EE102 Week 0, Lecture 1 (Fall 2025)

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1 Goals

- Logistics, grading, extensions, expectations
- Motivation to study signal processing
- Pre-requisites to signal processing: vectors and complex numbers

2 Why study signal processing?

Signal processing and linear systems theory is foundational in engineering. It has revolutionized engineering in more ways than we realize — machine learning/AI, RF amplifiers, satellite communications, airplanes, medical devices, automotive vehicles, MRI scans, and pretty much every other engineering and science discipline out there *directly* uses concepts from this course. This is fantastic but there is also a downside! Learning signal processing is dependent on many prerequisites as it builds on various other fundamental courses in engineering. As electrical engineers, you are required to learn signal processing and linear systems theory. Other engineering disciplines typically do not require such a course. This gives electrical engineers an edge because what you learn in this course is not just applicable to EE but also to all other areas. So, despite many pre-requisite requirements, I hope that you will be motivated to cross the technical barriers in learning signal processing.

2.1 Real-world significance

Count the number of questions you answer "yes" to from the list below:

- Do you enjoy music? Did you ever find songs that sounded very similar to each other? Would you like to be able to explain why that's the case (mathematically)?
- Would you like design your own electrical circuits that are able to meet performance specifications of your (future) clients?

- Do you anticipate that you will be working on radio frequency circuits in your career where you need to design circuits and systems that communicate at specific frequencies?
- Are you attracted to the rigor of electrical engineering? Or perhaps, put another way, are you looking forward to setting aside time to learn the mathematical underpinnings of electrical engineering?
- Would you like to gain a better understanding of how various "scanners" scan our body parts to provide useful medical information (like X-rays, MRI, CT scans, etc.)?
- Do you want to be able to explain to others how images and colors on any digital display are created and manipulated?
- Would you like to *mathematically* create new music that takes the best parts of some of the songs that you like? Or perhaps, create entirely new sounds that have never been heard before? By doing it mathematically, you will be making the process general, easily customizable, and reproducible.
- Do you anticipate that your career choice after graduation will involve image processing / machine learning / artificial intelligence?
- Are you interested in understanding the underpinnings of the controllers that are used in automotive vehicles, or robotics, or even in the design of precise drugs that target pathogens in the human body?
- Noise canceling in audio tech is a huge industry! Are you someone who fancies designing / understanding these systems?
- Do you want to understand how Shazam (or other apps that can recognize a song just based on a few beats) work?

If you counted more than a few "yes" answers, you are in the right place! This course will help you understand the mathematical underpinnings of many of these applications. Of course, this course will not go into the technical specifics of any of the applications. There won't be enough time for it. Other courses exist for such details. See below for what this course is not.

2.2 What signal processing is not?

In signal processing, you will **not** learn the fundamentals of circuit analysis, AC analysis, transistors, communication algorithms, design of RF circuits, or controller design. Most of those topics are already assigned to other specific courses. In signal processing and linear systems course, we will focus on the mathematical tools and techniques used to analyze and process signals and systems.

With that motivation, let us jump into a discussion about various pre-requisites that you will need to be familiar with to succeed in this course.

3 Pre-requisite #1: Vectors

When studying problems with many entities/observations, we structure our variables into vectors.

An n-dimensional vector $\mathbf{x}$ can be written as

$$\mathbf{x} = [\, x_1,\ x_2,\ \ldots,\ x_n \,], \qquad \mathbf{x} \in \mathbb{R}^n.$$

3.1 Matrices are transformations

If you transform a vector $\mathbf{x}$ to a new vector $\mathbf{y}$ such that all elements in $\mathbf{y}$ are linear combinations of elements in $\mathbf{x}$, then the transformation is called a matrix.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

such that

$$y_1 = \sum_{i=1}^{n} \alpha_{1i} x_i, \qquad y_2 = \sum_{i=1}^{n} \alpha_{2i} x_i, \quad \ldots, \quad y_m = \sum_{i=1}^{n} \alpha_{mi} x_i.$$

Then $A\mathbf{x} = \mathbf{y}$, where

$$A \in \mathbb{R}^{m \times n}, \qquad A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{bmatrix}.$$

We write

$$A:\ X \to Y,$$

where X is the vector space in $\mathbb{R}^n$ where $\mathbf{x}$ lies and Y is the vector space in $\mathbb{R}^m$ where $\mathbf{y}$ lies.

Recall

- Diagonal matrix
- Identity matrix
- Symmetric matrix
- Zero matrix
- Matrix transpose
- Matrix algebra ($+$, $-$, $\times$, inverse)

3.2 Real-world significance

Note that a transformation is called an "affine" transformation if it is linear

$$\mathbf{y} = A\mathbf{x} + \mathbf{b},$$

where A is a linear transformation (matrix) and $\mathbf{b}$ is a translation vector. Affine transformations are common in many practical applications such as image processing, computer-aided design in engineering, medical imaging, graphic design, and many more. On a lighter note, check this fun meme template out which uses matrix transformations at its core — the content aware scale gif and some related Reddit discussion on it. Creating memes often requires very specific image transforms (such as the Wide Keanu or the general Stretched Resolution meme)! On a more technical note, you can check out the Adobe Photoshop tool called "Transform" (or the equivalent rotate, scale, and skew tools in Microsoft Paint) — these tools allow users to manipulate images using affine transformations. The same concepts are at the core of many research-grade affine transform tools. Some examples are rasterio for geographical applications, flirt for affine transformations of MRI images and the RandomAffine tool for affine transformations in image augmentation used in machine learning applications.

In summary, vector and matrix algebra is the centerpiece in signal processing and you will see the mathematical preliminaries being used throughout the course.

4 Pre-requisite #2: Complex numbers

Although the usual way we learn about the complex unit "j" is as a convenient notation for a solution of

$$x^2 + 1 = 0 \quad \Rightarrow \quad x = \sqrt{-1} := \mathrm{j},$$

it is useful to recognize other places where this convenience is beneficial. In signal processing we are often looking for easy ways to analyze physical signals, not only to solve algebraic equations.

4.1 From vectors to a complex scalar.

Given a vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

define the (complex) scalar

$$z_{\mathbf{x}} = x_1 + \mathrm{j}x_2.$$

The entries x_1 and x_2 are not "added" in $\mathbb{R}$; they are bound only because they are components of the same vector. Writing $z_{\mathbf{x}}$ lets us treat the vector like a single scalar living in $\mathbb{C}$.

4.2 Inner products and linear dependence

In simple terms, the inner product between two vectors is a scalar quantity that quantifies a relationship between two vectors: how much they align with each other. In quantifying this, the inner product takes into account the lengths of the two vectors and the angle between them. The inner product can be used to define orthogonality (perpendicularity) — which is one the most fundamental concepts in signal processing.

Why? The key idea in EE 102 is that a linear combination of a set of orthogonal signals can be used to represent *any* signal (no matter how complicated), under some conditions, of course. So, understanding orthogonality, linear independence, and linear combinations is key to this course. Consequently, inner product is an important concept for this course.

If two vectors are orthogonal (that is, their inner product is zero), then these vectors are linearly independent. Indeed, we can prove that a set of non-zero mutually orthogonal vectors (say, $v_1, v_2, \ldots, v_n$) are linearly independent. You can show this by writing the linear combination of the vectors: $S = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ and showing that it is zero only if all constants α_i, $i = \{1, \ldots, n\}$ are equal to zero. To prove this linear independence, you can take the inner product of the above with any vector in the set (or any linear combination, thereof) say v_k:

$$S = v_k \cdot (c_1 v_1 + c_2 v_2 + ... + c_n v_n)$$
$$S = v_k \cdot c_1 v_1 + v_k \cdot c_2 v_2 + ... + v_k \cdot c_n v_n = 0$$

since the pairwise dot products (the inner product between each pair of vectors) are zero, we are only left with

$$c_k(v_k \cdot v_k) = 0$$

which is only possible if $c_k = 0$ since $v_k \cdot v_k$ is non-zero. Since this is true for any k, we have that all coefficients are zero. So, inner products play an important role in proving orthogonality (and thus, linear independence of vectors).

4.3 Inner product via complex numbers.

All the vector algebra can be tedious work! Complex numbers come to our rescue as we can represent a 2D vector as a complex number. For $\mathbf{x} = [x_1, x_2]^\top$ and $\mathbf{y} = [y_1, y_2]^\top$,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{y} = x_1 y_1 + x_2 y_2.$$

With the complex representations

$$z_{\mathbf{x}} = x_1 + \mathrm{j}x_2, \qquad z_{\mathbf{y}} = y_1 + \mathrm{j}y_2,$$

their product with conjugation is

$$\begin{aligned}\overline{z}_{\mathbf{x}}\, z_{\mathbf{y}} &= (x_1 - \mathrm{j}x_2)(y_1 + \mathrm{j}y_2) \\ &= (x_1 y_1 + x_2 y_2) \; + \; \mathrm{j}\,(x_1 y_2 - x_2 y_1).\end{aligned}$$

Taking the real part gives the vector inner product:

$$\Re(\overline{z}_{\mathbf{x}}\, z_{\mathbf{y}}) = x_1 y_1 + x_2 y_2 = \langle \mathbf{x}, \mathbf{y} \rangle.$$

Polar form viewpoint. Write $\mathbf{x}$ in polar coordinates with $r_x = \|\mathbf{x}\|$ and angle θ_x:

$$x_1 = r_x \cos\theta_x, \qquad x_2 = r_x \sin\theta_x,$$

so

$$z_{\mathbf{x}} = r_x(\cos\theta_x + \mathrm{j}\sin\theta_x) = r_x e^{\mathrm{j}\theta_x}.$$

Similarly $z_{\mathbf{y}} = r_y e^{\mathrm{j}\theta_y}$ from Euler's identity[1]. Then

$$\begin{aligned}\overline{z}_{\mathbf{x}}\, z_{\mathbf{y}} &= r_x e^{-\mathrm{j}\theta_x}\; r_y e^{\mathrm{j}\theta_y} = r_x r_y e^{\mathrm{j}(-\theta_x + \theta_y)} \\ &= r_x r_y \Big[\cos(-\theta_x + \theta_y) + \mathrm{j}\sin(-\theta_x + \theta_y)\Big],\end{aligned}$$

hence

$$\Re(\overline{z}_{\mathbf{x}}\, z_{\mathbf{y}}) = r_x r_y \cos(-\theta_x + \theta_y) = r_x r_y \cos(\theta_x - \theta_y) = \langle \mathbf{x}, \mathbf{y} \rangle.$$

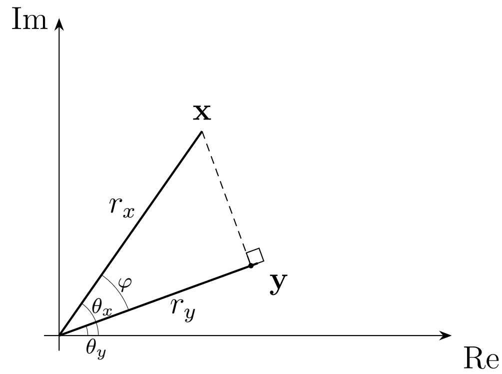


Figure 1: Geometric view of the inner product using polar form

Note that cosine is an even function, which allowed us to write the last equality above.

In Figure 1, observe that vectors $\mathbf{x}$ and $\mathbf{y}$ make angles θ_x and θ_y with the real axis and the angle between them is $\varphi = \theta_x - \theta_y$. You can make intuitive sense of the inner product in polar form by understanding its geometric interpretation (see Figure 1). Specifically, recall how we defined inner products in the previous section — a quantification of the alignment between two vectors. For our example in Figure 1, decompose $\mathbf{x}$ relative to $\mathbf{y}$: drop a perpendicular from the tip of $\mathbf{x}$ to the line span$\{\mathbf{y}\}$ (the direction spanned by $\mathbf{y}$). This splits $\mathbf{x}$ into a part *parallel* to $\mathbf{y}$ and a part *perpendicular* to $\mathbf{y}$:

$$\mathbf{x} = \underbrace{(\mathbf{x} \cdot \hat{\mathbf{y}})\, \hat{\mathbf{y}}}_{\text{projection onto span}\{\mathbf{y}\}\text{, parallel to }\mathbf{y}} + \underbrace{\Big(\mathbf{x} - (\mathbf{x} \cdot \hat{\mathbf{y}})\, \hat{\mathbf{y}}\Big)}_{\text{perpendicular (rejection) to }\mathbf{y}} .$$

The second part that is perpendicular to $\mathbf{y}$ is called the rejection because that's the part that is remaining (you can see that it is quite literally the remaining part as it is obtained by subtracting the projection from $\mathbf{x}$). Note that the inner product of $\mathbf{x}$ with the unit vector $\hat{\mathbf{y}}$ gives us the projection of x onto span$\{\mathbf{y}\}$. By computing the inner product, you can also check that the remaining (perpendicular part) of $\mathbf{x}$ is orthogonal to $\mathbf{y}$:

$$\mathrm{rej}_{\mathbf{y}}(\mathbf{x}) \cdot \hat{\mathbf{y}} = \mathbf{x}\cdot\hat{\mathbf{y}} - (\mathbf{x}\cdot\hat{\mathbf{y}})(\hat{\mathbf{y}}\cdot\hat{\mathbf{y}}) = \mathbf{x}\cdot\hat{\mathbf{y}} - \mathbf{x}\cdot\hat{\mathbf{y}} = 0.$$

Thus

$$\mathbf{x} = \mathrm{proj}_{\mathbf{y}}(\mathbf{x}) + \mathrm{rej}_{\mathbf{y}}(\mathbf{x}), \qquad \|\mathbf{x}\|^2 = \|\,\mathrm{proj}_{\mathbf{y}}(\mathbf{x})\|^2 + \|\,\mathrm{rej}_{\mathbf{y}}(\mathbf{x})\|^2.$$

In summary, the inner product measures how much of $\mathbf{x}$ points along $\mathbf{y}$, scaled by the length of $\mathbf{y}$. This is given by

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{y}\|\, (\mathbf{x} \cdot \hat{\mathbf{y}}) = \|\mathbf{x}\|\, \|\mathbf{y}\| \cos\varphi.$$

[1]You can practice proving Euler's identity that $e^{\mathrm{j}\theta} = \cos\theta + \mathrm{j}\sin\theta$ by expanding the left-hand side using the exponential series and collecting the real and imaginary parts together (the real part will be the cosine series and the imaginary part will be the sine series).

The sign of $\cos\varphi$ carries the orientation: it is positive when the angle is acute and negative when obtuse. Complex numbers in their polar form express the same geometric intuition:

$$\overline{z_{\mathbf{x}}}\, z_{\mathbf{y}} = r_x r_y e^{\mathrm{j}(\theta_x - \theta_y)}$$

so the real part matches the dot product:

$$\Re\!\left(\overline{z_{\mathbf{x}}}\, z_{\mathbf{y}}\right) = r_x r_y \cos\varphi = \mathbf{x} \cdot \mathbf{y}.$$

Therefore, the scalar projection of $\mathbf{x}$ onto $\mathbf{y}$ is $r_x \cos(\theta_x - \theta_y)$, which when multiplied by the absolute value of y gives the inner product (that is, $r_x r_y \cos(\theta_x - \theta_y)$ in complex polar form).

4.4 Real-world significance

We discussed three main topics in this section — vectors, complex numbers, and their products. Representing quantities as vectors has many advantages, which mirrors the advantage of using lists and arrays in computer programming. Inner products are useful in quantifying the alignment between vectors. A simple example is in machine learning, where the similarity between data points can be measured using inner products, which has applications in face detection, recommendation systems, and more. Finally, as discussed, complex numbers help us analyze vectors in a more nuanced way by providing a framework for understanding their magnitude and direction. You will see many more real-world application examples of complex numbers in signal processing. In Fourier analysis, complex exponentials are used as the orthogonal basis functions for representing signals in the frequency domain.

5 Pre-requisite #3: Circuits

Without going into the specific details about various electrical circuits, this section will briefly discuss the fundamental concepts of circuits that are relevant to signal processing. In signal processing, we will use circuits only as examples. In fact, equivalent examples can be devised that are relevant for other disciplines. For example, in circuit theory, Kirchoff's voltage and current laws are essential tools that are commonly used to analyze currents and voltages. These laws frame the conservation of energy for an electrical circuit setting. An equivalent mechanical engineering example is the analysis of forces in a static system (such as a spring-mass damper system), where conservation of energy provides a similar framework for understanding the system's behavior.

Historically, signal processing has been a field that has drawn heavily from electrical engineering concepts, particularly in the analysis and manipulation of signals. The design of the feedback amplifier in the 1920s is the prime example. Scientists and engineers were interested in maximizing the signal to noise ratio for amplifier circuits, which led to the use of many of the foundational mathematical theory that existed at the time. The formalization of these mathematical tools for electrical engineering applications led to the development of the field of signal processing. Since then, these tools have found applications in many other disciplines, including mechanical engineering, civil engineering, computer science, biomedical engineering, and more. But for better or for worse, the pedagogy of signal processing has remained set in that historical context. So, without challenging the years of history too much, we will continue to use circuit examples in this course. However, wherever possible, we will try to incorporate broader application examples too.

EE 102 Week 1, Lecture 1 (Fall 2025)

Instructor: Ayush Pandey

Date: September 3, 2025

1 Goals

- Introduction to signals: continuous-time and discrete-time
- Basic properties of signals: scaling, offset, linearity, and time invariance, and more
- Quantifying the energy and power of signals

2 What are signals?

A *signal* is a set of data or information. This is intentionally defined in a very broad manner. A simple way to understand signals: all mathematical functions that you have studied in your calculus classes are signals if you can attach a physical meaning to the function. Note that a signal need not be a function of time. It is often intuitive to think about functions of time and physical signals are functions of time (often, but not always!).

A *system* maps (that is, it processes) input signals into output signals. So, systems are characterized by their input-output relationships. See Figure 1 for a visual representation of signals and systems.

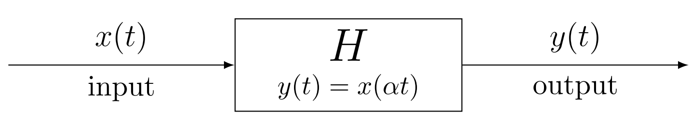


Figure 1: A system H that maps input signal $x(t)$ to output signal $y(t)$.

2.1 Continuous-time and discrete-time signals

Continuous-time domain is $\mathbb{R}$, and we write continuous-time signals as $x(t)$, if they are continuous functions of time (recall: continuous functions from your math classes). On the other hand, discrete-time domain is $\mathbb{Z}$, and a discrete-time signal is written as $x[n]$, where $n \in \mathbb{Z}$. This means that discrete-time signals are defined only at integer time indices.

2.2 Sketching signals

To sketch, draw and label axes, mark key values (peaks, zeros, discontinuities), and indicate any symmetry, periodicity, decay/growth, or piecewise structure (try to identify as many properties as you can before starting to sketch). The best way to start your sketch is to compute the values of the signal at "easy" points like, zero, the max time, etc.

3 Properties of signals

3.1 Scaling

Time scaling changes the horizontal axis by a constant α:

$$x_s(t) = x(\alpha t).$$

If $0 < \alpha < 1$, the signal expands in time whereas if $\alpha > 1$, it compresses the signal in time.

3.2 Offset

Time shifting offsets the horizontal axis by a constant T. A (right) delay of T seconds is defined by

$$x_d(t) = x(t - T).$$

Equivalently, $x_d(t_1 + T) = x(t_1)$ for every t_1.

3.3 Linearity of systems

A system H is *linear* if it satisfies additivity and homogeneity:

$$H\{x_1 + x_2\} = H\{x_1\} + H\{x_2\}, \qquad H\{k\, x\} = k\, H\{x\} \quad (\forall k \in \mathbb{C}).$$

Example: the exponential-weighting system $y(t) = e^{-at}x(t)$ is linear since

$$H\{x_1 + x_2\} = e^{-at}(x_1 + x_2) = e^{-at}x_1 + e^{-at}x_2 = H\{x_1\} + H\{x_2\}.$$

Remark. "Linear system" is a property of the *mapping*, not of the input/output signals. You should not confuse it with a straight-line graph of a scalar function, which you are used to thinking about when thinking about "linearity".

3.4 Time invariance of systems

A system H is *time invariant* if delaying the input by T produces the same delay at the output:

$$\text{If } y(t) = H\{x\}(t), \text{ then } H\{x(t-T)\} = y(t-T), \quad \forall T \in \mathbb{R}.$$

Intuition: If your opinion of a friend is dependent on the input about the friend, let's say that input is $x(t)$ (the friend descriptor signal), and seeing that input, you decide your opinion of your friend with an opinion signal called $y(t)$. Then, if your opinion about your friend does not change with time, that is, if you have the same opinion about your friend in the morning, in the evening (and even as the day changes), then your "opinion-defining" system (the one that outputs $y(t)$) is time-invariant! However, if your opinion of your friend keeps changing based on the time that you're meeting your friend, then you have a time-varying system of opinion generation (probably not a good trait!). Note that for time-invariant systems, the output is the "same response" delivered at the new time. It should not become, e.g., $k\,y(t)$ or $k\,y(t-T)$ depending on T.

3.5 A special signal — the unit step function

A unit step function is defined as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

It is a special signal because it models the "start" of something, or an "onset" of an event, or more simply, a "switching on" of a process. You can shift the time to $t - T$ to delay the start by T seconds, so it's a very versatile signal. Therefore, the unit step function finds use in various applications.

Quick check (in-class): Is the *unit step* $u(t)$ time-invariant? (Trick question: time invariance is a *system* property, not a signal property.)

4 Energy and power of signals

We quantify the "size" of signals using energy and (time-averaged) power. For continuous-time signals, we define

$$E_\infty \triangleq \int_{-\infty}^{+\infty} |x(t)|^2\,dt, \qquad P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt.$$

For discrete-time signals:

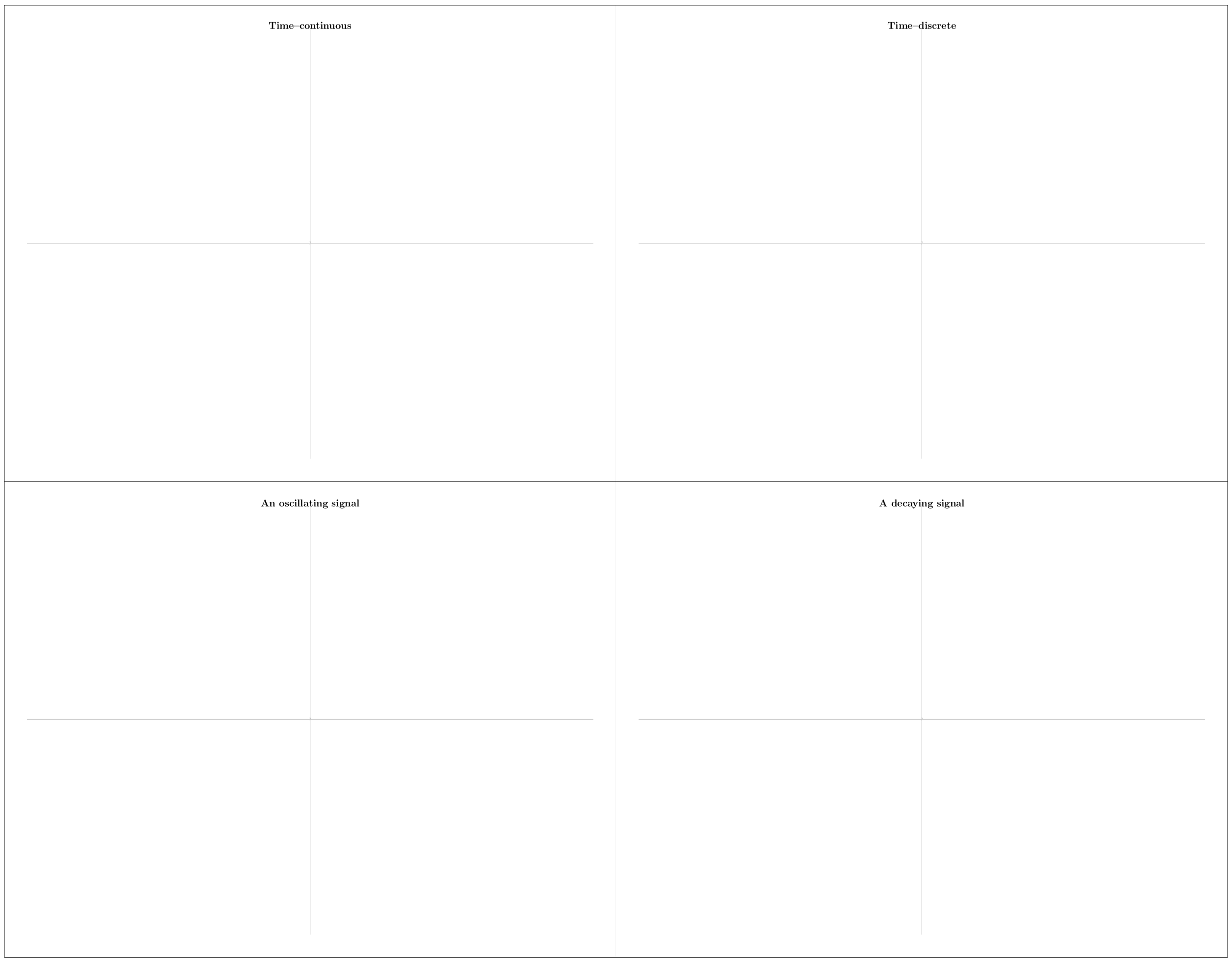
$$E_\infty \triangleq \sum_{n=-\infty}^{+\infty} |x[n]|^2, \qquad P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{+N} |x[n]|^2.$$

With a desired signal s and noise n, one practical signal-to-noise ratio is

$$\mathrm{SNR} = \frac{E_\infty(s)}{E_\infty(n)} \quad \text{(or } P_\infty \text{ for power signals).}$$

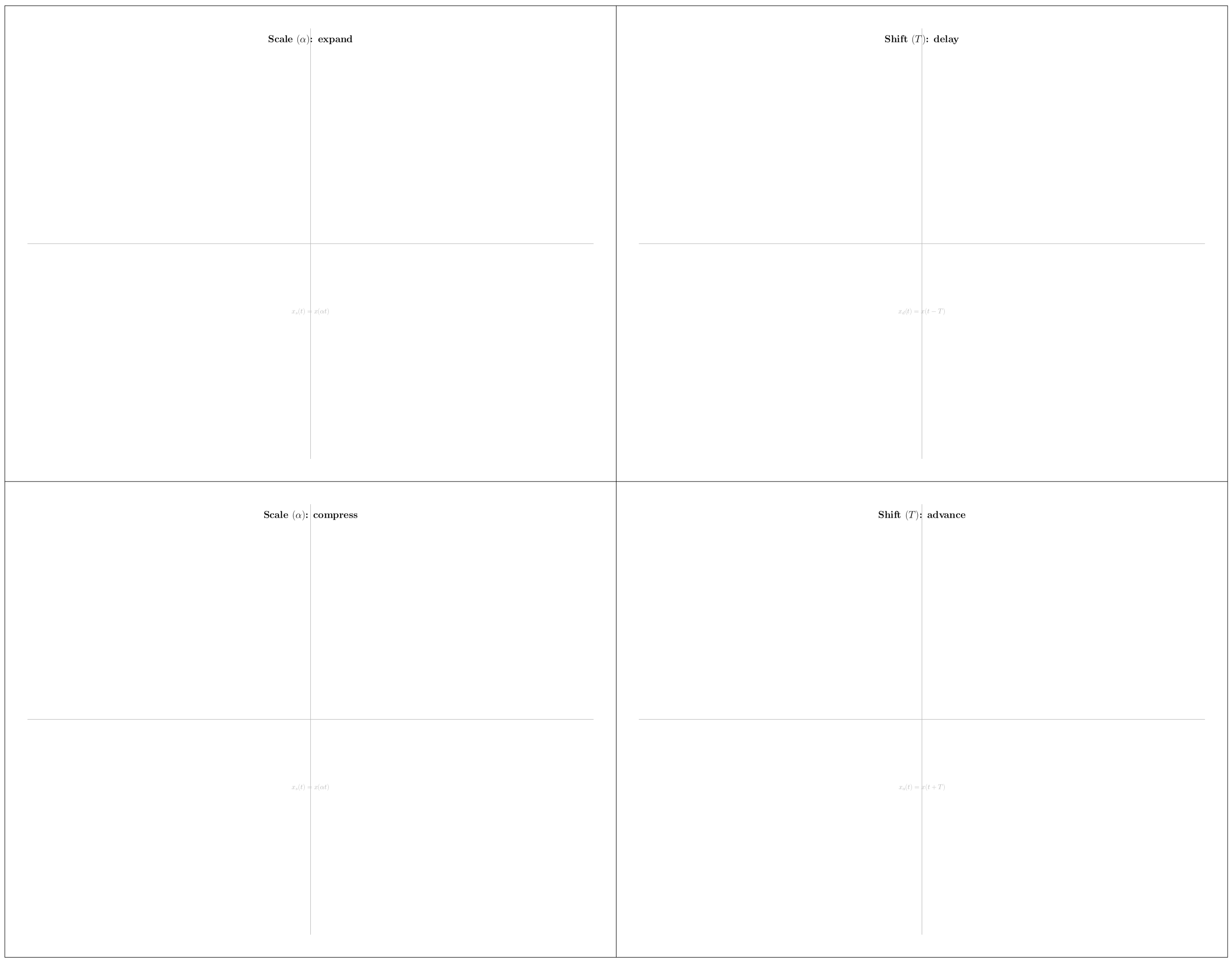
Worksheet #1: Sketching Signals (Part 1) — Groups of 4.

Each student takes one quadrant. Label axes clearly and annotate *what is your signal?*, *where is the signal likely to be observed?*, and key *properties of the signal*.



Worksheet #2: Transforming Signals (Part 2) — Groups of 2.

Each pair of students should scale and shift the two signals drawn by the other pair of students. Agree as a pair what scaling and shifting would mean and then draw it out. Clearly show the parameters and the transformed axes.



EE 102 Week 2, Lecture 1 (Fall 2025)

Instructor: Ayush Pandey

Date: September 8, 2025

1 Goals

- Review: time scaling, shifting, and combined operations on time-domain signals
- Review: energy and power — metrics to quantify signals
- Understand periodic signals using time shifting operations
- Derive the fundamental period of a signal
- Understand even and odd signals and their properties
- Apply signal operations to real-world signals using a guitar audio distortion example
- Next class: Complex exponentials, the unit impulse and step functions

2 Review: transforming signals

For a signal $x(t)$, common time operations include:

1. Reversal: $x(-t)$
2. Compression: $x(2t)$
3. Expansion: $x\left(\frac{t}{2}\right)$
4. Delay: $x(t-6)$
5. Advance: $x(t+6)$

2.1 How to sketch signal transformations?

To sketch signal transformations, first note down the key points on the X-axis (the time axis for time-domain signals). Then evaluate the value at the new domain (the shifted/scaled time) by looking at the values of the original signal at the corresponding time.

A quick summary: keep the vertical axis unchanged; apply horizontal changes only. For $x(at)$, compress if $|a| > 1$ and expand if $0 < |a| < 1$; for $x(t \pm T)$, shift right by T for $x(t-T)$ and left by T for $x(t+T)$; for $x(-t)$, reflect across the vertical axis.

Example: scaling and shifting a sinusoidal signal

Consider a sinusoidal signal $x(t) = \sin(t)$. A time-shifting transformation of $x(t)$ is given by

$$y(t) = x(t + t_0) = \sin(t + t_0)$$

where $t_0 \in \mathbb{R}$. For $t_0 > 0$, the signal is shifted to the left by t_0 while for $t_0 < 0$, the signal is shifted to the right by $|t_0|$. Similarly, a time-scaling transformation of $x(t)$ is given by

$$y(t) = x(\alpha t) = \sin(\alpha t)$$

where $\alpha \in \mathbb{R}$. For $|\alpha| > 1$, the signal is compressed by a factor of α while for $0 < |\alpha| < 1$, the signal is expanded by a factor of $\frac{1}{\alpha}$. If $\alpha < 0$, the signal is also reflected across the vertical axis.

Let us look at some specific values of t_0 and α to see how the signal is transformed.Figure 1 shows the original signal $x(t) = \sin(t)$ along with its time-shifted and time-scaled versions for different values of t_0 and α. Note that for $\alpha = -1$, we get a reflection of the original signal across the vertical axis. Expanding on this example, we can also combine time-shifting and time-scaling operations to get more complex transformations. For instance, consider the transformation

$$y(t) = x(\alpha t + t_0)$$

where both α and t_0 are non-zero. This transformation first scales the time by α and then shifts it by t_0. The order of operations matters here — if we were to shift first and then scale, we would have

$$y(t) = x(\alpha(t + t_0)) = x(\alpha t + \alpha t_0)$$

which is different from the previous transformation unless $\alpha = 1$.

If you intend to reverse the order of operations, you can redefine the shift parameter accordingly. For example, to achieve the same effect as $x(\alpha t + t_0)$ by shifting first and then scaling, you would need to use $x(\alpha(t + \frac{t_0}{\alpha}))$. Similarly, for the second combined transformation, you

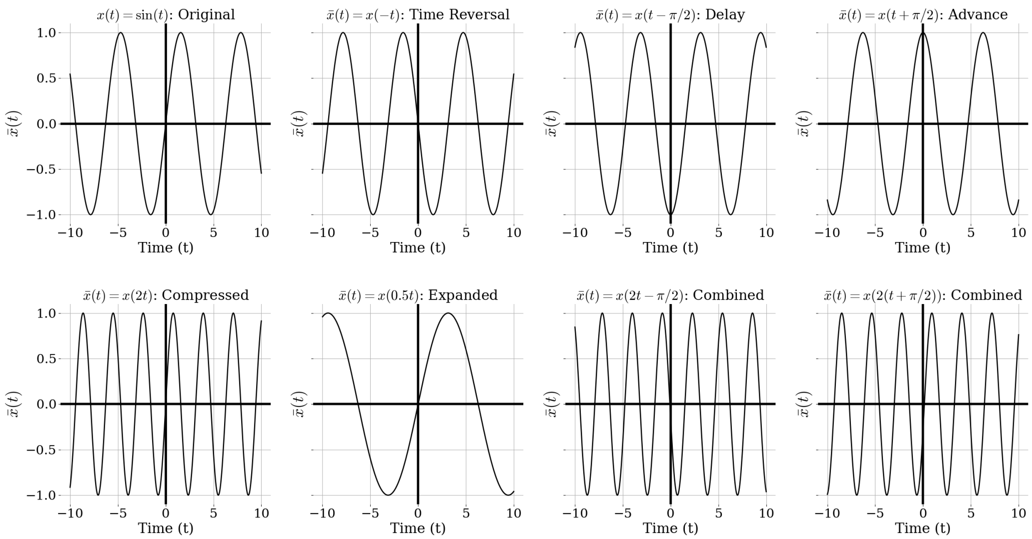


Figure 1: Time-shifting and time-scaling transformations of the signal $x(t) = \sin(t)$.

would need to use $x(\alpha t + \alpha t_0)$ to achieve the same effect as shifting first and then scaling by α next. Python code for generating signal transformations is available on Github[1].

2.2 Measuring the energy and power of signals

Previously, we defined energy as the integral of the squared magnitude of a signal over all time:

$$E_\infty(x) = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

and power as the time-averaged measure for a time period $[-T, T]$:

$$P_\infty(x) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt.$$

It is natural to wonder how we ended up with those specific definitions. You can build the intuition behind these definitions in two ways: (a) by considering an electrical signal $x(t)$ as a voltage across a 1-ohm resistor, and (b) by considering the mathematical convenience that these definitions provide. For (a), it is pretty clear that the energy dissipated in a resistor is given by the integral of the square of the voltage over time divided by the value of the resistor

[1] github.com/ee-ucmerced/ee102-signals-systems

(in this case, 1 ohm). To fully appreciate the mathematical meaning of these definitions, consider the following alternate definition of energy as the integral of the absolute value of the signal over all time:

$$E'_\infty(x) = \int_{-\infty}^{\infty} |x(t)| dt.$$

This would be a valid way to quantify the "energy" of a signal as well. Note that our goal is to not physically define "energy", the electrical engineering concept, rather we are interested in coming up with measures of signals that we can use to compare two different signals. Note that taking absolute value is *atleast* required to prevent the integral from being zero for signals that oscillate between positive and negative values. Despite this, the integral of the square of the absolute value (E) is preferred over just the integral of the absolute value (E') because it is the metric that lets us compare signals in the L^2 space, which is a Hilbert space[2]. Simply stated, this means that the space of signals with finite energy (i.e., $E_\infty(x) < \infty$) has mathematical properties that enable better analysis and manipulation of signals. For example, we can define an inner product between two signals as a metric that quantifies the similarity between two signals $x(t)$ and $y(t)$ as

$$\langle x, y \rangle = \int_{-\infty}^{\infty} x(t) y^*(t) dt,$$

where $y^*(t)$ is the complex conjugate of $y(t)$. This inner product allows us to define concepts like orthogonality and projection in the space of signals. Remember that being able to represent complex signals as linear combinations of standard signals is the core concept in signal processing — this is not possible without a clear notion of orthogonality! In fact, the definition of E above is simply the inner product of a signal with itself, i.e., $E_\infty(x) = \langle x, x \rangle$. The L^1 space (signals with finite E') does not have these properties, which is why we prefer to use E as our measure of energy. Power can then be defined as the time-averaged energy over a time period.

3 Periodic signals

Definition 1. A signal $x(t)$ is periodic if $\exists\, T_0 > 0$ such that $x(t + T_0) = x(t)$ for all t. The smallest such T_0 is the *fundamental period* of the signal.

In more verbose language, we say that a signal is a periodic signal if we can find a time shift T_0 such that shifting the signal by T_0 does not change the signal. If such a T_0 does not exist, then the signal is aperiodic. For example, $x(t) = \sin(t)$ is periodic. To find the fundamental period, we would have to find the smallest shift T_0 to satisfy $\sin(t + T_0) = \sin(t)$ for all t.

[2]Read more on Hilbert spaces here: https://en.wikipedia.org/wiki/Hilbert_space

Using the periodicity of the sine function, we know that $\sin(t + 2\pi) = \sin(t)$ for all t. Thus, $T_0 = 2\pi$ is the fundamental period of $\sin(t)$. Note that $T_0 = 4\pi$ also satisfies the periodicity condition, but it is not the fundamental period since it is not the smallest such T_0. In fact, any integer multiple of 2π would satisfy the periodicity condition, but we are only interested in the smallest such value for the fundamental period.

3.1 Why periodic signals?

Just like many other concepts in signal processing, periodic signals are also a mathematical convenience! Intuitively, it is clear that if something repeats over and over again, then we can analyze just one cycle of it and extend the results to the entire signal. Therefore, studying periodic signals is often a good starting point. But you may wonder — what if the signal is not periodic? Read on.

3.2 Periodic extensions

When a signal is defined on a finite interval (e.g., a single cycle), it is often useful to *periodically extend* it by repeating that interval end-to-end. This makes the time-averaged power well defined and makes symmetries/harmonics easier to see.

3.3 Example: Periodic or not?

Consider the following three signals. Our goal is to find out whether they are periodic or not. If they are periodic, we will report the fundamental period of these signals.

1. $x(t) = \cos(t)$
2. $x(t) = \cos(t)$ for $t \geq 0$ and $x(t) = -sin(t)$ for $t < 0$
3. $x(t) = e^{j\omega t}$ where $\omega \neq 0$

How to prove periodicity? We can simply *propose a T_0 and check* if the periodicity condition is satisfied. If you cannot find a T_0, that does not mean that the signal is aperiodic — it just means that you have not found the right T_0 yet! To prove aperiodicity, you have to show that no such T_0 exists. This is often done by contradiction. Proofs by contradiction is an important mathematical trick where you assume that the statement you want to prove is false and then show that this assumption leads to a contradiction (something that will be *obviously* incorrect). This implies that the original statement must be true. You will find the signal transformation properties useful in proving (a)periodicity. Finally, if you can exploit the properties of the given signal, then you will find a much easier path to the proof.

For the first example, we know that the cosine function is "oscillatory", which indicates that it is probably periodic. Let's try to find a T_0 such that $\cos(t + T_0) = \cos(t)$ for all t. You might propose a $T_0 = \pi$ and observe that $\cos(t + \pi) = -\cos(t)$, which is not what we were looking for. So, $T_0 = \pi$ is not a good choice for T_0. Let's try one more time. Propose $T_0 = 2\pi$. Then, compute $\cos(t + 2\pi) = \cos(t)$, which is a valid choice. To check if it is the fundamental period, we can see that any integer multiple of 2π would also satisfy the periodicity condition, but 2π is the smallest such value. Therefore, the fundamental period of $x(t) = \cos(t)$ is $T_0 = 2\pi$. Note that you can use trignometric identities to help you prove periodicity in an alternate way too.

For the second signal, sketch the signal to first get an intuition about whether it is periodic or not. You will see that the signal is a cosine wave for $t \geq 0$ and a negative sine wave for $t < 0$. The two parts do not match at $t = 0$, which indicates that the signal is not periodic. To prove this, we can use contradiction. Assume that the signal is periodic with period T_0. Then, we have $\cos(t+T_0) = x(t+T_0) = x(t)$ for all t. Now, consider the case when $t = -\frac{T_0}{2}$. Then, we have $\cos(-\frac{T_0}{2} + T_0) = \cos(\frac{T_0}{2}) = x(-\frac{T_0}{2}) = -\sin(-\frac{T_0}{2}) = \sin(\frac{T_0}{2})$. This is clearly not true! So, our assumption that the signal is periodic must be false. Therefore, the signal is aperiodic.

For the third signal, we can use the properties of the complex exponential function to prove periodicity. Write $x(t + T_0) = e^{j\omega(t+T_0)} = e^{j\omega t}e^{j\omega T_0}$. If we can find a T_0 such that $e^{j\omega T_0} = 1$, then we have periodicity. This is satisfied if $T_0 = \frac{2\pi}{\omega}$. Therefore, the fundamental period of $x(t) = e^{j\omega t}$ is $T_0 = \frac{2\pi}{\omega}$.

3.4 Power and energy of periodic signals

We can revisit the power and energy definitions for periodic signals. If x is periodic with period T_0, then the time-average power is well defined and can be computed over any interval of length T_0. Note that for a finite-duration input, E_∞ is finite and $P_\infty = 0$ (time average over an unbounded window goes to zero) whereas E_∞ is finite because we have finite-duration signal. For periodic signals, we have

$$P_\infty(x) = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2 dt \quad \text{(independent of } t_0\text{)}, \qquad E_\infty(x) = \infty \text{ unless } x \equiv 0.$$

Thus, periodic signals are *power signals* (finite power, infinite energy).

Energy and power for a periodic input

If $x(t)$ is periodic with fundamental period T_0, then $y_d(t)$ is also periodic with the *same* T_0 (memoryless mapping preserves period). Hence

$$E_\infty(y_d) = \int_{-\infty}^{\infty} |y_d(t)|^2 \, dt = \infty, \qquad P_\infty(y_d) = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |y_d(t)|^2 \, dt \text{ (finite).}$$

4 Even and odd signals

Recall that a mathematical function is called even if $f(-t) = f(t)$ for all t and odd if $f(-t) = -f(t)$ for all t. Examples of even functions include $\cos(t)$, t^2, and $|t|$. Examples of odd functions include $\sin(t)$, t^3, and the sign function $\mathrm{sgn}(t)$.

Definition 2. A signal $x(t)$ is even if $x(-t) = x(t)$ for all t and odd if $x(-t) = -x(t)$ for all t.

Proposition 1. *Any signal $x(t)$ can be uniquely decomposed as the sum of an even signal $x_e(t)$ and an odd signal $x_o(t)$.*

Proof. Let $x_e(t) = \frac{x(t)+x(-t)}{2}$ and $x_o(t) = \frac{x(t)-x(-t)}{2}$. Then, we have

$$x_e(-t) = \frac{x(-t) + x(t)}{2} = x_e(t)$$

so $x_e(t)$ is even. Similarly,

$$x_o(-t) = \frac{x(-t) - x(t)}{2} = -x_o(t).$$

which is is odd. Now, we can see that

$$x_e(t) + x_o(t) = \frac{x(t) + x(-t)}{2} + \frac{x(t) - x(-t)}{2} = x(t).$$

To prove uniqueness, we again use the proof by contradiction method. Assume, to the contrary, that there exist another pair of even and odd signals $x'_e(t)$ and $x'_o(t)$ such that $x(t) = x'_e(t) + x'_o(t)$. Then, we have

$$x_e(t) - x'_e(t) = x'_o(t) - x_o(t).$$

The left side is even (difference of two even functions), and the right side is odd (difference of two odd functions). The only function that is both even and odd is the zero function. Therefore, we have $x_e(t) - x'_e(t) = 0$ and $x'_o(t) - x_o(t) = 0$, which implies that $x_e(t) = x'_e(t)$ and $x_o(t) = x'_o(t)$. Hence, the decomposition is unique. □

5 Application demonstration: a guitar amplifier

An amplifier system can be modeled as $y(t) = \alpha x(t)$ where $\alpha > 1$ is the amplifier gain. However, real-world amplifiers have limits on the maximum and minimum output levels they can produce. When the input signal is too large, the output signal gets "clipped" at these limits. Although this is an undesirable effect for most audio applications, it is often used intentionally by musicians to create a distorted sound effect. This is common in many music genres such as rock, metal, and punk.

We can model a simple hard-clipping (overdrive) amplifier system as:

$$y_d(t) = \begin{cases} -\beta, & \alpha\, x(t) < -\beta, \\ \alpha\, x(t), & |\alpha\, x(t)| \le \beta, \\ \beta, & \alpha\, x(t) > \beta, \end{cases} \qquad \alpha > 0,\ \beta > 0.$$

This is a *memoryless* nonlinearity: at each t, $y_d(t)$ depends only on $x(t)$. It amplifies small inputs by α and saturates at $\pm\beta$ for large inputs. Here, the parameter α controls the amount of gain (loudness) and β controls the amount of distortion to apply. Vierinen has a YouTube demonstration for this effect[3].

The transfer curve for this system is shown in Figure 2.

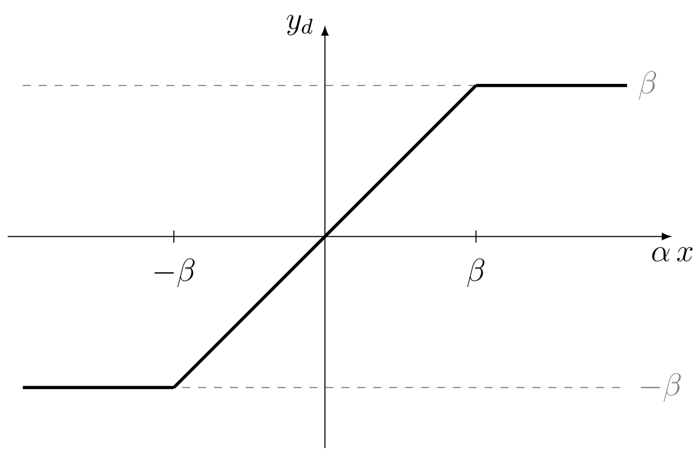


Figure 2: Hard-clipping nonlinearity: linear region $|\alpha x| \le \beta$, saturation outside.

