EE 122: Introduction to Control Systems

Lab #1: Modeling a Pendulum System

Released: January 26, 2026 **Due:** January 28 / 29, 2026

1 Objectives

This lab has three learning objectives:

a. become familiar with the controls lab equipment in SE1 172 and data acquisition workflow with MATLAB,

b. record open-loop angle data from a pendulum undergoing free decay, and

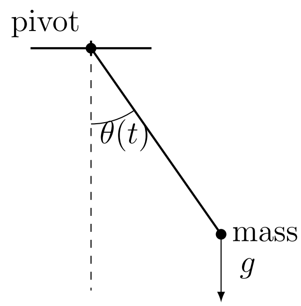
c. build a linear dynamical model from measurements.

The overall outcome is to derive a transfer function for the pendulum system that you have been assigned. Following the derivation, you must test your model to make a quantitative prediction of the period of the pendulum and compare that prediction to new measurements.

2 Pre-lab theory (must complete before lab)

2.1 The pendulum system

We model the pendulum as a rigid body rotating in a vertical plane about a pivot. Let $\theta(t)$ be the angular displacement from the stable downward equilibrium. To obtain a linear model, we restrict the analysis to small angle values (see derivation below to understand why). Therefore, in the lab experiment, you must release the pendulum from a small angle so that $\sin\theta \approx \theta$.



2.2 Model: equation of motion

We can write the equation of motion for the pendulum using Newton's second law for rotational systems, that is, *sum of torques = moment of inertia × angular acceleration*. Here, we have three kinds of torques acting on the pendulum: a gravitational torque that tends to restore the pendulum to the downward equilibrium ($-mg\ell\sin(\theta)$), a damping torque that opposes motion ($-b\dot{\theta}$) with b as the viscous damping coefficient, and an externally applied torque input $r(t)$. The equation of motion is therefore

$$J\,\ddot{\theta}(t) + b\,\dot{\theta}(t) + mg\ell\,\sin(\theta(t)) = r(t), \tag{1}$$

where J is the rotational inertia about the pivot, b is an (effective) viscous damping coefficient, m is an effective mass, ℓ is the distance from pivot to center of mass, g is gravitational acceleration, and $r(t)$ is an externally applied torque. In the lab experiment, we will observe the natural response of the system, so we will not externally apply $r(t)$ and observe the *free decay* after a release.
In the small-angle regime, $\sin(\theta) \approx \theta$, so (1) becomes the linear ODE

$$J\,\ddot{\theta}(t) + b\,\dot{\theta}(t) + mg\ell\,\theta(t) = r(t). \tag{2}$$

2.3 Pendulum transfer function

Taking the Laplace transform of equation (2) under zero initial conditions gives

$$\left(Js^2 + bs + mg\ell\right)\Theta(s) = R(s),$$

so the torque-to-angle transfer function is

$$G(s) \equiv \frac{\Theta(s)}{R(s)} = \frac{1}{Js^2 + bs + mg\ell}. \tag{3}$$

It is also common to rewrite it in standard second-order form,

$$G(s) = \frac{K\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4}$$

where

$$K = \frac{1}{mg\ell}, \quad \omega_n = \sqrt{\frac{mg\ell}{J}}, \quad \zeta = \frac{b}{2}\sqrt{\frac{1}{mg\ell J}}.$$

Pre-Lab Quiz 2.1: Make sure you're ready for the lab!

Show that the parameters K, ω_n, and ζ defined above lead from equation (3) to equation (4).

2.4 Identifying parameters from free-decay data

In the lab experiment, you will record $\theta(t)$ for a free-decay oscillation that is, with no input, $r(t) = 0$. Without the input, we cannot identify the transfer function fully, but we can identify the characteristic equation of the pendulum model (the denominator), since the natural response is determined by the poles. Recall that the characteristic equation is $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$, whose roots are the system poles.

