

Overview

In this note, we will introduce the fundamental objects of **vectors** and **matrices**, and discuss how to use them to represent systems of linear equations. We will further introduce the concept of **linearity**, and discuss how its formal definition relates to our intuitive understanding of “linear” functions. To illustrate all of these concepts, we will discuss the real-world technique of tomographic imaging.

1.1 What is Linear Algebra?

- Linear algebra is the study of vectors and their transformations.
- A lot of objects in EECS can be treated as vectors and studied with linear algebra.
- Linearity is a good first-order approximation to the complicated real world.
- There exist good fast algorithms to do many of these manipulations in computers.
- Linear algebra concepts are an important tool for modeling the real world.

As you will see in the homeworks and labs, these concepts can be used to do many interesting things in real-world-relevant application scenarios. In the previous note, we introduced the idea that all information devices and systems (1) take some piece of information from the real world, (2) convert it to the electrical domain for measurement, and then (3) process these electrical signals. Because so many efficient algorithms exist that perform linear algebraic manipulations with computers, linear algebra is often a crucial component of this processing step.

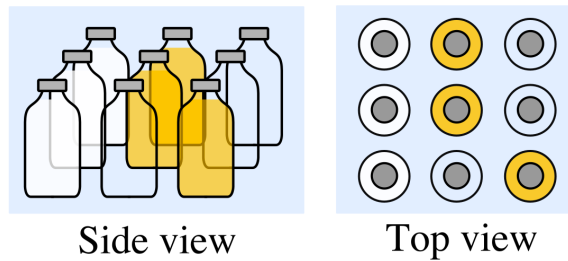
1.2 Application: Tomography

Throughout this course, we will motivate the introduction of concepts by considering a real-world application - this is the first one!

Tomography allows us to “see inside” a solid object, such as the human body or even the earth, by taking images section by section with a penetrating wave, such as X-rays. CT scans in medical imaging are perhaps the most famous such example — in fact, CT stands for “computed tomography.”

Let’s look at a specific toy example, using tomography to help with a (fairly unlikely!) real-world scenario.

A grocery store employee just had a truck load of bottles given to him. Each bottle is either empty, contains milk, or contains juice, and the bottles are packaged in boxes, with each box containing 9 bottles in a 3×3 grid. Inside a single box, it might look something like this:



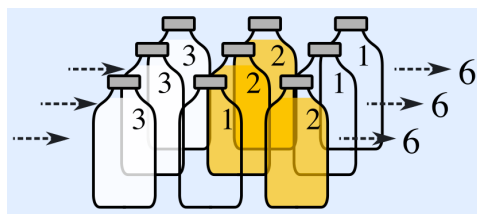
If we choose symbols such that M represents milk, J represents juice, and O represents an empty bottle, we can represent the stack of bottles shown above as follows:

$$\begin{array}{ccc} M & J & O \\ M & J & O \\ M & O & J \end{array} \quad (1)$$

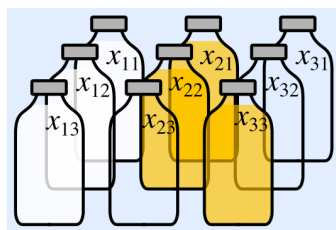
Imagine that our grocer cannot see directly into the box, but needs to determine its contents using a light source and light sensor. How can we help him do this?

Let the light source emit light with a certain known intensity. As the light passes through a bottle, its intensity diminishes by an amount that depends on the contents of the bottle - milk absorbs 3 units of light, juice absorbs 2 units of light and an empty bottle absorbs 1 unit of light. The box itself does not affect the intensity of the light. After the light emitted exits the box, we can use our sensor to measure the final intensity, and so determine the amount of light absorbed by each bottle.