Transforming the system: You can apply all the signal transformations to transform the system by transforming the transfer curve of the system shown above. Using Python, try to draw all time operations discussed above for $y_d(t)$ — the distorting amplifier system. Figure 3 shows the results.

[3]https://youtu.be/I30Mn_-yYF8.

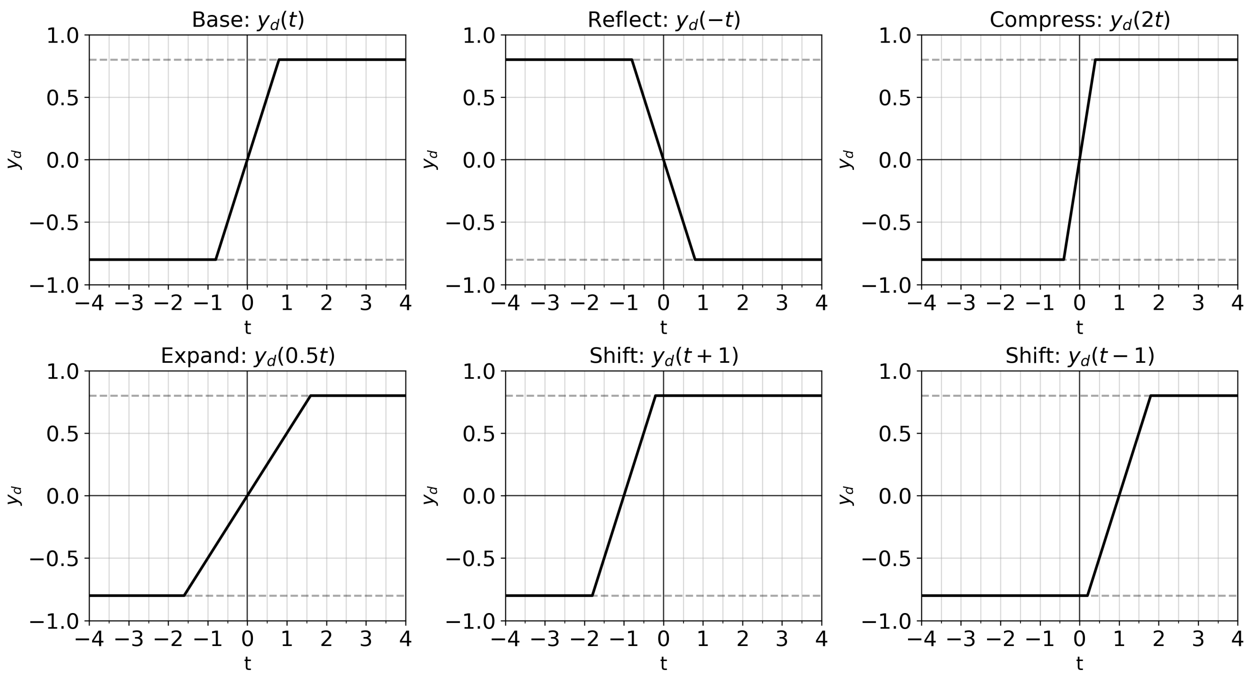


Figure 3: Time operations on the signal $y_d(t)$.

5.1 Distortion effects

By running the provided code, you can test various distortion effects by changing the parameters α and β. The code loads a .wav file for a sample guitar tone. Practice your Python (and music design) skills with this example!

5.2 Optional: Audio tone signal example and time operations

In the supplementary notes, you will find a Python notebook that creates a guitar-like audio tone. You can use computer programming to compute various time-transformed versions yourself.

Next class

The unit impulse $\delta(t)$ and step $u(t)$; convolution preview.

EE 102 Week 2, Lecture 2 (Fall 2025)

Instructor: Ayush Pandey

Date: September 10, 2025

1 Goals

- The timeless trio of signals: the complex exponential, the unit step, and impulse.
- The unit step function as a switch and an accumulator.
- The unit impulse function as an exciter and a sampler.
- The complex exponential signal as a sinusoid and a phasor.
- Applications of the timeless trio in real-world signal processing.

2 The unit step function

When we defined signals informally, we discussed how any mathematical function from your calculus class could be a signal as long as it represents something physical. Then it will not come as a surprise that for physical system applications, we would usually be interested only in positive values of time and we would want our signal to take the value of zero for all $t < 0$. Since we are extending the general concept of mathematical functions (which are defined for all t), it is important to have a mathematical way to write signals that are zero for $t < 0$. The unit step function does exactly that! Formally, we define the unit step function as

Definition 1. The unit step is a function defined as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}.$$

2.1 The unit step as a switch

As you can see in Figure 1, it is a discontinuous function that "steps" from 0 to 1 at $t = 0$. You may find the unit step function with different names: like the Heaviside function, or the ultrasensitive switch function.

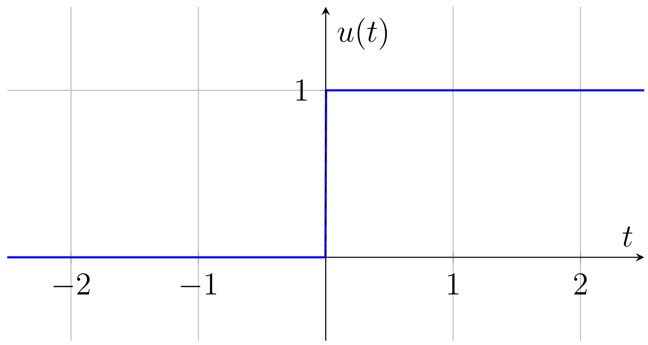


Figure 1: The unit step function $u(t)$.

So, any function that is zero for $t < 0$ can be written as the product of the unit step function and another function that defines the behavior of the signal for $t \geq 0$. For example, if we have a signal that is zero for $t < 0$ and equals $f(t)$ for $t \geq 0$, we can write it as $x(t) = f(t)u(t)$.

2.2 Example: A sinusoidal audio wave that starts at zero time

If $f(t) = Asin(\omega t + \phi)$, then we can write the signal as

$$x(t) = A\sin(\omega t + \phi)u(t).$$

This signal is zero for $t < 0$ and equals a sinusoidal wave for $t \geq 0$. The unit step function effectively "switches on" the sinusoidal wave at $t = 0$. But did we *really* need the step function here? You can argue that we could have defined all signals using two cases,

$$x(t) = \begin{cases} f(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

but this type of definition will quickly get cumbersome. So, yet again, we are introducing a mathematical object to make our lives easier, at least in the long run (at the moment, it may seem that we are making our life harder by learning another new function). For the specific sinusoidal example, the alternative way to define it is using a piecewise function that would need two different cases:

$$x(t) = \begin{cases} 0, & t < 0 \\ A\sin(\omega t + \phi), & t \geq 0 \end{cases}.$$

We would prefer $x(t) = A\sin(\omega t + \phi)u(t)$ over the piecewise definition so that we can work with just a single expression.

2.3 Unit step as a general switch

Beyond the switching behavior of unit step, we can also use it to define arbitrary "pulse" signals and also other piecewise continuous signals. For example, we can define a rectangular pulse of width τ as

$$p_\tau(t) = u(t) - u(t-\tau).$$

This pulse is 1 for $0 \leq t < \tau$ and zero otherwise (can you prove this without relying on sketching?). It is also possible to write the equation of the pulse signal by using a time reversal:

$$p_\tau(t) = u(t) + u(\tau - t) - 1$$

The sketch in both cases is the same: a pulse that starts at $t = 0$ and ends at $t = \tau$. More generally, we can write a pulse that starts at t_1 and ends at t_2 as

$$p_{t_1,t_2}(t) = u(t-t_1) - u(t-t_2).$$

See Figure 2 for a sketch of the rectangular pulse that starts at t_1 and ends at t_2.

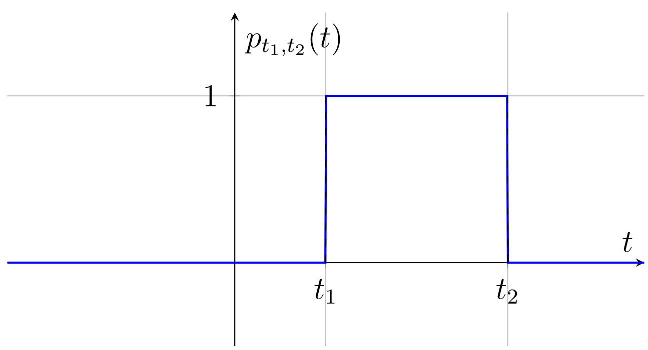


Figure 2: A rectangular pulse $p_{t_1,t_2}(t)$ that is 1 for $t \in [t_1, t_2)$ and 0 otherwise.

The pulse function provides us with more control over when the signal "turns on" and "turns off" — in Figure 2, it turns on at $t = t_1$ and turns off at $t = t_2$. We would expect such knobs of any modern switching system!

We can also define a triangular pulse of width 2τ that is zero for all $t < 0$ and $t > 2\tau$, and peaks at $t = \tau$. This triangular pulse would increase linearly from 0 to 1 in the interval $[0, \tau]$ and then decrease linearly from 1 to 0 in the interval $[\tau, 2\tau]$. By writing the equations of the two linear segments and then combining with unit step to decide when would each linear segment be "active", we can write the equation of the triangular pulse. Let's build it step-by-step. We write the two parts that "activate" the two linear segments using unit step functions:

$$w_1(t) = \underbrace{u(t) - u(t-\tau)}_{\text{active on } [0,\tau)}, \qquad w_2(t) = \underbrace{u(t-\tau) - u(t-2\tau)}_{\text{active on } [\tau,2\tau)}.$$

now, we define the two linear segments:

$$r(t) = \underbrace{\frac{t}{\tau}}_{\text{linear rise}}, \qquad f(t) = \underbrace{2 - \frac{t}{\tau}}_{\text{linear fall}}.$$

Then, the triangular pulse is obtained by gating each linear segment with its corresponding window and then adding the two gated segments:

$$tr_\tau(t) = \underbrace{r(t)}_{\text{rise}} \underbrace{w_1(t)}_{\text{gate }[0,\tau)} + \underbrace{f(t)}_{\text{fall}} \underbrace{w_2(t)}_{\text{gate }[\tau,2\tau)}.$$

$$tr_\tau(t) = \frac{t}{\tau}\big[u(t) - u(t-\tau)\big] + \left(2 - \frac{t}{\tau}\right)\big[u(t-\tau) - u(t-2\tau)\big].$$

See Figure 3 for a sketch of the triangular pulse.

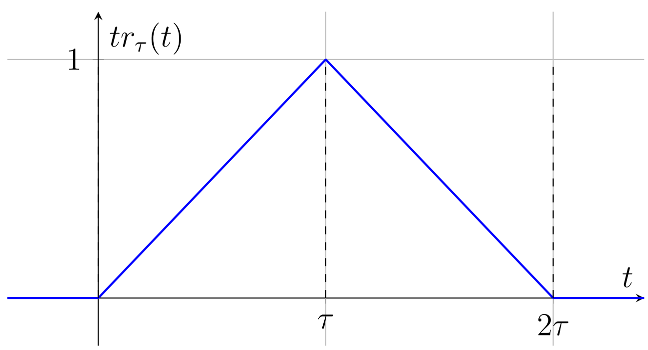


Figure 3: Triangular pulse of width 2τ: zero outside $[0, 2\tau]$, peak 1 at $t = \tau$.

The triangular pulse provides a smoother transition between the off and the on states compared to the rectangular pulse and is constructed using the unit step function as well. So, the unit step function is not just a simple switch it is also a building block for constructing more complex signals. The next "pulse" type signal that you should try is a trapezoidal pulse!

2.4 The unit step as an accumulator

The unit step function can also be viewed as an accumulator. If we integrate the unit step function, we obtain the ramp function (Problem 2.1 in HW #2):

$$r(t) = \int_{-\infty}^{t} u(\tau)d\tau = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}.$$

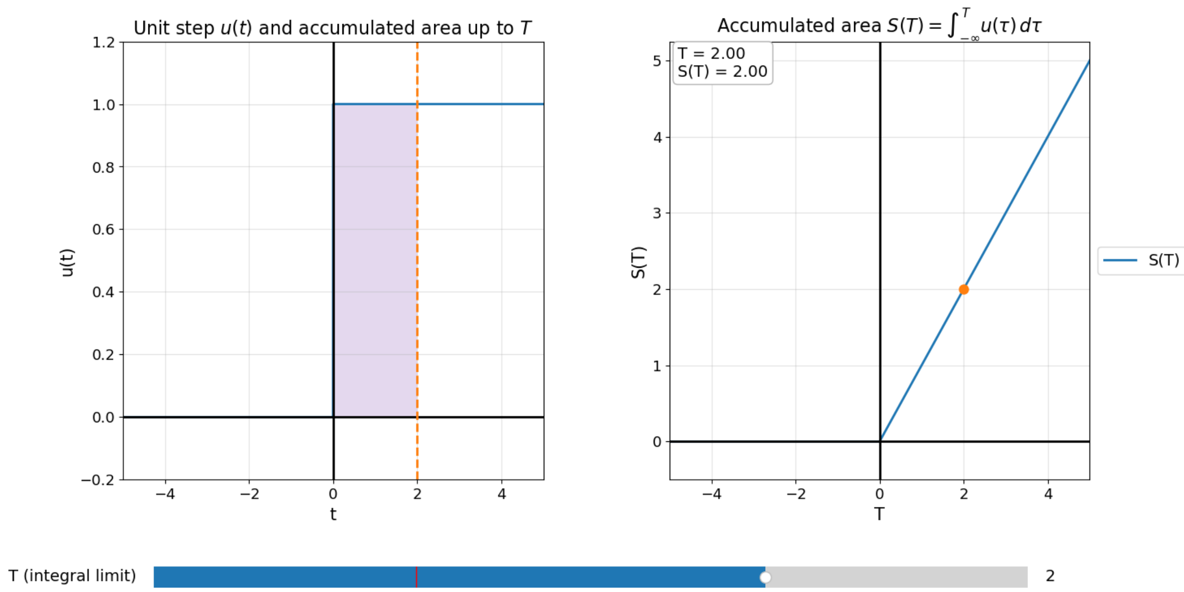


Figure 4: The unit step function $u(t)$ (left) and its accumulated area, the ramp function $r(t)$ (right). The slider below sets the upper limit of integration T.

This ramp function increases linearly for $t \geq 0$ and is zero for $t < 0$. It effectively accumulates the area under the unit step function. See Figure 4 for a sketch of the unit step function and its accumulated area (the ramp function) on the right side. A virtual manipulative is available for you to explore this concept interactively on the course Github page[1].

3 The unit impulse function

The unit impulse function is our second of the "timeless trio" of signals — one of the three most important signals in signal processing. In other fields, such as physics, it is also known as the Dirac delta function. In discrete mathematics, it is known as the Kronecker delta function. So, the same mysterious function has many different names! The reason is clear — it is a very useful mathematical object, while not even being a function in the traditional sense! We will define it formally later, but for now, let's understand it informally. You only need to remember two properties about the impulse function:

- It is zero at all times except at $t = 0$, that is $\delta(t) = 0$ for all $t \neq 0$.

[1]Here is the link for virtual manipulative on Github for unit step as an accumulator

- It has an area of 1 under its curve, that is, $\int_{-\infty}^{\infty} \delta(t)dt = 1$.

How is that possible? If a signal is zero everywhere except at one point, then there must be something unique happening at that point. To build intuition for this function, consider the rectangular pulse we defined earlier, with a slight modification. Let's define a rectangular pulse of width ϵ and height $\frac{1}{\epsilon}$ around the origin:

$$p_\epsilon(t) = \frac{1}{\epsilon}\left[u\left(t+\frac{\epsilon}{2}\right) - u\left(t-\frac{\epsilon}{2}\right)\right].$$

A sketch of this pulse is shown in Figure 5.

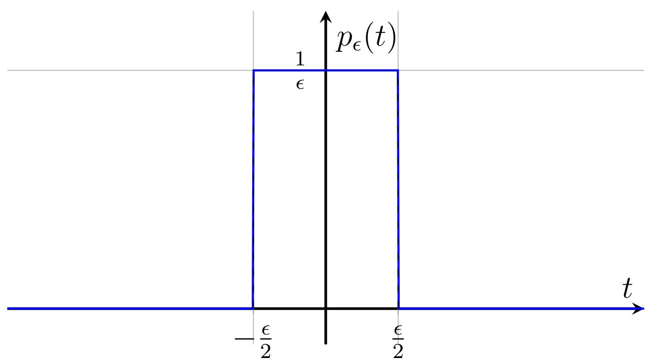


Figure 5: Rectangular pulse $p_\epsilon(t) = \frac{1}{\epsilon}\left[u\left(t+\frac{\epsilon}{2}\right) - u\left(t-\frac{\epsilon}{2}\right)\right]$ of width ϵ centered at the origin.

As you can see in the figure, the area of the rectangle is equal to 1 for all values of ϵ. But this does not satisfy the two properties of the impulse function listed above as there are points $t \neq 0$ where the pulse is non-zero. So, we need to modify this pulse further. As we make ϵ smaller, visualize how the rectangle becomes taller and narrower, while still maintaining an area of 1. You can interactively explore this concept using the virtual manipulative available on the course Github page[2].

3.1 Impulse as the limit of a rectangular pulse

Formally, we can write the unit impulse function as the limit of the rectangular pulse as ϵ approaches zero:

$$\delta(t) = \lim_{\epsilon \to 0} p_\epsilon(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[u\left(t+\frac{\epsilon}{2}\right) - u\left(t-\frac{\epsilon}{2}\right)\right].$$

Despite the definition above, it is not possible to write a closed-form expression for the impulse function because it is not defined at $t = 0$ and is zero everywhere else. The only

[2]Here is the link for virtual manipulative on Github for impulse approximation using a pulse. Additionally, you can approximate an impulse using a Gaussian function, see here.

quantifiable property that we know so far is that the area under the curve of a delta function is equal to 1. To visually describe an impulse function, we draw an arrow pointing upwards at the point at which the impuse is located. That is, if we have $\delta(t)$, we draw an arrow at $t = 0$; if we have $\delta(t - t_0)$, we draw an arrow at $t = t_0$. The height of the arrow is not important, but we label it with a number to indicate the area under the impulse. For example, if we have $A\delta(t - t_0)$, we draw an arrow at $t = t_0$ and label it with A to indicate that the area under the impulse is equal to A.

In practice, we can never generate a true impulse function, we can only get infinitesimally close to it as we keep making the width of the rectangular pulse infinitesimally close to zero and the height of the pulse close to infinity. Even though this may sound needlessly confusing, it provides us with a very powerful mathematical tool. One example is discussed next.

3.2 Impulse as a time-sampler

The impulse function can sample any test function at a specific point in time. Note that if we write $f(t)\delta(t)$, the product will be zero for all $t \neq 0$ because $\delta(t)$ is zero for all $t \neq 0$. The only point where the product is non-zero is at $t = 0$. So we can write $f(t)\delta(t) = f(0)\delta(t)$. This is also an impulse function located at $t = 0$ with an area of $f(0)$. Now if we integrate this product over all time, we get

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)\int_{-\infty}^{\infty} \delta(t)dt = f(0) \cdot 1 = f(0).$$

This property is known as the sifting property of the impulse function. By integrating any test function multiplied by an impulse function, we can extract the value of the test function at the location of the impulse $\rightarrow$ we have sampled that function! More generally, if we have an impulse located at $t = t_0$, we can write this impulse as $\delta(t - t_0)$. Then, by the same reasoning as above, we can show that

$$f(t_0) = \int_{-\infty}^{\infty} f(t)\delta(t - t_0)\, dt.$$

To prove the above, write the integral as

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)\, dt = f(t_0)\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = f(t_0) \cdot 1 = f(t_0).$$

We obtained the latter equality by observing that $\delta(t - t_0)$ is zero at every point other than $t = t_0$. Since $f(t_0)$ is independent of t, we can take it outside the integral. The remaining integral is equal to 1 because the area under the impulse function is equal to 1. This property makes the impulse function a powerful tool in signal processing and system analysis.

In formal mathematical analysis, the above is not seen as a property but is instead used to *define* the impulse function.

Definition 2. The unit impulse function $\delta(t)$ is defined as the function for which the area under curve of its product with any test function $f(t)$ that is continuous at $t = 0$, is equal to the value of the test function at the time instant at which the impulse is located ($t = 0$ for $\delta(t)$). That is,

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0). \tag{1}$$

3.3 Relationship between unit step and unit impulse

If you look back at the unit step function, you will notice that it is discontinuous at $t = 0$. So, can we define the differentiation of unit step function with time, du/dt? Generally, the answer will be no since the derivative of a function is not defined at points where the function is discontinuous. Let's try an alternate approach. Consider the following integral:

$$\int_{-\infty}^{\infty} f(t)\frac{du(t)}{dt}dt.$$

We can evaluate this integral using integration by parts to write

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\frac{du(t)}{dt}dt &= f(t)u(t)\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(t)\frac{df(t)}{dt}dt \\ &= f(\infty) - \int_{0}^{\infty} \frac{df(t)}{dt}dt \\ &= f(\infty) - [f(t)]_0^{\infty} \\ &= f(\infty) - f(\infty) + f(0).\end{aligned}$$

So, we derived that

$$\int_{-\infty}^{\infty} f(t)\frac{du(t)}{dt}dt = f(0), \tag{2}$$

which is the same as the definition of the impulse function discussed earlier — the area under the product of any test function and the impulse function is equal to the value of the test function at the location of the impulse. So, we can conclude by comparing equations (1) and (2) that

$$\frac{du(t)}{dt} = \delta(t).$$

Using a similar approach, you can also show that (HW #2)

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau.$$

4 Complex exponential signals

Continuous time

$$x(t) = A\, e^{j(\omega_0 t+\phi)} = A\cos(\omega_0 t + \phi) + j\, A\sin(\omega_0 t + \phi).$$

Real and imaginary parts are orthogonal sinusoids. Fundamental period $T_0 = \frac{2\pi}{\omega_0}$.

Discrete time

$$x[n] = A\, e^{j(\Omega_0 n+\phi)}.$$

This is periodic iff $\dfrac{\Omega_0}{2\pi} = \dfrac{M}{N}$ with integers M, N coprime. Then the fundamental period is $N_0 = N$. Otherwise, it is *aperiodic* on $\mathbb{Z}$.

Geometric phasor

The complex exponential traces a circle of radius A in the complex plane at angular speed ω_0 (continuous) or advances by a fixed angle Ω_0 per sample (discrete). The real part is the projection on the horizontal axis and the imaginary part is the vertical projection.

EE 102 Week 3, Lecture 1 (Fall 2025)

Instructor: Ayush Pandey

Date: September 15, 2025

Logistics

- HW #3 due: Mon, Sep 22 (new deadlines: every Monday at midnight). Midterm: Wed, Sep 24.
- Feedback from Week 2: new assignment deadline, more worked examples, will tie to book exercises, and the weekly HW.
- New office hours (Ayush): Mondays at 3.30pm in the library cafe. (Yaoyun, TBD): Fridays at 12pm.
- Midterm exam: in class, closed book, timed. One hour exam. Will be designed to be completed in 40 minutes. Will cover material up to Week 3 Lecture 2. A practice quiz will be posted on CatCourses. No programming problems on the exam.
- Suggested study approach (in order): lecture notes, concepts on the HW assignments, books as references (work on solved examples).

1 Goals for today

We will review EE 102 topics so far. This includes: signals, sketching signals, properties of signals, transformations of signals, and the three special signals (step, impulse, complex exponentials).

2 Icebreaker activity: comparing signal energy

You are driving an electric vehicle (EV) on a one-hour trip from point A to point B. There are many ways you can do this trip — which one uses the least energy?

Each team is assigned a simple signal trace $x(t)$ that represents how much battery is used up in an electric vehicle over $[0, 1]$ hour for a given scenario (assume arbitrary units such that the math works out). Your task is to compute the total energy E (in kWh) to determine who has the biggest energy consumption and predict your rank (1 = least energy spent). Among your team, you should confirm that everyone has the same answer and come to a consensus of the rank. Then, you will be asked to announce your energy and your guess of where you stand.

Key formula (use t in hours):

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

Team Prompts (pick one per team)

All signals are zero outside the intervals shown.

Team 1: Flat road, stop, flat road

$$x_1(t) = \begin{cases} 12, & 0 \le t < \frac{1}{3} \\ 0, & \frac{1}{3} \le t < \frac{2}{3} \\ 12, & \frac{2}{3} \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Team 1: Record your results
Energy $E =$ ____________ kWh   Predicted rank = ____________ (/8)

Team 2: Uphill, stop, downhill

$$x_2(t) = \begin{cases} 18, & 0 \le t < \frac{1}{3} & \text{(uphill)} \\ 0, & \frac{1}{3} \le t < \frac{2}{3} & \text{(stop)} \\ 6, & \frac{2}{3} \le t \le 1 & \text{(downhill)} \\ 0, & \text{otherwise} \end{cases}$$

Team 2: Record your results

Energy $E =$ ______________ kWh   Predicted rank = ______________ (/8)

Team 3: Rapid cruise (less efficient) to destination

$$x_3(t) = \begin{cases} 13.5, & 0 \le t < \frac{2}{3} \\ 0, & \frac{2}{3} \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Team 3: Record your results

Energy $E =$ ______________ kWh   Predicted rank = ______________ (/8)

Team 4: Dynamic speeding (ramp up, then ramp down)

Ramp up such that battery uses from 8 to 16 over the first 0.5 h, then ramp down to 8 over the next 0.5 h.

$$x_4(t) = \begin{cases} 8 + 16\,t, & 0 \le t < \frac{1}{2} \\ 24 - 16\,t, & \frac{1}{2} \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Team 4: Record your results

Energy $E =$ ______________ kWh   Predicted rank = ______________ (/8)

Team 5: Stop-and-go traffic (three bursts)

Scenario where $x(t) = 18$ for three 10 min segments, with 10 min stops between.

$$x_5(t) = \begin{cases} 18, & 0 \le t < \frac{1}{6},\ \ \frac{1}{3} \le t < \frac{1}{2},\ \ \frac{2}{3} \le t < \frac{5}{6} \\ 0, & \text{else on } [0,1] \\ 0, & \text{otherwise} \end{cases}$$

Team 5: Record your results

Energy $E =$ ______________ kWh   Predicted rank = ______________ (/8)

Team 6: Eco mode (slow and steady)

$$x_6(t) = \begin{cases} 9, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Team 6: Record your results

Energy $E =$ ________________ kWh   Predicted rank = ________________ (/8)

Team 7: Late starter, speeding later

$$x_7(t) = \begin{cases} 0, & 0 \le t < \frac{1}{2} \\ 16, & \frac{1}{2} \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Team 7: Record your results

Energy $E =$ ________________ kWh   Predicted rank = ________________ (/8)

Team 8 — Two hills then cruise

$$P_8(t) = \begin{cases} 24, & 0 \le t < \frac{1}{4} \\ 6, & \frac{1}{4} \le t < \frac{1}{2} \\ 10, & \frac{1}{2} \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Team 8: Record your results

Energy $E =$ ________________ kWh   Predicted rank = ________________ (/8)

3 Review: signals and basic properties

Refer to the previous week's notes for more details on signals. As a quick reminder: we defined signals as mathematical functions that describe physical quantities or represent information.

3.1 Basic properties of signals

- **Continuous-time and discrete-time:** Continuous-time signals are defined for every real-valued time t, while discrete time signals are defined only at discrete time instances n (usually integers).
- **Energy and power signals:** Energy signals have finite energy
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$
and zero average power, while power signals have finite, non-zero average power
$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt < \infty$$
and infinite energy.
- **Deterministic and random signals:** Deterministic signals can be precisely described by mathematical functions, while random signals exhibit uncertainty and are often characterized statistically.
- **Periodicity** Periodic signals repeat after a fixed interval T (i.e., $x(t) = x(t+T)$), while aperiodic signals do not exhibit such repetition. The smallest time-shift T_0 for which $x(t) = x(t+T_0)$ is called the fundamental period.
- **Even vs. odd:** Even signals satisfy $x(t) = x(-t)$, while odd signals satisfy $x(t) = -x(-t)$.

3.2 Signal transformations

Remember that the two most important types of transformations are time shifts and time scaling:
$$y(t) = x(t - t_0) \quad \text{(time shift)}, \qquad y(t) = x(at) \quad \text{(time scaling)}.$$
Here, for time-shift the t is replaced by $t - t_0$, and for time-scaling the t is replaced by at. So, if the signal is $x(t) = A\sin(\omega t + \phi)$, then the time-shifted signal is $y(t) = A\sin(\omega(t - t_0) + \phi)$ and the time-scaled signal is $y(t) = A\sin(\omega(at) + \phi)$.

3.3 Decomposition of signals with even and odd functions

In this course, we will often decompose signals into its fundamental components. There are many ways in which you can define these components. One of these is decomposing a signal into an even and an odd part. This helps us apply the properties of even and odd functions to analyze the signal, without having to analyze the entire signal at once.

Consider a signal $x(t)$,

$$x(t) = x_{\mathrm{e}}(t) + x_{\mathrm{o}}(t), \qquad x_{\mathrm{e}}(t) \triangleq \frac{x(t) + x(-t)}{2}, \quad x_{\mathrm{o}}(t) \triangleq \frac{x(t) - x(-t)}{2}.$$

Pop quiz Show that $x_{\mathrm{e}}(t)$ is an even function and $x_{\mathrm{o}}(t)$ is an odd function.

4 The "timeless trio" of signals

We noted three special signals: step, impulse, complex exponential and gave them a name (the "timeless trio"). We will now review these three signals in more detail.

4.1 The unit step as a switch

The unit step $u(t)$ is a fundamental building block for constructing piecewise signals. It acts as a switch that turns on at $t = 0$ and remains on thereafter. By combining and shifting unit steps, we can create complex piecewise functions.

The ideal unit step function is not realizable in practice. However, smooth *sigmoids* can approximate $u(t)$. You will find that these sigmoids are the main building blocks of neural networks! In a reductionist view, neural networks are just millions (or billions) of such sigmoids (unit steps) trained together to approximate any general function. That's why neural networks are sometimes also called *universal function approximators*. Someone who learns signal processing will be able to see the deeper picture: everything is a switch after all (as in electronics and computers).

You can write down the sigmoid functions as approximate unit steps:

$$\sigma_k(t) \;=\; \frac{1}{1 + e^{-kt}} \quad \text{and} \quad \tilde{\sigma}_k(t) \;=\; \frac{1}{2}\left(1 + \tanh(kt)\right), \qquad k \gg 1 \Rightarrow \sigma_k(t) \approx u(t).$$

Let's consider two examples to demonstrate the power of the unit step functions.

4.2 Example 1: A square wave using steps

The bipolar square pulse in Figure 1 can be written as a combination of unit step functions. Let's try to build it step-by-step. First, note that the first unit step (part 1 of the figure) turns on the signal at $t = 0$, so this can be written as $u(t)$. Next, we want the signal to turn off at $t = 1$ (part 2 of the figure). Consider the time-shifted signal $u(t-1)$, which turns on at $t = 1$. If we subtract this from the first step, we get a signal that is $+1$ on $[0, T)$ and 0 elsewhere. Next, we want the signal to turn negative at $t = 2T$ (part 3 of the figure). Consider the negative of the step function $-u(t-2T)$, which turns on at $t = 2T$. Adding this to the previous result gives a signal that is $+1$ on $[0, T)$, 0 on $[T, 2T)$, and -1 on $[2T, \infty)$. Finally, we want the signal to return to 0 at $t = 3T$ (part 4 of the figure). Consider the time-shifted step $u(t-3T)$, which turns on at $t = 3T$. Adding this to the previous result gives the desired square pulse. Thus, the complete expression for the square pulse is

$$x(t) \;=\; u(t) - u(t-T) \;-\; u(t-2T) + u(t-3T)$$

equals $+1$ on $[0, T)$, 0 on $[T, 2T)$, -1 on $[2T, 3T)$, and 0 otherwise. It is a sum of four shifted/scaled steps. Figure 1 overlays the four contributors.

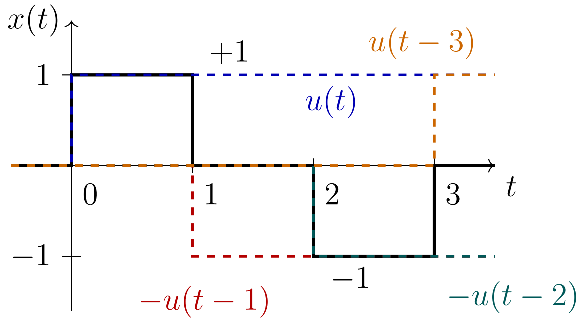


Figure 1: Square pulse constructed as $u(t) - u(t-T) - u(t-2T) + u(t-3T)$ (here $T = 1$). Colored dashed curves show the four step contributors.

Example 2 (piecewise levels using steps). A signal that is 0 for $t < 1$, jumps to 1 on $[1, 2)$, and then to 2 for $t \geq 2$ can be written *only* with steps:

$$x(t) \;=\; u(t-1) + u(t-2).$$

More generally, any piecewise-constant $x(t)$ can be expressed as $\sum_k a_k\, u(t-t_k)$ with appropriate increments a_k at the change points t_k.

4.3 The unit impulse as a sampler

We defined the impulse by its sifting property:

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt \;=\; f(t_0).$$

Think of δ as an idealized sampler that extracts the value of $f(t)$ at t_0.

A common approximation uses a narrow rectangle (area 1):

$$p_\varepsilon(t) \;=\; \frac{1}{\varepsilon}\Big[u\big(t+\tfrac{\varepsilon}{2}\big) - u\big(t-\tfrac{\varepsilon}{2}\big)\Big], \qquad \lim_{\varepsilon\to 0^+} p_\varepsilon(t) = \delta(t) \text{ in the distribution sense.}$$

You can visualize this approximation by running the `VM_delta_via_pulse.py` script available on course Github.

4.4 Complex exponentials

We have discussed the complex exponential function before: $e^{j\omega t}$. However, that is just a special case of the generalized complex exponential

$$x(t) \;=\; Ae^{st}, \qquad A\in\mathbb{C},\;\; s=\sigma+j\omega\in\mathbb{C}.$$

Why complex numbers / complex signals? With complex numbers, the central question is always: why did we even introduce the imaginary! Recall that when we worked with 2D vectors, we could represent them instead using a complex scalar. So, the vector operations, which can be tricky to account for become simpler with the standard scalar algebra that is applicable for complex numbers. We want the same convenience with signals. Additionally, as we will see, complex exponentials let us represent a *really* large class of signals such as sinusoids (of all kinds!), exponentials (both growing and decaying), and arbitrary combinations of these!

Pop Quiz 4.1: Check your understanding!

Recall that the utility of complex numbers is that they allow us to represent higher dimensions, in this case, two signals, using just one signal. Can you identify the two signals represented by $x(t) = e^{j\omega t}$?

Solution on page 11

To see that the general complex exponential can represent a large class of signals, write $A = |A|e^{j\phi}$ to obtain the *amplitude-phase* form

$$x(t) \;=\; |A|\,e^{\sigma t}\,e^{j(\omega t+\phi)} \;=\; e^{\sigma t}\Big(|A|\cos(\omega t+\phi) \;+\; j\,|A|\sin(\omega t+\phi)\Big).$$

Special cases emerge by selecting A and s:

1. A sinusoidal signal (pick A and s): Set $\sigma = 0$, $A = |A|e^{j\phi}$, $s = j\omega$:

$$x(t) = |A|e^{j(\omega t+\phi)} \;\Rightarrow\; \mathrm{Re}\{x(t)\} = |A|\cos(\omega t + \phi),\;\; \mathrm{Im}\{x(t)\} = |A|\sin(\omega t + \phi).$$

You can see that $e^{j\omega t}$ is periodic with $T = \frac{2\pi}{\omega}$ for $\omega \neq 0$.

Pop Quiz 4.2: Check your understanding!

When is Ae^{st} a constant k?

Solution on page 11

2. Real decaying exponential: Take $\omega = 0$, $\sigma < 0$, and $A \in \mathbb{R}$:

$$x(t) = Ae^{\sigma t}, \quad \text{monotone decay to 0 as } t \to \infty.$$

Pop Quiz 4.3: Check your understanding!

Derive the growing exponential using the complex exponential.

Solution on page 11

3. Exponentially damped sinusoid. Choose $\sigma < 0$, $\omega \neq 0$, $A = |A|e^{j\phi}$ and take the real part:

$$x(t) = |A|e^{\sigma t}\cos(\omega t + \phi).$$

Pop Quiz 4.4: Check your understanding!

Derive the exponentially growing cosine using the complex exponential.

Solution on page 11

4.4.1 Using complex algebra

You can derive all of the standard signals above from the general complex exponential also by expanding out the complex exponential by writing $A = a_1 + jb_1$ and $s = \sigma + j\omega$:

$$x(t) = (a_1 + jb_1)e^{(\sigma+j\omega)t} = (a_1 + jb_1)e^{\sigma t}(\cos(\omega t) + j\sin(\omega t)).$$

Expanding this out, we get

$$x(t) = e^{\sigma t}\big((a_1\cos(\omega t) - b_1\sin(\omega t)) + j(a_1\sin(\omega t) + b_1\cos(\omega t))\big).$$

Thus, the real and imaginary parts of $x(t)$ are

$$\mathrm{Re}\{x(t)\} = e^{\sigma t}(a_1 \cos(\omega t) - b_1 \sin(\omega t)), \quad \mathrm{Im}\{x(t)\} = e^{\sigma t}(a_1 \sin(\omega t) + b_1 \cos(\omega t)).$$

This shows that by choosing appropriate values for a_1, b_1, σ, and ω, we can generate a wide variety of signals, including sinusoids, exponentials, and combinations thereof.

Next steps

Next class: properties of systems (time invariance, linearity, causality, memory/memoryless, invertibility), responses of LTI systems, and how complex exponentials act as eigenfunctions of LTI systems.

Pop Quiz Solutions

Pop Quiz 4.1: Solution(s)

The two signals are the real and imaginary parts of the complex exponential: $\mathrm{Re}\{x(t)\} = \cos(\omega t)$ and $\mathrm{Im}\{x(t)\} = \sin(\omega t)$.

Pop Quiz 4.2: Solution(s)

Set $\omega = 0$ and $\sigma = 0$ to get $x(t) = A = k$. For real k, choose $A \in \mathbb{R}$.

Pop Quiz 4.3: Solution(s)

Set $\sigma > 0$, $\omega = 0$, and $A \in \mathbb{R}$ to get $x(t) = Ae^{\sigma t}$, which grows unbounded as $t \to \infty$.

Pop Quiz 4.4: Solution(s)

Set $\sigma > 0$, $\omega \neq 0$, and $A = |A|e^{j\phi}$ to get $x(t) = |A|e^{\sigma t}\cos(\omega t + \phi)$, which grows unbounded as $t \to \infty$.

EE 102 Week 3, Lecture 2 (Fall 2025)

Instructor: Ayush Pandey

Date: September 17, 2025

1 Goals for today

- Review (and wrap up) the introduction to signals and systems.
- Recall the differences between signals and systems and examples of each.
- Properties of signals to design systems: amplifier, frequency modulator, and more.
- Understand the properties of systems: time invariance, linearity, causality, memory, invertibility.
- Analyze system response using linearity and time-invariance.

2 Review: Signals and Systems

Recall that signals are mathematical functions that represent physical quantities, or information. Usually, we may think of signals as functions of time, but they can also be functions of space or other variables. A simple rule of thumb to remember is that signals are mathematical functions (that you can also visualize, if they are 1D or 2D).

On the other hand, you can easily distinguish systems from signals because systems are "processors" of signals. Every quantity/information/measurement is a signal, and systems can process a signal to produce an output signal. The signal that gets processed by the system is the input signal and the processed signal is the output signal. Due to this, a common abstraction of a system is a "black box" that takes in an input signal and produces an output signal. Formally, we write a system as an operator (or a mapping) that takes in a signal and produces another signal:

$$H : x(t) \mapsto y(t)$$

where $x(t)$ is the input signal, $y(t)$ is the output signal, and H is the system. Note that the above does not imply that $x(t) = y(t)$, and in fact, if the system is doing something meaningful, the output signal will be different from the input signal. System examples:

2.1 Example: A matrix as a system

Given an input vector $x \in \mathbb{R}^n$, a matrix $A \in \mathbb{R}^{m\times n}$ can be thought of as a system that produces an output vector $y \in \mathbb{R}^m$ as follows:

$$y = Ax$$

where A is the system that performs the matrix multiplication operation. The block diagram of the system can be represented as:

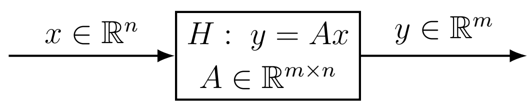


Figure 1: A system as a linear map implemented by matrix multiplication.

2.2 Example: An amplifier as a system

An amplifier is a system that takes in an input signal $x(t)$ and produces an output signal $y(t)$ by amplifying the input signal by a constant factor $\alpha \in \mathbb{R}$. The system can be represented as:

$$H : x(t) \mapsto y(t) = \alpha x(t)$$

where H is the system that amplifies the input signal by the factor α. We can draw a block diagram of the system as follows:

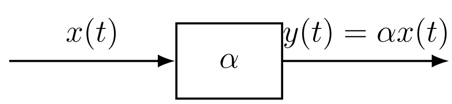


Figure 2: Amplifier system: multiply the input by a real gain α.

2.3 Example: A differential equation as a system

A system can also be represented by a differential equation that relates the input signal $x(t)$ and the output signal $y(t)$. For example, consider the following first-order linear differential equation:

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

where a and b are constants. This equation describes a system that takes in the input signal $x(t)$ and produces the output signal $y(t)$ by solving the differential equation. In this case, the block diagram of the system can be represented as:

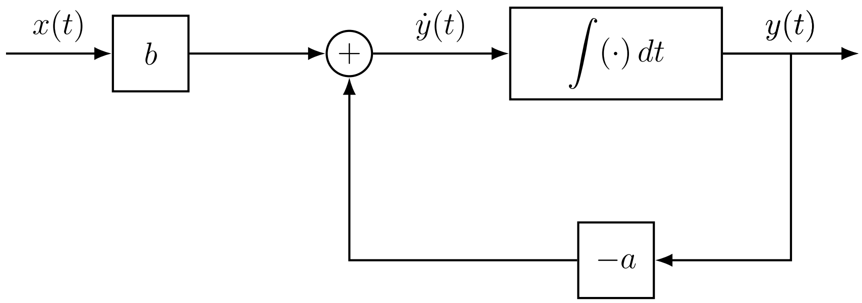


Figure 3: Realization of $\dot{y}(t) + a\, y(t) = b\, x(t)$.

2.4 Example: A frequency modulation system

A frequency modulation (FM) system is a system that takes in an input signal $x(t)$ and produces an output signal $y(t)$ by modulating the frequency of a carrier signal.

Pop Quiz 2.1: Check your understanding!

For a purely sinusoidal audio signal $x(t) = A\cos(\omega_0 t)$, define a frequency modulator system that produces an output signal $y(t)$ that has a different frequency than the input signal. What are various ways in which you can achieve this? Can you think of a general approach to create such a system by leveraging the properties of the complex exponential signal (see previous chapter of the notes)?

Solution on page 9

Consider a specific example where $x(t) = e^{j2\pi(10t)}$ and we send this input through a modulator system that multiplies $e^{j2\pi(32t)}$ to the input signal. The output can be obtained by computing the shift in frequency (exponents get added on multiplication). The output signal $y(t)$ for both the real and imaginary parts of the complex exponential modulator system is shown in Figure 4.

An audio signal can be thought of as a combination of many sinusoidal signals with different frequencies. So, we can leverage the properties of the general complex exponential signal to define the input audio signal $x(t)$ as a sum of complex exponentials:

$$x(t) = \sum_k \mathrm{Re}\left(A_k e^{j\omega_k t}\right)$$

where A_k and ω_k are the amplitude and frequency of the k-th sinusoidal component of the audio signal. Note that for simplicity, we are just considering the cosine component. To keep it general, we can continue to work with complex numbers as it allows us to take into account both sine and cosine components (orthogonally phase shifted by $\pi/2$, which is important in representing a general signal). If we define our system as a multiplication of the input signal

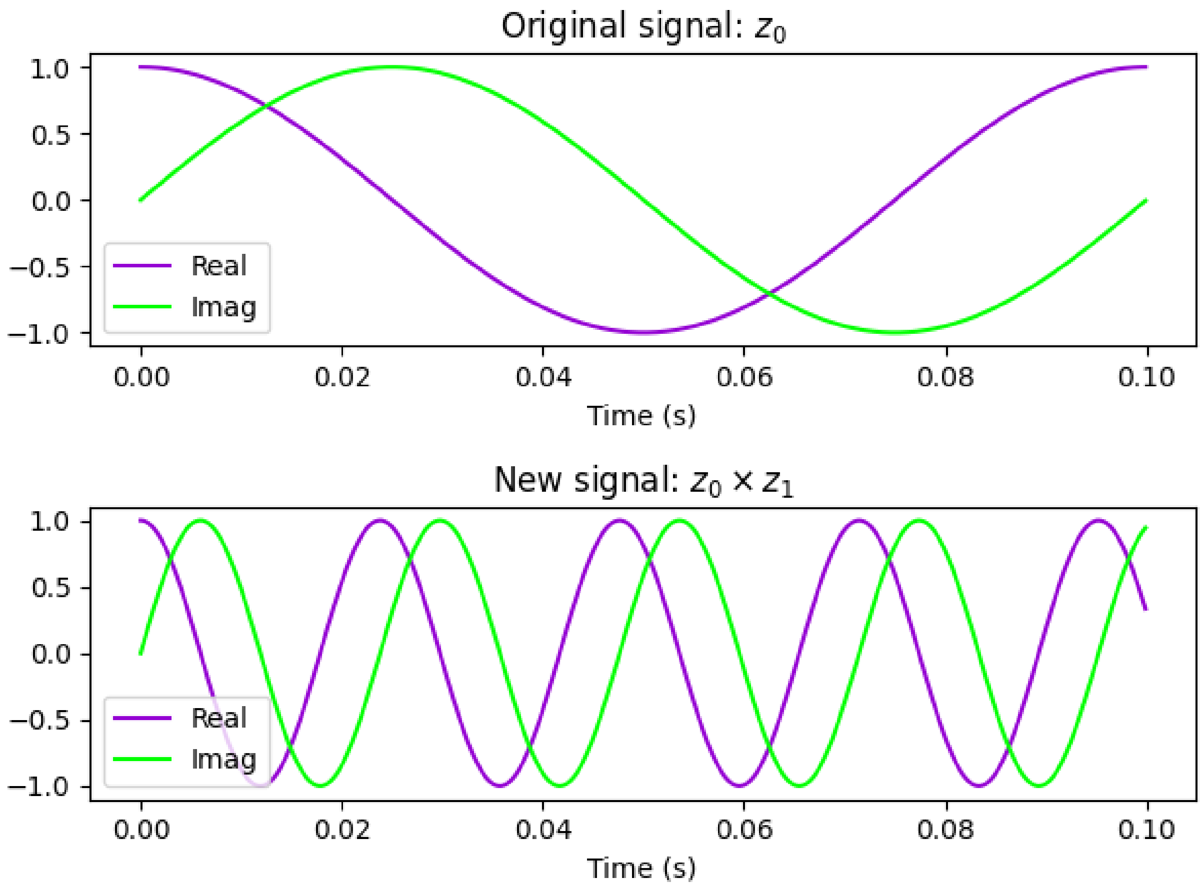


Figure 4: Frequency modulation of a sinusoidal signal using multiplication with a complex exponential.

with $\mathrm{Re}\{Ae^{j\omega_0 t}\}$, we will see that the output signal has a frequency that is shifted by ω_0 from the input signal. The system can be represented as:

$$H : x(t) \mapsto y(t) = x(t) \cdot \mathrm{Re}\{Ae^{j\omega_0 t}\}$$

where H is the system that modulates the frequency of the input signal by multiplying it with a carrier signal $\mathrm{Re}\{Ae^{j\omega_0 t}\}$. In this case, the output $y(t)$ can be written as a sum of sinusoidal signals with frequencies shifted by ω_0:

$$y(t) = \sum_k \mathrm{Re}\left(A_k e^{j(\omega_k + \omega_0)t}\right)$$

For a real audio signal that you can find on course Github: guitar_clean.wav, we can create the simple frequency modulator using the multiplication of a complex exponential. The input and the output signals are shown in Figure 5. You can interactively visualize and interpret the frequency of the output signal by running the virtual manipulator on frequency modulator using complex exponentials: VM_modulation.py.

