EE 102 Week 8, Lecture 1 (Fall 2025)

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1 Goals

The overall goal in EE 102 is to break down signals into its constituent fundamental components that can be easily analyzed. So far, we have achieved this in two ways: we have broken down any arbitrary signal into impulse signals, and we have broken down periodic signals into sinusoids. What's left? Signals that are not periodic. So, our goal is ...

By the end of this lecture, you should be able to break down aperiodic signals into their constituent frequency components using Fourier Transform.

2 Review: Decomposition of signals

In this section, we review how to decompose signals into simpler components. We have already seen two ways to do this: using impulses and using sinusoids. We will work through one example of each.

2.1 Breaking down signals using impulses

Choose a signal $x(t)$ — scribble something! Break it down so that it can be constructed using only impulses $\delta(t)$ with appropriate scales and shifts.

(a) Show it visually: On the left, sketch $x(t)$. On the right, place weighted impulses at sample locations corresponding to the value of $x(t)$ at those locations. Note that integer times are shown here, but you may choose any sampling interval (can be floating point because this is a continuous-time signal).

Sketch $x(t)$ here

$x(t)$

t

Put weights $x(n)$ beside $\delta(t-n)$

impulses

-4 -3 -2 -1 0 1 2 3 4 t
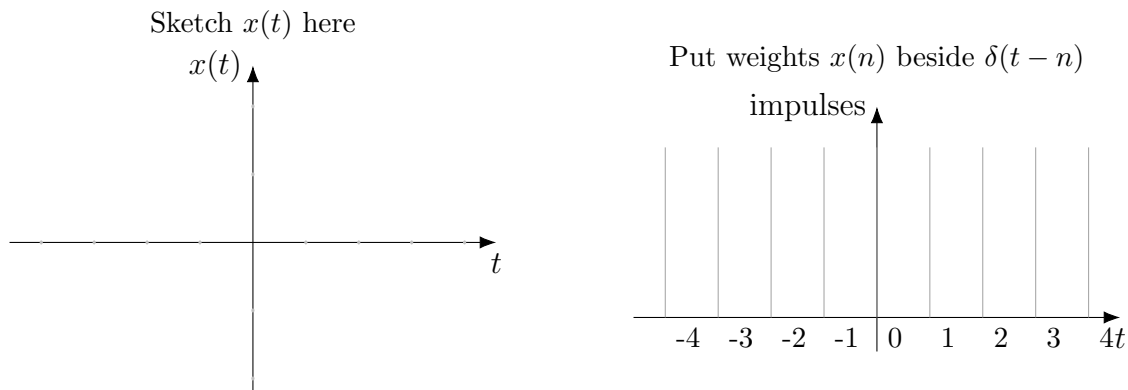
(b) Do it mathematically: Use the sifting property to write a formula for $x(t)$ using $\delta(\cdot)$.

Write your mathematical expression here:

Possible solution We sketch $x(t)$ as a continuous curve on the left, and place impulses at various locations, which is shown on the right.

$x(t)$

t

t
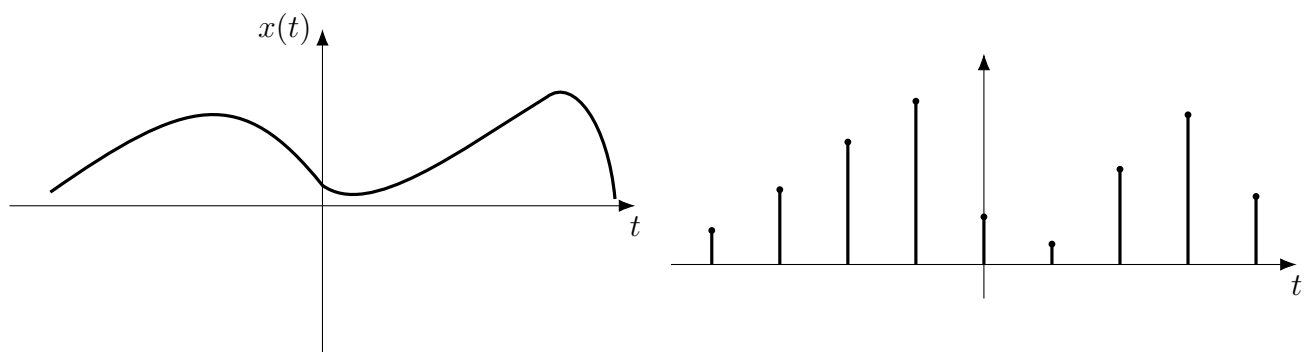
Mathematically, we have the sifting property of the impulse:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

which can be approximated using a sum over discrete time instants as

$$x(t) \approx \sum_{n=-\infty}^{\infty} x(n)\,\delta(t-n)\,.$$

2

2.2 Breaking down periodic signals using sinusoids

Draw an impulse train with period T_0. Then, sketch (approximately) how a finite sum of sinusoids could approximate this train over time. Finally, write the corresponding Fourier-series form and coefficients.

Write $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ and specify a_k. Optionally write the finite N-term partial sum you would draw.

