EE 122: Introduction to Control Systems

**Lab #2: Disturbance Response of a Pendulum System**

**Released:** Feb 3, 2026 **Due:** Feb 4/5, 2026

# 1 Objectives

This lab has four learning objectives:

a. understand the disturbance and step response of dynamical systems

b. quantify an external disturbance applied to a pendulum at rest by estimating an impulse magnitude,

c. connect time-domain properties to frequency-domain pole location, and

d. convert a second-order model to state space and generate a step response using `ct.step_response` (Python) or `step` (MATLAB).

**Reminder from Lab #1:** You already identified the characteristic equation of your pendulum setup by estimating $\zeta$ and $\omega_n$ from free-decay data. In this lab, you will fix $\zeta$ and $\omega_n$ to your Lab #1 values for your assigned setup and use them to quantify a disturbance event. Before proceeding further, make sure that you have your Lab #1 values ready.

**Pre-Lab Quiz 1.1: Make sure you're ready for the lab!**

Note down your Lab #1 values for $\zeta$ and $\omega_n$.

# 2 Pre-lab theory

## 2.1 Model

We model the pendulum as a rigid body rotating in a vertical plane about a pivot. Let $\alpha(t)$ be the angular displacement from the stable downward equilibrium. In the small-angle regime, the linearized equation of motion is

$$J\,\ddot{\alpha}(t) + b\,\dot{\alpha}(t) + mg\ell\,\alpha(t) = r(t), \tag{1}$$

where $r(t)$ is an externally applied torque (input).

We can write the second order transfer function in the standard form as

$$G(s) \equiv \frac{A(s)}{R(s)} = \frac{K\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \tag{2}$$

where the poles (and hence the natural response) are completely determined by $\zeta$ and $\omega_n$.

**Important:** In Lab #1, you did not apply a measured input $r(t)$, so you could not determine the gain $K$ from data. In Lab #2, we will focus on understanding how the poles of the system set the decay rate and the oscillation frequency when a disturbance input is applied. Remember that the poles only depend on $\zeta, \omega_n$ because they come from solving for the roots of the denominator of $G(s)$. Specifically, we will study how by measuring the oscillation frequency and time period, we can estimate the level of disturbance that was applied to the pendulum. In other words, you can estimate how hard you hit the pendulum!

## 2.2 Disturbance as an impulse

In this lab, you will start with the pendulum at rest near the downward equilibrium and apply a brief disturbance by tapping the pendulum (that is, hit it gently so it is disturbed from its rest position). We model this as a short torque pulse that is well-approximated by an impulse:

$$r(t) \approx I\,\delta(t),$$

where $I$ is the impulse magnitude of units: N·m·s and $\delta(t)$ is the Dirac delta function. Taking Laplace transforms, $\mathcal{L}\{\delta(t)\} = 1$, so the input is

$$R(s) = I.$$

Therefore, the output in the Laplace domain is

$$A(s) = G(s)\,R(s) = I\,G(s). \tag{3}$$

It is clear from this equation that the angle in frequency domain is simply the product of the transfer function and the impulse magnitude. In other words, what we see is that the impulse input excites the natural dynamics of the system, scaled by the impulse magnitude $I$.

## 2.3 Decay rate of the pendulum

The poles of $G(s)$ are the roots of

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0.$$

For the underdamped case $0 < \zeta < 1$, the poles are

$$s_{1,2} = -\zeta\omega_n \pm j\,\omega_d, \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}. \tag{4}$$

The real part

$$\sigma \equiv \zeta\omega_n$$

is the decay rate of the exponential envelope (we will discuss this in more detail next week during the lectures). Specifically, the oscillation envelope decays like $e^{-\sigma t}$. You can think of the decay rate as the value that sets how fast the oscillations die out. This is decided by the magnitude of the real part of the poles.

**Pre-Lab Quiz 2.1: Make sure you're ready for the lab!**

Using equation (4), show that the impulse response has an exponentially decaying envelope of the form $e^{-\sigma t}$, where $\sigma = \zeta\omega_n$.
Hint: To solve this problem, you may substitute numerical values for $\zeta$ and $\omega_n$ if you wish. Then, compute the inverse Laplace transform of $G(s)$ using partial fraction expansion as shown in the class. Once you obtain $y(t)$ you will see that the exponential decay term appears with the decay rate being equal to the real part of the poles.