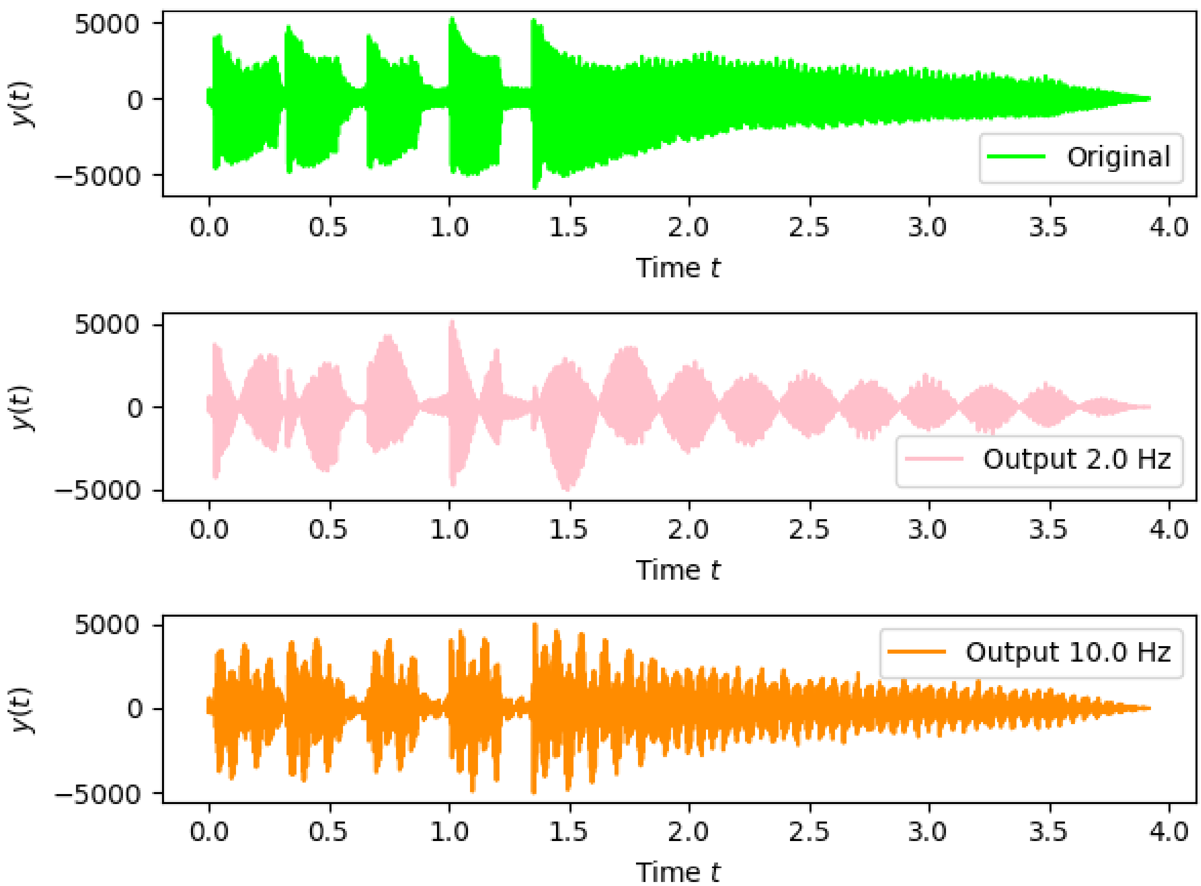


Figure 5: Frequency modulation of a guitar audio signal using multiplication with a complex exponential.

Pop Quiz 2.2: Check your understanding!

By using the virtual manipulator VM_modulation.py, test three different frequencies of the modulator system: 0.5 Hz, 5 Hz, and 20 Hz. Reflect on the expected sound for each modulation before listening to the output `guitar_modulated.wav`. Which one of the output signals is closest to the original audio signal? How would you describe the frequency modulation effect on audio?

Solution on page 9

Pop Quiz 2.3: Check your understanding!

Can you define a noise cancellation system that takes in a noisy signal and produces an output signal that cancels the noise? Assume that the noise is a sinusoidal signal with a known frequency and amplitude.

Solution on page 9

3 Properties of Systems

Systems can have various properties that define their behavior and characteristics. Some of the most common properties of systems are:

- **Linearity:** A system is linear if it satisfies the principles of superposition and homogeneity. This means that the output of the system for a linear combination of input signals is equal to the same linear combination of the outputs for each individual input signal.
- **Time-invariance:** A system is time-invariant if its behavior and characteristics do not change over time. This means that if the input signal is shifted in time, the output signal will also be shifted by the same amount.
- **Causality:** A system is causal if the output at any given time depends only on the current and past input values, and not on future input values. In other words, a causal system cannot anticipate future inputs.
- **Memory:** A system has memory if its output depends on past input values. A system without memory (also called memoryless) produces an output that depends only on the current input value.
- **Invertibility:** A system is invertible if there exists an inverse system that can recover the original input signal from the output signal. In other words, if you apply the inverse system to the output, you should get back the original input.
- **Stability:** A system is "BIBO" stable if bounded input signals produce bounded output signals. This means that if the input signal remains within a certain range, the output signal will also remain within a certain range (bounded input implies bounded output: BIBO).

Pop Quiz 3.1: Check your understanding!

Can you find a positive and a negative system example of each of the properties listed above? For example, for the memory property, the "positive" would be a system that has memory, and the "negative" would be a system that is memoryless.

Solution on page 9

A common type of system that we will end up studying is a causal stable LTI system. Causality is important because we will be studying time-domain signals. To ensure that signals don't blow up to ∞, we require stability. Finally, LTI systems, or linear time-invariant systems, will be the central class of systems we will study due to their nice properties and because many useful practical signals can be modeled as LTI systems.

3.1 Example: LTI system response

By using linearity and time-invariance, we can analyze the response of an LTI system to inputs and initial conditions. Consider the system with three different inputs: $x_1(t)$, $x_2(t)$, and $x_3(t)$. Define the output of the system for each of the signals as $y_1(t)$, $y_2(t)$, and $y_3(t)$ respectively. Let us assume that the system is also affected by its initial conditions $q_1(0)$ and $q_2(0)$.

We denote $y_i^{\text{ZS}}(t)$ as the zero-state output due to $x_i(t)$ with all initial conditions set to zero. Further, we denote $y_{q_1}(t)$ and $y_{q_2}(t)$ as the zero-input outputs produced by initial conditions in those two modes (with external inputs, x_i, set to zero).

Then, for any scalars $\alpha_1, \ldots, \alpha_5$,

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) + \alpha_3 x_3(t), \quad q(0) = \alpha_4\, q_1(0) + \alpha_5\, q_2(0)$$

$$\implies y(t) = \alpha_1 y_1^{\text{ZS}}(t) + \alpha_2 y_2^{\text{ZS}}(t) + \alpha_3 y_3^{\text{ZS}}(t) + \alpha_4 y_{q_1}(t) + \alpha_5 y_{q_2}(t).$$

Note that it is essential that y_i^{ZS} are computed with zero initial conditions otherwise the initial contributions would be double-counted. Likewise, y_{q_1}, y_{q_2} are computed with zero external input.

Further, if one of the inputs is shifted in time, say $x_3(t - t_0)$, then by the time-invariance property of the system, the output is also shifted by the same amount:

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) + \alpha_3 x_3(t - t_0), \quad q(0) = \alpha_4\, q_1(0) + \alpha_5\, q_2(0)$$

$$\implies y(t) = \alpha_1 y_1^{\text{ZS}}(t) + \alpha_2 y_2^{\text{ZS}}(t) + \alpha_3 y_3^{\text{ZS}}(t - t_0) + \alpha_4 y_{q_1}(t) + \alpha_5 y_{q_2}(t).$$

3.2 Example: LTI system properties to compute system response

Consider a continuous-time LTI system with input $x(t)$ and output $y(t)$. We have the following information about the system:

$$\begin{aligned} x_1(t) &= \cos(t) & &\longrightarrow & y_1(t) &= \tfrac{1}{10}\cos(t) - \tfrac{3}{10}\sin(t), \\ x_2(t) &= \cos(2t) & &\longrightarrow & y_2(t) &= \tfrac{1}{5}\cos(2t) + \tfrac{3}{10}\sin(2t), \\ x_3(t) &= \delta(t) & &\longrightarrow & y_3(t) &= \delta(t), \end{aligned}$$

Then, for a new input

$$x_{\text{new}}(t) = \cos(2t) - 10\sin(t+10),$$

our goal is to find $y_{\text{new}}(t)$ using only linearity and time invariance properties of the system.

Let us start by noting that since $\sin(t) = \cos\big(t - \frac{\pi}{2}\big)$, we can use time-invariance of the system to write the output for $\sin(t)$ (say $y_4(t)$) as

$$y_4(t) = y_1\Big(t - \frac{\pi}{2}\Big) = \tfrac{1}{10}\sin(t) + \tfrac{3}{10}\cos(t).$$

Then, the output to $\sin(t+10)$ is (say $y_5(t)$), again due to time-invariance is obtained by shifting $y_4(t)$ by 10:

$$y_5(t) = \tfrac{1}{10}\sin(t+10) + \tfrac{3}{10}\cos(t+10)$$

Putting it all together using the linearity of the system, we get the desired output $y_{\text{new}}(t)$:

$$y_{\text{new}}(t) = \tfrac{1}{5}\cos(2t) + \tfrac{3}{10}\sin(2t) \;-\; \sin(t+10) \;-\; 3\cos(t+10)\,.$$

Finally, if the system has an initial condition at $t = 0$, superpose the corresponding measured responses:

$$y_{\text{new,IC}}(t) = y_{\text{new}}(t) \;+\; a\,\delta(t)\,,$$

with $a \in \mathbb{R}$ set by the magnitude of the initial condition.

Pop Quiz Solutions

Pop Quiz 2.1: Solution(s)

If you define the system as a multiplication of the input signal with $\text{Re}\{Ae^{j\omega_0 t}\}$, you will see that the output signal has a frequency that is shifted by ω_0 from the input signal.

Pop Quiz 2.2: Solution(s)

The 0.5 Hz modulation is closest to the original audio signal. The frequency modulation effect on audio can be described as a shift in the frequency spectrum of the original signal, resulting in a change in the perceived pitch and timbre of the sound, which ends up sounding like a reverberation.

Pop Quiz 2.3: Solution(s)

See Problem 1 on Homework 3.

Pop Quiz 3.1: Solution(s)

Here are some examples:

- Linearity: Positive - Amplifier system $y(t) = \alpha x(t)$; Negative - Clipping system $y(t) = \text{clip}(x(t))$ is nonlinear.
- Time-invariance: Positive - Delay system $y(t) = x(t - t_0)$; Negative - Time-varying gain system $y(t) = tx(t)$.
- Causality: Positive - Delay system $y(t) = x(t - t_0)$; Negative - Anticipatory system $y(t) = x(t + t_0)$ for $t_0 > 0$.
- Memory: Positive - Integrator system $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$; Negative - Memoryless system $y(t) = x(t)^2$ only depends on the current instant of time.
- Invertibility: Positive - Scaling system $y(t) = 2x(t)$; Negative - Clipping system $y(t) = \text{clip}(x(t))$ cannot be inverted as all values lead to the same clipped behavior after a certain threshold.
- Stability: Positive - An RC circuit with decaying voltage: $y(t) = (1 - e^{-t/\tau})x(t)$; Negative - Unstable system (e.g., $y(t) = e^t x(t)$).

EE 102 Week 4, Lecture 1 (Fall 2025)

Instructor: Ayush Pandey

Date: September 22, 2025

1 Goals for today

To understand how to compute the output of a linear-time invariant system given any general input signal that is applied to the system.

2 Representing signals using impulses

We have discussed the three fundamental signals: the unit step, the unit impulse, and the complex exponential. What's common about all three of these functions? Why are these three signals so special? The answer is that these three signals can be used to represent any arbitrary signal (well, almost any!) — especially the signals that we come across in engineering practice. In the previous lectures, we have discussed how to write general signals only by using unit steps and complex exponentials. We have also briefly discussed how the definition of the unit impulse (in the sense of distributions), is in fact, a definition that says that each signal is made up of its samples at each time point and the impulse is the function that does the sampling. Let's try to break this down further with some examples.

2.1 Discrete-time step using impulse

To help you build intuition for this concept, consider a signal in discrete-time. For example, the unit step signal $u[n]$ is defined as

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

which can be visualized as a train of unit impulses as shown in Figure 1. It is clear that it is a signal that is made up of many impulses (see pop quiz below).

Pop Quiz 2.1: Check your understanding!

In discrete-time, write the unit step signal $u[n]$ as a sum of scaled and shifted unit impulses $\delta[n]$.

Solution on page 7

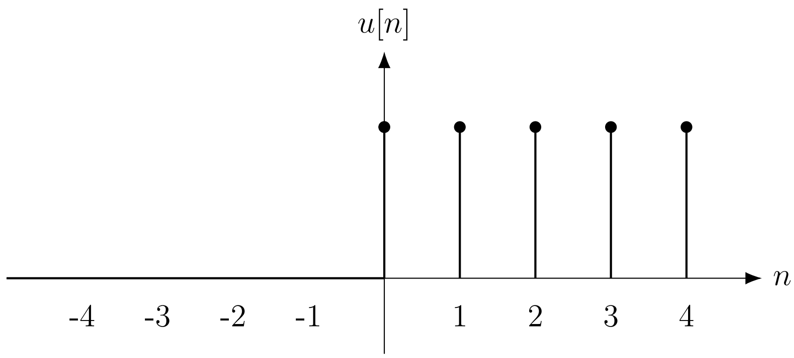


Figure 1: Unit step signal $u[n]$ visualized as a train of unit impulses.

We see that the unit step can be viewed as many impulses (one impulse at each time point). This is also true for every other signal — the simplest way to break down a signal is to view it as a sample at each time point. The unit impulse helps us pick out the samples at each time point so that we can write the logic above formally. A visualization of this idea is shown in Figure 2.

2.2 Impulse as a sampler in continuous-time

Recall the sampling property of the impulse in continuous-time:

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0).$$

This property tells us that the impulse function $\delta(t-t_0)$ acts as a sampler that picks out the value of the signal $x(t)$ at the specific time $t = t_0$. This means that we can represent any continuous-time signal $x(t)$ as an integral of scaled and shifted impulses.

2.3 Example: Representing a sinusoidal signal using impulses

Let's work through this with a specific example of a sinusoidal signal: $x(t) = \sin(t)$. We can write $\sin(0)$ as

$$\sin(0) = \int_{-\infty}^{\infty} \sin(\tau)\delta(\tau - 0)d\tau.$$

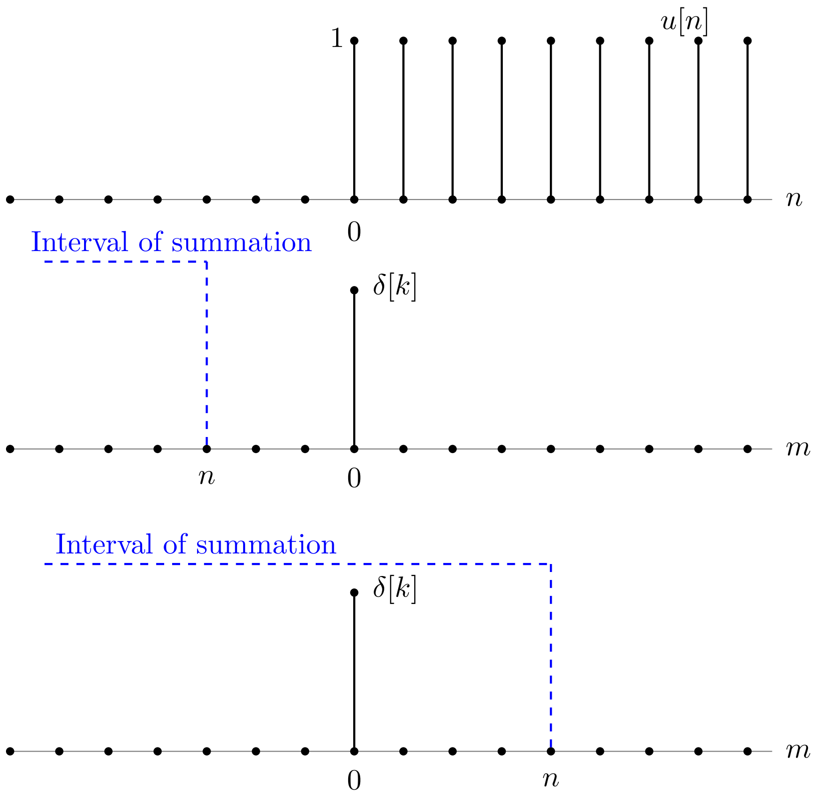


Figure 2: Visualization of the running sum representation of the unit step signal $u[n]$ as a sum of shifted impulses.

Similarly, we can write $\sin(t_1)$ for any arbitrary time t_1 as

$$\sin(t_1) = \int_{-\infty}^{\infty} \sin(\tau)\delta(\tau - t_1)d\tau.$$

This means that we can represent the entire signal $\sin(t)$ as an integral of scaled and shifted impulses:

$$\sin(t) = \int_{-\infty}^{\infty} \sin(\tau)\delta(\tau - t)d\tau.$$

Now, this might seem trivial or even circular! But the key insight here is that we are expressing the signal as a continuous superposition of impulses, each weighted by the value of the signal at that point in time.

2.4 General representation of signals using impulses

In general, any continuous-time signal $x(t)$ can be represented using impulses. Let's visualize an arbitrary signal $x(t)$ and see how it can be decomposed into impulses at each time point (see Figure 3).

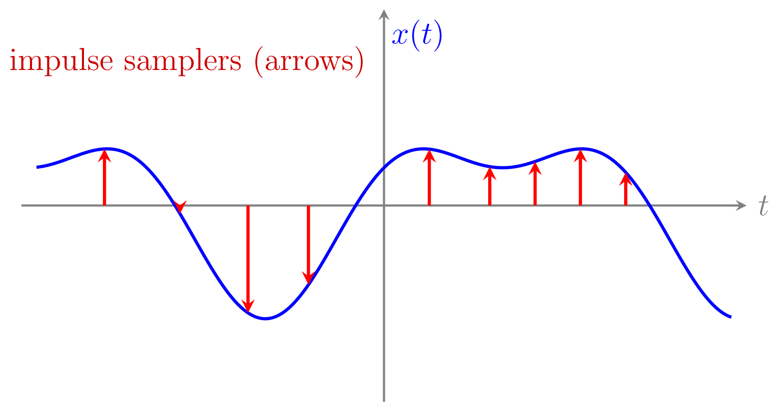


Figure 3: Impulse samplers drawn as arrows from the time axis to the signal $x(t)$ (arrowheads at the curve).

At each point, the impulse signal can be written by shifting the impulse to that point and scaling it by the value of the signal at that point. Therefore, we start with $\delta(t)$ as the impulse at $t = 0$. To get the impulse at an arbitrary time $t = \tau$, we shift the impulse to that point, which gives us $\delta(t - \tau)$. To scale this impulse by the value of the signal at that point, we multiply it by $x(\tau)$, resulting in $x(\tau)\delta(t - \tau)$. So, we have

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau.$$

3 General output of an LTI system using impulse response

We finally get to the main goal of this lecture. Our goal is to compute the output of systems given any arbitrary input signal. This is very important because in practice, we often encounter signals that are not as simple as the unit step or the complex exponential. We need a systematic way to compute the output of a system for any input signal. Think about the following examples, where we might want to compute the output of a system:

- An audio signal (your recorded voice singing a song, for example) that needs to be filtered to remove noise. How would you represent your filtered signal mathematically (to be able to analyze and report back its properties to your music producer!)?

- For an integrated circuit, you might want to compute the output voltage of a circuit given an arbitrary input voltage signal (in real-world, the voltage signal is never a perfect sinusoid!).
- In communications, you might want to compute the output of a communication channel given an arbitrary input signal (for example, a modulated signal carrying information).

To achieve all of the above, we start by describing the impulse response of the system.

3.1 Impulse response of an LTI system

We define the impulse response of a system as the output of the system when the input is an impulse signal. In continuous-time, if the system input is $\delta(t)$, then we call the output of this system, the impulse response and denote it as $h(t)$ (see Figure 4). Now, the question

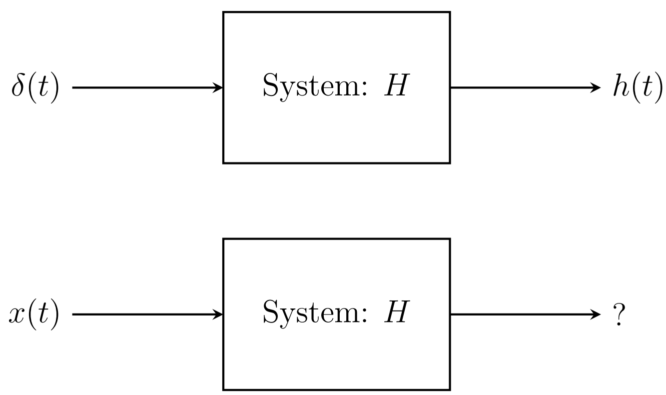


Figure 4: Impulse response of a system: output $h(t)$ when input is $\delta(t)$. What is the output for a general input $x(t)$?

is, what is the output of the system when the input is an arbitrary signal $x(t)$?

Pop Quiz 3.1: Check your understanding!

What is the output of the system for a shifted impulse input $\delta(t-\tau)$? Do you need additional assumptions about the system to answer this question?

Solution on page 7

Pop Quiz 3.2: Check your understanding!

What is the output of the system for a scaled impulse $k\delta(t)$? Do you need additional assumptions about the system to answer this question?

Solution on page 7

We start by writing the output of the system for a shifted and scaled impulse input. That is, if the input is $x(\tau)\delta(t-\tau)$, then the output will be $x(\tau)h(t-\tau)$ (only if, of course, the system is linear and time-invariant!). Now, we can write the output of the system for an arbitrary input $x(t)$ (see Figure 5) by using the integral representation of the input signal using impulses: So, using the integral representation of the input signal, we can write the

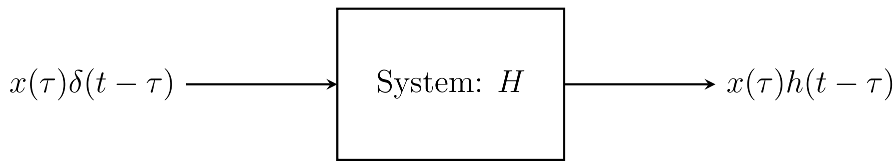


Figure 5: Impulse response of a system: output $h(t)$ when input is $\delta(t)$. What is the output for a general input $x(t)$?

output of the system as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

This integral is called the convolution integral, and it is denoted by the symbol $*$. Therefore, we can write the output of the system as

$$y(t) = x(t) * h(t).$$

Pop Quiz 3.3: Check your understanding!

Write the discrete-time version of the convolution integral.

Solution on page 7

4 Virtual manipulatives to understand convolution

In the next lecture, we will use virtual manipulatives to understand convolution better. We will also discuss some properties of convolution and how to compute convolution using graphical methods (with many practical examples!).

Pop Quiz Solutions

Pop Quiz 2.1: Solution(s)

Notice that for any given value of n, the unit step $u[n]$ is equal to 1 if $n \geq 0$ and 0 otherwise. This means that the unit step can be viewed as a sum of all the unit impulses $\delta[k]$ for k from $-\infty$ to n. So, we can write

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

for all $n \in \mathbb{Z}$. To represent the signal in terms of the general impulse $\delta[n]$, we can write:

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k].$$

You can see that for any given n, the sum will only include the impulses from $k = 0$ to $k = n$, which corresponds to the definition of the unit step.

Pop Quiz 3.1: Solution(s)

To answer this question, we need to assume that the system is time-invariant. This means that if the input is shifted in time, the output will also be shifted by the same amount. Therefore, if the input is $\delta(t-\tau)$, the output will be $h(t-\tau)$.

Pop Quiz 3.2: Solution(s)

To answer this question, we need to assume that the system is linear. This means that if the input is scaled by a constant factor, the output will also be scaled by the same factor. Therefore, if the input is $k\delta(t)$, the output will be $kh(t)$.

Pop Quiz 3.3: Solution(s)

In discrete-time, the convolution integral becomes a sum. Therefore, we can write the output of the system as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

This sum is called the convolution sum, and it is also denoted by the symbol $*$.

EE 102 Week 5, Lecture 1 (Fall 2025)

Instructor: Ayush Pandey

Date: September 29, 2025

1 Goals

2 Review: LTI systems and convolutions

Recall that using the linearity and time-invariance of the system, we can define the output, $y(t)$ of the system to any arbitrary input $x(t)$ in terms of the impulse response of the system $h(t)$ using the following integral:

$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau. \tag{1}$$

This integral is called the convolution integral.

Pop Quiz 2.1: Check your understanding!

Using the convolution integral, show that you recover the impulse response $h(t)$ when the input is $\delta(t)$. As a consequence, you will have proven that the convolution of a signal (in this case, $h(t)$) with an impulse is equal to the same signal.

Solution on page 8

3 Time-domain system response

In this section, we discuss the response of systems in time-domain. We focus our discussion on linear systems and their responses. The overarching message about linear systems is the following: if you *know* how the system responds under a given condition or an input, then you can construct the system output if any linear combination of the known conditions occur. In other words, you can use the isolated system response to obtain the output to a new input that is a linear combination of the inputs for which you have the data already. This is called the *principle of superposition* (note that the above is only an informal description).

We often find it useful to talk about two kinds of system outputs: (1) the "natural" or the characterisitic output of the system when there are no forcing inputs, and (2) the output of the system to forced inputs. For linear systems, we can add these two outputs to get the full output of the system when both are present simultaneously.

3.1 Building intuition for a system's responses

Consider yourself — a student invested in learning new things — as a system. As you stroll past the lakes in and around the campus, you must have observed that the number of birds increases through late fall and winter and then thins from late spring into the hot, dry summer. Without anyone explicitly teaching you the ecology of bird movement, you self-learn and update your knowledge about bird movement as you observe these patterns and correlate them with the season. This is your natural response as a "system" that is invested in learning new things. Simultaneously, if you enroll in an ecology class as you expand your general education, you might be "forced" to learn (due to the pressure of exams!) that the Central Valley lies on the Pacific Flyway. Large flocks of geese, cranes, and ducks concentrate here from roughly October through February and by early summer many waterfowl have departed north to breeding grounds. This learning will be your response to the external input (the instruction in the ecology course). Your overall learning (if your learning progresses linearly) is the sum of your self-learned concepts and the concepts from the course. In this case, a nonlinear learner can have an advantage — one who accentuates their overall learning by synthesizing new (extra) knowledge by combining the natural (self-learning) and forced learning in innovative ways.

We conclude by writing the output of any general linear system as a combination of the natural/characteristic/initial condition response and the forced response. For a system with initial condition x_0 and an input $u(t)$, we write the output $y(t)$ as

$$y(t) = y_0(t) + y_{\text{forced}}(t)$$

where $y_0(t)$ is the initial condition response to initial condition x_0. This is the characteristic response of the system without any forced inputs (the self-learning by seeing initial conditions, in the example above). Finally, $y_{\text{forced}}(t)$ is the forced response, that is, the output of the system when only the forced input $u(t)$ is present in isolation (the forced course-based learning, in the example above).

3.2 Example: Analyzing an RC circuit using convolution

Consider a series RC circuit shown in the diagram 1. Assume that we have an input voltage $u_{\text{in}}(t)$ that can be applied to the circuit using a function generator and the capacitor can have an initial voltage v_0 volts at time $t = 0$. We denote the output voltage for the circuit as

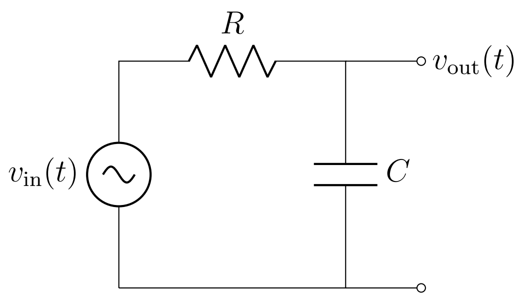


Figure 1: An RC circuit with input voltage $v_{\text{in}}(t)$ and output voltage $v_{\text{out}}(t)$.

the voltage across the capacitor $v_{\text{out}}(t)$. A typical analysis of such circuits uses differential equations to describe and compute the system response (recall pre-requisite #3 where you solved a differential equation to solve this circuit). Here, we will use convolution to find the output of the system to various common types of inputs:

- an impulse input at $t = 0$
- a step input voltage (modeling DC input)
- a sinusoidal voltage input (modeling AC input)
- a generalized complex exponential input signal

Pop Quiz 3.1: Check your understanding!

Before we jump into the application of convolution to the RC circuit example, it is important to ensure that the assumptions for the convolution integral to hold are still met. Your task is to state these assumptions (linearity and time-invariance) and prove that the RC circuit is linear and time-invariant. Additionally, also prove that the RC circuit is a causal system.

Solution on page 8

3.2.1 An impulse input to an RC circuit

An impulse at $t = 0$ is simply given by $v_{\text{in}}(t) = v_0\delta(t)$, where v_0 is the magnitude of the impulse (the area under the curve of this impulse is v_0). That is,

$$\int_{-\infty}^{\infty} v_{\text{in}}(t)dt = v_0.$$

From the pop quiz at the beginning of this lecture, you know that the output of the system (with zero initial conditions) to an impulse is just the impulse response. Since the system is linear (see pop quiz above), the output of the system to a scaled impulse $v_0\delta(t)$ will be equal to

$$y(t) = v_0 h(t)$$

where v_0 is the magnitude of the input impulse (the area under curve). To find the impulse response of the RC circuit, we will have to rely on our circuits knowledge — signal processing can only get us so far! From circuit theory, we know that an initial impulse on the circuit will cause a jump in the voltage and then an exponential decay (with a time constant of RC) through the resistor in the circuit. Full proof of this fact can be found in the solution of the pop-quiz below. We have the impulse response of an RC circuit (response to unit impulse)

$$h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t) \tag{4}$$

Therefore, the output to the scaled impulse is $y(t) = \frac{v_0}{RC} e^{-\frac{t}{RC}} u(t)$.

Pop Quiz 3.2: Check your understanding!

Prove that the impulse response of an RC circuit is given by equation (4).

Solution on page 9

3.2.2 A step input to an RC circuit

Let's compute the forced input response of the RC circuit for a scaled step input that models a DC voltage applied to the circuit $x(t) = v_{\mathrm{DC}} u(t)$. By applying the convolution integral, we can compute the system output $y(t)$ as

$$y(t) = \int_{-\infty}^{\infty} v_{\mathrm{DC}} u(\tau) h(t - \tau) d\tau$$

which can be evaluated as

$$y(t) = v_{\mathrm{DC}} \int_{0}^{t} h(t - \tau) d\tau$$

since the step function is zero for $\tau < 0$ and $h(t - \tau)$ is zero for $\tau > t$ (causality of the real-life RC circuit's voltage response). To further understand this, you can see that the step input of the DC voltage is only applied at time $t = 0$. So, anything for negative time is 0 (**important note:** we must account for this by multiplying a $u(t)$ to the final output expression!). Similarly, you will see that the impulse response of the system will be zero for any negative time. Therefore, we can limit the integration limits to 0 and t. Next, we can substitute the impulse response from equation (4) to get

$$y(t) = \frac{v_{\mathrm{DC}}}{RC} \int_0^t e^{-\frac{t-\tau}{RC}} d\tau.$$

Evaluating this integral, we get

$$y(t) = v_{\mathrm{DC}} \left(1 - e^{-\frac{t}{RC}}\right) u(t). \tag{5}$$

This is the forced response of the RC circuit to a step input voltage.

Pop Quiz 3.3: Check your understanding!

Verify the above equation from your circuit theory notes. Does it match up with the the output of an RC circuit when a DC voltage is applied as input?

Solution on page 10

3.2.3 A sinusoidal input to an RC circuit

We can compute the forced response of the RC circuit to a sinusoidal input voltage $x(t) = v_{\mathrm{AC}} \cos(\omega t) u(t)$. Using the convolution integral, we can write

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau.$$

Substituting the expressions for $x(t)$ and $h(t)$, we get

$$y(t) = \int_0^t v_{\mathrm{AC}} \cos(\omega\tau) \frac{1}{RC} e^{-\frac{t-\tau}{RC}} d\tau. \tag{6}$$

Evaluating this integral will give us the forced response of the RC circuit to the sinusoidal input. This is same as the pre-requisite #3 problem set! As you might remember, integrating the above requires a bit of effort as it is an integration by parts. Keep reading this section to learn a much easier way to do the same.

3.2.4 A generalized complex exponential signal input to an RC circuit

For a complex exponential signal, note that we are aiming for analysis of a broader variety of possible real signals. As we have seen before, complex exponentials can be used to define sinusoids, exponentials, and their combinations. By isolating the real and imaginary parts as needed, we can get a large class of signals as a subset of the general complex exponential signal $x(t) = Ae^{st}$ where A and s are complex numbers. We can apply the convolution integral (1) to compute the output of the signal $y(t)$ to this input. As expected, the output will also be a complex signal and its special cases will lead to the computation of the output of the system for the input signal represented in that special case. For zero initial conditions, we write the convolution equation for this input as

$$y(t) = \int_0^t Ae^{s\tau}\,\frac{1}{RC}e^{-\frac{t-\tau}{RC}}\,d\tau = \frac{A}{RC}\,e^{-t/(RC)}\int_0^t e^{\left(s+\frac{1}{RC}\right)\tau}\,d\tau.$$

Evaluating the integral,

$$y(t) = \frac{A}{RC}\,e^{-t/(RC)}\,\frac{e^{\left(s+\frac{1}{RC}\right)t}-1}{s+\frac{1}{RC}} = A\,\frac{e^{st}-e^{-t/(RC)}}{1+sRC}\,u(t).$$

By further manipulating the above, we can see that it is made up of two parts:

$$y(t) = Ae^{st}\,\frac{1}{1+sRC}\,u(t) - A\,\frac{e^{-t/(RC)}}{1+sRC}\,u(t). \tag{7}$$

This is an interesting result due to many properties that we can observe from the output:

Observation 1: The system is linear. We observe the output that the output has the same "shape" as the input. Because the system is linear, the complex exponential input Ae^{st} reappears in the output multiplied by a constant factor $\frac{1}{1+sRC}$. That first term,

$$y_{\text{forced}}(t) = A\,\frac{e^{st}}{1+sRC}\,u(t),$$

is the *forced (particular) response*. This is what you would derive by integrating the differential equation by parts! Here, we did the same but in two lines of algebra and a very simple integration instead — the power of exponentials!

Observation 2: The transient term decays to 0. The second term,

$$y_{\text{transient}}(t) = -A\,\frac{e^{-t/(RC)}}{1+sRC}\,u(t),$$

is the *transient (natural) response*. It always decays like $e^{-t/(RC)}$, so for large t the output approaches the forced response, the transient part will disappear to 0 as $t \to \infty$.

Observation 3: As promised, the complex exponential is useful because it specializes to many other simple cases we derived individually above.

Special case 1: **Unit step.** Note that by setting $s = 0$, $A = v_{\text{DC}}$, we get

$$y(t) = v_{\text{DC}}\big(1 - e^{-t/(RC)}\big)u(t),$$

which matches the step-response derived above in equation (5).

Special case 2: Real exponential. Although we did not derive this earlier, we can now easily compute the output of the RC circuit to an exponential input $x(t) = Ae^{\alpha t}$. For this, set $s = \alpha \in \mathbb{R}$, and choose A to be a real in the general form. Then, we have

$$y(t) = A\,\frac{e^{\alpha t} - e^{-t/(RC)}}{1 + \alpha RC}\,u(t).$$

Pop Quiz 3.4: Check your understanding!

Remind yourself and derive the sine signal from the complex exponential: Prove that a real input signal $x(t) = A_{\mathrm{AC}}\sin(\omega t)$ can be obtained from the general complex exponential $x_c(t) = Ae^{st}$ by taking an appropriate real (or imaginary) part. In particular, show that $x(t)$ is the imaginary part of the general complex exponential for a suitable choice of A and s.

Solution on page 10

Special case 3: Sinusoidal input. With $s = j\omega$, A real (see pop quiz above), take the real part of

$$y(t) = A\,\frac{e^{j\omega t} - e^{-t/(RC)}}{1 + j\omega RC}\,u(t).$$

For large t, the transient dies out and the output is a sinusoid with scaled amplitude and a phase lag, consistent with the earlier sinusoidal case (see equation (6)). In fact, note that we did not derive the output in closed form earlier (due to fear of the ugly integration by parts!). Now we have the answer to the same input but obtained in a much easier way.

Observation 4: If the input grows ($\mathrm{Re}\{s\} > 0$), the forced term grows accordingly; the RC remains stable (its transient decays), but a growing input produces a growing output.

Pop Quiz 3.5: Check your understanding!

Is the RC circuit bounded input bounded output stable?

Solution on page 10

4 Next class

In the next class, we will talk about an image processing example using convolution and will visualize convolution further.

Pop Quiz Solutions

Pop Quiz 2.1: Solution(s)

Substitute $x(t) = \delta(t)$ in equation (1) to write

$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} \delta(\tau)h(t-\tau)d\tau.$$

Since $\delta(\tau)$ is non-zero only when $\tau = 0$, we have

$$y(t) = h(t)\int_{-\infty}^{\infty} \delta(\tau)d\tau$$

with $h(t)$ out of the integral since it does not depend on τ (the integration variable). Notice that the integral is equal to 1 (definition of the impulse signal). So, we get the desired result $y(t) = h(t)$. Additionally, notice that $\delta(t) * x(t) = x(t)$ — holds true in general for any signal.

Pop Quiz 3.1: Solution(s)

Hints:
For linearity, prove that the principle of superposition holds for the system. Assume that for inputs u_1 and u_2, the outputs are y_1 and y_2. Then, write the system description using KVL (forward refer to the next pop quiz for a simple derivation of the following):

$$\dot{y}_1(t) + \frac{1}{RC}\, y_1(t) = \frac{1}{RC}\, u_1(t), \tag{8}$$

$$\dot{y}_2(t) + \frac{1}{RC}\, y_2(t) = \frac{1}{RC}\, u_2(t), \tag{9}$$

Now, for an input $k_1u_1+k_2u_2$, find the output y^* and show that it is equal to $k_1y_1+k_2y_2$ by using equations (8) and (9).
For time-invariance, consider one of the input-output pairs, say equation (8) and a time-shifted input $u_1(t-t_1)$. Prove that the output to the time-shifted input is the same as applying time-shift in the original output y_1.
For causality, note that the impulse response (voltage output to an impulse) is zero for all negative time in a RC circuit. You will need to closed-form expression for the impulse response to show this formally.

Pop Quiz 3.2: Solution(s)

Start by writing the KVL for the series RC circuit. The input is the applied source voltage $v_{\text{in}}(t)$ and the output is the capacitor voltage $v_{\text{out}}(t)$. KVL gives

$$v_{\text{in}}(t) = v_R(t) + v_C(t) = R\,i(t) + v_{\text{out}}(t),$$

where $i(t)$ is the series current and

$$v_{\text{out}}(t) = \frac{1}{C}\int i(t)\,dt,$$

with the constant of integration set by the initial capacitor voltage. Differentiating v_{out} yields $i(t) = C\,\dot{v}_{\text{out}}(t)$. Substituting into KVL gives a first-order ODE in one variable:

$$\dot{v}_{\text{out}}(t) + \frac{1}{RC}\,v_{\text{out}}(t) = \frac{1}{RC}\,v_{\text{in}}(t).$$

The homogeneous (natural) solution for arbitrary initial condition $v_{\text{out}}(0^-) = v_0$ is

$$v_{\text{out}}(t) = v_0\,e^{-t/(RC)}.$$

This is the *natural response* of the system due to the initial capacitor voltage.
To compute the *impulse response*, drive the circuit with a unit impulse input and set zero initial voltage:

$$v_{\text{in}}(t) = \delta(t), \qquad v_{\text{out}}(0^-) = 0.$$

Using the ODE above,

$$\dot{v}_{\text{out}}(t) + \frac{1}{RC}\,v_{\text{out}}(t) = \frac{1}{RC}\,\delta(t).$$

Using the integrating factor method, you can write

$$\frac{d}{dt}\big(e^{t/(RC)}\,v_{\text{out}}(t)\big) = \frac{1}{RC}\,e^{t/(RC)}\delta(t).$$

Since the system is causal, $v_{\text{out}}(t) = 0$ for $t < 0$ and the initial voltage is zero, so $v_{\text{out}}(0^-) = 0$. Integrating from 0^- to $t > 0$ gives

$$e^{t/(RC)}\,v_{\text{out}}(t) - 0 = \frac{1}{RC}\int_{0^-}^{t} e^{\tau/(RC)}\delta(\tau)\,d\tau = \frac{1}{RC}.$$

Hence, for $t > 0$, the impulse response is

$$h(t) = v_{\text{out}}(t) = \frac{1}{RC}\,e^{-t/(RC)}.$$

For a general definition of the impulse response that holds for all time, we can include causality using the unit step $u(t)$ function as

$$h(t) = \frac{1}{RC}\,e^{-t/(RC)}\,u(t).$$

Note that the units of the impulse response are 1/seconds so that the convolution $(h * x)(t)$ has units of volts (the units of output signal — $v_{\text{out}}(t)$).

Pop Quiz 3.3: Solution(s)

You should work on finding your circuit theory (ENGR/EE 065) notes and make sure that you understand why the two output expressions match.

Pop Quiz 3.4: Solution(s)

For $s = j\omega$ and $A = A_{\text{AC}}$ and taking the imaginary part, we get

$$\text{Im}\big\{A_{\text{AC}}e^{j\omega t}\big\} = \text{Im}\big\{A_{\text{AC}}\cos(\omega t) + jA_{\text{AC}}\sin(\omega t)\big\} = A_{\text{AC}}\sin(\omega t).$$

where we used the Euler's identity $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$ to expand the complex exponential. This is not the only way to obtain the sine function from the generalized complex exponential. An alternative is to set $s = j\omega$ and $A = -jA_{\text{AC}}$ and take the real part:

$$\begin{aligned}\text{Re}\big\{(-jA_{\text{AC}})e^{j\omega t}\big\} &= \text{Re}\big\{-jA_{\text{AC}}\cos(\omega t) - j^2A_{\text{AC}}\sin(\omega t)\big\}\\ &= \text{Re}\big\{-jA_{\text{AC}}\cos(\omega t) + A_{\text{AC}}\sin(\omega t)\big\}.\end{aligned}$$

The first term is purely imaginary, so its real part is 0, and the second term is real. Hence

$$\text{Re}\big\{(-jA_{\text{AC}})e^{j\omega t}\big\} = A_{\text{AC}}\sin(\omega t).$$

Pop Quiz 3.5: Solution(s)

For any bounded input $x(t)$, we have $|x(t)| < M$ for some finite M and all time t. The output is given by the convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

Since $|x(\tau)| < M$, we can write

$$|y(t)| \le \int_{-\infty}^{\infty} |x(\tau)||h(t-\tau)|d\tau < M\int_{-\infty}^{\infty} |h(t-\tau)|d\tau.$$

Since the impulse response of the RC circuit is given by equation (4), we can evaluate the integral to get

$$|y(t)| < M\int_{0}^{\infty} \frac{1}{RC}e^{-\frac{t-\tau}{RC}}d\tau = M.$$

Therefore, the output is bounded by M for all time t. Hence, the RC circuit is bounded input bounded output stable.
An alternative way is to start from equation (7) to show in a single step that $y(t)$ is bounded as long as the input Ae^{st} is bounded.

EE 102 Week 5, Lecture 2 (Fall 2025)

Instructor: Ayush Pandey

Date: September 29, 2025

1 Goals

The main goal of this lecture is to learn how to visualize the process of convolution using graphs.

2 Review: Convolution definition

Recall that in continuous-time, the output $y(t)$ of an LTI system with input $x(t)$ and impulse response $h(t)$ is given by the convolution integral:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau.$$

Pop Quiz 2.1: Check your understanding!

Prove that convolution is commutative, i.e., show that $x(t) * h(t) = h(t) * x(t)$.

Solution on page 15

3 Discrete time convolution

Similar to the derivation for continuous-time convolution, we can derive the discrete-time convolution sum. Consider a discrete-time LTI system with input $x[n]$, output $y[n]$, and impulse response $h[n]$. Note that for a discrete-time impulse $\delta[n]$, the output is $h[n]$. Recall the sifting property of the discrete-time impulse:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

Using linearity and time-invariance of the system, we can write the output as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

This is the discrete-time convolution sum, denoted by $y[n] = x[n] * h[n]$.

4 Example: A discrete-time echo system

An audio receiver system produces an echo. When excited by a unit impulse, it responds with an echo of magnitude 1 at $n = 0$ that decays exponentially as α^n for $\alpha \in (0, 1)$ until $n = 5$ (that is, for six seconds in total). You may assume that $\alpha = \frac{1}{2}$ for numerical parts. Answer the following:

(A) Sketch the impulse response $h[n]$ and label $h[0], h[1], \ldots, h[5]$.

(B) We want to understand the kind of echo that will be produced when the audio receiver system is excited by a pulse input of unit amplitude lasting three seconds, starting at $n = 0$ and staying at unit amplitude until $n = 3$. Find $y[n]$ for this input using convolution and show your steps.