Thus, if we shine light in a straight line through some bottles within the box, we can determine the total amount of light absorbed by the bottles as the sum of the light absorbed by each bottle. For instance, in our specific example, shining a light from left to right would look like this, with each row observed to absorb 6 total units of light:



In order to deal with this more generally, let's assign variables to the amount of light absorbed by each bottle:



This means that x_{11} would be the amount of light the top left bottle absorbs, x_{21} would be the amount of light the top middle bottle absorbs, and so forth. Shining the light from left to right for our specific example gives the following equations:

$$x_{11} + x_{21} + x_{31} = 6 \quad (2)$$

$$x_{12} + x_{22} + x_{32} = 6 \quad (3)$$

$$x_{13} + x_{23} + x_{33} = 6 \quad (4)$$

Similarly, we could consider shining a light from bottom to top:



Which would give the following equations:

$$x_{13} + x_{12} + x_{11} = 9 \quad (5)$$

$$x_{23} + x_{22} + x_{21} = 5 \quad (6)$$

$$x_{33} + x_{32} + x_{31} = 4 \quad (7)$$

Thus, we now know how to determine our observations given the contents of the box. But can we do the reverse? That is to say, given the amounts of light absorbed by each row and column of bottles, can we reconstruct the box's original contents?

From our above observations, one possible assignment of values to the x_{ij} (corresponding to the actual configuration of bottles) is

$$\begin{array}{ccc} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 3 & 1 & 2 \end{array} \quad (8)$$

However, the following assignment of values also works:

$$\begin{array}{ccc} 3 & 2 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{array} \quad (9)$$

which corresponds to a different configuration of bottles within the box. In other words, our observations are not sufficient to **uniquely** identify the configuration of bottles within the box. This is a problem!

Intuitively, if we can't identify an object in real-life from a set of observations, we make more observations - the same principle seems to apply here. To get these additional observations, we could shine light at different angles through the box.

This brings up some very natural questions: Would shining light through the diagonals of the box would provide us with enough information? If not, how many different directions do we need to shine light through before we are certain of the configuration of bottles? Do some measurements provide us with more information than others do? What happens as we vary the number of bottles in our box?

In Module 1 of this course, we will develop the tools to answer all of these questions.

1.3 What is a System of Equations?

A **system of equations** is nothing but a collection of one or more equations, expressed in terms of **functions**, **unknowns** (also called **variables**) and **constant terms**. In particular, if we are given functions $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ for $i = 1, \dots, m$ (this notation means that each f_i is a real function of n real variables) and constants $b_i \in \mathbb{R}$ for each $i = 1, \dots, m$, then we may write a corresponding system of equations

$$\begin{aligned} f_1(x_1, x_2, \dots, x_n) &= b_1 \\ f_2(x_1, x_2, \dots, x_n) &= b_2 \\ &\vdots \\ f_m(x_1, x_2, \dots, x_n) &= b_m, \end{aligned}$$

where x_1, \dots, x_n are the unknowns. A **solution** to the linear system is an assignment of values to the unknowns x_1, \dots, x_n such that all equations are simultaneously satisfied. In other words, when solving a system of equations, we try to answer the following question: *for what values x_1, \dots, x_n will the above equalities be satisfied?* Consider, for example, the tomography problem of the previous section. In the tomography problem, the functions f_i could be expressed as weighted sums of the input variables. The resulting system of equations was an example of what is called a **system of linear equations**. The key word here is “linear”, which distinguishes such systems from more general systems of equations. This is the topic of the next section.

1.4 What is a Linear Equation?

1.4.1 Intuition

In our bottle-sorting tomography example, we represented each measurement in a row or column as an equation. The collection of equations is an example of a **system of linear equations**, which summarizes the known relationships between the variables we want to solve for (x_{11}, x_{12}, x_{13} , etc.) and our measurements.

But what makes an equation “linear”? Essentially, a linear equation is one where each variable has degree 1. For instance, for unknowns x and y , the equation

$$5 \times x + 6 \times y = 7 \tag{10}$$

is made up of the two coefficients 5 and 6 multiplied by the two unknowns x and y respectively, summed together, that are then set equal to the scalar constant 7.

In contrast, the equation

$$y \times y = 5 \tag{11}$$

is not a linear equation, since we multiply our unknown y with itself, rather than a constant scalar.