## 2.4 Measuring the decay rate experimentally

Let $\alpha_k$ and $\alpha_{k+1}$ be the magnitudes of two successive same-sign peaks separated by one period $T$. For underdamped motion, the envelope is approximately exponential, so

$$|\alpha_{k+1}| \approx |\alpha_k|\, e^{-\sigma T}.$$

Taking logs of both sides, we get

$$\ln\left(\frac{|\alpha_k|}{|\alpha_{k+1}|}\right) \approx \sigma T.$$

Now, define the log decrement for one period (similar to Lab 1) as

$$\delta_k \equiv \ln\left(\frac{|\alpha_k|}{|\alpha_{k+1}|}\right).$$

Averaging over several peak pairs $(t_k, \alpha_k)$ for $k = 1, 2, \ldots, N$, we get

$$\bar{\delta} \approx \frac{1}{N-1} \sum_{k=1}^{N-1} \ln\left(\frac{|\alpha_k|}{|\alpha_{k+1}|}\right), \qquad T \approx \frac{1}{N-1} \sum_{k=1}^{N-1} (t_{k+1} - t_k).$$

The more same-sign peaks you average over, the more accurate your estimate of $\bar{\delta}$ and $T$ becomes. For the ECP setup, you may not be able to get many peaks, which is OK. You will be able to observe the error in modeling! Finally, the measured decay rate is

$$\sigma_{\text{meas}} \approx \frac{\bar{\delta}}{T}. \tag{5}$$

You will compare this to the theoretical decay rate

$$\sigma_{\text{theory}} = \zeta\omega_n$$

using your fixed Lab #1 values of $\zeta$ and $\omega_n$.

## 2.5 Settling time for a decaying oscillation

In a disturbance response experiment, the output returns to zero. We define settling time as the first time after which $\alpha(t)$ stays within a small band around zero. Let $\alpha_{\max}$ be the magnitude of the first (or largest) peak after the hit. For a given tolerance $\varepsilon$ (we will use 5% tolerance, so $\varepsilon = 0.05$, define the settling time $t_s$ as the smallest time such that

$$|\alpha(t)| \leq \varepsilon\, \alpha_{\max} \quad \text{for all } t \geq t_s.$$

Since the envelope decays like $e^{-\sigma t}$, a theoretical settling time estimate is obtained from

$$e^{-\sigma t_s} \approx \varepsilon \quad \Rightarrow \quad t_s \approx \frac{1}{\sigma} \ln\left(\frac{1}{\varepsilon}\right).$$

For $\varepsilon = 0.05$, $\ln(1/0.05) \approx 3$, so

$$t_{s,5\%} \approx \frac{3}{\sigma} = \frac{3}{\zeta\omega_n}. \tag{6}$$

**Pre-Lab Quiz 2.2: Make sure you're ready for the lab!**

Sketch an expected impulse (disturbance) response of a pendulum and indicate how you would measure the settling time $t_s$ from the time series. That is, clearly mark in your graph how the settling time is computed.

## 2.6 Estimating the impulse magnitude

From equation (3), the impulse response scales linearly with the impulse magnitude $I$. However, the absolute mapping from torque impulse to angle depends on the gain $K$, which you did not identify in Lab #1. Therefore, we can only estimate an effective impulse magnitude.

$$I_{\text{eff}} \equiv I\,K, \tag{7}$$

which is identifiable from output data using fixed $\zeta, \omega_n$. This is still a meaningful quantity because it measures how hard you hit the pendulum (in the units of the identified model).

**Curve fitting to find $I_{\text{eff}}$:** For $0 < \zeta < 1$, the impulse response of (2) has the form

$$\alpha(t) = I_{\text{eff}}\,\omega_n^2 \cdot \frac{1}{\omega_d}\,e^{-\sigma t}\sin(\omega_d t)\,u(t), \tag{8}$$

where $u(t)$ is the unit step, $\sigma = \zeta\omega_n$, and $\omega_d = \omega_n\sqrt{1-\zeta^2}$. With $\zeta, \omega_n$ fixed, the only unknown scaling in (8) is $I_{\text{eff}}$. You can estimate $I_{\text{eff}}$ by matching the measured amplitude of $\alpha(t)$ to this template. For example, using least squares over a time window after the hit you can estimate the value of $I_{\text{eff}}$ that minimizes the squared error between the measured and predicted amplitudes.

## 2.7 Step response

Following the derivation in class, we can obtain a state-space model from the normalized second-order ODE. We define the state vector $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \alpha(t) \\ \dot{\alpha}(t) \end{bmatrix}$. Then

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t), \tag{9}$$

with $y(t) = \alpha(t)$.