The impulse response of the system is

$$h[n] = \begin{cases} \alpha^n, & 0 \leq n \leq 5, \\ 0, & \text{otherwise}, \end{cases}$$

The input is a unit amplitude tone that starts at $n = 0$ and lasts three seconds. So, we can write the pulse signal for the input $x[n]$ as

$$x[n] = u[n] - u[n-3] = \begin{cases} 1, & n = 0, 1, 2, \\ 0, & \text{otherwise}. \end{cases}$$

Now, we can compute the output $y[n]$ using convolution:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k],$$

and give $y[n]$ explicitly for all n where it is nonzero. It is important that we are careful about all values of n for which $y[n]$ is nonzero. Echos can last longer than the original sound!

4.1 Convolution computation (without visualizing)

Let us compute the output for various values of n using the convolution sum directly:

n	$y[n]$
$-2:$	0
$-1:$	0
$0:$	1
$1:$	$1+\alpha$
$2:$	$1+\alpha+\alpha^2$
$3:$	$\alpha+\alpha^2+\alpha^3$
$4:$	$\alpha^2+\alpha^3+\alpha^4$
$5:$	$\alpha^3+\alpha^4+\alpha^5$
$6:$	$\alpha^4+\alpha^5+\alpha^6$
$7:$	$\alpha^5+\alpha^6$
$8:$	α^6
$9:$	0
$10:$	0

$$\Rightarrow \quad \text{with } \alpha = \tfrac{1}{2}: \; y[0..8] = \left[1, \tfrac{3}{2}, \tfrac{7}{4}, \tfrac{7}{8}, \tfrac{7}{16}, \tfrac{7}{32}, \tfrac{7}{64}, \tfrac{3}{64}, \tfrac{1}{64}\right].$$

So, we find that the output $y[n]$ is nonzero for $n = 0, 1, \ldots, 8$. In general, the output of convolution in discrete-time is equal to $N + M - 1$ where N and M are the lengths of the two signals being convolved. Here, the length of $x[n]$ is 3 and the length of $h[n]$ is 6, so the length of $y[n]$ is $3 + 6 - 1 = 8$. **This is important!**

4.2 Visualizing convolution (with graphs)

Now, we will solve this by using illustrations of convolution. For each index $n = 0, 1, \ldots,$ draw three plots in a row for each n:

$$x[k], \qquad h[n-k] \text{ (as a function of } k\text{)}, \qquad \text{and the resulting single sample } y[n],$$

so that the overlap of $x[k]$ and $h[n-k]$ and the accumulation giving $y[n]$ are visually clear.

Let's start by drawing $h[n]$ for $\alpha = \frac{1}{2}$:

4.3 Idea: Flip 'h' and slide through 'x'

Note the x-axis labels carefully! We have $x[k]$ and $h[k]$ because we need these for the convolution sum. We are interested in finding $y[n]$ for each value of n. For each n, we have a $x[k]$ and $h[k]$ that we use for all values of k to solve the convolution sum. Notice that $h[k]$

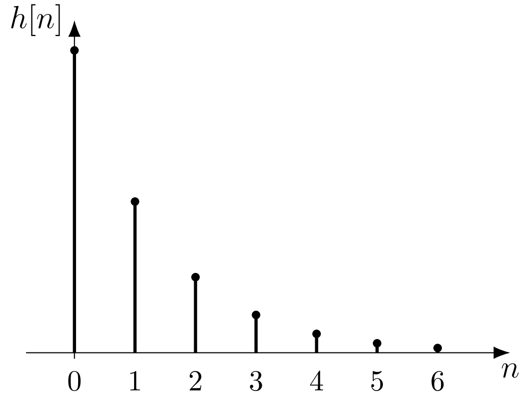


Figure 1: Impulse response $h[n]$ for $\alpha = \frac{1}{2}$.

is not directly used in the convolution sum, instead we have $h[n-k]$. This means that for each value of n, we need to flip $h[k]$ around the vertical axis and then shift it by n units to get $h[n-k]$.

For $n = 0$, the convolution is visualized in Figure 2.

Then, for $n = 1$, the convolution is visualized in Figure 3 and for all other values of n see Figures 4, 5, 6, 7, 8, 9, 10, and 11.

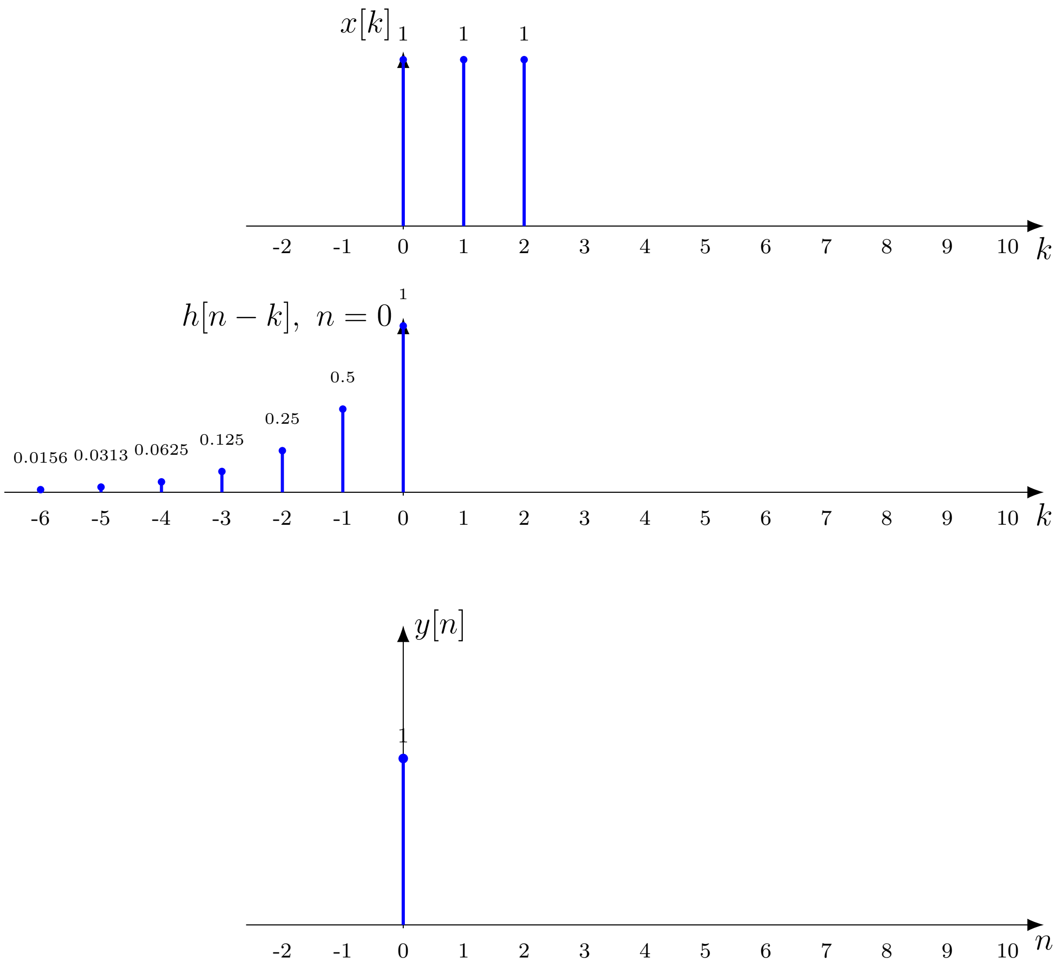


Figure 2: Convolution for $n = 0$.

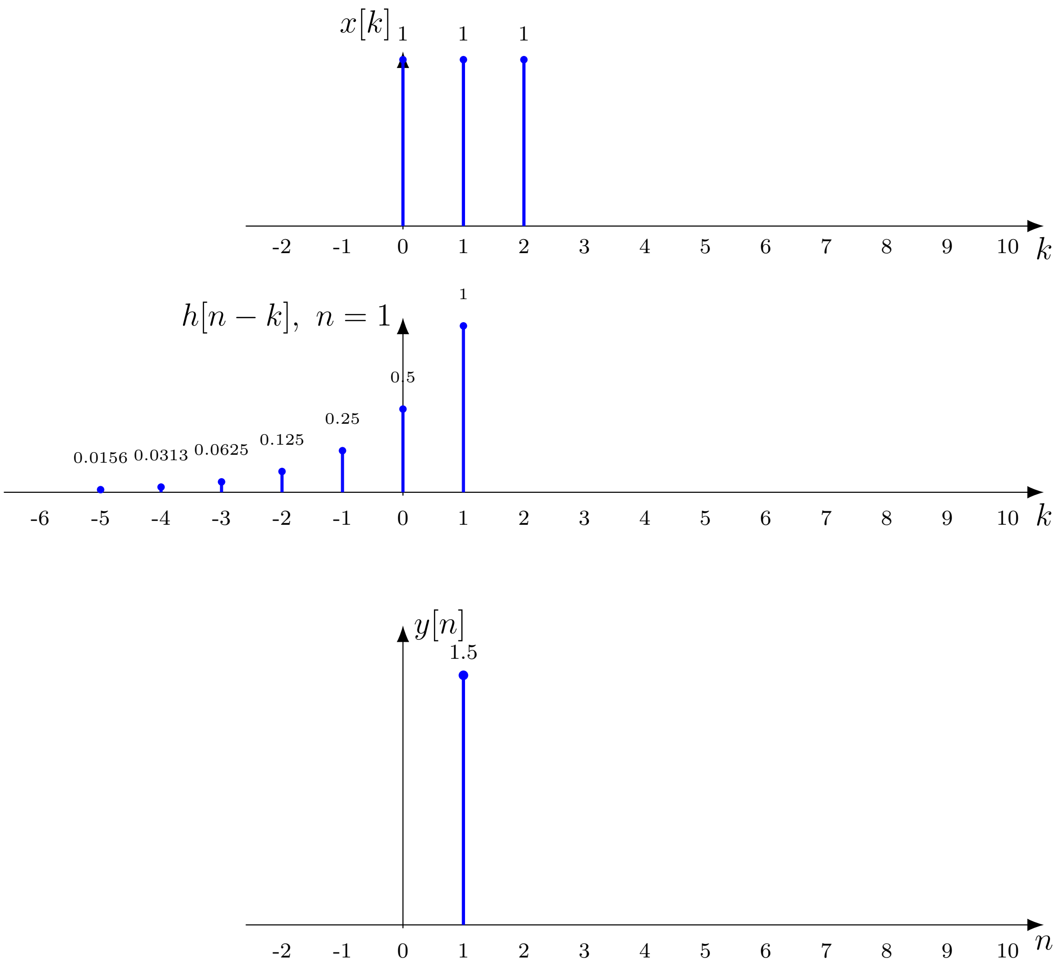


Figure 3: Convolution for $n = 1$.

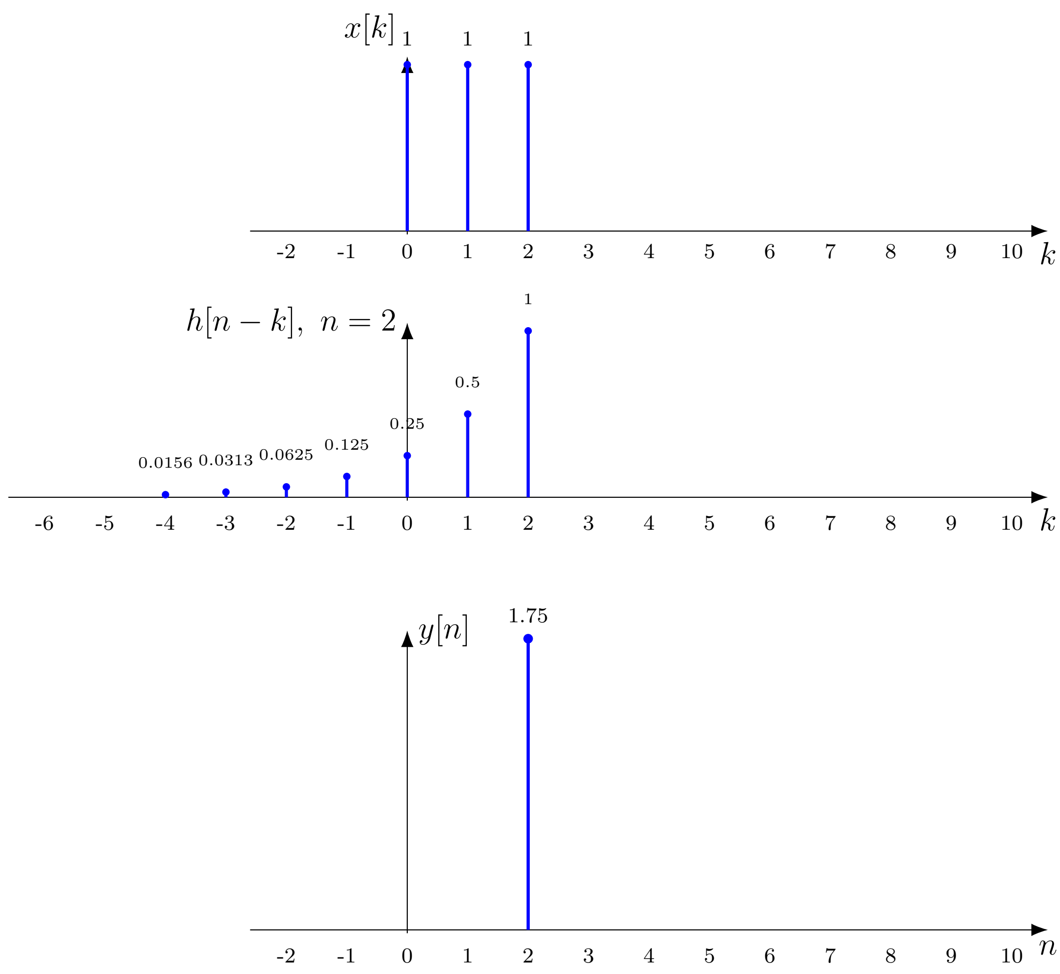


Figure 4: Convolution for $n = 2$.

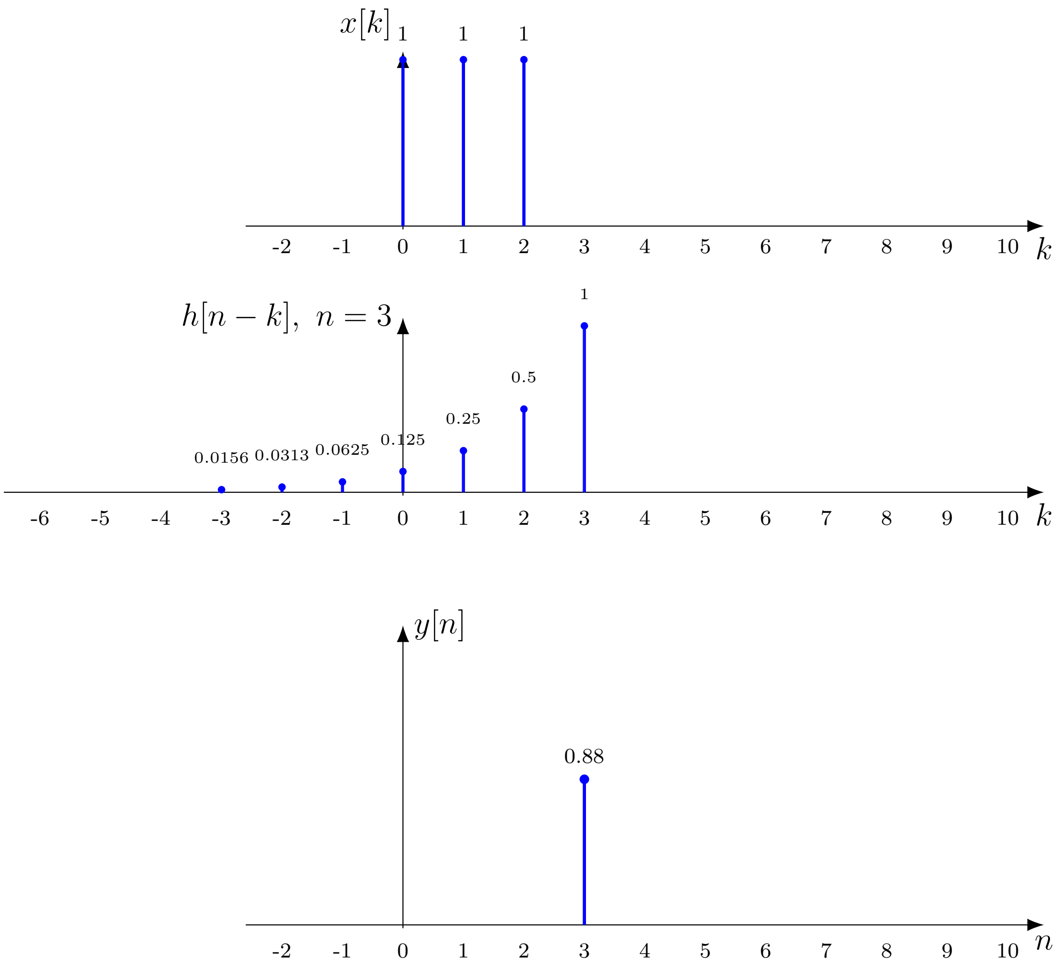


Figure 5: Convolution for $n = 3$.

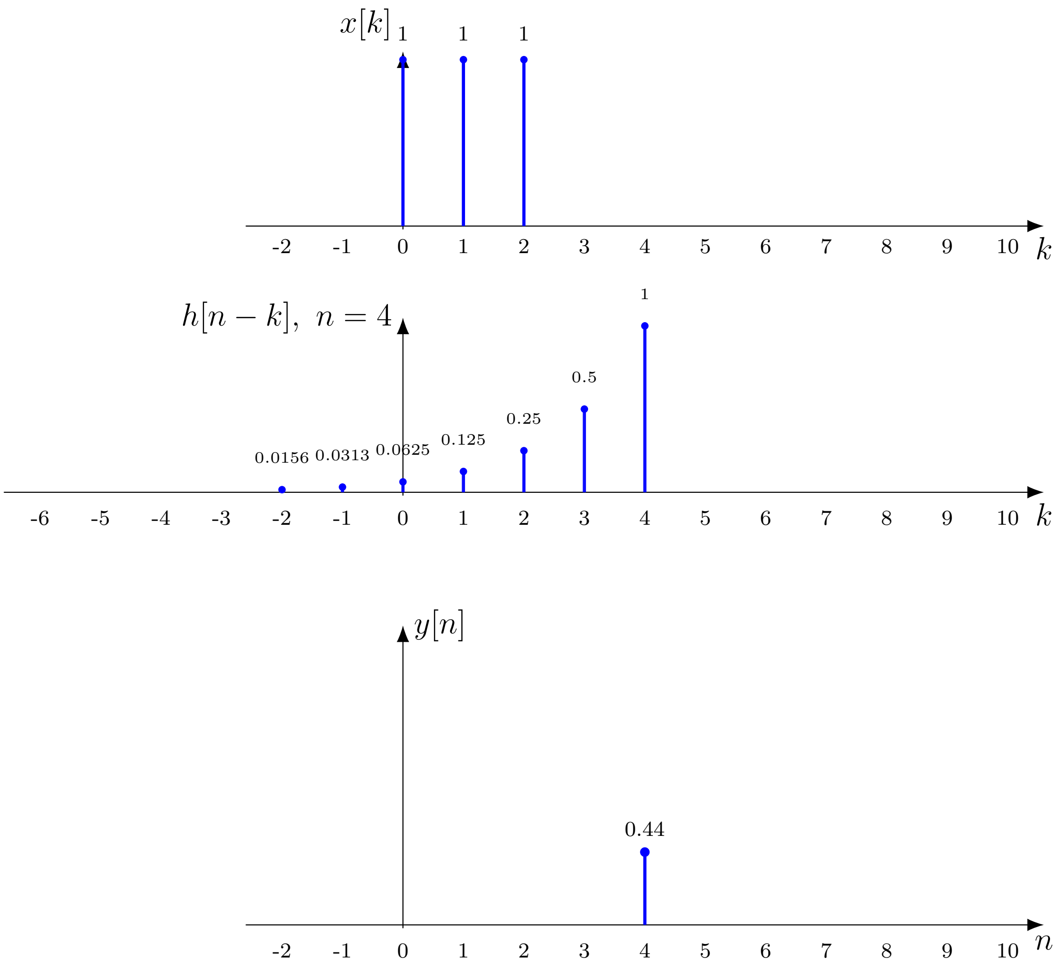


Figure 6: Convolution for $n = 4$.

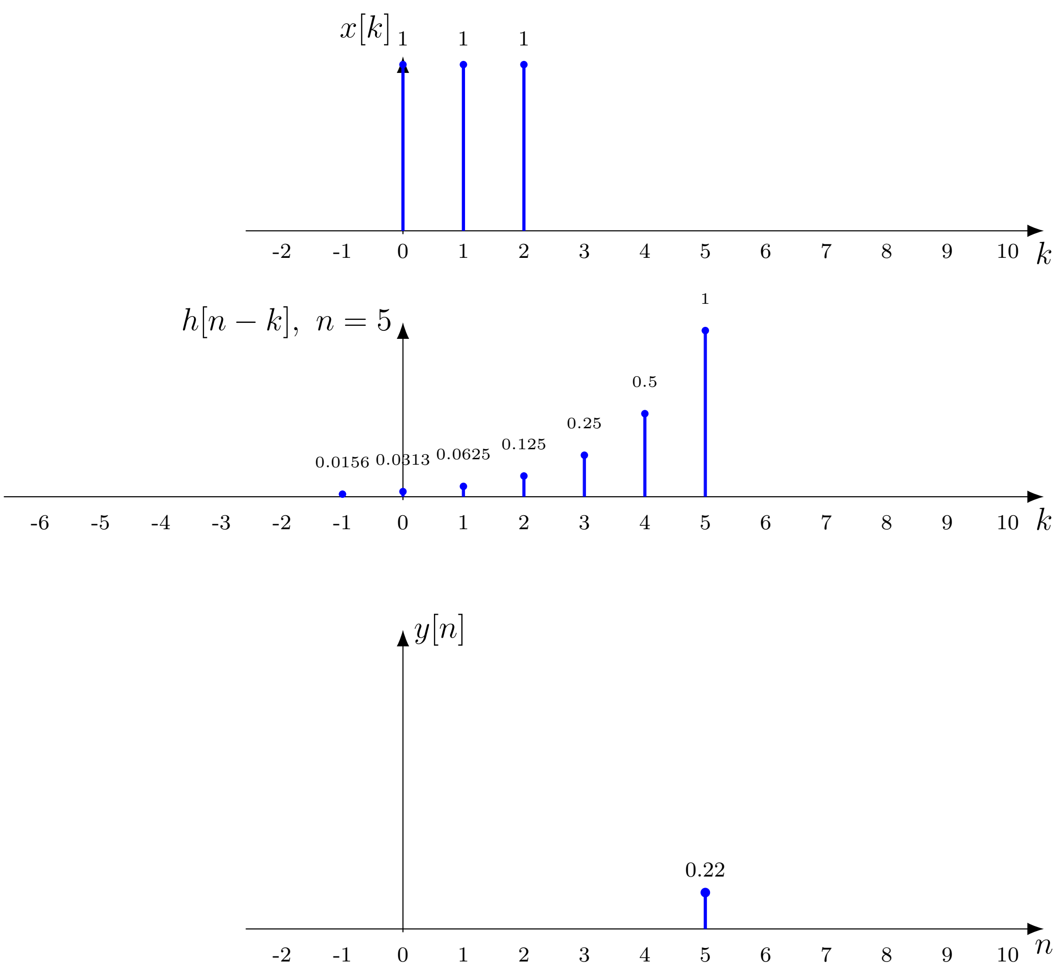


Figure 7: Convolution for $n = 5$.

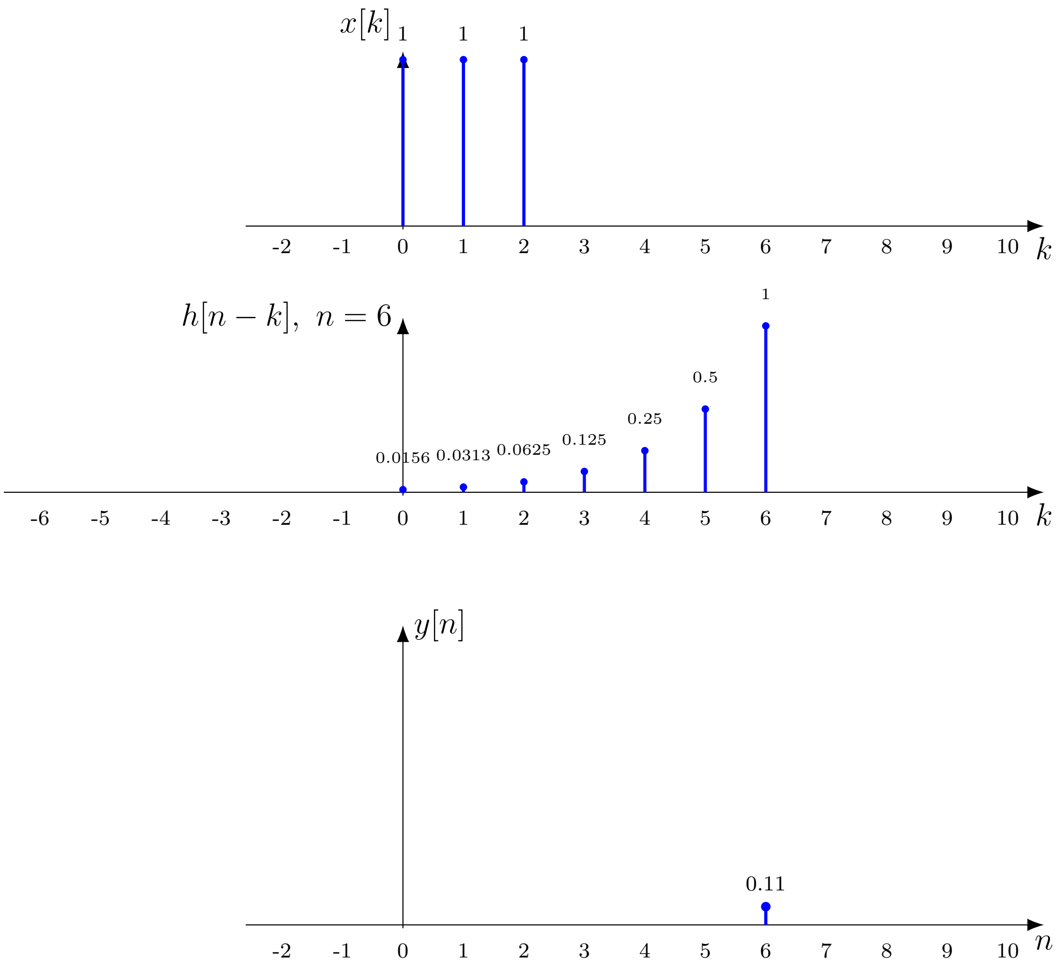


Figure 8: Convolution for $n = 6$.

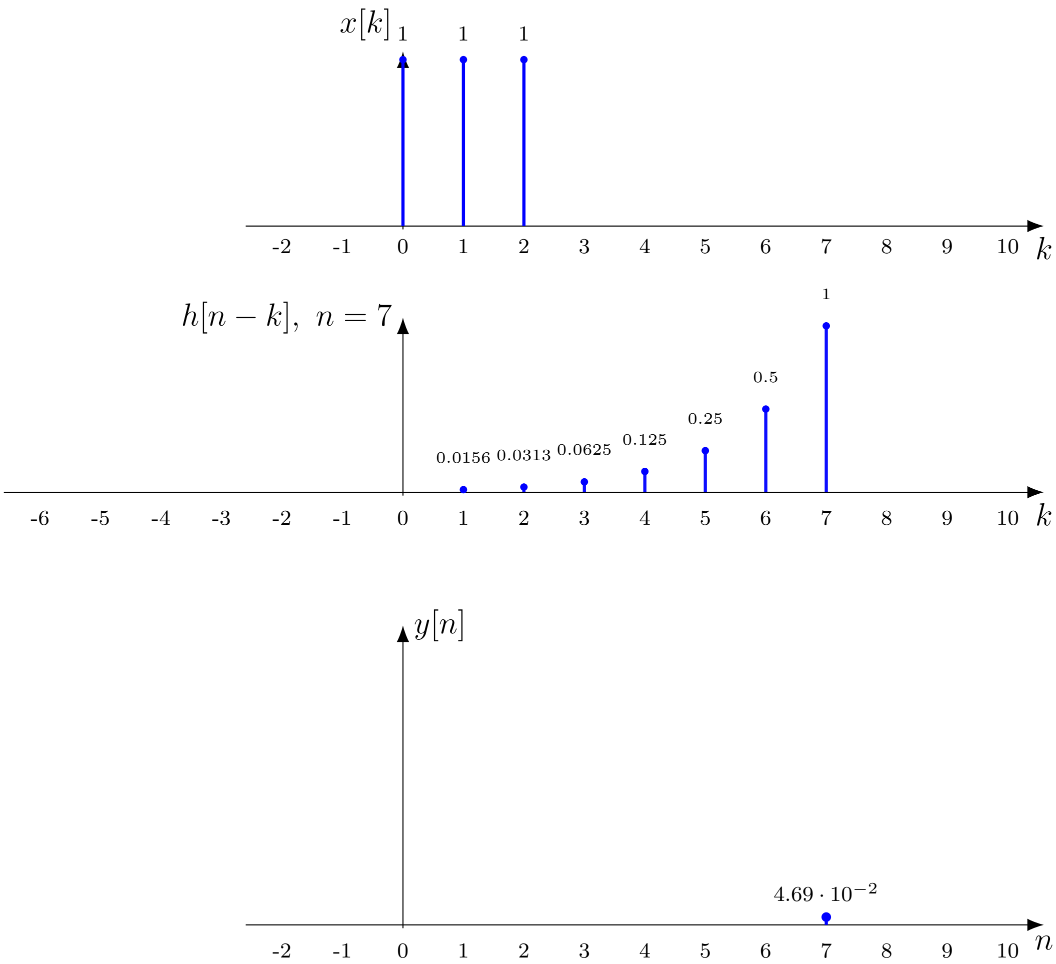


Figure 9: Convolution for $n = 7$.

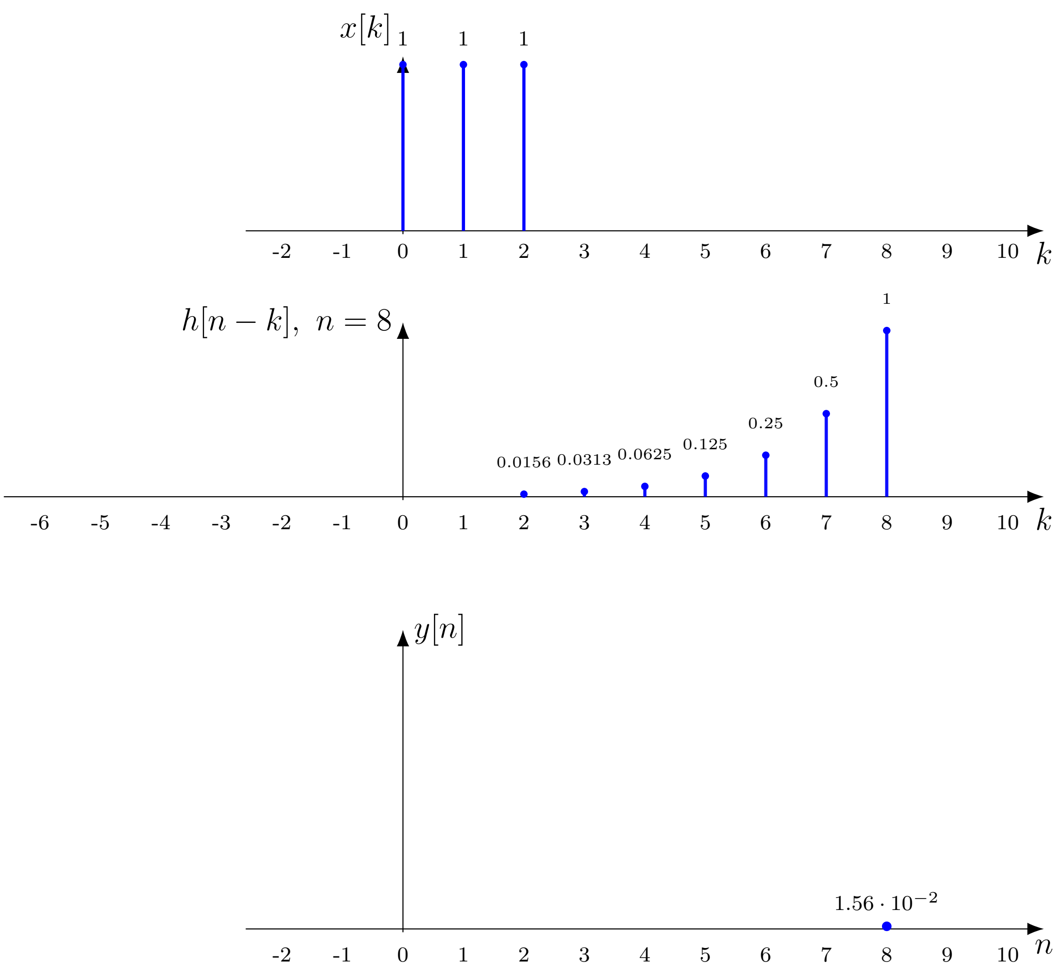


Figure 10: Convolution for $n = 8$.

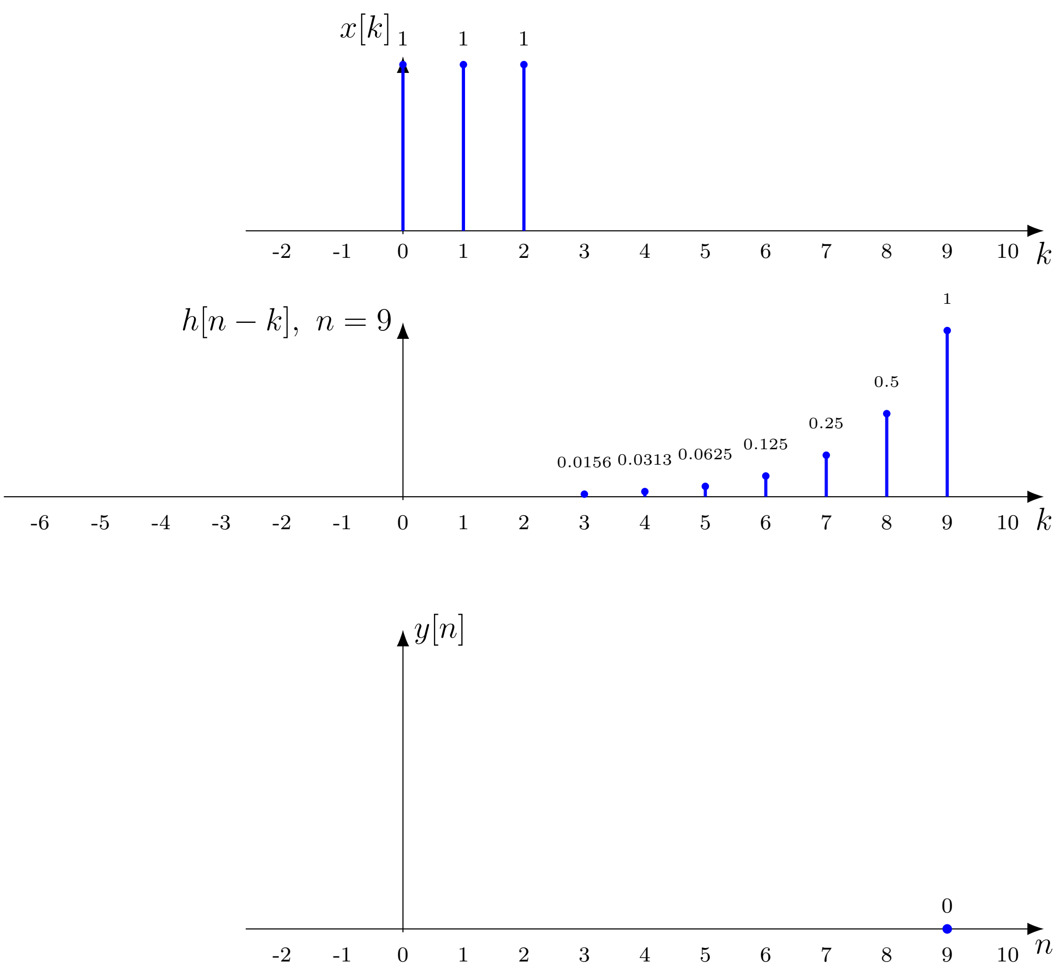


Figure 11: Convolution for $n = 9$.

Pop Quiz Solutions

Pop Quiz 2.1: Solution(s)

Write the convolution integral for left-hand side as

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau.$$

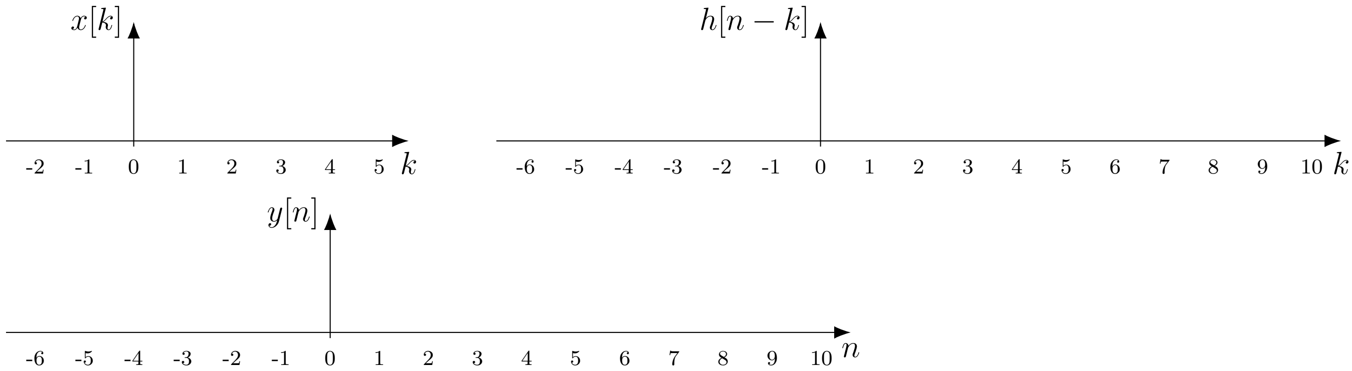
Now, let $s = t - \tau$. Then, $\tau = t - s$ and $d\tau = -ds$. When τ goes from $-\infty$ to ∞, s goes from ∞ to $-\infty$. Thus, we can rewrite the integral as

$$x(t) * h(t) = \int_{\infty}^{-\infty} x(t-s)h(s)(-ds) = \int_{-\infty}^{\infty} h(s)x(t-s)\, ds.$$

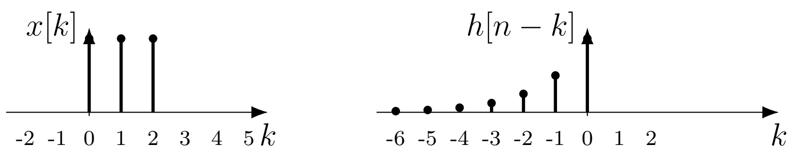
This is exactly the convolution integral for $h(t) * x(t)$. Hence, convolution is commutative.

Visualize convolution

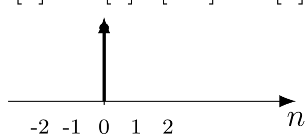
Graphically solve for $n = -1$



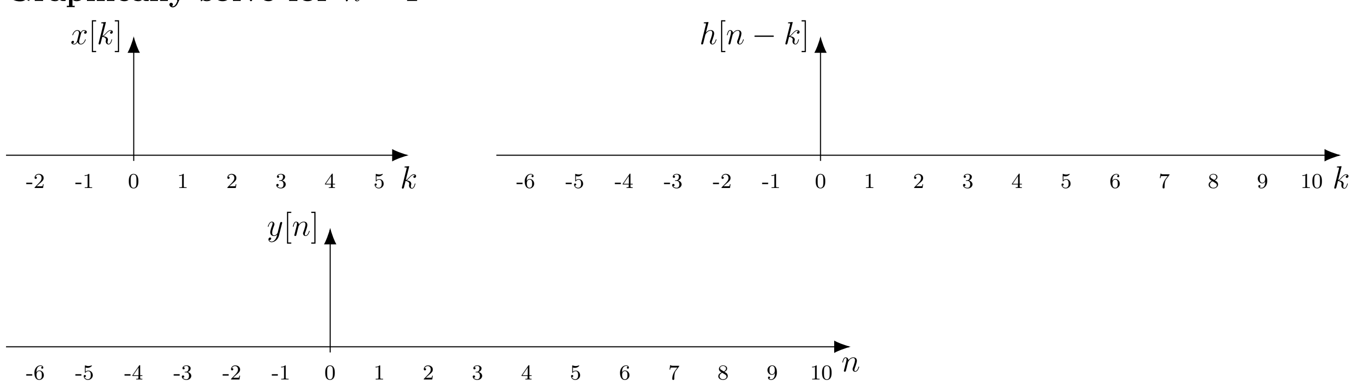
(Solved) Graphically show for $n = 0$



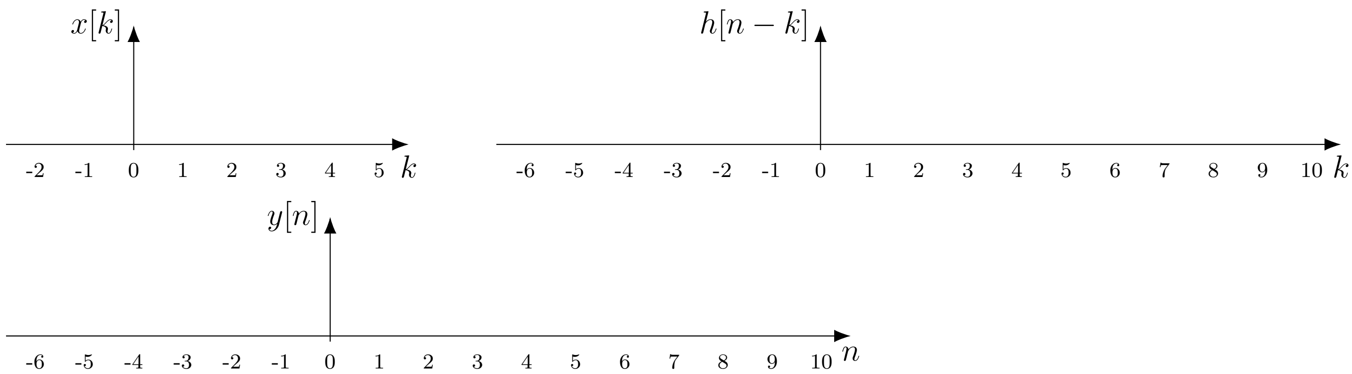
$$y[0] = \sum_k x[k]\, h[0-k] = x[0]\, h[0] + x[1]\, h[-1] + x[2]\, h[-2] = 1 + 0 + 0 = 1.$$



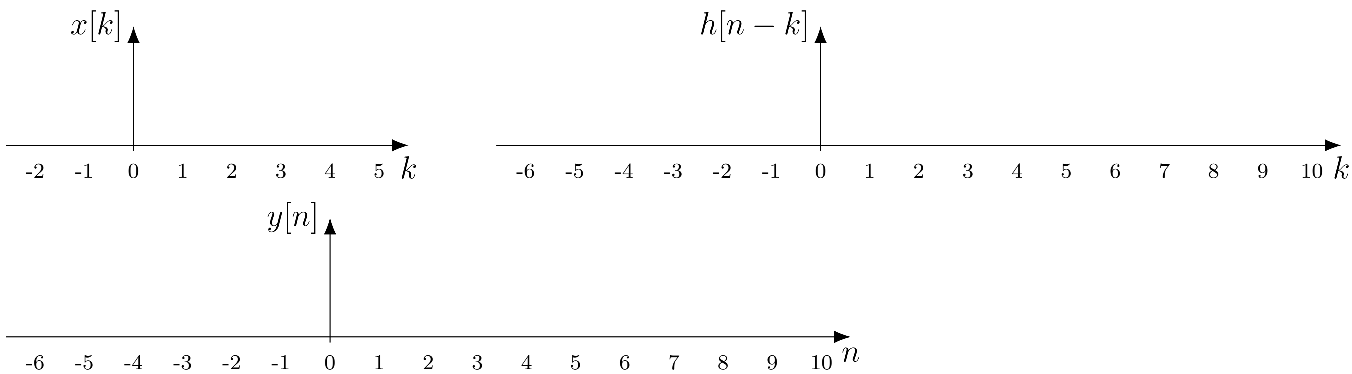
Graphically solve for $n = 1$



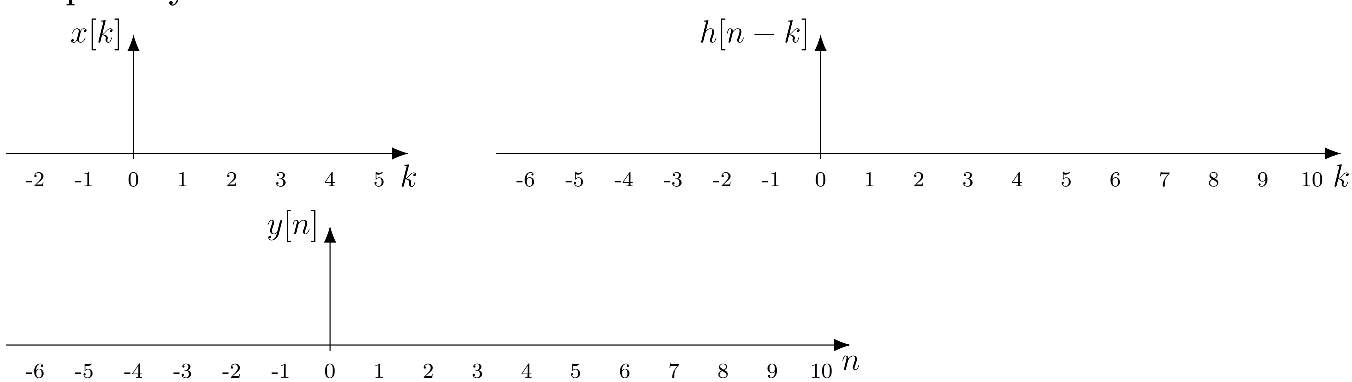
Graphically solve for $n = 2$



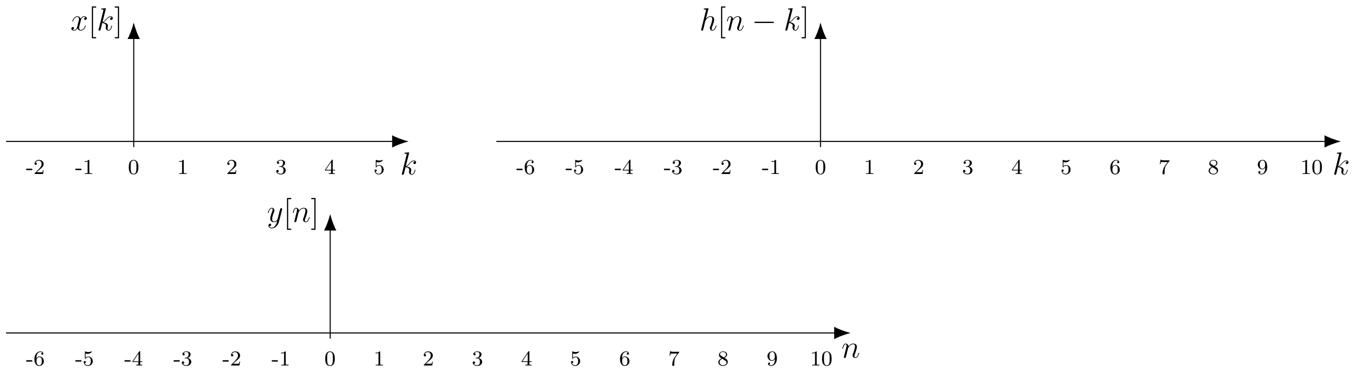
Graphically solve for $n = 3$



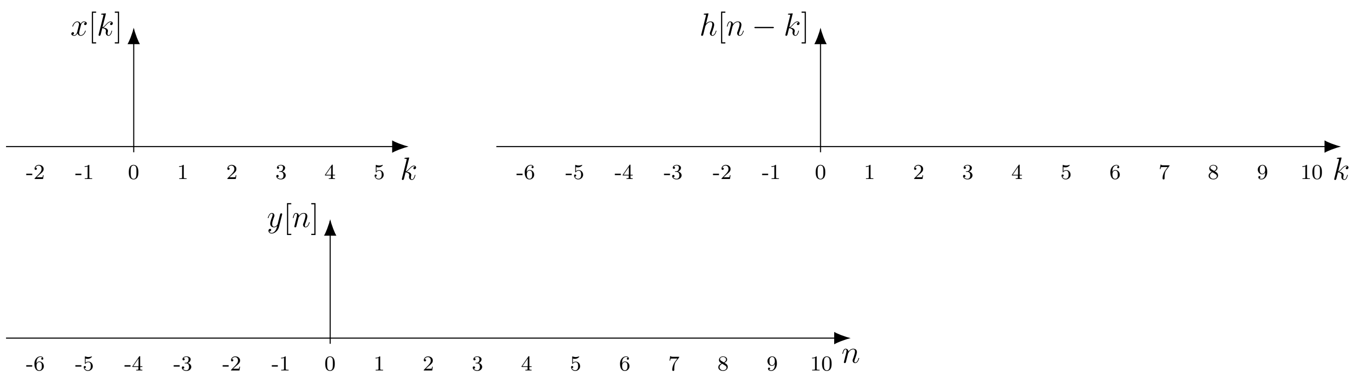
Graphically solve for $n = 4$



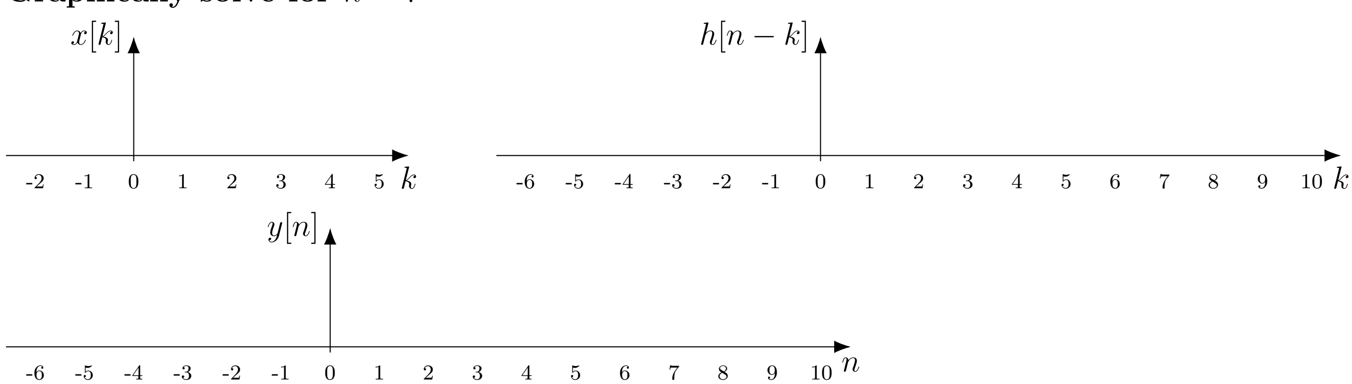
Graphically solve for $n = 5$



Graphically solve for $n = 6$



Graphically solve for $n = 7$



EE 102 Week 6, Lecture 1 (Fall 2025)

Instructor: Ayush Pandey

Date: October 6, 2025

1 Goals

The main goal of this lecture is to write the Fourier series representation of periodic signals. To motivate this, we will show how sinusoidal signals are eigenfunctions of LTI systems (that is, when a sinusoid is input to an LTI system, the output is also a sinusoid of the same frequency). This will lead us to the Fourier series representation of periodic signals.

