impulse train, for example?

To build intuition for this fact, recall that Fourier series for real signals is a sum of cosines. We start by writing the exponential FS:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where a_k are the complex Fourier coefficients. For real signals, we can use the property that $a_{-k} = a_k^*$ to rewrite this by breaking the sum into positive and negative k terms, and the DC term for $k=0$:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{-1} a_k e^{jk\omega_0 t}$$

Now, apply a change of variable $m = -k$ in the last sum:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k e^{jk\omega_0 t} + \sum_{m=1}^{\infty} a_{-m} e^{-jm\omega_0 t}$$

We can combine the two sums now as they both have the same limits. In combining the sums, we also recall the property that for real signals $a_{-k} = a_k^*$:

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t} \right)$$

Finally, we can use Euler's formula to rewrite the terms in the sum as cosines.

Case 1: Real and Even Let's consider a simple case. For real and even signals, the Fourier coefficients a_k are real and even. This means that $a_k = a_k^*$ and $a_{-k} = a_k$. Therefore, we can write:

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2a_k \cos(k\omega_0 t)$$

This is a sum of cosines with amplitudes $2a_k$!

Case 2: Complex a_k For general real signals, the Fourier coefficients a_k are complex. We can express a_k in polar form as $a_k = |a_k|e^{j\phi_k}$, where $|a_k|$ is the magnitude and ϕ_k is the phase. Then, we can write:

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2|a_k| \cos(k\omega_0 t + \phi_k)$$

This is a sum of cosines with amplitudes $2|a_k|$ and phase shifts ϕ_k.

Pop Quiz 3.1: Check your understanding!

On Desmos Graphing Calculator, explore how sums of cosines can approximate discontinuities. Create a visual graph that "looks" like a train of impulses.

Solution on page 6

3.1 Virtual manipulator: Gibbs Phenomena

For a train of impulses (HW #6 Problem 1), the Fourier coefficients are all equal to $1/T$. Therefore, the FS representation is:

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{jk\omega_0 t}$$

Explore the virtual manipulative on Fourier analysis of impulses by running `streamlit run VM_impulse_fourier_analysis.py` on your computer. Then, attempt to show the following by changing the knobs on the simulation:

- With two sinusoids being displayed (the last slider), draw on your notebook the points where constructive and destructive interference happens.
- Increase the number of harmonics (that is, how many high-frequency sinusoids are being added) iteratively and observe the behavior of the summed-up sinusoids near the discontinuities. What do you observe? This is called the Gibbs Phenomenon.

- What happens to the error energy as you increase the number of harmonics?
- Recall that an impulse is a signal with infinite height, zero width, and unit area. Can you explain how the sum of sinusoids is able to approximate such a signal?

4 Parseval's Theorem

As briefly discussed earlier, Fourier series gives us the optimal approximation of a periodic signal in terms of minimizing the error energy. This is formalized by Parseval's theorem, which states that the total energy of a periodic signal over one period is equal to the sum of the squares of its Fourier coefficients multiplied by the period. To prove this relation mathematically, we start with the FS synthesis equation and compute the signal energy over one period:

$$E = \int_{t_0}^{t_0+T} |x(t)|^2 dt$$

Substituting the FS synthesis equation into this integral, we have:

$$E = \int_{t_0}^{t_0+T} \left| \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right|^2 dt$$

Expanding the square using the definition of magnitude squared of a complex number z as zz^*, we get:

$$E = \int_{t_0}^{t_0+T} \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right) \left(\sum_{m=-\infty}^{\infty} a_m^* e^{-jm\omega_0 t} \right) dt$$

Bring the sums outside the integral:

$$E = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_k a_m^* \int_{t_0}^{t_0+T} e^{j(k-m)\omega_0 t} dt$$

The integral evaluates to T when $k = m$ and 0 otherwise, due to the orthogonality of the complex exponentials (the area under curve cancels out for sinusoids over one period). Therefore, we have:

$$E = \sum_{k=-\infty}^{\infty} |a_k|^2 T$$

Dividing both sides by T, we arrive at Parseval's theorem:

$$\frac{1}{T} \int_{t_0}^{t_0+T} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Interpretations:

Energy conservation: The total energy of the signal over one period is equal to the sum of the energies of its frequency components. Also, the energy in the time domain is equal to the energy in the frequency domain.

Ease of computation of energy: Parseval's theorem provides a convenient way to compute the energy of a signal in the frequency domain, which can be easier than computing it directly in the time domain (in some cases).

5 Recommended Practice Problems

- Drill 3.10 (on rectifier) in Lathi. This problem is very similar to your HW #7 problem (not identical).
- Example 3.14 in Lathi. Note that this is an advanced problem so you may want to read through it first.
- Table 3.1 in Oppenheim and Willsky and Table 3.1 in Lathi are both handy tables to screenshot and keep close!

Pop Quiz Solutions

Pop Quiz 3.1: Solution(s)

Keep on adding cosine sums with higher frequencies and observe how the approximation improves.