10.3.4.2.5

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Question: A fair coin is tossed three times. Find the PMF of the random variable using the sum of three independent Bernoulli trials. Verify through simulation.
Solution:
Let X be a discrete random variable
X = the number of heads in three tosses of a fair coin.

$$X = X_1 + X_2 + X_3 \tag{0.1}$$

where X_1, X_2, X_3 are independent Bernoulli trials.

$$X_i \sim \text{Bernoulli } (p = 0.5) \tag{0.2}$$
$$X \sim \text{Binomial } (n = 3, p = 0.5) \tag{0.3}$$
$$X_i = \begin{cases} 1, & \text{Outcome in Heads} \\ 0, & \text{Outcome in Tails} \end{cases} \tag{0.4}$$
$$\tag{0.5}$$

Compute the Moment-Generating Function (MGF) Using the Z-Transform :
The Z-transform of the PMF is given by;

$$M_{X_i}(z) = \sum_{n=-\infty}^{\infty} p_{X_i}(n) z^{-n} \tag{0.6}$$

Since X_i takes only two values (0 or 1):

$$M_{X_i}(z) = (1 - p) + pz^{-1} \tag{0.7}$$

since $X_1, X_2, X_3$are independent, their total MGF is:

$$M_X(z) = M_{X_1}(z) M_{X_2}(z) M_{X_3}(z) \tag{0.8}$$
$$\implies M_X(z) = ((1 - p) + pz^{-1})^3 \tag{0.9}$$
$$= \sum_{n=-\infty}^{\infty} {}^3C_n (1 - p)^{3-n} p^n z^{-n} \tag{0.10}$$
$$p_X(n) = {}^3C_n p^n (1 - p)^{3-n} \tag{0.11}$$
$$p_X(n) = \frac{{}^3C_n}{8} \tag{0.12}$$

The Probability Mass Function (PMF) for the given random variable is

$$p_X(n) = \begin{cases} \frac{1}{8}, & n = 0 \\ \frac{3}{8}, & n = 1 \\ \frac{3}{8}, & n = 2 \\ \frac{1}{8}, & n = 3 \end{cases} \tag{0.13}$$

Simulation: We simulate this process by generating uniform random numbers and classifying each trial as heads if the random number is less than $p = 0.5$, otherwise tails. The algorithm follows these steps:

1) Generate a uniform random number between $[0, 1)$.
2) Classify as heads if the number is less than 0.5, otherwise tails.
3) Repeat for three trials and count the number of heads.
4) Repeat this process for 10^5 simulations.
5) Count occurrences of each possible value (0, 1, 2, or 3 heads).
6) Divide by the total number of trials to get the probability estimate.

The below graph shows the comparision between theoretically calculated and simulated PMF of the given random variable.