Finally, consider a 50% duty cycle symmetric square wave of amplitude A and period $T_0 = \dfrac{2\pi}{\omega_0}$. Sketch one period on an axes, draw the first few nonzero harmonic magnitudes, and write the Fourier series formula you would use to represent this signal.

Write the FS synthesis and the coefficients for a 50% duty square wave of amplitude A.

Why/how is that useful?   After you have worked through the above examples, you are all set up to compute the output of any LTI system. To achieve that you would need the following additional information (one or the other):

- How does the system respond to an impulse input? (i.e., the impulse response of the system, $h(t)$)

- How does the system respond to sinusoidal inputs? (i.e., the frequency response of the system, $H(f)$)

Then, you can compute the output of the system for any arbitrary input signal $x(t)$ using either convolution or the Fourier synthesis equation (and LTI system properties!).

3 Fourier Transform: Decomposition of aperiodic signals

The **key idea** in this lecture is that aperiodic signals are also periodic signals, but with the time period $T \to \infty$. That is, the aperiodic signal also repeats itself, but only after an infinite amount of time has passed! This essentially means that it never repeats itself (inifinite is not defined).

So, let's try to apply that idea with an example. Consider a square wave of magnitude A, centered around 0, with a period of T. For the period between $-T/2$ and $T/2$, the signal is defined as:

$$x(t) = \begin{cases} A, & -T_1 \leq t \leq T_1, \\ 0, & \text{otherwise.} \end{cases}$$

We can find the Fourier series coefficients for this signal using the Fourier series analysis equation:

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt,$$

where $\omega_0 = \dfrac{2\pi}{T}$. Note that the limits of integration should be (ideally, according to the formula for a_k) over one period of the signal. However, since $x(t)$ is zero outside the interval $[-T_1, T_1]$, we can simplify the limits of integration to:

$$a_k = \frac{1}{T} \int_{-T_1}^{T_1} A e^{-jk\omega_0 t} dt.$$

Evaluating this integral, we get:

$$\begin{aligned} a_k &= \frac{A}{T} \int_{-T_1}^{T_1} e^{-jk\omega_0 t} dt \\ &= \frac{A}{T} \left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0} \right]_{-T_1}^{T_1} \\ &= \frac{A}{T} \left(\frac{e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}}{-jk\omega_0} \right) \\ &= \frac{A}{T} \left(\frac{-2j \sin(k\omega_0 T_1)}{-jk\omega_0} \right) \\ &= \frac{2A}{T} \cdot \frac{\sin(k\omega_0 T_1)}{k\omega_0}. \end{aligned}$$

For convenience, we define a new function called the **sinc function** as (pronounced "sink"):

$$\text{sinc}(x) = \frac{\sin(x)}{x}.$$

We can now rewrite the Fourier series coefficients as:

$$a_k = \frac{2AT_1}{T}\text{sinc}(k\omega_0 T_1).$$

Now, since $T \to \infty$, let's re-write the left-hand side as

$$Ta_k = 2AT_1\text{sinc}(k\omega_0 T_1).$$

Notice that as $T \to \infty$, the fundamental frequency $\omega_0 = \frac{2\pi}{T} \to 0$. The right hand side expression above is a function of frequency: $2AT_1\text{sinc}(k\omega_0 T_1)$. Since $\omega_0 \to 0$, we make a very important observation. The function of frequency can be defined as a continuous function in the limit. You can see this with the following re-definition:

$$\omega_0 := \Delta\omega$$

and then $k\omega_0 = k\Delta\omega$, which in turn we define as $\omega := \omega_k := k\Delta\omega$.

Intuitively, if you think of the frequency as the X-axis, then the Fourier series coefficients can be thought of as samples of a continuous function in the limit as $T \to \infty$. This is the key idea behind the Fourier transform: it allows us to represent aperiodic signals as a continuous sum of sinusoids, each with a specific frequency and amplitude.

For the square wave example above, we then have

$$Ta_k = 2AT_1 sinc(\omega T_1)$$

which can be further interpreted as

$$2\pi a_k = [2AT_1 sinc(\omega T_1)]\,\Delta\omega.$$

This frequency function is called the Fourier transform of the signal. What we have done is that we started from from a continuous-time (in time domain), which we have now *transformed* into a function of frequency (in frequency domain).

4 The Fourier Transform

For a general aperiodic signal $x(t)$, the Fourier transform can be derived using the Fourier series synthesis and analysis equations:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt.$$

As $T \to \infty$, we have:

$$\omega_0 = \frac{2\pi}{T} \to 0. \text{Thus, we can write:}$$

$$Ta_k = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt,$$

where we used the definition of frequency $\omega = k\omega_0$. Now, we can rewrite the synthesis equation as:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau \right] e^{j\omega t} d\omega.$$

We define the above as the Fourier transform pair:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt,$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega.$$

This pair of equations allows us to transform a time-domain signal $x(t)$ into its frequency-domain representation $X(\omega)$ and vice versa! We will work on specific examples in the next class.

5 Recommended reading and practice problems

- Solved example 3.5 in Oppenheim and Willsky, 2nd edition.

- Example 4.1 in Lathi.