2 Recap: The convolution integral

Recall that for a continuous-time LTI system with impulse response $h(t)$, the output $y(t)$ to an input $x(t)$ is given by the convolution integral:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

In discrete-time, we have an equivalent expression for the convolution sum. Here, for an input $x[n]$ and impulse response $h[n]$, the output $y[n]$ is given by:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

Pop Quiz 2.1: Check your understanding!

What is the maximum non-zero value of $y[n]$ if the length of $x[n]$ is M and the length of $h[n]$ is N? Note that in discrete-time for finite-length signals, the length of a signal is defined as the number of samples in the signal for which the signal is non-zero.

Solution on page 6

3 The importance of sinusoids

Historically, sinusoids have been very important in engineering and science. This is because many natural phenomena are periodic (for example, sound waves, light waves, etc.). In engineering, the motion of rotation objects (for example, wheels, gears, etc.), the alternating current (AC) in power systems, and the oscillations in electrical circuits are all periodic. Therefore, we focus our attention to this most fundamental class of periodic signals: sinusoids.

In defining convolution, the key idea was to express any given signal as a linear combination of shifted impulses. Similarly, we pose the following question: can we express any given periodic signal as a linear combination of sinusoids? The answer is yes, as you will see in the next few lectures.

3.1 Sinusoids are a special case of the general complex exponential

Recall that the general complex exponential is given by:

$$x_g(t) = A_z e^{st},$$

where A_z and s are complex numbers. We can write A_z and s in terms of their real and imaginary parts as:

$$A_z = a_1 + jb_1, \quad s = \sigma + j\omega,$$

where a_1, b_1, σ, ω are real numbers. Substituting these into the expression for $x(t)$, we have:

$$x(t) = (a_1 + jb_1)e^{(\sigma + j\omega)t} = (a_1 + jb_1)e^{\sigma t}e^{j\omega t}.$$

The general complex exponential can be used to represent a wide variety of signals by choosing special case values of A_z and s and then taking the real or the imaginary part, as needed.

Pop Quiz 3.1: Check your understanding!

For what values of A_z and s do we get a sinusoidal signal $x(t) = A\sin(\omega t)$?

Solution on page 6

Note that the sinusoidal signal is periodic with period $T_0 = \frac{2\pi}{\omega}$. A simple way to denote a sinusoidal signal is to use the complex exponential notation: $Ae^{j\omega t}$, and then take the real or imaginary part as needed. With the exponential function, we avoid the possibility of integration by parts (and we will be able to see more advantages next).

4 The eigenfunction of LTI systems

Let us consider an LTI system with impulse response $h(t)$ and an input $x(t) = Ae^{j\omega t}$. The output of the system is given by the convolution integral:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

by commutativity of convolution, we can also write:

$$\begin{aligned}
&= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)Ae^{j\omega(t-\tau)}d\tau \\
&= Ae^{j\omega t}\left[\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau\right]
\end{aligned}$$

This is a very interesting result!

Observation 1 The input appears exactly as it is in the output, except for a scaling factor that is within the square brackets. This means that when a complex exponential is input to an LTI system, the output is also a complex exponential signal.

A word of caution: the output above is only valid if the integral converges. This is true for physically realizable systems (for example, systems with stable impulse responses). More concretely, we need the impulse response to be absolutely integrable, that is,

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty$$

for the integral to converge. Then, the system is said to be BIBO (bounded-input bounded-output) stable since a bounded input will produce a bounded output (this is true, in general for any LTI system with absolutely integrable impulse response).

Observation 2 For a general complex exponential input, the above derivation still holds:

$$y(t) = Ae^{st}\left[\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau\right].$$

In the equation above, you might recognize the term in the square brackets as the Laplace transform of the impulse response $h(t)$ evaluated at s! So, the output of the system is the input scaled by the Laplace transform of the impulse response!

4.1 The meaning of the word "eigen"

Recall the definition of eigenvalues and eigenvectors from linear algebra. For a square matrix A, if there exists a non-zero vector v such that

$$Av = \lambda v,$$

where λ is a scalar, then v is called an eigenvector of A and λ is called the corresponding eigenvalue. The word "eigen" is a German word that means "own" or "self", or "characteristic". In the case of vectors, the eigenvectors are the special vectors that, when the matrix transformation A is applied to them, the output remains in the same direction as v, just scaled by λ. So, they remain their "own self" even after the matrix transformation is applied to them. Now, since v is a vector, it is called an eigenvector.

For our discussion on signals and systems, we do not have vectors. Instead, we have signals $x(t), y(t)$, and so on which are functions of time. So, we define a new term called **an eigenfunction**.

The meaning of eigenfunction is similar to that of eigenvectors. It is that special function that, when the system transformation is applied to it, the output remains the same function (up to a scaling factor). In our case, the complex exponential is that special function. When it is input to an LTI system, the output remains a complex exponential of the same frequency (up to a scaling factor). So, we say that complex exponentials are eigenfunctions of LTI systems.

5 Linear combinations of sinusoids

Since the system is linear, we can write the following for various combinations of inputs $a_i e^{s_i t}$:

$$\begin{aligned}
&\text{if } x_1(t) = a_1 e^{s_1 t}, \text{ then } y_1(t) = a_1 e^{s_1 t} H(s_1), \\
&\text{if } x_2(t) = a_2 e^{s_2 t}, \text{ then } y_2(t) = a_2 e^{s_2 t} H(s_2), \\
&\qquad \vdots \\
&\text{if } x_n(t) = a_n e^{s_n t}, \text{ then } y_n(t) = a_n e^{s_n t} H(s_n).
\end{aligned}$$

By linearity, if the input is a linear combination of the above inputs, that is,

$$x(t) = \sum_{k=1}^{n} a_k e^{s_k t}, \tag{1}$$

then the output is given by

$$y(t) = \sum_{k=1}^{n} a_k e^{s_k t} H(s_k).$$

This is a very powerful result! It means that if we can express any given signal as a linear combination of complex exponentials, then we can easily find the output of the system to that input.

Now what remains is to write the equation (1) for any given arbitrary signal. That is, can we represent any signal $x(t)$ as a linear combination of complex exponentials? This is where Fourier series comes in as it provides us a way to compute the coefficients a_k and the exponents s_k for periodic signals. We will see this in the next lecture.

6 Recommended Practice Problems

To practice the concepts learned in this lecture, here are the recommended examples and problems that you should practice:

1. Example 3.1 in Oppenheim and Willsky Signals and Systems textbook (2nd edition): the problem with $x(t) = e^{j2t}$ and the system description given by $y(t) = x(t-3)$
2. Problems 3.17 and 3.18 in Oppenheim and Willsky Signals and Systems textbook (2nd edition).
3. A challenging (but similar to midterm 1) problem: Problem 3.13 in Oppenheim and Willsky Signals and Systems textbook (2nd edition) on computing the output of a periodic signal with period $T_0 = 8$ and impulse response $H(j\omega)$ given to you.

Pop Quiz Solutions

Pop Quiz 2.1: Solution(s)

The length of $y[n]$ is $M+N-1$. To see this, use the graphical intuition of convolution. We flip the impulse response and slide it ($h[n-k]$) as n varies, across the input signal $x[k]$. Mathematically, the flip is making $h[k]$ to $h[-k]$ and the slide is making it $h[n-k]$ (the shift to the right by n, for positive n). The first non-zero value of $y[n]$ occurs when the last (right most) non-zero sample of $h[n-k]$ aligns with the first non-zero sample of $x[k]$ (since h was flipped). The last non-zero value of $y[n]$ occurs when the left-most non-zero sample of $h[n-k]$ aligns with the last non-zero sample of $x[k]$. So, the total number of non-zero samples in $y[n]$ is the sum of the lengths of $x[n]$ and $h[n]$ minus 1 (since the two end points are counted twice).

Pop Quiz 3.1: Solution(s)

For $A_z = A$ (a real number) and $s = j\omega$, that is, $\sigma = 0$, we have

$$x(t) = Ae^{j\omega t} = A\cos(\omega t) + jA\sin(\omega t).$$

Taking the imaginary part, we get $x(t) = \mathrm{Im}(x_g(t)) = A\sin(\omega t)$.

EE 102 Week 6, Lecture 2 (Fall 2025)

Instructor: Ayush Pandey

Date: October 8, 2025

1 Introduction and Review

At the end of the previous lecture, we established that $e^{j\omega t}$ is an eigenfunction of LTI systems. That is, for an input $x(t) = e^{j\omega t}$, the output $y(t)$ is also a complex exponential at the same frequency ω but scaled by a complex number $H(j\omega)$:

$$y(t) = H(j\omega)e^{j\omega t}.$$

where

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau.$$

As a result of this *very important* result, we proposed that if we are able to write any signal $x(t)$ as a linear combination of complex exponentials, then we can find the output $y(t)$ of an LTI system by simply scaling each complex exponential by $H(j\omega)$ and adding them up!

The previous line is a one-line summary of Fourier analysis and synthesis — something that we will be spending a lot of time on in the next few weeks.

We start this journey by positing the following problem: Given a T-periodic signal $x(t)$, can we write it as a linear combination of complex exponentials? If so, how? More concretely, can we write

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}, \qquad \omega_0 = \tfrac{2\pi}{T},$$

the complex Fourier series coefficients are $\{a_k\}$. Note that $a_k \in \mathbb{C}$ are complex numbers, in general. The big question is; how do we find a_k?

2 Goals

Represent any periodic signal $x(t)$ as a linear combination of complex exponentials:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}. \tag{1}$$

Find the coefficients a_k.

3 Introduction to Fourier Series

Since the first part of the goal is "any periodic signal $x(t)$" and we are claiming that $x(t)$ can be written as a linear combination of complex exponentials. So, it must be true that (and we should make sure of it) that $e^{jk\omega_0 t}$ is periodic for every integer k.

Pop Quiz 3.1: Check your understanding!

Prove that (a) $e^{j\omega_0 t}$, (b) $e^{jk\omega_0 t}$ for $k \in \mathbb{Z}$, and (c) $\sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$ are all periodic and find their fundamental periods.

Solution on page 8

Let's start to answer the second part of the goal: how do we find a_k? We will start with a simple example.

3.1 Example: A mix-sinusoidal audio signal

Consider the signal below that represents a combination of three sinusoids (added together). When you play a note of music at one specific frequency, you are playing one sinusoid. When you play a chord, you are playing multiple sinusoids at the same time by combining them together. So, in this example, we are representing an audio chord as a linear combination of complex exponentials to start our journey of representing *any* periodic signal as a linear combination of complex exponentials.

Consider

$$x(t) = \sin(6t) + \cos(2t) + \sin(12t), \qquad \omega_0 = 2.$$

Write $x(t)$ as a linear combination of $e^{jk\omega_0 t}$:

$$\sin(6t) = \frac{1}{2j}\left(e^{j6t} - e^{-j6t}\right) = \frac{1}{2j}\left(e^{j(3)\omega_0 t} - e^{-j(3)\omega_0 t}\right),$$

$$\cos(2t) = \frac{1}{2}\left(e^{j2t} + e^{-j2t}\right) = \frac{1}{2}\left(e^{j(1)\omega_0 t} + e^{-j(1)\omega_0 t}\right),$$

$$\sin(12t) = \frac{1}{2j}\left(e^{j12t} - e^{-j12t}\right) = \frac{1}{2j}\left(e^{j(6)\omega_0 t} - e^{-j(6)\omega_0 t}\right).$$

Hence

$$x(t) = \sum_{k=-6}^{6} a_k \, e^{jk\omega_0 t}, \qquad \omega_0 = 2,$$

with the nonzero Fourier series coefficients as

$$a_{\pm 2} = \frac{1}{2}, \qquad a_{\pm 3} = \pm\frac{1}{2j}, a_{\pm 6} = \pm\frac{1}{2j},$$

where the "$\pm$" pairs obey $a_{-k} = a_k^*$ for this real $x(t)$.

Here, the cos term contributes the even coefficients $a_{\pm 2}$; the sin terms contribute the odd, purely imaginary coefficients at $k = \pm 3, \pm 6$. All other a_k are zero.

4 The Trigonometric Form of Fourier Series

If the linear combination form in equation (1) is confusing and the fact that "we are representing everything as a linear combination of sinusoids" is not obvious to you, you can see how we can rewrite the Fourier series synthesis equation (1) in a more familiar trigonometric form. Although you might not find the formulation below much useful, it will at least convince you that we are indeed representing everything as a linear combination of sinusoids.

4.1 From exponentials to trigonometry

For real $x(t)$, $x^*(t) = x(t)$, and

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=1}^{\infty} \left(a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t} \right) + a_0.$$

Since $x(t)$ is real, we must have $a_{-k} = a_k^*$ (you can see that this is indeed the case in the example above with real signals). Therefore

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t} \right] = a_0 + 2 \sum_{k=1}^{\infty} \mathrm{Re}\left\{ a_k e^{jk\omega_0 t} \right\},$$

where we used the fact that for any complex number z, $z + z^* = 2\,\mathrm{Re}\{z\}$.

Writing $a_k = B_k + jC_k$ with $B_k, C_k \in \mathbb{R}$, we obtain the trigonometric form

$$x(t) = a_0 + 2 \sum_{k=1}^{\infty} \left[\, B_k \cos(k\omega_0 t) - C_k \sin(k\omega_0 t) \, \right]. \qquad (2)$$

Equation (2) shows that $x(t)$ is a linear combination of sinusoids at frequencies $k\omega_0$, $k = 1, 2, \ldots$ with real coefficients. The constant term a_0 is the DC component (average value) of $x(t)$ (as we will see again in the next section). It's also finally an equation without any complex numbers or the imaginary term j in it! So, it's hopefully more intuitive now. Let's continue towards our main goal — finding a_k.

4.2 The Fourier coefficients

To find a_k generally, let's start by multiplying both sides of equation (1) by $e^{-jn\omega_0 t}$ for some integer $n \in \mathbb{Z}$, we get

$$x(t)\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k\, e^{j(k-n)\omega_0 t}.$$

Next, let us integrate this equation over $[0, T)$,

$$\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}\, dt$$

Now, we need to evaluate the integral on the right-hand side. We have two cases:

- If $k = n$, then
$$\int_0^T e^{j(k-n)\omega_0 t}\, dt = \int_0^T 1\, dt = T$$
since $e^{j0} = 1$.
- If $k \neq n$, then we have
$$\int_0^T e^{j(k-n)\omega_0 t}\, dt$$
Using orthogonality over one period T, this integral is 0 (since integrating a sinusoid over one period will lead to the positive and negative areas canceling out). You can verify this by direct integration too:
$$\int_0^T e^{j(k-n)\omega_0 t}\, dt = \left[\frac{e^{j(k-n)\omega_0 t}}{j(k-n)\omega_0}\right]_0^T = \frac{e^{j(k-n)\omega_0 T} - 1}{j(k-n)\omega_0} = 0$$
since $e^{j(k-n)\omega_0 T} = e^{j(k-n)2\pi} = 1$ for every integer $k - n$ (from the pop-quiz above).

Hence, we have the Fourier series analysis equation (the equation that gives us values of a_k):

$$a_n = \frac{1}{T} \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt \qquad n \in \mathbb{Z},$$

if you replace n by k, you get the more familiar form:

$$a_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt, \qquad k \in \mathbb{Z},$$

which is the formula for the Fourier series coefficients. Note that for $k = 0$, we have

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt,$$

which is the average value (or DC component) of $x(t)$ over one period.

4.3 Properties of Fourier Series coefficients

There are many helpful properties that you should know about Fourier series coefficients. Here are a few of them (you can find the full list in the textbook):

Linearity For two periodic signals $x(t)$ and $y(t)$ with Fourier Series coefficients a_k and b_k, if we construct another signal by linear superposition, $z(t) = Ax(t) + By(t)$, then the Fourier Series coefficients for $z(t)$ satisfy

$$z(t) \iff \{\, Aa_k + Bb_k \,\}_{k \in \mathbb{Z}}.$$

Periodic convolution The Fourier series coefficients for the output $y(t)$ of a system with impulse response $h(t)$ to a periodic input $x(t)$ can be computed using periodic convolution.

Let $y(t) = (x * h)(t)$ denote periodic convolution with period T. We write the Fourier series expansion of $x(t)$ and $h(t)$ as

$$x(t) = \sum_{\ell=-\infty}^{\infty} a_\ell e^{j\ell\omega_0 t}, \qquad h(t) = \sum_{m=-\infty}^{\infty} b_m e^{jm\omega_0 t}, \qquad \omega_0 = \tfrac{2\pi}{T}.$$

Let $y(t) = (x * h)(t)$ denote the periodic convolution (integrate only for one period):,

$$y(t) = \int_0^T x(\tau)\, h(t - \tau)\, d\tau.$$

Let $\{c_k\}$ be the Fourier Series coefficients of $y(t)$, defined as

$$c_k = \frac{1}{T} \int_0^T y(t)\, e^{-jk\omega_0 t}\, dt.$$

We can find c_k in terms of a_k and b_k using convolution as follows. Start by substituting the expression for $y(t)$ into the definition of c_k:

$$c_k = \frac{1}{T}\int_0^T \left[\int_0^T x(\tau)\, h(t-\tau)\, d\tau\right] e^{-jk\omega_0 t}\, dt.$$

Now, using the following equations for the Fourier Series expansions of $x(\tau)$ and $h(t-\tau)$ (this is also an in-place proof for the time-shift property of Fourier Series!):

$$x(\tau) = \sum_{\ell} a_\ell e^{j\ell\omega_0\tau}$$

and

$$h(t-\tau) = \sum_m b_m e^{jm\omega_0(t-\tau)} = \sum_m b_m e^{jm\omega_0 t}\, e^{-jm\omega_0\tau},$$

we get

$$c_k = \frac{1}{T}\int_0^T\int_0^T \left(\sum_\ell a_\ell e^{j\ell\omega_0\tau}\right)\left(\sum_m b_m e^{jm\omega_0 t} e^{-jm\omega_0\tau}\right) e^{-jk\omega_0 t}\, d\tau\, dt.$$

Interchange sums and integrals and collect factors together to write,

$$\begin{aligned} c_k &= \frac{1}{T}\sum_\ell\sum_m a_\ell b_m \int_0^T\int_0^T e^{j\ell\omega_0\tau}\, e^{-jm\omega_0\tau}\, e^{jm\omega_0 t}\, e^{-jk\omega_0 t}\, d\tau\, dt \\ &= \frac{1}{T}\sum_\ell\sum_m a_\ell b_m \left[\int_0^T e^{j(m-k)\omega_0 t}\, dt\right]\left[\int_0^T e^{j(\ell-m)\omega_0\tau}\, d\tau\right]. \end{aligned}$$

Finally, note that periodic integral over one period is 0 unless the integrand is constant. So, for any integers p, $\int_0^T e^{jp\omega_0 t}\, dt = \begin{cases} T, & p = 0, \\ 0, & p \neq 0. \end{cases}$ Hence the t-integral is zero unless $m = k$, and the τ-integral is zero unless $\ell = m$:

$$\int_0^T e^{j(m-k)\omega_0 t}\, dt = T\,\delta_{m,k}, \qquad \int_0^T e^{j(\ell-m)\omega_0\tau}\, d\tau = T\,\delta_{\ell,m}.$$

So, only the terms with $\ell = m = k$ survive!

$$c_k = \frac{1}{T}\sum_\ell\sum_m a_\ell b_m\,(T\delta_{m,k})\,(T\delta_{\ell,m}) = \frac{1}{T}\,(T)(T)\,a_k b_k = T\,a_k b_k.$$

$$c_k = T\,a_k\,b_k\,, \qquad k \in \mathbb{Z}.$$

Thus, periodic convolution corresponds to a *line-by-line* product of Fourier Series coefficients: each harmonic k of the output equals T times the product of the input and impulse response harmonics at the same k.

Filtering of output using aperiodic impulse response If $h(t)$ is aperiodic (but LTI) and $x(t)$ is T-periodic with Fourier Series coefficients $\{a_k\}$, then

$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \sum_k a_k e^{jk\omega_0 t} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-jk\omega_0 \tau}\, d\tau}_{H(jk\omega_0)} .$$

Hence

$$y(t) = \sum_{k=-\infty}^{\infty} a_k\, H(jk\omega_0)\, e^{jk\omega_0 t}$$

i.e., each harmonic is scaled by the continuous-time frequency response $H(j\omega)$ evaluated at $\omega = k\omega_0$. So, the Fourier Series coefficients of $y(t)$ are

$$c_k = a_k\, H(jk\omega_0).$$

This will be very useful for your homework problems!

5 Practice Problems

1. Solved Example 3.6 in Oppenheim and Willsky (2nd Edition) — the square wave
2. Solved Example 3.7 in Oppenheim and Willsky (2nd Edition) — the ramp function
3. Work through the properties in Table 3.1 in Oppenheim and Willsky (2nd Edition)
4. Solved Example 3.5 in Oppenheim and Willsky (2nd Edition) — the square wave **(this is similar to HW 6 problem 1 and 2!)**

Pop Quiz Solutions

Pop Quiz 3.1: Solution(s)

We can just consider the general case: for $x(t) = e^{jk\omega_0 t}$ to be periodic with period T we need $x(t+T) = x(t)$, i.e.,

$$e^{jk\omega_0(t+T)} = e^{jk\omega_0 t} \iff e^{jk\omega_0 T} = 1 \iff k\omega_0 T = 2\pi m,\ m \in \mathbb{Z}.$$

Thus any $T = \dfrac{2\pi m}{k\omega_0}$ is a period and the smallest period (the fundamental period) is

$$T_0 = \frac{2\pi}{|k|\omega_0}.$$

Since each harmonic $e^{jk\omega_0 t}$ has a period that is an integer divisor of $\dfrac{2\pi}{\omega_0}$, the sum of all harmonics is periodic with the common (fundamental) period

$$T_0 = \frac{2\pi}{\omega_0}.$$

For $k = 1$, we get the simpler result for $e^{j\omega_0 t}$.
Moreover, note that $e^{jk2\pi} = \cos(2\pi k) + j\sin(2\pi k) = 1$ for every integer k, confirming periodicity.

EE 102 Week 7, Lecture 1 (Fall 2025)

Instructor: Ayush Pandey

Date: October 13, 2025

1 Goals

By the end of this lecture, you should be able to:

- Review EE 102 so far and discuss course goals.
- Appreciate the motivation behind signals and systems.
- Understand how linear combinations of sinusoids can approximate arbitrary signals.
- Analyze the Fourier series representation of discrete-time periodic signals.

2 Review of EE 102 So Far

2.1 Review: Signals

We started this course by introducing signals (as mathematical functions that convey information). In continuous-time, we denote signals as $x(t)$, where t is a continuous variable representing time and in discrete-time, we denote signals as $x[n]$, where n is an integer representing the sample index. Then, we set up many properties and transformations that can be applied to signals. Here's a reminder of some of the most notable ones:

- Time-shifting: $x(t - t_0)$ shifts the signal $x(t)$ to the right by t_0 units.
- Time-scaling: $x(at)$ compresses the signal if $a > 1$ and stretches it if $0 < a < 1$.
- Time-reversal: $x(-t)$ flips the signal around the vertical axis.
- Periodicity: A signal $x(t)$ is periodic with period T if $x(t + T) = x(t)$ for all t.
- Even and odd signals: A signal is even if $x(t) = x(-t)$ and odd if $x(t) = -x(-t)$.

2.2 Review: The trio of special signals

We spent quite a bit of time discussing three special signals that are fundamental to signal processing:

The unit impulse signal: The unit impulse signal, denoted as $\delta(t)$ in continuous-time and $\delta[n]$ in discrete-time, is defined as:

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

So, it is zero everywhere except at $t = 0$, where it is infinitely high! A useful way to define it is through its integral property:

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

In discrete-time, the unit impulse is defined as:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

and it has the summation property:

$$\sum_{n=-\infty}^{\infty} \delta[n] = 1$$

The unit step signal: The unit step signal, denoted as $u(t)$ in continuous-time and $u[n]$ in discrete-time, is defined as:

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

In discrete-time, the unit step is defined as:

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

We discussed how integration of the unit impulse gives the unit step:

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

and summation of the unit impulse gives the unit step in discrete-time:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

The complex exponential signal: The complex exponential signal, in its general form, is given by:

$$x(t) = Ae^{st}$$

where A and s are complex numbers. For $s = j\omega$ (purely imaginary) and $A \in \mathbb{R}$, we get sinusoidal signals (cosine and sine):

$$x(t) = Ae^{j\omega t} = A\cos(\omega t) + jA\sin(\omega t)$$

2.3 The overarching goal of signal processing

Although you can state the goal in many formal ways, let us try to simplify the goal down to something that we can make intuitive sense of. We define the trio of special signals above for a reason — the central hypothesis of signal processing is that **any arbitrary signal can be broken down into a sum of these special signals**. That is, you can write any signal $x(t)$ only using impulses, or only using steps, or only using complex exponentials! So, we set ourselves the following question:

How can we represent any arbitrary signal using only these special signals?

2.4 The sifting property to write signals using impulses

The sifting property of the impulse function allows us to express any continuous-time signal $x(t)$ as an integral of scaled and shifted impulses:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

In discrete-time, we can express any signal $x[n]$ as a sum of scaled and shifted impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

You can imagine that breaking a signal down into impulses is not that hard — you just need to sample the signal at literally every point in time and place an impulse at that point with the appropriate scaling. So, it is intuitive! Similarly, we can visualize easily how any arbitrary signal can be constructed using steps (imagine a staircase approximation of a signal). Mathematically, this follows from the fact that the step is the integral of the impulse. However, the last special signal — the complex exponential — is not as intuitive. How can we represent any arbitrary signal using only complex exponentials? We know that complex exponential, $e^{j\omega t}$ is an oscillatory signal. It is not that obvious how we can use oscillatory signals to represent arbitrary signals. We discuss this next.

2.5 Using complex exponentials to represent signals

The Fourier analysis is the answer to this question. We are currently engaged in building towards Fourier analysis that will let us express any arbitrary signal as a sum of complex exponentials. Since the idea is not that intuitive, the mathematical foundations are also not as straightforward as the sifting property. So, we simplify the task — we first ask a simpler question for signals with nicer properties. We ask "can we represent periodic signals using complex exponentials?" The answer is yes, and this is the Fourier series representation of periodic signals.

2.6 Introduction systems

To realize any application of signals and signal processing, we need to introduce the "processing" examples. This is what gets us to "systems". A system is an object that takes in an input signal, processes it, and produces an output signal. We denote systems using a block diagram as shown below:

$$\text{Input } x(t) \quad \longrightarrow \quad \boxed{\text{System}} \quad \longrightarrow \quad \text{Output } y(t)$$

2.7 Properties of systems

We discussed many properties of systems. Here are some of the most important ones:

- Linearity: A system is linear if it satisfies the principles of superposition and scaling. That is, for any inputs $x_1(t)$ and $x_2(t)$, and any scalars a and b, the system satisfies:

  $$y(t) = T\{ax_1(t) + bx_2(t)\} = ay_1(t) + by_2(t)$$

  where $y_1(t) = T\{x_1(t)\}$ and $y_2(t) = T\{x_2(t)\}$.

- Time-invariance: A system is time-invariant if a time shift in the input signal results in an identical time shift in the output signal. That is, if $y(t) = T\{x(t)\}$, then for any time shift t_0, we have:

  $$y(t - t_0) = T\{x(t - t_0)\}$$

For LTI systems, we can compute the output of the system to any arbitrary input using the convolution operation.

2.8 Impulse response and convolution

We define the impulse response of an LTI system as the output of the system when the input is an impulse signal. That is, if the input to the system is $\delta(t)$, then the output is $h(t)$, which is the impulse response. In block diagram form:

$$\delta(t) \quad \longrightarrow \quad \boxed{\text{LTI System}} \quad \longrightarrow \quad h(t)$$

The impulse response characterizes the behavior of the LTI system completely. For any arbitrary input signal $x(t)$, the output $y(t)$ can be computed using the convolution operation:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Now, we move on to the next question that remains unanswered — how can we represent signals using complex exponentials? and how can we use this to find out the output of LTI systems to arbitrary inputs?

We will spend the next 2-3 weeks answering this question using Fourier analysis.

Pop Quiz 2.1: Check your understanding!

In your own words, write the goals of signal processing. Then, describe what Fourier series is and write both the synthesis and analysis equations for Fourier series.

Solution on page 7

3 Properties of Fourier Series

We discussed the linearity and time-shifting properties of Fourier series last time. Here's a reminder:

- Linearity: If $x_1(t)$ and $x_2(t)$ have Fourier coefficients a_k and b_k respectively, then for any scalars A and B, the signal $x(t) = Ax_1(t) + Bx_2(t)$ has Fourier coefficients:

$$c_k = Aa_k + Bb_k$$

- Time-shifting: If $x(t)$ has Fourier coefficients a_k, then the time-shifted signal $x(t-t_0)$ has Fourier coefficients:

-

$$c_k = a_k e^{-jk\omega_0 t_0}$$

We also discussed the filtering property and derived it using convolution. For a system with impulse response $h(t)$ and a periodic input $x(t)$, the Fourier coefficients of the output $y(t)$ are given by:

$$c_k = a_k H(jk\omega_0)$$

where from the convolution equation, we found that

$$H(jk\omega_0) = \int_{-\infty}^{\infty} h(t)e^{-jk\omega_0 t}dt.$$

Note that this is called the *frequency response* of the system.

Pop Quiz 3.1: Check your understanding!

Prove the differentiation property of the Fourier series. That is, if $x(t)$ has Fourier coefficients a_k, then show that the derivative $x'(t)$ has Fourier coefficients $c_k = jk\omega_0 a_k$.

Solution on page 7

4 Discrete-time Fourier Series

In discrete-time, the Fourier series synthesis and analysis follows pretty much the same form as continuous-time. The synthesis equation is given by:

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}$$

where N is the period of the signal and $\omega_0 = \frac{2\pi}{N}$ is the fundamental frequency. The analysis equation is given by:

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\omega_0 n}$$

Note that the integration is replaced by a summation in discrete-time. Also, the sum in the synthesis equation is from 0 to $N-1$ because the Fourier coefficients are periodic with period N in discrete-time. You may also use any integer multiples of N in the limits of the summation, as you are only summing over one period.

5 Recommended Practice Problems

To practice the concepts learned in this lecture, here are the recommended examples and problems that you should practice:

Pop Quiz Solutions

Pop Quiz 2.1: Solution(s)

The Fourier series synthesis equation lets us *synthesize* a periodic signal $x(t)$ using a sum of complex exponentials:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where a_k are the Fourier coefficients and $\omega_0 = \frac{2\pi}{T}$ is the fundamental frequency for a signal with period T.
The Fourier series analysis equation lets us *analyze* a periodic signal $x(t)$ to find its Fourier coefficients:

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

where the integration is over a period.

Pop Quiz 3.1: Solution(s)

We start with the Fourier series synthesis equation for $x(t)$:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Differentiating both sides with respect to t, we get:

$$x'(t) = \sum_{k=-\infty}^{\infty} a_k \frac{d}{dt} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k (jk\omega_0) e^{jk\omega_0 t}$$

Thus, the Fourier coefficients of $x'(t)$ are:

$$c_k = jk\omega_0 a_k$$

You can also derive this using the analysis equation by integrating by parts (will be much harder!) or by using the filtering equation above by observing that differentiation in time domain corresponds to multiplication by $j\omega$ in frequency domain.

EE 102 Week 7, Lecture 1 (Fall 2025)

Instructor: Ayush Pandey

Date: October 13, 2025

1 Goals

By the end of this lecture, you should be able to:

- Understand that Fourier series is the optimal approximator of periodic signals. That is, it minimizes the error energy between the actual signal and the approximated signal.
- Apply Parseval's theorem and Fourier series property to analyze a real-world engineering system.

2 Review

Recall the Fourier Series (FS) synthesis equation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where we sum infinite number of terms to get the signal back *exactly*. But what if we only use a finite number of terms? We can define a truncated sum:

$$x_{\text{FS},N}(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

where N is a positive integer. This is an approximation of the original signal $x(t)$. We can define an error signal

$$e_N(t) = x(t) - x_{\text{FS},N}(t)$$

and the error energy over a period as

$$E_N = \int_{t_0}^{t_0+T} |e_N(t)|^2 dt$$

where T is the period of the signal.

The key idea is that FS gives us an approximation of the signal that minimizes this error energy E_N for a given N. This means that among all possible ways to approximate $x(t)$ using $2N+1$ terms, the FS approximation $x_{\mathrm{FS},N}(t)$ is the best in terms of minimizing the error energy. Further, in the limit of $N \to \infty$, the error energy E_N approaches zero, and the approximation becomes exact.

3 Intution for sums of sinusoids

We have noted many times that the FS gives us a way to write any periodic signal as a sum of sinusoids. But how would this be possible for signals with discontinuities? This is unintuitive because sinusoids are smooth, oscillating functions. So, how can they combine together to form a signal that is dicontinuous, like an impulse train, for example?

To build intuition for this fact, recall that Fourier series for real signals is a sum of cosines. We start by writing the exponential FS:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where a_k are the complex Fourier coefficients. For real signals, we can use the property that $a_{-k} = a_k^*$ to rewrite this by breaking the sum into positive and negative k terms, and the DC term for $k = 0$:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{-1} a_k e^{jk\omega_0 t}$$

Now, apply a change of variable $m = -k$ in the last sum:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k e^{jk\omega_0 t} + \sum_{m=1}^{\infty} a_{-m} e^{-jm\omega_0 t}$$

We can combine the two sums now as they both have the same limits. In combining the sums, we also recall the property that for real signals $a_{-k} = a_k^*$:

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t} \right)$$

Finally, we can use Euler's formula to rewrite the terms in the sum as cosines.

Case 1: Real and Even Let's consider a simple case. For real and even signals, the Fourier coefficients a_k are real and even. This means that $a_k = a_k^*$ and $a_{-k} = a_k$. Therefore, we can write:

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2a_k \cos(k\omega_0 t)$$

This is a sum of cosines with amplitudes $2a_k$!

Case 2: Complex a_k For general real signals, the Fourier coefficients a_k are complex. We can express a_k in polar form as $a_k = |a_k|e^{j\phi_k}$, where $|a_k|$ is the magnitude and ϕ_k is the phase. Then, we can write:

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2|a_k| \cos(k\omega_0 t + \phi_k)$$

This is a sum of cosines with amplitudes $2|a_k|$ and phase shifts ϕ_k.

Pop Quiz 3.1: Check your understanding!

On Desmos Graphing Calculator, explore how sums of cosines can approximate discontinuities. Create a visual graph that "looks" like a train of impulses.

Solution on page 6

3.1 Virtual manipulator: Gibbs Phenomena

For a train of impulses (HW #6 Problem 1), the Fourier coefficients are all equal to $1/T$. Therefore, the FS representation is:

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{jk\omega_0 t}$$

Explore the virtual manipulative on Fourier analysis of impulses by running `streamlit run VM_impulse_fourier_analysis.py` on your computer. Then, attempt to show the following by changing the knobs on the simulation:

- With two sinusoids being displayed (the last slider), draw on your notebook the points where constructive and destructive interference happens.
- Increase the number of harmonics (that is, how many high-frequency sinusoids are being added) iteratively and observe the behavior of the summed-up sinusoids near the discontinuities. What do you observe? This is called the Gibbs Phenomenon.

- What happens to the error energy as you increase the number of harmonics?
- Recall that an impulse is a signal with infinite height, zero width, and unit area. Can you explain how the sum of sinusoids is able to approximate such a signal?

4 Parseval's Theorem

As briefly discussed earlier, Fourier series gives us the optimal approximation of a periodic signal in terms of minimizing the error energy. This is formalized by Parseval's theorem, which states that the total energy of a periodic signal over one period is equal to the sum of the squares of its Fourier coefficients multiplied by the period. To prove this relation mathematically, we start with the FS synthesis equation and compute the signal energy over one period:

$$E = \int_{t_0}^{t_0+T} |x(t)|^2 dt$$

Substituting the FS synthesis equation into this integral, we have:

$$E = \int_{t_0}^{t_0+T} \left| \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right|^2 dt$$

Expanding the square using the definition of magnitude squared of a complex number z as zz^*, we get:

$$E = \int_{t_0}^{t_0+T} \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right) \left(\sum_{m=-\infty}^{\infty} a_m^* e^{-jm\omega_0 t} \right) dt$$

Bring the sums outside the integral:

$$E = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_k a_m^* \int_{t_0}^{t_0+T} e^{j(k-m)\omega_0 t} dt$$

The integral evaluates to T when $k = m$ and 0 otherwise, due to the orthogonality of the complex exponentials (the area under curve cancels out for sinusoids over one period). Therefore, we have:

$$E = \sum_{k=-\infty}^{\infty} |a_k|^2 T$$

Dividing both sides by T, we arrive at Parseval's theorem:

$$\frac{1}{T} \int_{t_0}^{t_0+T} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Interpretations:

Energy conservation: The total energy of the signal over one period is equal to the sum of the energies of its frequency components. Also, the energy in the time domain is equal to the energy in the frequency domain.

Ease of computation of energy: Parseval's theorem provides a convenient way to compute the energy of a signal in the frequency domain, which can be easier than computing it directly in the time domain (in some cases).

5 Recommended Practice Problems

- Drill 3.10 (on rectifier) in Lathi. This problem is very similar to your HW #7 problem (not identical).
- Example 3.14 in Lathi. Note that this is an advanced problem so you may want to read through it first.
- Table 3.1 in Oppenheim and Willsky and Table 3.1 in Lathi are both handy tables to screenshot and keep close!

Pop Quiz Solutions

Pop Quiz 3.1: Solution(s)

Keep on adding cosine sums with higher frequencies and observe how the approximation improves.

EE 102 Week 8, Lecture 1 (Fall 2025)

Instructor: Ayush Pandey

Date: October 20, 2025

1 Goals

The overall goal in EE 102 is to break down signals into its constituent fundamental components that can be easily analyzed. So far, we have achieved this in two ways: we have broken down any arbitrary signal into impulse signals, and we have broken down periodic signals into sinusoids. What's left? Signals that are not periodic. So, our goal is ...

By the end of this lecture, you should be able to break down aperiodic signals into their constituent frequency components using Fourier Transform.

2 Review: Decomposition of signals

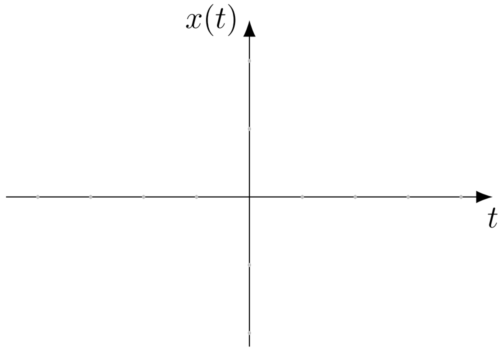
In this section, we review how to decompose signals into simpler components. We have already seen two ways to do this: using impulses and using sinusoids. We will work through one example of each.

2.1 Breaking down signals using impulses

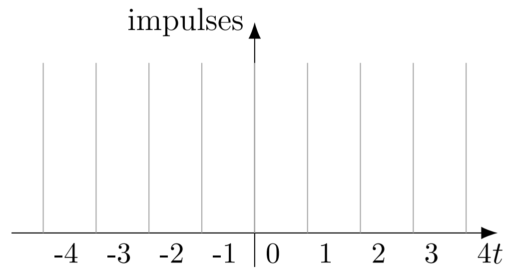
Choose a signal $x(t)$ — scribble something! Break it down so that it can be constructed using only impulses $\delta(t)$ with appropriate scales and shifts.

(a) Show it visually: On the left, sketch $x(t)$. On the right, place weighted impulses at sample locations corresponding to the value of $x(t)$ at those locations. Note that integer times are shown here, but you may choose any sampling interval (can be floating point because this is a continuous-time signal).

Sketch $x(t)$ here



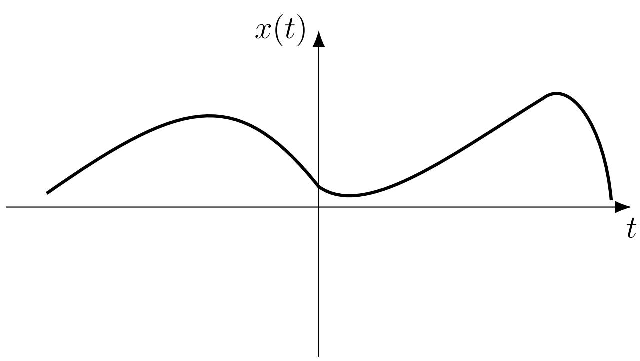
Put weights $x(n)$ beside $\delta(t-n)$

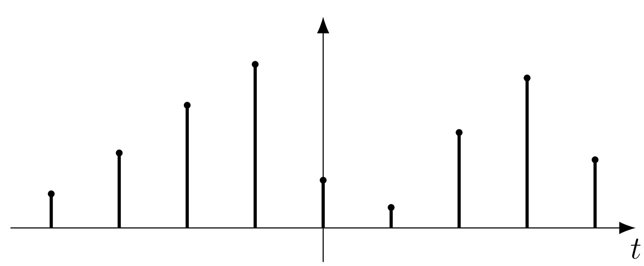


(b) Do it mathematically: Use the sifting property to write a formula for $x(t)$ using $\delta(\cdot)$.

Write your mathematical expression here:

Possible solution We sketch $x(t)$ as a continuous curve on the left, and place impulses at various locations, which is shown on the right.





Mathematically, we have the sifting property of the impulse:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

which can be approximated using a sum over discrete time instants as

$$x(t) \approx \sum_{n=-\infty}^{\infty} x(n)\,\delta(t-n)\,.$$

2.2 Breaking down periodic signals using sinusoids

Draw an impulse train with period T_0. Then, sketch (approximately) how a finite sum of sinusoids could approximate this train over time. Finally, write the corresponding Fourier-series form and coefficients.

Write $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ *and specify* a_k. *Optionally write the finite* N*-term partial sum you would draw.*

Finally, consider a 50% duty cycle symmetric square wave of amplitude A and period $T_0 = \frac{2\pi}{\omega_0}$. Sketch one period on an axes, draw the first few nonzero harmonic magnitudes, and write the Fourier series formula you would use to represent this signal.

Write the FS synthesis and the coefficients for a 50% *duty square wave of amplitude* A.