**Pre-Lab Quiz 2.3: Make sure you're ready for the lab!**

Starting from the ODE, write down $A, B, C, D$ for the state-space model in eq. (9) with $y = \alpha$.

## 2.8 Before proceeding any further...

Show your pre-lab work, including your solutions to all pre-lab quizzes, to your TA. Your TA will assign points to you out of 30 for the pre-lab work. But if you have 0/30 points, you will not be allowed to proceed with the lab.

| **Pre-Lab Grade and Experimental Steps** | | |
|---|---|---|
| **Due:** Feb 4/5, 2026 | **Name:** ____________ | **Grade:** _____ |

# 1 Experimental setup and data collection

The SE1 172 lab has three pendulum setups: the rotary pendulum with servo (Quanser), the cart pendulum (Quanser), or the ECP pendulum apparatus. In this lab, you will record the pendulum angle $\alpha(t)$ as a function of time after an externally applied disturbance (a brief hit/tap) while the system is otherwise in open loop.
Before powering up, verify that the pendulum can swing freely, that cables are not in the path of motion, and that you are familiar with the emergency procedures.

**Step 0:** Open MATLAB / ECP and set path to EE 122 lab files. These steps are dependent on your setup. Follow the specific instructions for your assigned setup for your test run and then follow the directions in Steps 1 to 7 to collect data for your analysis.

**Rotary pendulum setup**: Go to C:\ Desktop \ EE 122 \ Rotary Inverted Pendulum \Software \. Open `setup_rotpen.m` file to see the setup parameters and **run** the file. Make sure that the voltage amplifier is powered ON (green light on in the front). Open the Simulink model `q_rotpen_mdl_student_lab1.slx` and **click** on "Monitor & Tune", then on "Build for Monitoring". After building, **click** on "Connect" and then "Start". Open the Scope window showing the pendulum angle ($\alpha$). Hold the servo angle still (someone else should hold it) for consistent measurements.

**Cart pendulum setup**: Open `setup_ip02_sip.m` file and **run** it. Then, follow the same steps as the rotary pendulum setup in the Simulink model.

**ECP pendulum setup**: Open ECP software. Set the pendulum at rest and go to Utility and **click** on "Zero Position". This setup requires you to apply a calibration formula to get angles in degrees from the counts that you observe: `1 degree is 44.4 counts`. After zero positioning, set up data acquisition for the encoder angle and export raw data after your run.

**Step 1:** Confirm the sign convention and the units (you must use angles in radians — so if degrees are reported convert them!). With a few trial runs, understand how the data stream is being stored in MATLAB / ECP software.
Divide clear roles in your group: who will control the software, who will apply the disturbance, who will record the time of the hit, who will do the calculations, and

who will fill out the worksheet.

**Step 2:** Bring the pendulum to the downward equilibrium and let it come to rest. Your goal is to apply a brief disturbance **starting from rest**. Wait until the angle trace is approximately constant (near zero) before applying the hit.

**Step 3:** Start recording $\alpha(t)$. Note the sampling rate $f_s$ (Hz).

**Step 4:** Apply a brief disturbance (a quick tap/hit) to the pendulum. The disturbance should be short compared to the oscillation period, and the resulting motion should remain in the small-angle regime (target peak magnitude less than 15°). For the ECP setup, you may go beyond the 15° so that you get enough peaks (you can expect inaccuracies to creep in because you are allowing this nonlinear behavior).

**Step 5:** Continue recording until the oscillations visibly decay and the pendulum settles back near zero. Record long enough so that you can capture at least 4-6 same-sign peaks (for better averaging).

**Step 6:** Save your data! Email/send the data to yourself and move on from the experimental setup so that another team can perform the data collection.

**Step 7:** From your recorded data, identify same-sign peaks $(t_k, \alpha_k)$ for $k = 1, 2, \ldots$ after the hit. Use at least 4-5 peak pairs so you can average the decay and the period. Then:

- compute $\sigma_{\text{meas}}$ from your data,
- compute $\sigma_{\text{theory}} = \zeta\omega_n$ from your Lab #1 values,
- compute and compare settling times (measured vs theory),
- estimate $I_{\text{eff}}$ as the disturbance impulse scale.

