EE 102: Signal Processing and Linear Systems

Homework #10: Filter Design

Name: ______________ **Submission Date:** ______________

Problem 1 An underdamped series RLC circuit is excited by an input at time t_0. The circuit outputs a voltage that is a damped oscillation at its natural frequency. This output signal is modeled as

$$x(t) = A\, e^{-\alpha\,(t-t_0)} \cos\bigl(\omega_0\,(t-t_0)\bigr)\, u(t-t_0), \qquad \alpha > 0,\ \omega_0 > 0,$$

where $u(\cdot)$ is the unit step.

(a) **(6 points)** Write two time-domain properties of $x(t)$ that you can observe directly from the signal equation or by graphing it.

(b) **(2 points)** Write a signal $\bar{x}(t)$ that when shifted by t_0 gives $x(t)$.

(c) **(10 points)** Compute the Fourier transform $X(\omega)$ of $x(t)$.

Hint: Use Euler's formula to express the cosine as a sum of complex exponentials — don't solve integration by parts!

(d) **(5 points)** Compute the Fourier transform $\bar{X}(\omega)$ of $\bar{x}(t)$. Using the time-shift property, prove that $X(\omega) = e^{-j\omega t_0}\bar{X}(\omega)$ to verify your result.

(e) **(5 points)** Sketch $|X(\omega)|$. Indicate the spectral peaks (the frequency peaks) at $\pm\omega_0$.

(f) **(6 points)** Define an ideal low-pass filter as follows:

$$H_{\mathrm{LP}}(\omega) = \begin{cases} 1, & |\omega| \le \Omega_c, \\ 0, & |\omega| > \Omega_c. \end{cases}$$

Sketch $H_{\mathrm{LP}}(\omega)$ for positive and negative ω. Label $\pm\Omega_c$ on your graph.

(g) **(8 points)** By using the inverse Fourier transform, compute the impulse response of the low-pass filter

Hint: use even symmetry of $H_{\mathrm{LP}}(\omega)$. Solved example 4.5 in Oppenheim & Willsky 2nd Edition has the answer.

(h) (10 points) We know that if we 'apply' the filter system to the input. That is, if there is a system with impulse response $h_{\text{LP}}(t)$, and we apply an input $x(t)$ to this system to obtain an output $y(t)$. Using the convolution theorem, we know that in the frequency domain, we have that $Y(\omega) = X(\omega)\, H_{\text{LP}}(\omega)$. Compute $Y(\omega)$ and discuss the properties of $Y(\omega)$.

(i) (10 points) Now, let's do the same in time-domain. Write the convolution integral

$$y(t) = (x * h_{\text{LP}})(t) = \int_{-\infty}^{\infty} x(\tau)\, h_{\text{LP}}(t-\tau)\, d\tau.$$

Solve the convolution integral to compute $y(t)$ in time-domain. Compute the inverse Fourier transform of $Y(\omega)$ to obtain $y(t)$ and verify your result.

Important: You will not be able to solve the convolution integral in closed-form. Instead, express your answer as an integral expression that can be evaluated numerically. Things can be quite difficult in time-domain!

(j) (10 points) Using Parseval's theorem, we can define a new type of energy in the frequency domain that is band-limited to $|\omega| \le \Omega_c$:

$$E_{\text{LP}} = \frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} |X(\omega)|^2\, d\omega,$$

this is the energy contained in the signal $x(t)$ that lies within the passband of the low-pass filter.

Compute E_{LP} in terms of $\alpha, \omega_0, \Omega_c$.

Hint: use $\left|\alpha + j(\omega-\omega_0)\right|^{-2} = \frac{1}{\alpha^2+(\omega-\omega_0)^2}$ and $\int \frac{d\omega}{a^2+(\omega-b)^2} = \frac{1}{a}\arctan\frac{\omega-b}{a}$. You may find it useful to ignore cross terms if $\alpha \ll \omega_0$. How do you justify this approximation physically?

(k) (10 points) Explore other possible filters in frequency domain: low-pass, band-pass, and high-pass. Comment on which one might better isolate the ring-down mode in the RLC circuit. Refer to sketches, $h(t)$, and band-energy calculations. You don't need to solve it out completely but make convincing arguments using Fourier analysis properties.

Problem 2 Validate Problem 1 using computer programs. You may start on the coding problem before finishing Problem 1. It is **highly recommended that you start early with the homework so that you can get help during TA's Friday office hour**.

With a sampling rate F_s Hz, we have a time interval of $\Delta t = 1/F_s$ and we can write the continuous-time signal above by substituting $t = n\Delta t$ as a discrete-time signal:

$$x[n] = A\, e^{-\alpha(n-n_0)\Delta t} \cos\big(\omega_0 (n-n_0)\Delta t\big)\, u[n-n_0].$$

(a) **(6 points)** Choose parameters $(A, \alpha, \omega_0, t_0)$ and F_s. Implement the equation above in a function. Then, plot $x[n]$ versus $n\Delta t$. Label your graph.

(b) **(8 points)** Re-use the previous homework assignment code for DFT implementation to compute the DFT with length N for the signal above. Plot $|X[k]|$ versus discrete frequency. Indicate indices corresponding to $\pm\omega_0$. You may need to center the spectrum for better visualization.

(c) **(8 points)** Implement the ideal low-pass filter by masking $|\omega| \leq \Omega_c$ in the DFT domain. Compute $y[n]$ by inverse DFT. Plot the filtered signal $y[n]$ and comment on the filter's operation.

(d) **(10 points)** Implement at least one other filter type (high-pass or band-pass, or something else of your choice). Provide the DFT mask, impulse response, and plot the filtered signal. Comment on how well the filter isolates the ring-down mode in the RLC circuit.

(e) **(10 points)** Estimate band-limited input energy by summing $|X[k]|^2$ over passband indices to verify your answer in Problem 1. Find out how well your filter isolates the ringing mode for your chosen parameters. Experiment with different parameters.