Why/how is that useful? After you have worked through the above examples, you are all set up to compute the output of any LTI system. To achieve that you would need the following additional information (one or the other):

- How does the system respond to an impulse input? (i.e., the impulse response of the system, $h(t)$)
- How does the system respond to sinusoidal inputs? (i.e., the frequency response of the system, $H(f)$)

Then, you can compute the output of the system for any arbitrary input signal $x(t)$ using either convolution or the Fourier synthesis equation (and LTI system properties!).

3 Fourier Transform: Decomposition of aperiodic signals

The **key idea** in this lecture is that aperiodic signals are also periodic signals, but with the time period $T \to \infty$. That is, the aperiodic signal also repeats itself, but only after an infinite amount of time has passed! This essentially means that it never repeats itself (inifinite is not defined).

So, let's try to apply that idea with an example. Consider a square wave of magnitude A, centered around 0, with a period of T. For the period between $-T/2$ and $T/2$, the signal is defined as:

$$x(t) = \begin{cases} A, & -T_1 \leq t \leq T_1, \\ 0, & \text{otherwise.} \end{cases}$$

We can find the Fourier series coefficients for this signal using the Fourier series analysis equation:

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt,$$

where $\omega_0 = \dfrac{2\pi}{T}$. Note that the limits of integration should be (ideally, according to the formula for a_k) over one period of the signal. However, since $x(t)$ is zero outside the interval $[-T_1, T_1]$, we can simplify the limits of integration to:

$$a_k = \frac{1}{T} \int_{-T_1}^{T_1} A e^{-jk\omega_0 t} dt.$$

Evaluating this integral, we get:

$$\begin{aligned} a_k &= \frac{A}{T} \int_{-T_1}^{T_1} e^{-jk\omega_0 t} dt \\ &= \frac{A}{T} \left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0} \right]_{-T_1}^{T_1} \\ &= \frac{A}{T} \left(\frac{e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}}{-jk\omega_0} \right) \\ &= \frac{A}{T} \left(\frac{-2j \sin(k\omega_0 T_1)}{-jk\omega_0} \right) \\ &= \frac{2A}{T} \cdot \frac{\sin(k\omega_0 T_1)}{k\omega_0}. \end{aligned}$$

For convenience, we define a new function called the **sinc function** as (pronounced "sink"):

$$\text{sinc}(x) = \frac{\sin(x)}{x}.$$

We can now rewrite the Fourier series coefficients as:

$$a_k = \frac{2AT_1}{T}\text{sinc}(k\omega_0 T_1).$$

Now, since $T \to \infty$, let's re-write the left-hand side as

$$Ta_k = 2AT_1\text{sinc}(k\omega_0 T_1).$$

Notice that as $T \to \infty$, the fundamental frequency $\omega_0 = \frac{2\pi}{T} \to 0$. The right hand side expression above is a function of frequency: $2AT_1\text{sinc}(k\omega_0 T_1)$. Since $\omega_0 \to 0$, we make a very important observation. The function of frequency can be defined as a continuous function in the limit. You can see this with the following re-definition:

$$\omega_0 := \Delta\omega$$

and then $k\omega_0 = k\Delta\omega$, which in turn we define as $\omega := \omega_k := k\Delta\omega$.

Intuitively, if you think of the frequency as the X-axis, then the Fourier series coefficients can be thought of as samples of a continuous function in the limit as $T \to \infty$. This is the key idea behind the Fourier transform: it allows us to represent aperiodic signals as a continuous sum of sinusoids, each with a specific frequency and amplitude.

For the square wave example above, we then have

$$Ta_k = 2AT_1 sinc(\omega T_1)$$

which can be further interpreted as

$$2\pi a_k = [2AT_1 sinc(\omega T_1)]\,\Delta\omega.$$

This frequency function is called the Fourier transform of the signal. What we have done is that we started from from a continuous-time (in time domain), which we have now *transformed* into a function of frequency (in frequency domain).

4 The Fourier Transform

For a general aperiodic signal $x(t)$, the Fourier transform can be derived using the Fourier series synthesis and analysis equations:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt.$$

As $T \to \infty,$ we have:

$$\omega_0 = \frac{2\pi}{T} \to 0.\text{Thus, we can write:}$$

$$Ta_k = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt,$$

where we used the definition of frequency $\omega = k\omega_0$. Now, we can rewrite the synthesis equation as:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau\right]e^{j\omega t}d\omega.$$

We define the above as the Fourier transform pair:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt,$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega.$$

This pair of equations allows us to transform a time-domain signal $x(t)$ into its frequency-domain representation $X(\omega)$ and vice versa! We will work on specific examples in the next class.

5 Recommended reading and practice problems

- Solved example 3.5 in Oppenheim and Willsky, 2nd edition.
- Example 4.1 in Lathi.

$_{a}nswer$

EE 102 Week 8, Lecture 2 (Fall 2025)

Instructor: Ayush Pandey

Date: August 20, 2025

1 Announcements

- HW #8 is due on Mon Oct 27. This is also your practice for the midterm exam #2 as the problems cover the material that will be on the exam.
- Midterm exam #2 will be held on Wed Oct 29 during regular class time (4.30pm - 5.45pm) in our usual classroom (COB2 175).
- HW #9 will be due the following week but this will be a shorter homework because we will only have one lecture next week.

2 Goals

By the end of this lecture, you should be able to understand Fourier Transforms (FT) of standard signals and appreciate the value of the frequency domain in understanding signals.

3 Review: Fourier analysis of aperiodic signals

Recall that we started our "Fourier journey" by arguing that it would be very useful if we could represent any arbitrary signal using only basic building blocks of sine and cosine functions. So far, we have seen that this is indeed possible for periodic signals using Fourier Series (FS). We proposed that any periodic signal $x(t)$ with period T can be represented as a linear combination of complex exponentials as follows:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \tag{1}$$

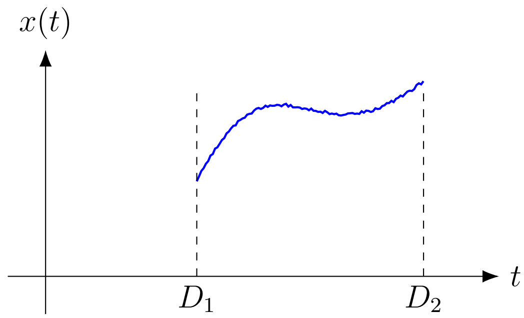


Figure 1: An aperiodic signal $x(t)$ defined over a finite interval.

where $\omega_0 = \frac{2\pi}{T}$ is the fundamental frequency and the Fourier coefficients a_k are given by

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \tag{2}$$

where the integral is taken over any interval of length T.

Equations (**??**) and (**??**) are known as the synthesis and analysis equations of Fourier Series, respectively.

3.1 From Fourier Series to Fourier Transform

To extend the Fourier Series representation to aperiodic signals, we consider the limit as the period T approaches infinity. The intuition here is that as T becomes very large, the periodic signal $x(t)$ will resemble an aperiodic signal over any finite interval.

So, consider an aperiodic signal as shown in Figure **??**.

To apply Fourier Series to this aperiodic signal, we note that this is also a periodic signal BUT with an infinite time period. That is, every ∞ seconds, the signal repeats itself! An infinite number of seconds is not measurable and therefore, the signal never actually repeats itself in any finite time interval. But this mathematical trick allows us to use the Fourier Series representation for this aperiodic signal. So, $x(t)$ can be defined for all time as:

$$x(t) = \begin{cases} x(t), \text{ (the given function)}, & D_1 \leq t \leq D_2 \\ 0, & \text{otherwise} \end{cases}$$

This trick manifests itself in many ways that change the FS synthesis and analysis equations (**??**) and (**??**). Let's work through these steps one by one.

First, recall that

$$T = \frac{2\pi}{\omega_0} \implies \omega_0 = \frac{2\pi}{T}$$

So, as $T \to \infty$, we have $\omega_0 \to 0$. This means that the fundamental frequency becomes infinitesimally small. Since this is an infinitesimally small quantity, we rename it as $\Delta\omega$. So, we have $\Delta\omega \to 0$. Next, in FS equations, we have $k\omega_0$. With the renamed variable for ω_0, we have

$$k\omega_0 = k\Delta\omega := \omega$$

where the last step is a definition of a new variable ω that we set equal to $k\Delta\omega$. You should note that as $\Delta\omega \to 0$, we multiple it by k which takes all integer values from $-\infty$ to ∞. Therefore, the variable ω takes all real values from $-\infty$ to ∞. This is interesting because even though $\Delta\omega$ is infinitesimally small (very very close to zero), it is not exactly zero. And so, by multiplying it with all integer values of k, we can get all real values of ω. A small, very small quantity, can also have a big impact! A life lesson here (never stop going for that big impactful outcome even if you feel small and insignificant)!

With the bookkeeping done above (and life lessons learned on the way), we are now ready to rewrite the FS analysis equation (**??**) in the limit as $T \to \infty$. We have

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

as $T \to \infty$, integral limits become $-\infty$ to ∞ because the signal is aperiodic and is zero outside of the interval $[D_1, D_2]$.

$$Ta_k = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Key observation here is that the right hand side is a function of frequency (time gets integrated over). So, we define yet another thing. A function of frequency called $X(\omega)$. Let $X(\omega) = Ta_k$. Then,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{3}$$

Equation (**??**) is defined as the Fourier Transform (FT) of the signal $x(t)$. It is the frequency domain representation of the time domain signal $x(t)$. So far, we have defined how to transform the signal $x(t)$ into a function that characterizes the signal in the frequency domain (using a function of frequency). But we have not yet shown how $x(t)$, an aperiodic signal, can be broken down into its frequency components (or in other words, into sinusoidal signals).

To show how $x(t)$ can be synthesized from its frequency components, we start with the FS synthesis equation (**??**):

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

As $T \to \infty$, we have $\omega_0 \to \Delta\omega$ and $k\omega_0 = \omega$. Therefore, we can rewrite the synthesis equation as:

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{X(\omega)}{T} e^{j\omega t}$$

where we used the definition of $X(\omega)$ from above and the definition of frequency ω as $k\Delta\omega$. Since $T = \frac{2\pi}{\Delta\omega}$, we have

$$x(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega) e^{j\omega t} \Delta\omega.$$

As $\Delta\omega \to 0$, the summation becomes an integral over all real values of ω:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \tag{4}$$

Equation (**??**) is the synthesis equation for Fourier Transform. It shows how any aperiodic signal $x(t)$ can be synthesized from its frequency components represented by $X(\omega)$.

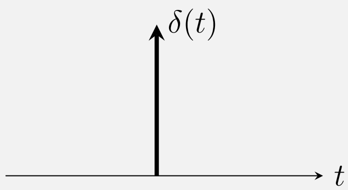
4 Fourier analysis of standard signals

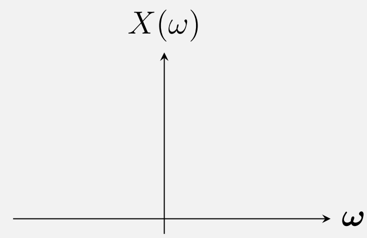
We have set up many standard signals in this class so far: impulse, step, complex exponential, sinusoid, square wave, and more. Now that we are equipped with applying the Fourier analysis to *any* signal[1] (whether it is periodic or aperiodic), we can get many insights about the utility of the Fourier analysis.

Pop Quiz 4.1: Check your understanding!

Without computing the Fourier Transform, predict the frequency domain representation of a pure impulse signal $\delta(t)$ and sketch it out in a graph (on the right):

[1] In EE 102, we are not going to discuss the specific mathematical conditions needed for Fourier analysis to apply.





Hint: On Desmos Graphing Calculator[a], try graphing cosines added together. For example, start with $\cos(x)$, then try $\cos(x)+\cos(2x)$, then $\cos(x)+\cos(2x)+\cos(3x)$, and so on. What happens as you keep adding more cosine terms? How many frequencies would you need to add to approximate an impulse at $x = 0$?

Solution on page **??**

[a] https://www.desmos.com/calculator

4.1 Fourier Transform of an impulse

For $x(t) = \delta(t)$, we have

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

Using the sifting property of the impulse, we get

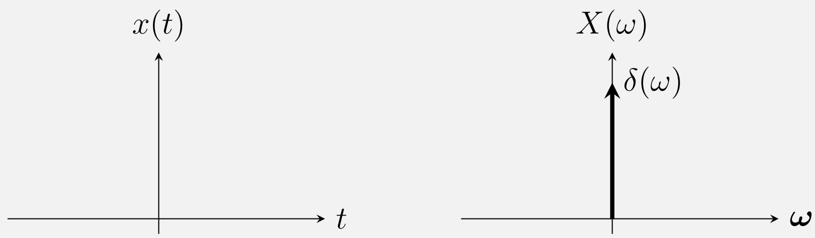
$$X(\omega) = e^{-j\omega \cdot 0} = 1$$

This confirms our intuition from the pop quiz above. The Fourier Transform of an impulse signal is a constant function equal to 1 for all frequencies ω.

4.2 Inverse Fourier Transform of an impulse in frequency domain

Pop Quiz 4.2: Check your understanding!

Without computing the inverse Fourier Transform, predict the time domain representation of a frequency domain impulse signal $X(\omega) = \delta(\omega)$ and sketch it out in a graph on the left:



Solution on page **??**

For $X(\omega) = \delta(\omega)$, we have

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega$$

Using the sifting property of the impulse, we get

$$x(t) = \frac{1}{2\pi} e^{j0 \cdot t} = \frac{1}{2\pi}$$

This confirms our intuition from the pop quiz above. The inverse Fourier Transform of an impulse in frequency domain is a constant function equal to $\frac{1}{2\pi}$ for all time t. By solving it out, we can now see that the magnitude of the constant function is $\frac{1}{2\pi}$. This is a DC voltage signal (if we were talking about voltages). You can relate this with concepts from your circuits class. Whenever you talk about DC signals, you say that it is a signal with 0 frequency. Here, we see that a signal with only 0 frequency component (an impulse at 0 frequency) is indeed a DC signal in time-domain.

So, an impulse $2\pi\delta(\omega)$ in frequency domain would correspond to a unit DC signal in time-domain.

4.3 Shifted impulse in frequency domain

For $X(\omega) = \delta(\omega - \omega_0)$, we have

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega$$

Using the sifting property of the impulse, we get

$$x(t) = \frac{1}{2\pi} e^{j\omega_0 t}$$

This shows that a shifted impulse in frequency domain corresponds to a complex exponential signal (consisting of both a sine and a cosine term) in time domain. The frequency of the complex exponential is determined by the location of the impulse in frequency domain.

This also relates to your circuits class. Whenever we say that a signal has ONE frequency, we mean that we have a pure sinusoid at that frequency (either a cosine or a sine). Taking the real or imaginary part of the complex exponential above gives us a cosine or sine signal, respectively, both with frequency ω_0.

5 Recommended reading and practice problems

- Solved Example 4.2 in Lathi (Fourier Transform of a rectangular pulse)
- Solved Example 4.10 in Lathi (Fourier Transform of a sign function)
- Solve the problem 4.3-15 in Lathi (Fourier transform of the differentiation operation)

Pop Quiz Solutions

Pop Quiz 4.1: Solution(s)

The impulse in time-domain contains all infinite frequencies. Therefore, the frequency domain representation $X(\omega)$ is a constant function equal to 1 for all ω. This means that the impulse signal has equal contributions from all frequency components. Note that whenever we are looking for a frequency domain representation of a signal, we are looking for a function of frequency (X-axis is frequency).

Pop Quiz 4.2: Solution(s)

One impulse at 0 frequency is a DC signal (constant signal) in time domain. Therefore, the time domain representation $x(t)$ is a constant function for all t.

EE 102 Week 9, Lecture 1 (Fall 2025)

Instructor: Ayush Pandey

Date: October 27, 2025

1 Announcements

- HW #8 is due on Mon Oct 27. This is also your practice for the midterm exam #2 as the problems cover the material that will be on the exam.
- Midterm exam #2 will be held on Wed Oct 29 during regular class time (4.30pm - 5.45pm) in our usual classroom (COB2 175).
- HW #9 will be due AFTER A WEEK's break on Nov 10.

2 Goals

By the end of this lecture, you should be able to connect Fourier Transforms to at least one real-world application.

3 Quiz: Knowledge check (so far)

Pop Quiz 3.1: Check your understanding!

If the impulse response of a LTI system is h(t). What is the output of the system to a step input u(t)?

- $\int_{-\infty}^{\infty} h(\tau)d\tau$
- $\int_{-\infty}^{t} h(\tau)d\tau$
- $\frac{d}{dt}h(t)$
- $\int_{0}^{t} h(\tau)d\tau$

Solution on page 8

Pop Quiz 3.2: Check your understanding!

What is $2[\cos(t) + \sin(t)]$ in complex exponential notation?

- $(-1 + j)e^{jt} + (1 + j)e^{-jt}$
- $(1 + j)e^{jt} + (-1 + j)e^{-jt}$
- $(1 + j)e^{jt} + (1 - j)e^{-jt}$
- $(1 - j)e^{jt} + (1 + j)e^{-jt}$

Solution on page 8

Pop Quiz 3.3: Check your understanding!

What is the DC term in the Fourier Series (a_0) for the signal: $1 + \cos(2t)$?

- 0
- 1
- $\frac{1}{2j}$
- -1

Solution on page 9

Pop Quiz 3.4: Check your understanding!

For $\omega_0 = 1$, what are the non-zero complex Fourier Series coefficients (a_k) of the signal: $1 + \cos(t) + \sin(2t)$? That is, find the values for a_{-2}, a_{-1}, a_0, a_1, and a_2.

- $1/2j, 1/2, 0, 1/2, 1/2j$
- $-1/2j, 1/2, 1, 1/2, 1/2j$
- $1/2, 1/2, 0, 1/2, 1/2$
- $1/2, 1/2, 1, 1/2, 1/2$

Solution on page 9

4 Review: Fourier analysis of aperiodic signals

For any[1] aperiodic signal $x(t)$, we can represent it as a superposition of complex exponentials using the Fourier Transform:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt,\ \omega \in \mathbb{R}$$

where $X(\omega)$ is the Fourier Transform of $x(t)$. This is a function of frequency. So, the idea is that we are able to represent the original signal $x(t)$ into a new dimension where X-axis is the frequency ω instead of time t. What is this frequency? Whose frequency?

Since we are representing each signal as a combination of complex exponentials of the form $e^{j\omega t}$, we are implicitly saying that the signal is made up of sines and cosines. There is only one parameter that defines a sine or cosine wave: its frequency. So, instead of repeating sine and cosine, we start referring to their frequency (a number). So, for example, we might say that $1+\cos(2t)$ is made up of a frequency 0 component and a frequency 2 component. That's it — two numbers define the signal. This is the idea of frequency domain representation of signals.

4.1 What is so special about sinusoids?

Nothing other than the fact that it is our choice of basis functions to represent signals that are commonly found in engineering. You may relate the concept of basis functions to the

[1] Under suitable conditions on the signal that we are not explicitly discussing in this course

concept of basis vectors that you work with in your linear algebra and vector calculus class. Just like any vector in 3D space can be represented as a linear combination of the basis vectors $[1,0,0]$, $[0,1,0]$, and $[0,0,1]$, any signal can be represented as a linear combination of sinusoids of different frequencies.

Similar to vector spaces, the choice of basis functions is not unique. Indeed, quantum physicists use other basis functions that are useful for their applications. Like, wavelet functions are used in describing atoms and molecules. The math behind the Fourier transform still plays out in the same way — that is the beauty and the brilliance of this mathematical tool developed by Fourier.

4.2 Fourier transform of real and even signals

We will prove the following:

Proposition 1. *For a real and even signal $x(t)$, its Fourier Transform $X(\omega)$ is also real and even.*

Proof. Since $x(t)$ is real and even, we have:

$$x(t) = x(-t), \quad x(t) \in \mathbb{R}.$$

The Fourier Transform of $x(t)$ is given by:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

To show that $X(\omega)$ is even, we compute $X(-\omega)$:

$$X(-\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt.$$

By changing the variable of integration, let $u = -t$, so $dt = -du$. The limits of integration change accordingly: when $t = -\infty, u = \infty$; and when $t = \infty, u = -\infty$. Thus, we have:

$$X(-\omega) = \int_{\infty}^{-\infty} x(-u)e^{-j\omega u}(-du) = \int_{-\infty}^{\infty} x(u)e^{-j\omega u}du = X(\omega).$$

Therefore, $X(\omega)$ is even. To show that $X(\omega)$ is real, we compute the complex conjugate of $X(\omega)$:

$$X^*(\omega) = \left(\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right)^* = \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt = X(-\omega).$$

Since we have already shown that $X(-\omega) = X(\omega)$, it follows that:

$$X^*(\omega) = X(\omega).$$

Therefore, $X(\omega)$ is real. $\square$

Proposition 2. *For a real and even signal $x(t)$, its Fourier Transform $X(\omega)$ can be rewritten as a Fourier Cosine Transform.*

Proof. Write $X(\omega)$ as:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

Using Euler's formula, we can express the complex exponential as:

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t).$$

Substituting this into the expression for $X(\omega)$, we get:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)[\cos(\omega t) - j\sin(\omega t)]dt.$$

Since the function is even, let us break down the integral into two parts: $-\infty$ to 0 and 0 to ∞:

$$X(\omega) = \int_{-\infty}^{0} x(t)[\cos(\omega t) - j\sin(\omega t)]dt + \int_{0}^{\infty} x(t)[\cos(\omega t) - j\sin(\omega t)]dt.$$

By changing the variable of integration in the first integral, let $u = -t$, so $dt = -du$. The limits of integration change accordingly: when $t = -\infty, u = \infty$; and when $t = 0, u = 0$. Thus, we have:

$$X(\omega) = \int_{\infty}^{0} x(-u)[\cos(-\omega u) - j\sin(-\omega u)](-du) + \int_{0}^{\infty} x(t)[\cos(\omega t) - j\sin(\omega t)]dt.$$

Using the evenness of $x(t)$, we have $x(-u) = x(u)$. Also, $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$. Therefore, the first integral becomes:

$$X(\omega) = \int_{0}^{\infty} x(u)[\cos(\omega u) + j\sin(\omega u)]du + \int_{0}^{\infty} x(t)[\cos(\omega t) - j\sin(\omega t)]dt.$$

Combining the two integrals, we get:

$$X(\omega) = \int_{0}^{\infty} x(t)[\cos(\omega t) + j\sin(\omega t) + \cos(\omega t) - j\sin(\omega t)]dt = 2\int_{0}^{\infty} x(t)\cos(\omega t)dt.$$

Thus, we have shown that for a real and even signal $x(t)$, its Fourier Transform $X(\omega)$ can be expressed as:

$$X(\omega) = 2\int_{0}^{\infty} x(t)\cos(\omega t)dt.$$

□

5 Properties of Fourier Transform

5.1 Time shifting property

If $x(t)$ has the Fourier Transform $X(\omega)$, then the Fourier Transform of the time-shifted signal $x(t-t_0)$ is given by:

$$X_{\text{shifted}}(\omega) = X(\omega)e^{-j\omega t_0}.$$

Proof. The Fourier Transform of the time-shifted signal $x(t-t_0)$ is given by:

$$X_{\text{shifted}}(\omega) = \int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}dt.$$

By changing the variable of integration, let $u = t - t_0$, so $dt = du$. The limits of integration remain the same: when $t = -\infty, u = -\infty$; and when $t = \infty, u = \infty$. Thus, we have:

$$X_{\text{shifted}}(\omega) = \int_{-\infty}^{\infty} x(u)e^{-j\omega(u+t_0)}du = e^{-j\omega t_0}\int_{-\infty}^{\infty} x(u)e^{-j\omega u}du.$$

Recognizing that the integral is the Fourier Transform of $x(t)$, we get:

$$X_{\text{shifted}}(\omega) = X(\omega)e^{-j\omega t_0}.$$

□

All other properties of the Fourier transform are listed and proved in the textbooks: Section 4.6 in Oppenheim and Willsky, "Signals and Systems", 2nd edition and Table 4.2 in Lathi.

6 Looking forward to the applications of Fourier Transform

Take this fact to be true for now: We will be "multiplying" filters to our signals to achieve desired effects. That is, if in the frequency domain, you have a signal with a "value" of 0.5 at 100 Hz. Now, if you multiple this signal by a frequency domain signal (the filter) that is 0 all across but has a value of 10 at 100Hz. Then, your original signal becomes 5 at 100Hz. This is the idea of filtering signals in the frequency domain.

With this broad idea, draw your concept of the following filters:

- Design the $H(\omega)$ for a filter that passes all frequencies that lie between 50 Hz to 500 Hz. Remember that $\omega = 2\pi f$.
- Design a personalized filter that only allows your personal voice notes (voice frequencies) to pass through and blocks all other sounds.

7 Recommended reading and practice problems

To study for the midterm exam #2, please work on the EE 102 pre-midterm MCQ on CatCourses and HW #8.

Pop Quiz Solutions

Pop Quiz 3.1: Solution(s)

The output of a LTI system to an input $x(t)$ is given by the convolution of the input with the impulse response $h(t)$:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

For a step input $u(t)$, we have:

$$y(t) = u(t) * h(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau.$$

Since $u(\tau) = 0$ for $\tau < 0$ and $u(\tau) = 1$ for $\tau \geq 0$, the limits of integration change to:

$$y(t) = \int_{0}^{\infty} h(t-\tau)d\tau.$$

By changing the variable of integration, $s = t - \tau$, we have $ds = -d\tau$. The limits change accordingly: when $\tau = 0, s = t$; and when $\tau = \infty, s = -\infty$. Thus, we get:

$$y(t) = \int_{t}^{-\infty} h(s)(-ds) = \int_{-\infty}^{t} h(s)ds.$$

Pop Quiz 3.2: Solution(s)

We can use Euler's formula to express cosine and sine in terms of complex exponentials:

$$\cos(t) = \frac{e^{jt} + e^{-jt}}{2}, \quad \sin(t) = \frac{e^{jt} - e^{-jt}}{2j}.$$

Therefore,

$$2[\cos(t) + \sin(t)] = 2\left[\frac{e^{jt} + e^{-jt}}{2} + \frac{e^{jt} - e^{-jt}}{2j}\right].$$

Simplifying this expression, we get:

$$2[\cos(t) + \sin(t)] = e^{jt} + e^{-jt} + \frac{1}{j}[e^{jt} - e^{-jt}].$$

Since $\frac{1}{j} = -j$, we have:

$$2[\cos(t) + \sin(t)] = (1-j)e^{jt} + (1+j)e^{-jt}.$$

Pop Quiz 3.3: Solution(s)

The DC term in the Fourier Series, denoted as a_0, represents the average value of the signal over one period. For the signal $x(t) = 1+\cos(2t)$, we can calculate a_0 as follows:

$$a_0 = \frac{1}{T}\int_0^T x(t)dt,$$

where T is the period of the signal. The period of $\cos(2t)$ is $\frac{2\pi}{2} = \pi$. Therefore, we have:

$$a_0 = \frac{1}{\pi}\int_0^\pi (1+\cos(2t))dt.$$

Evaluating the integral:

$$\int_0^\pi (1+\cos(2t))dt = \int_0^\pi 1dt + \int_0^\pi \cos(2t)dt = \pi + 0 = \pi.$$

Thus,

$$a_0 = \frac{1}{\pi}\cdot\pi = 1.$$

Pop Quiz 3.4: Solution(s)

The complex Fourier Series coefficients a_k for a periodic signal $x(t)$ with fundamental frequency ω_0 are given by:

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt,$$

where $T = \frac{2\pi}{\omega_0}$ is the period of the signal. For $\omega_0 = 1$, we have $T = 2\pi$. The signal is $x(t) = 1+\cos(t)+\sin(2t)$. We can express $\cos(t)$ and $\sin(2t)$ in terms of complex exponentials:

$$\cos(t) = \frac{e^{jt}+e^{-jt}}{2}, \quad \sin(2t) = \frac{e^{j2t}-e^{-j2t}}{2j}.$$

Therefore, the signal can be rewritten as:

$$x(t) = 1+\frac{e^{jt}+e^{-jt}}{2}+\frac{e^{j2t}-e^{-j2t}}{2j}.$$

This gives us the following non-zero coefficients:

$$a_0 = 1, \quad a_1 = \frac{1}{2}, \quad a_{-1} = \frac{1}{2}, \quad a_2 = \frac{1}{2j}, \quad a_{-2} = -\frac{1}{2j}.$$

EE 102 Week 10, Lecture 1 (Fall 2025)

Instructor: Ayush Pandey

Date: November 3, 2025

1 Announcements

- HW #9 is due on Mon Nov 10.
- Midterm exam #2 take-home version is due Nov 3 at midnight. The midterm score s_m will be computed as follows:

  $$s_m = \frac{s_i + \max(s_i, s_t)}{2}$$

  where s_i is the in-class midterm score and s_t is the take-home midterm score. So, you won't lose any points on the exam if you choose to not submit the take-home version.

2 Goals

By the end of this lecture, you will be able to compute the output of an LTI system to any input signal...in frequency domain.

This goal has two parts: you already know how to achieve the first part — the convolution integral gives you the output of an LTI system to any input signal if you know the impulse response of the system. The second part is what we will focus on in this lecture: how to compute the output of an LTI system in frequency domain, and how that might be useful in engineering.

3 Review: Output of LTI systems using convolution

Consider a system H shown in the block diagram below:

$$x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$$

Note that H is a notation we are using for the system. It is not a function, or a signal. It is the abstract representation of the system itself. We usually model the system using its impulse response $h(t)$. That is, the response (output) of the system when the input is $\delta(t)$ is $h(t)$. For the input $x(t)$, what is the output $y(t)$ of the system? The answer is given by the convolution integral:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

This is the time-domain representation of the output of an LTI system.

4 Review: Breaking down signals

We have seen many ways of breaking down signals into simpler components in this class: breaking down a signal into shifted and scaled impulses using the sifting property of the delta function, breaking down a signal into step functions, breaking down a signal into complex exponentials using Fourier series and Fourier transforms, etc. We review the last two methods below.

4.1 Periodic signals can be broken down using Fourier series

For a periodic signal $x(t)$ with period T, we can write:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \tag{1}$$

where $\omega_0 = \frac{2\pi}{T}$ and

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt. \tag{2}$$

4.2 Aperiodic signals can be broken down using Fourier transforms

If a signal $x(t)$ is aperiodic, we can break it down into complex exponentials using the Fourier transform. The Fourier transform is also a representation of the signal $x(t)$ in the frequency domain (with frequency variable ω as the X-axis). We have

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt, \tag{3}$$

and the inverse Fourier transform (Fourier synthesis) is given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega. \tag{4}$$

5 Computing the output of LTI systems

We start this section with a pop-quiz where you will be asked to apply your knowledge of LTI systems (superposition) and Fourier series to compute the output of an LTI system to a periodic input signal.

Pop Quiz 5.1: Check your understanding!

For an LTI system, we know that the output of the system to an input $e^{j\omega t}$ is given by $H(j\omega)e^{j\omega t}$, where $H(j\omega)$ is the frequency response of the system (observe the eigen-function property as the input appears again in the output). Using the Fourier series representation of a periodic signal, compute the output of an LTI system to a periodic input signal $x(t)$ with period T.

Solution on page 7

For a periodic signal $x(t)$ with period T, the output of an LTI system to the input $x(t)$ is given by

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t},$$

where a_k are the Fourier series coefficients of $x(t)$ and $\omega_0 = \frac{2\pi}{T}$. The Fourier series coefficients of the output signal $y(t)$ are given by

$$b_k = a_k H(jk\omega_0).$$

Observe how the Fourier Series coefficients of the output signal (this is the frequency domain representation of the periodic output signal) are obtained by multiplying the Fourier series coefficients of the input signal with the frequency response of the system evaluated at discrete frequencies $k\omega_0$.

5.1 Output of LTI systems to aperiodic signals in the frequency domain

Now, we move on to finding the output of an LTI system to aperiodic input signals. But before we jump into it, it is important to remind ourselves of the time-shifting property of the Fourier transform.

Pop Quiz 5.2: Check your understanding!

If the Fourier transform of a signal $x(t)$ is $X(\omega)$, prove that the Fourier transform of the time-shifted signal $x(t-t_0)$ is given by $X(\omega)e^{-j\omega t_0}$.

Solution on page 7

To write the output of an LTI system to an aperiodic input signal, we start with the convolution equation

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

The Fourier transform of the output signal $y(t)$ is given by using the Fourier transform equation (3):

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \right) e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dt \right) d\tau \end{aligned}$$

where we wrote the last step by interchanging the order of integration: we brought the $d\tau$ integral outside and kept all the terms involving τ inside. We kept the t terms where they were and we will now integrate over t first. From the previous pop-quiz, we know that

$$\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dt = H(\omega)e^{-j\omega\tau}.$$

Substituting this in the previous equation, we get

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} x(\tau)H(\omega)e^{-j\omega\tau}d\tau \\ &= H(\omega)\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau \\ &= H(\omega)X(\omega). \end{aligned}$$

Thus, we have shown that the output signal (in frequency domain) is given by

$$Y(\omega) = H(\omega)X(\omega)$$

the product of the frequency response of the system and the Fourier transform of the input signal. Just the multiplication! No integration!

This is a VERY IMPORTANT result. For starters, it lets us compute the output of an LTI system to any input signal using simple multiplication in frequency domain. Second, it gives us insights into how different frequency components of the input signal are affected by the system. For example, if $H(\omega)$ is small for large values of ω, then we know that high-frequency components of the input signal will be attenuated in the output signal. It allows us to *design* systems with a desired $H(\omega)$. For example, if there are two (or however many) frequencies that I absolutely dislike (my ears hurt when I hear the screeching sound of a chalk on a blackboard of frequency ω_1!). Then, I can design a system such that the frequency response $H(\omega)$ has very small values at ω_1: $H(\omega_1) \approx 0$. This system will attenuate the frequency component ω_1 in the output signal. Such systems are called notch filters.

The fundamental concept we have derived above is called the Convolution Theorem (or the convolution property of Fourier transforms).

6 Example: First-order decay systems

A lot of things in real-life respond with a first-order decay to an impulse input. That is, if you suddenly push someone (an impulse input), they might be affected at $t = 0$ but then they will slowly come to rest (decay) over time. Mathematically, the impulse response of such a system is given by

$$h(t) = e^{-at}u(t),$$

where $a > 0$ is a constant that determines how fast the system decays. The unit step function $u(t)$ ensures that the impulse response is causal (the output cannot start before the input is applied at $t = 0$).

Now, what if there is a new input from a particularly annoying classmate who pushes you for a while but slowly decays their push. The input signal can be modeled as

$$x(t) = e^{-bt}u(t),$$

where $b > 0$ is another constant. What is the output of the system to this input signal?

To compute the output of the system to this input signal, we first compute the Fourier transforms of the input signal and the impulse response.

$$\begin{aligned} X(\omega) &= \int_0^\infty e^{-bt}e^{-j\omega t}dt \\ &= \int_0^\infty e^{-(b+j\omega)t}dt \\ &= \left[\frac{e^{-(b+j\omega)t}}{-(b+j\omega)}\right]_0^\infty \\ &= \frac{1}{b+j\omega}, \end{aligned}$$

and

$$\begin{aligned}
H(\omega) &= \int_0^\infty e^{-at} e^{-j\omega t} dt \\
&= \int_0^\infty e^{-(a+j\omega)t} dt \\
&= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_0^\infty \\
&= \frac{1}{a+j\omega}.
\end{aligned}$$

Using the convolution theorem, the output of the system in frequency domain is given by

$$Y(\omega) = H(\omega)X(\omega) = \frac{1}{(a+j\omega)(b+j\omega)}.$$

It's that easy! Now, we can already analyze the frequency properties of the output without bothering to apply the convolution integral or the inverse Fourier transform. For example, we can see that as $\omega \to \infty$, $Y(\omega) \to 0$. Thus, high-frequency components of the input signal are attenuated in the output signal.

With the convolution theorem at hand, we can now study many many kinds of engineering systems because this is what enables many areas of electrical engineering: frequency modulation, amplitude modulation, communication systems, control systems, signal processing, image filtering, audio processing, and more! Soon...

7 Recommended reading and practice problems

Section 4.5.1 in Oppenheim and Willsky, 2nd edition is an interesting section to read to learn more about the applications of Fourier transforms in communication.

Pop Quiz Solutions

Pop Quiz 5.1: Solution(s)

Using the Fourier series representation of $x(t)$ from (4), we have

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

where $\omega_0 = \frac{2\pi}{T}$. Using the superposition property of LTI systems, the output of the system to the input $x(t)$ is given by

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}.$$

The Fourier series coefficients of the output signal $y(t)$ are given by $b_k = a_k H(jk\omega_0)$.

Pop Quiz 5.2: Solution(s)

Using the definition of the Fourier transform, we have

$$\mathcal{F}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt.$$

Apply the most common integration solving trick: change of variables (colloquially known as u-sub). Let $u = t - t_0$. Then, $du = dt$, and $t = u + t_0$. Substituting these in the integral, we get

$$\begin{aligned}
\mathcal{F}\{x(t-t_0)\} &= \int_{-\infty}^{\infty} x(u) e^{-j\omega(u+t_0)} du \\
&= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(u) e^{-j\omega u} du \\
&= e^{-j\omega t_0} X(\omega).
\end{aligned}$$

EE 102 Week 10, Lecture 2 (Fall 2025)

Instructor: Ayush Pandey

Date: November 5, 2025

1 Announcements

- EE 122 Lab 02L is open and registration spots are available.
- HW #9 is due on Mon Nov 10.

2 Goals

By the end of this lecture, you will be able to understand the Fourier Transform of discrete-time signals.

3 Review: Fourier Transform in Continuous-Time

The Fourier Transform (FT) of a continuous-time signal $x(t)$ is defined as:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

where ω is the angular frequency in radians per second. The Fourier transform provides a frequency-domain representation of the signal, allowing us to analyze its frequency components. That is, the signal, originally visualized with time as the X-axis, is now possible to be visualized with frequency (ω) as the X-axis.

On the other hand, if you know how the signal behaves in the frequency domain, you can recover the time-domain signal using the Inverse Fourier Transform (IFT):

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

4 Fourier Transform of Discrete-Time Signals

For discrete-time signals, the Fourier Transform is defined similarly. What will you change in the two equations above? First, the variable t is replaced by n, which represents discrete time indices (integers). That is, $n \in \mathbb{Z}$ while $t \in \mathbb{R}$ (all real numbers). Second, the integral will be replaced by a summation since we are dealing with discrete values (integration works for continuous values). Therefore, the Fourier Transform of a discrete-time signal $x[n]$ is given by:

$$X(\dots) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{1}$$

Why the "$\dots$" in the equation above?

Pop Quiz 4.1: Check your understanding!

Is the Fourier Transform of a discrete-time signal a function of continuous frequency ω or discrete frequency k?

Solution on page 11

Notice that in the equation (1), we still have ω in the equation ($\omega \in \mathbb{R}$). So, the Fourier Transform of a discrete-time signal is a continuous function of frequency ω (in radians). This transform is called the Discrete-Time Fourier Transform (DTFT). Additionally, there are two reasons for **not** denoting it as $X(\omega)$. First, not a significant issue, but not denoting it as $X(\omega)$ helps to avoid confusion with the Fourier Transform of continuous-time signals (which is denoted as $X(\omega)$). The second issue is more important. Work on the following pop-quiz to understand it.

Pop Quiz 4.2: Check your understanding!

Prove the following:

- The continuous-time complex exponential signal $x(t) = e^{j\omega t}$ is periodic *in time* with period $T = \frac{2\pi}{\omega}$.
- The discrete-time complex exponential signal $x[n] = e^{j\omega n}$ is **periodic *in frequency*** with period 2π.

Solution on page 11

The discrete-time complex exponential signal $x[n] = e^{j\omega n}$ is periodic in frequency with period 2π. This may be counter-intuitive – why don't we have such a property in continuous-time? The key change from continuous-time to discrete-time is that time is now discrete (i.e., n takes only integer values, rather than taking all values in $\mathbb{R}$). So, after every 2π increase in frequency, the discrete time complex exponential signal repeats itself because n is an integer and any integer multiple of 2π in the exponent results in a full rotation around the unit circle. To see this more concretely, you can use Euler's identity to express the discrete-time complex exponential signal as:

$$x[n] = e^{j\omega n} = \cos(\omega n) + j\sin(\omega n)$$

After a 2π increase in frequency, we have:

$$x[n] = e^{j(\omega+2\pi)n} = e^{j\omega n}e^{j2\pi n} = e^{j\omega n}(\cos(2\pi n) + j\sin(2\pi n)) = e^{j\omega n}(1 + j\cdot 0) = e^{j\omega n}$$

So, you see the periodicity! If we follow through the same analysis in continuous-time, we will see that the periodicity does not hold for frequency. Given $x(t) = e^{j\omega t}$, after a 2π increase in frequency, we have:

$$x(t) = e^{j(\omega+2\pi)t} = e^{j\omega t}e^{j2\pi t} = e^{j\omega t}(\cos(2\pi t) + j\sin(2\pi t))$$

The term $\cos(2\pi t) + j\sin(2\pi t)$ does not equal 1 for all real values of t, so the periodicity does not hold in continuous-time.

FINALLY, we can replace the "..." in equation (1). We have established two properties of the DTFT: it is a continuous function of frequency and it is periodic with period 2π. Due to these two properties, we dentoe the DTFT as $X(e^{j\omega})$ to emphasize that it is a function of continuous frequency ω and is periodic with period 2π. Therefore, the final expression for the Discrete-Time Fourier Transform (DTFT) of a discrete-time signal $x[n]$ is:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{2}$$

Since the DTFT is a continuous function of frequency, the inverse DTFT will not involve summation over frequency (continuous variables are integrated). Therefore, the Inverse Discrete-Time Fourier Transform (IDTFT) is given by:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \tag{3}$$

The integration limit indicates that the integration is performed over one period of the DTFT, which is from 0 to 2π (or any other interval of length 2π) since the DTFT is periodic with period 2π.

5 Properties of DTFT

5.1 Periodicity

We have discussed how the discrete-time complex exponential signal is periodic in frequency with period 2π. Using that, we can show that the DTFT is also periodic with period 2π. To see this, consider the DTFT expression in equation (2):

$$X(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2\pi)n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}e^{-j2\pi n} = X(e^{j\omega})$$

Thus, the DTFT is periodic with period 2π.

5.2 Linearity

Just like the continuous-time Fourier Transform, the DTFT is also linear. That is, if you have two discrete-time signals $x_1[n]$ and $x_2[n]$ with DTFTs $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$ respectively, then for any constants a and b, the DTFT of the linear combination $y[n] = ax_1[n] + bx_2[n]$ is given by:

$$Y(e^{j\omega}) = aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

5.3 Time Shifting

If a discrete-time signal $x[n]$ has a DTFT $X(e^{j\omega})$, then the DTFT of the time-shifted signal $x[n-n_0]$ is given by:

$$X_{\text{shifted}}(e^{j\omega}) = X(e^{j\omega})e^{-j\omega n_0}$$

This property indicates that a shift in the time domain corresponds to a multiplication by a complex exponential in the frequency domain.

5.4 Frequency Shifting

If a discrete-time signal $x[n]$ has a DTFT $X(e^{j\omega})$, then the DTFT of the frequency-shifted signal, with shifted frequency: $\omega_{\text{shifted}} = \omega - \omega_0$, is given by $X(e^{j(\omega-\omega_0)})$. This can be written as

$$X(e^{j(\omega-\omega_0)}) = X(e^{j\omega})e^{j\omega_0 n}$$

5.5 Convolution

The convolution property, as discussed in the previous lecture, is among the most important property for signal processing applications. For a system with impulse response $h[n]$ and input signal $x[n]$, the output signal $y[n]$ is given by the convolution of $x[n]$ and $h[n]$:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

The DTFT of the output signal $y[n]$ is given by the product of the DTFTs of the input signal and the impulse response:

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

This is useful because now you get full control over the design of the system by designing the frequency response $H(e^{j\omega})$ to achieve the desired output characteristics.

6 Examples: Filtering

Let's start with a simple example. Consider the design of an audio processing system. So, our goal is to design $h(t)$ in time-domain (or if it is easier, we might design $H(j\omega)$ in frequency-domain) to achieve the desired output characteristics. Let's say that there is an audio input signal $x(t)$ and we only want to keep the three most dominant frequency components of the signal and remove all other frequency components in the output. If the three most dominant frequency components are at frequencies ω_1, ω_2, and ω_3, then we can design the frequency response of the system as follows:

$$H(j\omega) = \begin{cases} 1, & \text{if } \omega \in (\omega_1 - \Delta\omega, \omega_1 + \Delta\omega) \\ 1, & \text{if } \omega \in (\omega_2 - \Delta\omega, \omega_2 + \Delta\omega) \\ 1, & \text{if } \omega \in (\omega_3 - \Delta\omega, \omega_3 + \Delta\omega) \\ 0, & \text{otherwise} \end{cases}$$

where $\Delta\omega$ is a small frequency range around each dominant frequency component. This design allows only the desired frequency components to pass through while attenuating all other frequencies (because of the convolution property).

6.1 Filtering an audio mix

Consider an audio signal where you have three frequency tones mixed together:

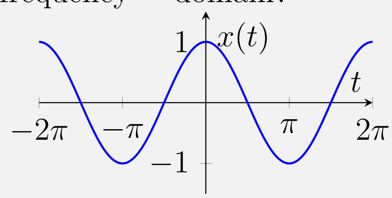
$$x(t) = \cos(t) + \cos(5t) + \cos(100t)$$

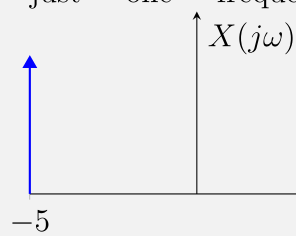
The three frequency components are at $\omega_1 = 1$ rad/s, $\omega_2 = 5$ rad/s, and $\omega_3 = 100$ rad/s. Our goal is to design a filter that only allows the frequency component at $\omega_2 = 5$ rad/s to pass through while attenuating the other two frequency components. We can achieve this by designing the frequency response of the audio processing system. Then, by using the convolution property, the output Fourier Transform will be the product of the input Fourier Transform and the designed frequency response. Finally, we can use the Inverse Fourier Transform to recover the time-domain output signal.

The first step is to compute the Fourier Transform of the input signal $x(t)$.

Pop Quiz 6.1: Check your understanding!

What is the Fourier Transform of $x(t) = \cos(\omega_0 t)$? Start by predicting what you expect the $\cos(t)$ signal to look like in the frequency domain? Hint: It's just one frequency component.





Solution on page 12

To see the Fourier transform of a cosine signal formally, compute the inverse Fourier Transform of the following signal:

$$X(\omega) = 2\pi\delta(\omega - \omega_0)$$

The inverse Fourier Transform is given by:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

(you can see the reason why we have 2π in front of the delta function — so it cancels out with the $\frac{1}{2\pi}$ term in front of the integral in the inverse Fourier Transform equation).

Substituting $X(\omega) = 2\pi\delta(\omega - \omega_0)$ into the inverse Fourier Transform equation, we have:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega$$

now, using the sifting property of the delta function, we can evaluate the integral:

$$x(t) = e^{j\omega_0 t}$$

since the area under curve of the impulse is 1, we get $e^{j\omega_0 t}$.

