EECS 16A Designing Information Devices and Systems I
Spring 2015

Practice problems

Vectors, Vector Spaces, Bases

1. True or False?

   (a) Let $\mathbf{v}_1 \cdots \mathbf{v}_n$ be a set of vectors in an n-dimensional vector space. $\mathbf{v}_1$ is the zero vector, and $\mathbf{v}_1 \cdots \mathbf{v}_n$ form a basis.

   (b) If $\sum_{i=1}^{N} \alpha_1 v_i = 0$ and v_i are length-n vectors, then $\alpha_i = 0$, $\forall i$.

2. Given any vector $\mathbf{v}$ in $\mathbb{R}^2$, how would you find another vector which is perpendicular to it?

3. What is the general form of any vector on the line $ax + by = c$? What is the general form of the vector normal to it?

4. Find the condition for r and s such that the vectors $\begin{bmatrix} r & 2 & s \end{bmatrix}^T$, $\begin{bmatrix} r+1 & 2 & 1 \end{bmatrix}^T$ and $\begin{bmatrix} 3 & s & 1 \end{bmatrix}^T$ form a basis for $\mathbb{R}^3$.

5. Let $(\mathbb{V}, \mathbb{R})$ be a vector space in $\mathbb{R}^N$. Show that any set of vectors $\{v_1, v_2, \ldots, v_r\} \in \mathbb{V}$ where $r > N$ can never form a basis for the vector space.

6. Find the angle between the vectors $\begin{bmatrix} 2 & 5 \end{bmatrix}^T$ and $\begin{bmatrix} -1 & 3 \end{bmatrix}^T$. Show that the vectors $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$ and $\begin{bmatrix} 5 & -1 & -1 \end{bmatrix}^T$ are perpendicular. Show that the vectors $\begin{bmatrix} 1 & -3 \end{bmatrix}^T$ and $\begin{bmatrix} -2 & 6 \end{bmatrix}^T$ are parallel. Find a unit vector in the direction of $\mathbf{u} = \begin{bmatrix} 1 & 3 & 4 \end{bmatrix}^T$, as well as a unit vector in the opposite direction of $\mathbf{u}$.

7. A new robot arm has three infinitely extendable arm segments such that each segment continues after another. The first segment can extend along the direction $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$, the second in the direction of $\begin{bmatrix} 1/2 & 1/3 & 0 \end{bmatrix}$ and the third in the direction of $\begin{bmatrix} 0 & 2 & 2 \end{bmatrix}$.

   (a) What is the vector space that models the reachable points?

   (b) If we want to reach point $\begin{bmatrix} 10 & 7 & 3 \end{bmatrix}$, how much should each arm segment be extended?

   (c) If the second segment is broken such that it is fixed at length 1, what is the new space that models the reachable points?

   (d) If we are allowed to add a fixed-length fourth segment to the case of part (c) to make the model a valid vector space, what direction and what length should the fourth segment be? What is the resulting vector space?

8. Consider two bases in $\mathbb{R}^3$. The first one, $B = \{\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T, \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}^T, \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T\}$ and the second basis $C = \{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}^T, \begin{bmatrix} 0 & 0 & 3 \end{bmatrix}^T\}$ The coordinates of a vector $\mathbf{v}$ relative to B are (x, y, z). The coordinates of a vector $\mathbf{v}$ relative to C are (x', y', z'). Write the relation between these coordinates in matrix notation.

Matrices

1. Come up with set of matrix transformations for the following

   (a) Convert a square matrix to a diagonal matrix. For instance,

   $$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \rightarrow \begin{bmatrix} A_{1,1} & 0 \\ 0 & A_{2,2} \end{bmatrix}$$

   (b) Mirror a vector in $\mathbb{R}^N$ about the first coordinate. For instance in $\mathbb{R}^2$, $\begin{bmatrix} x & y \end{bmatrix}^T \rightarrow \begin{bmatrix} x & -y \end{bmatrix}^T$

   (c) Mirror a vector about the $x = y$ line in $\mathbb{R}^2$.

2. True or False?

   (a) If $AB = AC$, then $B = C$

   (b) All matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ commute i.e., $AB = BA$.

3. Can you think of matrices A and B that do commute?

4. Find α such that $A^2 = B$ when $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$.

5. Invert the following matrices:

   (a) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & -4 \\ 2 & -8 \end{bmatrix}$

   (d) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

   (e) $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6. Find the eigenvalues of the matrices in the previous part.

7. What are the conditions on K for the following matrix to be invertible: $\begin{bmatrix} 1 & 0 & -K \\ 2 & 1 & 3 \\ K & 0 & 1 \end{bmatrix}$

Systems of Linear Equations

1. For what values of a and b does the system of linear equations have no solutions:

   $$\begin{aligned} ax + y + 2z &= 0 \\ x + 2y + z &= b \\ 2x + y + az &= 0 \end{aligned}$$

2. Show that there is no solution for the following system of linear equations:

$$\begin{aligned}x+4y-2z&=3\\3x+y+5z&=7\\2x+3y+z&=5\end{aligned}$$

3. The sum of three numbers is 6. If we multiply the third number by 3 and add the second number to it, we get 11. By adding the first and third number, we get the double of the second number. Represent this algebraically and find the numbers.

4. Solve the set of equations:

$$\begin{aligned}x+2y+z&=2\\2x-3y+4z&=1\\3x+6y+3z&=6\end{aligned}$$