**Lab Worksheet**

**Due:** Feb 4/5, 2026 **Name:** ______________ **Grade:** _____

## Setup information

Your setup name (cart-pole, rotary, ECP): ________________________

Sampling rate $f_s$ (Hz): ____________

Your Lab #1 values for this setup:

$\zeta =$ ______________ $\omega_n$ (rad/s) $=$ ______________

## Peak table

Record at least 4-6 same-sign peaks after the disturbance. Use magnitudes $|\alpha_k|$ so the decay ratio is positive:

$$\delta_k = \ln\left(\frac{|\alpha_k|}{|\alpha_{k+1}|}\right).$$

| Peak $k$ | time $t_k$ (s) | peak $\lvert\alpha_k\rvert$ (rad) | $\delta_k$ |
|---|---|---|---|
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| 5 | | | |
| 6 | | | – |

## Measured decay rate vs theoretical decay rate

Compute the average period $T$ using same-sign peaks (in the equations below we assume 6 peaks, which gives 5 periods. You should adjust it accordingly if you have a different number of peaks):

$$T \approx \frac{1}{5}\sum_{k=1}^{5}(t_{k+1} - t_k), \qquad \omega_d = \frac{2\pi}{T}.$$

Compute the average log decay $\bar{\delta}$:

$$\bar{\delta} \approx \frac{1}{5} \sum_{k=1}^{5} \ln\left( \frac{|\alpha_k|}{|\alpha_{k+1}|} \right).$$

Then compute the measured decay rate: $\sigma_{\text{meas}} \approx \frac{\bar{\delta}}{T}$.. Compute the theoretical decay rate using Lab #1 values: $\sigma_{\text{theory}} = \zeta\omega_n$..

**Your computed values:**
$T$ (s): ________________, $\bar{\delta}$: ________________, $\sigma_{\text{meas}}$ (1/s): ________________

$\sigma_{\text{theory}}$ (1/s): ________________, $\omega_d$ (rad/s): ________________

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| Compare $\sigma_{\text{meas}}$ and $\sigma_{\text{theory}}$. Report a percent error. |

## Settling time

Define $\alpha_{\text{max}}$ as the magnitude of the first (or largest) peak after the hit. Choose a tolerance band $\varepsilon = 0.05$ (corresponding to 5%).

$$\text{Threshold} = \varepsilon\, \alpha_{\text{max}}.$$

**Measured settling time:** from your time series, find the first time $t_s$ after which $|\alpha(t)| \leq \varepsilon\alpha_{\text{max}}$ for all later times.

**Theoretical settling time:** use

$$t_{s,5\%} \approx \frac{3}{\sigma_{\text{theory}}} = \frac{3}{\zeta\omega_n}.$$

**Experimental Observations:**
$\alpha_{\text{max}}$ (rad): ________________, Threshold $0.05\alpha_{\text{max}}$ (rad): ________________

$t_{s,\text{meas}}$ (s): ________________, $t_{s,5\%,\text{theory}}$ (s): ________________

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| Describe how you measured settling time from your data. Compare measured vs theory and comment on agreement of the settling time. |

## Estimate the disturbance impulse magnitude

To estimate the effective impulse magnitude $I_{\text{eff}} = IK$, which scales the impulse response, use the fixed $\zeta, \omega_n$, and compute:

$$\sigma_{\text{theory}} = \zeta\omega_n, \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}.$$

The impulse response template (up to scale) is:

$$\alpha(t) \approx I_{\text{eff}}\,\omega_n^2 \cdot \frac{1}{\omega_d}\,e^{-\sigma_{\text{theory}}t}\sin(\omega_d t).$$

Estimate $I_{\text{eff}}$ by matching this template to your measured $\alpha(t)$ after the hit. Use least squares curve fitting, for example, to minimize the squared error over a time window after the hit.

**Estimated value:**

$$I_{\text{eff}} = \underline{\hspace{6cm}}$$

## State space and step response

From the pre-lab, write $A, B, C, D$:

$$A = \begin{bmatrix} \underline{\hspace{3cm}} & \underline{\hspace{3cm}} \\ \underline{\hspace{3cm}} & \underline{\hspace{3cm}} \end{bmatrix}, \quad B = \begin{bmatrix} \underline{\hspace{3cm}} \\ \underline{\hspace{3cm}} \end{bmatrix},$$

$$C = \begin{bmatrix} \underline{\hspace{3cm}} & \underline{\hspace{3cm}} \end{bmatrix}, \quad D = \underline{\hspace{3cm}}.$$

**Generate a step response plot using $A, B, C, D$:**

- In Python, use `control` (`ct`) and `ct.step_response`.
- In MATLAB, use `ss(A,B,C,D)` then `step(sys)`.

Use your Lab #1 values of $\zeta$ and $\omega_n$. Show your plot to the TA.

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| Report the percent overshoot and the settling time of the normalized step response by computing these values from the step response plot. |