9.2.1

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Question: Check if the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$ has a solution $y = e^x + 1$ for $y(0) = 2$ and $y'(0) = 1$.

1) **Theoretical Solution:**

$$\text{Taking } \frac{dy}{dx} = t; \tag{1.1}$$

$$\frac{dt}{dx} - t = 0 \tag{1.2}$$

$$\int \frac{1}{t} dt = \int dx \tag{1.3}$$

$$ln(t) = x + k \tag{1.4}$$

$$t = Ce^x \text{ but } y'(0) = 1 \implies \frac{dy}{dx} = e^x \tag{1.5}$$

$$\int dy = \int e^x dx \tag{1.6}$$

$$\implies y = e^x + k \tag{1.7}$$

$$\text{but } y(0) = 2 \implies k = 1 \tag{1.8}$$

$$\therefore y = e^x + 1 \text{ is the solution under given conditions.} \tag{1.9}$$

2) **Using Trapezoidal Rule:**

$$\text{For } \frac{dy}{dx} = f(x, y) \tag{2.1}$$

$$\int_{y_n}^{y_{n+1}} dy = \int_{x_n}^{x_{n+1}} f(x, y)\, dx \tag{2.2}$$

$$y_{n+1} - y_n \approx \frac{h}{2} \left(f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right) \tag{2.3}$$

$$\implies y_{n+1} = y_n + \frac{h}{2} \left(f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right) \tag{2.4}$$

$$\text{where } h = x_{n+1} - x_n \tag{2.5}$$

To solve the differential equation $y'' - y' = 0$ numerically using the trapezoidal rule, we first need to rewrite it as a system of first-order differential equations, let $y' = v$ and $v' = v$
For $y' = v$, applying trapezoidal rule, gives

$$y_{n+1} = y_n + \frac{h}{2} (v_n + v_{n+1}) \tag{2.6}$$

For $v' = v$, applying trapezoidal rule, gives

$$v_{n+1} = v_n + \frac{h}{2}(v_n + v_{n+1}) \tag{2.7}$$

$$\implies v_{n+1} = \frac{1+\frac{h}{2}}{1-\frac{h}{2}} v_n \tag{2.8}$$

$$\therefore \text{The final difference equations are} y_{n+1} = y_n + \frac{h}{2}(v_n + v_{n+1}) \tag{2.9}$$

$$v_{n+1} = \frac{1+\frac{h}{2}}{1-\frac{h}{2}} v_n \tag{2.10}$$

3) **Using Bilinear Transform:**
The bilinear transform maps the continuous-time derivative operator $\frac{d}{dt}$ to a discrete time operator in the z-domain as:

$$\frac{d}{dt} \longrightarrow \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}} \tag{3.1}$$

$$\text{here, h is the step size and } z^{-1} \text{ is a delay operator which represents;} \tag{3.2}$$

$$z^{-1}y[n] = y[n-1] \tag{3.3}$$

$$\text{For } v' = v \text{ we get } \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}} V(z) = V(z) \tag{3.4}$$

$$\implies z^{-1} = \frac{\frac{2}{h}-1}{\frac{2}{h}+1} \tag{3.5}$$

$$\text{Thus, the difference equation that we get is } v[n] = \alpha v[n-1] \tag{3.6}$$

$$\alpha = \frac{\frac{2}{h}-1}{\frac{2}{h}+1} \tag{3.7}$$

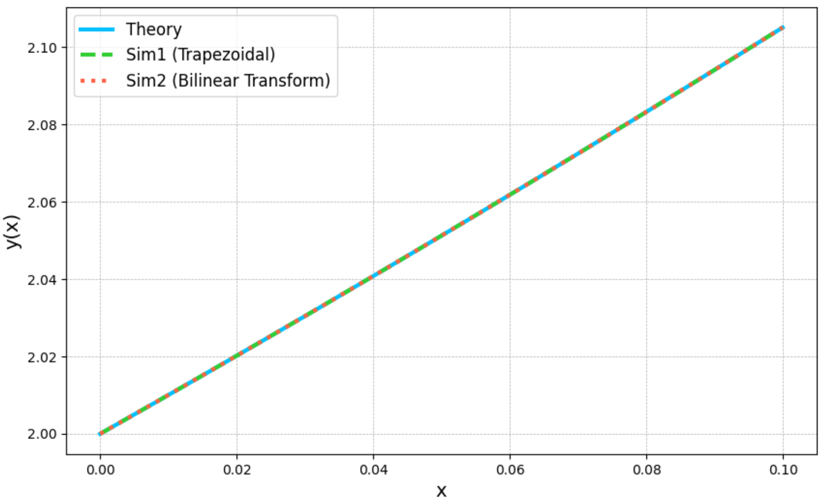
$$\text{Similarly, for } y' = v;\ \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}} V(z) = Y(z) \tag{3.8}$$

$$\text{Applying inverse z-transform, we get } y[n+1] = y[n] + \frac{h}{2}(v[n] + v[n+1]) \tag{3.9}$$

$$\text{Therefore, the final difference equations are } v[n] = \alpha v[n-1]; \alpha = \frac{\frac{2}{h}-1}{\frac{2}{h}+1} \tag{3.10}$$

$$y[n+1] = y[n] + \frac{h}{2}(v[n] + v[n+1]) \tag{3.11}$$

The below graph shows the comparison between the curve that is obtained theoretically and the simulation curve(numerically generated points through iterations).


Theory
Sim1 (Trapezoidal)
Sim2 (Bilinear Transform)
y(x)
x
2.00
2.02
2.04
2.06
2.08
2.10
0.00
0.02
0.04
0.06
0.08
0.10