Observe that equations such as

$$8x = 4y,$$

or

$$8x - 4y = 0,$$

or

$$2x - y = 0$$

are all examples of linear equations. Notice that the scalar constant is allowed to be 0.

1.4.2 Linear Equations

A system of equations is **linear** if each of the functions involved is *linear* in its variables. This statement seems a bit self-referential, but in fact it makes good sense once we define what it means for a function to be linear in its variables. To this end, we make the following definition:

Definition 1.1 (Linear Functions): A real-valued function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is *linear* if for all real-valued $\alpha, \beta, y_1, \dots, y_n, z_1, \dots, z_n$, the following identity holds:

$$f(\alpha y_1 + \beta z_1, \alpha y_2 + \beta z_2, \dots, \alpha y_n + \beta z_n) = \alpha f(y_1, \dots, y_n) + \beta f(z_1, \dots, z_n). \tag{12}$$

In words, a linear function satisfies the following two properties:

- i) *Homogeneity*: scaling the input to the function scales the output by the same amount.
- ii) *Superposition*: the function evaluated on the sum of two choices of input variables is equal to the sum of the function evaluated on each choice of input variables separately.

An extremely useful fact is that all linear functions can be represented as a weighted sum of the input variables. Let's state this important observation as a Theorem.

Theorem 1.1: If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is linear, then there exist coefficients c_1, c_2, \dots, c_n (i.e., real constants, not depending on the input to the function) such that

$$f(x_1, \dots, x_n) = c_1 x_1 + c_2 x_2 + \dots + c_n x_n \quad \text{for all } x_1 \in \mathbb{R}, \dots, x_n \in \mathbb{R}. \tag{13}$$

The expression on the right hand side of (13) is called a **linear combination** of the x_i 's. Hence, any linear function can be expressed as a linear combination of its inputs.

Why is this true? Well, the only thing we know at this point is the definition of linearity, so let's use it. In particular, we can start by cleverly rewriting $f(x_1, \dots, x_n)$ as

$$f(x_1, \dots, x_n) = f(\underbrace{x_1 \times 1 + 1 \times 0}_{x_1}, \underbrace{x_1 \times 0 + 1 \times x_2}_{x_2}, \underbrace{x_1 \times 0 + 1 \times x_3}_{x_3}, \dots, \underbrace{x_1 \times 0 + 1 \times x_n}_{x_n}). \tag{14}$$

Conveniently, this is precisely in the form of (12) with $\alpha = x_1$, $\beta = 1$, $(y_1, y_2, \dots, y_n) = (1, 0, \dots, 0)$ and $(z_1, z_2, \dots, z_n) = (0, x_2, \dots, x_n)$. Thus, we apply the definition of linearity to conclude

$$f(x_1, \dots, x_n) = x_1 f(1, 0, \dots, 0) + f(0, x_2, \dots, x_n). \quad (15)$$

We can apply the same trick again to see that

$$f(0, x_2, \dots, x_n) = x_2 f(0, 1, 0, \dots, 0) + f(0, 0, x_3, \dots, x_n) \quad (16)$$

and so forth. Putting it all together, we have

$$f(x_1, \dots, x_n) = x_1 f(1, 0, \dots, 0) + x_2 f(0, 1, 0, \dots, 0) + \dots + x_n f(0, 0, \dots, 0, 1). \quad (17)$$

Now, for each $i = 1, \dots, n$, we define $c_i := f(0, 0, \dots, 0, 1, 0, \dots, 0)$, where the 1 is in the i th position. This is just a real number, not depending on x_1, \dots, x_n , and therefore we have convinced ourselves that (13) is indeed true.

A **linear equation** is simply an equation of the form $f(x_1, \dots, x_n) = b$, where $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a linear function, $b \in \mathbb{R}$ is a given constant, and the x_1, \dots, x_n are real-valued unknowns (i.e., variables).

Let's consider a few examples. For variables x and y , the statement

$$5 \times x + 6 \times y = 7 \quad (18)$$

is a linear equation. Indeed, the left hand side is a linear function of x and y .

In contrast, the equation

$$y \times y = 5 \quad (19)$$

is not a linear equation since the function $f(y) = y^2$ is not linear (it is a good exercise to convince yourself of this fact).

