EE 122/ME 141 Week 2, Lecture 2 (Spring 2026)

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1 Goals

1.1 Homogeneous solution

The homogeneous solution is the response when the input is zero. For a second-order system,

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0,$$

the homogeneous solution is determined by the characteristic equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0.$$

1.2 Particular solution

The particular solution is any one solution that matches the forcing input. For example, a constant (step) input often yields a constant steady-state output, while a sinusoidal input yields a sinusoidal steady-state output at the same frequency, with an amplitude and phase determined by the transfer function.

1.3 Impulse and step as standard test inputs

Two canonical inputs are:

$$u(t) = \delta(t) \quad \text{(impulse)}, \qquad u(t) = u(t) \quad \text{(unit step)}.$$

The impulse response is the output to $\delta(t)$ with zero initial conditions. The step response is the output to a unit step with zero initial conditions. These are forced responses. In contrast, a free-decay pendulum release is a zero-input response driven by nonzero initial conditions.

Pop Quiz 1.1: Check your understanding!

A second-order system has transfer function $G(s) = \frac{k}{s^2+2\zeta\omega_n s+\omega_n^2}$. Write an expression for $Y(s)$ for (i) an impulse input and (ii) a unit-step input. Do not invert the transforms.

Solution on page 5

2 Natural frequency, damping, and damped frequency

Second-order dynamics are everywhere: suspensions, doors, elevators, robots, and even sensor filtering. The parameters ω_n and ζ are a compact way to describe what the system does.

2.1 Natural frequency ω_n

Interpretation. Roughly, ω_n sets the time scale of oscillation. For lightly damped systems, the oscillations are close to frequency ω_n.

Example interpretation. Two car suspensions can have the same damping ratio but different ω_n. The one with larger ω_n responds faster and oscillates more rapidly.

2.2 Damping and damping ratio ζ

Interpretation. Damping describes how quickly oscillations die out. The dimensionless damping ratio ζ classifies the response:

$$0 < \zeta < 1 \text{ underdamped (oscillatory)}, \qquad \zeta = 1 \text{ critically damped},$$

$$\zeta > 1 \text{ overdamped (non-oscillatory)}.$$

Example interpretation. A door closer is designed to avoid oscillation. Its effective damping ratio is typically at or above 1 so the door returns smoothly without bouncing.

2.3 Damped frequency ω_d and its dependence on ζ

For the characteristic equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0,$$

the roots are

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}.$$

When $0 < \zeta < 1$, the roots are complex:

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}.$$

The oscillation frequency is the imaginary part:

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}.$$

As damping increases (larger ζ), the oscillation frequency decreases.

Pop Quiz 2.1: Check your understanding!

A system has $\omega_n = 10$ rad/s. Compute ω_d for (i) $\zeta = 0.1$, (ii) $\zeta = 0.6$, and (iii) $\zeta = 1.2$. State which cases oscillate.

Solution on page 5

3 Poles as roots of the characteristic equation

For a transfer function

$$G(s) = \frac{N(s)}{D(s)},$$

the poles are the roots of $D(s)$. Poles determine the natural response and the transient behavior. In the second-order case,

$$D(s) = s^2 + 2\zeta\omega_n s + \omega_n^2,$$

so the pole locations encode both the decay rate (real part) and oscillation frequency (imaginary part). This is why identifying ζ and ω_n from a free-decay experiment is still a meaningful modeling task: it identifies the dominant poles of the dynamics.

Pop Quiz Solutions

Pop Quiz 1.1: Solution(s)

For an impulse, $U(s) = 1$, so

$$Y(s) = G(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

For a unit step, $U(s) = 1/s$, so

$$Y(s) = \frac{k}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}.$$

Pop Quiz 2.1: Solution(s)

For $0 < \zeta < 1$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$. (i) $\omega_d = 10\sqrt{1-0.01} = 10\sqrt{0.99} \approx 9.95$ rad/s, oscillatory. (ii) $\omega_d = 10\sqrt{1-0.36} = 10\sqrt{0.64} = 8$ rad/s, oscillatory. (iii) $\zeta > 1$ means no oscillation; the poles are real and the response is overdamped.