So, we have the Fourier Transform pair:

$$X(\omega) = 2\pi\delta(\omega - \omega_0) \quad \leftrightarrow \quad x(t) = e^{j\omega_0 t}$$

Now, using Euler's formula, we can express the cosine function in terms of complex exponentials:

$$\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

So, the Fourier Transform of $\cos(\omega_0 t)$ can be derived by taking the Fourier Transform of each exponential term separately and adding them together by using the linearity property of the Fourier Transform.

$$\mathcal{F}\{\cos(\omega_0 t)\} = \frac{1}{2}\left(\mathcal{F}\{e^{j\omega_0 t}\} + \mathcal{F}\{e^{-j\omega_0 t}\}\right)$$

We get,

$$\mathcal{F}\{\cos(\omega_0 t)\} = \frac{1}{2}\left(2\pi\delta(\omega - \omega_0) + 2\pi\delta(\omega + \omega_0)\right) = \pi\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)$$

So, now we can write $X(\omega)$ for our input signal $x(t) = \cos(t) + \cos(5t) + \cos(100t)$:

$$X(\omega) = \pi\left(\delta(\omega - 1) + \delta(\omega + 1)\right) + \pi\left(\delta(\omega - 5) + \delta(\omega + 5)\right) + \pi\left(\delta(\omega - 100) + \delta(\omega + 100)\right)$$

Our goal is to design a filter that only allows the frequency component at $\omega = 5$ rad/s to pass through while attenuating the other two frequency components. Therefore, we can design the frequency response of the audio processing system as follows:

$$H(j\omega) = \begin{cases} 1, & \text{if } \omega \in (5 - \Delta\omega, 5 + \Delta\omega) \\ 0, & \text{otherwise} \end{cases}$$

where $\Delta\omega$ is a small frequency range around 5 rad/s. Visually, this frequency response is shown in Figure 1. The output Fourier Transform $Y(\omega)$ is given by the product of the input

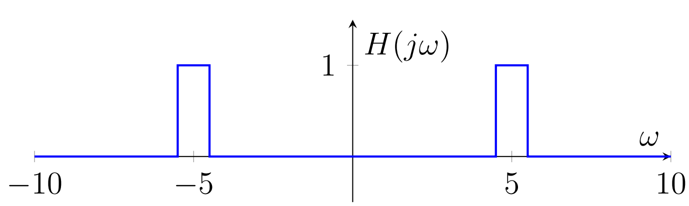


Figure 1: Frequency response of the designed filter.

Fourier Transform $X(\omega)$ and the designed frequency response $H(j\omega)$:

$$Y(\omega) = X(\omega)H(j\omega) = \pi\left(\delta(\omega - 5) + \delta(\omega + 5)\right)$$

because all other impulses are multiplied by zero.

Finally, we can use the Inverse Fourier Transform to recover the time-domain output signal $y(t)$:

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi \left(\delta(\omega - 5) + \delta(\omega + 5)\right) e^{j\omega t} d\omega$$
$$= \frac{1}{2}\left(e^{j5t} + e^{-j5t}\right) = \cos(5t)$$

Thus, the output signal $y(t)$ contains only the frequency component at $\omega = 5$ rad/s, as desired.

6.2 Discrete-time example: filtering

Now let's consider a discrete-time example. Although it might seem that the story will proceed similarly, there are some important differences due to the periodicity of the DTFT! We have a discrete-time signal where three frequency tones are mixed together:

$$x[n] = \cos(n) + \cos(5n) + \cos(100n)$$

Let's assume that we want an ideal filter that removes the component at 100 rad/s (after wrapping into $(-\pi, \pi]$) while keeping the components at $\omega = \pm 1$ and $\omega = \pm 5$.

Write the DTFT

$$\mathcal{F}\{\cos(\omega_0 n)\} \;=\; \pi\big[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\big],$$

The tones at $\omega = 1$ and $\omega = 5$ already lie in $(-\pi, \pi]$. The tone at 100 must be wrapped into $(-\pi, \pi]$:

$$100 - 32\pi \;\approx\; -0.53 \text{ rad}.$$

Hence, over one principal period $(-\pi, \pi]$, the three tones appear at

$$\omega = \pm 1, \qquad \omega = \pm 5, \qquad \omega = \pm 0.53.$$

Therefore,

$$X(e^{j\omega}) = \pi \sum_{\omega_0 \in \{1,\,5,\,0.53\}} \big[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\big]$$

We want to keep $\omega = \pm 1, \pm 5$ and reject $\omega = \pm 0.53$. Even though the frequency we wanted to remove was 100 rad/s, after wrapping into $(-\pi, \pi]$, it became -0.53 rad/s — so, if you thought that you could design a low-pass filter to remove high frequencies, you would be mistaken! In this case, we instead need a high-pass filter!

So, let's design a high-pass filter with cutoff between 0.53 and 1; for instance,

$$\omega_c = 0.5$$

Define

$$H(e^{j\omega}) \;=\; \begin{cases} 0, & |\omega| < \omega_c, \\ 1, & \omega_c \le |\omega| \le \pi, \end{cases} \qquad \text{extended } 2\pi\text{-periodically.}$$

Now, using the convolution property, the output DTFT is

$$Y(e^{j\omega}) \;=\; H(e^{j\omega})\, X(e^{j\omega}).$$

The impulses at $\omega = \pm 0.53$ lie in $(-\omega_c, \omega_c)$. So, these go away! While the impulses at $\omega = \pm 1, \pm 5$ are passed through. We have,

$$Y(e^{j\omega}) = \pi\big[\delta(\omega-1) + \delta(\omega+1) + \delta(\omega-5) + \delta(\omega+5)\big].$$

Taking the inverse DTFT by inspection,

$$y[n] \;=\; \cos(n) \;+\; \cos(5n).$$

Finally, we can derive the impulse response using the inverse Fourier transform (if needed).

7 Computational implementation of the DTFT

Although the previous example worked fine, there are some practical issues with implementing the DTFT on a computer. First, the DTFT integral in equation (2) is defined over an infinite range (from $-\infty$ to ∞). Computers only have finite computations! Second, the DTFT is a continuous function of frequency, but in practice, we can only compute it at discrete frequency points (everything in computers is finite). To address these issues, we sample the Discrete-Time Fourier Transform at specific frequencies. This leads to a new Fourier transform called the Discrete Fourier Transform (DFT). The DFT is defined for a finite-length discrete-time signal $x[n]$ of length N as:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, \quad k = 0, 1, 2, \ldots, N-1 \tag{4}$$

where k represents the discrete frequency index. The DFT provides a frequency-domain representation of the finite-length discrete-time signal, allowing us to analyze its frequency components at specific discrete frequencies.

What changed?

- You have a time-domain signal of finite length N (from $n = 0$ to $n = N - 1$). This is your computational array that stores the signal.
- You compute the Fourier Transform at discrete frequency points $k = 0, 1, 2, \ldots, N-1$, not in radians but in terms of indices. Notice how the total number of frequency points is also N.
- The frequency resolution is determined by the length of the signal N. Due to this, we are able to write ω as $\omega = \frac{2\pi}{N}k$ in the DFT equation, that is integer multiples of the fundamental frequency spacing $\frac{2\pi}{N}$.
- The frequency spacing between adjacent DFT bins is given by $\Delta f = \frac{1}{N}$ (in cycles per sample) or $\Delta\omega = \frac{2\pi}{N}$ (in radians per sample).

From the DFT, you can recover the time-domain signal using the Inverse Discrete Fourier Transform (IDFT):

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}, \quad n = 0, 1, 2, \ldots, N-1 \tag{5}$$

The IDFT allows you to reconstruct the original finite-length discrete-time signal from its DFT representation.

Equations (4) and (5) form the basis for many signal processing applications and these are the two computationally feasible equations you are expected to apply in Homework #9.

8 Recommended Reading

Read the Wikipedia page on Discrete Fourier Transform (DFT) – it can be confusing! But as a student in EE 102, you might now be one of those few people in the world who are able to *understand* this difficult topic!

Pop Quiz Solutions

Pop Quiz 4.1: Solution(s)

It is a continuous function of frequency ω (in radians) because n gets summed over and $e^{-j\omega n}$ is defined for all real values of ω.

Pop Quiz 4.2: Solution(s)

For the continuous-time complex exponential signal, we apply the continuous time periodicity condition that must hold true for all periodic signals:

$$x(t) = x(t + T)$$

Substituting $x(t) = e^{j\omega t}$ into the periodicity condition, we have:

$$e^{j\omega t} = e^{j\omega(t+T)}$$

Simplifying the right-hand side, we get:

$$e^{j\omega t} = e^{j\omega t}e^{j\omega T}$$

The term $e^{j\omega T}$ must equal 1 for the equality to hold for all t. This leads to the condition:

$$e^{j\omega T} = 1$$

This occurs when $T = \frac{2\pi k}{\omega}$ for any integer k. The fundamental period corresponds to $k = 1$, giving us:

$$T = \frac{2\pi}{\omega}$$

For the discrete-time complex exponential signal, we need to check periodicity in frequency. So, the condition that we need to check is:

$$e^{j\omega n} = e^{j(\omega+\Omega)n}$$

for some period Ω. We must find out what Ω is. Simplifying the right-hand side, we have:

$$e^{j\omega n} = e^{j\omega n}e^{j\Omega n}$$

The term $e^{j\Omega n}$ must equal 1 for the equality to hold. This leads to the condition that $\Omega = 2\pi k$ for any integer k. The fundamental period corresponds to $k = 1$, giving us:

$$\Omega = 2\pi$$

Pop Quiz 6.1: Solution(s)

Intuitively, we expect the Fourier Transform of $\cos(\omega_0 t)$ to consist of impulse at the frequency components $\omega = \omega_0$. However, because we work with complex exponentials, we actually have two impulses: one at $\omega = \omega_0$ and another at $\omega = -\omega_0$. The negative frequency arises due to the Euler's formula representation of cosine (which converts the complex to the real):

$$\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

So, it's clear that there is a frequency at ω_0 and another frequency at $-\omega_0$.

EE 102 Week 11, Lecture 1 (Fall 2025)

Instructor: Ayush Pandey

Date: November 10, 2025

1 Announcements

- HW #9 is due on Mon Nov 10.
- HW #10 will be due on Mon Nov 17.
- Rest of the semester: HW #11 on Mon Nov 24, HW #12 on Dec 8, Final exam on Dec 16 (Tue) from 9am to 11am.

2 Review: Fourier Transform (FT), Inverse FT (IFT)

Let $x(t) \in \mathbb{C}$ be absolutely integrable. We use the continuous-time Fourier transform (CTFT)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt, \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega.$$

Basic properties we will use/have proved before:

$$\mathcal{F}\{x^*(t)\} = X^*(-\omega), \quad \mathcal{F}\{x(t-t_0)\} = e^{-j\omega t_0}X(\omega), \quad \mathcal{F}\{x(at)\} = \tfrac{1}{|a|}X(\tfrac{\omega}{a}).$$

3 Quantifying Signal Energy

We have defined the energy of a signal, $x(t)$, using an $L2$ norm metric (that is, squared magnitude integrated over time):

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt.$$

Remember that this is a metric of how "big" a signal is in time domain. It does not correspnd, necessarily, to the physical energy of a system. To compute the physical energy you would usually need to ensure that the units of $x(t)$ match up with the units of the system you are analyzing. The importance of the formula above is that it lets us "give a number" associated with the signal that defines how big it is so that multiple signals can be compared. Otherwise, it is hard to say if one signal is bigger than another. But the problem with the above formula is that it requires you to integrate for all time and often signals can be quite complicated, which makes the integration hard to compute. So, we propose that we can compute the energy of a signal in the frequency domain instead. Why? Because usually what we will find is that signals in the frequency domain are much simpler to analyze. For example, many signals have energy concentrated in a few frequencies, which makes it easy to compute the energy just for the most dominant frequencies rather than the entire signal in time-domain.

To formalize the above, we build the equivalent of Parseval's theorem for CTFT next.

3.1 Parseval's Theorem for CTFT

Apply the inverse Fourier Transform equation, to manipulate E_x to derive the Parseval's theorem for CTFT. Note that the general identity that connects the energy ($L2$ norm) of a signal in time domain to the energy of the signal in frequency domain is called Plancherel-Parseval identity, to give credit to both mathematicians who contributed to its development. For our setting, we start to prove this by starting with the computation for E_x by substitution the inverse FT expression for $x(t)$ so that we can find out the energy as a function of $X(\omega)$:

$$\begin{aligned} E_x &= \int_{-\infty}^{\infty} |x(t)|^2 \, dt \\ &= \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt \\ &= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \right) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(u)\, e^{-jut}\, du \right) dt \end{aligned}$$

where we used two key steps: (i) the definition of squared magnitude of a complex number is equal to the number times its complex conjugate. For signals, we have $|x(t)|^2 = x(t)\, x^*(t)$; (ii) we substituted the inverse FT expression for both $x(t)$ and $x^*(t)$, where we used a dummy variable u for the second integral to avoid confusing it with ω, the first dummy variable for integration. Now, we can rearrange the above expression as

$$\begin{aligned} E_x &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(\omega)\, X^*(u) \left(\int_{-\infty}^{\infty} e^{j(\omega-u)t}\, dt \right) d\omega\, du \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(\omega)\, X^*(u) \cdot 2\pi\delta(\omega - u)\, d\omega\, du \end{aligned}$$

How did we write the last step? Let's work on it step by step. The inner integral is

$$\int_{-\infty}^{\infty} e^{j(\omega-u)t}\, dt.$$

This integration is not convergent in the usual sense. But using our knowledge of Fourier analysis, we can recognize that this is the Fourier transform of a delta function. Let's build that step-by-step. Start by answering the following pop-quiz:

Pop Quiz 3.1: Check your understanding!

What is the inverse Fourier transform of a delta function $2\pi\delta(\omega - \omega_0)$? How do you interpret it?

Solution on page 7

So, from the pop-quiz above, we can see that the Fourier transform of $e^{j\omega_0 t}$ is $2\pi\delta(\omega - \omega_0)$. Let's write this out explicitly using the equation for the continuous-timeFourier transform:

$$2\pi\delta(\omega - \omega_0) = \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{j(\omega_0 - \omega)t}\, dt.$$

Notice how similar this is to our inner integral above! So, we can write

$$\int_{-\infty}^{\infty} e^{j(\omega - u)t}\, dt = 2\pi\delta(\omega - u).$$

Using this result, we can continue our derivation of E_x:

$$\begin{aligned} E_x &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(\omega)\, X^*(u) \cdot 2\pi\delta(\omega - u)\, d\omega\, du \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \left(\int_{-\infty}^{\infty} X^*(u)\delta(\omega - u)\, du \right) d\omega \end{aligned}$$

Using the sifting property of delta function, we have

$$\begin{aligned} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) X^*(\omega)\, d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega. \end{aligned}$$

This is the Parseval's theorem for CTFT. It tells us that we can compute the energy of a signal by simply integrating the squared magnitude of its Fourier transform over frequency, scaled by a factor of $\frac{1}{2\pi}$.

You may wonder that nothing much changed – it is still an integration with limits going from $-\infty$ to ∞. But the key point is that in many practical scenarios, $X(\omega)$ only has significant values over a small range of frequencies and is usually zero for most other frequencies. So, in practice, you end up integrating over just the range of frequencies that are most dominant / you are interested in. This type of modular analysis was not possible in the time-domain.

4 Understanding Power Spectral Density (PSD)

For signals that are random processes (like EEG)[1], we define a quantity that is an extension of the energy computation in the previous section, called the *power spectral density* (PSD). The PSD tells us how the power of a signal is distributed across different frequencies. In fact, whenever you see the word "spectrum" in engineering, it is likely referring to some kind of frequency-domain representation of a signal! So, Fourier transforms are more common around you than you think.

We now extend our previous discussion to an application example of EEG.

4.1 What is EEG?

EEG stands for electroencephalogram. It is a recording of the electrical activity of the brain, typically measured using electrodes placed on the scalp. EEG signals are used in various applications, including medical diagnosis (e.g., epilepsy), sleep studies, brain-computer interfaces, and cognitive research. Fourier transforms (specifically, PSD analysis) is one of the main tools used by doctors all around the world to study the properties of the EEG signal! Fourier transform is far reaching!

4.2 EEG Signal Model and PSD

We model the EEG signal $x(t)$ as a wide-sense stationary (WSS) random process modeled as an autocorrelation function $R_{xx}(\tau)$:

$$R_{xx}(\tau) = \mathbb{E}(x(t)x(t+\tau))$$

where $\mathbb{E}(\cdot)$ is the expectation operator (average over many realizations of the random process).

Pop Quiz 4.1: Check your understanding!

From a quick overview of EEG signals using the simulator at `https://bionichaos.com/EEGSynth/`, what are the main frequency bands present in an EEG signal? What brain activities do they correspond to?

Solution on page 7

[1]Don't worry if you are not comfortable yet / do not understand what random processes are. Think of this as a signal for this class.

4.3 How to quantify EEG signals?

For doctors, the most important task is to always compare the data of their patient to the data of a healthy subject. In the case of EEG signal (which looks quite complicated in time domain), a comparison is not easy because the doctor would have to carefully monitor many peaks, valleys, and all other features and compare them to a healthy subject. This is not practical. So, engineers have come to the rescue (as always :)! Specifically, electrical engineers have built methods that can quantify the EEG signal in a way that is easy to compare. This is done using the *power spectral density* (PSD) of the EEG signal.

This is so popular that MATLAB has a built-in function to compute the PSD of a signal[2]. The advantage of learning signal processing is that you become one of the few people in the world who can understand what is going on under the hood of such computations so that you can innovate the design of these systems and change the healthcare world for the better!

To formally write the PSD, we start with the Fourier transform of the EEG signal autocorrelation model $R_{xx}(\tau)$:

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau.$$

The $S_{xx}(\omega)$ is called the *power spectral density* (PSD) of the EEG signal $x(t)$ – as simple as the Fourier transform of the autocorrelation model of the EEG signal. The PSD tells us how the power of the EEG signal is distributed across different frequencies. Computing it is a different issue — you need to do it in discrete-time domain because computers are discrete (refer to Week 10 Lecture 2 for more details).

4.4 Applying Parseval's Theorem to EEG PSD

From Parseval's theorem for CTFT, we can see that the energy of a signal can be computed in frequency domain as

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega.$$

Since we can only compute for finite-time, we define a time window of duration T. This is a rectangular window on $[0, T))$ so our limits of the integration change accordingly. We define the finite-time Fourier transform (still in continuous-time) as

$$X_T(\omega) = \int_{0}^{T} x(t)\, e^{-j\omega t}\, dt, \qquad \widehat{S}_{xx}(\omega) = \frac{1}{2\pi T}\, |X_T(\omega)|^2 \quad \text{[power/Hz]},$$

[2]See the periodogram function periodogram and another related tutorial PSD with FFT for more details

so that $\frac{1}{2\pi}\int \widehat{S}_{xx}(\omega)\,d\omega \approx \frac{1}{T}\int_0^T |x(t)|^2 dt$ the time-average power over $[0, T)$. Now, in our computers, we need to implement this in discrete-time. So, we sample at F_s Hz over duration T, giving $N = F_s T$ samples (recall DFT from Week 10). The discrete Fourier transform (DFT) is

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/N}, \qquad f_k = \frac{kF_s}{N}, \quad \Delta f = \frac{F_s}{N}, \quad k = 0, \ldots, N-1.$$

The Parseval's theorem for DFT changes to be

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2.$$

Therefore, the final quantity that gets computed usually is the *periodogram* estimate of the PSD:

$$\widehat{S}^{(2)}_{xx}(f_k) = \frac{1}{NF_s}\,|X[k]|^2,$$

so that $\sum_{k=0}^{N-1} \widehat{S}^{(2)}_{xx}(f_k)\,\Delta f = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$ (the average power).

On the EEG simulator page, you see the $\widehat{S}_{xx}(f)$ versus f (Hz) where peaks indicate frequencies with high power. So, all you do when you are asked to compare a diseased brain with a healthy brain is to compare the powers for your frequencies of interest: does the diseased brain have the same "power" during deep sleep and full concentration modes as a healthy brain? If not, doctors then prescribe treatments and drugs to help the patient recover — further away the patients PSD is from ideal the doctor might prescribe stronger drugs!

Pop Quiz Solutions

Pop Quiz 3.1: Solution(s)

If in the frequency domain, we have a delta function $2\pi\delta(\omega - \omega_0)$, then we only have a non-zero value at frequency ω_0. This means that in time domain, we have a complex exponential signal $e^{j\omega_0 t}$. We can show this formally:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}\, d\omega = e^{j\omega_0 t}.$$

Pop Quiz 4.1: Solution(s)

An EEG signal is typically composed of multiple frequency bands, each associated with different brain activities, plus artifacts from eye blinks (EOG) and muscle activity (EMG). The main EEG bands are:

- Delta (0.5-4 Hz): deep sleep.
- Theta (4-8 Hz): drowsiness.
- Alpha (8-13 Hz): relaxed, eyes closed.
- Beta (13-30 Hz): alert, task engagement.
- Artifacts: EOG (eye blinks) and EMG (muscle activity).
- Higher frequencies (¿30 Hz) are often considered noise.

EE 102 Week 11, Lecture 2 (Fall 2025)

Instructor: Ayush Pandey

Date: November 12, 2025

1 Announcements

- HW #10 will be due on Mon Nov 17.
- Rest of the semester: HW #11 on Mon Nov 24, HW #12 on Dec 8, Final exam on Dec 16 (Tue) from 9am to 11am.
- Dec 1: Guest Lecture by Prof. Xiaofan Yu on 2D Fourier Transforms and image processing applications. Dec 3: Guest lecture by Yaoyun Zhou on Fast Fourier Transform algorithms.

2 Today's Learning Goals

- Understand frequency-selective filtering using the properties of Fourier analysis.
- Design ideal and realizable low-pass, high-pass, and band-pass filters.

3 Low Pass Filters

Filters are systems that take in an input signal and modify its frequency content to achieve a desired effect in the generated output. A low-pass filter is a system that allows low-frequency components to pass through while attenuating high-frequency components.

Pop Quiz 3.1: Check your understanding!

Sketch a low-pass filter in the frequency domain.

Solution on page 9

We can sketch an ideal low-pass filter in the frequency domain with cutoff frequency ω_c as a pulse shaped function between $-\omega_c$ and ω_c: The low-pass filter in Figure 1 allows frequencies below $|\omega_c|$ to pass through unchanged (because the magnitude of the frequency response is

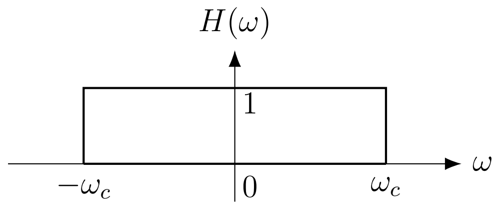


Figure 1: Ideal low-pass filter in the frequency domain.

1) while attenuating frequencies above ω_c because the frequency response is 0. This is true because of the convolution property.

The low-pass filter is an LTI system with frequency response $H_{\text{LP}}(\omega)$. It can be mathematically written as

$$H_{\text{LP}}(\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

or in other words,

$$H_{\text{LP}}(\omega) = \begin{cases} 1, & \omega < \omega_c \land \omega > -\omega_c \\ 0, & \omega \geq \omega_c \lor \omega \leq -\omega_c \end{cases}$$

where $\land$ is the logical AND operator and $\lor$ is the logical OR operator. The passband is the range of frequencies that are allowed to pass through the filter without attenuation, while the stopband is the range of frequencies that are attenuated. In the case of the low-pass filter shown above, the passband is $-\omega_c \leq \omega \leq \omega_c$ and the stopband is $\omega < -\omega_c$ or $\omega > \omega_c$.

Why does the filter turn the output 0 for frequencies outside the passband? This is where the convoltion property of Fourier analysis comes in — one of the most important results that we have derived in this class! It says that the frequency response (the Fourier transform) of the output signal is equal to the multiplication of the frequency response of the input signal and the frequency response of the filtering system:

$$Y(\omega) = X(\omega) \cdot H_{\text{LP}}(\omega)$$

which, in the case of a low-pass filter, means that the frequency response of the output signal will be exactly equal to 0 for frequencies outside the passband and will pass the input as it is (without change) within the passband since the magnitude of the frequency response of the filter is 1 in the passband.

This is all great, but what about the time-domain representation of this filter? After all, anything that we will implement will be implemented in real-life systems where time response will be a consideration. So, we need to find the impulse response of the filter, $h_{\text{LP}}(t)$, which is the inverse Fourier transform of $H_{\text{LP}}(\omega)$.

Applying the inverse Fourier transform, we get:

$$h_{\text{LP}}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{\text{LP}}(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}d\omega$$

Evaluating the integral, we have:

$$h_{\text{LP}}(t) = \frac{1}{2\pi}\left[\frac{e^{j\omega t}}{jt}\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi jt}\left(e^{j\omega_c t} - e^{-j\omega_c t}\right)$$

Using Euler's formula, this simplifies to:

$$h_{\text{LP}}(t) = \frac{\sin(\omega_c t)}{\pi t}$$

Let us plot the impulse response to understand the ideal low-pass filter in the time domain. From Figure 2, we can observe that it extends for all time: $(-\infty, \infty)$. But the intuition

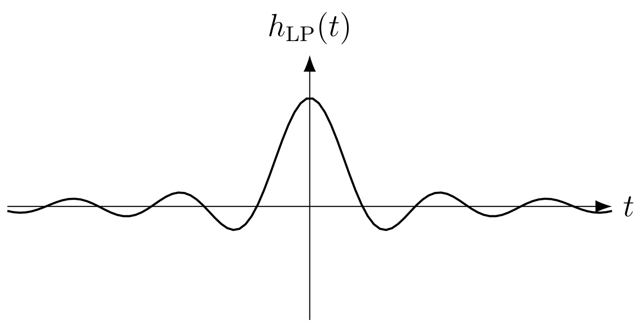


Figure 2: Impulse response of the ideal low-pass filter.

behind "low-passing" is not quite apparent just from the time-domain. What about the computation of the output of the system, $y(t)$? We know that the output of an LTI system in time-domain is given by the convolution integral. So, we can compute $y(t)$ as

$$y(t) = x(t) * h_{\text{LP}}(t) = \int_{-\infty}^{\infty} x(\tau)h_{\text{LP}}(t-\tau)d\tau$$

On the other hand, the frequency response of the output is given by the product of the frequency responses of the input and the filter:

$$Y(\omega) = X(\omega) \cdot H_{\text{LP}}(\omega)$$

It is clear that in the frequency domain the output is easily computed by multiplication, which is often simpler than convolution in the time domain. But the time-domain does reveal three "problems" with the ideal low-pass filter:

1. The impulse response $h_{\mathrm{LP}}(t)$ is not zero for $t < 0$. This means that the ideal low-pass filter is non-causal and therefore, it cannot be implemented in real-time systems.

2. The impulse response extends infinitely in time, which means the filter has an infinite duration impulse response and is not time-limited. Not practical for real-time implementation.

3. We observe a ring-y behavior in the impulse response. Notice the trailing oscillations — these are a direct result of the sharp cut-off in the frequency domain. In fact, as you would recall, the sharp cutoff is a negative impulse and impulse signal is made up of all frequencies from $-\infty$ to ∞. So, we get the Gibbs phenomenon which show up as oscillations in the time domain.

Why are oscillations a problem? A common application of low-pass filtering is in automotives — design of suspension systems. The bumps and jerks that you feel on the road (if your suspension systems are not perfect) are high frequency signals. So, suspension systems are designed on the concept of a low-pass filter that smooths out these high frequency disturbances, providing a more comfortable ride. If the ideal low-pass filter is used with ring-y impulse response, we observe that the output will have oscillations (due to Gibbs phenomena) that can cause discomfort and instability in the system, defeating the purpose of the suspensions. So, we discuss non-ideal, realizable filter designs next.

4 Realizable Low-pass Filters

We have discussed a real-life low-pass filter before: the RC circuit. It is a simple example of a first-order low-pass filter that is causal (addressed the first issue above) and has an exponentially decaying impulse response, which is more practical for real-time implementation.

Recall that the impulse response of an RC circuit is given by

$$h_{\mathrm{RC}}(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$$

where $u(t)$ is the unit step function, ensuring causality. Let's substitute $1/RC = a$ for simplicity. Then, the impulse response becomes

$$h_{\mathrm{RC}}(t) = a e^{-at} u(t)$$

The frequency response of this filter can be found by taking the Fourier Transform of the impulse response:

$$H_{\mathrm{RC}}(\omega) = \int_{-\infty}^{\infty} h_{\mathrm{RC}}(t) e^{-j\omega t} dt = \int_{0}^{\infty} a e^{-at} e^{-j\omega t} dt$$

which can be evaluated as

$$H_{\text{RC}}(\omega) = \int_0^{\infty} ae^{-(a+j\omega)t}dt = \frac{a}{a+j\omega}$$

Pop Quiz 4.1: Check your understanding!

By computing the magnitude of the frequency response, comment on what happens when $\omega \to 0$ and when $\omega \to \infty$.

Solution on page 9

4.1 The cut-off frequency and bandwidth

This frequency response shows that the RC circuit acts as a low-pass filter because as $\omega \to 0$, the magnitude approaches 1, and as $\omega \to \infty$, the magnitude approaches 0. You can see this by evaluating the magnitude response

$$|H_{\text{RC}}(\omega)| = \frac{a}{\sqrt{a^2+\omega^2}}$$

which decreases as ω increases, confirming the low-pass behavior. Let's sketch this frequency response and compare it with the ideal low-pass filter on the same sketch. See Figure 3.

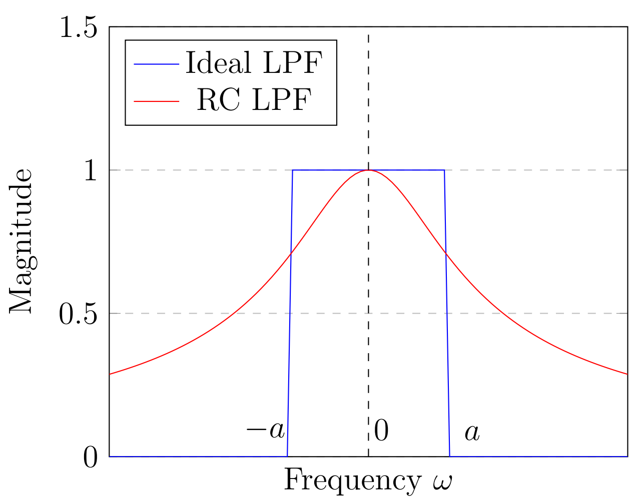


Figure 3: Comparison of the magnitude response of the ideal low-pass filter and the RC low-pass filter. The cut-off frequency of the ideal filter is at $\omega = a$.

Find out the value of the RC low-pass filter at $\omega = a$ and $\omega = -a$. We have

$$|H_{\text{RC}}(a)| = |H_{\text{RC}}(-a)| = \frac{a}{\sqrt{a^2+a^2}} = \frac{a}{\sqrt{2a^2}} = \frac{1}{\sqrt{2}} \approx 0.707$$

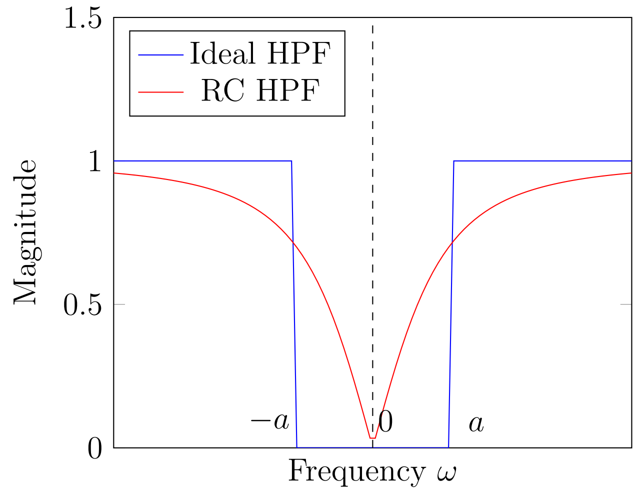


Figure 4: Comparison of the magnitude response of the ideal high-pass filter and the RC high-pass filter. The cut-off frequency of the ideal filter is at $\omega = a$.

This has been given a special name and is treated as the unique point called the *cut-off frequency* or *3 dB point* of the filter, where the output power drops to half of the input power (the frequency response therefore drops by 1 over the square root of 2 — remember Parseval's theorem).

So, the cut-off frequency of the RC filter is defined to be at $\omega_c = a = \frac{1}{RC}$. This also defines the bandwidth of the filter. In this case, the bandwidth is ω_c.

5 High-pass and Band-pass Filters

We can design other filters from the low-pass filter by using Fourier transform properties. First, let us create a high-pass filter from the low-pass filter by applying a frequency shift and inversion. The frequency response of a high-pass filter can be expressed as

$$H_{\text{HP}}(\omega) = 1 - H_{\text{RC}}(\omega) = 1 - \frac{a}{a + j\omega} = \frac{j\omega}{a + j\omega}$$

This filter attenuates low frequencies and allows high frequencies to pass, which is the opposite behavior of the low-pass filter. See Figure 4.

Similarly, a band-pass filter can be created by combining a low-pass and a high-pass filter, allowing frequencies within a certain range to pass while attenuating frequencies outside that range. The design and analysis of these filters follow from the properties of their frequency responses.

By using Fourier transform properties, we can design other filters. Recall that a frequency shift is a multiplication by complex exponential:

$$\mathcal{F}\{x(t)e^{j\omega_\ell t}\} = X(\omega - \omega_\ell), \qquad \mathcal{F}\{x(t)e^{-j\omega_\ell t}\} = X(\omega + \omega_\ell).$$

Consider a signal $x(t)$ with Fourier Transform $X(\omega)$. Now, we desire a new "frequency-selective" filter where we can tunably pick a frequency band that the filter passes. Let us call this tunable frequency ω_ℓ.

We start by considering a frequency shift in the original signal to obtain: $y(t) = x(t)e^{j\omega_\ell t} \Rightarrow Y(\omega) = X(\omega - \omega_\ell)$. Then, if you pass $y(t)$ through the ideal low-pass filter we discussed above, we get $w(t) = y(t) * h(t) \Rightarrow W(\omega) = H(\omega)X(\omega - \omega_\ell)$. Finally, if you shift back, you get $f(t) = w(t)e^{-j\omega_\ell t} \Rightarrow F(\omega) = W(\omega + \omega_\ell)$.

This cascade implements a band-pass filter that is now centered at ω_ℓ using a fixed low-pass filter and two modulators. The pipeline is "shift frequency by tunable amount $\rightarrow$ select the band using the low-pass $\rightarrow$ shift back" (see Section 4.5.1 in Oppenheim and Willsky for more details). See Figure 5 for a visual illustration.

6 Frequency-selective Filter Design

Therefore, with the same low-pass filter as above and not changing anything in it, we were able to obtain a band-pass filter by carefully applying the shifts in the frequency domain! The overall operation is still a linear and time-invariant system.

7 Recommended Reading

The content of this lecture is directly applicable to the design of amplitude modulation systems. Here are two solved examples that you should work on to make sure you understand the concepts thoroughly:

1. Solved Example 4.21 in *Signals and Systems* by Oppenheim and Willsky 2nd Edition
2. Solved Example 4.22 in *Signals and Systems* by Oppenheim and Willsky 2nd Edition

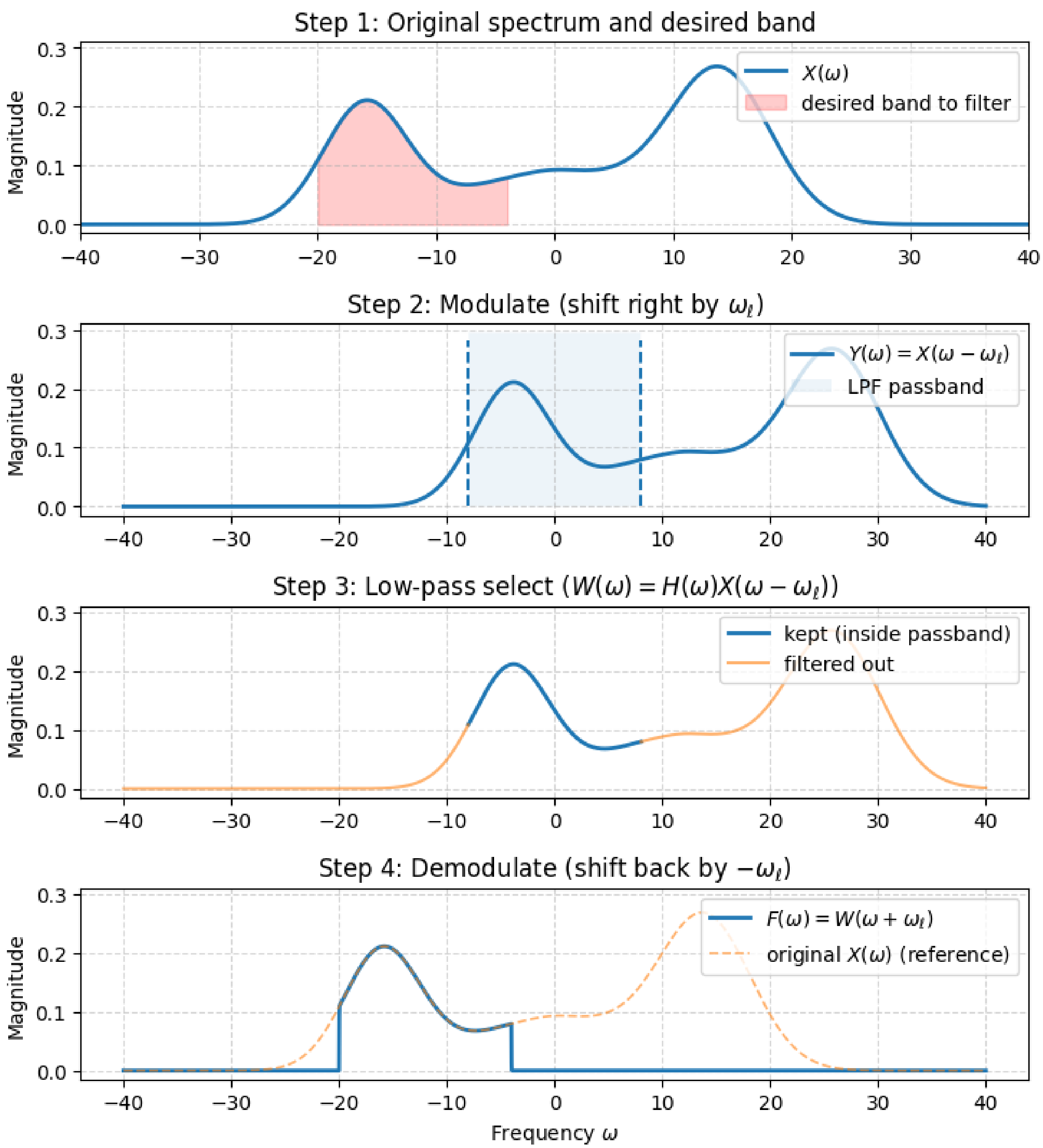


Figure 5: Frequency-selective filtering process: shifting the spectrum $X(\omega)$ to get $Y(\omega)$, applying a low-pass filter to get $W(\omega)$, and shifting back to obtain a band-pass filter centered at ω_ℓ, $F(\omega)$.

Pop Quiz Solutions

Pop Quiz 3.1: Solution(s)

See Figure 1.

Pop Quiz 4.1: Solution(s)

We see that as $\omega \to 0$, the magnitude approaches 1, and as $\omega \to \infty$, the magnitude approaches 0.

EE 102 Week 11, Lecture 1 (Fall 2025)

Note: The last two sections of this document will be covered in lecture 2.

Instructor: Ayush Pandey

Date: November 17, 2025

1 Announcements

- HW #10 is due on Mon Nov 17.
- HW #11 will be released Tue Nov 18 and due on Mon Nov 24.

2 Goals

By the end of this lecture, you should be able to understand the process of sampling a continuous-time signal and how the original signal can be reconstructed from the discrete samples. You will learn to distinguish between the sampling rate, the frequency of the original signal, and the frequencies in the sampled signal. Finally, you will be able to understand that if you keep on increasing the number of samples for a signal, your reconstructed signal will get closer and closer to the original signal. The lower number of samples you take, the more distorted your reconstructed signal will be. But there is a minimum number of samples you must take in order to capture all frequency information of the original signal without distortion. This minimum sampling rate is called the Nyquist rate.

3 Introduction to Sampling: A Cosine Example

In this section we connect what you saw in the in-class activity (sampling different signals at different rates) to the frequency-domain view of sampling and the Nyquist rate. We will use a single cosine as our starting point:

$$x(t) = \cos(\omega_0 t),$$

where ω_0 is the angular frequency in rad/s.

The process that we will follow will start by writing the sampled signal in time-domain using impulses. Then, to meet our learning goal of understanding the frequency-domain view of sampling — that is, we want to find out the minimum number of samples needed to capture all frequency information of the original signal without distortion — we will derive the Fourier transform of the sampled signal. Using the frequency domain representation of the sampled signal, we will be able to comment on the minimum sampling rate needed and any other extra steps needed to ensure that we can reconstruct the original signal's frequencies without distortion.

3.1 Sampling in time using an impulse train

A continuous-time signal $x(t)$ is sampled every T_s seconds. The sampling instants are

$$t_n = nT_s, \quad n = 0, \pm 1, \pm 2, \ldots$$

The sampled signal can be written in continuous time as a train of weighted impulses:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s). \tag{1}$$

This expression comes directly from the **sifting property** of the impulse that we have discussed before:

$$\int_{-\infty}^{\infty} x(t)\,\delta(t - t_0)\,dt = x(t_0).$$

Each term $x(nT_s)\,\delta(t - nT_s)$ is zero everywhere except at $t = nT_s$, and if we integrate $x_s(t)$, only those points contribute. So, the summation above holds.

Pop Quiz 3.1: Check your understanding!

Define the sampling period T_s, the seconds between samples, on a graph. Find out the sampling frequency F_s (in Hz) and the sampling angular frequency ω_s (in rad/s) using T_s.

Solution on page 8

Our goal is to reconstruct the original frequency content of $x(t)$, in this case, the cosine frequency ω_0, from the sampled signal $x_s(t)$. To do this, we will analyze the frequency-domain representation of the sampled signal $x_s(t)$ next.

Fourier transform of the sampled signal

The signal that we are interested in finding the frequency domain representation of is the sampled signal: $x_s(t)$. Spec ifically, we want to see whether the frequency domain representation of the sampled signal $X_s(\omega)$ contains the original frequency ω_0 of the cosine signal $x(t) = \cos(\omega_0 t)$, and if so, under what conditions (how many sampled do we need to ensure that we get the original frequency ω_0 in $X_s(\omega)$ without any distortion). We will do this in several steps.

3.1.1 Fourier transform of the original cosine

First, we look at the Fourier transform of the original cosine signal $x(t) = \cos(\omega_0 t)$. Using Euler's relation we can write the cosine as a sum of complex exponentials:

$$\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right).$$

Now, using the CTFT definition

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt,$$

we know the transform of complex exponentials:

$$\mathcal{F}\{e^{j\omega_0 t}\} = 2\pi\,\delta(\omega - \omega_0), \qquad \mathcal{F}\{e^{-j\omega_0 t}\} = 2\pi\,\delta(\omega + \omega_0).$$

Therefore, for $x(t) = \cos(\omega_0 t)$,

$$X(\omega) = \frac{1}{2}\Big[2\pi\,\delta(\omega - \omega_0) + 2\pi\,\delta(\omega + \omega_0)\Big] = \pi\Big[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\Big].$$

So the spectrum of a cosine consists of two impulses, one at $+\omega_0$ and one at $-\omega_0$.

3.1.2 Fourier transform of a train of impulses

Now we need the Fourier transform of a periodic impulse train, which we use to model the sampling process. Note that we have the train of impulse signal

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

multiplied with the original signal $x(t)$ to get the sampled signal $x_s(t)$ (see equation (1)). This is a train of impulses spaced T_s seconds apart. We observe that $p(t)$ is periodic in time with period T_s since

$$p(t + T_s) = p(t) \quad \text{for all } t.$$

Pop Quiz 3.2: Check your understanding!

Plot the train of impulse signal $p(t)$. Visually and mathematically verify that $p(t)$ is periodic with period T_s.

Solution on page 8

Because $p(t)$ is periodic, we can represent it as a sum of complex exponentials using its Fourier series. This will help us compute its Fourier transform more easily. We have

$$p(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{jk\omega_s t}, \quad \text{with} \quad \omega_s = \frac{2\pi}{T_s},$$

and the Fourier series coefficients are

$$c_k = \frac{1}{T_s}\int_0^{T_s} p(t)\, e^{-jk\omega_s t}\, dt,$$

Recall from Homework #6 Problem 1 that the Fourier series coefficients for a periodic impulse train is

$$c_k = \frac{1}{T_s} \quad \text{for all } k, \quad \text{so} \quad p(t) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} e^{jk\omega_s t}.$$

Now we take the Fourier transform of $p(t)$. Using linearity:

$$P(\omega) = \mathcal{F}\{p(t)\} = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} \mathcal{F}\{e^{jk\omega_s t}\}.$$

From Pop quiz 3.1 in week 11 lecture 1, we know that the Fourier transform of a complex exponential is

$$\mathcal{F}\{e^{j\omega_0 t}\} = 2\pi\,\delta(\omega - \omega_0).$$

Here, $\omega_0 = k\omega_s$, so

$$\mathcal{F}\{e^{jk\omega_s t}\} = 2\pi\,\delta(\omega - k\omega_s).$$

Therefore

$$P(\omega) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s).$$

So, finally, we can write that the Fourier transform of a periodic impulse train with period T_s is a train of impulses in the frequency domain (how convenient!)! The impulses in the frequency domain are spaced by $\omega_s = \dfrac{2\pi}{T_s}$ — the sampling frequency. So, we have the pair:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \quad \Longleftrightarrow \quad P(\omega) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s).$$

Now are ready to compute the Fourier transform of the sampled signal $x_s(t)$.

Fourier transform of the sampled signal: multiplication of $p(t)$ and $x(t)$

Sampling a signal $x(t)$, as in equation (1), can be viewed as multiplying the original signal $x(t)$ by the impulse train $p(t)$. How? See below:

$$x_s(t) = x(t)\, p(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(t)\, \delta(t - nT_s).$$

Using the sifting property inside the sum:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(t - nT_s),$$

which matches our original expression in equation (1). Now, our goal is to compute the frequency domain representation of the sampled signal $x_s(t)$ by computing its Fourier transform.