Observe that equations such as

$$8x = 4y,$$

or

$$8x - 4y = 0,$$

or

$$2x - y = 0$$

are all examples of linear equations. Notice that the constant term in a linear equation is allowed to be 0.

1.4.3 Affine Functions

What about functions like

$$f_3(x) = 2x + 1, \quad x \in \mathbb{R}?$$

Plotting this function, we see that it is a line. But it doesn't seem to fit into the form $f(x) = cx$, so is it linear? A simple check, if we're ever unsure about the behavior of a function, is to plug in some simple input values

and see how the output behaves. Let's do that here, for $x = 1$ and $x = 2$. We see that

$$f_3(1) = 3 \text{ and } f_3(2) = 5,$$

so doubling the input value from 1 to 2 changes the output by a factor of $5/3$. Thus, this function is not linear, *even though* it describes the equation of a line. This motivates the following definition: A function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be an **affine function** if it can be written in the form

$$g(x_1, \dots, x_n) = f(x_1, \dots, x_n) + c_0 \quad \text{for all } x_1 \in \mathbb{R}, \dots, x_n \in \mathbb{R},$$

for some linear function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and constant term $c_0 \in \mathbb{R}$. By applying Theorem 1.1, we conclude that any affine function can be written as

$$g(x_1, \dots, x_n) = c_0 + c_1x_1 + c_2x_2 + \dots + c_nx_n.$$

Notice that the definition of affine functions includes all linear functions (by setting the scalar constant to 0), so every linear function is also affine, though not vice-versa. Nevertheless, *a system of equations involving all affine functions is still a system of linear equations.* (why?)

These definitions mean that while all functions describing a line can be shown to be affine, not all of them are linear. This has the unfortunate consequence that, in informal conversation, *affine* functions may be called *linear*, since both describe a line. This usage, though common, is **wrong**, as we saw with the example of f_3 .

1.5 Vectors and Matrices

We will now introduce some new notation that will help us deal with systems of linear equations in a more compact form.

1.5.1 Vectors

Definition 1.2 (Vector):

A vector is an ordered list of numbers. Suppose we have a collection of n real numbers: x_1, x_2, \dots, x_n . This collection can be written as a single point in an n -dimensional space, denoted as:

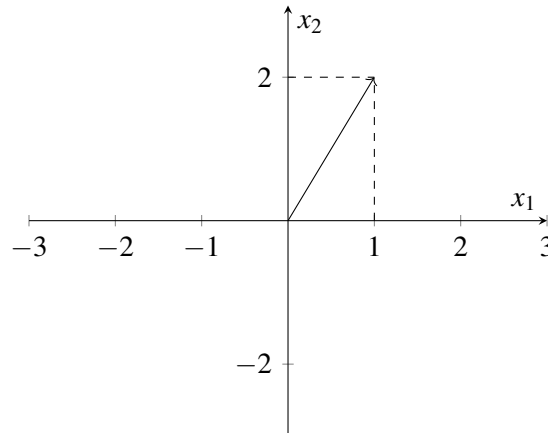
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \quad (20)$$

We call \vec{x} a **vector**. Because \vec{x} contains n real numbers, we can use the \in ("in" — i.e., is a member of) symbol to write $\vec{x} \in \mathbb{R}^n$ (\mathbb{R} represents the set of real numbers). If the elements of \vec{x} were complex numbers, we would write $\vec{x} \in \mathbb{C}^n$. Each x_i (for i between 1 and n) is called a **component**, or **element**, of the vector. The **size** of a vector is the number of components it contains (so the example vector is of size n , and the example below is of size two).

Example 1.1 (Vector of size two):

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

In the above example, \vec{x} is a vector with two components. Because the components are both real numbers, $\vec{x} \in \mathbb{R}^2$. We can represent the vector graphically on a 2-D plane, using the first element, x_1 , to denote the horizontal position of the vector and the second element, x_2 , to denote its vertical position:



Additional Resources For more on vectors, read pages 1-6 of *Strang* and try Problem Set 1.1.
Extra: Try reading the portions on linear combinations which generate a “space.”