Approach: Let θ_k be the magnitude of the kth peak (same sign peaks, i.e., peak-to-peak one period apart) and let t_k be the time at which that peak occurs. Then, we define the following quantities that lead us to estimates of ζ and ω_n, which fully specify the characteristic equation.

(A) Period and damped natural frequency: If T is the average time between same-sign peaks, then we can estimate the damped natural frequency as

$$T \approx \frac{1}{N-1}\sum_{k=1}^{N-1}(t_{k+1} - t_k), \qquad \omega_d = \frac{2\pi}{T}.$$

Note that $t_{k+1} - t_k$ is one period of oscillation, so we average over $N-1$ such intervals if we have N peaks. This gives us an averaged estimate of the period T and hence ω_d (better than just observing one period).

(B) Damping ratio: For underdamped motion, successive peak magnitudes satisfy approximately exponential decay. Therefore, we can write,

$$\theta_{k+1} \approx \theta_k\, e^{-\delta}, \qquad \delta = \ln\left(\frac{\theta_k}{\theta_{k+1}}\right).$$

The δ quantity defines how much the pendulum amplitude has decayed over one period T. This is the same as $\zeta\omega_n T$, that is, natural frquency scaled by the damping ratio.
Now, average the computed δ over several peak pairs to reduce the noise in your measurements:

$$\bar{\delta} = \frac{1}{N-1}\sum_{k=1}^{N-1} \ln\left(\frac{\theta_k}{\theta_{k+1}}\right).$$

Now, we know that the damped frequency ω_d for a 2nd order system is related to the natural frequency ω_n and the damping ratio ζ by

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

and we have $T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$. In one time period, the decrement is $\delta = \zeta\omega_n T_d = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$, which leads to the damping ratio as

$$\zeta = \frac{\bar{\delta}}{\sqrt{4\pi^2+\bar{\delta}^2}}. \tag{5}$$

(C) Natural frequency: Finally, for 2nd order systems, the damped and undamped natural frequencies are related by

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}}. \tag{6}$$

Pre-Lab Quiz 2.2: Make sure you're ready for the lab!

Using the inertia for a point-mass pendulum, $J = m\ell^2$, find out an expression for the viscous damping coefficient b in terms of ζ, ω_n, m, and ℓ.

Hint: Write the ODE as

$$\ddot{\theta} + \frac{b}{J}\dot{\theta} + \frac{mg\ell}{J}\theta = 0$$

and compare with the standard form $\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0$.

2.5 Before proceeding any further...

Show your pre-lab work, including your solutions to both pre-lab quizzes, to your TA. Your TA will assign points to you out of 30 for the pre-lab work. But if you have 0/30 points, you will not be allowed to proceed with the lab.

Pre-Lab Grade and Experimental Steps

Due: January 28 / 29, 2026 **Name:** ______________ **Grade:** _____

The SE1 172 lab has three pendulum setups: the rotary pendulum with servo (Quanser), the cart pendulum (Quanser), or the ECP pendulum apparatus. The measurements you need is the pendulum angle $\theta(t)$ as a function of time in open loop. The model-fitting steps are identical irrespective of the setup you are assigned, however, the answers will differ because the pendulum setups are mechanically different. Before powering up, verify that the pendulum can swing freely, that cables are not in the path of motion, and that you are familiar with the emergency procedures.

Step 0: Open MATLAB / ECP and set path to EE 122 lab files. These steps are dependent on your setup.

Rotary pendulum setup: Go to C:\ Desktop \ EE 122 \ Rotary Inverted Pendulum \Software \. Open setup_rotpen.m file to see the setup parameters and run the file. Make sure that the voltage amplifier is powered ON (green light on in the front). Open the Simulink model `q_rotpen_bal_student.mdl` and build for monitoring. After building, click on connect and then start. You should see the Scope windows showing the pendulum angle.

Cart pendulum setup: Open setup_ip02_sip.m file and run it. Then, follow the same steps as the rotary pendulum setup.