In the frequency domain, multiplication in time corresponds to *convolution* in frequency. We have not seen this explicitly but you may get the intuition for why this holds true from the duality between time and frequency domains. Specifically, recall that when we convolve two signals in time domain, that is, if we have $y(t) = x(t) * h(t)$, then in the frequency domain, we have $Y(\omega) = X(\omega)H(\omega)$ — multiplication in frequency! So, by duality, it holds true that multiplication in time corresponds to convolution in frequency. You may find a proof of this property in Chapter 4 of Oppenheim and Willsky's Signals and Systems textbook (2nd Edition). We use this property here to compute the Fourier transform of the sampled signal $x_s(t)$. We have

$$X_s(\omega) = \frac{1}{2\pi}\big[X(\omega) * P(\omega)\big],$$

where $X(\omega)$ and $P(\omega)$ are the Fourier transforms of $x(t)$ and $p(t)$ respectively (the $1/2\pi$ factor comes from the multiplication property of the Fourier transform) Now, substituting $P(\omega)$, we have

$$X_s(\omega) = \frac{1}{2\pi} X(\omega) * \left(\frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \right) = \frac{1}{T_s} \left[X(\omega) * \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \right].$$

Again, recall a property of Fourier transforms that the convolution with a shifted delta shifts the function. So,

$$X(\omega) * \delta(\omega - a) = X(\omega - a).$$

$$X_s(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s).$$

Therefore, we get that the frequency domain representation of the sampled signal $X_s(\omega)$ is a scaled sum of shifted copies of the original spectrum $X(\omega)$ (not ideal!), shifted by multiples of ω_s.

For our cosine example, we have already computed $X(\omega)$:

$$X(\omega) = \pi\big[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\big]$$

which gives us

$$X_s(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} \pi\big[\delta(\omega - k\omega_s - \omega_0) + \delta(\omega - k\omega_s + \omega_0)\big].$$

So $X_s(\omega)$ has impulses at

$$\omega = \pm\omega_0 + k\omega_s, \qquad k \in \mathbb{Z}.$$

The original two impulses at $\pm\omega_0$ are now repeated (copied) at integer multiples of the sampling frequency ω_s!!! This is not great, because now the sampled signal contains many more frequency components than the original signal (these are the distortions that you hear when you don't sample correctly!). How can we remove these additional impulses and recover the original cosine frequency ω_0? Low pass filtering! We will discuss this in more detail next time.

Next class: Filtering to remove distortions

The original continuous-time cosine only had two impulses at $\omega = \pm\omega_0$. After sampling, $X_s(\omega)$ contains infinitely many impulses, at

$$\omega = \pm\omega_0, \quad \pm\omega_0 \pm \omega_s, \quad \pm\omega_0 \pm 2\omega_s, \quad \ldots$$

If ω_s is large enough, these copies are separated in frequency and do not overlap. But if ω_s is too small, the shifted impulses can collide or cross into each other. In other words, impulses from different "copies" of $X(\omega)$ may end up in the same frequency location. This leads to distortion in the reconstructed signal called aliasing.

What is aliasing? When the sampling frequency is too low, different frequency components in the original signal produce overlapping or indistinguishable contributions in the sampled spectrum. In the reconstructed signal, these contributions appear as *fake* or *misplaced* frequencies. This phenomenon is called **aliasing**: different original frequencies become indistinguishable after sampling.

For a single cosine, aliasing means that a high-frequency cosine can "look like" a lower-frequency cosine once sampled, because their impulses line up after shifts by ω_s.

As we said above, to reconstruct the original continuous-time signal $x(t)$ from its samples, we may design a low-pass filter to remove the extra copies in $X_s(\omega)$. This will work only if the copies do not overlap with the original spectrum. An ideal low-pass filter has a cutoff frequency ω_c that passes only the frequency band containing the original $X(\omega)$ (from $-\omega_c$ to $+\omega_c$) and removes all higher frequencies will work well.

For a single cosine at ω_0, we can choose the cutoff frequency ω_c such that

$$\omega_0 < \omega_c < \omega_s - \omega_0.$$

This is possible *only if* the original impulse at $+\omega_0$ is strictly separated from the nearest shifted impulse at $\omega_s - \omega_0$. That gives the inequality

$$\omega_0 < \omega_s - \omega_0 \quad \iff \quad \omega_s > 2\omega_0.$$

Under this condition, an ideal low-pass filter with passband $|\omega| < \omega_c$ will keep the original impulses at $\pm\omega_0$ and completely remove all impulses from the shifted copies. In the time domain, this low-pass filter reconstructs the original continuous-time cosine $x(t) = \cos(\omega_0 t)$ from the samples (our overall goal!). The condition that we just obtained is called the **Nyquist sampling condition**.

We have shown that to avoid overlap between the original impulses at $\pm\omega_0$ and the nearest replicated impulses (at $\pm\omega_0 \pm \omega_s$), we must have

$$\omega_s > 2\omega_0.$$

We can obtain an equivalent relation in Hz. Recall $\omega_s = \dfrac{2\pi}{T_s}$ and $\omega_0 = 2\pi f_0$ where f_0 is the ordinary frequency in Hz. In terms of the sampling frequency $F_s = 1/T_s$:

$$\omega_s > 2\omega_0 \quad \iff \quad 2\pi F_s > 2(2\pi f_0) \quad \iff \quad F_s > 2f_0.$$

So, the Nyquist rate for a single cosine:

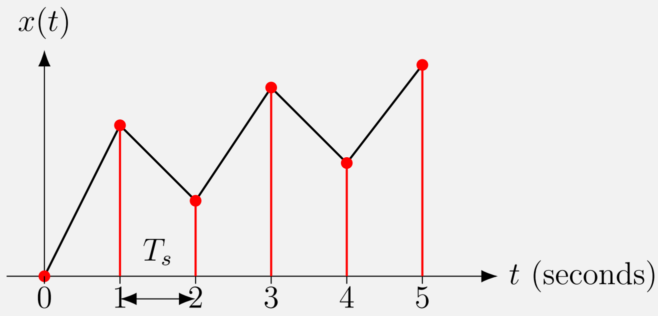
$$F_s > 2f_0, \qquad f_0 = \frac{\omega_0}{2\pi}.$$

If we sample *above* this Nyquist rate, we can in principle use an ideal low-pass filter to remove the extra copies in $X_s(\omega)$ and reconstruct the original cosine exactly from its samples. This is the key result of this week's theoretical topics.

Pop Quiz Solutions

Pop Quiz 3.1: Solution(s)

The sampling period T_s is the time interval between two consecutive samples. See the graph below for reference:



The sampling frequency F_s is the number of samples taken per second, which is the reciprocal of the sampling period:

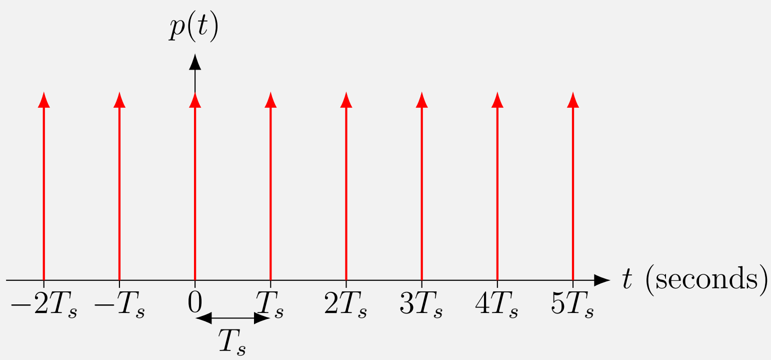
$$F_s = \frac{1}{T_s} \quad \text{[samples/second]}.$$

The sampling angular frequency is

$$\omega_s = 2\pi F_s = \frac{2\pi}{T_s}.$$

Pop Quiz 3.2: Solution(s)

The plot of the train of impulse signal $p(t)$ is shown below:



To verify that $p(t)$ is periodic with period T_s, we check that $p(t+T_s) = p(t)$ for all t:

$$p(t+T_s) = \sum_{n=-\infty}^{\infty} \delta(t+T_s-nT_s) = \sum_{n=-\infty}^{\infty} \delta(t-(n-1)T_s).$$

By changing the index of summation from n to $m = n-1$, we have

$$p(t+T_s) = \sum_{m=-\infty}^{\infty} \delta(t-mT_s) = p(t).$$

EE 102 Week 11, Lecture 1 (Fall 2025)

Instructor: Ayush Pandey

Date: November 17, 2025

Note: Week 12 Lecture 1 and Lecture 2 have the "mostly" the same set of lecture notes. The last section in lecture 2 is the only extra section.

1 Announcements

- HW #10 is due on Mon Nov 17.
- HW #11 will be released Tue Nov 18 and due on Mon Nov 25.

2 Goals

By the end of this lecture, you should be able to understand the process of sampling a continuous-time signal and how the original signal can be reconstructed from the discrete samples. You will learn to distinguish between the sampling rate, the frequency of the original signal, and the frequencies in the sampled signal. Finally, you will be able to understand that if you keep on increasing the number of samples for a signal, your reconstructed signal will get closer and closer to the original signal. The lower number of samples you take, the more distorted your reconstructed signal will be. But there is a minimum number of samples you must take in order to capture all frequency information of the original signal without distortion. This minimum sampling rate is called the Nyquist rate.

3 Introduction to Sampling: A Cosine Example

In this section we connect what you saw in the in-class activity (sampling different signals at different rates) to the frequency-domain view of sampling and the Nyquist rate. We will use a single cosine as our starting point:

$$x(t) = \cos(\omega_0 t),$$

where ω_0 is the angular frequency in rad/s.

The process that we will follow will start by writing the sampled signal in time-domain using impulses. Then, to meet our learning goal of understanding the frequency-domain view of sampling — that is, we want to find out the minimum number of samples needed to capture all frequency information of the original signal without distortion — we will derive the Fourier transform of the sampled signal. Using the frequency domain representation of the sampled signal, we will be able to comment on the minimum sampling rate needed and any other extra steps needed to ensure that we can reconstruct the original signal's frequencies without distortion.

3.1 Sampling in time using an impulse train

A continuous-time signal $x(t)$ is sampled every T_s seconds. The sampling instants are

$$t_n = nT_s, \quad n = 0, \pm 1, \pm 2, \ldots$$

The sampled signal can be written in continuous time as a train of weighted impulses:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s). \tag{1}$$

This expression comes directly from the **sifting property** of the impulse that we have discussed before:

$$\int_{-\infty}^{\infty} x(t)\,\delta(t - t_0)\,dt = x(t_0).$$

Each term $x(nT_s)\,\delta(t - nT_s)$ is zero everywhere except at $t = nT_s$, and if we integrate $x_s(t)$, only those points contribute. So, the summation above holds.

Pop Quiz 3.1: Check your understanding!

Define the sampling period T_s, the seconds between samples, on a graph. Find out the sampling frequency F_s (in Hz) and the sampling angular frequency ω_s (in rad/s) using T_s.

Solution on page 9

Our goal is to reconstruct the original frequency content of $x(t)$, in this case, the cosine frequency ω_0, from the sampled signal $x_s(t)$. To do this, we will analyze the frequency-domain representation of the sampled signal $x_s(t)$ next.

Fourier transform of the sampled signal

The signal that we are interested in finding the frequency domain representation of is the sampled signal: $x_s(t)$. Spec ifically, we want to see whether the frequency domain representation of the sampled signal $X_s(\omega)$ contains the original frequency ω_0 of the cosine signal $x(t) = \cos(\omega_0 t)$, and if so, under what conditions (how many sampled do we need to ensure that we get the original frequency ω_0 in $X_s(\omega)$ without any distortion). We will do this in several steps.

3.1.1 Fourier transform of the original cosine

First, we look at the Fourier transform of the original cosine signal $x(t) = \cos(\omega_0 t)$. Using Euler's relation we can write the cosine as a sum of complex exponentials:

$$\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right).$$

Now, using the CTFT definition

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt,$$

we know the transform of complex exponentials:

$$\mathcal{F}\{e^{j\omega_0 t}\} = 2\pi\,\delta(\omega - \omega_0), \qquad \mathcal{F}\{e^{-j\omega_0 t}\} = 2\pi\,\delta(\omega + \omega_0).$$

Therefore, for $x(t) = \cos(\omega_0 t)$,

$$X(\omega) = \frac{1}{2}\big[2\pi\,\delta(\omega - \omega_0) + 2\pi\,\delta(\omega + \omega_0)\big] = \pi\big[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\big].$$

So the spectrum of a cosine consists of two impulses, one at $+\omega_0$ and one at $-\omega_0$.

3.1.2 Fourier transform of a train of impulses

Now we need the Fourier transform of a periodic impulse train, which we use to model the sampling process. Note that we have the train of impulse signal

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

multiplied with the original signal $x(t)$ to get the sampled signal $x_s(t)$ (see equation (1)). This is a train of impulses spaced T_s seconds apart. We observe that $p(t)$ is periodic in time with period T_s since

$$p(t + T_s) = p(t) \quad \text{for all } t.$$

Pop Quiz 3.2: Check your understanding!

Plot the train of impulse signal $p(t)$. Visually and mathematically verify that $p(t)$ is periodic with period T_s.

Solution on page 9

Because $p(t)$ is periodic, we can represent it as a sum of complex exponentials using its Fourier series. This will help us compute its Fourier transform more easily. We have

$$p(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{jk\omega_s t}, \quad \text{with} \quad \omega_s = \frac{2\pi}{T_s},$$

and the Fourier series coefficients are

$$c_k = \frac{1}{T_s} \int_0^{T_s} p(t)\, e^{-jk\omega_s t}\, dt,$$

Recall from Homework #6 Problem 1 that the Fourier series coefficients for a periodic impulse train is

$$c_k = \frac{1}{T_s} \quad \text{for all } k, \quad \text{so} \quad p(t) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t}.$$

Now we take the Fourier transform of $p(t)$. Using linearity:

$$P(\omega) = \mathcal{F}\{p(t)\} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \mathcal{F}\{e^{jk\omega_s t}\}.$$

From Pop quiz 3.1 in week 11 lecture 1, we know that the Fourier transform of a complex exponential is

$$\mathcal{F}\{e^{j\omega_0 t}\} = 2\pi\, \delta(\omega - \omega_0).$$

Here, $\omega_0 = k\omega_s$, so

$$\mathcal{F}\{e^{jk\omega_s t}\} = 2\pi\, \delta(\omega - k\omega_s).$$

Therefore

$$P(\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s).$$

So, finally, we can write that the Fourier transform of a periodic impulse train with period T_s is a train of impulses in the frequency domain (how convenient!)! The impulses in the frequency domain are spaced by $\omega_s = \dfrac{2\pi}{T_s}$ — the sampling frequency. So, we have the pair:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \quad \Longleftrightarrow \quad P(\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s).$$

Now are ready to compute the Fourier transform of the sampled signal $x_s(t)$.

Fourier transform of the sampled signal: multiplication of $p(t)$ and $x(t)$

Sampling a signal $x(t)$, as in equation (1), can be viewed as multiplying the original signal $x(t)$ by the impulse train $p(t)$. How? See below:

$$x_s(t) = x(t)\,p(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s) = \sum_{n=-\infty}^{\infty}x(t)\,\delta(t-nT_s).$$

Using the sifting property inside the sum:

$$x_s(t) = \sum_{n=-\infty}^{\infty}x(nT_s)\,\delta(t-nT_s),$$

which matches our original expression in equation (1). Now, our goal is to compute the frequency domain representation of the sampled signal $x_s(t)$ by computing its Fourier transform.

In the frequency domain, multiplication in time corresponds to *convolution* in frequency. We have not seen this explicitly but you may get the intuition for why this holds true from the duality between time and frequency domains. Specifically, recall that when we convolve two signals in time domain, that is, if we have $y(t) = x(t) * h(t)$, then in the frequency domain, we have $Y(\omega) = X(\omega)H(\omega)$ — multiplication in frequency! So, by duality, it holds true that multiplication in time corresponds to convolution in frequency. You may find a proof of this property in Chapter 4 of Oppenheim and Willsky's Signals and Systems textbook (2nd Edition). We use this property here to compute the Fourier transform of the sampled signal $x_s(t)$. We have

$$X_s(\omega) = \frac{1}{2\pi}\big[X(\omega) * P(\omega)\big],$$

where $X(\omega)$ and $P(\omega)$ are the Fourier transforms of $x(t)$ and $p(t)$ respectively (the $1/2\pi$ factor comes from the multiplication property of the Fourier transform) Now, substituting $P(\omega)$, we have

$$X_s(\omega) = \frac{1}{2\pi}X(\omega) * \left(\frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_s)\right) = \frac{1}{T_s}\left[X(\omega) * \sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_s)\right].$$

Again, recall a property of Fourier transforms that the convolution with a shifted delta shifts the function. So,

$$X(\omega) * \delta(\omega-a) = X(\omega-a).$$

$$X_s(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty}X(\omega-k\omega_s).$$

Therefore, we get that the frequency domain representation of the sampled signal $X_s(\omega)$ is a scaled sum of shifted copies of the original spectrum $X(\omega)$ (not ideal!), shifted by multiples of ω_s.

For our cosine example, we have already computed $X(\omega)$:

$$X(\omega) = \pi\big[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\big]$$

which gives us

$$X_s(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} \pi\big[\delta(\omega - k\omega_s - \omega_0) + \delta(\omega - k\omega_s + \omega_0)\big].$$

So $X_s(\omega)$ has impulses at

$$\omega = \pm\omega_0 + k\omega_s, \qquad k \in \mathbb{Z}.$$

The original two impulses at $\pm\omega_0$ are now repeated (copied) at integer multiples of the sampling frequency ω_s!!! This is not great, because now the sampled signal contains many more frequency components than the original signal (these are the distortions that you hear when you don't sample correctly!). How can we remove these additional impulses and recover the original cosine frequency ω_0? Low pass filtering! We will discuss this in more detail next time.

Filtering to remove distortions

The original continuous-time cosine only had two impulses at $\omega = \pm\omega_0$. After sampling, $X_s(\omega)$ contains infinitely many impulses, at

$$\omega = \pm\omega_0, \quad \pm\omega_0 \pm \omega_s, \quad \pm\omega_0 \pm 2\omega_s, \quad \ldots$$

If ω_s is large enough, these copies are separated in frequency and do not overlap. But if ω_s is too small, the shifted impulses can collide or cross into each other. In other words, impulses from different "copies" of $X(\omega)$ may end up in the same frequency location. This leads to distortion in the reconstructed signal called aliasing.

What is aliasing? When the sampling frequency is too low, different frequency components in the original signal produce overlapping or indistinguishable contributions in the sampled spectrum. In the reconstructed signal, these contributions appear as *fake* or *misplaced* frequencies. This phenomenon is called **aliasing**: different original frequencies become indistinguishable after sampling.

For a single cosine, aliasing means that a high-frequency cosine can "look like" a lower-frequency cosine once sampled, because their impulses line up after shifts by ω_s.

As we said above, to reconstruct the original continuous-time signal $x(t)$ from its samples, we may design a low-pass filter to remove the extra copies in $X_s(\omega)$. This will work only if the copies do not overlap with the original spectrum. An ideal low-pass filter has a cutoff frequency ω_c that passes only the frequency band containing the original $X(\omega)$ (from $-\omega_c$ to $+\omega_c$) and removes all higher frequencies will work well.

For a single cosine at ω_0, we can choose the cutoff frequency ω_c such that

$$\omega_0 < \omega_c < \omega_s - \omega_0.$$

This is possible *only if* the original impulse at $+\omega_0$ is strictly separated from the nearest shifted impulse at $\omega_s - \omega_0$. That gives the inequality

$$\omega_0 < \omega_s - \omega_0 \quad \Longleftrightarrow \quad \omega_s > 2\omega_0.$$

Under this condition, an ideal low-pass filter with passband $|\omega| < \omega_c$ will keep the original impulses at $\pm\omega_0$ and completely remove all impulses from the shifted copies. In the time domain, this low-pass filter reconstructs the original continuous-time cosine $x(t) = \cos(\omega_0 t)$ from the samples (our overall goal!). The condition that we just obtained is called the **Nyquist sampling condition**.

We have shown that to avoid overlap between the original impulses at $\pm\omega_0$ and the nearest replicated impulses (at $\pm\omega_0 \pm \omega_s$), we must have

$$\omega_s > 2\omega_0.$$

We can obtain an equivalent relation in Hz. Recall $\omega_s = \dfrac{2\pi}{T_s}$ and $\omega_0 = 2\pi f_0$ where f_0 is the ordinary frequency in Hz. In terms of the sampling frequency $F_s = 1/T_s$:

$$\omega_s > 2\omega_0 \quad \Longleftrightarrow \quad 2\pi F_s > 2(2\pi f_0) \quad \Longleftrightarrow \quad F_s > 2f_0.$$

So, the Nyquist rate for a single cosine:

$$F_s > 2f_0, \qquad f_0 = \frac{\omega_0}{2\pi}.$$

If we sample *above* this Nyquist rate, we can in principle use an ideal low-pass filter to remove the extra copies in $X_s(\omega)$ and reconstruct the original cosine exactly from its samples. This is the key result of this week's theoretical topics.

4 Nyquist Sampling Theorem

We end with the formal statement of the Nyquist Sampling Theorem for general bandlimited signals (not just a single cosine).

Theorem 1 (Nyquist Sampling Theorem)**.** *If a continuous-time signal $x(t)$ contains no frequencies higher than B Hz, that is, if its Fourier transform $X(\omega)$ is zero for $|\omega| > 2\pi B$, then $x(t)$ is completely determined by its samples taken at a sampling frequency F_s greater than $2B$ Hz. In this case, $x(t)$ can be reconstructed from its samples using an ideal low-pass filter.*

5 Recommended Reading

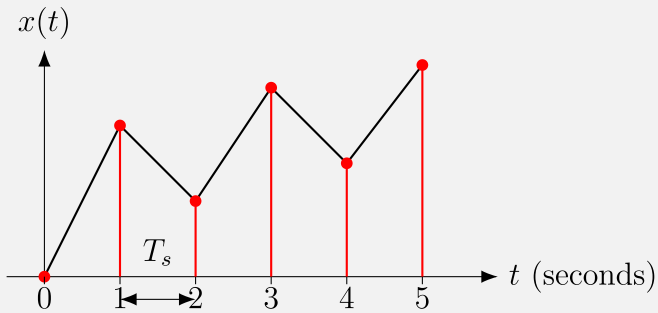
Run `VM_aliasing.ipynb` notebook that is available on course GitHub. Try to achieve the following using the code:

- Visualize aliasing with $x(t) = \sin(2\pi f_0 t)$ for different values of f_0 and different sampling frequencies F_s.
- Observe how increasing the number of samples improves the reconstruction of the original signal for the sine wave.
- Apply the sampling theorem to adequately sample and reconstruct your recorded voice from Homework #10.

Pop Quiz Solutions

Pop Quiz 3.1: Solution(s)

The sampling period T_s is the time interval between two consecutive samples. See the graph below for reference:



The sampling frequency F_s is the number of samples taken per second, which is the reciprocal of the sampling period:

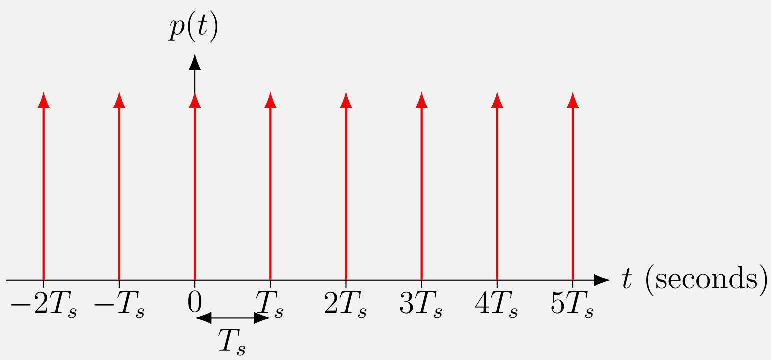
$$F_s = \frac{1}{T_s} \quad \text{[samples/second]}.$$

The sampling angular frequency is

$$\omega_s = 2\pi F_s = \frac{2\pi}{T_s}.$$

Pop Quiz 3.2: Solution(s)

The plot of the train of impulse signal $p(t)$ is shown below:



To verify that $p(t)$ is periodic with period T_s, we check that $p(t + T_s) = p(t)$ for all t:

$$p(t + T_s) = \sum_{n=-\infty}^{\infty} \delta(t + T_s - nT_s) = \sum_{n=-\infty}^{\infty} \delta(t - (n-1)T_s).$$

By changing the index of summation from n to $m = n - 1$, we have

$$p(t + T_s) = \sum_{m=-\infty}^{\infty} \delta(t - mT_s) = p(t).$$

EE 102: In-class activity on understanding sampling

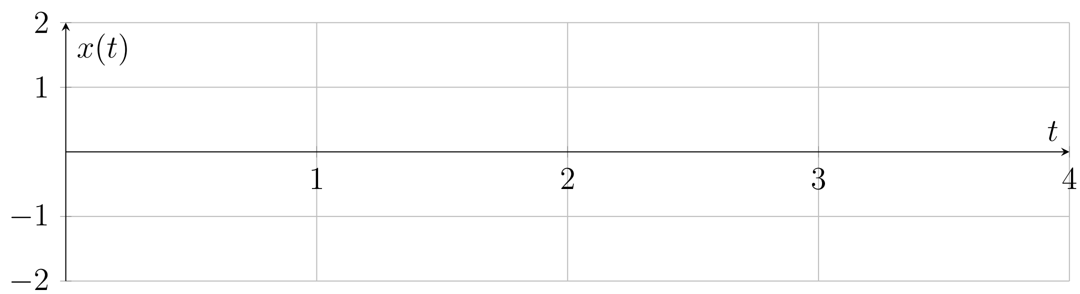
Team members: ________________ + ________________

This is a two-team activity. Each team will design a sampling puzzle for the other team to solve. That is, you will provide samples of a continuous-time signal to the other team, and they will try to reconstruct the original signal from those samples. You will rate how well they did and do it again with higher number of samples.

Original continuous-time signal: $x(t) = \cos(2\pi t) + \sin(4\pi t)$.

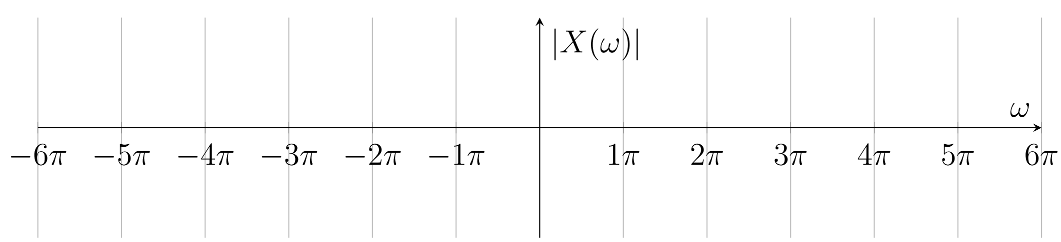
Given information The first component $\cos(2\pi t)$ has angular frequency $\omega_1 = 2\pi$, period $T_1 = \dfrac{2\pi}{2\pi} = 1$. Frequency in Hz is 1 Hz. The second component $\sin(4\pi t)$ has angular frequency $\omega_2 = 4\pi$, period $T_2 = \dfrac{2\pi}{4\pi} = \dfrac{1}{2}$. Frequency in Hz is 2 Hz. So, overall signal period: $T_0 = 1$.

Plot $x(t)$ Use Desmos or any other graphing tool / code to plot $x(t)$ over the time interval $t = 0$ to $t = 4$. DO NOT show this plot to the other team.



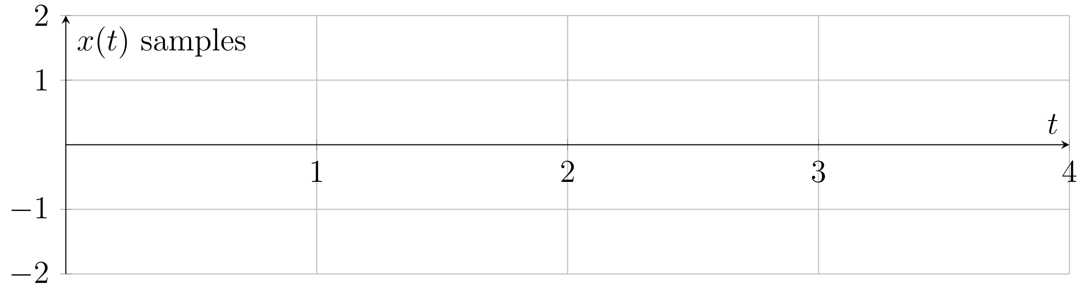
Frequency domain: $X(\omega) = \pi\big[\delta(\omega - 2\pi) + \delta(\omega + 2\pi)\big] - j\pi\big[\delta(\omega - 4\pi) - \delta(\omega + 4\pi)\big]$.

Sketch $|X(\omega)|$: Use the axes below to sketch the locations (in ω) of the nonzero components of $|X(\omega)|$. You may ignore the exact complex amplitudes for now and focus on the positions at $\omega = \pm 2\pi$ and $\omega = \pm 4\pi$.



Sampling Puzzle Design #1: Provide 8 Samples

You will provide 8 samples of $x(t)$ over the *fixed* time interval $t \in [0, 4]$ seconds. Mark an "X" at each sample point that you provide to the other team. Place samples uniformly.



Help the other team by showing them the sampling frequency computation. Since the total number of samples is $N_1 = 8$ and the time interval is from $t = 0$ to $t = 4$, we have the sampling period (time between samples) (FILL IN THE BLANKS):

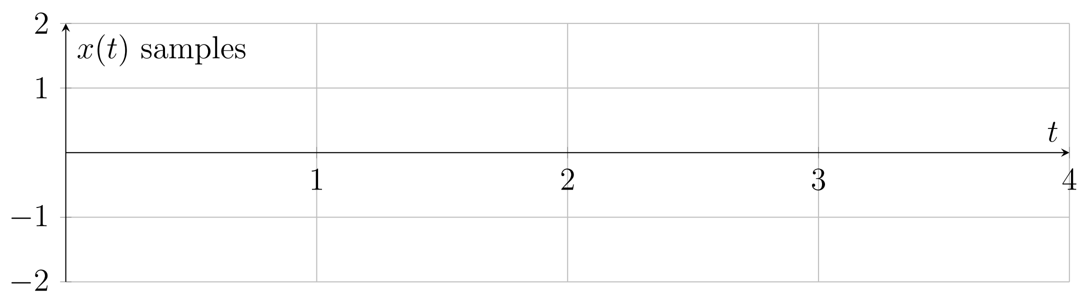
$$T_{s,1} = \frac{t_{\text{end}} - t_{\text{start}}}{N_1} = \underline{\hspace{2cm}}. \quad \text{Sampling frequency: } F_{s,1} = \frac{1}{T_{s,1}} = \underline{\hspace{2cm}} \text{ Hz.}$$

Rate the reconstruction: What did you get right? ________ What did you miss? ________

Sampling Puzzle Design #2: Provide 16 Samples

Mark an "X" at each sample point that you provide to the other team. Place samples uniformly. Since the total number of samples is $N_2 = 16$ and the time interval is from $t = 0$ to $t = 4$, we have the sampling period (time between samples) (FILL IN THE BLANKS):

$$T_{s,2} = \frac{t_{\text{end}} - t_{\text{start}}}{N_2} = \underline{\hspace{2cm}}. \quad \text{Sampling frequency: } F_{s,2} = \frac{1}{T_{s,2}} = \underline{\hspace{2cm}} \text{ Hz.}$$



Rate the reconstruction: What did you get right? ________ What did you miss? ________

EE 102: In-class activity on understanding sampling

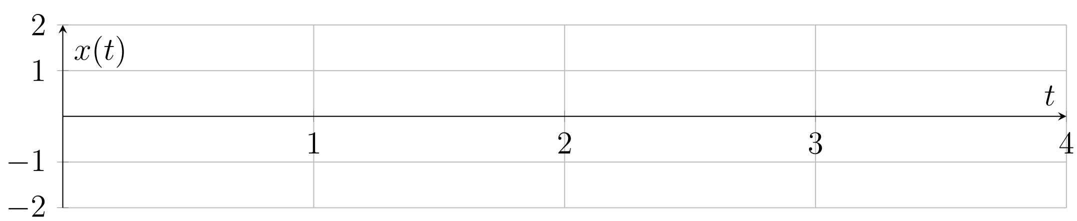
Team members: ____________________ + ____________________

This is a two-team activity. Each team will design a sampling puzzle for the other team to solve. That is, you will provide samples of a continuous-time signal to the other team, and they will try to reconstruct the original signal from those samples. You will rate how well they did and do it again with higher number of samples.

Original continuous-time signal: $x(t) = \cos(2\pi t) + \sin(4\pi t) + 0.8\cos(6\pi t)$.

Given information The first component $\cos(2\pi t)$ has angular frequency $\omega_1 = 2\pi$, period $T_1 = \frac{2\pi}{2\pi} = 1$. Frequency in Hz is 1 Hz. The second component $\sin(4\pi t)$ has angular frequency $\omega_2 = 4\pi$, period $T_2 = \frac{2\pi}{4\pi} = \frac{1}{2}$. Frequency in Hz is 2 Hz. The third component $0.8\cos(6\pi t)$ has angular frequency $\omega_3 = 6\pi$, period $T_3 = \frac{2\pi}{6\pi} = \frac{1}{3}$. Frequency in Hz is 3 Hz.

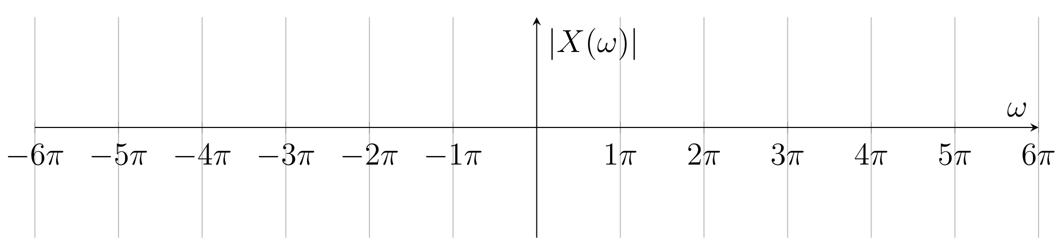
Plot $x(t)$ Use Desmos or any other tool to plot $x(t)$ over $t = 0$ to $t = 4$. DO NOT show this plot to the other team.



Frequency domain:

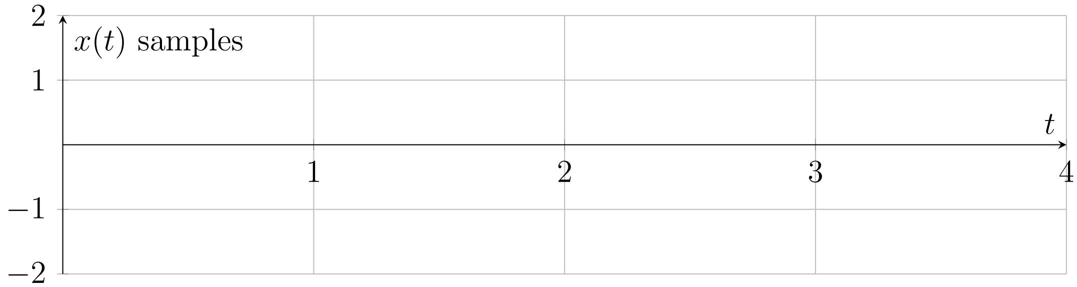
$$X(\omega) = \pi\big[\delta(\omega-2\pi)+\delta(\omega+2\pi)\big] - j\pi\big[\delta(\omega-4\pi)-\delta(\omega+4\pi)\big] + 0.8\pi\big[\delta(\omega-6\pi)+\delta(\omega+6\pi)\big].$$

Sketch $|X(\omega)|$: Use the axes below to sketch the locations (in ω) of the nonzero components of $|X(\omega)|$. Focus on the positions at $\omega = \pm 2\pi$, $\omega = \pm 4\pi$, and $\omega = \pm 6\pi$.



Sampling Puzzle Design #1: Provide 8 Samples

You will provide 8 samples of $x(t)$ over the *fixed* time interval $t \in [0,\, 4]$ seconds. Mark an "X" at each sample point that you provide to the other team. Place samples uniformly.



Help the other team by showing them the sampling frequency computation. Since the total number of samples is $N_1 = 8$ and the time interval is from $t = 0$ to $t = 4$, we have the sampling period (time between samples) (FILL IN THE BLANKS):

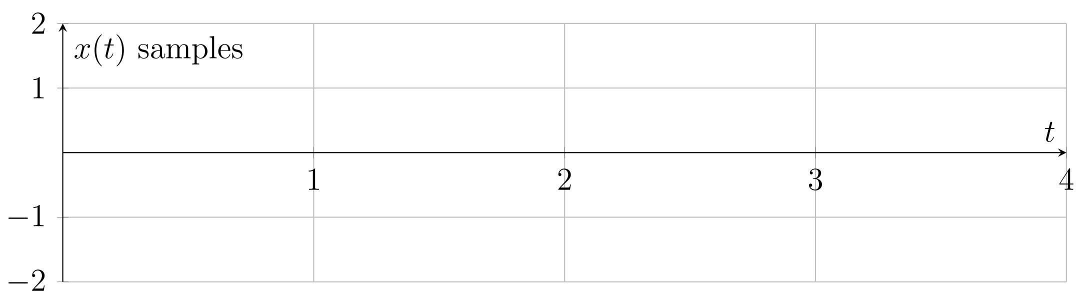
$$T_{s,1} = \frac{t_{\text{end}} - t_{\text{start}}}{N_1} = \underline{\hspace{3cm}}. \quad \text{Sampling frequency: } F_{s,1} = \frac{1}{T_{s,1}} = \underline{\hspace{3cm}} \text{ Hz.}$$

Rate the reconstruction: What did you get right? __________ What did you miss? __________

Sampling Puzzle Design #2: Provide 16 Samples

Mark an "X" at each sample point that you provide to the other team. Place samples uniformly. Since the total number of samples is $N_2 = 16$ and the time interval is from $t = 0$ to $t = 4$, we have the sampling period (time between samples) (FILL IN THE BLANKS):

$$T_{s,2} = \frac{t_{\text{end}} - t_{\text{start}}}{N_2} = \underline{\hspace{3cm}}. \quad \text{Sampling frequency: } F_{s,2} = \frac{1}{T_{s,2}} = \underline{\hspace{3cm}} \text{ Hz.}$$



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EE 102: In-class activity on understanding sampling

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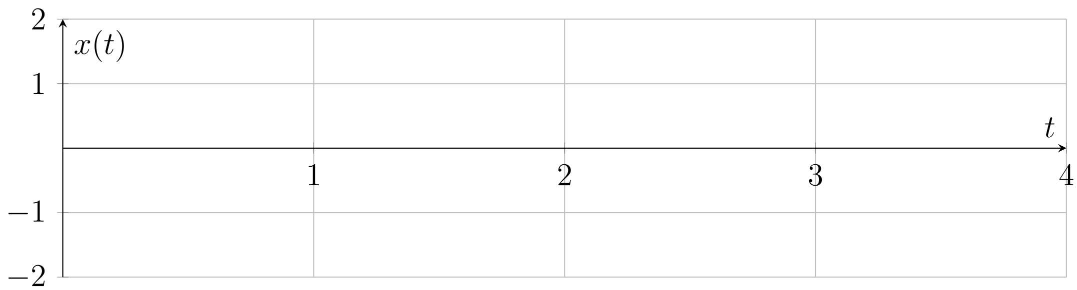
Original continuous-time signal: $x(t) = 0.5\cos(\pi t) + \cos(2\pi t)$.

Given information The first component $0.5\cos(\pi t)$ has angular frequency $\omega_1 = \pi$, period $T_1 = \dfrac{2\pi}{\pi} = 2$. Frequency in Hz is 0.5 Hz. The second component $\cos(2\pi t)$ has angular frequency $\omega_2 = 2\pi$, period $T_2 = \dfrac{2\pi}{2\pi} = 1$. Frequency in Hz is 1 Hz. So, overall signal period: $T_0 = 2$.

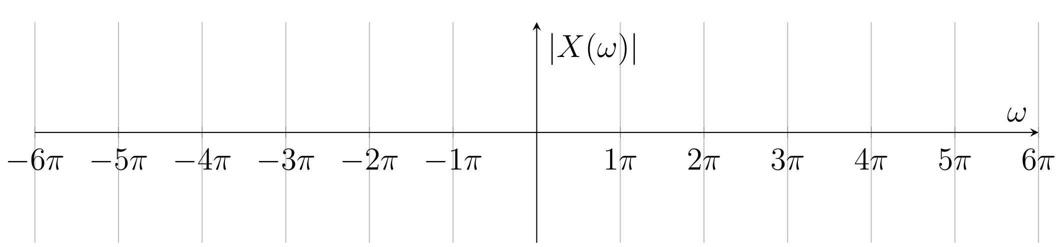
Plot $x(t)$ Use Desmos or any other graphing tool / code to plot $x(t)$ over the time interval $t = 0$ to $t = 4$. DO NOT show this plot to the other team.



Frequency domain:

$$X(\omega) = 0.5\pi\big[\delta(\omega-\pi)+\delta(\omega+\pi)\big] + \pi\big[\delta(\omega-2\pi)+\delta(\omega+2\pi)\big].$$

Sketch $|X(\omega)|$: Use the axes below to sketch the locations (in ω) of the nonzero components of $|X(\omega)|$. Focus on the positions at $\omega = \pm\pi$ and $\omega = \pm 2\pi$.



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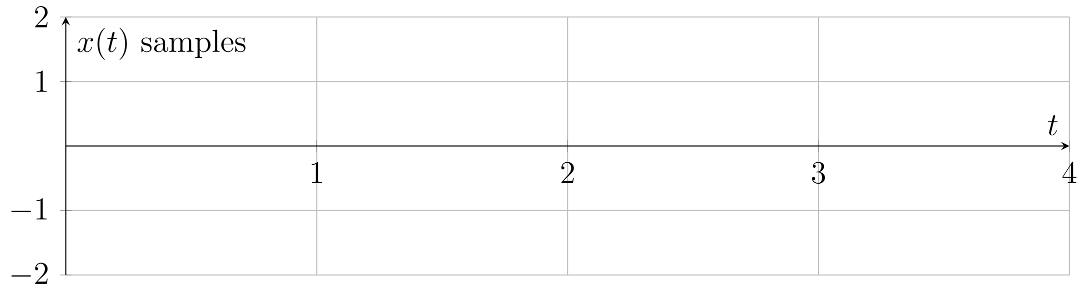

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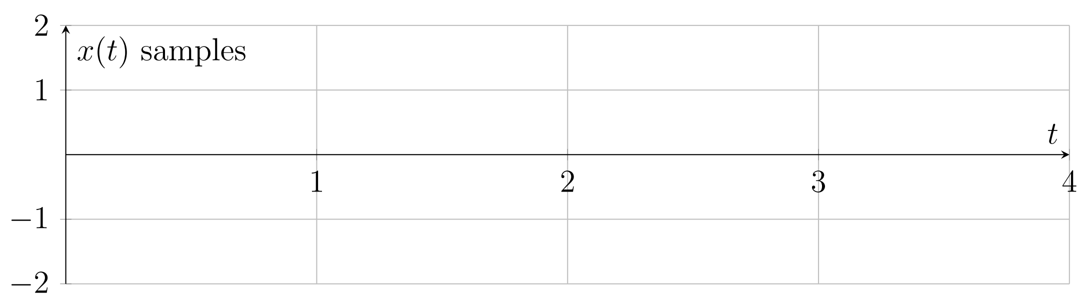

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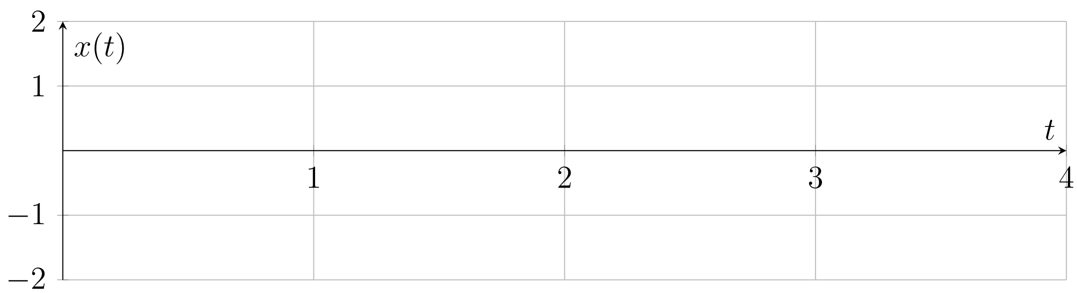

Frequency domain:

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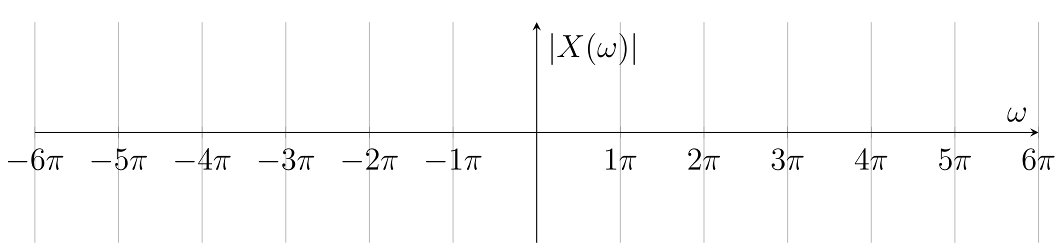

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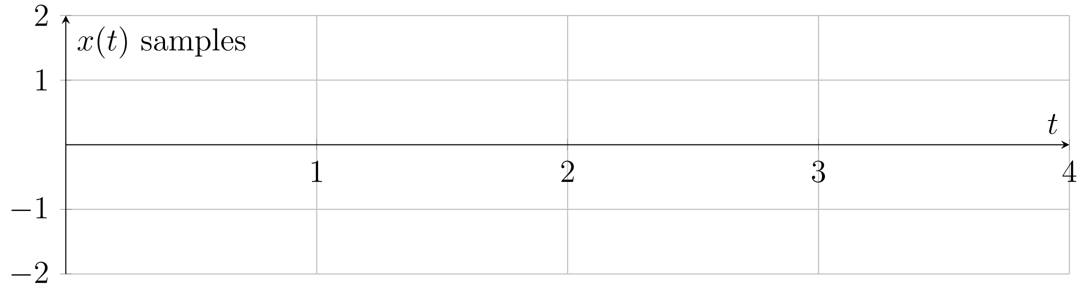

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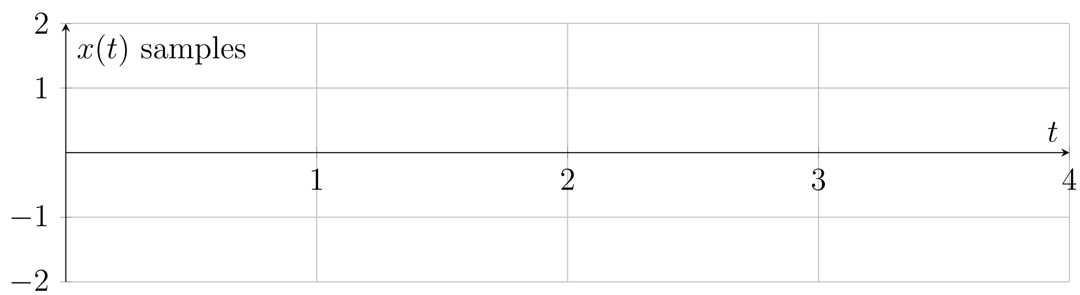

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