Read more on vectors in *Schaum’s* on pages 1-3 and try Problems 1.1 to 1.6.

1.5.2 Matrices

Definition 1.3 (Matrix): A **matrix** is a rectangular array of numbers, written as:

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \quad (21)$$

Each A_{ij} (where i is the row index and j is the column index) is a **component**, or **element** of the matrix A .

Example 1.2 (4×3 Matrix):

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \\ 4 & 8 & 12 \end{bmatrix}$$

In the example above, A has $m = 4$ rows and $n = 3$ columns (a 4×3 matrix).

1.6 Representing Linear Systems using Matrices

In its most general form, a system of m linear equations involving n variables can be written as

$$\begin{aligned} f_1(x_1, x_2, \dots, x_n) &= b_1 \\ f_2(x_1, x_2, \dots, x_n) &= b_2 \\ &\vdots \\ f_m(x_1, x_2, \dots, x_n) &= b_m, \end{aligned}$$

where each f_i is a linear function, and b_i is a constant. From Theorem 1.1, we know that each linear function f_i can be expressed as a linear combination:

$$f_i(x_1, \dots, x_n) = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n$$

for appropriate coefficients a_{i1}, \dots, a_{in} . Hence, any system of m linear equations in n variables can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m. \end{aligned} \tag{22}$$

Generally, the a_{ij} 's and b_i 's are fixed constants, and we aim to find a solution x_1, \dots, x_n to the above system of equations (i.e., a choice of x_1, \dots, x_n that satisfies each of the above equations). Indeed, a good exercise is to note that the tomography problem fits into this framework. There, the a_{ij} 's are determined by the configuration of bottles we shine the light through, and the b_i 's are the light intensity we measure at the sensor. Moreover, each linear equation corresponds to a different measurement.

1.6.1 Augmented Matrices

Observe that the general expression for a system of linear equations given in (22) is a bit tedious to write. To address this, we introduce a notational device known as an **augmented matrix** to simplify things. The importance of this notational device goes beyond convenience, it will allow us to view the process of solving systems of equations as a sequence of simple modifications of the corresponding augmented matrix. More specifically, we arrange the coefficients a_{ij} and constants b_i in the system (22) into an array as follows:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right] \tag{23}$$

In the above, the entry in row i and column $j \leq n$ corresponds to the coefficient preceding variable x_j in linear equation i in (22). Similarly, the entry in row i and column n is b_i , corresponding to the constant in linear equation i . This representation of a linear system is known as the **augmented matrix representation**. An interesting thing to notice about this representation is that the symbols corresponding to our unknowns have vanished entirely! This reinforces what we know already: the label we assign to any given variable is

immaterial (e.g., we can call our variables x_1, \dots, x_n or y_1, \dots, y_n or u, v, w, \dots, z , the specific choice of label doesn't matter).

When representing a system of linear equations as an augmented matrix, we always draw a line before the last column as done above, since that column contains the constants, not the coefficients, corresponding to our system of linear equations. This is important: it distinguishes an augmented matrix (a notational device used to represent a system of equations) from a matrix (a mathematical object).

1.6.2 Matrix-Vector Form

Intuitively, we can imagine that augmented matrices convey the “underlying” linear system corresponding to our problem, without involving the actual variable names that we chose. Still, we might want to preserve the variable names, since without them we are losing some information about our problem. To do so, we can write our above system of equations in the following form:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{31} \\ x_{32} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \\ 9 \\ 5 \\ 4 \end{bmatrix}.$$

What's going on here? Compared to the augmented matrix form, notice that the constant terms have not been included into the main matrix (known as the *coefficient matrix*), but instead have been placed in a vector on the right-hand-side of an equality. In addition, observe that the unknowns have been placed in a vector that is right-multiplied with the coefficient matrix on the left-hand-side of the equality. Often, the letter A is used to represent the coefficient matrix, \vec{x} for the vector of unknowns, and \vec{b} for the vector of constants.