ECP pendulum setup Open ECP software. Go to Data, and then Setup Data Acquisition. Add encoder position 2 to the left side and set two cycles. Then, go to Command and hit Execute (make sure that normal data sampling is checked). You should see the angle data being plotted in real-time.

Step 1: Familiarize yourself with the setup and ask your TA to show you where you can record the data. Confirm the sign convention and the units (you must use angles in radians — so if degrees are reported convert them!). With a few trial runs, understand how the data stream is being stored in MATLAB / ECP software.
Divide clear roles in your group: who will control the software, who will perform the release, who will record the data, who will do the calculations, and who will fill out the worksheet.

Step 2: Record your data for long enough so that you can capture at least 6-8 same-sign peaks (for better averaging). Note the sampling rate f_s (Hz) at which data is recorded.

Step 3: Bring the pendulum to a small displacement from the downward equilibrium (target less than $15°$). Hold it still for one second to establish a clean initial time, then release without pushing (we are observing the natural response).

Step 4: Record $\theta(t)$ from release through visible decay. If you observe strong damping and you you are unable to understand the peaks, repeat by adjusting the release angle. Save your data! Email/send the data to yourself and move on from the experimental setup so that another team can perform the data collection. You may return back if you need to re-collect data (that is, if your initial run did not capture enough peaks or was noisy).

Step 5: From your recorded data, identify same-sign peaks (t_k, θ_k) for $k = 1, 2, \ldots$. Use at least 5 peak pairs so you can average the decay and the period.

Step 6: Use your first few peaks to compute $\bar{\delta}$, ζ, T, ω_d, and ω_n. Then predict later peaks and compare to your measured values.

Lab Worksheet

Due: January 28 / 29, 2026 **Name:** ______________ **Grade:** _____

Your setup name (cart-pole, rotary, ECP): ________________________

Sampling rate f_s (Hz): ____________

Release angle magnitude (approx): ____________

Record at least 6 same-sign peaks. Use magnitudes $|\theta_k|$ so the decay ratio is positive.

Peak k	time t_k (s)	peak $|\theta_k|$ (rad)	δ_k
1			
2			
3			
4			
5			
6			–

Parameter Estimates

Compute the average period T using same-sign peaks:

$$T \approx \frac{1}{5}\sum_{k=1}^{5}(t_{k+1} - t_k) \qquad , \qquad \omega_d = \frac{2\pi}{T}.$$

Compute the average log decay $\bar{\delta}$:

$$\bar{\delta} \approx \frac{1}{5}\sum_{k=1}^{5} \ln\left(\frac{|\theta_k|}{|\theta_{k+1}|}\right).$$

Then compute

$$\zeta = \frac{\bar{\delta}}{\sqrt{4\pi^2 + \bar{\delta}^2}}, \qquad \omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}}.$$

Your computed values:

T (in seconds): ______________ , $\bar{\delta}$: ______________ , ζ: ______________ ,

ω_d (in rad/s): ______________ , ω_n (in rad/s): ______________ .

Transfer function (with numerical values) $G(s) = \Theta(s)/R(s) = 1/(Js^2 + bs + mg\ell)$

Validate your experimental model

Use your estimates of T and $\bar{\delta}$ to predict future peak times and magnitudes from earlier peaks. Use peak 1 as the reference. Use:

$$t_k^{\text{pred}} \approx t_1 + (k-1)T, \qquad |\theta_k|^{\text{pred}} \approx |\theta_1|\, e^{-(k-1)\bar{\delta}}.$$

Peak k	t_k^{meas}	t_k^{pred}	$\lvert\theta_k\rvert^{\text{meas}}$	$\lvert\theta_k\rvert^{\text{pred}}$
3				
4				
5				

Validation: Does your model predict peak *timing* well? Does it predict peak *decay* well? Report the error and discuss a reason for mismatch (if any):