After these substitutions, the above equation becomes simply

$$\begin{aligned}
 A &= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \\
 \vec{x} &= \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{31} \\ x_{32} \\ x_{33} \end{bmatrix} \\
 \vec{b} &= \begin{bmatrix} 6 \\ 6 \\ 6 \\ 9 \\ 5 \\ 4 \end{bmatrix} \\
 \implies A\vec{x} &= \vec{b}.
 \end{aligned}$$

The above representation is fundamentally an alternative shorthand for systems of linear equations that preserves the variable names of the original system.

1.7 Practice Problems

1.7.1 Mechanical Practice

Mechanical practice problems are also available in an interactive form on the course website, along with their solutions.

1. Is $x + 2y = 4z$ linear?
2. Is $\sin x - 2 = 6$ linear?
3. Is $\sum_{i=1}^{50} i \cdot x - e^{-3}y = \sin \frac{\pi}{3}$ linear?
4. Write $\begin{cases} 2x - 3y = 1 \\ 3x + y = -2 \end{cases}$ in matrix form.

$$(a) \begin{bmatrix} 2 & -3 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{aligned} \text{(b)} \quad & \begin{bmatrix} 2 & 3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ \text{(c)} \quad & \begin{bmatrix} 2 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ \text{(d)} \quad & \begin{bmatrix} 2 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

1.7.2 Homework / Exam Practice

1. Tyler's Optimal Tea

Learning Goal: *Recognize a problem that can be cast as a system of linear equations.*

Tyler's Optimal Tea has a unique way of serving its customers. To ensure the best customer experience, each customer gets a combination drink personalized to their tastes. Tyler knows that a lot of customers don't know what they want, so when customers walk up to the counter, they are asked to taste four standard combination drinks that each contain a different mixture of the available pure teas.

Each combination drink (Classic, Roasted, Mountain, and Okinawa) is made of a mixture of pure teas (Black, Oolong, Green, and Earl Grey), with the total amount of pure tea in each combination drink always the same, and equal to one cup. The table below shows the quantity of each pure tea (Black, Oolong, Green, and Earl Grey) contained in each of the four standard combination drinks (Classic, Roasted, Mountain, and Okinawa).

Tea [cups]	Classic	Roasted	Mountain	Okinawa
Black	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{2}{3}$
Oolong	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$
Green	0	$\frac{1}{3}$	$\frac{3}{5}$	0
Earl Grey	$\frac{1}{3}$	0	0	0

Initially, the customer's ratings for each of the pure teas are unknown. Tyler's goal is to determine how much the customer likes each of the pure teas, so that an optimal combination drink can then be made. By letting the customer taste and score each of the four standard combination drinks, Tyler can use linear algebra to determine the customer's initially unknown ratings for each of the pure teas. After a customer gives a score (all of the scores are real numbers) for each of the four standard combination drinks, Tyler then calculates how much the customer likes each pure tea and mixes up a special combination drink that will maximize the customer's score.

The score that a customer gives for a combination drink is a linear combination of the ratings of the constituent pure teas, based on their proportion. For example, if a customer's rating for black tea is 6 and oolong tea is 3, then the total score for the Okinawa Tea drink would be $6 \cdot \frac{2}{3} + 3 \cdot \frac{1}{3} = 5$ because Okinawa has $\frac{2}{3}$ black tea and $\frac{1}{3}$ Oolong tea.

Professor Ranade was thirsty after giving the first lecture, so Professor Ranade decided to take a drink break at Tyler's Optimal Tea. Professor Ranade walked in and gave the following ratings:

Combination Drink	Score
Classic	7
Roasted	7
Mountain	$7\frac{2}{5}$
Okinawa	$6\frac{1}{3}$

1. What were Professor Ranade's ratings for each tea? **Work this problem out by hand in terms of the steps. You may use a calculator to do algebra.**
2. What mystery tea combination could Tyler put in Professor Ranade's personalized drink to maximize the customer's score? If there is more than one correct answer, state that there are many answers, and give one such combination. What score would Professor Ranade give for the answer you wrote down? Assume the total amount of tea must be one cup.